ADVANCED MATHEMATICS

Trigonometry (Adv), T3 Trig Functions and Graphs (Adv)

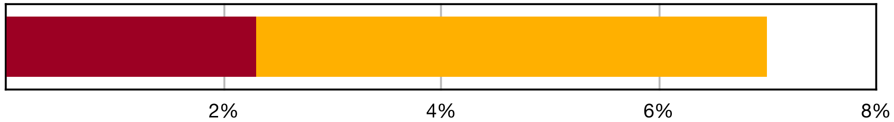
Trig Graphs (Y12)

Trig Applications (Y12)

Teacher: Cathyanne Horvat

Exam Equivalent Time: 90 minutes *(based on allocation of 1.5 minutes per mark)*

T3 Trig Functions and Graphs



Trig Graphs
Trig Applications

***SmarterMaths analytics based on the average contribution to new syllabus Advanced Maths exams since 2020.**

HISTORICAL CONTRIBUTION

- *T3 Trig Functions and Graphs* has contributed a meaningful 7.0% per new syllabus *Adv* exam since it was introduced in 2020.
- This topic has been split into two sub-topics for analysis purposes: *1-Trig Graphs (2.3%), and 2-Trig Applications (4.3%).*
- This analysis looks at the sub-topic *Trig Graphs*.

HSC ANALYSIS - What to expect and common pitfalls

- *Trig Graphs* have been examined in each of the new syllabus exams with a multiple choice in 2020 and longer answer questions in 2021-22 (worth 2 marks each).
- The database size reflects our view that this area will be consistently examined going forward and numerous examples of all trig graph variations are covered.
- We recommend close revision of *T3 EQ-Bank 3 and 5* which are informed by the question style and difficulty level of NESA's sample questions. Also, special attention should be given to *2013 HSC 6 MC* which was surprisingly poorly answered.
- Note that more than half of students answered the 2016 multiple choice question on a tan function graph's *period* incorrectly. Deserves attention.

Questions

1. Trigonometry, 2ADV T3 2020 HSC 6 MC

Which interval gives the range of the function $y = 5 + 2\cos 3x$?

A. $[2, 8]$

B. $[3, 7]$

C. $[4, 6]$

D. $[5, 9]$

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2. Trigonometry, 2ADV T3 SM-Bank 18 MC

The period of the function $f(x) = \tan\left(\dfrac{\pi x}{2}\right)$ is

A. 2

B. 4

C. 2π

D. 4π

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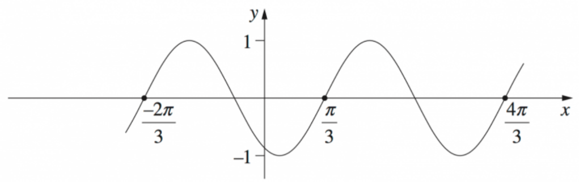
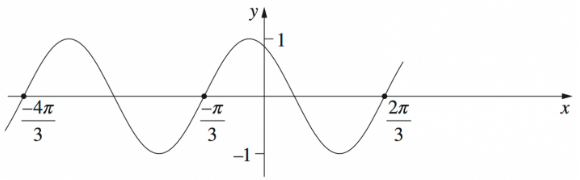
3. Trigonometry, 2ADV T3 2013 HSC 6 MC

Which diagram shows the graph $y = \sin\left(2x + \frac{\pi}{3}\right)$?

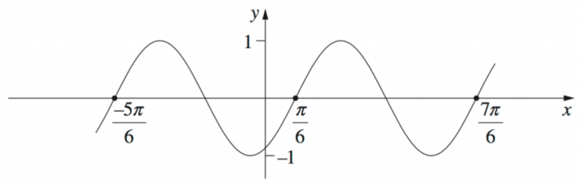
(A)



(B)



(C)



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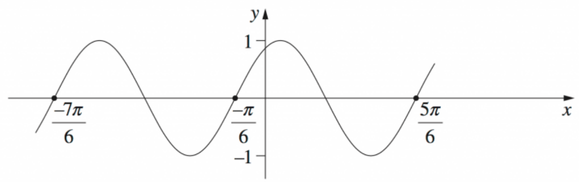


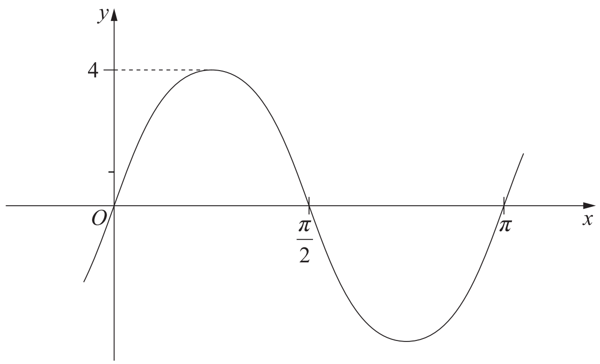
4. Trigonometry, 2ADV T3 EQ-Bank 5

The function $f(x) = \sin x$ is transformed into the function $g(x) = \dfrac{\sin(4x)}{3}$.

Describe in words how the amplitude and period have changed in this transformation. ***(2 marks)***

5. Trigonometry, 2ADV T3 2010 HSC 8c

The graph shown is $y = A\sin bx$.



i. Write down the value of A. *(1 mark)*

ii. Find the value of b. *(1 mark)*

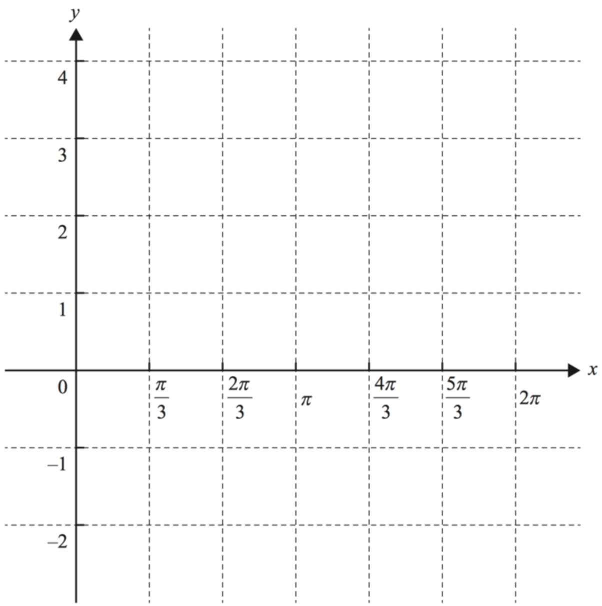
iii. On the same set of axes, draw the graph $y = 3\sin x + 1$ for $0 \leq x \leq \pi$. *(2 marks)*

6. Trigonometry, 2ADV T3 SM-Bank 9

Let $f(x) = 2\cos(x) + 1$ for $0 \leq x \leq 2\pi$.

i. Solve the equation $2\cos(x) + 1 = 0$ for $0 \leq x \leq 2\pi$. *(2 marks)*

ii. Sketch the graph of the function $f(x)$ on the axes below. Label the endpoints and local minimum point with their coordinates. *(3 marks)*

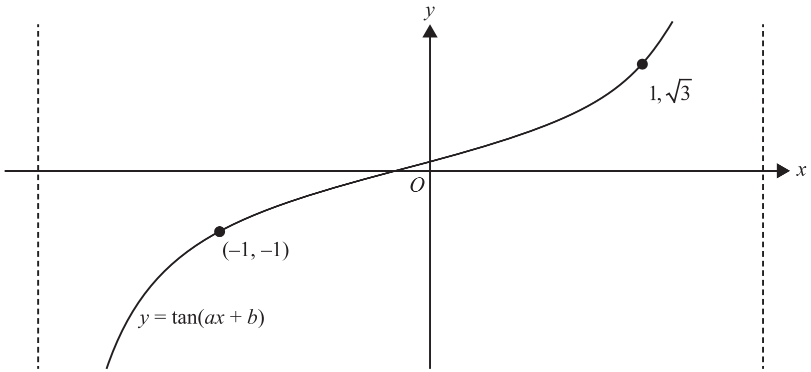


7. Trigonometry, 2ADV T3 2017 HSC 14a

Sketch the curve $y = 4 + 3\sin 2x$ for $0 \le x \le 2\pi$. *(3 marks)*

8. Trigonometry, 2ADV T3 2010 MET1 3

Shown below is part of the graph of a period of the function of the form $y = \tan(ax + b)$.



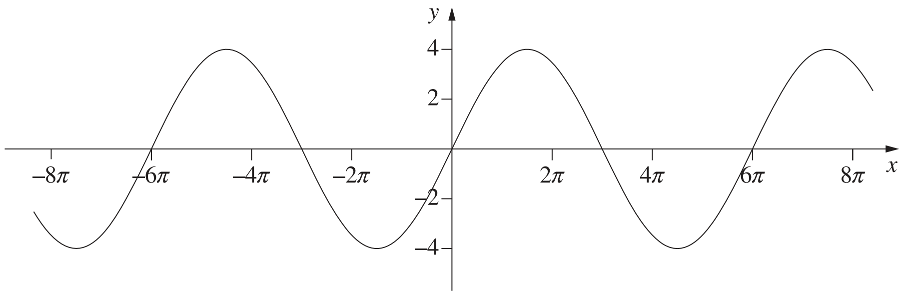
Find the value of a and the value of b, where $a > 0$ and $0 < b < 1$. *(3 marks)*

9. Trigonometry, 2ADV T3 2021 HSC 20

For what values of x, in the interval $0 \leq x \leq \frac{\pi}{4}$, does the line $y = 1$ intersect the graph of $y = 2\sin 4x$? *(2 marks)*

10. Trigonometry, 2ADV T3 2022 HSC 14

The graph of $y = k\sin(ax)$



What are the values of k and a? *(2 marks)*

11. Trigonometry, 2ADV T3 SM-Bank 13

On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), \quad 0 \leq t \leq 24$$

where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

i. Find the minimum depth of the water in the river. *(1 mark)*

ii. Find the values of t for which $h(t) = 10$. *(2 marks)*

12. Trigonometry, 2ADV T3 2018 HSC 15a

The length of daylight, $L(t)$, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation

$$L(t) = 12 + 2\cos\left(\frac{2\pi t}{366}\right),$$

where t is the number of days after 21 December 2015, for $0 \le t \le 366$.

i. Find the length of daylight on 21 December 2015. ***(1 mark)***

ii. What is the shortest length of daylight? ***(1 mark)***

iii. What are the two values of t for which the length of daylight is 11? ***(2 marks)***

13. Trigonometry, 2ADV T3 SM-Bank 10

The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2018.

i. Find the period and amplitude of the function n. ***(2 marks)***

ii. Find the maximum and minimum populations of wombats in this location. ***(2 marks)***

iii. Find $n(10)$. ***(1 mark)***

iv. Over the 12 months from 1 March 2018, find the fraction of time when the population of wombats in this location was less than $n(10)$. ***(2 marks)***

14. Trigonometry, 2ADV T3 2013 HSC 13a

The population of a herd of wild horses is given by

$$P(t) = 400 + 50\cos\left(\frac{\pi}{6}t\right)$$

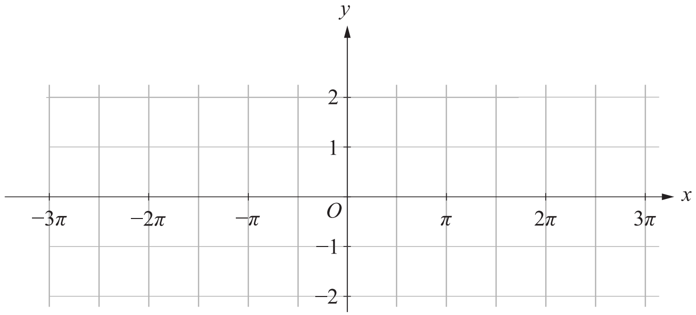
where t is time in months.

i. Find all times during the first 12 months when the population equals 375 horses. ***(2 marks)***

ii. Sketch the graph of $P(t)$ for $0 \leq t \leq 12$. ***(2 marks)***

15. Trigonometry, 2ADV T3 EQ-Bank 3

By drawing graphs on the number plane, show how many solutions exist for the equation $\cos x = \left|\dfrac{x-\pi}{4}\right|$ in the domain $(-\infty, \infty)$ *(3 marks)*



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16. Trigonometry, 2ADV T3 SM-Bank 8

$f(x) = 2\sin(2x)$ is defined in the domain $\left\{x\colon \dfrac{\pi}{8} \le x < \dfrac{\pi}{3}\right)$

What is the range of the function $f(x)$? *(2 marks)*

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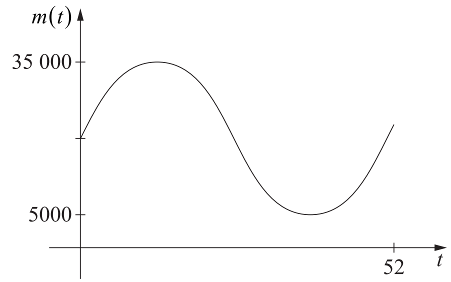
17. Trigonometry, 2ADV T3 2020 HSC 31

The population of mice on an isolated island can be modelled by the function.

$$m(t) = a\sin\left(\frac{\pi}{26}t\right) + b,$$

where t is the time in weeks and $0 \le t \le 52$. The population of mice reaches a maximum of 35 000 when $t = 13$ and a minimum of 5000 when $t = 39$. The graph of $m(t)$ is shown.



a. What are the values of a and b? *(2 marks)*

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b. On the same island, the population of cats can be modelled by the function

$$c(t) = -80\cos\left(\frac{\pi}{26}(t-10)\right) + 120$$

Consider the graph of $m(t)$ and the graph of $c(t)$.
Find the values of $t,\ 0 \le t \le 52$, for which both populations are increasing. *(3 marks)*

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c. Find the rate of change of the mice population when the cat population reaches a maximum. ***(2 marks)***

18. Trigonometry, 2ADV T3 2009 HSC 7b

Between 5 am and 5 pm on 3 March 2009, the height, h, of the tide in a harbour was given by

$$h = 1 + 0.7\sin\left(\frac{\pi}{6}t\right) \quad \text{for} \quad 0 \le t \le 12$$

where h is in metres and t is in hours, with $t = 0$ at 5 am.

i. What is the period of the function h? ***(1 mark)***

ii. What was the value of h at low tide, and at what time did low tide occur? ***(2 marks)***

iii. A ship is able to enter the harbour only if the height of the tide is at least 1.35 m.
Find all times between 5 am and 5 pm on 3 March 2009 during which the ship was able to enter the harbour. ***(3 marks)***

Worked Solutions

1. Trigonometry, 2ADV T3 2020 HSC 6 MC

$-1 \leq \cos 3x \leq 1$
$-2 \leq 2\cos 3x \leq 2$
$3 \leq 5 + 2\cos 3x \leq 7$
$\therefore$ Range $[3, 7]$
$\Rightarrow B$

2. Trigonometry, 2ADV T3 SM-Bank 18 MC

$n = \dfrac{\pi}{2}$

$\text{Period} = \dfrac{\pi}{n} = \dfrac{\pi}{\frac{\pi}{2}} = 2$

$\Rightarrow A$

3. Trigonometry, 2ADV T3 2013 HSC 6 MC

♦♦ *Mean mark 34%*

At $x = 0,\ y = \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$
$\Rightarrow$ It cannot be A or C

Find x when $y = 0$,

$\sin\left(2x + \dfrac{\pi}{3}\right) = 0$

$\therefore\ 2x + \dfrac{\pi}{3} = 0 \quad (\sin 0 = 0)$

$2x = -\dfrac{\pi}{3}$

$x = -\dfrac{\pi}{6}$

$\Rightarrow D$

4. Trigonometry, 2ADV T3 EQ-Bank 5

$g(x) = \dfrac{1}{3}\sin(4x)$
$\Rightarrow$ The new amplitude is one third of the original amplitude.

$\text{Period} = \dfrac{2\pi}{n} \Rightarrow n = \dfrac{1}{4}$
$\Rightarrow$ The new period is one quarter of the original period.

Worked Solutions

5. Trigonometry, 2ADV T3 2010 HSC 8c

i. $A = 4$

ii. Since the graph passes through $\left(\frac{\pi}{4}, 4\right)$

Substituting into $y = 4\sin bx$

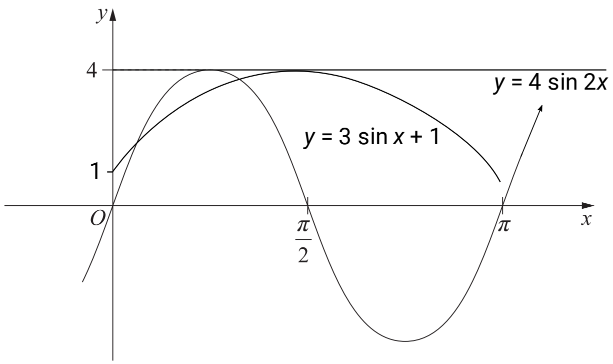
$$4\sin\left(b \times \frac{\pi}{4}\right) = 4$$

$$\sin\left(b \times \frac{\pi}{4}\right) = 1$$

$$b \times \frac{\pi}{4} = \frac{\pi}{2}$$

$$\therefore b = 2$$

iii.



MARKER'S COMMENT: Graphs are consistently drawn too small by many students. Aim to make your diagrams 1/3 to 1/2 of a page.

6. Trigonometry, 2ADV T3 SM-Bank 9

i.

$$2\cos(x) + 1 = 0$$

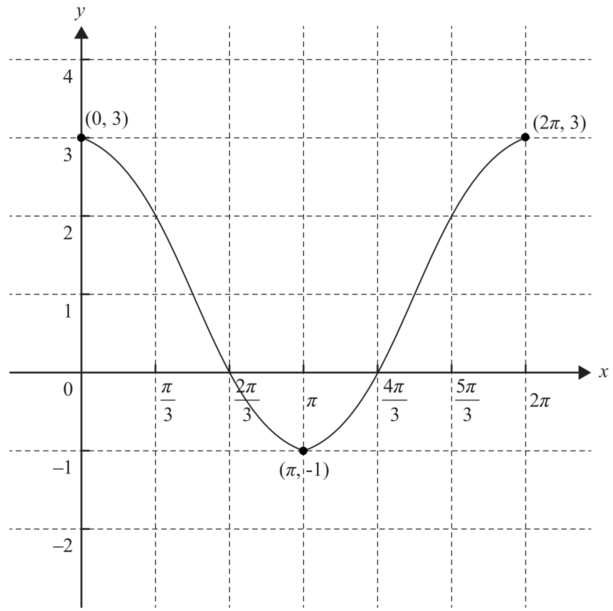
$$\cos(x) = -\frac{1}{2}$$

$\Rightarrow \cos\frac{\pi}{3} = \frac{1}{2}$ and cos is negative in 2nd/3rd quadrant

$$\therefore x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}, \frac{4\pi}{3}$$

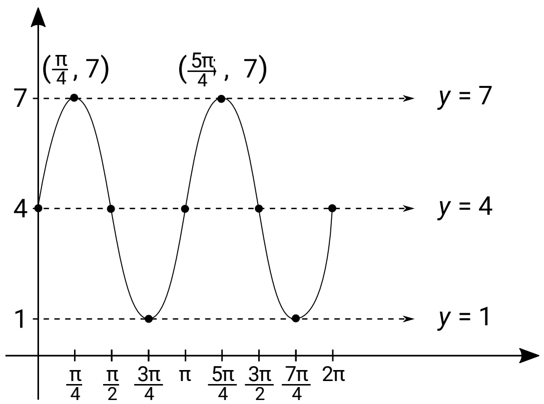
ii.



7. Trigonometry, 2ADV T3 2017 HSC 14a

$y = 4 + 3\sin 2x$

$\Rightarrow$ Amplitude of 3 about $y = 4$



8. Trigonometry, 2ADV T3 2010 MET1 3

$y = \tan(ax + b)$

Substitute $\left(1, \sqrt{3}\right), (-1, -1)$ into equation:

$$\tan(a+b) = \sqrt{3}$$
$$\tan(b-a) = -1$$
$$a + b = \frac{\pi}{3} \ \dots (1)$$
$$b - a = -\frac{\pi}{4} \ \dots (2)$$

Add (1) + (2):

$$2b = \frac{\pi}{3} - \frac{\pi}{4}$$
$$b = \frac{\pi}{24}$$

Substitute into (1):

$$a + \frac{\pi}{24} = \frac{\pi}{3}$$
$$a = \frac{7\pi}{24}$$

9. Trigonometry, 2ADV T3 2021 HSC 20

Find x such that:

$$2\sin 4x = 1$$
$$\sin 4x = \frac{1}{2}$$
$$4x = \sin^{-1}\frac{1}{2}$$
$$4x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \dots$$
$$\therefore x = \frac{\pi}{24}, \frac{5\pi}{24} \quad \left(0 \le x \le \frac{\pi}{4}\right)$$

10. Trigonometry, 2ADV T3 2022 HSC 14

Amplitude $= 4$

$\Rightarrow k = 4$

Period $= 6\pi$

$$\frac{2\pi}{a} = 6\pi$$
$$6\pi a = 2\pi$$
$$\Rightarrow a = \frac{1}{3}$$

11. Trigonometry, 2ADV T3 SM-Bank 13

i. h_{min} occurs when $\sin\left(\frac{\pi t}{12}\right) = -1$

$$\therefore h_{\text{min}} = 14 - 8$$
$$= 6 \text{ m}$$

MARKER'S COMMENT: Students who used calculus to find the minimum were less successful.

ii. $14 + 8\sin\left(\frac{\pi}{12}t\right) = 10$

$$\sin\left(\frac{\pi}{12}t\right) = -\frac{1}{2}$$

Solve in general:

$$\frac{\pi}{12}t = \frac{7\pi}{6} + 2\pi n \quad \text{or} \quad \frac{\pi}{12}t = \frac{11t}{6} + 2\pi n,$$
$$t = 14 + 24n \qquad\qquad t = 22 + 24n$$

Substitute integer values for n,

$\therefore t = 14$ or 22, $\quad (0 \le t \le 24)$

12. Trigonometry, 2ADV T3 2018 HSC 15a

i. $L(t) = 12 + 2\cos\left(\dfrac{2\pi t}{366}\right)$

On 21 Dec 2015 $\Rightarrow t = 0$

$$\begin{aligned}\therefore L(0) &= 12 + 2\cos 0 \\ &= 14 \text{ hours}\end{aligned}$$

ii. Shortest length of daylight occurs when

$$\cos\left(\frac{2\pi t}{366}\right) = -1$$

♦ *Mean mark 43%.*

$$\begin{aligned}\therefore \text{Shortest length} &= 12 + 2(-1) \\ &= 10 \text{ hours}\end{aligned}$$

iii. Find t such that $L(t) = 11$:

$$11 = 12 + 2\cos\left(\frac{2\pi t}{366}\right)$$

$$\cos\left(\frac{2\pi t}{366}\right) = -\frac{1}{2}$$

$$\frac{2\pi t}{366} = \frac{2\pi}{3} \quad \text{or} \quad \frac{2\pi t}{366} = \frac{4\pi}{3}$$

$$t = \frac{366}{3} \qquad t = \frac{366 \times 2}{3}$$

$$= 122 \qquad = 244$$

$$\therefore t = 122 \text{ or } 244$$

13. Trigonometry, 2ADV T3 SM-Bank 10

i. $\text{Period} = \dfrac{2\pi}{n} = \dfrac{2\pi}{\frac{\pi}{3}} = 6 \text{ months}$

$\text{Amplitude} = 400$

ii. Max: $1200 + 400 = 1600$ wombats

Min: $1200 – 400 = 800$ wombats

iii.
$$\begin{aligned}n(10) &= 1200 + 400\cos\left(\frac{10\pi}{3}\right) \\ &= 1200 + 400\cos\left(\frac{2\pi}{3}\right) \\ &= 1200 - 400 \times \frac{1}{2} \\ &= 1000 \text{ wombats}\end{aligned}$$

iv. Find t when $n(t) = 1000$

$$1000 = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$$

$$\cos\left(\frac{\pi t}{3}\right) = -\frac{1}{2}$$

$$\frac{\pi t}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \ldots$$

$$t = 2, 4, 8, 10$$

Since $n(0) = 1600$,

$\Rightarrow n(t)$ drops below 1000 between $t = 2$ and $t = 4$,

and between $t = 8$ and $t = 10$.

$$\begin{aligned}\therefore \text{Fraction} &= \frac{2+2}{12} \\ &= \frac{1}{3} \text{ year}\end{aligned}$$

14. Trigonometry, 2ADV T3 2013 HSC 13a

i. $P(t)=400+50\cos\left(\frac{\pi}{6}t\right)$

Need to find t when $P(t)=375$

$$375=400+50\cos\left(\frac{\pi}{6}t\right)$$

$$50\cos\left(\frac{\pi}{6}t\right)=-25$$

$$\cos\left(\frac{\pi}{6}t\right)=-\frac{1}{2}$$

Since $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$, and cos is negative in $2^{\text{nd}}/3^{\text{rd}}$ quadrants:

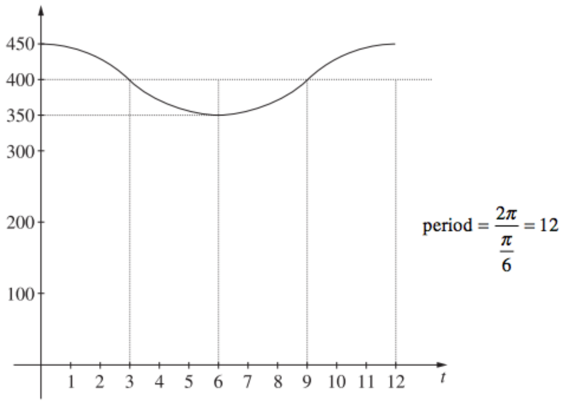
$$\Rightarrow \frac{\pi}{6}t=\left(\pi-\frac{\pi}{3}\right),\ \left(\pi+\frac{\pi}{3}\right),\ \left(3\pi-\frac{\pi}{3}\right)$$

$$=\frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{8\pi}{3},\ \ldots$$

$$\therefore t=4,\ 8,\ 16,\ \ldots$$

$\therefore$ In the 1st 12 months, $P(t)=375$ when $t=4$ months and 8 months.

ii.



♦ *Mean mark 39%*

15. Trigonometry, 2ADV T3 EQ-Bank 3

Sketch:

$y=\cos x$

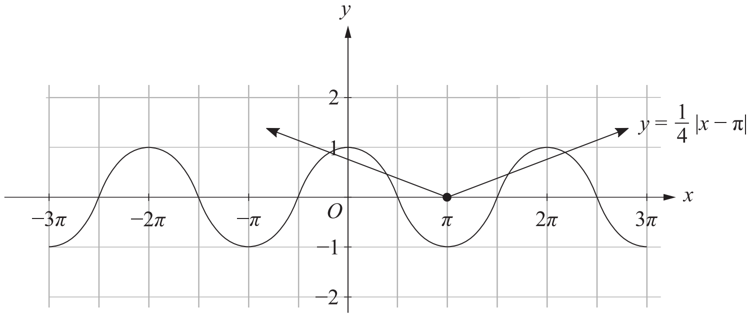
$y=\left|\frac{x-\pi}{4}\right|$

Translate π units to the right:

$y=|x| \ \Rightarrow\ y=|x-\pi|$

Multiply by $\frac{1}{4}$:

$y=|x-\pi| \ \Rightarrow\ y=\frac{1}{4}|x-\pi|=\left|\frac{x-\pi}{4}\right|$



$\therefore$ There are 4 solutions.

16. Trigonometry, 2ADV T3 SM-Bank 8

♦ Mean mark 45%.

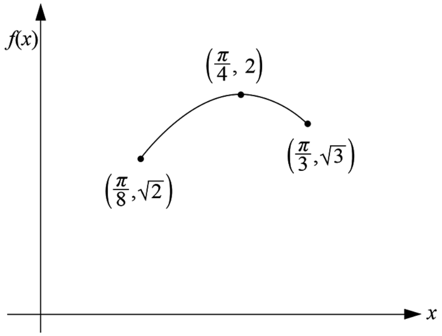
$\sin(2x)_{\max}$ occurs when $x = \dfrac{\pi}{4}$ (within domain)

$\Rightarrow f(x)_{\max} = 2\sin\left(\dfrac{\pi}{2}\right) = 2$

Checking endpoints:

When $x = \dfrac{\pi}{8} \Rightarrow y = 2\sin\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

When $x = \dfrac{\pi}{3} \Rightarrow y = 2\sin\left(\dfrac{2\pi}{3}\right) = \sqrt{3}$



$\therefore$ Range $= \left[\sqrt{2}, 2\right]$,

17. Trigonometry, 2ADV T3 2020 HSC 31

a. $b = \dfrac{35\ 000 + 5000}{2}$

$= 20\ 000$

$a =$ amplitude of sin graph

$= 35\ 000 – 20\ 000$

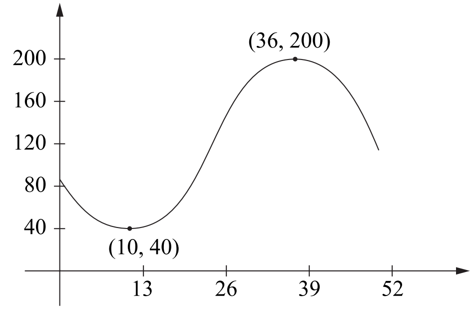
$= 15\ 000$

b. By inspection of the $m(t)$ graph

♦♦ Mean mark part (b) 30%.

$m\prime(t) > 0$ when $0 \le t < 13$ and $39 < t \le 52$

Sketch $c(t)$:



Minimum $(\cos 0)$ when $t = 10$

Maximum $(\cos\pi)$ when $t = 36$

$\therefore c\prime(t) > 0$ when $10 < t < 36$

$\therefore$ Both populations are increasing when $10 < t < 13$

c. $c(t)$ maximum when $t = 36$

♦♦♦ Mean mark part (c) 27%.

$$m(t) = 15\ 000\sin\left(\frac{\pi}{26}t\right) + 20\ 000$$

$$m\prime(t) = \frac{15\ 000\pi}{26}\cos\left(\frac{\pi}{26}t\right)$$

$$m\prime(36) = \frac{15\ 000\pi}{26} \cdot \cos\left(\frac{36\pi}{26}\right)$$

$$= -\ 642.7$$

$\therefore$ Mice population is decreasing at 643 mice per week.

18. Trigonometry, 2ADV T3 2009 HSC 7b

i. $h = 1 + 0.7\sin\left(\frac{\pi}{6}t\right) \quad \text{for } 0 \le t \le 12$

$$T = \frac{2\pi}{n} \quad \text{where } n = \frac{\pi}{6}$$
$$= 2\pi \times \frac{6}{\pi}$$
$$= 12 \text{ hours}$$

$\therefore$ The period of h is 12 hours.

ii. Find h at low tide

$\Rightarrow h$ will be a minimum when

$$\sin\left(\frac{\pi}{6}t\right) = -1$$

$$\therefore\ h_{\text{min}} = 1 + 0.7(-1)$$
$$= 0.3 \text{ metres}$$

IMPORTANT: Using $\sin x = -1$ for a minimum here is very effective and time efficient. This property of trig functions is ***often very useful*** in harder questions.

Since $\sin x = -1$ when $x = \frac{3\pi}{2}$

$$\frac{\pi}{6}t = \frac{3\pi}{2}$$
$$t = \frac{3\pi}{2} \times \frac{6}{\pi}$$
$$= 9 \text{ hours}$$

$\therefore$ Low tide occurs at 2pm (5 am + 9 hours)

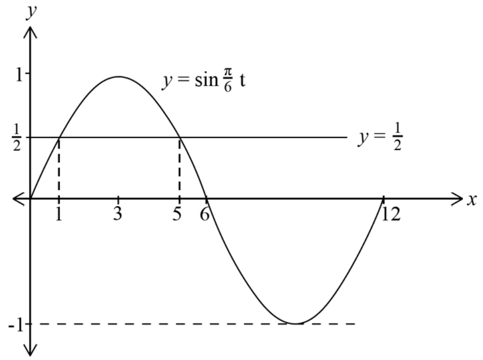
iii. Find t when $h \ge 1.35$

$$1 + 0.7\sin\left(\frac{\pi}{6}t\right) \ge 1.35$$
$$0.7\sin\left(\frac{\pi}{6}t\right) \ge 0.35$$
$$\sin\left(\frac{\pi}{6}t\right) \ge \frac{1}{2}$$

$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$ when

$$\frac{\pi}{6}t = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{13\pi}{6},\ \text{etc}\ldots$$

$$t = 1,\ 5 \qquad (0 \le t \le 12)$$



From the graph,

$$\sin\left(\frac{\pi}{6}t\right) \ge \frac{1}{2} \quad \text{when} \quad 1 \le t \le 5$$

$\therefore$ Ship can enter the harbour between 6 am and 10 am.