ADVANCED MATHEMATICS
Functions (Adv), F2 Graphing (Adv)

Non-Calculus Graphing (Y12)
Transformations (Y12)

Teacher: Cathyanne Horvat
Exam Equivalent Time: 60 minutes *(based on allocation of 1.5 minutes per mark)*

F2 Graphing

Transformations
Non-Calculus Graphing

***SmarterMaths analytics based on the average contribution to new syllabus Advanced Maths exams since 2020.**

HISTORICAL CONTRIBUTION

- *F2 Graphing Techniques* has contributed an average of 4.0% per Adv exam since the new syllabus was introduced in 2020.
- We have split the topic into 2 categories for analysis purposes: *1-Transformations (2.7%) and 2-Non-Calculus Graphing (1.3%).*
- This analysis looks at *Transformations*.

HSC ANALYSIS - What to expect and common pitfalls

- *Transformations* represents new syllabus content that explicitly looks at translations and dilations of several function types, including the introduction of the aforementioned terminology.
- The *2022 Adv* exam required students to calculate translations and dilations in three separate steps, producing a mean mark of 51%. This question is on the back of *2021 Q21* which combined vertical and horizontal dilations and similarly caused problems with a 48% state mean mark. Revision attention here goes without saying.
- The NESA sample HSC exam, released in March 2020, has been instructive in developing this challenging database area. Pay careful attention to *F2 EQ-Bank* questions.
- There have been some examples in past HSC exams that looked at similar content. Please review of *F2 2013 HSC 15c* which proved very challenging for a majority of students.
- We note that Trig transformations, which we regard as an extremely important transformation sub-topic, are covered separately under *T3 Trig Graphs*.
- This topic area provides scope for examiners to ask both low and high difficulty questions, with a variety of underlying functions.

Questions

1. Functions, 2ADV F2 SM-Bank 9 MC

The graph of the function $f(x)=\dfrac{3x+2}{5-x}$, has asymptotes at

A. $x=-5,\ y=\dfrac{3}{2}$

B. $x=\dfrac{2}{3},\ y=-3$

C. $x=5,\ y=3$

D. $x=5,\ y=-3$

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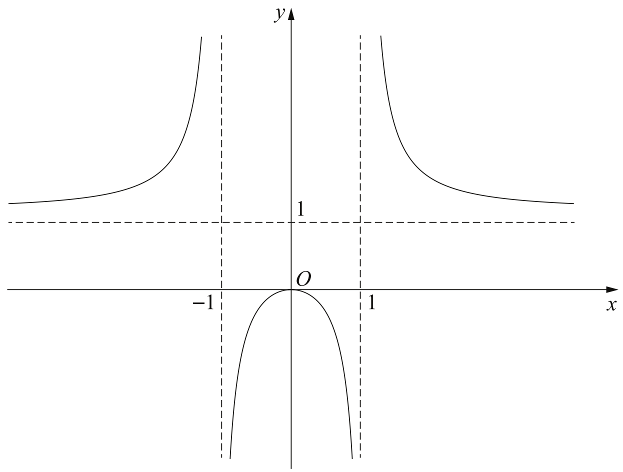
2. Functions, 2ADV' F2 2019 HSC 4 MC

The diagram shows the graph of $y=f(x)$.



Which equation best describes the graph?

A. $y=\dfrac{x}{x^2-1}$

B. $y=\dfrac{x^2}{x^2-1}$

C. $y=\dfrac{x}{1-x^2}$

D. $y=\dfrac{x^2}{1-x^2}$

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3. Functions, 2ADV' F2 2015 HSC 5 MC

What are the asymptotes of $y = \dfrac{3x}{(x+1)(x+2)}$

(A) $y = 0, \quad x = -1, \quad x = -2$

(B) $y = 0, \quad x = 1, \quad x = 2$

(C) $y = 3, \quad x = -1, \quad x = -2$

(D) $y = 3, \quad x = 1, \quad x = 2$

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4. Functions, 2ADV F2 SM-Bank 8 MC

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis.

B. dilation by a factor of 2 from the x-axis.

C. dilation by a factor of $\dfrac{1}{2}$ from the x-axis.

D. dilation by a factor of $\dfrac{1}{2}$ from the y-axis.

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5. Functions, 2ADV F2 SM-Bank 1

i. Draw the graph $y = \ln x$. ***(1 mark)***

ii. Explain how the above graph can be transformed to produce the graph

$$y = 3\ln(x+2)$$

and sketch the graph, clearly identifying all intercepts. ***(3 marks)***

6. Functions, 2ADV F1 SM-Bank 35

i. Sketch the function $y = f(x)$ where $f(x) = (x-1)^3$ on a number plane, labelling all intercepts. *(1 mark)*

ii. On the same graph, sketch $y = -f(-x)$. Label all intercepts. *(2 marks)*

7. Functions, 2ADV F2 EQ-Bank 16

$y = -\dfrac{(x+2)^4}{3}$ has been produced by three successive transformations: a translation, a dilation and then a reflection.

i. Describe each transformation and state the equation of the graph after each transformation. *(2 marks)*

ii. Sketch the graph. *(1 mark)*

8. Functions, 2ADV' F2 2012 HSC 13b

i. Find the horizontal asymptote of the graph $y = \dfrac{2x^2}{x^2+9}$. *(1 mark)*

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ii. Without the use of calculus, sketch the graph $y = \dfrac{2x^2}{x^2+9}$, showing the asymptote found in part (i). *(2 marks)*

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9. Functions, 2ADV F2 EQ-Bank 14

List a set of transformations that, when applied in order, would transform $y = x^2$ to the graph with equation $y = 1 - 6x - x^2$. *(3 marks)*

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10. Functions, 2ADV F2 SM-Bank 16

Let $f(x) = x^2 - 4$

Let the graph of $g(x)$ be a transformation of the graph of $f(x)$ where the transformations have been applied in the following order:

• dilation by a factor of $\dfrac{1}{2}$ from the vertical axis (parallel to the horizontal axis)

• translation by two units to the right (in the direction of the positive horizontal axis

Find $g(x)$ and the coordinates of the horizontal axis intercepts of the graph of $g(x)$. ***(3 marks)***

11. Functions, 2ADV F2 EQ-Bank 13

The curve $y = kx^2 + c$ is subject to the following transformations

- Translated 2 units in the positive x-direction
- Dilated in the positive y-direction by a factor of 4
- Reflected in the y-axis

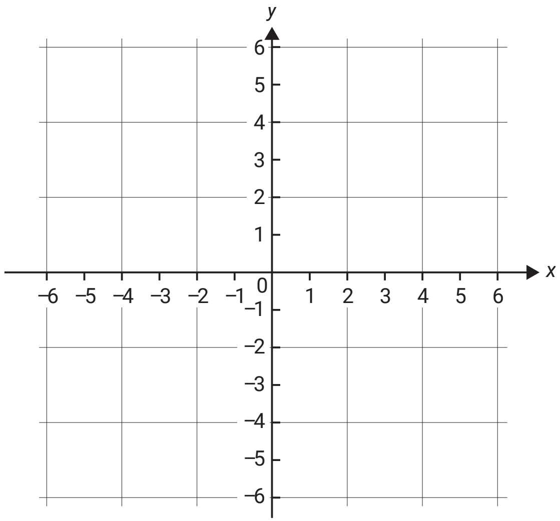
The final equation of the curve is $y = 8x^2 + 32x - 8$.

i. Find the equation of the graph after the dilation. ***(1 mark)***

ii. Find the values of k and c. ***(2 marks)***

12. Functions, 2ADV F2 SM-Bank 20

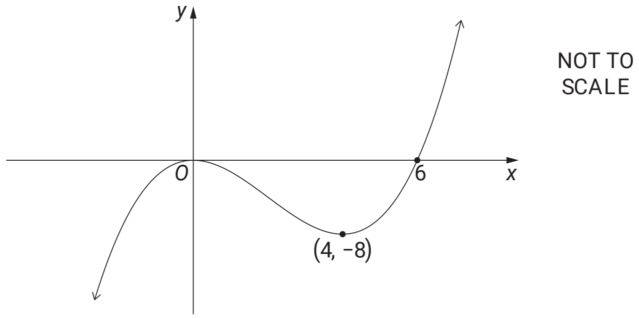
a. Sketch the graph of $y = 1 - \dfrac{2}{x-2}$ on the axes below. Label asymptotes with their equations and axis intercepts with their coordinates. ***(3 marks)***



b. Find the values of x for which $1 - \dfrac{2}{x-2} \geq 3$. ***(1 mark)***

13. Functions, 2ADV F2 2021 HSC 21

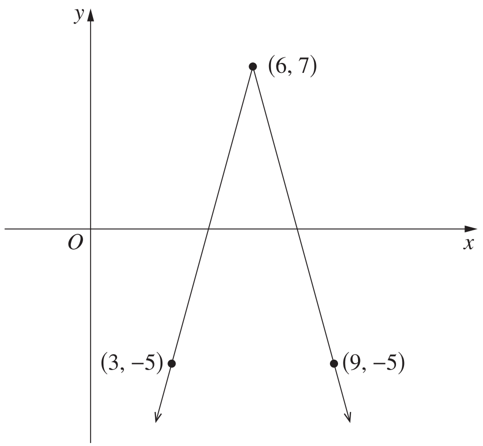
Consider the graph of $y = f(x)$ as shown.



Sketch the graph of $y = 4f(2x)$ showing the x-intercepts and the coordinates of the turning points. ***(2 marks)***

14. Functions, 2ADV F2 2023 HSC 27

The graph of $y = f(x)$, where $f(x) = a|x - b| + c$, passes through the points $(3, -5)$, $(6, 7)$ and $(9, -5)$ as shown in the diagram.



a. Find the values of a, b and c. ***(3 marks)***

b. The line $y = mx$ cuts the graph of $y = f(x)$ in two distinct places.

Find all possible values of m. ***(2 marks)***

15. Functions, 2ADV F2 2013 HSC 15c

i. Sketch the graph $y = |2x - 3|$. ***(1 mark)***

ii. Using the graph from part (i), or otherwise, find all values of m for which the equation $|2x - 3| = mx + 1$ has exactly one solution. ***(2 marks)***

Worked Solutions

1. Functions, 2ADV F2 SM-Bank 9 MC

$$\begin{aligned} f(x) &= \frac{3x+2}{5-x} \\ &= \frac{-(15-3x)+17}{5-x} \\ &= -3+\frac{17}{5-x} \end{aligned}$$

Vertical asymptote: $x=5$

Horizontal asymptote: $y=-3$

$\Rightarrow D$

2. Functions, 2ADV' F2 2019 HSC 4 MC

By elimination:

Graph is an even function

$\Rightarrow f(x)=f(-x)$

$\therefore$ Eliminate A and C

When $-1<x<1,\quad y\leq 0$

$\therefore$ Eliminate D

$\Rightarrow B$

3. Functions, 2ADV' F2 2015 HSC 5 MC

$$y=\frac{3x}{(x+1)(x+2)}$$

Asymptotes at $x=-1$ and $x=-2$

As $x\to\infty, y\to 0^+$

As $x\to-\infty, y\to 0^-$

$\therefore$ Horizontal asymptote at $y=0$

$\Rightarrow A$

4. Functions, 2ADV F2 SM-Bank 8 MC

Let $f(x)=\sqrt{8x^3+1}$

$$f\left(\frac{1}{2}x\right)=\sqrt{8\left(\frac{1}{2}x\right)^3+1}$$
$$=\sqrt{x^3+1}$$

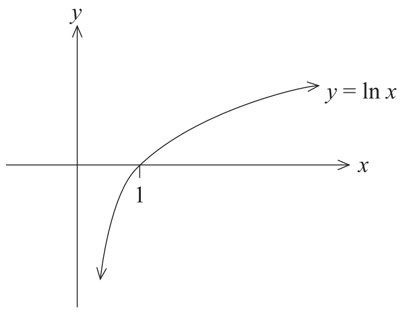
$\therefore$ Transformation correct when x is swapped for $\dfrac{x}{2}$

i.e. graph is dilated by factor of 2 from y-axis

$\Rightarrow A$

5. Functions, 2ADV F2 SM-Bank 1

i.

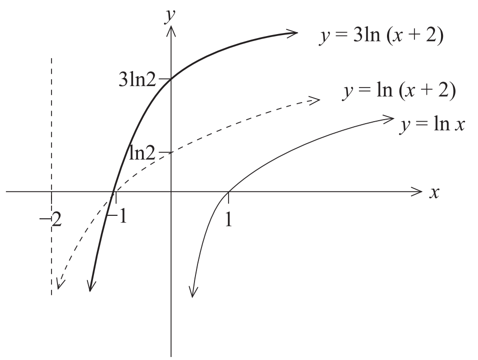


ii. Transforming $y=\ln x \Rightarrow y=\ln(x+2)$

$y=\ln x \Rightarrow$ shift 2 units to left.

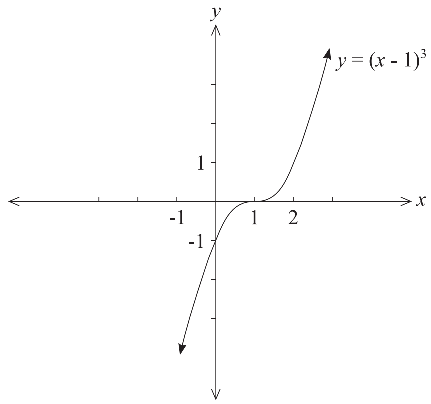
Transforming $y=\ln(x+2)$ to $y=3\ln(x+2)$

$\Rightarrow$ increase each y value by a factor of 3



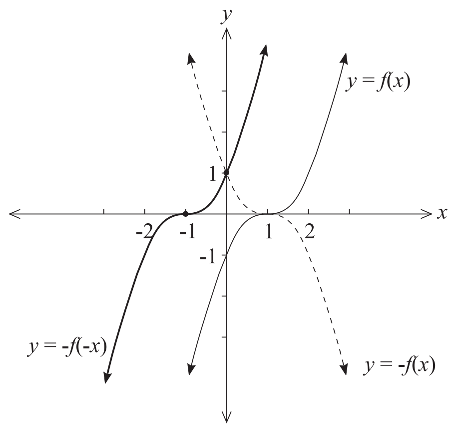
6. Functions, 2ADV F1 SM-Bank 35

i. $y=(x-1)^3 \Rightarrow y=x^3$ shifted 1 unit to the right.



ii. $y=-f(x) \Rightarrow$ reflect $y=(x-1)^3$ in x-axis.

$y=-f(-x) \Rightarrow$ reflect $y=-f(x)$ in y-axis.



7. Functions, 2ADV F2 EQ-Bank 16

i. Transformation 1:

Translate $y = x^4$ 2 units to the left.

$y = x^4 \ \Rightarrow \ y = (x+2)^4$

Transformation 2:

Dilate $y = (x+2)^4$ by a factor of $\dfrac{1}{3}$ from the x-axis

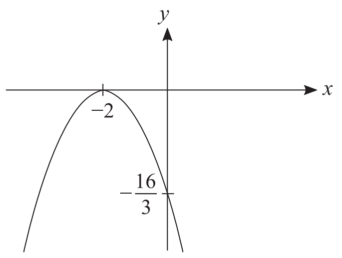
$y = (x+2)^4 \ \Rightarrow \ y = \dfrac{(x+2)^4}{3}$

Transformation 3:

Reflect $y = \dfrac{(x+2)^4}{3}$ in the x-axis.

$y = \dfrac{(x+2)^4}{3} \ \Rightarrow \ y = -\dfrac{(x+2)^4}{3}$

ii.



8. Functions, 2ADV' F2 2012 HSC 13b

i. $y = \dfrac{2x^2}{x^2+9}$

$= \dfrac{2}{1+\frac{9}{x^2}}$

As $x \to \infty,\ y \to 2$

As $x \to -\infty,\ y \to 2$

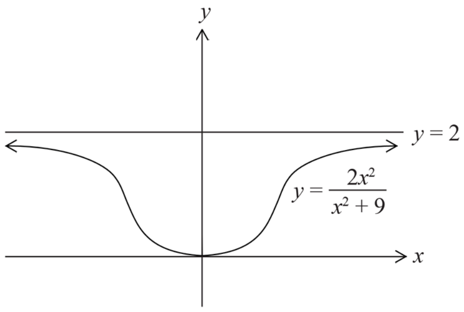
$\therefore$ Horizontal asymptote at $y = 2$

ii. At $x = 0,\ y = 0$

$f(x) = \dfrac{2x^2}{x^2+9} \geq 0$ for all x

$f(-x) = \dfrac{2(-x)^2}{(-x)^2+9} = \dfrac{2x^2}{x^2+9} = f(x)$

Since $f(x) = f(-x) \ \Rightarrow$ EVEN function



9. Functions, 2ADV F2 EQ-Bank 14

$y = x^2$

Transformation 1:
Translate 3 units in negative x-direction

$y = (x+3)^2$

$y = x^2 + 6x + 9$

Transformation 2:
Translate 10 units in negative y-direction

$y = x^2 + 6x - 1$

Transformation 3:
Reflect in the x-axis

$y = -\left(x^2 + 6x - 1\right)$

$\quad = 1 - 6x - x^2$

10. Functions, 2ADV F2 SM-Bank 16

1st transformation

Dilation by a factor of $\dfrac{1}{2}$ from the y-axis:

$x^2 - 4 \ \Rightarrow \ \left(\dfrac{x}{\frac{1}{2}}\right)^2 - 4 = 4x^2 - 4$

2nd transformation
Translation by 2 units to the right:

$4x^2 - 4 \ \Rightarrow \ g(x) = 4(x-2)^2 - 4$

x-axis intercept of $g(x)$:

$$\begin{aligned} 4(x-2)^2 - 4 &= 0 \\ (x-2)^2 &= 1 \\ x - 2 &= \pm 1 \end{aligned}$$

$x - 2 = 1 \ \Rightarrow \ x = 3$
$x - 2 = -1 \ \Rightarrow \ x = 1$

$\therefore$ Horizontal axis intercepts occur at $(1, 0)$ and $(3, 0)$.

11. Functions, 2ADV F2 EQ-Bank 13

i. $y = kx^2 + c$

Translate 2 units in positive x-direction.

$y = kx^2 + c \ \Rightarrow \ y = k(x-2)^2 + c$

Dilate in the positive y-direction by a factor of 4.

$y = k(x-2)^2 + c \ \Rightarrow \ y = 4k(x-2)^2 + 4c$

ii. $y = 4k\left(x^2 - 4x + 4\right) + 4c$

$\quad = 4kx^2 - 16kx + 16k + 4c$

Reflect in the y-axis.

$\Rightarrow$ Swap: $x \to -x$

$y = 4k(-x)^2 - 16k(-x) + 16k + 4c$

$\quad = 4kx^2 + 16kx + 16k + 4c$

COMMENT: Using "swap" terminology for reflections in the y-axis is simpler and more intelligible for students in our view.

Equating co-efficients:

$4k = 8$

$\therefore k = 2$

$$\begin{aligned} 16k + 4c &= -8 \\ 4c &= -40 \\ \therefore c &= -10 \end{aligned}$$

12. Functions, 2ADV F2 SM-Bank 20

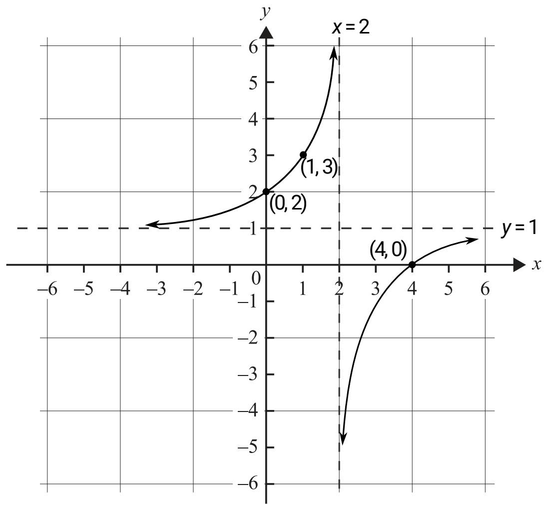
a. Asymptotes:

$x = 2$

As $x \to \pm\infty,\ y \to 1 \ \Rightarrow$ Asymptote at $y = 1$

y-intercept at $(0, 2)$

x-intercept at $(4, 0)$

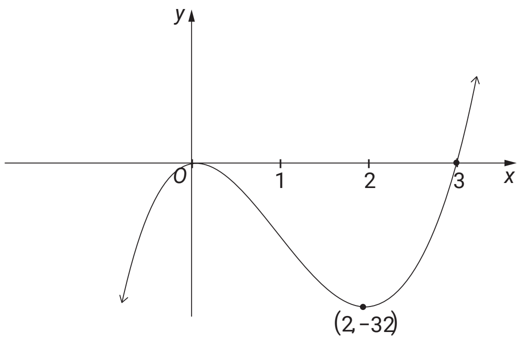


b. By inspection of the graph:

$$1 - \frac{2}{x-2} \geq 3 \ \text{ for } \ x \in [1, 2)$$

13. Functions, 2ADV F2 2021 HSC 21

♦ *Mean mark 48%.*



14. Functions, 2ADV F2 2023 HSC 27

a. Consider the transformation of $y = -|x|$

Translate 6 units to the right

$y = -|x| \ \ \to \ \ y = -|x-6|$

$\therefore b = 6$

Translate 7 units vertically up

$y = -|x-6| \ \ \to \ \ y = -|x-6| + 7$

$\therefore c = 7$

$f(x) = a|x-6| + 7$ passes through $(3, -5)$:

$-5 = a|3-6| + 7$

$-5 = 3a + 7$

$3a = -12$

$\therefore a = -4$

♦♦♦ Mean mark (b) 23%.

b. $y = mx$ passes through $(0, 0)$

One solution when $y = mx$ passes through $(0, 0)$ and $(6, 7)$

$$m = \frac{7-0}{6-0} = \frac{7}{6}$$

As graph gets flatter and turns negative $\Rightarrow$ 2 solutions

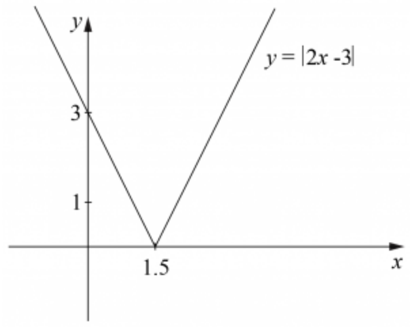
2 solutions continue until $y = mx$ is parallel to the line joining $(6, 7)$ to $(9, -5)$, where:

$$m = \frac{7-(-5)}{6-9} = -\frac{12}{3} = -4$$

$\therefore$ 2 solutions when $-4 < m < 7/6$

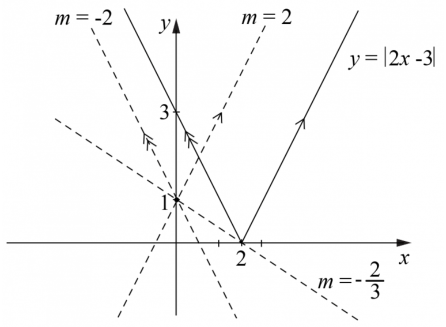
15. Functions, 2ADV F2 2013 HSC 15c

i.



♦ *Mean mark 49%*
MARKER'S COMMENT: Many students drew diagrams that were "too small", didn't use rulers or didn't use a consistent scale on the axes!

ii.



Line of intersection $y = mx + 1$ passes through $(0, 1)$
If it also passes through $(1.5, 0) \Rightarrow$ 1 solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{1-0}{0-\frac{3}{2}}$$
$$= -\frac{2}{3}$$

♦♦ *Mean mark 25%.*
COMMENT: Students need a clear graphical understanding of what they are finding to solve this very challenging, Band 6 question.

Gradients of $y = |\ 2x - 3\ |$ are 2 or -2

Considering a line through $(0, 1)$:
If $m \geq 2$, only intersects once.

Similarly,
If $m < -2$, only intersects once.

$\therefore$ Only one solution when $m = -\dfrac{2}{3},\ \ m \geq 2$ or $m < -2$