2023 BORED OF STUDIES TRIAL EXAMINATION

Mathematics Extension 1

Solutions

Section I

Answers

1	D	**6**	B
2	D	**7**	C
3	C	**8**	D
4	D	**9**	A
5	D	**10**	B

Brief explanations

1 If $y = f^{-1}(x)$ then

$$\begin{aligned}
x &= f(y) \\
\frac{dx}{dy} &= f'(y) \\
\frac{dy}{dx} &= \frac{1}{f'(f^{-1}(x))} \qquad \text{let } g(x) = f^{-1}(x) \\
g'(x) &= \frac{1}{f'(g(x))} \\
g'(0) &= \frac{1}{f'(g(0))} \qquad \text{but } g(0) = 1 \\
&= \frac{1}{f'(1)} \\
&= \frac{1}{2}
\end{aligned}$$

Hence, the answer is (D).

2 Given the divisor and remainder

$$\begin{aligned}
P(x) &= D(x)cx + d \\
acx^3 + bcx + d &\equiv D(x)cx + d \\
cx(ax^2 + b) &\equiv D(x)cx + d \\
D(x) &= ax^2 + b
\end{aligned}$$

Hence, the answer is (D).

3 Since $x^2 \geq 0$ and $\sin^{-1} x$ is an increasing function then

$$\begin{aligned} 1 + 2x^2 &\geq 1 \\ -\sqrt{1+2x^2} &\leq -1 \\ 1 - \sqrt{1+2x^2} &\leq 0 \\ \sin^{-1}\left(1 - \sqrt{1+2x^2}\right) &\leq 0 \end{aligned}$$

However, note that the lower bound of $1 - \sqrt{1+2x^2}$ must be -1 so the range must be $\left[-\frac{\pi}{2}, 0\right]$. Hence, the answer is (C).

4 The dot product must be zero for the vectors to be perpendicular so $ac + bd = 0$. A necessary condition is that ac and bd must be opposite in sign. The only option which has this property is (D).

5 The general term is $c_k = \binom{8}{k} 3^k$ so numerically evaluating each option

$$\begin{aligned} c_1 &= \binom{8}{1} \times 3^1 = 24 \\ c_3 &= \binom{8}{3} \times 3^3 = 1512 \\ c_5 &= \binom{8}{5} \times 3^5 = 13608 \\ c_7 &= \binom{8}{7} \times 3^7 = 17496 \end{aligned}$$

Hence, the answer is (D).

Remark: Note that this is actually *not* the largest coefficient of the entire binomial expansion. However, it is the largest coefficient of the given options.

6 Rearranging gives

$$\begin{aligned} R\cos(x+\alpha) &= -\sqrt{2}\sin x - \cos\left(x + \frac{\pi}{4}\right) \\ R\cos x\cos\alpha - R\sin x \sin\alpha &= -\sqrt{2}\sin x - \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x \\ &= -\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x \\ R\cos\alpha = -\frac{1}{\sqrt{2}} \quad &\text{and} \quad R\sin\alpha = \frac{1}{\sqrt{2}} \\ \tan\alpha &= -1 \\ \alpha &= \frac{3\pi}{4} \end{aligned}$$

Noting that since $R > 0$ then the only quadrant which has positive value for $\sin\alpha$ and negative value for $\cos\alpha$ is the second quadrant. Hence, the answer is (B).

7 First consider the case when $x = 0$ then as y increases to a large positive value, the slope is almost flat. This similarly occurs when y decreases to a large negative value. The only options which have this property are (C) and (D).

Notice there is a flat slope at $(-1, 1)$. Option (C) gives $\dfrac{dy}{dx} = 0$ at this point whereas option (D) gives $\dfrac{dy}{dx} = -1 + e^{-2}$ which is negative at this point.

Only option (C) is consistent with this observation.

8 There are two cases to consider.

Case 1: The couple sit in the two seat row

There are 2! ways for the couple to arrange in the two seat row. There are 4! ways to arrange in the four seat row. The total number of arrangements is $4! \times 2!$.

Case 2: The couple sit in the four seat row

In the two seat row there are $\binom{4}{2}$ possible groups and 2! ways to arrange within that row. In the four seat row, there are 2! ways for the couple to arrange within themselves. Treating the couple as one group there are 3! ways to arrange all members of the four seat row. Therefore, the total number of arrangements is $\binom{4}{2} \times 2! \times 2! \times 3!$.

In total there are $4! \times 2! + \binom{4}{2} \times 2! \times 2! \times 3!$, or equivalently, 192 ways to arrange the group. Hence, the answer is (D).

9 Notice that

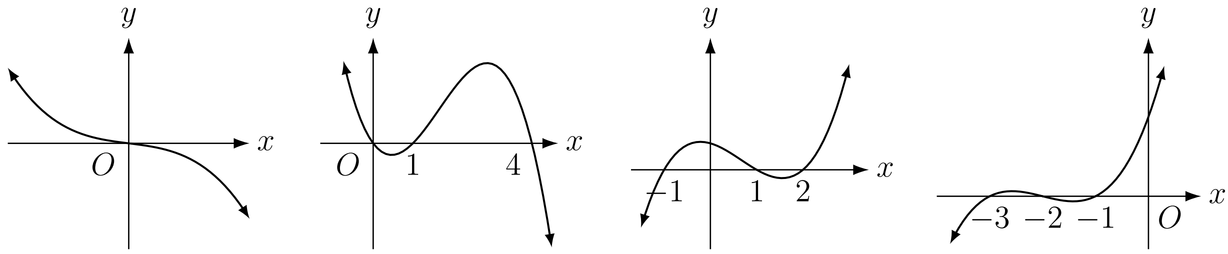
$$\begin{aligned}
f(x) &= 8\cos^6 x - 12\cos^4 x + 6\cos^2 x - 1 \\
&= 8\cos^6 x - 8\cos^4 x + 2\cos^2 x - 4\cos^4 x + 4\cos^2 x - 1 \\
&= 2\cos^2 x(4\cos^4 x - 4\cos^2 x + 1) - (4\cos^4 x - 4\cos^2 x + 1) \\
&= (2\cos^2 x - 1)(4\cos^4 x - 4\cos^2 x + 1) \\
&= (2\cos^2 x - 1)^3 \\
&= \cos 2x \cos^2 2x \\
&= \frac{1}{2}\cos 2x(\cos 4x + 1) \\
&= \frac{1}{4}(\cos(4x - 2x) + \cos(4x + 2x)) + \frac{1}{2}\cos 2x \\
&= \frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x \\
\int f(x)\,dx &= \int \left(\frac{1}{4}\cos 6x + \frac{3}{4}\cos 2x\right) dx \\
&= \frac{1}{24}\sin 6x + \frac{3}{8}\sin 2x + c
\end{aligned}$$

Hence, the answer is (A).

10 A necessary condition for the integral to be zero is that the polynomial must have a mix of positive and negative function values in the domain $x > 0$.

Consider the sketches of each option below.

(A) $\quad$ (B) $\quad$ (C) $\quad$ (D)



$y = -x(x^2+1)$ $\quad$ $y = -x(x-1)(x-4)$ $\quad$ $y = (x+1)(x-1)(x-2)$ $\quad$ $y = (x+1)(x+2)(x+3)$

For option (A), when $x > 0$ then $-x(x^2+1) < 0$ so it is not possible for the integral to be zero.

For option (D), when $x > 0$ then $(x+1)(x+2)(x+3) > 0$ so it is not possible for the integral to be zero.

For option (C), it can be seen that $\displaystyle\int_0^1 (x+1)(x-1)(x-2)\,dx > 0$. So now consider

$$\begin{aligned}\int_0^2 (x+1)(x-1)(x-2)\,dx &= \int_0^2 (x^3 - 2x^2 - x + 2)\,dx \\ &= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_0^2 \\ &= \frac{2}{3}\end{aligned}$$

This means that the entire 'negative' region is insufficient to offset the 'positive' region, so the answer cannot be (C).

For option (B), it can be seen that $\displaystyle\int_0^1 -x(x-1)(x-4)\,dx < 0$. So now consider

$$\begin{aligned}\int_0^4 -x(x-1)(x-4)\,dx &= \int_0^4 (-x^3 + 5x^2 - 4x)\,dx \\ &= \left[-\frac{x^4}{4} + \frac{5x^3}{3} - 2x^2\right]_0^4 \\ &= \frac{32}{3}\end{aligned}$$

This means that the 'positive' region is more than sufficient enough to offset the 'negative' region. This means there must exist some $1 < a < 4$ where $\displaystyle\int_0^a -x(x-1)(x-4)\,dx = 0$. Hence, the answer is (B).

Question 11

(a) First find a vector parallel to the line $2x + 3y + 6 = 0$. Use the x-intercept $(-3, 0)$ and y-intercept $(0, -2)$ as convenient points which can construct a vector $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$. Let $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j}$ and so

$$\begin{aligned}
\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} \\
&= \frac{1}{\sqrt{13}}(\underset{\sim}{i} - 2\underset{\sim}{j}) \cdot (3\underset{\sim}{i} - 2\underset{\sim}{j})\hat{\underset{\sim}{b}} \\
&= \frac{7}{\sqrt{13}}\hat{\underset{\sim}{b}} \\
&= \frac{7}{13}(3\underset{\sim}{i} - 2\underset{\sim}{j})
\end{aligned}$$

(b) Using the fact that $\sin 2x = 2\sin x \cos x$

$$\begin{aligned}
\text{LHS} &= \cos 20^\circ \cos 40^\circ \cos 80^\circ \\
&= \frac{\sin 40^\circ}{2\sin 20^\circ} \times \frac{\sin 80^\circ}{2\sin 40^\circ} \times \frac{\sin 160^\circ}{2\sin 80^\circ} \\
&= \frac{\sin 160^\circ}{8\sin 20^\circ} \qquad \text{but } \sin(180^\circ - x) = \sin x \\
&= \frac{1}{8} \\
&= \text{RHS}
\end{aligned}$$

(c) (i) Since $P(x)$ has a double root then this double root satisfies $P'(x) = 0$

$$\begin{aligned}
3x^2 - 3 &= 0 \\
x &= \pm 1
\end{aligned}$$

$P(-1) = 4$ and $P(1) = 0$ so the double root is $x = 1$. Let the remaining root be α. By sum of roots

$$\begin{aligned}
1 + 1 + \alpha &= 0 \\
\alpha &= -2
\end{aligned}$$

The roots of $P(x)$ are $1, 1, -2$.

(ii) Solving the inequality

$$
\begin{aligned}
\frac{x^2 - x + 2}{4x} &\le \frac{1}{x+1} \\
\frac{x^2 - x + 2}{4x} - \frac{1}{x+1} &\le 0 \\
\frac{(x+1)(x^2 - x + 2) - 4x}{4x(x+1)} &\le 0 \\
\frac{x^3 - x^2 + 2x + x^2 - x + 2 - 4x}{x(x+1)} &\le 0 \\
x(x+1)(x^3 - 3x + 2) &\le 0 \\
x(x+1)(x-1)^2(x+2) &\le 0 \quad \text{by factor theorem using roots from part (i)}
\end{aligned}
$$

Since $(x-1)^2 \ge 0$ for all real x then $x = 1$ satisfies the inequality. When $x \ne 1$ then $(x-1)^2 > 0$ which implies $x(x+1)(x+2) \le 0$.

Noting that $x \ne 0$ and $x \ne -1$, the solutions are $x = 1, x \le -2$ or $-1 < x < 0$.

(d) (i) Note that $\dfrac{dR}{dt} = -2(R-5)$.

The general solution to the shifted exponential decay equation is $R = 5 + Be^{-2t}$ for some constant B.

When $t = 0, R = 1$ so $B = -4$.

Hence, the specific solution is $R = 5 - 4e^{-2t}$.

(ii) Let A be the area of the disc so $A = \pi R^2$ so using related rates

$$
\begin{aligned}
\frac{dA}{dt} &= \frac{dA}{dR} \times \frac{dR}{dt} \\
&= 4\pi R(5 - R)
\end{aligned}
$$

The rate of change is the disc's area is described by the concave down parabola $y = 4\pi x(5-x)$. This has a global maximum when $x = \frac{5}{2}$. Substitute this into R to find t

$$
\begin{aligned}
\frac{5}{2} &= 5 - 4e^{-2t} \\
e^{-2t} &= \frac{5}{8} \\
t &= \frac{1}{2}\ln\frac{8}{5}
\end{aligned}
$$

(e) (i) Let the velocity vector be $\underset{\sim}{v} = v_x\underset{\sim}{i} + v_y\underset{\sim}{j}$. Since the velocity is constant then at time t then

$$\begin{aligned}\underset{\sim}{r} &= -5\underset{\sim}{i} + 10\underset{\sim}{j} + t(v_x\underset{\sim}{i} + v_y\underset{\sim}{j}) \\ &= (-5 + tv_x)\underset{\sim}{i} + (10 + tv_y)\underset{\sim}{j}\end{aligned}$$

When $t = 3, \underset{\sim}{r} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$ so

$$\begin{aligned}-5 + (3)v_x &= 4 \Rightarrow v_x = 3 \\ 10 + (3)v_y &= -2 \Rightarrow v_y = -4\end{aligned}$$

Hence, the displacement vector is $\underset{\sim}{r} = (-5 + 3t)\underset{\sim}{i} + (10 - 4t)\underset{\sim}{j}$.

(ii) From the working in part (i), the velocity vector is $\underset{\sim}{v} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$. It's magnitude is $|\underset{\sim}{v}| = \sqrt{3^2 + 4^2} = 5$.

The direction of the vector in true bearing is the angle the vector makes to the positive vertical (as north).

The vector $3\underset{\sim}{i} - 4\underset{\sim}{j}$ lies in the fourth quadrant so if θ is the angle to the positive horizontal then $\tan\theta = \dfrac{4}{3}$. The true bearing is $90^\circ + \tan^{-1}\dfrac{4}{3} \approx 143^\circ$T.

Hence, the velocity of the boat is 5km per hour directed at 143°T.

Question 12

(a) (i) Let X be the total number questions the student answers correctly. Let M be the total marks attained by the student, which is related by

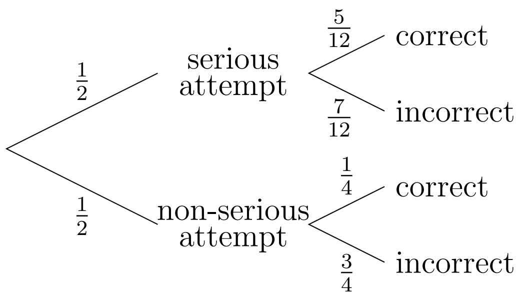
$$\begin{aligned} M &= 3X - (25 - X) \\ &= 4X - 25 \end{aligned}$$

Given $X \sim B\left(25, \frac{1}{4}\right)$ then

$$\begin{aligned} P(M = 15) &= P(4X - 25 = 15) \\ &= P(X = 10) \\ &= \binom{25}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^{15} \end{aligned}$$

(ii) Let S be a Bernoulli random variable which represents whether the student makes a serious or non-serious attempt at the question, so $S \sim B\left(1, \frac{1}{2}\right)$.

Consider a probability tree of all the outcomes.



The unconditional probability of getting the answer correct for a question is

$$\begin{aligned} P(X = 1) &= P(X = 1|S = 0) + P(X = 1|S = 1) \\ &= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{5}{12} \\ &= \frac{1}{3} \end{aligned}$$

The overall probability of getting the correct answer has improved to $\dfrac{1}{3}$ so now $X \sim B\left(25, \frac{1}{3}\right)$ which means

$$\begin{aligned} P(M \geq 27) &= P(4X - 25 \geq 27) \\ &= P(X \geq 13) \\ &= P\left(\frac{X - 25\left(\frac{1}{3}\right)}{\sqrt{25\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}} \geq \frac{13 - 25\left(\frac{1}{3}\right)}{\sqrt{25\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}}\right) \\ &\approx P(Z \geq 1.98) \quad \text{where } Z \sim N(0,1) \\ &\approx P(Z \geq 2) \\ &\approx 0.025 \end{aligned}$$

Noting that from the reference sheet, $P(-2 \leq Z \leq 2) = 0.95$, which implies $P(Z \geq 2) = 0.025$, by symmetry of the standard normal curve.

(b) (i) Consider the vector $\overrightarrow{AP}$ which related to $\overrightarrow{AO}$ by a scalar. Let $\hat{\underset{\sim}{u}}$ be the unit vector in the direction of $\overrightarrow{AO}$.

$$\begin{aligned}\overrightarrow{AO} &= \frac{1}{2}\underset{\sim}{i} + \frac{1}{2}\underset{\sim}{j} \\ &= \frac{1}{2}(\underset{\sim}{i} + \underset{\sim}{j}) \\ \hat{\underset{\sim}{u}} &= \frac{1}{\sqrt{2}}(\underset{\sim}{i} + \underset{\sim}{j})\end{aligned}$$

Use this to derive $\overrightarrow{AP}$.

$$\begin{aligned}\overrightarrow{AP} &= |\overrightarrow{AP}|\hat{\underset{\sim}{u}} \\ &= \left(|\overrightarrow{AO}| + |\overrightarrow{OP}|\right)\hat{\underset{\sim}{u}} \quad \text{but } |\overrightarrow{OP}| = \frac{1}{2} \text{ and } |\overrightarrow{AO}| = \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)(\underset{\sim}{i} + \underset{\sim}{j}) \\ &= \frac{\sqrt{2}+2}{4}(\underset{\sim}{i} + \underset{\sim}{j})\end{aligned}$$

Hence

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{AP} - \overrightarrow{AB} \\ &= \frac{\sqrt{2}+2}{4}(\underset{\sim}{i} + \underset{\sim}{j}) - \underset{\sim}{i} \\ &= \left(\frac{\sqrt{2}-2}{4}\right)\underset{\sim}{i} + \left(\frac{\sqrt{2}+2}{4}\right)\underset{\sim}{j}\end{aligned}$$

(ii) First note that

$$\begin{aligned} OP &= OQ \quad \text{(equal radii)} \\ \therefore\ &\triangle POQ \text{ is isosceles} \\ \Rightarrow\ &\angle OPQ = \angle OQP \end{aligned}$$

Finding $\angle OPQ$ using the unit vector derived earlier in working for (i)

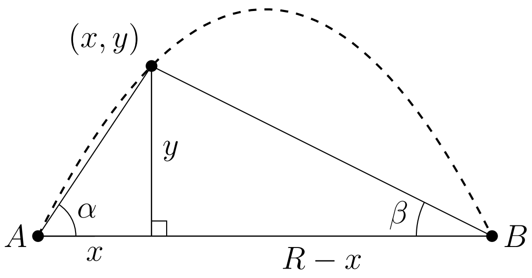
$$\begin{aligned} \cos\angle OPQ &= \frac{(-\underset{\sim}{\hat{u}})\cdot\overrightarrow{PB}}{|\underset{\sim}{\hat{u}}||\overrightarrow{PB}|} \\ &= \frac{\underset{\sim}{\hat{u}}\cdot\overrightarrow{BP}}{|\overrightarrow{BP}|} \\ \underset{\sim}{\hat{u}}\cdot\overrightarrow{BP} &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}-2}{4}\right)+\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}+2}{4}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} |\overrightarrow{BP}| &= \sqrt{\left(\frac{\sqrt{2}-2}{4}\right)^2+\left(\frac{\sqrt{2}+2}{4}\right)^2} \\ &= \frac{1}{4}\sqrt{2-2\sqrt{2}+4+2+2\sqrt{2}+4} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \cos\angle OPQ &= \frac{1}{\sqrt{3}} \\ \cos\angle POQ &= \cos(\pi - 2\angle OPQ) \\ &= -\cos 2\angle OPQ \\ &= -(2\cos^2\angle OPQ - 1) \qquad \text{but } \cos\angle OPQ = \frac{1}{\sqrt{3}} \\ &= \frac{1}{3} \end{aligned}$$

(c) (i) The displacement of the ball has coordinates (x, y) at time t as shown below.



$$
\begin{aligned}
\tan\alpha &= \frac{y}{x}\\
&= \frac{ut\sin\theta - \dfrac{gt^2}{2}}{ut\cos\theta}\\
&= \tan\theta - \frac{gt}{2u\cos\theta}
\end{aligned}
$$

Since $u >$, $g > 0$ and $0 < \theta < \dfrac{\pi}{2}$ then $\tan\alpha$ is a decreasing linear function of t.

(ii) First note that the range R is a fixed value which occurs with some time of flight (call this T). This means that $R = uT\cos\theta$ and so

$$
\begin{aligned}
\tan\beta &= \frac{y}{R-x}\\
&= \frac{ut\sin\theta - \dfrac{gt^2}{2}}{uT\cos\theta - ut\cos\theta}\\
&= \frac{2ut\sin\theta - gt^2}{2u(T-t)\cos\theta}
\end{aligned}
$$

However, T can be evaluated directly by solving for $y = 0$

$$
\begin{aligned}
uT\sin\theta - \frac{gT^2}{2} &= 0\\
T &= \frac{2u\sin\theta}{g} \qquad \text{noting that } T > 0
\end{aligned}
$$

Noting that $t < T$ when the ball is in flight then

$$
\begin{aligned}
\tan\beta &= \frac{gtT - gt^2}{2u(T-t)\cos\theta}\\
&= \frac{gt(T-t)}{2u(T-t)\cos\theta}\\
&= \frac{gt}{2u\cos\theta}\\
\tan\alpha + \tan\beta &= \tan\theta - \frac{gt}{2u\cos\theta} + \frac{gt}{2u\cos\theta}\\
&= \tan\theta
\end{aligned}
$$

Hence, $\tan\alpha + \tan\beta$ is independent of time.

(d) (i) Using sum to product and double angle results

$$\begin{aligned}
\text{LHS} &= \sin 7\theta + \sin\theta \\
&= \sin(4\theta + 3\theta) + \sin(4\theta - 3\theta) \\
&= 2\sin 4\theta \cos 3\theta \\
&= 2(2\sin 2\theta\cos 2\theta)(4\cos^3\theta - 3\cos\theta) \quad \text{using the given result} \\
&= 8\sin\theta\cos\theta(1 - 2\sin^2\theta)(4\cos^3\theta - 3\cos\theta) \\
&= 8\sin\theta\cos^2\theta(1 - 2\sin^2\theta)(4\cos^2\theta - 3) \\
&= 8\sin\theta(1 - \sin^2\theta)(1 - 2\sin^2\theta)(4(1 - \sin^2\theta) - 3) \\
&= 8\sin\theta(1 - \sin^2\theta)(1 - 2\sin^2\theta)(1 - 4\sin^2\theta) \\
&= 8\sin\theta(1 - \sin^2\theta)(1 - 6\sin^2\theta + 8\sin^4\theta) \\
&= 8\sin\theta(1 - 6\sin^2\theta + 8\sin^4\theta - \sin^2\theta + 6\sin^4\theta - 8\sin^6\theta) \\
&= 8\sin\theta(1 - 7\sin^2\theta + 14\sin^4\theta - 8\sin^6\theta) \\
&= 8\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta \\
&= \text{RHS}
\end{aligned}$$

(ii) Rearranging the result in part (i) gives

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

Let $x = \sin\theta$, so the roots of $64x^6 - 112x^4 + 56x^2 - 7$ can be interpreted as solving the equation

$$\begin{aligned}
64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7 &= 0 \\
-\frac{\sin 7\theta}{\sin\theta} &= 0 \\
\sin 7\theta &= 0 \quad \text{where } \sin\theta \neq 0 \\
7\theta &= \pm\pi, \pm 2\pi, \pm 3\pi \\
\theta &= \pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7},
\end{aligned}$$

Note that other solutions (such as $\theta = 0$) either contradict the condition that $\sin\theta \neq 0$ or give the same value of $\sin\theta$ as the existing solutions.

The roots are $\pm\sin\frac{\pi}{7}, \pm\sin\frac{2\pi}{7}$ and $\pm\sin\frac{3\pi}{7}$.

(iii) The 6th degree polynomial $64x^6 - 112x^4 + 56x^2 - 7$ can also be interpreted as a cubic polynomial by letting $u = x^2$, which reduces it to $64u^3 - 112u^2 + 56u - 7$.

Equivalently, this is a cubic polynomial in $u = \sin^2 \theta$.

From part (ii) it has the roots $\sin^2 \dfrac{\pi}{7}, \sin^2 \dfrac{2\pi}{7}$ and $\sin^2 \dfrac{3\pi}{7}$.

Let $\alpha = \sin^2 \dfrac{\pi}{7}, \beta = \sin^2 \dfrac{2\pi}{7}$ and $\gamma = \sin^2 \dfrac{3\pi}{7}$.

$$\begin{aligned}
\text{LHS} &= \operatorname{cosec}^2\frac{\pi}{7} + \operatorname{cosec}^2\frac{2\pi}{7} + \operatorname{cosec}^2\frac{3\pi}{7} \\
&= \frac{1}{\sin^2 \frac{\pi}{7}} + \frac{1}{\sin^2 \frac{2\pi}{7}} + \frac{1}{\sin^2 \frac{3\pi}{7}} \\
&= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \\
&= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \\
&= \frac{56}{64} \times \frac{64}{7} \\
&= 8 \\
&= \text{RHS}
\end{aligned}$$

Question 13

(a) First note that $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ is an even function and when $x \to \infty$ then $\dfrac{1-x^2}{1+x^2} \to -1$ and when $x = 0$, $\dfrac{1-x^2}{1+x^2} = 1$.

This suggests that the domain of of $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ is $(-1, 1]$ and its range is $[0, \pi)$.

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$ with $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.

As a general solution, for some integer k

$$\begin{aligned}
\cos 2\theta &= \frac{1-x^2}{1+x^2} \quad \text{(by } t \text{ formula)} \\
2\theta &= 2k\pi \pm \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \\
\theta &= k\pi \pm \frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad \text{but } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \pm\frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)
\end{aligned}$$

When $x \geq 0$ then $0 \leq \theta < \dfrac{\pi}{2} \Rightarrow 0 \leq 2\theta < \pi$ which aligns with the range of $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, hence for $x \geq 0$

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

When $x < 0$ then $-\dfrac{\pi}{2} < \theta < 0 \Rightarrow -\pi < 2\theta < 0$ which aligns with the range of $-\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, hence for $x < 0$

$$\theta = -\frac{1}{2}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

Altogether, this means

$$\tan^{-1} x = \begin{cases} -\dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{for } x < 0 \\[2ex] \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{for } x \geq 0. \end{cases}$$

(b) (i) The parametric equations of the circle $x^2 + y^2 = r^2$ for some parameter θ are

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\frac{dx}{d\theta} &= -r\sin\theta \\
&= -y \\
\frac{dy}{d\theta} &= r\cos\theta \\
&= x
\end{aligned}$$

Differentiate both sides of the given property with respect to θ, noting that r is a constant and independent of θ.

$$\begin{aligned}
g(r) &= f(x)f(y) \\
\frac{d}{d\theta}\left[f(x)f(y)\right] &= 0 \\
f(x)\frac{d}{d\theta}f(y) + f(y)\frac{d}{d\theta}f(x) &= 0 \\
f(x)\frac{d}{dy}f(y)\frac{dy}{d\theta} + f(y)\frac{d}{dx}f(x)\frac{dx}{d\theta} &= 0 \\
f(x)f'(y)x - f(y)f'(x)y &= 0 \\
\frac{f'(x)}{xf(x)} &= \frac{f'(y)}{yf(y)}
\end{aligned}$$

Noting that $f(x) > 0$ and $f(y) > 0$ as $f(z)$ is a probability density function. Also, note that $x \neq 0$ and $y \neq 0$.

(ii) The LHS and RHS of the result in part (i) are two identical expressions except x is interchanged with y. Since this is true for all (x, y) on the circle $x^2 + y^2 = r^2$ then it must be that they must equal a constant value.

However, note that $f(x)$ has a global maximum turning point at $x = 0$. This means that when $x < 0$ then $f'(x) > 0$ which implies $xf'(x) < 0$. Similarly when $x > 0$ then $f'(x) < 0$ which also implies $xf'(x) < 0$. Since $f(x) > 0$ for all x (because it is a probability density function) then the constant must be negative since it is a positive value divided by negative value.

Hence, $\dfrac{f'(x)}{xf(x)} = -k$ for some constant $k > 0$.

(iii) Solving the differential equation

$$\begin{aligned}
\frac{f'(x)}{xf(x)} &= -k \\
\frac{f'(x)}{f(x)} &= -kx \\
\int \frac{f'(x)}{f(x)}\,dx &= -k\int x\,dx \\
\ln f(x) &= -\frac{kx^2}{2} + c \qquad \text{when } x = 0, f(x) = 1 \Rightarrow c = 0 \\
f(x) &= e^{-\frac{kx^2}{2}}
\end{aligned}$$

The final step is to find a specific value for the constant k. Since $f(x)$ is a probability density function on all real x then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1 \\
\int_{-\infty}^{\infty} e^{-\frac{kx^2}{2}}\,dx &= 1
\end{aligned}$$

Let $u = x\sqrt{k} \Rightarrow du = \sqrt{k}\,dx$. When $x \to \infty$ then $u = \infty$ and when $x \to -\infty$ then $u \to -\infty$ so

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{kx^2}{2}}\,dx &= \frac{1}{\sqrt{k}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du \\
1 &= \frac{1}{\sqrt{k}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du
\end{aligned}$$

However, recall that for a standard normal probability density function

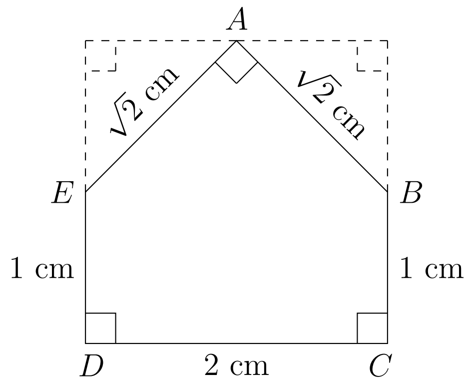
$$\int_{-\infty}^{\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,du = 1$$

Hence, $k = 2\pi$ so the specific solution is $f(x) = e^{-\pi x^2}$.

(c) (i) Fix any random point inside the pentagon as the first point. If the second point does not lie on perimeter of the pentagon, then it is always possible to increase the distance to the fixed point by moving that second point further out towards the pentagon's perimeter. Noting that the first and second points are interchangeable, then this means the longest distance between any two points must be between two points on the perimeter of the pentagon.

Now consider if the first point is fixed on an edge of the pentagon. If the second point also lies on an edge then it is always possible to increase the distance further by moving along the edge in a certain direction. This does not necessarily apply if that second point lies on a vertex instead. Noting that the first and second points are interchangeable, then this means the longest distance must be between any two vertices. In particular, non-adjacent vertices because the distance can always be increased if the points lie on two adjacent vertices.

(ii) Label the vertices A, B, C, D and E as below.

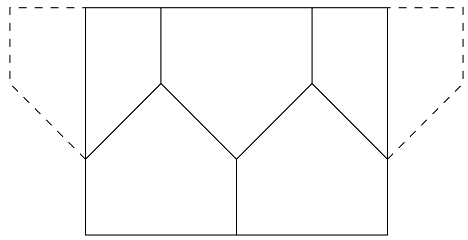


Consider all the possible pairwise distances between non-adjacent vertices.

$$AC = \sqrt{2^2 + 1^2} = \sqrt{5}\,\text{cm}$$
$$AD = \sqrt{2^2 + 1^2} = \sqrt{5}\,\text{cm}$$
$$BD = \sqrt{1^2 + 2^2} = \sqrt{5}\,\text{cm}$$
$$BE = 2\,\text{cm}$$
$$CA = \sqrt{2^2 + 1^2} = \sqrt{5}\,\text{cm}$$
$$CE = \sqrt{1^2 + 2^2} = \sqrt{5}\,\text{cm}$$

Note that by symmetry there are similar results when considering DB, DA, EB and EC. From the above, the longest distance between any two points within the pentagon is $\sqrt{5}$ cm.

(iii) Split the 3 cm by 4 cm rectangle into the five regions below using the pentagon from part (ii).



When six points are placed at random, at least two must exist in the same region by the pigeonhole principle. In part (ii), it was shown that the largest possible distance between any two points of such a region was $\sqrt{5}$ cm. This means that the distance between these two points in the same region are within $\sqrt{5}$ cm of each other.

(d) When $n = 1$

$$\begin{aligned}
\text{LHS} &= \frac{\sin x}{\cos x + \cos 3x} \\
&= \frac{\sin(2x - x)}{\cos(2x - x) + \cos(2x + x)} \\
&= \frac{\sin 2x \cos x - \cos 2x \sin x}{2\cos x \cos 2x} \\
&= \frac{\sin 2x}{2\cos 2x} - \frac{\sin x}{2\cos x} \\
&= \frac{\tan 2x - \tan x}{2} \\
&= \text{RHS}
\end{aligned}$$

Hence, the statement is true for $n = 1$.

Assume the statement is true for $n = k$.

$$\frac{\sin x}{\cos x + \cos 3x} + \frac{\sin x}{\cos x + \cos 5x} + \cdots + \frac{\sin x}{\cos x + \cos(2k+1)x} = \frac{\tan(k+1)x - \tan x}{2}$$

Required to prove the statement is true for $n = k + 1$.

$$\frac{\sin x}{\cos x + \cos 3x} + \frac{\sin x}{\cos x + \cos 5x} + \cdots + \frac{\sin x}{\cos x + \cos(2k+3)x} = \frac{\tan(k+2)x - \tan x}{2}$$

$$\begin{aligned}
\text{LHS} &= \frac{\sin x}{\cos x + \cos 3x} + \frac{\sin x}{\cos x + \cos 5x} + \cdots + \frac{\sin x}{\cos x + \cos(2k+1)x} + \frac{\sin x}{\cos x + \cos(2k+3)x} \\
&= \frac{\tan(k+1)x - \tan x}{2} + \frac{\sin x}{\cos x + \cos(2k+3)x} \quad \text{(by assumption)} \\
&= \frac{\tan(k+1)x}{2} + \frac{\sin x}{\cos[(k+2)-(k+1)]x + \cos[(k+2)+(k+1)]x} - \frac{\tan x}{2} \\
&= \frac{\sin(k+1)x}{2\cos(k+1)x} + \frac{\sin x}{2\cos(k+2)x\cos(k+1)x} - \frac{\tan x}{2} \\
&= \frac{\sin(k+1)x\cos(k+2)x + \sin x}{2\cos(k+2)x\cos(k+1)x} - \frac{\tan x}{2} \\
&= \frac{\sin(k+1)x\cos(k+2)x + \sin[(k+2)-(k+1)]x}{2\cos(k+2)x\cos(k+1)x} - \frac{\tan x}{2} \\
&= \frac{\sin(k+1)x\cos(k+2)x + \sin(k+2)x\cos(k+1)x - \cos(k+2)x\sin(k+1)x}{2\cos(k+2)x\cos(k+1)x} - \frac{\tan x}{2} \\
&= \frac{\sin(k+2)x\cos(k+1)x}{2\cos(k+2)x\cos(k+1)x} - \frac{\tan x}{2} \\
&= \frac{\tan(k+2)x - \tan x}{2} \\
&= \text{RHS}
\end{aligned}$$

Since the statment is true for $n = 1$, then by induction it is true all positive integers n.

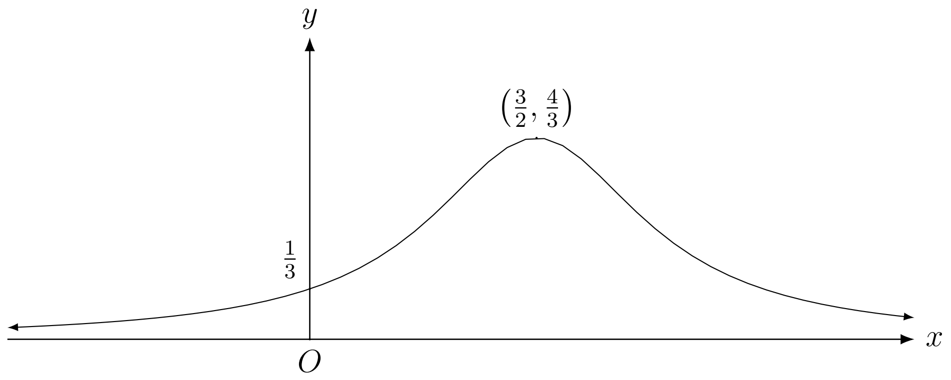
Question 14

(a) (i) First consider $y = x^2 - 3x + 3$. It's discriminant is $\Delta = -3 < 0$ so it is positive definite. It has a minimum turning point at $\left(\frac{3}{2}, \frac{3}{4}\right)$.

For the graph of $y = \dfrac{1}{x^2 - 3x + 3}$, when $x \to \pm\infty$ then $y \to 0$.

Since $x^2 - 3x + 3$ is positive definite then $y > 0$.

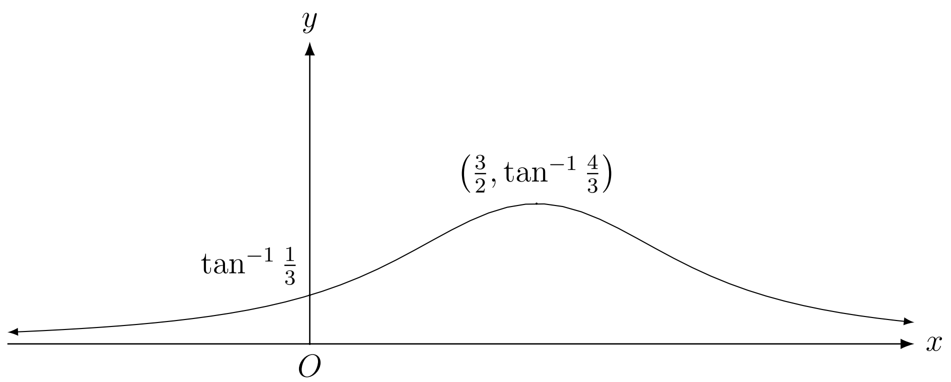
It also has a maximum turning point at $\left(\frac{3}{2}, \frac{4}{3}\right)$.



(ii) For the graph of $y = \tan^{-1}\left(\dfrac{1}{x^2 - 3x + 3}\right)$, when $x \to \pm\infty$ then $\dfrac{1}{x^2 - 3x + 3} \to 0$ so $y \to 0$.

Since $x^2 - 3x + 3$ is positive definite then $\dfrac{1}{x^2 - 3x + 3} > 0$ so $y > 0$.

It also has a maximum turning point at $\left(\frac{3}{2}, \tan^{-1}\frac{4}{3}\right)$.



(iii) Consider the graphs of $f(x) = x$ and $g(x) = \tan^{-1} x$.

$$f'(x) = 1$$
$$g'(x) = \frac{1}{1+x^2}$$

For $x > 0$, $f'(x) > g'(x)$. Since $f'(0) = g'(0) = 1$ then $f(x) > g(x)$ for $x > 0$. Hence, $x > \tan^{-1} x$ for $x > 0$.

Note that since $\dfrac{1}{x^2 - 3x + 3} > 0$ then the equivalent argument applies that $\dfrac{1}{x^2 - 3x + 3} > \tan^{-1}\left(\dfrac{1}{x^2 - 3x + 3}\right)$. Hence, the graphs never intersect.

(iv) Differentiating with respect to x

$$\begin{aligned}
&\frac{d}{dx}\left[(x+c)\tan^{-1}(x+c) - \ln\left(\sqrt{1+(x+c)^2}\right)\right] \\
&= \frac{d}{dx}\left[(x+c)\tan^{-1}(x+c) - \frac{1}{2}\ln(1+(x+c)^2)\right] \\
&= \tan^{-1}(x+c) + \frac{x+c}{1+(x+c)^2} - \frac{1}{2} \times \frac{2(x+c)}{1+(x+c)^2} \\
&= \tan^{-1}(x+c)
\end{aligned}$$

(v) First note that

$$\begin{aligned}
\frac{1}{x^2 - 3x + 3} &= \frac{1}{x^2 - 3x + 2 + 1} \\
&= \frac{1}{(x-1)(x-2)+1} \\
&= \frac{x - 1 + 2 - x}{1 - (x-1)(2-x)}
\end{aligned}$$

Let $\alpha = \tan^{-1}(x-1)$ and $\beta = \tan^{-1}(2-x)$ so

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
&= \frac{x - 1 + 2 - x}{1 - (x-1)(2-x)} \\
&= \frac{1}{x^2 - 3x + 3} \\
\alpha + \beta &= \tan^{-1}\left(\frac{1}{x^2 - 3x + 3}\right) \\
\tan^{-1}(x-1) + \tan^{-1}(2-x) &= \tan^{-1}\left(\frac{1}{x^2 - 3x + 3}\right)
\end{aligned}$$

Let A be the area between the curves.

$$
\begin{aligned}
A &= \int_1^2 \frac{dx}{x^2-3x+3} - \int_1^2 \tan^{-1}\left(\frac{1}{x^2-3x+3}\right) dx \\
&= \int_1^2 \frac{dx}{x^2-3x+\frac{9}{4}+\frac{3}{4}} - \int_1^2 \tan^{-1}(x-1)\,dx - \int_1^2 \tan^{-1}(2-x)\,dx \\
&= \int_1^2 \frac{dx}{\left(x-\frac{3}{2}\right)^2+\frac{3}{4}} - \int_1^2 \tan^{-1}(x-1)\,dx + \int_1^2 \tan^{-1}(x-2)\,dx
\end{aligned}
$$

Evaluating each integral individually and using the result in part (iv)

$$
\begin{aligned}
\int_1^2 \frac{dx}{\left(x-\frac{3}{2}\right)^2+\frac{3}{4}} &= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2\left(x-\frac{3}{2}\right)}{\sqrt{3}}\right)\right]_1^2 \\
&= \frac{2\pi}{3\sqrt{3}} \\
\int_1^2 \tan^{-1}(x-1)\,dx &= \left[(x-1)\tan^{-1}(x-1) - \ln\left(\sqrt{1+(x-1)^2}\right)\right]_1^2 \\
&= \frac{\pi}{4} - \ln\sqrt{2} \\
\int_1^2 \tan^{-1}(x-2)\,dx &= \left[(x-2)\tan^{-1}(x-2) - \ln\left(\sqrt{1+(x-2)^2}\right)\right]_1^2 \\
&= -\frac{\pi}{4} + \ln\sqrt{2} \\
A &= \frac{2\pi}{3\sqrt{3}} - \frac{\pi}{2} + 2\ln\sqrt{2}
\end{aligned}
$$

(b) To rth goal is fixed as the last goal which ends the training session. The soccer player must score $(r-1)$ goals in the prior games, with a maximum of $(n-1)$ games allowed.

If she scores $(r-1)$ goals within $(r-1)$ games there are $\binom{r-1}{r-1}$ possible ways.

If she scores $(r-1)$ goals within r games there are $\binom{r}{r-1}$ possible ways.

...

If she scores $(r-1)$ goals within $(n-1)$ games there are $\binom{n-1}{r-1}$ possible ways.

The total number of possibilities is the LHS.

However, this can also be interpreted as arranging r goals within n games in any order. The last goal must always be the end of the training session and the subsequent 'non-goals' can be ignored, as this does not change the total number of possibilities. There are $\binom{n}{r}$ possibilities, which is the RHS. Hence,

$$
\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \cdots + \binom{n-1}{r-1} = \binom{n}{r}.
$$

(c) (i) For integer $n \geq 4$

$$\begin{aligned}
\text{RHS} &= 24\binom{n}{4} + 36\binom{n}{3} + 14\binom{n}{2} + \binom{n}{1} \\
&= \frac{24n!}{4!(n-4)!} + \frac{36n!}{3!(n-3)!} + \frac{14n!}{2!(n-2)!} + \frac{n!}{1!(n-1)!} \\
&= n(n-1)(n-2)(n-3) + 6n(n-1)(n-2) + 7n(n-1) + n \\
&= (n^2-n)(n^2-5n+6) + 6n(n^2-3n+2) + 7n^2 - 7n + n \\
&= n^4 - 5n^3 + 6n^2 - n^3 + 5n^2 - 6n + 6n^3 - 18n^2 + 12n + 7n^2 - 7n + n \\
&= n^4 \\
&= \text{LHS}
\end{aligned}$$

(ii) Using the result in part (c)(i)

$$\begin{aligned}
4^4 &= 24\binom{4}{4} + 36\binom{4}{3} + 14\binom{4}{2} + \binom{4}{1} \\
5^4 &= 24\binom{5}{4} + 36\binom{5}{3} + 14\binom{5}{2} + \binom{5}{1} \\
6^4 &= 24\binom{6}{4} + 36\binom{6}{3} + 14\binom{6}{2} + \binom{6}{1} \\
&\ldots \\
n^4 &= 24\binom{n}{4} + 36\binom{n}{3} + 14\binom{n}{2} + \binom{n}{1}
\end{aligned}$$

Also, from part (b)

$$\begin{aligned}
\binom{n+1}{5} &= \binom{4}{4} + \binom{5}{4} + \binom{6}{4} + \cdots + \binom{n}{4} \\
\binom{n+1}{4} &= \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{n}{3} \\
\binom{n+1}{3} &= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} \\
\binom{n+1}{2} &= \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1}
\end{aligned}$$

Applying these results

LHS

$$
\begin{aligned}
&= 1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 + \cdots + n^4 \\
&= 1^4 + 2^4 + 3^4 + 24\binom{n+1}{5} + 36\left[\binom{n+1}{4} - \binom{3}{3}\right] \\
&\quad + 14\left[\binom{n+1}{3} - \binom{3}{2} - \binom{2}{2}\right] + \binom{n+1}{2} - \binom{3}{1} - \binom{2}{1} - \binom{1}{1} \\
&= 24\binom{n+1}{5} + 36\binom{n+1}{4} + 14\binom{n+1}{3} + \binom{n+1}{2} \\
&\quad + 1^4 + 2^4 + 3^4 - 36\binom{3}{3} - 14\binom{3}{2} - 14\binom{2}{2} - \binom{3}{1} - \binom{2}{1} - \binom{1}{1} \\
&= 24\binom{n+1}{5} + 36\binom{n+1}{4} + 14\binom{n+1}{3} + \binom{n+1}{2} \\
&= \frac{24(n+1)!}{5!(n-4)!} + \frac{36(n+1)!}{4!(n-3)!} + \frac{14(n+1)!}{3!(n-2)!} + \frac{(n+1)!}{2!(n-1)!} \\
&= \frac{(n+1)n(n-1)(n-2)(n-3)}{5} + \frac{3(n+1)n(n-1)(n-2)}{2} \\
&\quad + \frac{7(n+1)n(n-1)}{3} + \frac{(n+1)n}{2} \\
&= \frac{1}{30}n(n+1)[6(n-1)(n-2)(n-3) + 45(n-1)(n-2) + 70(n-1) + 15] \\
&= \frac{1}{30}n(n+1)[6(n-1)(n^2-5n+6) + 45(n^2-3n+2) + 70n - 70 + 15] \\
&= \frac{1}{30}n(n+1)[6n^3 - 30n^2 + 36n - 6n^2 + 30n - 36 + 45n^2 - 135n + 90 + 70n - 70 + 15] \\
&= \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1) \\
&= \frac{1}{30}n(n+1)(6n^3 + 9n^2 + 3n - 2n - 1) \\
&= \frac{1}{30}n(n+1)(3n(2n^2 + 3n + 1) - (2n+1)) \\
&= \frac{1}{30}n(n+1)(3n(2n+1)(n+1) - (2n+1)) \\
&= \frac{1}{30}n(n+1)(2n+1)(3n(n+1) - 1) \\
&= \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1) \\
&= \text{RHS}
\end{aligned}
$$