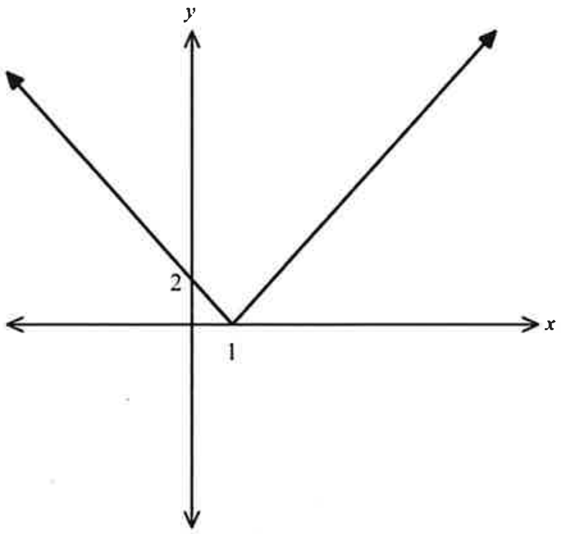
Name: ____________________ Due: Solutions

Year 12 Advanced Term 2 – Assignment 1

Section 1 – Multiple Choice

1. The graph of $y = f(x)$ is shown below.



Horizontal translation 3 L

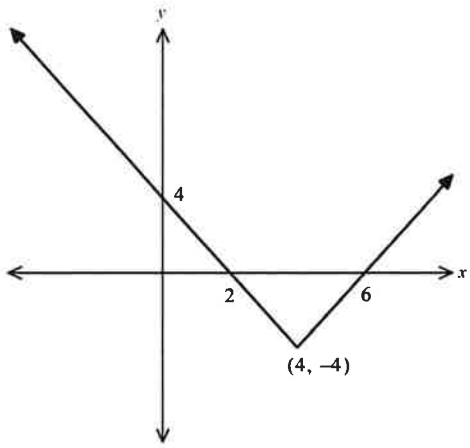
Vertical translation 4 up

∴ Point (cusp) is

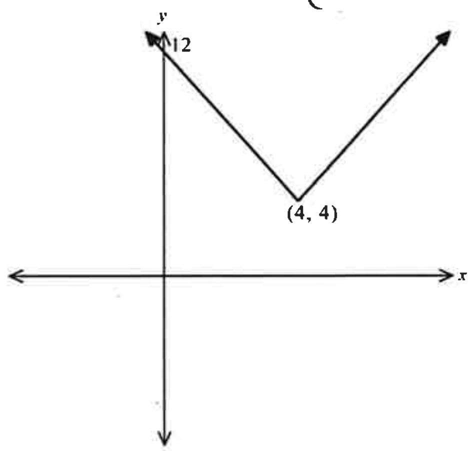
$(1-3,\ 0+4) = (-2, 4)$

Which of the graphs below represents $y = f(x + 3) + 4$?

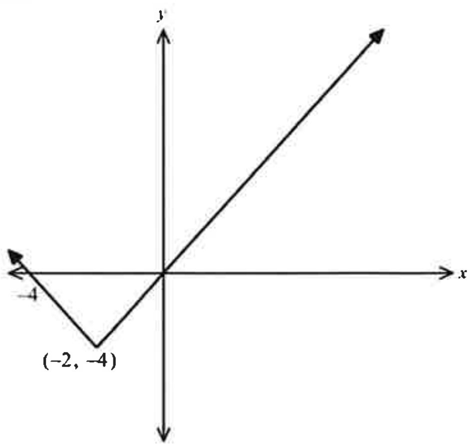
A.



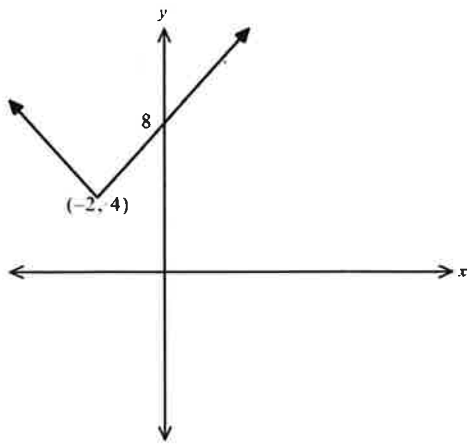
B.



C.



D.



2. Which of the following is equivalent to $\frac{d}{dx}\left(\frac{\sin x}{x^2}\right)$?

A. $\frac{\cos x}{2x}$

B. $\frac{x^2 \sin x - 2x \cos x}{x^4}$

C. $\frac{x \cos x + 2\sin x}{x^3}$

D. $\frac{x \cos x - 2\sin x}{x^3}$

Use Quotient rule:

$$\frac{x^2(\cos x) - (\sin x)(2x)}{(x^2)^2}$$

$$= \frac{x^2(\cos x) - 2x \sin x}{x^4}$$

$$= \frac{x(\cos x) - 2\sin x}{x^3}$$

$\Longrightarrow$ D

3. What is $\int 6x^2(4x^3-5)^3\,dx$?

A. $\frac{(4x^3-5)^4}{8} + C$

B. $12x(4x^3-5)^4 + C$

C. $2x^3(4x^3-5)^4 + C$

D. $\frac{2x^3(4x^3-5)^4}{8} + C$

major player is $(4x^3-5)^3$ 4

Consider $(4x^3-5)^4$

Now $\frac{d}{dx}(4x^3-5)^4$

$= 4(4x^3-5)^3 \times 12x^2$

$= 48x^2(4x^3-5)^3$

$\therefore \int 6x^2(4x^3-5)^3 = \frac{1}{8}\int 48x^2(4x^3-5)^3$

$= \frac{1}{8}(4x^3-5)^4 + C$

$\Longrightarrow$ A

4. Which is the complete solution set to the equation $\sin^2\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$, for the domain $0 \le x \le 2\pi$?

A. $x = \frac{\pi}{12}, \frac{13\pi}{12}$

B. $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

C. $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

D. $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

$\sin\left(x+\frac{\pi}{6}\right) = \pm\frac{1}{\sqrt{2}}$

$\therefore x+\frac{\pi}{6} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}$, etc

$\frac{\pi}{6} \le x+\frac{\pi}{6} \le 2\pi+\frac{\pi}{6}$

$\frac{\pi}{6} \le x+\frac{\pi}{6} \le \frac{13\pi}{6}$

$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \cancel{\frac{25\pi}{12}}$

$\Longrightarrow$ C

Section 2 – Show full working!

1. Express $\frac{2x}{x^2-4} - \frac{x+1}{x^2-x-2}$ as a single algebraic fraction, in simplest form.

$\frac{2x}{(x+2)(x-2)} - \frac{x+1}{(x-2)(x+1)}$

$\frac{2x(x+1) - (x+1)(x+2)}{(x+2)(x-2)(x+1)}$

$= \frac{(x+1)(2x-(x+2))}{(x+2)(x-2)(x+1)}$

$= \frac{\cancel{(x+1)}\cancel{(x-2)}}{(x+2)\cancel{(x-2)}\cancel{(x+1)}} = \frac{1}{x+2}$

2. Find the equation of the normal to the curve $y = x^4 - 3x^2 + 18x + 24$ at the point where $x = -2$.

$\frac{dy}{dx} = 4x^3 - 6x + 18$

at $x = -2$, $m_T = 4(-2)^3 - 6(-2) + 18$

$= -32 + 12 + 18$

$m_T = -2$

$\therefore m_N = \frac{1}{2}$

at $x = -2$, $y = (-2)^4 - 3(-2)^2 + 18(-2) + 24 = -8$

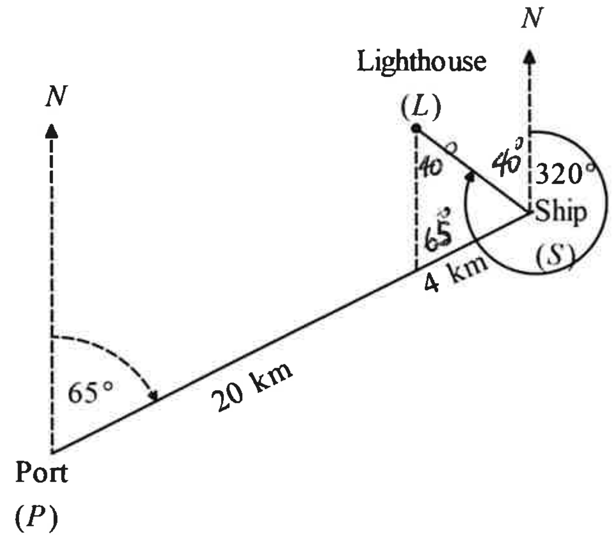
Eq^n of normal: $y + 8 = \frac{1}{2}(x + 2)$

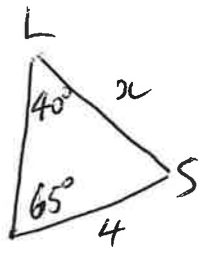
$2y + 16 = x + 2$

$x - 2y - 14 = 0$

3. A ship leaves port travelling on a bearing of 065°. After travelling 20 kilometres, the ship is due south of a lighthouse.

The ship continues on this bearing for a further 4 kilometres, then measures the bearing of the lighthouse to be 320°.





Calculate the distance from the ship to the lighthouse at this time.

$\frac{x}{\sin 65^\circ} = \frac{4}{\sin 40^\circ}$

$x = \frac{4 \sin 65^\circ}{\sin 40^\circ}$

$x \doteq 5.63985$

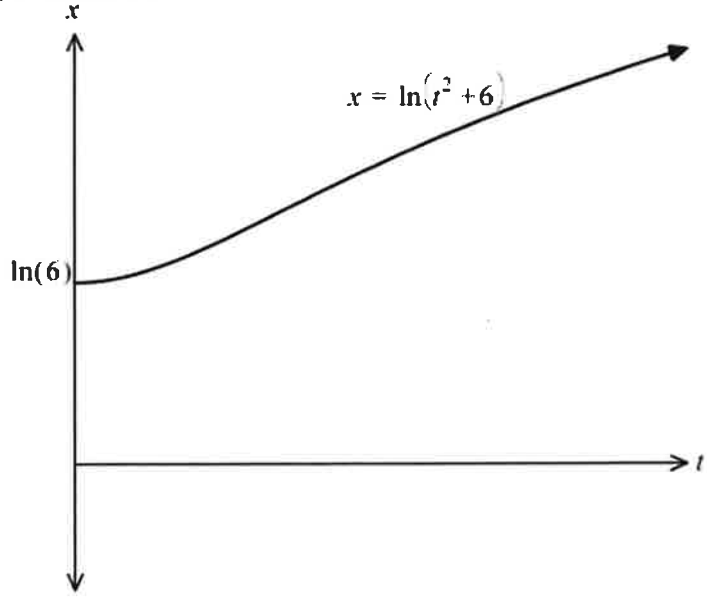
$\therefore x \doteq 5.6$ km

$\therefore$ Ship is 5.6 km from lighthouse

4. A particle moves on the x-axis so that it's displacement in metres from the origin at a time t seconds is given by the equation $x = \ln(t^2 + 6)$.

The particle starts from rest at the point $x = \ln(6)$ and accelerates in a positive direction as shown in the distance-time graph below.

When $t=0$, $x=\ln 6$
$v=0$
$a>0$



Determine when the acceleration of the particle becomes zero and find the velocity at this time.

$$x = \ln(t^2+6)$$

$$v = \frac{2t}{t^2+6}$$

$$a = \frac{(t^2+6)\times 2 - (2t)(2t)}{(t^2+6)^2}$$

$$= \frac{2t^2+12-4t^2}{(t^2+6)^2}$$

$$a = \frac{12-2t^2}{(t^2+6)^2}$$

Require $a=0 \implies 12-2t^2=0$

$$6-t^2=0$$

$$t=\sqrt{6} \text{ only } (t\geq 0)$$

When $t=\sqrt{6}$, $v = \frac{2\times\sqrt{6}}{\sqrt{6}^2+6} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$

$\therefore$ When $t=\sqrt{6}$, a is zero and $v=\frac{\sqrt{6}}{6}$ m/s

5. Use calculus to determine and verify the nature of the stationary points, find local maxima and minima and points of inflection (horizontal or otherwise) and hence sketch the graph of the function $y = 12x^5 - 15x^4 - 40x^3$. Accurate values for all x-intercepts are not required.

$y' = 60x^4 - 60x^3 - 120x^2$

For SP, $y' = 0 \Rightarrow 60x^4 - 60x^3 - 120x^2 = 0$

$\cancel{60}x^2(x^2 - x - 2) = 0$

$x^2(x-2)(x+1) = 0$

$x = 0, 2, -1$

$\therefore (0,0), (2,-176) \;\&\; (-1,13)$ are SP

$y'' = 240x^3 - 180x^2 - 240x$

at $x = 0$, $y'' = 0$ $\therefore (0,0)$ is a possible HPOI.

$x = 2$, $y'' = 720 > 0$ $\therefore$ concave up

$\therefore$ local min at $(2,-176)$

$x = -1$, $y'' = -180 < 0$ $\therefore$ concave down

$\therefore$ local max at $(-1,13)$

x	-0.5	0	1
y''	45	0	-180

since concavity changes about $(0,0)$, then a HPOI exists at $(0,0)$

For POI, $y'' = 0$: $240x^3 - 180x^2 - 240x = 0$

$4x^3 - 3x^2 - 4x) = 0$

$x(4x^2 - 3x - 4) = 0$

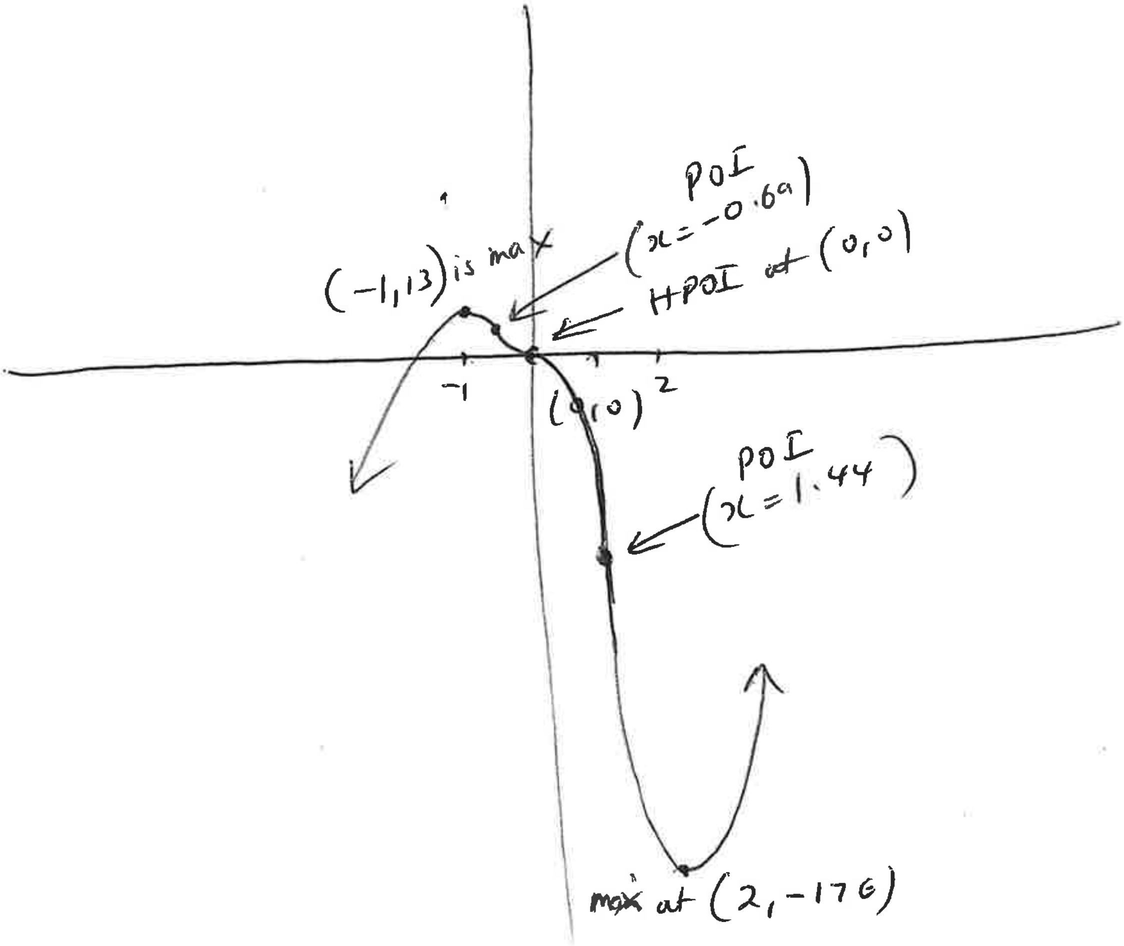
$x = 0$, $x = \frac{3 \pm \sqrt{73}}{8} \doteqdot -0.69, 1.44$

x	-1	-0.69	$-\frac{1}{2}$	0	1	1.44	2
y''	-180	0	45	0	-180	0	720

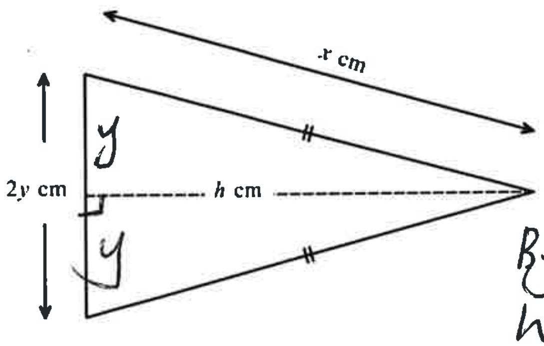
$\therefore$ concavity changes about $x = 0$, about $x = -0.69$ and $x = 1.44$

$\therefore$ POI at $x = -0.69$ & $x = 1.44$

$$y = 12x^5 - 15x^4 - 40x^3$$
$$= x^3(12x^2 - 15x - 40)$$



6. A banner is designed as an isosceles triangle, with equal sides of length x cm and base of length $2y$ cm, as shown.



By PT

$h = \sqrt{x^2 - y^2}$

$= \sqrt{x^2 - (20-x)^2}$

$= \sqrt{x^2 - (400 - 40x + x^2)}$

$= \sqrt{\cancel{x^2} - 400 + 40x - \cancel{x^2}}$

$h = \sqrt{40x - 400}$ *

The total perimeter of the triangle is 40 cm.

(a) Show that the area of the triangle in terms of x can written as:

$$A = (20 - x)(40x - 400)^{\frac{1}{2}}$$

$2y + 2x = 40$

$y + x = 20$ or $y = 20 - x$

$\therefore A = \frac{1}{\cancel{2}} \times \cancel{2}y \times h = (20 - x)\sqrt{40x - 400}$, OR

$A = (20 - x)(40x - 400)^{\frac{1}{2}}$

(b) Use calculus to find the values of x and y which give a maximum area and find this area.

For max area, (SP) $A' = 0$:

$A' = (20 - x)\frac{1}{\cancel{2}}(40x - 400)^{-\frac{1}{2}} \times \cancel{40}^{20} + (40x - 400)^{\frac{1}{2}}(-1)$

$= (40x - 400)^{-\frac{1}{2}}\left[20(20 - x) - (40x - 400)^{1}\right]$

$= (40x - 400)^{-\frac{1}{2}}\left[400 - 20x - 40x + 400\right]$

$A' = (40x - 400)^{-\frac{1}{2}}\left[800 - 60x\right]$

$\therefore A' = 0$ when $800 - 60x = 0$

$80 = 6x$

$\frac{40}{3} = x$

Test nature of SP using A'

x	12	$\frac{40}{3}$	14
A'	$\frac{80}{\sqrt{4400}}$	0	$\frac{-40}{\sqrt{160}}$

$\therefore$ max at $x = \frac{40}{3}$

$\therefore$ when $x = \frac{40}{3}$,

$y = 20 - \frac{40}{3} = \frac{40}{3}$

$\therefore$ Max area when $x = \frac{40}{3}, y = \frac{40}{3}$