EXAMPLE 16.1 PRIMITIVE FUNCTIONS

2 First expand $f'(x)$.

$$f'(x)=(x-1)(x-2)$$
$$=x^2-3x+2$$
$$f(x)=\frac{x^{2+1}}{2+1}-\frac{3x^{1+1}}{1+1}+2x+C$$
$$=\frac{x^3}{3}-\frac{3x^2}{2}+2x+C$$

(a) correct

(b) correct

(c) incorrect

(d) incorrect

4 $\text{LHS}=\dfrac{(2x+1)^3}{6}$

$$=\frac{8x^3+3\times 4x^2+3\times 2x+1}{6}$$
$$=\frac{8x^3}{6}+\frac{12x^2}{6}+\frac{6x}{6}+\frac{1}{6}$$
$$=\frac{4x^3}{3}+2x^2+x+\frac{1}{6}$$
$$=\frac{4x^3}{3}+2x^2+x+C$$
$$=\text{RHS}$$

where $C=\dfrac{1}{6}$

$$\therefore \frac{(2x+1)^3}{6}=\frac{4x^3}{3}+2x^2+x+C$$

6 **(a)** $f(x)=x^3-3x^2$

$$F(x)=\frac{x^4}{4}-\frac{3x^3}{3}+C$$
$$=\frac{x^4}{4}-x^3+C$$

(b) $f(x)=\dfrac{x^2}{5}-\dfrac{x^3}{4}$

$$F(x)=\frac{1}{5}\times\frac{x^3}{3}-\frac{1}{4}\times\frac{x^4}{4}+C$$
$$=\frac{x^3}{15}-\frac{x^4}{16}+C$$

 ISBN 978 1 4886 1829 1

(c) $f(x) = x^2(1-3x)$

$= x^2 - 3x^3$

$F(x) = \frac{x^3}{3} - \frac{3x^4}{4} + C$

(d) $f(x) = (x^2-1)(x^2+1)$

$= x^4 - 1$

$F(x) = \frac{x^5}{5} - x + C$

(e) $f(x) = x^{-\frac{3}{2}} + x^{-\frac{5}{2}}$

$$F(x) = \frac{x^{-\frac{3}{2}+\frac{2}{2}}}{-\frac{3}{2}+\frac{2}{2}} + \frac{x^{-\frac{5}{2}+\frac{2}{2}}}{-\frac{5}{2}+\frac{2}{2}} + C$$

$$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} + C$$

$$= -2x^{-\frac{1}{2}} - \frac{2x^{-\frac{3}{2}}}{3} + C$$

$$= -\frac{2}{\sqrt{x}} - \frac{2}{3x\sqrt{x}} + C$$

(f) $f(x) = \sqrt{x} + \sqrt[3]{x}$

$$= x^{\frac{1}{2}} + x^{\frac{1}{3}}$$

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C$$

$$= \frac{2x^{\frac{3}{2}}}{3} + \frac{3x^{\frac{4}{3}}}{4} + C$$

$$= \frac{2x\sqrt{x}}{3} + \frac{3x\sqrt[3]{x}}{4} + C$$

8 C

$f'(x) = 4x^2 - 3x$

$f(x) = \frac{4x^3}{3} - \frac{3x^2}{2} + C$

$f(-1) = 3$

$$\frac{4\times(-1)^3}{3} - \frac{3\times(-1)^2}{2} + C = 3$$

$$-\frac{4}{3} - \frac{3}{2} + C = 3$$

$$-\frac{17}{6} + C = 3$$

$$C = 3 + \frac{17}{6} = \frac{35}{6}$$

$$\therefore f(x) = \frac{4x^3}{3} - \frac{3x^2}{2} + \frac{35}{6}$$

 ISBN 978 1 4886 1829 1

10 $f'(x) = 3x^2 - 2x + 3$

$$\begin{aligned} f(x) &= \frac{3x^3}{3} - \frac{2x^2}{2} + 3x + C \\ &= x^3 - x^2 + 3x + C \end{aligned}$$

$f(3) = 3$

$3^3 - 3^2 + 3 \times 3 + C = 3$
$C = -24$

$\therefore f(x) = x^3 - x^2 + 3x - 24$

12 $\dfrac{dy}{dx} = 2x + b$

When $x = 3$, $\dfrac{dy}{dx} = 2$

$2 = 2 \times 3 + b$
$b = -4$

$\dfrac{dy}{dx} = 2x - 4$

$$\begin{aligned} y &= \frac{2x^2}{2} - 4x + C \\ &= x^2 - 4x + C \end{aligned}$$

When $x = 3$, $y = -3$

$-3 = 3^2 - 4 \times 3 + C$
$C = 0$

$\therefore y = x^2 - 4x$

14 $\dfrac{dy}{dx} = (x-1)(x+2)$

$= x^2 + x - 2$

$y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + C$

When $x = 0$, $y = 4$

$4 = \dfrac{0^3}{3} + \dfrac{0^2}{2} - 2 \times 0 + C$
$C = 4$

$\therefore y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 4$

 ISBN 978 1 4886 1829 1

16 $\frac{ds}{dt} = 12t^2 - 6t + 1$

$$s = \frac{12t^3}{3} - \frac{6t^2}{2} + t + C$$
$$= 4t^3 - 3t^2 + t + C$$

When $t = 1$, $s = 4$

$$4 = 4\times1^3 - 3\times1^2 + 1 + C$$
$$C = 2$$

$$\therefore s = 4t^3 - 3t^2 + t + 2$$

18 $\frac{dd}{dt} = v$

$$= 3t^2 + 4$$
$$d = \frac{3t^3}{3} + 4t + C$$
$$= t^3 + 4t + C$$

When $t = 0$, $d = 0$

$$0 = \frac{3\times0^3}{3} + 4\times0 + C$$
$$C = 0$$

$$d = t^3 + 4t$$

EXERCISE 16.2 INDEFINITE INTEGRALS

2 **(a)** $\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= \frac{2x^{\frac{3}{2}}}{3} + C$$
$$= \frac{2x\sqrt{x}}{3} + C$$

(b) $\int \frac{1}{x^2}dx = \int x^{-2}dx$

$$= \frac{x^{-1}}{-1} + C$$
$$= -\frac{1}{x} + C$$

(c) $\int\left(1+\sqrt{x}+x\right)dx=\int\left(1+x^{\frac{1}{2}}+x\right)dx$

$$=x+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^2}{2}+C$$

$$=x+\frac{2x^{\frac{3}{2}}}{3}+\frac{x^2}{2}+C$$

$$=x+\frac{2x\sqrt{x}}{3}+\frac{x^2}{2}+C$$

(d) $\int\left(x+\frac{1}{x^2}\right)dx=\int\left(x+x^{-2}\right)dx$

$$=\frac{x^2}{2}+\frac{x^{-1}}{-1}+C$$

$$=\frac{x^2}{2}-\frac{1}{x}+C$$

(e) $\int\left(x+\frac{1}{x}\right)^2 dx=\int\left(x^2+2+x^{-2}\right)dx$

$$=\frac{x^3}{3}+2x+\frac{x^{-1}}{-1}+C$$

$$=\frac{x^3}{3}+2x-\frac{1}{x}+C$$

(f) $\int\left(1-\sqrt{x}\right)^2 dx=\int\left(1-2x^{\frac{1}{2}}+x\right)dx$

$$=x-\frac{2x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^2}{2}+C$$

$$=x-\frac{4x^{\frac{3}{2}}}{3}+\frac{x^2}{2}+C$$

$$=x-\frac{4x\sqrt{x}}{3}+\frac{x^2}{2}+C$$

4 $\frac{dy}{dx}=1+x+3x^2$

$$y=x+\frac{x^2}{2}+\frac{3x^3}{3}+C$$

$$=x+\frac{x^2}{2}+x^3+C$$

At $(2,6)$,

$$6=2+\frac{2^2}{2}+2^3+C$$

$$C=-6$$

$$\therefore y=x+\frac{x^2}{2}+x^3-6$$

EXERCISE 16.3 PRIMITIVES OF TRIGONOMETRIC FUNCTIONS

2 D

$$f'(x) = 3\cos\frac{x}{3}$$

$$f(x) = 3 \times \frac{\sin\frac{x}{3}}{\frac{1}{3}} + C$$

$$= 3 \times 3 \times \sin\frac{x}{3} + C$$

$$= 9\sin\frac{x}{3} + C$$

4 (a) Let $f(x) = 1 - \cos x$

$$f'(x) = \sin x$$

$$\int \frac{\sin x}{1-\cos x}dx = \int \frac{f'(x)}{f(x)}dx$$

$$= \ln\left(f(x)\right) + C$$

$$= \ln\left(1-\cos x\right) + C$$

(b) Let $f(x) = \sin x$

$$f'(x) = \cos x$$

$$\int \frac{\cos x}{\sin x}dx = \int \frac{f'(x)}{f(x)}dx$$

$$= \ln\left(f(x)\right) + C$$

$$= \ln\left(\sin x\right) + C$$

EXERCISE 16.4 INTEGRATING THE EXPONENTIAL FUNCTION

2 (a) $\int e^{-x}dx = \frac{1}{-1}e^{-x} + C$

$$= -e^{-x} + C$$

(b) $\int e^{\frac{x}{2}}dx = \frac{1}{\frac{1}{2}}e^{\frac{x}{2}} + C$

$$= 2e^{\frac{x}{2}} + C$$

(c) $\int e^{-3x}dx = \frac{1}{-3}e^{-3x} + C$

$$= -\frac{1}{3}e^{-3x} + C$$

(d) $\int \left(e^{-t} - 1\right)dt = \frac{1}{-1}e^{-t} - t + C$

$$= -e^{-t} - t + C$$

(e) $\int \left(e^{2u} + u^2\right)du = \frac{1}{2}e^{2u} + \frac{u^3}{3} + C$

(f) $$\int\left(e^{-2.5x}+e^{0.4x}\right)dx=-\frac{1}{2.5}e^{-2.5x}+\frac{1}{0.4}e^{0.4x}+C$$
$$=-\frac{2}{5}e^{-2.5x}+\frac{5}{2}e^{0.4x}+C$$

EXERCISE 16.5 INTEGRALS RESULTING IN LOGARITHMIC FUNCTIONS

2 **C**

$$\frac{dy}{dx}=\frac{1}{x}$$
$$y=\log_e x+C$$

When $x=2$, $y=0$

$$0=\log_e 2+C$$
$$C=-\log_e 2$$

$$y=\log_e x-\log_e 2$$
$$=\log_e\left(\frac{x}{2}\right)$$

4 Let $g(x)=x^2+9$.

$$g'(x)=2x$$

$$f'(x)=\frac{x}{x^2+9}$$
$$=\frac{1}{2}\times\frac{2x}{x^2+9}$$
$$f(x)=\int\frac{x}{x^2+9}dx$$
$$=\frac{1}{2}\int\frac{2x}{x^2+9}dx$$
$$=\frac{1}{2}\int\frac{g'(x)}{g(x)}dx$$
$$=\frac{1}{2}\ln\left(g(x)\right)+C$$
$$=\frac{1}{2}\ln\left(x^2+9\right)+C$$

ISBN 978 1 4886 1829 1

$$f(0) = \log_e 3$$
$$\frac{1}{2}\log_e(0+9) + C = \log_e 3$$
$$\frac{1}{2}\log_e 9 + C = \log_e 3$$
$$\log_e \sqrt{9} + C = \log_e 3$$
$$\log_e 3 + C = \log_e 3$$
$$C = 0$$

$$f(x) = \frac{1}{2}\log_e\left(x^2+9\right)$$

(a) correct **(b)** correct

(c) incorrect **(d)** correct

6 **(a)** $\frac{d}{dx}\left(a^x\right) = a^x \log_e a$

$\frac{d}{dx}\left(2^x\right) = 2^x \log_e 2$

(b) $f(x) = x + 10^x$

$f'(x) = 1 + 10^x \log_e 10$

(c) $f(x) = e^x + 5^x$

$f'(x) = e^x + 5^x \log_e 5$

(d) Use the chain rule where $u = x^2$ so $\frac{du}{dx} = 2x$ and $5^{x^2} = 5^u$.

$$\frac{d5^u}{dx} = \frac{d5^u}{du} \times \frac{du}{dx}$$
$$\frac{d5^u}{dx} = 5^u \log_e 5 \times 2x$$
$$\frac{d5^{x^2}}{dx} = 2x \times 5^{x^2} \log_e 5 \times$$

(e) Recall that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

Use the chain rule where $u = \sqrt{x}$ so $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ and $a^{\sqrt{x}} = a^u$.

$$\frac{da^u}{dx} = \frac{da^u}{du} \times \frac{du}{dx}$$
$$\frac{da^u}{dx} = a^u \log_e a \times \frac{1}{2\sqrt{x}}$$
$$\frac{da^{\sqrt{x}}}{dx} = \frac{a^{\sqrt{x}} \log_e a}{2\sqrt{x}}$$

 ISBN 978 1 4886 1829 1

CHAPTER REVIEW 16

2 (a) $\dfrac{dy}{dx} = 5x + 4$

$$y = \frac{5x^2}{2} + 4x + C$$

(b) $\dfrac{dy}{dx} = 5 - 4x + 3x^2 + x^3$

$$\begin{aligned} y &= 5x - \frac{4x^2}{2} + \frac{3x^3}{3} + \frac{x^4}{4} + C \\ &= 5x - 2x^2 + x^3 + \frac{x^4}{4} + C \end{aligned}$$

(c) $\dfrac{dy}{dx} = 2x + \sqrt{x} + 3$

$$\begin{aligned} y &= \frac{2x^2}{2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3x + C \\ &= x^2 + \frac{2x\sqrt{x}}{3} + 3x + C \end{aligned}$$

4 (a) $\dfrac{dV}{dt} = 140 + 13t - t^2$

$$V = 140t + \frac{13t^2}{2} - \frac{t^3}{3} + C$$

When $t = 0$, $V = 0$

$$V = 140t + \frac{13t^2}{2} - \frac{t^3}{3}$$

(b) When $t = 12$,

$$\begin{aligned} V &= 140 \times 12 + \frac{13 \times 12^2}{2} - \frac{12^3}{3} \\ &= 2040 \end{aligned}$$

There are 2040 litres after 12 minutes.

 ISBN 978 1 4886 1829 1

6 **(a)** $\int 3\sin\frac{x}{2}\,dx = 3\times\frac{1}{\frac{1}{2}}\times\left(-\cos\frac{x}{2}\right)+C$

$= -6\cos\frac{x}{2}+C$

(b) $\int\left(x+\sec^2 2x\right)dx = \frac{x^2}{2}+\frac{1}{2}\tan 2x + C$

(c) $\int\frac{\cos t}{\sin t}\,dt = \ln\left(\sin t\right)+C$