2020 **BORED OF STUDIES TRIAL EXAMINATION**

Mathematics Extension 1

Solutions

Section I

Answers

1	C	**6**	B
2	C	**7**	B
3	D	**8**	D
4	D	**9**	C
5	C	**10**	A

Brief explanations

1 When $x \to \pm\infty$, the slope is very close to 0. The only option with this property is (C).

2 Using $\theta = \cos^{-1}\left(\dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}\right) \Rightarrow \theta = \cos^{-1}\left(\dfrac{3}{5}\right)$.

This is equivalent to $\sin^{-1}\left(\dfrac{4}{5}\right)$, so the answer is (C).

3 Since the polynomial is monic and has a maximum turning point at the origin then it has the form $P(x) = x^2(x-\alpha)$ for some constant α. Since the leading coefficient of $P(x)$ is positive, the maximum turning point of the curve must occur at a smaller x-value than the minimum turning point. Also, since the minimum turning point cannot be higher in y-value than the maximum turning point, then it must be located in the 4^{th} quadrant. Hence, the answer is (D).

4 Since $y^2 \geq 0$ then $\dfrac{dy}{dx} \geq 1$. When $y = 0$, $\dfrac{dy}{dx} = 1$.

This means that the curve has the smallest gradient when $y = 0$ and the gradient is larger elsewhere. The only option with this property is (D).

5 The only quadratic polynomials with reciprocals that integrate to inverse tangent functions are those that have no real solution. Hence, the discriminant of $P(x)$ needs to be negative, so $a^2 - 4b < 0$. This implies that $b > \frac{a^2}{4}$. Since $a^2 > 0$ for non-zero values of a, then it is necessary that $b > 0$. Hence, the answer is (C).

6 The unit vector corresponding to the direction $\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j}$ is $\frac{\sqrt{3}}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}$.

The net force vector is $\underset{\sim}{j} + 4\left(\frac{\sqrt{3}}{2}\underset{\sim}{i} - \frac{1}{2}\underset{\sim}{j}\right)$, which is equivalent to $2\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j}$.

The magnitude of the net force is $\sqrt{(2\sqrt{3})^2 + 1^2} = \sqrt{13}$, so the answer is (B).

7 By polynomial long division, $\frac{P(x)}{Q(x)}$ can be rewritten as $A(x) + \frac{R(x)}{Q(x)}$, where $A(x)$ and $R(x)$ are polynomials of degree 1 or 0. Regardless of the degrees of $A(x)$ and $R(x)$, the polynomial $A(x)$ is an asymptote of the graph.

Hence, the minimum number of asymptotes is 1, so the answer is (B).

8 If $x < 1$ then $|x - 3| > 0 > \frac{1}{x-1}$, so $x < 1$ is part of the set. Hence, the answer is (D).

9 Note that for $0 < t < \frac{\pi}{2}$, the y-value of the displacement is negative.
After leaving the origin, the next time the particle crosses the x-axis is $t = \frac{\pi}{2}$.

The acute angle at which the particle crosses the x-axis is given by $\tan\theta = \left|\frac{\dot{y}}{\dot{x}}\right|$.

When $t = \frac{\pi}{2}$ then $\tan\theta = \left|\frac{\frac{\pi}{2}\sin\frac{\pi}{2} - \cos\frac{\pi}{2}}{\sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2}}\right|$. Hence, $\tan\theta = \frac{\pi}{2}$ or $\theta = \tan^{-1}\frac{\pi}{2}$,

so the answer is (C).

10 Let X be the number of heads. It is binomial distributed with

$$E(X) = 50 \quad \text{and } \mathrm{Var}(X) = 25$$

The probability of getting at least 60 heads or at least 60 tails is given by

$$\begin{aligned} P(X \geq 60) + P(X \leq 40) &= P\left(\frac{X - 50}{\sqrt{25}} \geq \frac{60 - 50}{\sqrt{25}}\right) + P\left(\frac{X - 50}{\sqrt{25}} \leq \frac{40 - 50}{\sqrt{25}}\right) \\ &= P(Z \geq 2) + P(Z \leq -2) \end{aligned}$$

Since the number of trials is large, X can be approximated by a normal distribution, where $Z = \frac{X - np}{\sqrt{np(1-p)}}$ follows a standard normal distribution.

From the reference sheet, approximately 95% of z-scores lie between -2 and 2. This implies that $P(-2 < Z < 2) \approx 0.95$.

Since $P(Z \geq 2) + P(Z \leq -2) = 1 - P(-2 < Z < 2)$ then the probability required is approximately 0.05. Hence, the answer is (A).

Question 11

(a) Using the cosine double angle formula and converting products of trig functions to sums

$$\begin{aligned}\int \cos^2 4x \cos^2 x\, dx &= \int \left(\frac{1+\cos 8x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\ &= \frac{1}{4}\int (1 + \cos 2x \cos 8x + \cos 2x + \cos 8x)\ dx \\ &= \frac{1}{4}\int \left(1 + \frac{\cos 10x + \cos 6x}{2} + \cos 2x + \cos 8x\right) dx \\ &= \frac{1}{4}\left(x + \frac{\sin 10x}{20} + \frac{\sin 6x}{12} + \frac{\sin 2x}{2} + \frac{\sin 8x}{8}\right) + c\end{aligned}$$

(b) The distance from the launch point $(0,0)$ to a given position (x,y) is found by

$$\begin{aligned}D^2 &= x^2 + y^2 \\ &= V^2t^2\cos^2\theta + \left(Vt\sin\theta - \frac{gt^2}{2}\right)^2 \\ &= V^2t^2\cos^2\theta + V^2t^2\sin^2\theta - gVt^3\sin\theta + \frac{g^2t^4}{4} \\ &= V^2t^2 - gVt^3\sin\theta + \frac{g^2t^4}{4}\end{aligned}$$

Using the chain rule

$$\begin{aligned}\frac{dD^2}{dt} &= 2V^2t - 3gVt^2\sin\theta + g^2t^3 \\ \frac{dD^2}{dD} \times \frac{dD}{dt} &= t(g^2t^2 - 3gVt\sin\theta + 2V^2) \\ \frac{dD}{dt} &= \frac{t(g^2t^2 - 3gVt\sin\theta + 2V^2)}{2D}\end{aligned}$$

If D is increasing then $\frac{dD}{dt} > 0$. Since $t > 0$ and $D > 0$ then $g^2t^2 - 3gVt\sin\theta + 2V^2 > 0$ is needed to ensure this condition holds. This is a quadratic polynomial in t, so the leading coefficient is needed to be positive and the discriminant is needed to be negative.

Note that $g^2 > 0$ and the discriminant is given by

$$\begin{aligned}\Delta &= 9g^2V^2\sin^2\theta - 8g^2V^2 \\ &= g^2V^2(9\sin^2\theta - 8)\end{aligned}$$

For $\Delta < 0$ to hold, we must have $\sin^2\theta < \frac{8}{9}$.

(c) Let $u = x\sqrt{x}$, then $du = \frac{3}{2}\sqrt{x}\,dx \Rightarrow \sqrt{x}\,dx = \frac{2}{3}du$.
When $x = 0, u = 0$ and when $x \to \infty, u \to \infty$.

$$\begin{aligned}
\int_0^\infty \sqrt{\frac{x}{e^{x^3}}}\,dx &= \int_0^\infty \frac{1}{\sqrt{e^{u^2}}} \times \frac{2}{3}\,du \\
&= \frac{2}{3}\int_0^\infty e^{-\frac{u^2}{2}}\,du \\
&= \frac{1}{3}\int_{-\infty}^\infty e^{-\frac{u^2}{2}}\,du \\
&= \frac{\sqrt{2\pi}}{3}
\end{aligned}$$

Since $e^{-\frac{u^2}{2}}$ is an even function then $\int_{-\infty}^\infty e^{-\frac{u^2}{2}}\,du = 2\int_0^\infty e^{-\frac{u^2}{2}}\,du$.

Also, recall that the standard normal probability density function is $f(u) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}$.
Hence, it follows that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-\frac{u^2}{2}}\,du = 1.$$

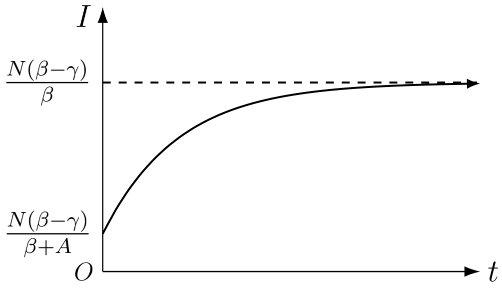
(d) (i) Substitute $S = N - I$

$$\begin{aligned}
\frac{dI}{dt} &= \frac{\beta SI}{N} - \gamma I \\
&= \frac{\beta I(N-I) - \gamma NI}{N} \\
\frac{dt}{dI} &= \frac{N}{I((\beta-\gamma)N - \beta I)} \\
&= \frac{1}{\beta-\gamma} \times \frac{(\beta-\gamma)N - \beta I + \beta I}{I((\beta-\gamma)N - \beta I)} \\
&= \frac{1}{\beta-\gamma}\left(\frac{1}{I} + \frac{\beta}{N(\beta-\gamma) - \beta I}\right)
\end{aligned}$$

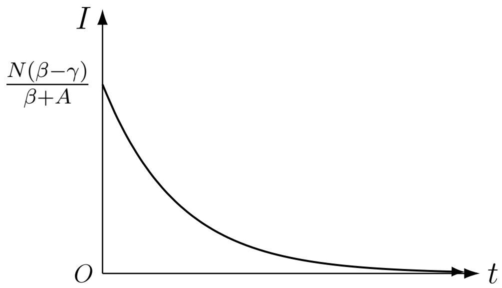
(ii) By integration of part (i)

$$\begin{aligned}
t &= \frac{1}{\beta-\gamma}\int\left(\frac{1}{I} + \frac{\beta}{N(\beta-\gamma) - \beta I}\right)dI \\
&= \frac{1}{\beta-\gamma}\left[\ln I - \ln\left(N(\beta-\gamma) - \beta I\right)\right] + c \\
(\beta-\gamma)(t-c) &= \ln\left(\frac{I}{N(\beta-\gamma) - \beta I}\right) \\
e^{-(\beta-\gamma)(t-c)} &= \frac{N(\beta-\gamma) - \beta I}{I} \\
\beta + e^{(\beta-\gamma)c}e^{-(\beta-\gamma)t} &= \frac{N(\beta-\gamma)}{I} \\
I &= \frac{N(\beta-\gamma)}{\beta + Ae^{-(\beta-\gamma)t}} \quad \text{where } A = e^{(\beta-\gamma)c} \text{ which is a constant}
\end{aligned}$$

(iii) Let $k = \beta - \gamma$. When $k > 0$ then as $t \to \infty$, $e^{-kt} \to 0$, hence $I \to \frac{N(\beta-\gamma)}{\beta}$. This means that I increases from its initial value $\frac{N(\beta-\gamma)}{\beta+A}$ to its limiting value $\frac{N(\beta-\gamma)}{\beta}$.



When $k < 0$ then as $t \to \infty$, $e^{-kt} \to \infty$, hence $I \to 0$. This means that I decreases from its initial value $\frac{N(\beta-\gamma)}{\beta+A}$ towards its limiting value of zero.



(iv) When $\frac{\beta}{\gamma} > 1$ or $\beta > \gamma$, the number of infections increases rapidly before stabilising towards a population infection number close to $\frac{N(\beta-\gamma)}{\beta}$.

However, when $\frac{\beta}{\gamma} < 1$ or $\beta < \gamma$, the number of infections decreases rapidly before stabilising close to zero infections.

Remark: The ratio $\frac{\beta}{\gamma}$ represents the reproductive number (or R_0) of the virus for this population. As shown above, whether R_0 is less than or greater than 1 leads to drastically different long term infection outcomes for the population.

Question 12

(a) Differentiating both sides of the relation with respect to t and using the chain rule

$$\begin{aligned}
\frac{1}{R_E} &= \frac{1}{R_A} + \frac{1}{R_B} \\
\frac{d}{dt}\left(\frac{1}{R_E}\right) &= \frac{d}{dt}\left(\frac{1}{R_A}\right) + \frac{d}{dt}\left(\frac{1}{R_B}\right) \\
\frac{d}{dR_E}\left(\frac{1}{R_E}\right)\frac{dR_E}{dt} &= \frac{d}{dR_A}\left(\frac{1}{R_A}\right)\frac{dR_A}{dt} + \frac{d}{dR_B}\left(\frac{1}{R_B}\right)\frac{dR_B}{dt} \\
-\frac{1}{R_E^2}\frac{dR_E}{dt} &= -\frac{1}{R_A^2}\frac{dR_A}{dt} - \frac{1}{R_B^2}\frac{dR_B}{dt}
\end{aligned}$$

But $\dfrac{dR_A}{dt} = 1$ and $\dfrac{dR_B}{dt} = -1$ so

$$\begin{aligned}
-\frac{1}{R_E^2}\frac{dR_E}{dt} &= -\frac{1}{R_A^2} + \frac{1}{R_B^2} \\
\frac{dR_E}{dt} &= R_E^2\left(\frac{1}{R_A^2} - \frac{1}{R_B^2}\right) \\
&= R_E^2\left(\frac{1}{R_A} + \frac{1}{R_B}\right)\left(\frac{1}{R_A} - \frac{1}{R_B}\right) \qquad \text{but } \frac{1}{R_E} = \frac{1}{R_A} + \frac{1}{R_B} \\
&= R_E\left(\frac{1}{R_A} - \frac{1}{R_B}\right)
\end{aligned}$$

(b) Let $G(y)$ and $g(y)$ be the cumulative distribution and probability density functions of Y respectively. Since $Y = F(X)$ then $y \in [0, 1]$ as $F(x) \in [0, 1]$. Also, note that cumulative distribution functions are non-decreasing and so have a unique inverse.

$$\begin{aligned}
G(y) &= P(Y \leq y) \\
&= P(F(X) \leq y) \\
&= P(X \leq F^{-1}(y)) \qquad \text{but } F(x) = P(X \leq x) \\
&= F(F^{-1}(y)) \\
&= y \\
g(y) &= G'(y) \\
&= 1
\end{aligned}$$

Hence, the probability density function of Y represents a uniform distribution.

(c) Consider a reference face with N edges. Each edge is shared with another face.
Hence, there must be at least N other faces in the polyhedron.
These N other faces have a number of edges which can range from 3 to N.
By the pigeonhole principle, there exists at least two faces with the same number of edges.

Remark: A stronger conclusion that can be drawn from the same information is that there exists at least three faces with the same number of edges OR there exists at least two faces with k_1 edges, and at least two faces with k_2 edges, for distinct integers k_1, k_2.

(d) **Method 1:** Consider the sum of the numerator and denominator.
Use the following auxiliary angle transformation

$$\begin{aligned}\sin\theta+\cos\theta&=\sqrt{2}\left(\cos\theta\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sin\theta\right)\\&=\sqrt{2}\cos\left(45^\circ-\theta\right)\end{aligned}$$

$$\begin{aligned}&\sin 1^\circ+\sin 2^\circ+\sin 3^\circ+\cdots+\sin 44^\circ+\cos 1^\circ+\cos 2^\circ+\cos 3^\circ+\cdots+\cos 44^\circ\\&=(\sin 1^\circ+\cos 1^\circ)+(\sin 2^\circ+\cos 2^\circ)+(\sin 3^\circ+\cos 3^\circ)+\cdots+(\sin 44^\circ+\cos 44^\circ)\\&=\sqrt{2}\cos 44^\circ+\sqrt{2}\cos 43^\circ+\sqrt{2}\cos 42^\circ+\cdots+\sqrt{2}\cos 1^\circ\end{aligned}$$

Hence

$$\sin 1^\circ+\sin 2^\circ+\sin 3^\circ+\cdots+\sin 44^\circ=(\sqrt{2}-1)(\cos 1^\circ+\cos 2^\circ+\cos 3^\circ+\cdots+\cos 44^\circ)$$
$$\frac{\sin 1^\circ+\sin 2^\circ+\sin 3^\circ+...+\sin 44^\circ}{\cos 1^\circ+\cos 2^\circ+\cos 3^\circ+...+\cos 44^\circ}=\sqrt{2}-1$$

Method 2: Converting the trigonometric sums to trigonometric products using

$$\sin\theta+\sin(45^\circ-\theta)=2\sin 22.5^\circ\cos(22.5^\circ-\theta)$$

$$\cos\theta+\cos(45^\circ-\theta)=2\cos 22.5^\circ\cos(22.5^\circ-\theta)$$

From the numerator

$$\begin{aligned}&\sin 1^\circ+\sin 2^\circ+\sin 3^\circ+\cdots+\sin 44^\circ\\&=(\sin 1^\circ+\sin 44^\circ)+(\sin 2^\circ+\sin 43^\circ)+(\sin 3^\circ+\sin 42^\circ)+\cdots+(\sin 22^\circ+\sin 23^\circ)\\&=2\sin 22.5^\circ\left[\cos(22.5^\circ-1^\circ)+\cos(22.5^\circ-2^\circ)+\cos(22.5^\circ-3^\circ)+\cdots+\cos(22.5^\circ-22^\circ)\right]\end{aligned}$$

Similarly for the denominator

$$\begin{aligned}&\cos 1^\circ+\cos 2^\circ+\cos 3^\circ+\cdots+\cos 44^\circ\\&=2\cos 22.5^\circ\left[\cos(22.5^\circ-1^\circ)+\cos(22.5^\circ-2^\circ)+\cos(22.5^\circ-3^\circ)+\cdots+\cos(22.5^\circ-22^\circ)\right]\end{aligned}$$

This implies that

$$\frac{\sin 1^\circ+\sin 2^\circ+\sin 3^\circ+...+\sin 44^\circ}{\cos 1^\circ+\cos 2^\circ+\cos 3^\circ+...+\cos 44^\circ}=\tan 22.5^\circ$$

Using the tangent double angle

$$\begin{aligned}\tan 45^\circ&=\frac{2\tan 22.5^\circ}{1-\tan^2 22.5^\circ}\\ \tan^2 22.5^\circ+2\tan 22.5^\circ-1&=0\\ (\tan 22.5^\circ+1)^2&=2\\ \tan 22.5^\circ+1&=\sqrt{2}\quad\text{noting that }\tan 22.5^\circ+1>0\\ \tan 22.5^\circ&=\sqrt{2}-1\end{aligned}$$

Hence

$$\frac{\sin 1^\circ+\sin 2^\circ+\sin 3^\circ+...+\sin 44^\circ}{\cos 1^\circ+\cos 2^\circ+\cos 3^\circ+...+\cos 44^\circ}=\sqrt{2}-1$$

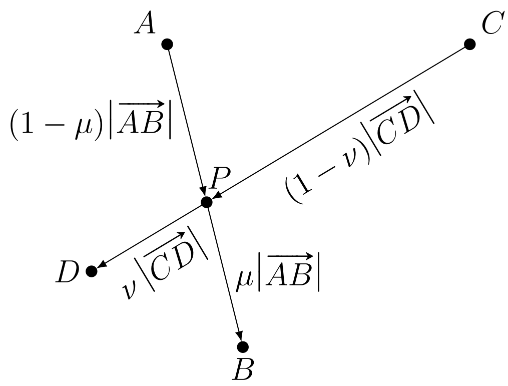
(e) (i) Since P is a point on the interval AB then $\overrightarrow{AP}$ and $\overrightarrow{AB}$ point in the same direction but have different lengths.

$$\begin{aligned}\overrightarrow{AP} &= (1-\mu)\overrightarrow{AB}\\ \underset{\sim}{p} - \underset{\sim}{a} &= (1-\mu)(\underset{\sim}{b} - \underset{\sim}{a})\\ \underset{\sim}{p} &= (1-\mu)\underset{\sim}{b} - (1-\mu)\underset{\sim}{a} + \underset{\sim}{a}\\ \underset{\sim}{p} &= \mu\underset{\sim}{a} + (1-\mu)\underset{\sim}{b}\end{aligned}$$

(ii) Since $0 < \mu < 1$ then the ratio $\frac{1-\mu}{\mu}$ can be any positive number. In fact, μ represents the percentage of the interval AB that the sub-interval PB takes up.

Since P can be any point on the interval AB, let it also be the point where AB and CD intersect. Since P divides both AB and CD there exists some parameters μ and say ν such that

$$\frac{AP}{PB} = \frac{1-\mu}{\mu} \quad \text{and} \quad \frac{CP}{PD} = \frac{1-\nu}{\nu}$$



From part (i) this means that

$$\begin{aligned}\underset{\sim}{p} &= \mu\underset{\sim}{a} + (1-\mu)\underset{\sim}{b}\\ \underset{\sim}{p} &= \nu\underset{\sim}{c} + (1-\nu)\underset{\sim}{d}\\ \mu\underset{\sim}{a} + (1-\mu)\underset{\sim}{b} - \nu\underset{\sim}{c} - (1-\nu)\underset{\sim}{d} &= \underset{\sim}{0}\end{aligned}$$

Let $\lambda_1 = \mu$, $\lambda_2 = 1-\mu$, $\lambda_3 = -\nu$ and $\lambda_4 = -(1-\nu)$. Notice that

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= \mu + 1 - \mu - \nu - (1-\nu)\\ &= 0\end{aligned}$$

(f) Let $t = \tan\theta$

$$\begin{aligned}
a\sin 4\theta + b\cos 4\theta &= c \\
2a\sin 2\theta\cos 2\theta + b(2\cos^2 2\theta - 1) &= c \\
2a\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right) + b\left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) &= c \\
4at(1-t^2) + 2b(1-t^2)^2 - b(1+t^2)^2 &= c(1+t^2)^2 \\
4at - 4at^3 + 2bt^4 - 4bt^2 + 2b - bt^4 - 2bt^2 - b &= ct^4 + 2ct^2 + c \\
(b-c)t^4 - 4at^3 - 2(3b+c)t^2 + 4at + b - c &= 0
\end{aligned}$$

This is a quartic equation in $t = \tan\theta$ with the roots $\tan\theta_1, \tan\theta_2, \tan\theta_3$ and $\tan\theta_4$. Using the relationships between roots and coefficients

$$\begin{aligned}
S &= \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4 \\
&= \frac{4a}{b-c} \\
T &= \tan\theta_2\tan\theta_3\tan\theta_4 + \tan\theta_1\tan\theta_3\tan\theta_4 + \tan\theta_1\tan\theta_2\tan\theta_4 + \tan\theta_1\tan\theta_2\tan\theta_3 \\
&= -\frac{4a}{b-c}
\end{aligned}$$

Hence $S + T = 0$.

Question 13

(a) (i)

$$\begin{aligned}
f(x) &= \tan^{-1} x \\
f'(x) &= \frac{1}{x^2+1} \\
g(x) &= \frac{x}{x^2+1} \\
g'(x) &= \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} \\
&= \frac{1-x^2}{(x^2+1)^2}
\end{aligned}$$

Consider

$$\begin{aligned}
f'(x) - g'(x) &= \frac{1}{x^2+1} - \frac{1-x^2}{(x^2+1)^2} \\
&= \frac{x^2+1-1+x^2}{(x^2+1)^2} \\
&= \frac{2x^2}{(x^2+1)^2} \\
&> 0 \quad \text{when } x > 0
\end{aligned}$$

Hence $f'(x) > g'(x)$ for the domain $x > 0$.

(ii) First note that $f(0) = 0$ and $g(0) = 0$. Since $f(x)$ and $g(x)$ share a common point at the origin and $f'(x) > g'(x)$ for $x > 0$ then it must be that $f(x) > g(x)$, since $f(x)$ has a larger gradient than $g(x)$. Hence

$$\begin{aligned}
\tan^{-1} x &> \frac{x}{x^2+1} \qquad \text{note that } f(x) > 0 \text{ and } g(x) > 0 \text{ for } x > 0 \\
\sqrt{\tan^{-1} x} &> \sqrt{\frac{x}{x^2+1}} \\
\frac{\sqrt{\tan^{-1} x}}{x} &> \frac{1}{\sqrt{x(x^2+1)}}
\end{aligned}$$

Noting that taking the square root of both sides of the inequality preserves the sign as it is an increasing function.

Remark: Another way to think of it is as follows.
Let a and b be positive numbers such that $a > b$, so

$$\begin{aligned}
a - b &> 0 \\
(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) &> 0 \\
\sqrt{a} - \sqrt{b} &> 0 \qquad \text{noting that } \sqrt{a} + \sqrt{b} > 0. \\
\sqrt{a} &> \sqrt{b}
\end{aligned}$$

(iii) Investigating $y = \dfrac{\sqrt{\tan^{-1} x}}{x}$ noting the domain of $x > 0$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\dfrac{x}{2(1+x^2)\sqrt{\tan^{-1} x}} - \sqrt{\tan^{-1} x}}{x^2} \\ &= \frac{x - 2(1+x^2)\tan^{-1} x}{2x^2(1+x^2)\sqrt{\tan^{-1} x}}\end{aligned}$$

From part (ii)

$$\begin{aligned}\frac{\sqrt{\tan^{-1} x}}{x} &> \frac{1}{\sqrt{x(x^2+1)}} \\ \frac{(1+x^2)\tan^{-1} x}{x} &> 1 \\ 2(1+x^2)\tan^{-1} x &> 2x \\ x - 2(1+x^2)\tan^{-1} x &< -x\end{aligned}$$

But $x > 0$ is the domain so $-x < 0$ which implies that $x - 2(1+x^2)\tan^{-1} x < 0$. Since $2x^2(1+x^2)\sqrt{\tan^{-1} x} > 0$ for $x > 0$ then the curve is decreasing.

As $x \to \infty$ then $\dfrac{\sqrt{\tan^{-1} x}}{x} \to 0$ since $\tan^{-1} x \to \dfrac{\pi}{2}$.

Note that as $x \to 0$ then $\dfrac{1}{\sqrt{x(x^2+1)}} \to \infty$.

Since the curve $y = \dfrac{\sqrt{\tan^{-1} x}}{x}$ is always above $y = \dfrac{1}{\sqrt{x(x^2+1)}}$ from part (ii), then it must be that $\dfrac{\sqrt{\tan^{-1} x}}{x} \to \infty$ as $x \to 0$ as well.

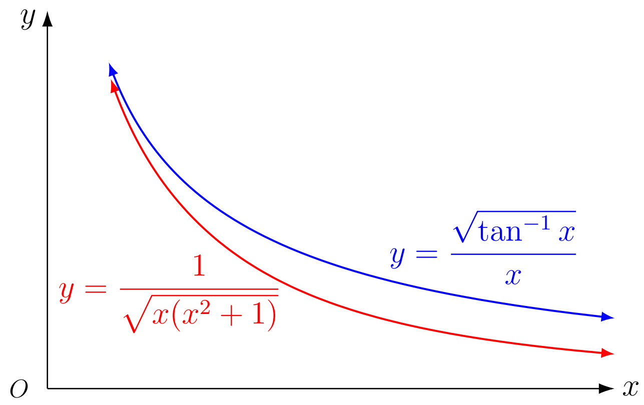
Investigating $y = \dfrac{1}{\sqrt{x(x^2+1)}}$ noting the domain of $x > 0$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{2}(x^3+x)^{-\frac{3}{2}}(3x^2+1) \\ &= -\frac{3x^2+1}{2(x^3+x)^{\frac{3}{2}}} \\ &< 0 \quad \text{since } 2(x^3+x)^{\frac{3}{2}} > 0 \text{ and } 3x^2+1 > 0 \text{ for } x > 0\end{aligned}$$

This implies that the curve is decreasing for $x > 0$.

Also as $x \to \infty$ then $\dfrac{1}{\sqrt{x(x^2+1)}} \to 0$ and as $x \to 0$ then $\dfrac{1}{\sqrt{x(x^2+1)}} \to \infty$.

Using this information to sketch the curves gives the following.



(iv) Let V be the volume of the solid

$$
\begin{aligned}
V &= \pi \int_1^{\sqrt{3}} \left(\frac{\sqrt{\tan^{-1} x}}{x} \right)^2 dx - \pi \int_1^{\sqrt{3}} \left(\frac{1}{\sqrt{x(x^2+1)}} \right)^2 dx \\
&= \pi \int_1^{\sqrt{3}} \left(\frac{\tan^{-1} x}{x^2} - \frac{1}{x(x^2+1)} \right) dx
\end{aligned}
$$

But $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$ and $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{x^2+1}$ so

$$
\begin{aligned}
V &= \pi \int_1^{\sqrt{3}} \left(-\tan^{-1} x \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{1}{x}\frac{d}{dx}(\tan^{-1} x) \right) dx \\
&= \pi \int_1^{\sqrt{3}} \frac{d}{dx}\left(-\frac{1}{x} \times \tan^{-1} x \right) dx \\
&= \pi \left[-\frac{\tan^{-1} x}{x} \right]_1^{\sqrt{3}} \\
&= \pi \left(\frac{\pi}{4} - \frac{\frac{\pi}{3}}{\sqrt{3}} \right) \\
&= \pi^2 \left(\frac{1}{4} - \frac{1}{3\sqrt{3}} \right) \quad \text{cubic units}
\end{aligned}
$$

(b) (i) For $n = 0$

$$\begin{aligned} LHS &= \alpha^0(p - \beta q) \\ &= p - \beta q \\ RHS &= x_1 - \beta x_0 \quad \text{but } x_0 = q, x_1 = p \\ &= p - \beta q \end{aligned}$$

$LHS = RHS$ so the statement is true for $n = 0$.

Assume the statement is true $n = k$ so

$$\alpha^k(p - \beta q) = x_{k+1} - \beta x_k$$

Required to prove the statement is true for $n = k + 1$

$$\alpha^{k+1}(p - \beta q) = x_{k+2} - \beta x_{k+1}$$

Since α and β are the roots of the quadratic equation $x^2 = ax + b$ then

$$\alpha + \beta = a \quad \text{and} \quad \alpha\beta = -b$$

Using this result and the assumption for the induction proof

$$\begin{aligned} LHS &= \alpha^{k+1}(p - \beta q) \\ &= \alpha(x_{k+1} - \beta x_k) \quad \text{by assumption} \\ &= \alpha x_{k+1} - \alpha\beta x_k \\ &= (a - \beta)x_{k+1} + bx_k \\ &= ax_{k+1} + bx_k - \beta x_{k+1} \quad \text{but } x_n = ax_{n-1} + bx_{n-2} \\ &= x_{k+2} - \beta x_{k+1} \\ &= RHS \end{aligned}$$

Hence, by induction the statement is true for all integers $n \geq 0$.

(ii) For part (i), the roots α and β are interchangeable because the relation makes no distinction on which particular root is used (it just uses one of the roots). Hence

$$\begin{aligned} \alpha^n(p - \beta q) &= x_{n+1} - \beta x_n \\ \beta^n(p - \alpha q) &= x_{n+1} - \alpha x_n \\ \alpha^n(p - \beta q) - \beta^n(p - \alpha q) &= (\alpha - \beta)x_n \\ (\alpha^n - \beta^n)p + (\alpha\beta^n - \alpha^n\beta)q &= (\alpha - \beta)x_n \quad \text{but } \alpha\beta = -b \\ x_n &= \frac{(\alpha^n - \beta^n)p + (\alpha^{n-1} - \beta^{n-1})bq}{\alpha - \beta} \end{aligned}$$

Question 14

(a) (i) Since $\text{Var}(X) = E[X - E(X)]^2$ then the expression to prove can be rewritten as

$$E[X - E(X)]^2 \geq rP\left([X - E(X)]^2 \geq r\right)$$

Let $Y = [X - E(X)]^2$ and it suffices to prove first that $E(Y) \geq rP\left(Y \geq r\right)$.

Let the outcomes of Y be $\{y_1, y_2, ..., y_n\}$ where $y_1 \leq y_2 \leq y_3 \leq \cdots \leq y_n$ (without loss of generality) and their respective probabilities of occurring are $\{p_1, p_2, ..., p_n\}$.

First consider the case where $y_1 \leq r \leq y_n$.

Suppose that $y_1 \leq \cdots \leq y_{r-1} \leq r \leq y_r \leq \cdots \leq y_n$ for some values of y_r and y_{r-1}. This means that

$$\begin{aligned} E(Y) &= p_1y_1 + \cdots + p_{r-1}y_{r-1} + p_ry_r + \cdots + p_ny_n \\ &\geq 0 \times y_1 + \cdots + 0 \times y_{r-1} + p_r \times r + \cdots + p_n \times r \\ &= r(p_r + p_{r+1} + \cdots + p_n) \\ &= rP(Y \geq r) \end{aligned}$$

Since the probabilities $\{p_1, p_2, ..., p_{r-1}\}$ are all non-negative and the values of $\{y_r, y_{r+1}, ..., y_n\}$ are all greater than or equal to r. Also, the sum $p_r+p_{r+1}+\cdots+p_n$ is the probability that Y is at least r.

Consider the case where $r > y_n$ then $P(Y \geq r) = 0$. Since Y is non-negative then $E(Y) \geq 0$. Hence, $E(Y) \geq rP\left(Y \geq r\right)$ for $r > y_n$.

Consider the case where $r < y_1$ then $P(Y \geq r) = 1$. This also suggests that $E(Y) \geq r$ since the average value of Y cannot be larger than a number below its minimum value. Hence, $E(Y) \geq rP\left(Y \geq r\right)$ for $r < y_1$.

By letting $Y = [X - E(X)]^2$ and noting that $r > 0$ then

$$\text{Var}(X) \geq rP\left(|X - E(X)| \geq \sqrt{r}\right)$$

(ii) Let $X = \hat{p}$ where $\hat{p}$ is the proportion of people in the sample who own a bike. It is known that

$$E(\hat{p}) = p \quad \text{and } \text{Var}(\hat{p}) = \frac{p(1-p)}{n}$$

Using part (i), let $\epsilon = \sqrt{r}$ and note that $P(|\hat{p} - p| \geq \epsilon) = 1 - P(|\hat{p} - p| < \epsilon)$

$$\frac{p(1-p)}{n} \geq \epsilon^2 P(|\hat{p} - p| \geq \epsilon)$$

$$P(|\hat{p} - p| < \epsilon) \geq 1 - \frac{p(1-p)}{n\epsilon^2}$$

Hence, an appropriate value for L is $\left(1 - \dfrac{p(1-p)}{n\epsilon^2}\right)$.

(iii) When $n \to \infty$ then $\dfrac{p(1-p)}{n\epsilon^2} \to 0$, hence $L \to 1$. However, $P(|\hat{p} - p| \geq \epsilon) \leq 1$ as it is a probability.

Since the lower bound is increasing towards 1 then it must be that

$$\lim_{n\to\infty} P(|\hat{p} - p| < \varepsilon) = 1$$

Note that ϵ can be any positive number. This means that as the sample size n grows larger, the probability that the absolute difference between the sample proportion $\hat{p}$ and the population proportion p being less than *every* positive value (no matter how large or small) is almost certain.

This implies that the bigger the sample size, the more likely that the sample estimate will accurately represent the population proportion.

Remark: This limit result is a mathematical expression of the *law of large numbers*, which basically says that the bigger the sample size, the more likely the sample will be similar to the population.

(b) (i) There are 6! ways to colour the 6 faces of the cube. This is an overcount because identical configurations from rotating the cube are being counted more than once.

Consider a reference face which is in a fixed position on the top of the cube. This cube is the same cube after rotating about each of the 4 square faces facing the sides. Altogether, there are 6×4 identical configurations of the cube.

Hence, it can be deduced that there are $\frac{6!}{6\times 4} = 30$ colourings.

Alternative Method: Select one of the 6 faces to colour first, but the face can be rotated to any of the 5 other faces. Further note that the second colour can be in any of the 5 other faces, but can be rotated to 4 locations only. Finally, the other 4 faces can be coloured but their location is now fixed, so there are a total of $\frac{6}{6} \times \frac{5}{4} \times 4! = 30$ colourings.

(ii) Applying a similar analogy to the cube, there are 8! ways to colour the 8 faces of the octahedron. This is an overcount because identical configurations from rotating the octahedron are being counted more than once.

Consider a reference vertex which is in a fixed position on the top of the octahedron. The octahedron is the same octahedron after rotating about each of the 4 equilateral triangle faces facing the sides. There are 6 vertices, so there are 6×4 identical configurations of the octahedron.

Hence, it can be deduced that there are $\frac{8!}{6\times 4} = 1680$ colourings.

Alternative Method 1: Select one of the 8 faces to colour first, but the face can be rotated to any of the 7 other faces. Further note that the second colour can be in any of the 7 other faces, but can be rotated to 3 locations only. Finally, the other 6 faces can be coloured and their location is now fixed, so there are a total of $\frac{8}{8} \times \frac{7}{3} \times 6! = 1680$ colourings.

Alternative Method 2: Suppose instead of colouring each face, colour the centre of each face. These centres happen to form a cube and the problem is equivalent to colouring the vertices of a cube.

Select one of the 8 vertices to colour first, but the vertex can be rotated to any of the 7 other vertices. Note that the second colour can be in any of the 7 other vertices, but can be rotated to 3 locations only. Finally, the other 6 vertices can be coloured and their location is now fixed, so there are a total of $\frac{8}{8} \times \frac{7}{3} \times 6! = 1680$ colourings.

(c) (i) For each path that gets to the $(n-1)^{\text{th}}$ step, the only way to finish is with a "step". For each path that gets to the $(n-2)^{\text{th}}$ step, the only way to finish is with a "lunge", since taking two steps will result in a path that passes through the $(n-1)^{\text{th}}$ step, which has already been counted.

Hence, $\psi(n) = \psi(n-1) + \psi(n-2)$ for all $n \geq 3$.

(ii) For $n = 1$, there is only 1 way to climb the staircase (one "step"), so $\psi(1) = 1$. For $n = 2$, there are 2 ways (two "steps" or one "lunge"), so $\psi(2) = 2$.

From the definition of F_k, it follows that $F_0 = 0, F_1 = 1, F_2 = 1$ and $F_3 = 2$. Hence, $\psi(1) = F_2$ and $\psi(2) = F_3$.

As $\psi(n)$ and F_k share the same general relation (the sum of the last two values) but n and k are related by $k = n+1$, then it can be concluded that $\psi(n) = F_{n+1}$.

(iii) If Leonardo wishes to lunge exactly k times then he will cover $2k$ out of the n stairs. The remaining $n - 2k$ are covered by a "step". The number of ways of arranging this sequence of $(n-2k)$ "steps" and k "lunges" is $\frac{(n-2k+k)!}{(n-2k)!k!}$, or equivalently, $\binom{n-k}{k}$.

(iv) Consider the case where n is even. The minimum number of lunges is 0 and the maximum number of lunges is $\frac{n}{2}$. Hence, using part (iii) the total number of ways Leonardo can ascend an n-stair staircase is given by

$$\begin{aligned}\psi(n) &= \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-\frac{n}{2}}{\frac{n}{2}}\\ &= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n}{2}}{\frac{n}{2}}\end{aligned}$$

When n is odd, the minimum number of lunges is also 0 and the maximum number of lunges is $\frac{n-1}{2}$. Hence, using part (iii) the total number of ways Leonardo can ascend an n-stair staircase is given by

$$\begin{aligned}\psi(n) &= \binom{n-0}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-\frac{n-1}{2}}{\frac{n-1}{2}}\\ &= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}}\end{aligned}$$

Since $\psi(n) = F_{n+1}$ then

$$F_{n+1} = \begin{cases} \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n}{2}}{\frac{n}{2}} & \text{if } n \text{ is even.}\\ \\ \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}} & \text{if } n \text{ is odd.}\end{cases}$$