ADVANCED MATHEMATICS

Trigonometry (Adv), T2 Trig Functions and Identities (Adv)

Exact Trig Ratios (Y11)

Trig Identities and Harder Equations (Y11)

Trigonometry (Adv), T3 Trig Functions and Graphs (Adv)

Trig Graphs (Y12)

Trig Applications (Y12)

Teacher: Cathyanne Horvat

Exam Equivalent Time: 75 minutes *(based on allocation of 1.5 minutes per mark)*

Questions

1. Trigonometry, 2ADV T3 EQ-Bank 5

   The function $f(x) = \sin x$ is transformed into the function $g(x) = \dfrac{\sin(4x)}{3}$.

   Describe in words how the amplitude and period have changed in this transformation. ***(2 marks)***

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2. Trigonometry, 2ADV T2 2009 HSC 1e

   Find the exact value of θ such that $2\cos\theta = 1$, where $0 \leq \theta \leq \dfrac{\pi}{2}$. ***(2 marks)***

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3. Trigonometry, 2ADV T2 SM-Bank 43

   Find the exact value of

   $$\cot\left(-\frac{5\pi}{6}\right).$$ ***(2 marks)***

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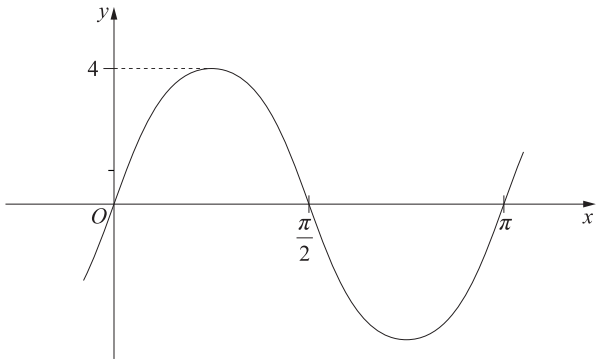
4. Trigonometry, 2ADV T3 2010 HSC 8c

   The graph shown is $y = A\sin bx$.

   i. Write down the value of A. ***(1 mark)***

   ii. Find the value of b. ***(1 mark)***

   iii. Copy or trace the graph into your writing booklet.
   On the same set of axes, draw the graph $y = 3\sin x + 1$ for $0 \leq x \leq \pi$. ***(2 marks)***

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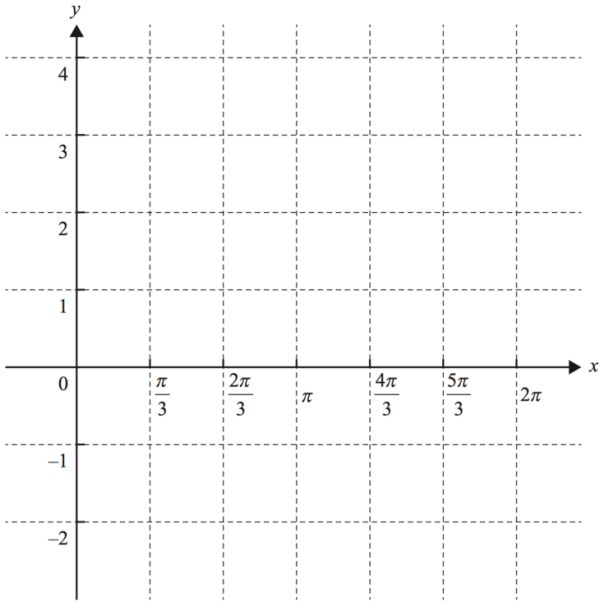
5. Trigonometry, 2ADV T3 SM-Bank 9

Let $f(x) = 2\cos(x) + 1$ for $0 \leq x \leq 2\pi$.

i. Solve the equation $2\cos(x) + 1 = 0$ for $0 \leq x \leq 2\pi$. *(2 marks)*

ii. Sketch the graph of the function $f(x)$ on the axes below. Label the endpoints and local minimum point with their coordinates. *(3 marks)*



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6. Trigonometry, 2ADV T2 2016 HSC 11g

Solve $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ for $0 \leq x \leq 2\pi$. *(2 marks)*

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7. Trigonometry, 2ADV T2 2020 HSC 19

Prove that $\sec\theta - \cos\theta = \sin\theta\tan\theta$. *(2 marks)*

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8. Trigonometry, 2ADV T2 SM-Bank 42

Prove that

$$\frac{1 - \sin^2 x\cos^2 x}{\sin^2 x} = \cot^2 x + \sin^2 x.$$ *(2 marks)*

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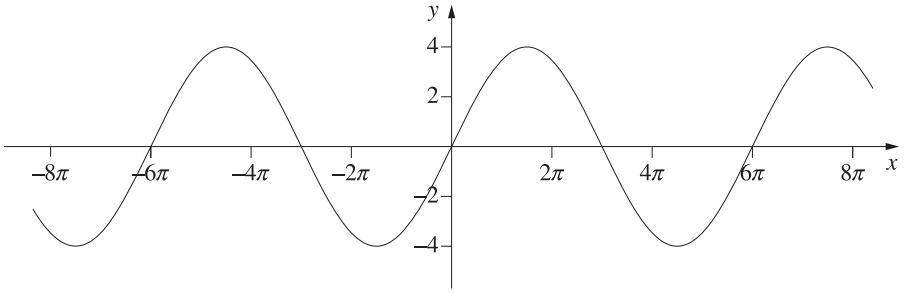
9. Trigonometry, 2ADV T3 2022 HSC 14

The graph of $y = k\sin(ax)$



What are the values of k and a? *(2 marks)*

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10. Trigonometry, 2ADV T3 SM-Bank 12

State the range and period of the function

$$h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right).$$ *(2 marks)*

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11. Trigonometry, 2ADV T3 SM-Bank 13

On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), \quad 0 \leq t \leq 24$$

where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

i. Find the minimum depth of the water in the river. *(1 mark)*

ii. Find the values of t for which $h(t) = 10$. *(2 marks)*

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12. Trigonometry, 2ADV T2 2023 HSC 20

Find all the values of θ, where $0° \leq \theta \leq 360°$, such that

$$\sin(\theta - 60°) = -\frac{\sqrt{3}}{2}$$ *(3 marks)*

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13. Trigonometry, 2ADV T3 2018 HSC 15a

The length of daylight, $L(t)$, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation

$$L(t) = 12 + 2\cos\left(\frac{2\pi t}{366}\right),$$

where t is the number of days after 21 December 2015, for $0 \leq t \leq 366$.

i. Find the length of daylight on 21 December 2015. *(1 mark)*

ii. What is the shortest length of daylight? *(1 mark)*

iii. What are the two values of t for which the length of daylight is 11? *(2 marks)*

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14. Trigonometry, 2ADV T2 2004 HSC 8a

i. Show that $\cos\theta\tan\theta = \sin\theta$. *(1 mark)*

ii. Hence solve $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\,\theta$ for $0 \leq \theta \leq 2\pi$. *(2 marks)*

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15. Trigonometry, 2ADV T3 2022 HSC 23

The depth of water in a bay rises and falls with the tide. On a particular day the depth of the water, d metres, can be modelled by the equation

$$d = 1.3 - 0.6\cos\left(\frac{4\pi}{25}t\right),$$

where t is the time in hours since low tide.

a. Find the depth of water at low tide and at high tide. *(2 marks)*

b. What is the time interval, in hours, between two successive low tides? *(1 mark)*

c. For how long between successive low tides will the depth of water be at least 1 metre? *(3 marks)*

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16. Trigonometry, 2ADV T2 2014 HSC 15a

Find all solutions of $2\sin^2 x + \cos x - 2 = 0$, where $0 \leq x \leq 2\pi$. *(3 marks)*

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17. Trigonometry, 2ADV T2 SM-Bank 40

Let $(\tan\theta - 1)\left(\sin\theta - \sqrt{3}\cos\theta\right)\left(\sin\theta + \sqrt{3}\cos\theta\right) = 0$.

i. State all possible values of $\tan\theta$. *(1 mark)*

ii. Hence, find all possible solutions for $(\tan\theta - 1)\left(\sin^2\theta - 3\cos^2\theta\right) = 0$, where $0 \leq \theta \leq \pi$. *(2 marks)*

Worked Solutions

1. Trigonometry, 2ADV T3 EQ-Bank 5

$g(x) = \frac{1}{3}\sin(4x)$

$\Rightarrow$ **The new amplitude is one third of the original amplitude.**

$\text{Period} = \frac{2\pi}{n} \Rightarrow n = \frac{1}{4}$

$\Rightarrow$ **The new period is one quarter of the original period.**

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2. Trigonometry, 2ADV T2 2009 HSC 1e

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\therefore\ \theta = \frac{\pi}{3}, \quad 0 \leq \theta \leq \frac{\pi}{2}$

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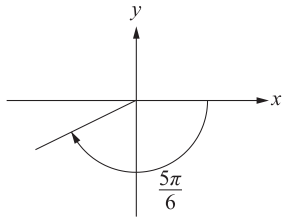
3. Trigonometry, 2ADV T2 SM-Bank 43



$$\begin{aligned}\cot\left(-\frac{5\pi}{6}\right) &= \frac{1}{\tan\left(-\frac{5\pi}{6}\right)} \\ &= \frac{1}{\tan\left(\frac{\pi}{6}\right)} \\ &= \frac{1}{\frac{1}{\sqrt{3}}} \\ &= \sqrt{3}\end{aligned}$$

Worked Solutions

4. Trigonometry, 2ADV T3 2010 HSC 8c

i. $A = 4$

ii. Since the graph passes through $\left(\frac{\pi}{4}, 4\right)$

Substituting into $y = 4\sin bx$

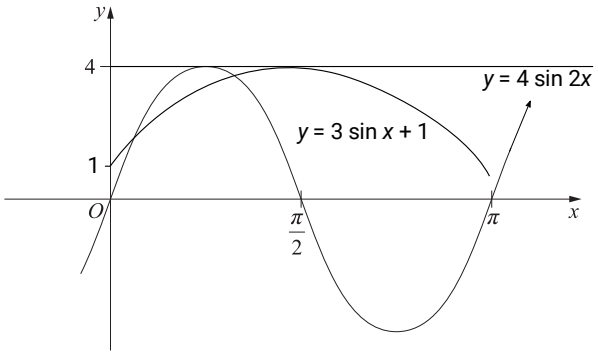
$$4\sin\left(b \times \frac{\pi}{4}\right) = 4$$

$$\sin\left(b \times \frac{\pi}{4}\right) = 1$$

$$b \times \frac{\pi}{4} = \frac{\pi}{2}$$

$$\therefore b = 2$$

iii.



MARKER'S COMMENT: Graphs are consistently drawn too small by many students. Aim to make your diagrams 1/3 to 1/2 of a page.

5. Trigonometry, 2ADV T3 SM-Bank 9

i. $2\cos(x) + 1 = 0$

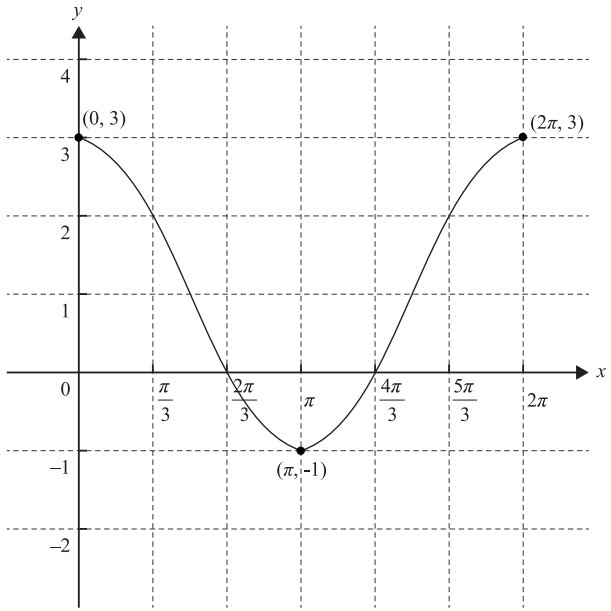
$$\cos(x) = -\frac{1}{2}$$

$\Rightarrow \cos\frac{\pi}{3} = \frac{1}{2}$ and cos is negative in 2nd/3rd quadrant

$$\therefore x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}, \frac{4\pi}{3}$$

ii.



6. Trigonometry, 2ADV T2 2016 HSC 11g

$$\sin\frac{x}{2}=\frac{1}{2} \text{ for } 0\le x\le 2\pi$$

$$\Rightarrow \text{Base angle}=\frac{\pi}{6},$$

$$\frac{x}{2}=\frac{\pi}{6},\pi-\frac{\pi}{6},2\pi+\frac{\pi}{6},\ldots$$

$$=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\ldots$$

$$\therefore x=\frac{\pi}{3},\ \frac{5\pi}{3} \text{ for } 0\le x\le 2\pi$$

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7. Trigonometry, 2ADV T2 2020 HSC 19

$$\begin{aligned}\text{LHS}&=\frac{1}{\cos\theta}-\cos\theta\\&=\frac{1-\cos^2\theta}{\cos\theta}\\&=\frac{\sin^2\theta}{\cos\theta}\\&=\sin\theta\cdot\frac{\sin\theta}{\cos\theta}\\&=\sin\theta\tan\theta \ldots \text{ as required}\end{aligned}$$

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8. Trigonometry, 2ADV T2 SM-Bank 42

$$\begin{aligned}\text{RHS}&=\frac{\cos^2x}{\sin^2x}+\sin^2x\\&=\frac{\cos^2x+\sin^4x}{\sin^2x}\\&=\frac{\cos^2x+\sin^2x\left(1-\cos^2x\right)}{\sin^2x}\\&=\frac{\cos^2x+\sin^2x-\sin^2x\cos^2x}{\sin^2x}\\&=\frac{1-\sin^2x\cos^2x}{\sin^2x}\\&=\text{LHS}\end{aligned}$$

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9. Trigonometry, 2ADV T3 2022 HSC 14

$$\text{Amplitude}=4$$

$$\Rightarrow k=4$$

$$\text{Period}=6\pi$$

$$\frac{2\pi}{a}=6\pi$$

$$6\pi a=2\pi$$

$$\Rightarrow a=\frac{1}{3}$$

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10. Trigonometry, 2ADV T3 SM-Bank 12

$$-1\le\cos\left(\frac{\pi x}{2}\right)\le 1$$

$$-3\le 3\cos\left(\frac{\pi x}{2}\right)\le 3$$

$$1\le 4+3\cos\left(\frac{\pi x}{2}\right)\le 7$$

$$\therefore \text{ Range: } 1\le y\le 7$$

$$\text{Period }=\frac{2\pi}{n}=\frac{2\pi}{\frac{\pi}{2}}=4$$

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11. Trigonometry, 2ADV T3 SM-Bank 13

i. $h_{\min}$ occurs when $\sin\left(\frac{\pi t}{12}\right)=-1$

$$\therefore h_{\min}=14-8$$

$$=6 \text{ m}$$

MARKER'S COMMENT: Students who used calculus to find the minimum were less successful.

ii. $14+8\sin\left(\frac{\pi}{12}t\right)=10$

$$\sin\left(\frac{\pi}{12}t\right)=-\frac{1}{2}$$

Solve in general:

$$\frac{\pi}{12}t=\frac{7\pi}{6}+2\pi n \quad \text{or} \quad \frac{\pi}{12}t=\frac{11t}{6}+2\pi n,$$

$$t=14+24n \qquad t=22+24n$$

Substitute integer values for n,

$$\therefore t=14 \text{ or } 22, \quad (0\le t\le 24)$$

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12. Trigonometry, 2ADV T2 2023 HSC 20

$\sin 60° = \dfrac{\sqrt{3}}{2} \Rightarrow$ Base angle $= 60°$

$\Rightarrow$ sin is negative in 3rd and 4th quadrants

$\sin(\theta - 60°) = 180 + 60, 360 - 60$

$= 240°, 300°$

$\theta - 60° = 240° \Rightarrow \theta = 300°$

$\theta - 60° = 300° \Rightarrow \theta = 360°$

Consider $\theta = 0°$

$\sin(0 - 60°) = \sin(-60°) = -\dfrac{\sqrt{3}}{2}$

$\therefore \theta = 0°, 300°$ and $360°$

13. Trigonometry, 2ADV T3 2018 HSC 15a

♦ Mean mark 43%.

i. $L(t) = 12 + 2\cos\left(\dfrac{2\pi t}{366}\right)$

On 21 Dec 2015 $\Rightarrow t = 0$

$\therefore L(0) = 12 + 2\cos 0$

$= 14$ hours

ii. Shortest length of daylight occurs when

$\cos\left(\dfrac{2\pi t}{366}\right) = -1$

$\therefore$ Shortest length $= 12 + 2(-1)$

$= 10$ hours

iii. Find t such that $L(t) = 11$:

$11 = 12 + 2\cos\left(\dfrac{2\pi t}{366}\right)$

$\cos\left(\dfrac{2\pi t}{366}\right) = -\dfrac{1}{2}$

$\dfrac{2\pi t}{366} = \dfrac{2\pi}{3}$ or $\dfrac{2\pi t}{366} = \dfrac{4\pi}{3}$

$t = \dfrac{366}{3}$ $t = \dfrac{366 \times 2}{3}$

$= 122$ $= 244$

$\therefore t = 122$ or 244

14. Trigonometry, 2ADV T2 2004 HSC 8a

i. Prove $\cos\theta\tan\theta = \sin\theta$

$$\begin{aligned}
\text{LHS} &= \cos\theta\tan\theta \\
&= \cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) \\
&= \sin\theta \\
&= \text{RHS}
\end{aligned}$$

ii. $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\,\theta$

$$\begin{aligned}
\therefore 8\sin\theta(\sin\theta) &= \operatorname{cosec}\,\theta, \quad \text{(part (i))} \\
8\sin^2\theta &= \frac{1}{\sin\theta} \\
8\sin^3\theta &= 1 \\
\sin^3\theta &= \frac{1}{8} \\
\sin\theta &= \frac{1}{2} \\
\therefore \theta &= \frac{\pi}{6}, \frac{5\pi}{6}. \quad (\text{for } 0 \leq \theta \leq 2\pi)
\end{aligned}$$

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15. Trigonometry, 2ADV T3 2022 HSC 23

a. Since $-1 \leq \cos\left(\frac{4\pi}{25}t\right) \leq 1$:

$$\begin{aligned}
\text{Low Tide} &= 1.3 - 0.6(1) = 0.7 \text{ m} \\
\text{High Tide} &= 1.3 - 0.6(-1) = 1.9 \text{ m}
\end{aligned}$$

b. Time between two low tides = Period of equation (n)

♦ Mean mark part (b) 47%.

$$\begin{aligned}
\frac{2\pi}{n} &= \frac{4\pi}{25} \\
\frac{n}{2\pi} &= \frac{25}{4\pi} \\
n &= \frac{25}{2} \text{ hours}
\end{aligned}$$

c. Find t when $d = 1$:

♦ Mean mark part (c) 46%.

$$\begin{aligned}
1.3 - 0.6\cos\left(\frac{4\pi}{25}t\right) &= 1 \\
-0.6\cos\left(\frac{4\pi}{25}t\right) &= -0.3 \\
\cos\left(\frac{4\pi}{25}t\right) &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{4\pi}{25}t &= \frac{\pi}{3}, \ \frac{5\pi}{3} \\
t &= \frac{25}{12}, \ \frac{125}{12}
\end{aligned}$$

$\therefore$ Time between low tides where water depth ≥ 1 m

$$\begin{aligned}
&= \frac{125}{12} - \frac{25}{12} \\
&= \frac{100}{12} \\
&= \frac{25}{3} \text{ hours}
\end{aligned}$$

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16. Trigonometry, 2ADV T2 2014 HSC 15a

♦ *Mean mark 42%*

$$2\sin^2 x + \cos x - 2 = 0$$
$$2(1 - \cos^2 x) + \cos x - 2 = 0$$
$$2 - 2\cos^2 x + \cos x - 2 = 0$$
$$-2\cos^2 x + \cos x = 0$$
$$\cos x(-2\cos x + 1) = 0$$

$\therefore -2\cos x + 1 = 0$ or $\cos x = 0$

$2\cos x = 1$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos x = \frac{1}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Since cos is positive in $1^{st}/4^{th}$ quadrants,

$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$ for $0 \le x \le 2\pi$

17. Trigonometry, 2ADV T2 SM-Bank 40

i. $(\tan\theta - 1)\left(\sin\theta - \sqrt{3}\cos\theta\right)\left(\sin\theta + \sqrt{3}\cos\theta\right) = 0$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \sin\theta - \sqrt{3}\cos\theta = 0$

♦ *Mean mark 42%.*

$\sin\theta = \sqrt{3}\cos\theta$

$\tan\theta = \sqrt{3}$

$\Rightarrow \sin\theta + \sqrt{3}\cos\theta = 0$

$\sin\theta = -\sqrt{3}\cos\theta$

$\tan\theta = -\sqrt{3}$

$\therefore \tan\theta = 1$ or $\tan\theta = \pm\sqrt{3}$

ii. $(\tan\theta - 1)\left(\sin^2\theta - 3\cos^2\theta\right) = 0$

Using part a:

$(\tan\theta - 1)\left(\sin\theta - \sqrt{3}\cos\theta\right)\left(\sin\theta + \sqrt{3}\cos\theta\right) = 0$

♦ *Mean mark 42%.*

$\Rightarrow \tan\theta = 1$ or $\tan\theta = \pm\sqrt{3}$

$\theta = \frac{\pi}{4}$ $\quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

$\therefore \theta = \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{2\pi}{3}$ $\quad (0 \le \theta \le \pi)$