EXERCISE 17.1 AREA UNDER A CURVE

2 **(a)** Follow Example 2.

$y = x^2$

There is one subinterval is from $x = 1$ to $x = 3$. The subintervals will have a width of 2.

x	1	3
y	1	9

The smallest value is $x = 1,\ y = 1,\ A = 2 \times 1 = 2$

The largest value is $x = 3,\ y = 9,\ A = 9 \times 1 = 18$

Hence $2 < A < 18$.

Taking the average of these two measurements gives $A \approx \frac{2+18}{2} = 10$.

(b) There are two subintervals from $x = 1$ to $x = 3$. The subintervals will have a width of 1.

x	1	2	3
y	1	4	9

For $1 \le x \le 2,$ the smallest value is $x = 1,\ y = 1,$ the largest value is $x = 2,\ y = 4.$

For $2 \le x \le 3,$ the smallest value is $x = 2,\ y = 4$, the largest value is $x = 3,\ y = 9.$

$$\begin{aligned}&(1+4)\times 1 < A < (4+9)\times 1\\&5 < A < 13\end{aligned}$$

Taking the average of these two measurements gives $A \approx \frac{5+13}{2} = 9$.

(c) Four subintervals from $x = 1$ to $x = 3$. The subintervals will have a width of 0.5.

x	1	1.5	2	2.5	3
y	1	2.25	4	6.25	9

For $1 \le x \le 1.5,$ the smallest value is $x = 1,\ y = 1,$ the largest value is $x = 1.5,\ y = 2.25.$

For $1.5 \le x \le 2,$ the smallest value is $x = 1.5,\ y = 2.25,$ the largest value is $x = 2,\ y = 4.$

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For $2 \le x \le 2.5$, the smallest value is $x = 2$, $y = 4$, the largest value is $x = 2.5$, $y = 6.25$.

For $2.5 \le x \le 3$, the smallest value is $x = 2.5$, $y = 6.25$, the largest value is $x = 3$, $y = 9$.

$$\left(1+2.25+4+6.25\right)\times 0.5 \le A \le \left(2.25+4+6.25+9\right)\times 0.5$$
$$6.75 \le A \le 10.75$$

Taking the average of these two measurements gives $A \approx \dfrac{6.75+10.75}{2} = 8.75$.

4 **(a)** $y = x+1$

There is one subinterval from $x=1$ to $x=3$. The subintervals will have a width of 2.

x	1	3
y	2	4

The smallest value is $x=1$, $y=2$, $A = 2\times 2 = 4$

The largest value is $x=3$, $y=4$, $A = 4\times 2 = 8$

Hence $4 < A < 8$.

Taking the average of these two measurements gives $A \approx \dfrac{4+8}{2} = 6$.

(b) There are two subintervals from $x=1$ to $x=3$. The subintervals will have a width of 1.

x	1	2	3
y	2	3	4

For $1 \le x \le 2$, the smallest value is $x=1$, $y=2$, the largest value is $x=2$, $y=3$.

For $2 \le x \le 3$, the smallest value is $x=2$, $y=3$, the largest value is $x=3$, $y=4$.

$$\left(2+3\right)\times 1 < A < \left(3+4\right)\times 1$$
$$5 < A < 7$$

Taking the average of these two measurements gives $A \approx \dfrac{5+7}{2} = 6$.

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(c) Four subintervals from $x=1$ to $x=3$. The subintervals will have a width of 0.5.

x	1	1.5	2	2.5	3
y	2	2.5	3	3.5	4

For $1\le x\le 1.5$, the smallest value is $x=1$, $y=2$, the largest value is $x=1.5$, $y=2.5$.

For $x=1.5$, $y=2$, the smallest value is $x=1.5$, $y=2.5$, the largest value is $x=2$, $y=3$.

For $2\le x\le 2.5$, the smallest value is $x=2$, $y=3$, the largest value is $x=2.5$, $y=3.5$.

For $2.5\le x\le 3$, the smallest value is $x=2.5$, $y=3.5$, the largest value is $x=3$, $y=4$.

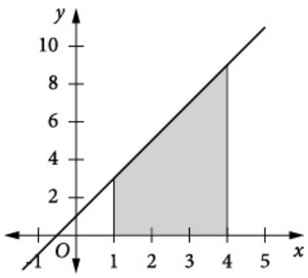
$$\left(2+2.5+3+3.5\right)\times 0.5\le A\le\left(2.5+3+3.5+4\right)\times 0.5$$
$$11\le A\le 13$$

Taking the average of these two measurements gives $A\approx\dfrac{11+13}{2}=6$.

EXERCISE 17.2 THE DEFINITE INTEGRAL AND THE AREA UNDER A CURVE

2 Write the x values of the ends of the integral in the correct notation, the upper end on top and the lower end beneath.

$$\int_1^4\left(2x+1\right)dx$$



The required area is a trapezium. Its width (or 'height') is $4-1=3$ units and the parallel sides are the y values at the ends of the interval.

When $x=1$, $y=2\times 1+1=3$, and when $x=4$, $y=2\times 4+1=9$.

$$A=\frac{a+b}{2}\times h=\frac{3+9}{2}\times 3=18\ \text{units}^2$$

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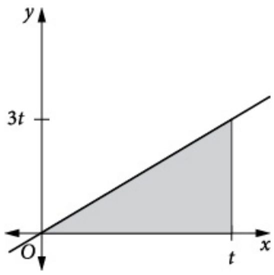
4 Write the x values of the ends of the integral in the correct notation, the upper end (t) on top and the lower end beneath.

$$\int_0^t 3x\,dx$$

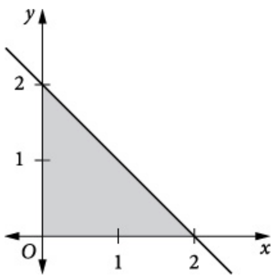


The required area is a triangle, base t and height $y = 3x = 3t$.

$$A = \frac{1}{2} \times t \times 3t = \frac{3t^2}{2} \text{ units}^2$$

6 Write the x values of the ends of the integral in the correct notation, the upper end on top and the lower end beneath.

$$\int_0^2 (2 - x)\,dx$$



The required area is a triangle.

$$A = \frac{1}{2} \times 2 \times 2 = 2 \text{ units}^2$$

8 Write the x values of the ends of the integral in the correct notation, the upper end on top and the lower end beneath.

$$\int_{-1}^5 \sqrt{9 - (x-2)^2}\,dx$$

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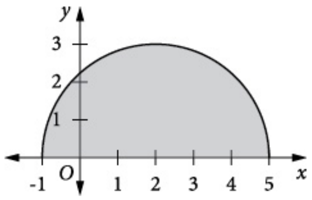
If $y = \sqrt{9-(x-2)^2}$,

$$y^2 = 9-(x-2)^2$$
$$(x-2)^2 + y^2 = 9$$

This is the equation of a circle, centre $(2,\ 0)$ and radius 3 units.

Since we are using the positive square root, the curve will be the top half of this circle.



The required area is a semi-circle.

$$A = \frac{1}{2}\times\pi\times3^2 = \frac{9\pi}{2}\ \text{units}^2$$

EXERCISE 17.3 THE DEFINITE INTEGRAL AND THE PRIMITIVE FUNCTION

2 $\displaystyle\int_1^2 x^2dx = \left[\frac{x^3}{3}\right]_1^2$

$$= \frac{2^3}{3} - \frac{1^3}{3}$$
$$= 2\frac{1}{3}$$

$$\int_2^3 x^2dx = \left[\frac{x^3}{3}\right]_2^3$$
$$= \frac{3^3}{3} - \frac{2^3}{3}$$
$$= 6\frac{1}{3}$$

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$$\begin{aligned}\int_1^3 x^2dx &= \left[\frac{x^3}{3}\right]_1^3\\ &= \frac{3^3}{3}-\frac{1^3}{3}\\ &= 8\frac{2}{3}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \int_1^2 x^2dx + \int_2^3 x^2dx\\ &= 2\frac{1}{3}+6\frac{1}{3}\\ &= 8\frac{2}{3}\\ &= \int_1^3 x^2dx\\ &= \text{LHS}\end{aligned}$$

$$\therefore \int_1^3 x^2dx = \int_1^2 x^2dx + \int_2^3 x^2dx$$

4 $$\begin{aligned}\int_0^3\left(4x^2+8x\right)dx &= \left[\frac{4x^3}{3}+\frac{8x^2}{2}\right]_0^3\\ &= \left[\frac{4x^3}{3}+4x^2\right]_0^3\\ &= \left(\frac{4\times 3^3}{3}+4\times 3^2\right)-0\\ &= 72\end{aligned}$$

$$\begin{aligned}4\int_0^3\left(x^2+2x\right)dx &= 4\left[\frac{x^3}{3}+\frac{2x^2}{2}\right]_0^3\\ &= 4\left[\frac{x^3}{3}+x^2\right]_0^3\\ &= 4\left[\left(\frac{3^3}{3}+3^2\right)-0\right]\\ &= 4\times 18\\ &= 72\end{aligned}$$

$$\int_0^3\left(4x^2+8x\right)dx = 72 = 4\int_0^3\left(x^2+2x\right)dx$$

6 B

$$\begin{aligned}\int_0^5 \left(6+8x-3x^2\right)dx &= \left[6x+\frac{8x^2}{2}-\frac{3x^3}{3}\right]_0^5 \\ &= \left[6x+4x^2-x^3\right]_0^5 \\ &= \left(6\times5+4\times5^2-5^3\right)-0 \\ &= 5\end{aligned}$$

8 **(a)** $$\begin{aligned}\int_b^a 3x^2dx &= \left[\frac{3x^3}{3}\right]_b^a \\ &= \left[x^3\right]_b^a \\ &= a^3-b^3\end{aligned}$$

(b) $$\begin{aligned}\int_0^2 \left(ax^2+bx+c\right)dx &= \left[\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right]_0^2 \\ &= \left(\frac{a\times2^3}{3}+\frac{b\times2^2}{2}+c\times2\right)-0 \\ &= \frac{8a}{3}+2b+2c\end{aligned}$$

(c) $$\begin{aligned}\int_0^a \left(x^2-2x\right)dx &= \left[\frac{x^3}{3}-\frac{2x^2}{2}\right]_0^a \\ &= \left[\frac{x^3}{3}-x^2\right]_0^a \\ &= \left(\frac{a^3}{3}-a^2\right)-0 \\ &= \frac{a^3}{3}-a^2\end{aligned}$$

10 Since the coefficient of x^2 is negative, the parabola will be concave down.

Find the x-intercepts.

$$\begin{aligned}2-x-x^2&=0 \\ (2+x)(1-x)&=0\end{aligned}$$

$x=-2,\ x=1$

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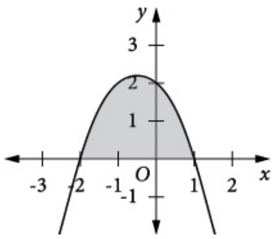


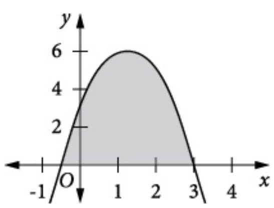
$$
\begin{aligned}
A &= \int_{-2}^{1}\left(2-x-x^2\right)dx \\
&= \left[2x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-2}^{1} \\
&= \left(2\times 1-\frac{1^2}{2}-\frac{1^3}{3}\right)-\left(2\times -2-\frac{(-2)^2}{2}-\frac{(-2)^3}{3}\right) \\
&= \frac{7}{6}+\frac{10}{3} \\
&= \frac{9}{2}\text{ units}^2
\end{aligned}
$$

12 Since the coefficient of x^2 is negative, the parabola will be concave down.

Find the x-intercepts.

$$
\begin{aligned}
3+5x-2x^2 &= 0 \\
(3-x)(1+2x) &= 0
\end{aligned}
$$

$$x=3,\ x=-\frac{1}{2}$$



$$A=\int_{-0.5}^{3}\left(3+5x-2x^2\right)dx$$
$$=\left[3x+\frac{5x^2}{2}-\frac{2x^3}{3}\right]_{-0.5}^{3}$$
$$=\left(3\times3+\frac{5\times3^2}{2}-\frac{2\times3^3}{3}\right)-\left(3\times-0.5+\frac{5\times(-0.5)^2}{2}-\frac{2\times(-0.5)^3}{3}\right)$$
$$=\frac{27}{2}+\frac{19}{24}$$
$$=\frac{343}{24}$$
$$=14\frac{7}{24}\text{ units}^2$$

14 $\int_{-1}^{a}xdx=\left[\frac{x^2}{2}\right]_{-1}^{a}$

$$=\frac{a^2}{2}-\frac{(-1)^2}{2}$$
$$=\frac{a^2}{2}-\frac{1}{2}$$

If $\int_{-1}^{a}xdx=0$, then

$$\frac{a^2}{2}-\frac{1}{2}=0$$
$$a^2=1$$

Since a is the upper value of x, a must be greater than 0.

$a=1$

16 Sketch the graph of the cubic.

The coefficient of x^3 is negative, so as $x\to+\infty,\ y\to-\infty$ and as $x\to-\infty,\ y\to+\infty$.

Find the x-intercepts.

$$x^2(3-x)=0$$
$$x=0,\ x=3$$

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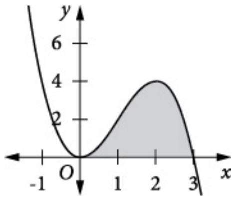
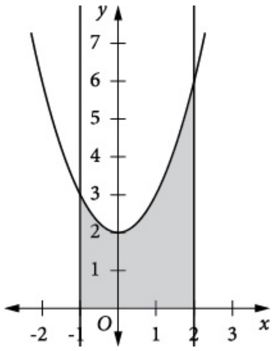


EXERCISE 17.4 MORE AREAS

2 Sketch the graph of $y = x^2 + 2$, which is the parabola $y = x^2$ raised 2 units.



$$\begin{aligned}\int_{-1}^{2}\left(x^2+2\right)dx &= \left[\frac{x^3}{3}+2x\right]_{-1}^{2}\\ &= \left(\frac{2^3}{3}+2\times 2\right)-\left(\frac{(-1)^3}{3}+2\times(-1)\right)\\ &= \frac{20}{3}+\frac{7}{3}\\ &= 9\end{aligned}$$

4 Sketch the graph of $y = 4 - x^2$, which is the parabola $y = -x^2$ raised 4 units.

Find the x-intercepts.

$$\begin{aligned}4 - x^2 &= 0\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$

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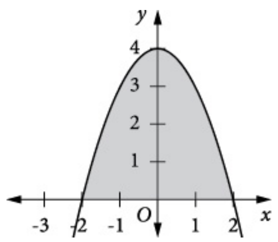
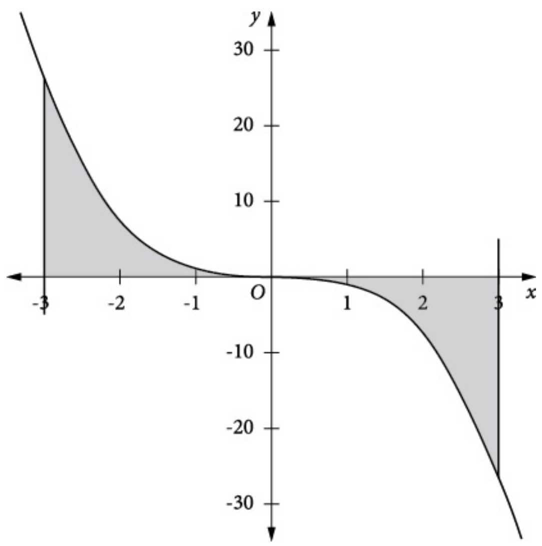


This function is even, so $\int_{-2}^{2}\left(4-x^2\right)dx = 2\int_{0}^{2}\left(4-x^2\right)dx$

$$\begin{aligned}2\int_{0}^{2}\left(4-x^2\right)dx &= 2\left[4x-\frac{x^3}{3}\right]_0^2 \\ &= 2\left[\left(4\times 2-\frac{2^3}{3}\right)-0\right] \\ &= \frac{32}{3} \\ &= 10\frac{2}{3}\ \text{units}^2\end{aligned}$$

6



This function is odd, so area $= 2\int_{-3}^{0}\left(-x^3\right)dx = 2\left|\int_{0}^{3}\left(-x^3\right)dx\right|$.

$$A = 2\left|\left[-\frac{x^4}{4}\right]_0^3\right|$$

$$= 2\left|\left[\left(-\frac{(3)^4}{4}\right) - 0\right]\right|$$

$$= \left|-\frac{81}{2}\right|$$

$$= \frac{81}{2}$$

$$= 40.5$$

Alternatively, using the positive half of the area,

$$A = 2\int_0^2 \left(4 - x^2\right) dx$$

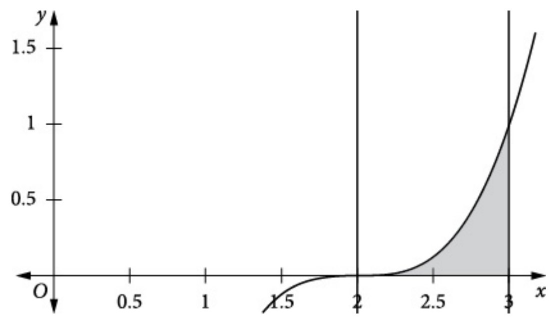
$$= 2\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 2\left[\left(4 \times 2 - \frac{2^3}{3}\right) - 0\right]$$

$$= \frac{32}{3}$$

$$= 10\frac{2}{3}$$

8 $f\left(x\right)$ is the cubic $y = x^3$ translated 2 units to the right.



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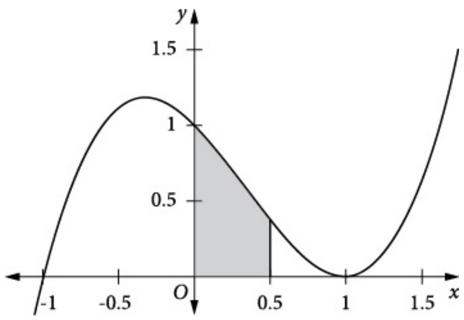
$$\int_2^3 (x-2)^3\,dx = \left[\frac{(x-2)^4}{4}\right]_2^3$$
$$= \frac{(3-2)^4}{4} - \frac{(2-2)^4}{4}$$
$$= \frac{1}{4}$$

10 (a) Sketch the curve.

The coefficient of x^3 is positive, so as $x \to +\infty,\ y \to -\infty$ and as $x \to -\infty,\ y \to +\infty$.

Find the x-intercepts.

$$(x+1)(x-1)^2 = 0$$
$$x = -1,\ x = 1$$



$$\int_0^{0.5} (x+1)(x-1)^2\,dx = \int_0^{0.5} (x+1)(x^2-2x+1)\,dx$$
$$= \int_0^{0.5} (x^3 - 2x^2 + x + x^2 - 2x + 1)\,dx$$
$$= \int_0^{0.5} (x^3 - x^2 - x + 1)\,dx$$
$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^{0.5}$$
$$= \left(\frac{0.5^4}{4} - \frac{0.5^3}{3} - \frac{0.5^2}{2} + 0.5\right) - 0$$
$$= \frac{67}{192}$$

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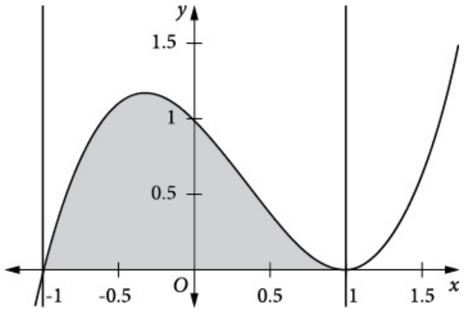
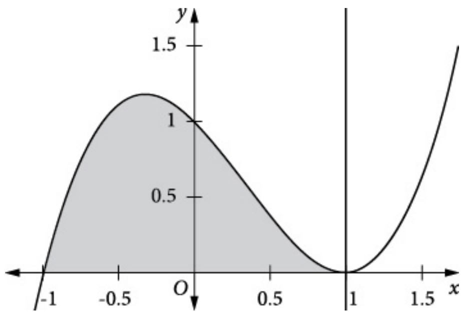
(b) From our knowledge of the curve, this will be the area between -1 and 1.



$$\begin{aligned}\int_{-1}^{1}(x+1)(x-1)^2\,dx &= \int_{-1}^{1}\left(x^3-x^2-x+1\right)dx\\ &= \left[\frac{x^4}{4}-\frac{x^3}{3}-\frac{x^2}{2}+x\right]_{-1}^{1}\\ &= \left(\frac{1^4}{4}-\frac{1^3}{3}-\frac{1^2}{2}+1\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^3}{3}-\frac{(-1)^2}{2}-1\right)\\ &= \frac{5}{12}+\frac{11}{12}\\ &= \frac{4}{3}\\ &= 1\frac{1}{3}\end{aligned}$$

(c) From our knowledge of the curve, this will be the area between 0 and 1.



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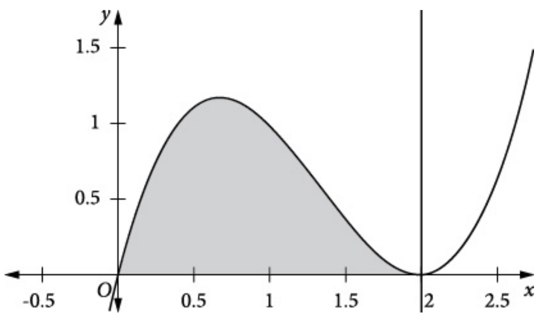
$$\begin{aligned}\int_0^1 (x+1)(x-1)^2\,dx &= \int_0^1 \left(x^3 - x^2 - x + 1\right)dx \\ &= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^1 \\ &= \left(\frac{1^4}{4} - \frac{1^3}{3} - \frac{1^2}{2} + 1\right) - 0 \\ &= \frac{5}{12}\end{aligned}$$

12 Sketch the curve.

The coefficient of x^3 is positive, so as $x \to +\infty,\ y \to -\infty$ and as $x \to -\infty,\ y \to +\infty$.

Find the x-intercepts.

$$x^2(x-2) = 0$$
$$x = 0,\ x = 2$$



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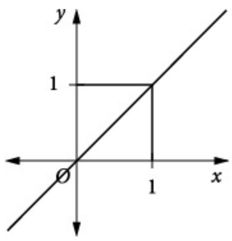
$$\begin{aligned}\int_0^2 x(x-2)^2\,dx &= \int_0^2 x\left(x^2-4x+4\right)dx\\ &= \int_0^2\left(x^3-4x^2+4x\right)dx\\ &= \left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{4x^2}{2}\right]_0^2\\ &= \left[\frac{x^4}{4}-\frac{4x^3}{3}+2x^2\right]_0^2\\ &= \left(\frac{2^4}{4}-\frac{4\times 2^3}{3}+2\times 2^2\right)-0\\ &= \frac{4}{3}\\ &= 1\frac{1}{3}\end{aligned}$$

14 Area of the square is 1 unit^2.

(a)

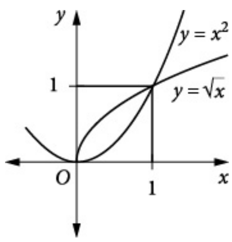


The area of the bottom triangle is given by

$$\begin{aligned}\int_0^1 x\,dx &= \left[\frac{x^2}{2}\right]_0^1\\ &= \frac{1}{2}-0\\ &= \frac{1}{2}\end{aligned}$$

This is half the area of the square, and so the square is bisected by the line $y = x$.

(b) Sketch the curves and the square.



Find the bottom area, the area under $y = x^2$.

$$\begin{aligned}\int_0^1 x^2 dx &= \left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3}\end{aligned}$$

This is one third the area of the square.

Find the area under the other curved line, $y = \sqrt{x}$.

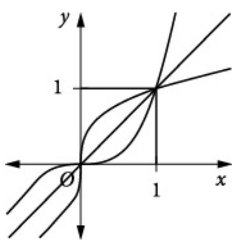
$$\begin{aligned}\int_0^1 \sqrt{x} dx &= \int_0^1 x^{\frac{1}{2}} dx \\ &= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^1 \\ &= \frac{2\times 1}{3} - 0 \\ &= \frac{2}{3}\end{aligned}$$

This is two-thirds the area of the square, and since the bottom part is one third the area of the square, the middle section must also be one third the area of the square.

The top area is $1 - \frac{2}{3} = \frac{1}{3}$ the area of the square.

Therefore the square is trisected.

(c) Sketch the curves and the square.



Find the bottom area, the area under $y = x^3$.

$$\int_0^1 x^3 dx = \left[\frac{x^4}{3}\right]_0^1$$
$$= \frac{1}{4} - 0$$
$$= \frac{1}{4}$$

The bottom part is one quarter the area of the square.

We know from part **(a)** that the area under the line is half the area of the square, so each of the bottom two areas is one quarter the area of the square.

$$\int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1$$
$$= \frac{1}{2} - 0$$
$$= \frac{1}{2}$$

Find the area under the top curve, $y = \sqrt[3]{x}$.

$$\int_0^1 \sqrt[3]{x} dx = \int_0^1 x^{\frac{1}{3}} dx$$
$$= \left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^1$$
$$= \frac{3 \times 1}{4} - 0$$
$$= \frac{3}{4}$$

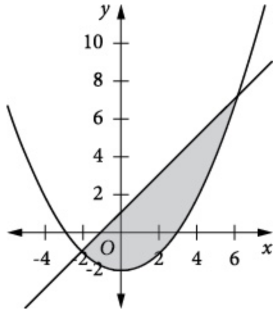
The area second from the top must also be one quarter the area of the square, and the top area is $1 - \frac{3}{4} = \frac{1}{4}$ the area of the square.

Therefore the square is divided into four equal parts.

EXERCISE 17.5 AREA BETWEEN TWO CURVES

2 Find the x values of the points of intersection of the two graphs.

$$x+1=\frac{x^2}{4}-2$$
$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$
$$x=-2,\ x=6$$



The straight line is continuously greater than the parabola over this interval, so the area is as follows:

$$\begin{aligned}\int_{-2}^{6}\left[(x+1)-\left(\frac{x^2}{4}-2\right)\right]dx &= \int_{-2}^{4}\left(x-\frac{x^2}{4}+3\right)dx\\ &=\left[\frac{x^2}{2}-\frac{x^3}{12}+3x\right]_{-2}^{6}\\ &=\left(\frac{6^2}{2}-\frac{6^3}{12}+3\times 6\right)-\left(\frac{(-2)^2}{2}-\frac{(-2)^3}{12}+3\times(-2)\right)\\ &=18+\frac{10}{3}\\ &=\frac{64}{3}\\ &=21\frac{1}{3}\end{aligned}$$

The area of the region is $21\frac{1}{3}$ units^2.

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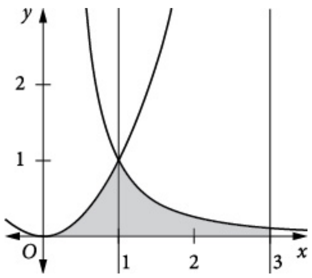
4 Find the x values of the points of intersection of the two graphs.

$$x^2 = \frac{1}{x^2}$$
$$x^4 = 1$$
$$x = \pm 1$$

In this case we need only consider $x = 1$.

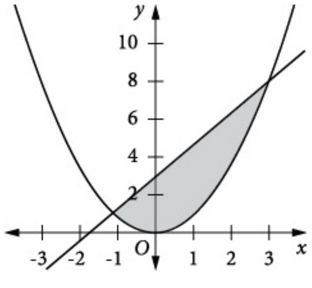


$$\int_0^1 x^2 dx + \int_1^3 \frac{1}{x^2} dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^{-1}}{-1}\right]_1^3$$
$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{1}{-x}\right]_1^3$$
$$= \left(\frac{1}{3} - 0\right) + \left(\frac{1}{-3} - \frac{1}{-1}\right)$$
$$= 1$$

The area of the region is 1 units^2.

6 Find the x values of the points of intersection of the two graphs.

$$2x + 3 = x^2$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = -1,\ x = 3$$

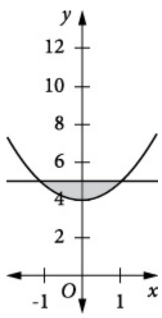


$$\begin{aligned}\int_{-1}^{3}\left(2x+3-x^2\right)dx &= \left[x^2+3x-\frac{x^3}{3}\right]_{-1}^{3}\\ &= \left(3^2+3\times 3-\frac{3^3}{3}\right)-\left((-1)^2+3\times(-1)-\frac{(-1)^3}{3}\right)\\ &= 9+\frac{5}{3}\\ &= \frac{32}{3}\\ &= 10\frac{2}{3}\end{aligned}$$

The area of the region is $10\frac{2}{3}$ units^2.

8 Find the x values of the points of intersection of the two graphs.

$$\begin{aligned}x^2+4&=5\\ x^2&=1\\ x&=\pm 1\end{aligned}$$



$$\begin{aligned}\int_{-1}^{1}\left(5-\left(x^2+4\right)\right)dx &= 2\int_{0}^{1}\left(1-x^2\right)dx\\ &= 2\left[x-\frac{x^3}{3}\right]_{0}^{1}\\ &= 2\left(1-\frac{1}{3}\right)\\ &= \frac{4}{3}\\ &= 1\frac{1}{3}\end{aligned}$$

The area of the region is $1\frac{1}{3}$ units^2.

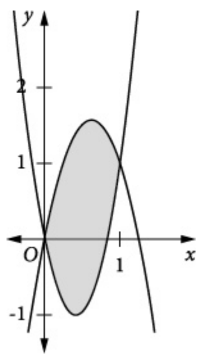
Alternatively, you can treat the area under $y=5$ as a rectangle.

$$\begin{aligned}5\times 2-\int_{-1}^{1}\left(x^2+4\right)dx &= 10-2\int_{0}^{1}\left(x^2+4\right)dx\\ &= 10-2\left[\frac{x^3}{3}+4x\right]_0^1\\ &= 10-2\left(\frac{1}{3}+4-0\right)\\ &= \frac{4}{3}\\ &= 1\frac{1}{3}\ \text{unit}^2\end{aligned}$$

The area of the region is $1\frac{1}{3}$ units^2.

10 Find the x values of the points of intersection of the two graphs.

$$\begin{aligned}6x^2-5x &= 5x-4x^2\\ 10x^2-10x &= 0\\ 10x(x-1) &= 0\\ x &= 0,\ x=1\end{aligned}$$

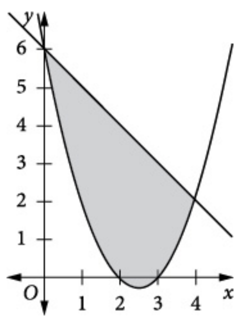


$$\begin{aligned}\int_0^1 \left(5x-4x^2-\left(6x^2-5x\right)\right)dx &= \int_0^1 \left(10x-10x^2\right)dx\\ &= \left[\frac{10x^2}{2}-\frac{10x^3}{3}\right]_0^1\\ &= \left[5x^2-\frac{10x^3}{3}\right]_0^1\\ &= \left(5-\frac{10}{3}\right)-0\\ &= \frac{5}{3}\\ &= 1\frac{2}{3}\ \text{unit}^2\end{aligned}$$

The area of the region is $1\frac{2}{3}$ units^2.

12 Find the x values of the points of intersection of the two graphs.

$$\begin{aligned}6-x &= x^2-5x+6\\ x^2-4x &= 0\\ x(x-4) &= 0\\ x &= 0,\ x=4\end{aligned}$$

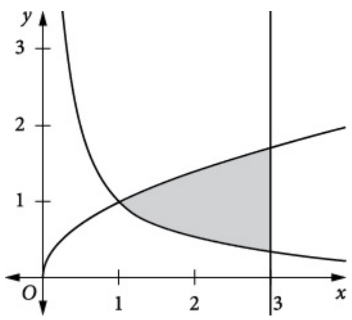


$$\begin{aligned}\int_0^4\left(6-x-\left(x^2-5x+6\right)\right)dx &= \int_0^4\left(4x-x^2\right)dx \\ &= \left[\frac{4x^2}{2}-\frac{x^3}{3}\right]_0^4 \\ &= \left[2x^2-\frac{x^3}{3}\right]_0^4 \\ &= \left(2\times 4^2-\frac{4^3}{3}\right)-0 \\ &= \frac{32}{3} \\ &= 10\frac{2}{3}\text{ unit}^2\end{aligned}$$

The area of the region is $10\frac{2}{3}$ units^2.

14 (a) Find the x values of the points of intersection of the first two graphs.

$$\begin{aligned}\sqrt{x} &= \frac{1}{x} \\ x^{\frac{3}{2}} &= 1 \\ x &= 1\end{aligned}$$



For $x=1$, $\sqrt{x} > \frac{1}{x}$

Therefore, the area is $\int_1^3\left(\sqrt{x}-\frac{1}{x}\right)dx$.

(b) Let $y = \sqrt{x} - \frac{1}{x}$

x	1	2	3
y	0	$\sqrt{2} - \frac{1}{2}$	$\sqrt{3} - \frac{1}{3}$

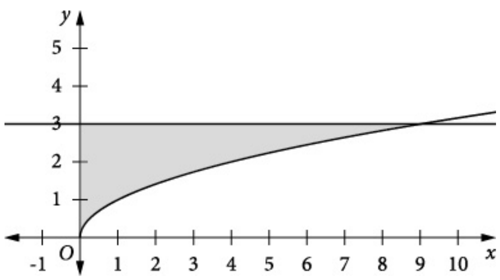
Using the trapezoidal rule:

$$\begin{aligned}\int_1^3 \left(\sqrt{x} - \frac{1}{x}\right) dx &= \frac{1}{2} \times 1 \times \left(0 + \sqrt{2} - \frac{1}{2}\right) + \frac{1}{2} \times 1 \times \left(\sqrt{2} - \frac{1}{2} + \sqrt{3} - \frac{1}{3}\right)\\ &= 1.61\end{aligned}$$

The area of the region is $10\frac{2}{3}$ units^2.

EXERCISE 17.6 AREA BOUNDED BY THE *Y*-AXIS

2



$y = \sqrt{x}$

$x = y^2$

$$\begin{aligned}\int_{y=0}^{3} x\,dy &= \int_0^3 y^2\,dy\\ &= \left[\frac{y^3}{3}\right]_0^3\\ &= \frac{3^3}{3} - 0\\ &= 9\end{aligned}$$

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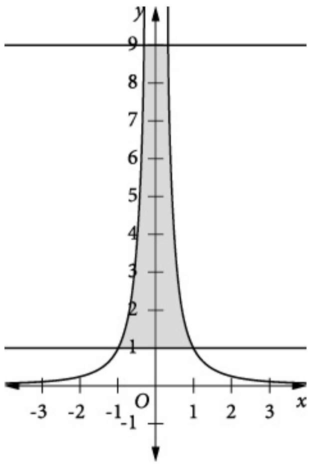
4



$$y = \frac{1}{x^2}$$

$$x^2 = \frac{1}{y}$$

$$x = \pm\frac{1}{\sqrt{y}}$$

$$\begin{aligned}\int_1^9 -y^{-\frac{1}{2}}dy + \int_1^9 y^{-\frac{1}{2}}dy &= 2\int_1^9 y^{-\frac{1}{2}}dy \\ &= 2\left[2y^{\frac{1}{2}}\right]_1^9 \\ &= 2\left(2\times\sqrt{9} - 2\times\sqrt{1}\right) \\ &= 8\end{aligned}$$

6 **(a)** $y = x^2 + 1$

$$\frac{dy}{dx} = 2x$$

At $x = 2$, $m_T = 2\times 2 = 4$

$y = 2^2 + 1 = 5$

Equation of tangent at $(2, 5)$:

$$\begin{aligned}y - 5 &= 4(x-2) \\ y - 5 &= 4x - 8 \\ y &= 4x - 3\end{aligned}$$

(b) $y = 4x - 3$ crosses the y-axis at -3.

$y = x^2 + 1$ crosses the y-axis at 1.

$$4x = y + 3$$
$$x = \frac{y+3}{4}$$

$$y = x^2 + 1$$
$$x^2 = y - 1$$
$$x = \pm\sqrt{y-1}$$

$$\int_{-3}^{5}\left(\frac{y+3}{4}\right)dy - \int_{1}^{5}\sqrt{y-1}\,dy = \frac{1}{4}\left[\frac{y^2}{2} + 3y\right]_{-3}^{5} - \left[\frac{2(y-1)^{\frac{3}{2}}}{3}\right]_{1}^{5}$$
$$= \frac{1}{4}\left[\left(\frac{5^2}{2} + 3\times 5\right) - \left(\frac{(-3)^2}{2} + 3\times(-3)\right)\right] - \left[\left(\frac{2(5-1)^{\frac{3}{2}}}{3}\right) - \left(\frac{2(1-1)^{\frac{3}{2}}}{3}\right)\right]$$
$$= \frac{1}{4}\times 32 - \frac{16}{3}$$
$$= \frac{8}{3}$$
$$= 2\frac{2}{3}$$

Note: This area can also (and more easily) be calculated finding areas between 0 and 2 on the x-axis.

$$\int_0^2\left(x^2 + 1 - (4x-3)\right)dx = \int_0^2\left(x^2 - 4x + 4\right)dx$$
$$= \left[\frac{x^3}{3} - \frac{4x^2}{2} + 4x\right]_0^2$$
$$= \left[\frac{x^3}{3} - 2x^2 + 4x\right]_0^2$$
$$= \left[\frac{2^3}{3} - 2\times 2^2 + 4\times 2 - 0\right]_0^2$$
$$= \frac{8}{3}$$
$$= 2\frac{2}{3}$$

(c) $y = 4x - 3$ crosses the x-axis at $\frac{3}{4}$.

Subtract the bottom triangle, which has a base of $\frac{3}{4}$ units and height 3 units.

Area is $2\frac{2}{3} - \frac{1}{2} \times \frac{3}{4} \times 3 = 1\frac{13}{24}$.

8 **(a)** $x^2 = 4 - x^2$

$$2x^2 = 4$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

$$\begin{aligned}\int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - x^2 - x^2\right)dx &= 2\int_0^{\sqrt{2}} \left(4 - 2x^2\right)dx \\ &= 2\left[4x - \frac{2x^3}{3}\right]_0^{\sqrt{2}} \\ &= 2\left[\left(4 \times \sqrt{2} - \frac{2 \times \left(\sqrt{2}\right)^3}{3}\right) - 0\right] \\ &= 2\left(4\sqrt{2} - \frac{4\sqrt{2}}{3}\right) \\ &= \frac{16\sqrt{2}}{3}\end{aligned}$$

(b) $y = 4 - x^2$ crosses the x-axis at ± 2.

$$\begin{aligned}2\left\{\int_0^{\sqrt{2}} x^2 dx + \int_{\sqrt{2}}^{2} \left(4 - x^2\right)dx\right\} &= 2\left\{\left[\frac{x^3}{3}\right]_0^{\sqrt{2}} + \left[4x - \frac{x^3}{3}\right]_{\sqrt{2}}^{2}\right\} \\ &= 2\left[\left(\frac{\left(\sqrt{2}\right)^3}{3} - 0\right) + \left(4 \times 2 - \frac{2^3}{3}\right) - \left(4 \times \sqrt{2} - \frac{\left(\sqrt{2}\right)^3}{3}\right)\right] \\ &= 2\left(\frac{2\sqrt{2}}{3} + \frac{16}{3} - \frac{10\sqrt{2}}{3}\right) \\ &= 2\left(\frac{16 - 8\sqrt{2}}{3}\right) \\ &= \frac{16\left(2 - \sqrt{2}\right)}{3}\end{aligned}$$

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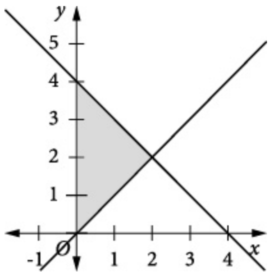
10 Find the x values of the intersection of the lines and sketch the situation.

$$\begin{aligned} 4-x &= x \\ 2x &= 4 \\ x &= 2 \end{aligned}$$



$$\begin{aligned} \int_0^2 y\,dy + \int_2^4 (4-y)\,dy &= \left[\frac{y^2}{2}\right]_0^2 + \left[4y - \frac{y^2}{2}\right]_2^4 \\ &= \frac{2^2}{2} - 0 + \left(4\times 4 - \frac{4^2}{2}\right) - \left(4\times 2 - \frac{2^2}{2}\right) \\ &= 4 \end{aligned}$$

This can also be done geometrically, perhaps as a check. The (vertical) base of the triangle is 4 units and its height is 2 units.

Area $= \frac{1}{2}\times 4\times 2 = 4$ units.

12 (a) $y \le 3x$ is the area on and under the line $y = 3x$.

Draw $y = 3x$ with a solid line and consider the area underneath this line.

$y \ge x^2 - 4$ is the area on and above the parabola $y = x^2 - 4$, which is the parabola $y = x^2$ translated 4 units down.

Draw $y = x^2 - 4$ with a solid line and consider the area above this line.

Shade the region which is below $y = 3x$ and above $y = x^2 - 4$.

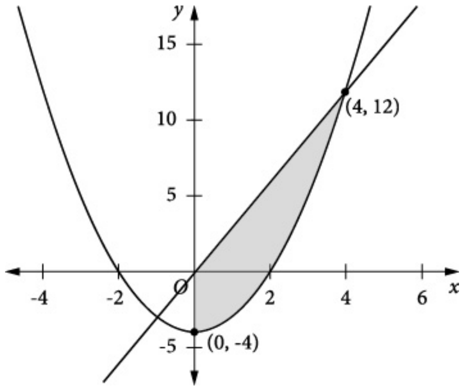
Find the points of intersection for the line and the parabola.

$$\begin{aligned} x^2 - 4 &= 3x \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \end{aligned}$$

We are only concerned with $x = 4$.

When $x = 4$, $y = 3\times 4 = 12$.

The parabola crosses the y-axis at $(0,\ -4)$ and the line passes through the origin.



(b) $y = 3x$

$$x = \frac{y}{3}$$

$$\begin{aligned} y &= x^2 - 4 \\ x^2 &= y + 4 \\ x &= \pm\sqrt{y+4} \end{aligned}$$

When $x = 4$, $y = 12$

$$\begin{aligned} \int_{-4}^{12} \left(\sqrt{y+4}\right) dy - \int_{0}^{12} \frac{y}{3}\, dy &= \left[\frac{2(y+4)^{\frac{3}{2}}}{3}\right]_{-4}^{12} - \left[\frac{y^2}{6}\right]_{0}^{12} \\ &= \left(\frac{2(12+4)^{\frac{3}{2}}}{3} - \frac{2(-4+4)^{\frac{3}{2}}}{3}\right) - \left(\frac{12^2}{6} - 0\right) \\ &= \frac{128}{3} - 24 \\ &= \frac{56}{3} \\ &= 18\frac{2}{3} \end{aligned}$$

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This can (as a check) be also calculated relative to the x-axis.

$$\begin{aligned}\int_0^4 \left(3x-\left(x^2-4\right)\right)dx &= \int_0^4 \left(3x-x^2+4\right)dx \\ &= \left[\frac{3x^2}{2}-\frac{x^3}{3}+4x\right]_0^2 \\ &= \left[\frac{3\times 4^2}{2}-\frac{4^3}{3}+4\times 4-0\right]_0^2 \\ &= \frac{56}{3} \\ &= 18\frac{2}{3}\end{aligned}$$

EXERCISE 17.7
DEFINITE INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

2 **(a)** $\int_0^{\pi} \sin x dx = \left[-\cos x\right]_0^{\pi}$

$$\begin{aligned}&= -\cos\pi - \left(-\cos 0\right) \\ &= -\left(-1\right)-\left(-1\right) \\ &= 2\end{aligned}$$

(b) $\int_0^{\frac{\pi}{3}} \sec^2 x dx = \left[\tan x\right]_0^{\frac{\pi}{3}}$

$$\begin{aligned}&= \tan\frac{\pi}{3}-\left(\tan 0\right) \\ &= \sqrt{3}-0 \\ &= \sqrt{3}\end{aligned}$$

(c) $\int_{\frac{\pi}{3}}^{\pi} \cos\frac{x}{2} dx = \left[2\sin\frac{x}{2}\right]_{\frac{\pi}{3}}^{\pi}$

$$\begin{aligned}&= 2\sin\frac{\pi}{2}-\left(2\sin\frac{\pi}{6}\right) \\ &= 2\times 1-2\times\frac{1}{2} \\ &= 1\end{aligned}$$

(d) $\int_0^{\frac{\pi}{2}} \cos x dx = \left[\sin x\right]_0^{\frac{\pi}{2}}$

$$\begin{aligned}&= \sin\frac{\pi}{2}-\left(\sin 0\right) \\ &= 1-0 \\ &= 1\end{aligned}$$

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(e) $\int_0^{\pi}(\sin x+\cos x)\,dx=[-\cos x+\sin x]_0^{\pi}$

$$\begin{aligned}&=(-\cos\pi+\sin\pi)-(-\cos 0+\sin 0)\\&=1+0-(-1+0)\\&=2\end{aligned}$$

(f) $\int_{-\pi}^{\pi}\sin x dx=[-\cos x]_{-\pi}^{\pi}$

$$\begin{aligned}&=-\cos\pi-(-\cos(-\pi))\\&=-(-1)-1\\&=0\end{aligned}$$

(g) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\cos x-\sin 2x)\,dx=\left[\sin x+\frac{1}{2}\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$$\begin{aligned}&=\left(\sin\frac{\pi}{2}+\frac{1}{2}\cos\left(2\times\frac{\pi}{2}\right)\right)-\left(\sin\frac{\pi}{6}+\frac{1}{2}\cos\left(2\times\frac{\pi}{6}\right)\right)\\&=\left(1+\frac{1}{2}\times(-1)\right)-\left(\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\right)\\&=-\frac{1}{4}\end{aligned}$$

(h)

$$\begin{aligned}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(\sin 2x-\cos 2x)\,dx&=\left[-\frac{1}{2}\cos 2x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}\\&=\left(-\frac{1}{2}\cos\left(2\times\frac{\pi}{4}\right)-\frac{1}{2}\sin\left(2\times\frac{\pi}{4}\right)\right)-\left(-\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right)-\frac{1}{2}\sin\left(2\times\frac{\pi}{8}\right)\right)\\&=\left(0-\frac{1}{2}\right)-\left(-\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)\\&=-\frac{1}{2}-\left(-\frac{1}{\sqrt{2}}\right)\\&=\frac{\sqrt{2}-1}{2}\end{aligned}$$

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(i) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\sin x + \sin 2x)\,dx = \left[-2\cos x - \frac{1}{2}\cos 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$$= \left(-2\cos\frac{\pi}{4} - \frac{1}{2}\cos\left(2\times\frac{\pi}{4}\right)\right) - \left(-2\cos\left(-\frac{\pi}{4}\right) - \frac{1}{2}\cos\left(2\times-\frac{\pi}{4}\right)\right)$$
$$= \left(-2\times\frac{1}{\sqrt{2}} - 0\right) - \left(-2\times\frac{1}{\sqrt{2}} - 0\right)$$
$$= 0$$

(j) $\int_0^{\pi} \left(\sin\frac{x}{4} + \cos\frac{x}{4}\right)dx = \left[-4\cos\frac{x}{4} + 4\sin\frac{x}{4}\right]_0^{\pi}$

$$= \left(-4\cos\frac{\pi}{4} + 4\sin\frac{\pi}{4}\right) - (-4\cos 0 + 4\sin 0)$$
$$= \left(-4\times\frac{1}{\sqrt{2}} + 4\times\frac{1}{\sqrt{2}}\right) - (-4\times 1 + 0)$$
$$= 4$$

(k) $\int_0^{\frac{\pi}{3}} \left(3\cos 3x - \frac{\sin 2x}{2}\right)dx = \left[\sin 3x + \frac{\cos 2x}{4}\right]_0^{\frac{\pi}{3}}$

$$= \left(\sin\left(3\times\frac{\pi}{3}\right) + \frac{\cos\left(2\times\frac{\pi}{3}\right)}{4}\right) - \left(\sin 0 + \frac{\cos 0}{4}\right)$$
$$= \left(0 - \frac{1}{8}\right) - \left(0 + \frac{1}{4}\right)$$
$$= -\frac{3}{8}$$

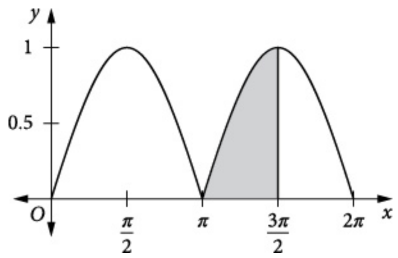
(l) $\int_{-\pi}^{\pi} \left(\frac{\sin x}{2} - \cos x\right)dx = \left[-\frac{\cos x}{2} - \sin x\right]_{-\pi}^{\pi}$

$$= \left(-\frac{\cos\pi}{2} - \sin\pi\right) - \left(-\frac{\cos(-\pi)}{2} - \sin(-\pi)\right)$$
$$= \left(\frac{1}{2} - 0\right) - \left(\frac{1}{2} - 0\right)$$
$$= 0$$

4 Note that the surd in the equation is positive.

$$f(x) = \sqrt{1-\cos^2 x} = \sqrt{\sin^2 x} = |\sin x|$$

That is the graph of $y = \sin x$, but with the negative part reflected in the x-axis. The graph with the required region is drawn below.



$$\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\cos^2 x}dx = \int_{\pi}^{\frac{3\pi}{2}} |\sin x| dx$$

Since $\sin x < 0$ for $0 \le x \le 2\pi$,

$$\begin{aligned}\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\cos^2 x}dx &= \int_{\pi}^{\frac{3\pi}{2}} -\sin x dx \\ &= \left[\cos x\right]_{\pi}^{\frac{3\pi}{2}} \\ &= \cos\frac{3\pi}{2} - \cos\pi \\ &= 0-(-1) \\ &= 1\end{aligned}$$

6 **(a)** Let $f(x) = 1+\sin x$ so $f'(x) = \cos x$.

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}dx &= \int_0^{\frac{\pi}{6}} \frac{f'(x)}{f(x)}dx \\ &= \left[\ln\left(f(x)\right)\right]_0^{\frac{\pi}{6}} \\ &= \left[\ln\left(1+\sin x\right)\right]_0^{\frac{\pi}{6}} \\ &= \ln\left(1+\sin\frac{\pi}{6}\right) - \ln\left(1+\sin 0\right) \\ &= \ln\frac{3}{2} - 0 \\ &= \ln\frac{3}{2}\end{aligned}$$

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(b) Let $f(x) = 2 - \cos x$ so $f'(x) = \sin x$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin x}{2 - \cos x} dx &= \int_0^{\frac{\pi}{2}} \frac{f'(x)}{f(x)} dx \\
&= \Big[\ln\big(f(x)\big)\Big]_0^{\frac{\pi}{2}} \\
&= \Big[\ln\big(2 - \cos x\big)\Big]_0^{\frac{\pi}{2}} \\
&= \ln\left(2 - \cos\frac{\pi}{2}\right) - \ln\big(2 - \cos 0\big) \\
&= \ln 2 - \ln 1 \\
&= \ln 2
\end{aligned}$$

8 $\cos 2x \geq 0$ for $0 \leq x \leq \frac{\pi}{4}$, so the region will be entirely above the x-axis.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \cos 2x dx &= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\sin\left(2 \times \frac{\pi}{4}\right) - \frac{1}{2}\sin 0 \\
&= \frac{1}{2}
\end{aligned}$$

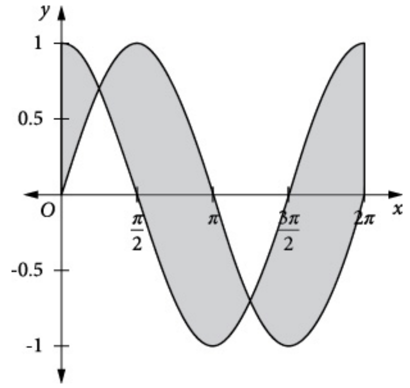
10 First find the values of θ where the two curves intersect.

$$\begin{aligned}
&\sin\theta = \cos\theta \\
&\tan\theta = 1 \\
&\theta = \frac{\pi}{4},\ \frac{5\pi}{4}
\end{aligned}$$

When $\theta = \frac{\pi}{4}$, $y = \frac{1}{\sqrt{2}}$

When $\theta = \frac{5\pi}{4}$, $y = -\frac{1}{\sqrt{2}}$

The points of intersection of the curves $y = \sin\theta$ and $y = \cos\theta$ are $\left(\frac{\pi}{4},\ \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4},\ -\frac{1}{\sqrt{2}}\right)$.



$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx + \int_{\frac{5\pi}{4}}^{2\pi} (\cos x - \sin x)\,dx \\
&= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}} + \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + \left[\sin x + \cos x\right]_{\frac{5\pi}{4}}^{2\pi} \\
&= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - 1 + \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) + \left(\sin 2\pi + \cos 2\pi\right) - \left(\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}\right) \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + 1 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\
&= \frac{8}{\sqrt{2}} \\
&= \frac{8}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= 4\sqrt{2}
\end{aligned}
$$

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EXERCISE 17.8 DEFINITE INTEGRALS INVOLVING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

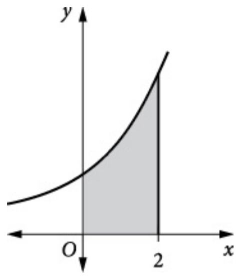
2 $I=\int_{-1}^{1}\left(e^{x}-e^{-x}\right)^{2}dx$

$$=\int_{-1}^{1}\left(e^{2x}-2\times e^{x}\times e^{-x}+e^{-2x}\right)dx$$
$$=\int_{-1}^{1}\left(e^{2x}-2+e^{-2x}\right)dx$$
$$=\left[\frac{e^{2x}}{2}-2x+\frac{e^{-2x}}{-2}\right]_{-1}^{1}$$
$$=\left[\frac{e^{2x}}{2}-2x-\frac{e^{-2x}}{2}\right]_{-1}^{1}$$
$$=\left(\frac{e^{2}}{2}-2-\frac{e^{-2}}{2}\right)-\left(\frac{e^{-2}}{2}+2-\frac{e^{2}}{2}\right)$$
$$=2\left(\frac{e^{2}}{2}-2-\frac{e^{-2}}{2}\right)$$
$$=e^{2}-4-e^{-2}$$

(a) correct **(b)** correct

(c) correct **(d)** incorrect

4 **(a)** Draw a diagram.



$$\int_{0}^{2}e^{x}dx=\left[e^{x}\right]_{0}^{2}$$
$$=e^{2}-e^{0}$$
$$=e^{2}-1$$

(b) $y=e^{x}$

$$\frac{dy}{dx}=e^{x}$$

At $x = 2$,

$$m_T = e^2$$
$$y = e^2$$

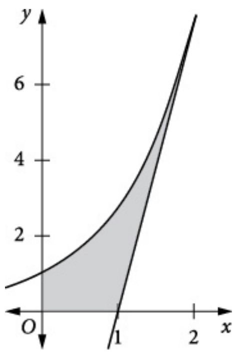
Equations of tangent at $x = 2$:

$$\begin{aligned} y - e^2 &= e^2(x-2) \\ y - e^2 &= e^2 x - 2e^2 \\ y &= e^2 x - e^2 \\ e^2 x - y - e^2 &= 0 \end{aligned}$$

(c) Find where the tangent cuts the x-axis, then draw the diagram.

When $y = 0$,

$$\begin{aligned} e^2 x - 0 - e^2 &= 0 \\ e^2 x &= e^2 \\ x &= 1 \end{aligned}$$

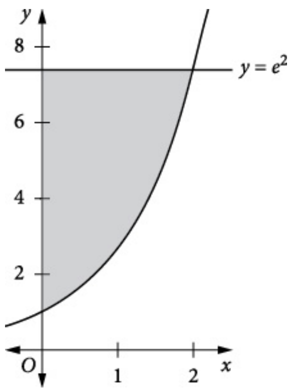


$$\begin{aligned} \int_0^2 e^x dx - \int_1^2 \left(e^2 x - e^2\right) dx &= \left[e^x\right]_0^2 - \left[\frac{e^2 x^2}{2} - e^2 x\right]_1^2 \\ &= e^2 - e^0 - \left[\left(\frac{e^2 \times 2^2}{2} - e^2 \times 2\right) - \left(\frac{e^2}{2} - e^2\right)\right] \\ &= e^2 - 1 - \left[\left(2e^2 - 2e^2\right) - \left(-\frac{e^2}{2}\right)\right] \\ &= e^2 - 1 - \frac{e^2}{2} \\ &= \frac{e^2}{2} - 1 \end{aligned}$$

The same result can be obtained by

$$\begin{aligned}\int_0^1 e^x dx + \int_1^2 \left(e^x - \left(e^2 x - e^2\right)\right) dx &= \int_0^1 e^x dx + \int_1^2 \left(e^x - e^2 x + e^2\right) dx \\ &= \left[e^x\right]_0^1 + \left[e^x - \frac{e^2 x^2}{2} + e^2 x\right]_1^2 \\ &= e^1 - e^0 + \left[\left(e^2 - \frac{e^2 \times 2^2}{2} + e^2 \times 2\right) - \left(e^1 - \frac{e^2}{2} + e^2\right)\right] \\ &= e^1 - 1 + e^2 - 2e^2 + 2e^2 - e^1 + \frac{e^2}{2} - e^2 \\ &= \frac{e^2}{2} - 1\end{aligned}$$

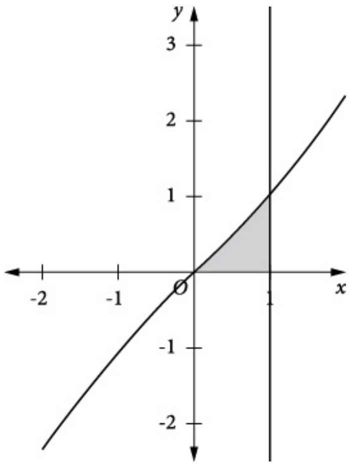
(d)



Take the difference of the two areas.

$$\begin{aligned}A &= 2 \times e^2 - \int_0^2 e^x dx \\ &= 2e^2 - \left(e^2 - 1\right) \\ &= e^2 + 1\end{aligned}$$

6



$$\int_0^1 \left(e^{0.5x} - e^{-0.5x}\right) dx = \left[\frac{e^{0.5x}}{0.5} - \frac{e^{-0.5x}}{-0.5}\right]_0^1$$
$$= \left[2e^{0.5x} + 2e^{-0.5x}\right]_0^1$$
$$= 2e^{0.5} + 2e^{-0.5} - \left(2e^0 + 2e^0\right)$$
$$= 2\left(e^{0.5} + e^{-0.5} - 2\right)$$

8 (a) $$\int_1^3 \left(x^2 + \frac{1}{x}\right) dx = \left[\frac{x^3}{3} + \ln x\right]_1^3$$
$$= \left(\frac{3^3}{3} + \ln 3\right) - \left(\frac{1^3}{3} + \ln 1\right)$$
$$= 9 + \ln 3 - \frac{1}{3}$$
$$= \frac{26 + 3\ln 3}{3}$$

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(b) Let $f(x) = x^2 + 1$ so that $f'(x) = 2x$.

$$\begin{aligned}
\int_{-3}^{3} \frac{x}{x^2+1}\,dx &= \int_{-3}^{3} \frac{f'(x)}{f(x)}\,dx \\
&= \left[\frac{1}{2}\ln\left(f(x)\right)\right]_{-3}^{3} \\
&= \left[\frac{1}{2}\ln\left(x^2+1\right)\right]_{-3}^{3} \\
&= \left(\frac{1}{2}\ln\left(3^2+1\right)\right) - \left(\frac{1}{2}\ln\left((-3)^2+1\right)\right) \\
&= \frac{1}{2}\ln 10 - \frac{1}{2}\ln 10 \\
&= 0
\end{aligned}$$

(c)
$$\begin{aligned}
\int_{2}^{4} \frac{x^2-1}{x}\,dx &= \int_{2}^{4} \left(x - \frac{1}{x}\right) dx \\
&= \left[\frac{x^2}{2} - \ln x\right]_{2}^{4} \\
&= \left(\frac{4^2}{2} - \ln 4\right) - \left(\frac{2^2}{2} - \ln 2\right) \\
&= 6 - \ln 2
\end{aligned}$$

(d)
$$\begin{aligned}
\int_{2}^{4} \left(\frac{1}{x} + \frac{1}{x^2}\right) dx &= \int_{2}^{4} \left(\frac{1}{x} + x^{-2}\right) dx \\
&= \left[\ln x + \frac{x^{-1}}{-1}\right]_{2}^{4} \\
&= \left(\ln 4 - \frac{1}{4}\right) - \left(\ln 2 - \frac{1}{2}\right) \\
&= \ln 2 + \frac{1}{4}
\end{aligned}$$

(e)
$$\begin{aligned}
\int_{2}^{3} \left(x + \frac{1}{x-1}\right) dx &= \left[\frac{x^2}{2} + \ln\left(x-1\right)\right]_{2}^{3} \\
&= \left(\frac{3^2}{2} + \ln\left(3-1\right)\right) - \left(\frac{2^2}{2} + \ln\left(2-1\right)\right) \\
&= \ln 2 + \frac{5}{2}
\end{aligned}$$

(f) $\int_1^2 \left(x - \frac{1}{x^2}\right)^2 dx = \int_1^2 \left(x^2 - \frac{2}{x} + x^{-4}\right) dx$

$$= \left[\frac{x^3}{3} - 2\ln x + \frac{x^{-3}}{-3}\right]_1^2$$
$$= \left(\frac{2^3}{3} - 2\ln 2 - \frac{2^{-3}}{3}\right) - \left(\frac{1^3}{3} - 2\ln 1 - \frac{1^{-3}}{3}\right)$$
$$= \frac{8}{3} - 2\ln 2 - \frac{1}{24} - \frac{1}{3} + 0 + \frac{1}{3}$$
$$= \frac{64-1}{24} - 2\ln 2$$
$$= \frac{21}{8} - 2\ln 2$$

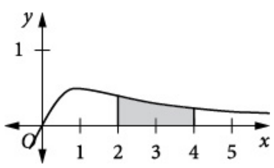
(g) $\int_1^3 \left(e^x + \frac{1}{x}\right) dx = \left[e^x + \ln x\right]_1^3$

$$= \left(e^3 + \ln 3\right) - \left(e^1 + \ln 1\right)$$
$$= e^3 - e + \ln 3$$

(h) $\int_1^4 \left(\sqrt{x} - \frac{1}{x}\right) dx = \left[\frac{2x^{\frac{3}{2}}}{3} - \ln x\right]_1^4$

$$= \left(\frac{2 \times 4^{\frac{3}{2}}}{3} - \ln 4\right) - \left(\frac{2 \times 1}{3} - \ln 1\right)$$
$$= \frac{14}{3} - 2\ln 2$$

10



Let $f(x) = x^2 + 1$ so that $f'(x) = 2x$.

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$$\begin{aligned}\int_2^4 \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int_2^4 \frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\int_2^4 \frac{f'(x)}{f(x)}\,dx \\ &= \frac{1}{2}\Big[\ln\big(f(x)\big)\Big]_2^4 \\ &= \frac{1}{2}\Big[\ln\left(x^2+1\right)\Big]_2^4 \\ &= \frac{1}{2}\left(\ln\left(4^2+1\right)-\ln\left(2^2+1\right)\right) \\ &= \frac{1}{2}\left(\ln 17-\ln 5\right) \\ &= \frac{1}{2}\ln\frac{17}{5}\end{aligned}$$

12 $y = x+\dfrac{1}{x}$

Asymptotes: $x = 0$

When $x \to \pm\infty$, $\dfrac{1}{x} \to 0$, $y \to x$.

$y = x$ is an asymptote.

$$\frac{dy}{dx} = 1-\frac{1}{x^2}$$
$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

Turning points occurs when $\dfrac{dy}{dx} = 0$.

$$\begin{aligned}1-\frac{1}{x^2} &= 0 \\ x^2 &= 1 \\ x &= \pm 1\end{aligned}$$

At $x = -1$, $y = -2$.

$$\frac{d^2y}{dx^2} = \frac{2}{(-1)^3} = -2$$

$\therefore (-1,\ -2)$ is a maximum turning point.

At $x = 1$, $y = 2$.

$$\frac{d^2y}{dx^2} = \frac{2}{1^3} = 2$$

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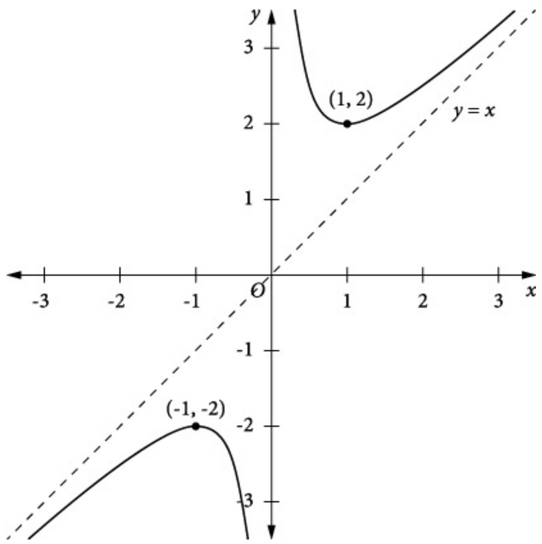
$\therefore (1,\ 2)$ is a minimum turning point.



$$\int_1^2 \left(x + \frac{1}{x} \right) dx = \left[\frac{x^2}{2} + \ln x \right]_1^2$$
$$= \left(\frac{2^2}{2} + \ln 2 \right) - \left(\frac{1^2}{2} + \ln 1 \right)$$
$$= \frac{3}{2} + \ln 2$$

14 Let $f(x) = 1 + e^x$ so that $f'(x) = e^x$.

$$\int_0^1 \frac{e^x}{1+e^x} dx = \log_e c$$
$$\int_0^1 \frac{f'(x)}{f(x)} dx = \log_e c$$
$$\left[\log_e \left(f(x) \right) \right]_0^1 = \log_e c$$
$$\left[\log_e \left(1 + e^x \right) \right]_0^1 = \log_e c$$
$$\log_e \left(1 + e^1 \right) - \log_e \left(1 + e^0 \right) = \log_e c$$
$$\log_e (1 + e) - \log_e 2 = \log_e c$$
$$\log_e \left(\frac{1+e}{2} \right) = \log_e c$$
$$c = \frac{1+e}{2}$$

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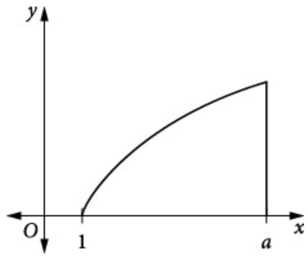
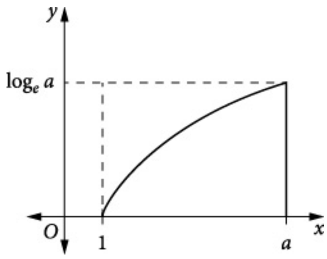
16



Since we cannot yet integrate the logarithmic function, we must find the areas relative to the y-axis.

$$y = \log_e x$$
$$e^y = x$$

Where $x = a,\ y = \log_e a.$



$$\begin{aligned}\int_1^a \log_e x dx &= a \times \log_e a - \int_0^{\log_e a} e^y dy \\ &= a \ln a - \left[e^y\right]_0^{\log_e a} \\ &= a \ln a - \left(e^{\ln a} - e^0\right) \\ &= a \ln a - a + 1\end{aligned}$$

EXERCISE 17.9 APPLICATIONS INVOLVING DEFINITE INTEGRALS

2 (a) (i) $t = 0$

$$\frac{dQ}{dt}(0) = 2(0) + 1 = 1$$

Initial rate: 1 item/min

(ii) $t = 10$

$$\frac{dQ}{dt}(10) = 2(10) + 1 = 21$$

Rate after 10 minutes: 21 items/min

(b) $\int_0^{10} (2x+1)\, dx = \left[x^2 + x\right]_0^{10}$

$$\begin{aligned}&= \left(10^2 + 10\right) - \left(0^2 + 0\right) \\ &= 110\end{aligned}$$

110 items

4 $\int_0^{180}\left[\frac{1}{10\ 000}\left(200t-t^2\right)\right]\ dx=\frac{1}{10\ 000}\int_0^{180}\left(200t-t^2\right)\ dx$

$$=\frac{1}{10\ 000}\left[100t^2-\frac{t^3}{3}\right]_0^{180}$$

$$=\frac{1}{10\ 000}\left[\left(100(180)^2-\frac{(180)^3}{3}\right)-\left(100(0)^2-\frac{(0)^3}{3}\right)\right]$$

$V=129.6$

(a) incorrect **(b)** correct (line 1)

(c) incorrect **(d)** correct (line 4)

6 **(a)** $v=t^2-4t^3$

$$x=\int\left(t^2-4t^3\right)\ dt$$

$$=\frac{t^3}{3}-t^4+C$$

The body starts at *O*, so when $t=0,\ x=0$.

$$0=\frac{0^3}{3}-0^4+C$$

$$C=0$$

$$x=\frac{t^3}{3}-t^4$$

(b) $v=t^2-4t^3$

$$a=\frac{dv}{dt}=2t-12t^2$$

8 **(a)** $a=10-2t$

$$v=\int a\ dt=\int\left(10-2t\right)\ dt$$

$$=10t-t^2+C$$

Initial velocity is $11\ \text{m s}^{-1}$.

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$$11 = 10 \times 0 - 0^2 + C$$
$$C = 11$$
$$v = 11 + 10t - t^2$$

$$x = \int v \ dt = \int \left(11 + 10t - t^2\right) \ dt$$
$$= 11t + 5t^2 - \frac{t^3}{3} + C$$

Initial displacement is 0 m.

$$0 = 11 \times 0 + 5 \times 0^2 - \frac{0^3}{3} + C$$
$$C = 0$$
$$x = 11t + 5t^2 - \frac{t^3}{3}$$

(b) $v = 0 = 11 + 10t - t^2$

$$0 = -\left(t^2 - 10t - 11\right)$$
$$0 = -\left(t - 11\right)\left(t + 1\right)$$
$$t = -1, \ 11$$

The body has zero velocity after 11 s.

When $t = 11$, $x = 11\left(11\right) + 5\left(11\right)^2 - \frac{\left(11\right)^3}{3} = 283\frac{1}{3}$

The displacement at this time is $283\frac{1}{3}$ m.

10 (a) $a = -10$

$$v = \int a \ dt = \int -10 \ dt = -10t + C$$

Initial velocity: 25 m s^{-1}.

$$25 = -10(0) + C$$
$$C = 25$$
$$v = -10t + 25$$

$$h = \int v \; dt = \int \left(-10t + 25\right) \; dt$$
$$h = -5t^2 + 25t + C$$

Initial displacement: 0 m (from O)

$$0 = -5(0)^2 + 25(0) + C$$
$$C = 0$$
$$h = -5t^2 + 25t$$

(b) Maximum height occurs when the velocity is zero.

$$0 = -10t + 25$$
$$10t = 25$$
$$t = 2.5$$

When $t = 2.5 \Rightarrow x = -5\left(2.5\right)^2 + 25\left(2.5\right) = 31.25$

The maximum height reached by the particle is 31.25 m.

(c) $\dfrac{25}{2} = -10t + 25$

$$10t = \frac{25}{2}$$
$$t = 1.25$$

1.25 s

12 $v\left(t\right) = 12t^2 - 6t + 1$

$$s\left(t\right) = \frac{12t^3}{3} - \frac{6t^2}{2} + t + C$$
$$= 4t^3 - 3t^2 + t + C$$

Given $s\left(1\right) = 4$

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$$4 = 4(1)^3 - 3(1)^2 + (1) + C$$
$$4 = 2 + C$$
$$C = 2$$
$$\therefore s(t) = 4t^3 - 3t^2 + t + 2$$

14 (a) $v_A = \frac{dx_A}{dt} = 50 - 40t$

$$s_A = |50 - 40t|$$
$$v_B = \frac{dx_B}{dt} = 160t + 20$$
$s_B = |160t + 20| = 160t + 20$ since $t \geq 0$.

$$s_A(0) = |50 - 40(0)| = 50$$
$$s_B(0) = 160(0) + 20 = 20$$

Car A speed at point O: 50 km h^{-1}

Car B speed at point O: 20 km h^{-1}

(b) $|50 - 40t| = 160t + 20$

$$50 - 40t = 160t + 20$$
$$t = 0.15$$
$$50 - 40t = -(160t + 20)$$
$$50 - 40t = -160t - 20$$
$$120t = -20 - 50$$
$$t < 0$$

$t = 0.15$ hours, or $t = 0.15 \times 60 = 9$ min.

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(c) The cars are at the same point at the same time t if

$$50t - 20t^2 = 80t^2 + 20t$$
$$100t^2 - 30t = 0$$
$$10t(10t - 3) = 0$$
$$t = 0,\ 0.3$$

When $t = 0$, $x_A = x_B = 0$, so they are both at O.

When $t = 0.3$, $x_A = 50t - 20t^2 = 50 \times 0.3 - 20 \times 0.3^2 = 13.2$

The distance from O to Q is 13.2 km.

(d) Car C is travelling at constant velocity, so $x_C = mt + c$.

When $t = 0$, A and B are at O, and C is 2 km ahead, so $x_C = +2$.

$$x_C = mt + c$$
$$2 = m \times 0 + c$$
$$c = 2$$
$$x_C = mt + 2$$

C arrives at Q (13.2 km) at $t = 0.3$

$$x_C = mt + 2$$
$$13.2 = m \times 0.3 + 2$$
$$0.3m = 11.2$$
$$m = \frac{11.2}{0.3} = \frac{112}{3}$$
$$\therefore x_c = \frac{112t}{3} + 2$$

16 D

If its velocity is positive it must be increasing with time.

If its acceleration is negative, velocity and hence the gradient will be decreasing, in other words the graph will be getting less steep.

These correspond to graph **D**.

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18 (a) Point A: $v(0) = 60 + 40e^0 = 100 \text{ m s}^{-1}$

Point B: $v(3) = 60 + 40e^{-3} \approx 62 \text{ m s}^{-1}$

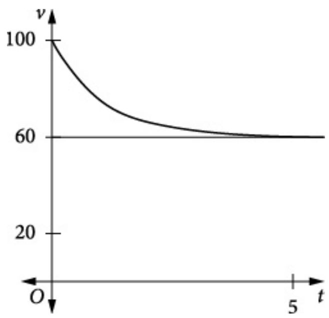
(b) (i) $a = \dfrac{dv}{dt} = -40e^{-t}$

(ii) $v = 60 + 40e^{-t}$

$$v = 60 - \left(-40e^{-t}\right)$$
$$v = 60 - a$$
$$v + a = 60$$
$$a = 60 - v$$

(c) Plot the points for $t = 0$ and $t = 3$ and note that as $t \to \infty, v \to 60 + 0 = 60$.

$v = 60$ is an asymptote.



As t increases, the velocity decreases towards 60 km h^{-1}.

(d) $\int_0^3 \left(60 + 40e^{-t}\right) dt = \left[60t - 40e^{-t}\right]_0^3$

$= \left[60(3) - 40e^{-(3)}\right] - \left[60(0) - 40e^{-(0)}\right]$

≈ 218

The distance from *A* to *B* is about 218 km, to the nearest whole number.

20 $a = 2\sin t$

$$v = \int a\ dt = \int (2\sin t)\ dt$$
$$= -2\cos t + C$$

Where $t = 0, v = 1$

$$1 = -2\cos(0) + C$$
$$1 = -2 + C$$
$$C = 3$$
$$\therefore v = 3 - 2\cos t$$

$$x = \int v\ dt = \int (3 - 2\cos t)\ dt$$
$$= 3t - 2\sin t + C$$

Where $t = 0, x = 1$

$$1 = 3(0) - 2\sin(0) + C$$
$$C = 1$$
$$\therefore x = 1 + 3t - 2\sin t$$

22 (a) $x(0) = 4\log_e(1+0) - 2(0) = 0$

(b) $v = \dfrac{dx}{dt} = \dfrac{4}{1+t} - 2$

$$v(0) = \frac{4}{1+0} - 2 = 2$$

(c) $a = \dfrac{dv}{dt} = -\dfrac{4}{(1+t)^2}$

As $(1+t)^2 > 0$ for $t \ge 0$, $a < 0$ for all $t \ge 0$

(d) $\dfrac{4}{1+t} - 2 = 0$

$$\frac{4}{1+t} = 2$$
$$2(1+t) = 4$$
$$1 + t = 2$$
$$t = 1$$

24 (a) $a = \dfrac{dv}{dt}$

$$= 49\left(0 - -0.5 \times e^{-0.5t}\right)$$
$$= \frac{49e^{-0.5t}}{2}$$

(b) $v(0) = 49\left(1 - e^{-0.5(0)}\right) = 49(1-1) = 0$

(c) As $t \to \infty,\ e^{-0.5t} \to 0,\ v \to 49(1-0) = 49$

(d) $\int_0^5 49\left(1 - e^{-0.5t}\right)\,dt = 49\left[t - \dfrac{e^{-0.5t}}{-0.5}\right]_0^5$

$$= 49\left[t + 2e^{-0.5t}\right]_0^5$$
$$= 49\left[\left(5 + 2e^{-0.5(5)}\right) - \left(0 + 2e^{-0.5(0)}\right)\right]$$
$$\approx 155$$

The distance fallen in the first 5 seconds is about 155 units.

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EXERCISE 17.10
APPROXIMATE METHODS OF INTEGRATION—TRAPEZOIDAL RULE

2 (a) Make a table of values, using $f(x)=\dfrac{1}{x^2+1}$.

x	0	0.5	1
$f(x)$	1	$\dfrac{1}{1.25}=\dfrac{4}{5}$	$\dfrac{1}{2}$

$$h=\frac{1-0}{2}=\frac{1}{2}$$

$$\begin{aligned}\int_0^1\frac{dx}{x^2+1}&\approx\frac{h}{2}\left(f(0)+2f(0.5)+f(1)\right)\\&=\frac{1}{4}\left(1+2\times\frac{4}{5}+\frac{1}{2}\right)\\&=\frac{31}{40}\end{aligned}$$

(b) Make a table of values, using $f(x)=\dfrac{1}{x^2+1}$.

x	0	0.25	0.5	0.75	1
$f(x)$	1	$\dfrac{1}{1.0625}=\dfrac{16}{17}$	$\dfrac{1}{1.25}=\dfrac{4}{5}$	$\dfrac{1}{1.5625}=\dfrac{16}{25}$	$\dfrac{1}{2}$

$$h=\frac{1-0}{4}=\frac{1}{4}$$

$$\begin{aligned}\int_0^1\frac{dx}{x^2+1}&\approx\frac{h}{2}\left(f(0)+2\left(f(0.25)+f(0.5)+f(0.75)\right)+f(1)\right)\\&=\frac{1}{8}\times\left(1+2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{25}\right)+\frac{1}{2}\right)\\&=\frac{5323}{6800}\\&=0.783\text{ (3 d.p.)}\end{aligned}$$

4 (a) $\displaystyle\int_1^2 x^3dx=\left[\frac{x^4}{4}\right]_1^2$

$$= \frac{2^4}{4} - \frac{1^4}{4}$$
$$= \frac{15}{4}$$
$$= 3\frac{3}{4}$$

(b) Make a table of values, using $f(x) = x^3$.

x	1	$\frac{3}{2}$	2
$f(x)$	1	$\frac{27}{8}$	8

$$h = \frac{1-0}{2} = \frac{1}{2}$$

$$\int_1^2 x^3 dx \approx \frac{h}{2}\left(f(1) + 2f(1.5) + f(2)\right)$$
$$= \frac{1}{4}\left(1 + \frac{27}{8} + 2\right)$$
$$= \frac{1}{4} \times \frac{63}{4}$$
$$= \frac{63}{16}$$
$$= 3\frac{15}{16}$$

6 Make a table of values, using $f(x) = 4^x$.

x	−1	−0.5	0	0.5	1
$f(x)$	$4^{-1} = \frac{1}{4}$	$4^{-0.5} = \frac{1}{2}$	1	$4^{0.5} = 2$	4

$$h = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}$$

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$$\begin{aligned}\int_{-1}^{1} 4^x\,dx &\approx \frac{h}{2}\big(f(-1)+2\big(f(-0.5)+f(0)+f(0.5)\big)+f(1)\big)\\ &=\frac{1}{4}\times\left(\frac{1}{4}+2\left(\frac{1}{2}+1+2\right)+1\right)\\ &=\frac{45}{16}\\ &=2.8125\end{aligned}$$

8 Make a table of values, using $f(x)=\sqrt{1+x^3}$.

x	0	1	2
$f(x)$	1	$\sqrt{2}$	3

$$h=\frac{2-0}{2}=1$$

$$\begin{aligned}\int_0^2 \sqrt{1+x^3}\,dx &\approx \frac{h}{2}\big(f(0)+2f(1)+f(2)\big)\\ &=\frac{1}{2}\left(1+2\sqrt{2}+3\right)\\ &=2+\sqrt{2}\\ &=3.414 \text{ (3 d.p.)}\end{aligned}$$

10 Make a table of values, using $f(x)=\sqrt{1+x^4}$.

x	0	1	2	3	4
$f(x)$	1	$\sqrt{2}$	$\sqrt{17}$	$\sqrt{82}$	$\sqrt{257}$

$$h=\frac{4-0}{4}=1$$

$$\begin{aligned}\int_0^4 \sqrt{1+x^4}\,dx &\approx \frac{h}{2}\big(f(0)+2\big(f(1)+f(2)+f(3)\big)+f(4)\big)\\ &=\frac{1}{2}\times\left(1+2\left(\sqrt{2}+\sqrt{17}+\sqrt{82}\right)+\sqrt{257}\right)\\ &=23.11 \text{ (2 d.p.)}\end{aligned}$$

12 (a) Make a table of values, using $f(x) = \sin^2 x$.

The middle x value will be $\frac{3\pi}{6} = \frac{\pi}{2}$

x	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$
$f(x)$	$\frac{1}{4}$	1	$\frac{1}{4}$

$$h = \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \frac{1}{2} \times \frac{4\pi}{6} = \frac{\pi}{3}$$

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x\,dx &\approx \frac{h}{2}\left(f\left(\frac{\pi}{6}\right) + 2f\left(\frac{\pi}{2}\right) + f\left(\frac{5\pi}{6}\right)\right)\\ &= \frac{\pi}{6}\left(\frac{1}{4} + 2 + \frac{1}{4}\right)\\ &= \frac{5\pi}{12}\\ &\approx 1.31\end{aligned}$$

(b) Make a table of values, using $f(x) = \cos^2 x$.

The middle x value will be $\frac{3\pi}{6} = \frac{\pi}{2}$

x	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$
$f(x)$	$\frac{1}{4}$	1	$\frac{1}{4}$

$$h = \frac{1}{2}\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = \frac{1}{2} \times \frac{2\pi}{3} = \frac{\pi}{3}$$

$$\begin{aligned}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2 x dx &\approx \frac{h}{2}\left(f\left(-\frac{\pi}{3}\right) + 2f(0) + f\left(\frac{\pi}{3}\right)\right)\\ &= \frac{\pi}{6}\left(\frac{1}{4} + 2 + \frac{1}{4}\right)\\ &= \frac{5\pi}{12}\\ &\approx 1.31\end{aligned}$$

14 Make a table of values, using $f(x) = \dfrac{4}{1+x^2}$.

x	0	0.5	1
$f(x)$	4	$\dfrac{4}{1.25} = \dfrac{16}{5}$	2

$$h = \frac{1-0}{2} = \frac{1}{2}$$

$$\begin{aligned}\int_0^1 \frac{dx}{x^2+1} &\approx \frac{h}{2}\left(f(0) + 2f(0.5) + f(1)\right)\\ &= \frac{1}{4}\left(4 + \frac{32}{5} + 2\right)\\ &= 3.1\\ &\approx \pi\end{aligned}$$

16 Make a table of values, using $f(x) = \sqrt{1+\cos^2 x}$.

x	0	$\dfrac{\pi}{4}$	$\dfrac{\pi}{2}$	$\dfrac{3\pi}{4}$	π
$f(x)$	$\sqrt{2}$	$\sqrt{\dfrac{3}{2}}$	1	$\sqrt{\dfrac{3}{2}}$	$\sqrt{2}$

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

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$$\begin{aligned}\int_0^{\pi}\sqrt{1+\cos^2 x}\,dx &\approx \frac{h}{2}\left(f(0)+2\left(f\left(\frac{\pi}{4}\right)+f\left(\frac{\pi}{2}\right)+f\left(\frac{3\pi}{4}\right)\right)+f(\pi)\right)\\ &=\frac{\pi}{8}\left(\sqrt{2}+2\left(\sqrt{\frac{3}{2}}+1+\sqrt{\frac{3}{2}}\right)+\sqrt{2}\right)\\ &=\frac{\pi}{8}\left(2\sqrt{2}+2\left(2\sqrt{\frac{3}{2}}+1\right)\right)\\ &=\frac{\pi}{4}\left(\sqrt{2}+\sqrt{6}+1\right)\\ &\approx 3.82\end{aligned}$$

18 Make a table of values, using $f(x)=xe^{0.4x}$.

x	1	2	3	4	5
$f(x)$	$e^{0.4}$	$2e^{0.4\times 2}$	$3e^{0.4\times 3}$	$4e^{0.4\times 4}$	$5e^{0.4\times 5}$

$$h=\frac{5-1}{4}=1$$

$$\begin{aligned}\int_1^5 xe^{0.4x}dx &\approx \frac{h}{2}\left(f(1)+2\left(f(2)+f(3)+f(4)\right)+f(5)\right)\\ &=\frac{1}{2}\left(e^{0.4}+2\left(2e^{0.8}+3e^{1.2}+4e^{1.6}\right)+5e^{2}\right)\\ &\approx 53.44\end{aligned}$$

EXERCISE 17.11 AVERAGE VALUE OF A FUNCTION—AN APPLICATION OF INTEGRATION

2 $f_{\text{ave}} = \dfrac{1}{2-(-2)}\int_{-2}^{2} x^4 dx$

$$= \frac{1}{4} \times \left[\frac{x^5}{5}\right]_{-2}^{2}$$
$$= \frac{1}{4} \times \left(\frac{2^5}{5} - \frac{(-2)^5}{5}\right)$$
$$= \frac{16}{5}$$
$$= 3\frac{1}{5}$$

4 D

$$f_{\text{ave}} = \frac{1}{3-(-1)}\int_{-1}^{3}\left(x^2 - 2x + 1\right) dx$$
$$= \frac{1}{4} \times \left[\frac{x^3}{3} - x^2 + x\right]_{-1}^{3}$$
$$= \frac{1}{4} \times \left[\left(\frac{3^3}{3} - 3^2 + 3\right) - \left(\frac{(-1)^3}{3} - (-1)^2 + (-1)\right)\right]$$
$$= \frac{4}{3}$$

6 $\dfrac{1}{7-0}\int_{0}^{7}\dfrac{12}{\sqrt{x+2}}dx = \dfrac{1}{7} \times \left[\dfrac{12(x+2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_{0}^{7}$

$$= \frac{1}{7} \times \left[24\sqrt{x+2}\right]_{0}^{7}$$
$$= \frac{1}{7} \times \left(24\sqrt{7+2} - 24\sqrt{0+2}\right)$$
$$= \frac{24\left(3-\sqrt{2}\right)}{7}$$

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8 **(a)** $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 e^x dx$

$$= \frac{1}{2} \times \left[e^x\right]_0^2$$
$$= \frac{1}{2} \times \left(e^2 - e^0\right)$$
$$= \frac{1}{2}\left(e^2 - 1\right)$$

(b) $f_{\text{ave}} = \frac{1}{2-0}\int_0^2 e^{-x} dx$

$$= \frac{1}{2} \times \left[-e^{-x}\right]_0^2$$
$$= \frac{1}{2} \times \left(-e^{-2} + e^0\right)$$
$$= \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$$
$$= \frac{1}{2}\left(\frac{e^2 - 1}{e^2}\right)$$
$$= \frac{e^2 - 1}{2e^2}$$

(c) $f_{\text{ave}} = \frac{1}{5-1}\int_1^5 e^{2x} dx$

$$= \frac{1}{4} \times \left[\frac{e^{2x}}{2}\right]_1^5$$
$$= \frac{1}{4} \times \left(\frac{e^{2\times5}}{2} - \frac{e^{2\times1}}{2}\right)$$
$$= \frac{1}{8}\left(e^{10} - e^2\right)$$
$$= \frac{e^2}{8}\left(e^8 - 1\right)$$

(d) $f_{\text{ave}} = \frac{1}{2-(-2)}\int_{-2}^2 \left(e^x + e^{-x}\right) dx$

$$= \frac{1}{4} \times \left[e^x - e^{-x}\right]_{-2}^2$$
$$= \frac{1}{4} \times \left[\left(e^2 - e^{-2}\right) - \left(e^{-2} - e^2\right)\right]$$
$$= \frac{1}{4}\left(2e^2 - 2e^{-2}\right)$$
$$= \frac{1}{2}\left(e^2 - e^{-2}\right)$$
$$= \frac{1}{2}\left(\frac{e^4 - 1}{e^2}\right)$$
$$= \frac{e^4 - 1}{2e^2}$$

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10 $f_{\text{ave}} = \dfrac{1}{\dfrac{\pi}{2} - \left(-\dfrac{\pi}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos x + 3\sin x)\, dx$

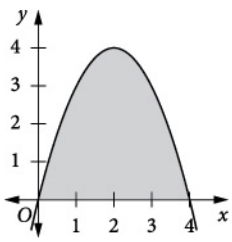
$$= \frac{1}{\pi} \times \left[2\sin x - 3\cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \frac{1}{\pi} \times \left[\left(2\sin\frac{\pi}{2} - 3\cos\frac{\pi}{2}\right) - \left(2\sin\left(-\frac{\pi}{2}\right) - 3\cos\left(-\frac{\pi}{2}\right)\right)\right]$$
$$= \frac{1}{\pi} \times \left[(2-0) - (2\times -1 - 0)\right]$$
$$= \frac{4}{\pi}$$

CHAPTER REVIEW 17

2 Since the coefficient of x^2 is negative, the parabola will be concave down.

Find the x-intercepts.

$$4x - x^2 = 0$$
$$x(4-x) = 0$$
$$x = 0,\ x = 4$$



$$\int_0^4 (4x - x^2)\, dx = \left[\frac{4x^2}{2} - \frac{x^3}{3}\right]_0^4$$
$$= \left(\frac{4\times 4^2}{2} - \frac{4^3}{3}\right) - 0$$
$$= \frac{32}{3}$$
$$= 10\frac{2}{3}$$

4 $y = x^2$

$$\frac{dy}{dx} = 2x$$

At $x = 2$, $m_T = 2 \times 2 = 4$

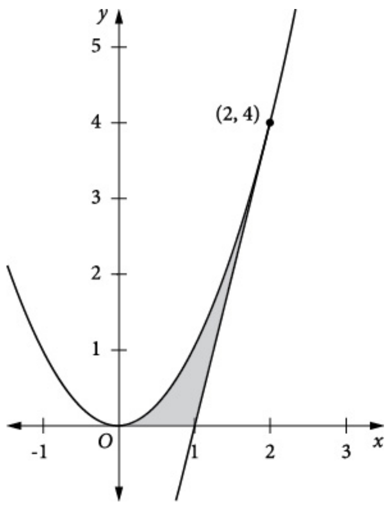
$y = 2^2 = 4$

Equation of tangent:

$$\begin{aligned} y - 4 &= 4(x-2) \\ y - 4 &= 4x - 8 \\ y &= 4x - 4 \end{aligned}$$

Find where the tangent crosses the x-axis.

$y = 0 \Rightarrow 0 = 4x - 4 \Rightarrow x = 1$



$$\begin{aligned} A &= \int_0^2 x^2 dx - \int_1^2 (4x-4)\,dx \\ &= \left[\frac{x^3}{3}\right]_0^2 - \left[2x^2 - 4x\right]_1^2 \\ &= \left(\frac{2^3}{3} - 0\right) - \left[\left(2\times 2^2 - 4\times 2\right) - \left(2\times 1^2 - 4\times 1\right)\right] \\ &= \frac{8}{3} - (0+2) \\ &= \frac{2}{3} \end{aligned}$$

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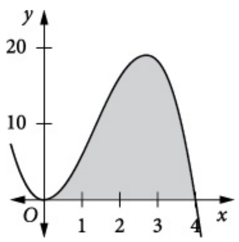
6 Sketch the graph of the cubic.

The coefficient of x^3 is negative, so as $x \to +\infty,\ y \to -\infty$ and as $x \to -\infty,\ y \to +\infty$.

Find the x-intercepts.

$$2x^2(4-x)=0$$
$$x=0,\ x=4$$



$$\begin{aligned}
A &= \int_0^4 2x^2(4-x)\,dx \\
&= \int_0^4 \left(8x^2-2x^3\right)dx \\
&= \left[\frac{8x^3}{3}-\frac{x^4}{2}\right]_0^4 \\
&= \left(\frac{8\times 4^3}{3}-\frac{4^4}{2}\right)-0 \\
&= \frac{128}{3} \\
&= 42\frac{2}{3}
\end{aligned}$$

8 $f'(x)=\left(\frac{1}{2x}+x\right)^2$

$$\begin{aligned}
&= \frac{1}{4x^2}+1+x^2 \\
&= \frac{1}{4}\times x^{-2}+1+x^2
\end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{1}{4}\times\frac{x^{-1}}{-1}+x+\frac{x^3}{3}+C \\
&= -\frac{1}{4x}+x+\frac{x^3}{3}+C
\end{aligned}$$

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$$f(2) = 6$$

$$-\frac{1}{4\times 2} + 2 + \frac{2^3}{3} + C = 6$$

$$-\frac{1}{8} + 2 + \frac{8}{3} + C = 6$$

$$C = 4 + \frac{1}{8} - \frac{8}{3} = \frac{35}{24}$$

$$\therefore f(x) = \frac{x^3}{3} + x - \frac{1}{4x} + \frac{35}{24}$$

10 $f_{\text{ave}} = \frac{1}{9-1}\int_1^9 \left(1+\sqrt{x}\right)dx$

$$= \frac{1}{8}\left[x + \frac{2x^{\frac{3}{2}}}{3}\right]_1^9$$

$$= \frac{1}{8}\left[\left(9 + \frac{2\times 9^{\frac{3}{2}}}{3}\right) - \left(1 + \frac{2\times 1^{\frac{3}{2}}}{3}\right)\right]$$

$$= \frac{1}{8}\left(27 - \frac{5}{3}\right)$$

$$= \frac{19}{6}$$

12 (a) $4x - x^2 > 0$

$$x(4-x) > 0$$

$$0 < x < 4$$

(b) Since the coefficient of x^2 is negative, the parabola will be concave down.

The parabola is positive between 0 and 4, so it is positive between 1 and 2.

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$$\begin{aligned}A &= \int_1^2 \left(4x - x^2\right) dx \\ &= \left[2x^2 - \frac{x^3}{3}\right]_1^2 \\ &= \left(2\times 2^2 - \frac{2^3}{3}\right) - \left(2\times 1^2 - \frac{1^3}{3}\right) \\ &= \frac{16}{3} - \frac{5}{3} \\ &= \frac{11}{3} \\ &= 3\frac{2}{3}\end{aligned}$$

(c) $y = 4x - x^2$

$$\frac{dy}{dx} = 4 - 2x$$

At $x = 1,\ m_T = 4 - 2\times 1 = 2$

$\tan\theta = 2 \Rightarrow \theta \approx 63°26'$

14 Make a table of values, using $f(x) = \dfrac{1}{x^2+1}$.

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{5}$	$\frac{1}{2}$	1	$4^{0.5} = 2$	$\frac{1}{5}$

$$h = \frac{2-(-2)}{4} = \frac{4}{4} = 1$$

$$\begin{aligned}\int_{-2}^{2} \frac{dx}{1+x^2} &= \frac{h}{2}\Big(f(-2) + 2\big(f(-1) + f(0) + f(1)\big) + f(2)\Big) \\ &= \frac{1}{2}\left(\frac{1}{5} + 2\left(\frac{1}{2} + 1 + \frac{1}{2}\right) + \frac{1}{5}\right) \\ &= \frac{11}{5} \\ &= 2.2\end{aligned}$$

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16 $f_{\text{ave}} = \dfrac{1}{\frac{\pi}{2}-\frac{\pi}{4}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(2\cos 2x+\sin 2x\right)dx$

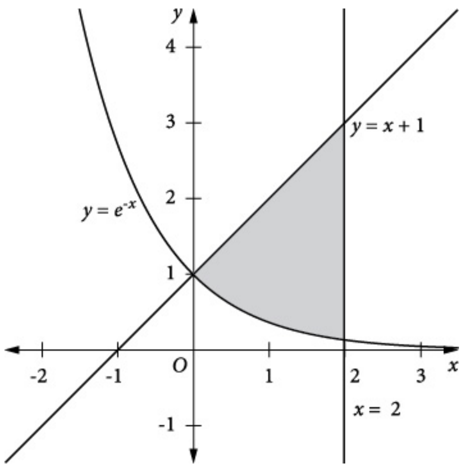
$$= \frac{4}{\pi}\left[\sin 2x - \frac{\cos 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{4}{\pi}\left[\left(\sin\left(2\times\frac{\pi}{2}\right)-\frac{\cos\left(2\times\frac{\pi}{2}\right)}{2}\right)-\left(\sin\left(2\times\frac{\pi}{4}\right)-\frac{\cos\left(2\times\frac{\pi}{4}\right)}{2}\right)\right]$$

$$= \frac{4}{\pi}\left[\left(0+\frac{1}{2}\right)-\left(1-0\right)\right]$$

$$= -\frac{2}{\pi}$$

18 Sketch the graphs. $y = e^x$ and $y = x+1$ both have y -intercepts of 1, so they intersect at $\left(0,\ 1\right)$.



$$A = \int_0^2\left(x+1-e^{-x}\right)dx$$

$$= \left[\frac{x^2}{2}+x+e^{-x}\right]_0^2$$

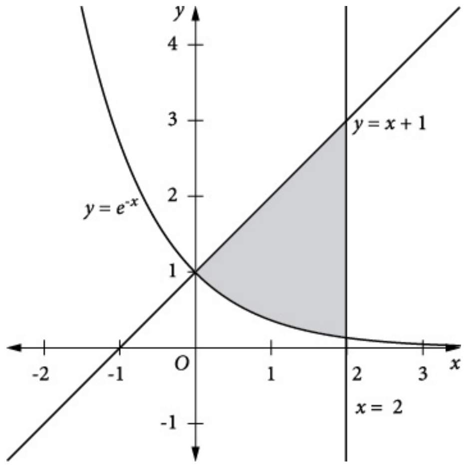
$$= \left(\frac{2^2}{2}+2+e^{-2}\right)-\left(e^0\right)$$

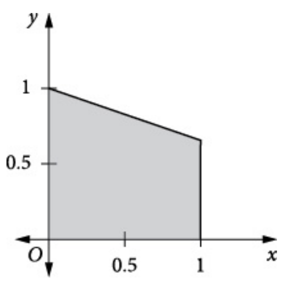
$$= 4+\frac{1}{e^2}-1$$

$$= 3+\frac{1}{e^2}$$

$$= \frac{3e^2+1}{e^2}$$

20





$$\int_0^1 e^{-\frac{x}{2}}dx = \left[-2e^{-\frac{x}{2}}\right]_0^1$$
$$= \left(-2e^{-\frac{1}{2}}\right) - \left(-2e^0\right)$$
$$= -2e^{-\frac{1}{2}} + 2$$
$$= 2\left(1 - e^{-\frac{1}{2}}\right)$$
$$\approx 0.7869$$

22 (a) Use the chain rule.

Let $u = \sin x$ so that $\frac{du}{dx} = \cos x$ and $\frac{d}{dx}\left(\log_e\left(\sin x\right)\right) = \frac{d}{dx}\left(\log_e\left(u\right)\right)$.

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$$\begin{aligned}\frac{d}{dx}\left(\log_e(\sin x)\right) &= \frac{d}{dx}\left(\log_e(u)\right)\\ &= \frac{d}{du}\left(\log_e(u)\right)\times\frac{du}{dx}\\ &= \frac{1}{u}\times\cos x\\ &= \frac{1}{\sin x}\times\cos x\\ &= \frac{\cos x}{\sin x}\end{aligned}$$

$$\therefore \frac{d}{dx}\left(\log_e(\sin x)\right) = \cot x$$

(b) $$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot dx - \frac{1}{2}\times\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\times\frac{\pi}{4} &= \left[\log_e(\sin x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \frac{\pi^2}{32}\\ &= \log_e\left(\sin\frac{\pi}{2}\right) - \log_e\left(\sin\frac{\pi}{4}\right) - \frac{\pi^2}{32}\\ &= \log_e 1 - \log_e\frac{1}{\sqrt{2}} - \frac{\pi^2}{32}\\ &= \log_e\sqrt{2} - \frac{\pi^2}{32}\\ &= \frac{1}{2}\log_e 2 - \frac{\pi^2}{32}\end{aligned}$$

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24 $a = 6 - 8t,\ t \geq 0$

$v = \int (6 - 8t)\ dt$
$v = 6t - 4t^2 + C$

When $t = 0,\ v = 10$

$6(0) - 4(0)^2 + C = 10$
$C = 10$
$\therefore v = 6t - 4t^2 + 10$

$x = \int (6t - 4t^2 + 10)\ dx$
$x = 3t^2 - \frac{4}{3}t^3 + 10 + C$

When $t = 0,\ x = 0$

$3(0)^2 - \frac{4}{3}(0)^3 + 10(0) + C = 0$
$C = 0$
$\therefore x = 3t^2 - \frac{4}{3}t^3 + 10$

$v = 0 \Rightarrow 6t - 4t^2 + 10 = 0$
$-2(2t^2 - 6t - 5) = 0$
$-2(t + 1)(2t - 5) = 0$
$t = 2.5$

When $t = 2.5,\ x = 3(2.5)^2 - \frac{4}{3}(2.5)^3 + 10(2.5) = 22\frac{11}{12}$

The velocity is 0 at $t = 2.5$ seconds.

The displacement at this time is $22\frac{11}{12}$ m.

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26 (a) $v = 6t^2 - 4t + 1$

$s = 2t^3 - 2t^2 + t + C$

$a = \dfrac{dv}{dt} = 12t - 4$

(b) $v = 6t^2 - 4t + 1 = 3$

$6t^2 - 4t - 2 = 0$

$2(3t^2 - 2t - 1) = 0$

$2(3t+1)(t-1) = 0$

$t = 1$

$a = 12(1) - 4 = 8 \text{ m s}^{-2}$

(c) $a = 12t - 4 = 20$

$12t = 24$

$t = 2$

$v = 6(2)^2 - 4(2) + 1$

$v = 17 \text{ m s}^{-2}$

28 (a) $\ddot{x} = 9\sin 3t$

$$\begin{aligned}\dot{x} &= \int (9\sin 3t)\ dt\\ &= -3\cos 3t + C\end{aligned}$$

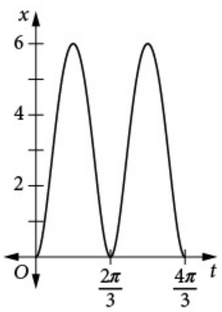
When $t = 0,\ \dot{x} = 0$

$$\begin{aligned}-3\cos(3\times 0) + C &= 0\\ -3 + C &= 0\\ C &= 3\end{aligned}$$

$\therefore \dot{x} = 3 - 3\cos 3t$

(b) The period is $\dfrac{2\pi}{3}$ and the amplitude is 3.

The graph will consist of two cycles.



The particle first comes to rest at $t = \dfrac{2\pi}{3}$.

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(c) The velocity is always positive or zero.

$$\begin{aligned}\text{Distance} &= \int_0^{\frac{2\pi}{3}} (3-3\cos 3t)\,dt \\ &= \left[3t-\sin 3t\right]_0^{\frac{2\pi}{3}} \\ &= 2\pi\end{aligned}$$

The particle travels from 0 to 6 and back to 0, a total distance of 12 units.

(d) $x = \int \dot{x}\ dt$

$$= \int (3-3\cos 3t)\ dt$$

$$x = 3t - \sin 3t + C$$

If the particle is originally at the origin, then when $t = 0,\ x = 0.$

$$0 = 3\times 0 - \sin(3\times 0) + C \Rightarrow C = 0$$

$$x = 3t - \sin 3t$$

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