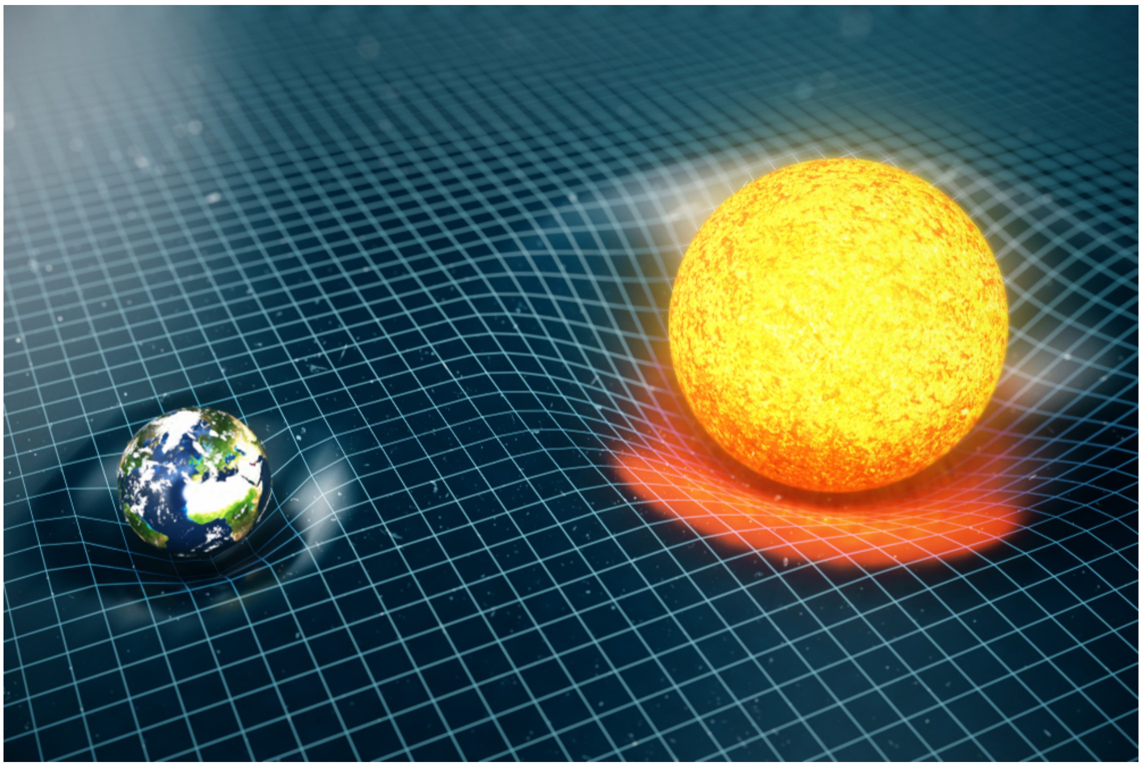
Module 5: Advanced Mechanics

Part 3: Motion in Gravitational Fields



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Contents

Syllabus content: Advanced Mechanics

Motion in Gravitational Fields

Inquiry question: How does the force of gravity determine the motion of planets and satellites?

Students:

- apply qualitatively and quantitatively Newton's Law of Universal Gravitation to:
  - determine the force of gravity between two objects $F = \frac{GMm}{r^2}$
  - investigate the factors that affect the gravitational field strength $g = \frac{GM}{r^2}$
  - predict the gravitational field strength at any point in a gravitational field, including at the surface of a planet (ACSPH094, ACSPH095, ACSPH097)
- investigate the orbital motion of planets and artificial satellites when applying the relationships between the following quantities:
  - gravitational force
  - centripetal force
  - centripetal acceleration
  - mass
  - orbital radius
  - orbital velocity
  - orbital period
- predict quantitatively the orbital properties of planets and satellites in a variety of situations, including near the Earth and geostationary orbits, and relate these to their uses (ACSPH101)
- investigate the relationship of Kepler's Laws of Planetary Motion to the forces acting on, and the total energy of, planets in circular and non-circular orbits using: (ACSPH101)
  - $v = \frac{2\pi r}{T}$
  - $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$
- derive quantitatively and apply the concepts of gravitational force and gravitational potential energy in radial gravitational fields to a variety of situations, including but not limited to:
  - the concept of escape velocity $v_{esc} = \sqrt{\frac{GM}{2r}}$

- total potential energy of a planet or satellite in its orbit $U = -\frac{GMm}{r}$
- total energy of a planet or satellite in its orbit $U + K = -\frac{GMm}{2r}$
- energy changes that occur when satellites move between orbits
- Kepler's Laws of Planetary Motion

Kepler's laws of Planetary Motion

Figure 1: Portrait of Kepler by an unknown artist, 1610

Johannes Kepler was a German astronomer who developed three empirical laws describing planetary motion, partly published in 1609 in "Astronomia nova" and a complete version in a 1617 textbook entitled "Epitome of Copernican Astronomy". These laws were based on the extremely precise measurements he had access to as an assistant to Tycho Brahe.

1. The Law of Orbits: All planets move in elliptical orbits, with the sun at one focus.
2. The Law of Areas: A line that connects a planet to the sun sweeps out equal areas in equal times.
3. The Law of Periods: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

$$r^3 \propto T^2$$

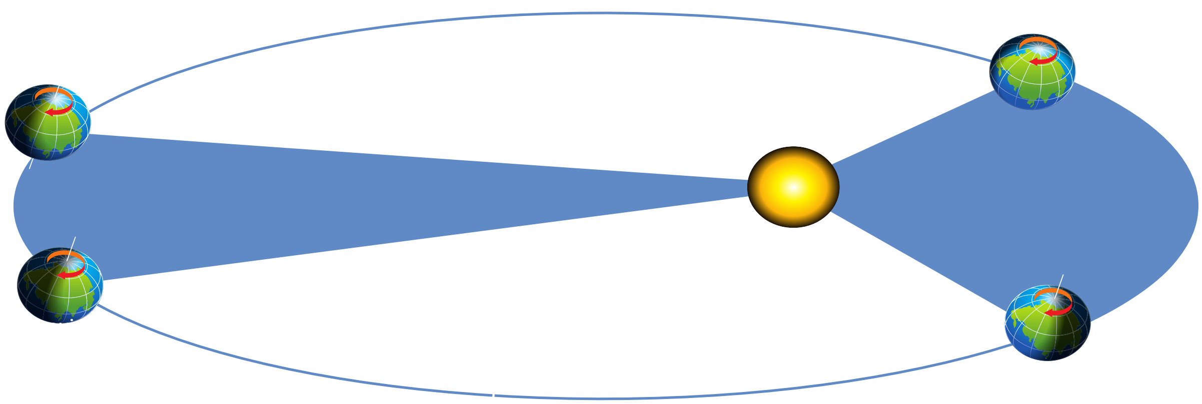

Figure 2: According to Kepler's laws of planetary motion, a planet (here the Earth) moves in an elliptical trajectory, with the sun at one focus. It sweeps out equal areas (along a line connecting the planet to the sun) in equal times, such that it travels faster when it is nearer the sun and slower when further away.

Newton's law of Universal Gravitation

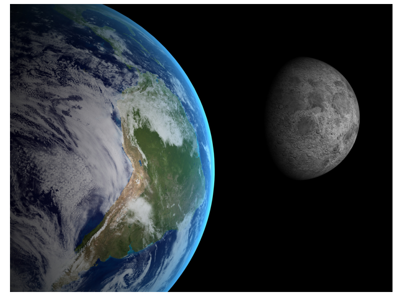

Figure 3: Earth and the moon (not to scale!)

Historical development

> "(In 1665) I began to think of gravity extending to the orb of the Moon, and having found out how to estimate the force with which [a] globe revolving within a sphere presses the surface of the sphere, from Kepler's Rule of the periodical times of the Planets being in a sesquialterate proportion of their distances from the centers of their orbs I deduced that the forces which keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from the centers about which they revolve: and thereby compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the earth, and found them answer pretty nearly." **- Isaac Newton**

A quote from Newton, in the preface to "A catalogue of the Portsmouth collection of books and papers written by or belonging to Sir Isaac Newton : the scientific portion of which has been presented by the Earl of Portsmouth to the University of Cambridge" by Newton, Isaac, Sir, 1642-1727; Luard, Henry Richards, 1825-1891, inscriber.

In the above quote, Isaac Newton describes how he used Kepler's third law (that "the periodical times of the Planets being in a sesquialterate proportion of their distances from the centers of their orbs") to develop the hypothesis that the force of gravity must vary as the inverse square of the distance between masses.

To see how this must be the case (for circular orbits at least!), we first note that the speed at which the mass is moving around the orbit must be the distance travelled in one orbit divided by the time taken to complete an orbit (the period, T)

$$v = \frac{2\pi r}{T}$$

Rearranging this equation we have an expression for $\frac{1}{T^2}$ as

$$\frac{1}{T^2} = \frac{v^2}{4\pi^2 r^2}$$

From Kepler's third law $r^3 \propto T^2$ so we can write:

$$\frac{1}{r^3} \propto \frac{1}{T^2}$$

so substituting our expression for $\frac{1}{T^2}$ we have

$$\frac{1}{r^3} \propto \frac{v^2}{r^2}$$

Cancelling an r from both sides we obtain an expression with centripetal acceleration on the right hand side.

$$\frac{1}{r^2} \propto \frac{v^2}{r}$$

Finally, we note that the gravitational force must be the source of the centripetal force so that the planet continues to orbit around the sun, so

$$F \propto a_c = \frac{v^2}{r}$$

which means that

$$F \propto \frac{1}{r^2}$$

Newton then used his knowledge of the radius of the earth (the circumference of the earth was measured by Eratosthenes around 230BC), the distance of the moon from earth (about 60 earth radii, first measured by Hipparchus around 190 BC) and its centripetal acceleration towards earth ($a_{moon} = 0.00272\text{ms}^{-2}$, which can be calculated from knowing its period of orbit around earth, about 27.3 days) to test his hypothesis.

If the force of gravity did follow an inverse square law then the ratio of the acceleration of the moon towards earth and the acceleration of an object near the surface of the earth (e.g. an apple!) should be in the same proportion as the inverse of their distances squared.

The ratio of the accelerations of the moon to that of an object falling near the surface of the earth (such as an apple) is

$$\frac{a_{moon}}{a_{apple}} = \frac{0.00272}{9.8} \simeq \frac{1}{3600} = \frac{1}{60^2}$$

The ratio of the inverse distances squared is

$$\frac{d^2_{apple}}{d^2_{moon}} = \frac{1}{60^2}$$

In this way Newton "compared the force requisite to keep the Moon in her orb with the force of gravity at the surface of the earth, and found them answer pretty nearly", providing experimental support for his law of universal gravitation.

Newton's law of Universal Gravitation

Newton had established that the force of gravity is proportional to the inverse square of the distance between two masses.

As larger masses feel a larger force of gravity, the force must be proportional to mass of the object m. By Newton's 3rd law, force that one mass M exerts on mass m must be the same as the force m exerts on M, so the force due to gravity must depend upon *both* masses

$$F \propto \frac{Mm}{r^2}$$

Taking the constant of proportionality to be G (where $G = 6.67 \times 10^{-11}\mathrm{m^3kg^{-1}s^{-2}}$, which we now call the Universal Gravitational constant), Newton obtained

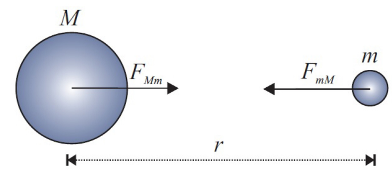

Figure 4: Newton's law of Universal Gravitation

The law of Universal Gravitation

$$F = \frac{GMm}{r^2} \tag{1}$$

Example 1.

(a) Use Newton's law of universal gravitation to calculate the gravitational force which acts on a 1.0kg mass located on the surface of earth. The earth has a mass if $M_E = 6.0 \times 10^{24}$kg and a radius of 6.4×10^6m.

(b) What is the gravitational force which acts on the earth due to the 1.0kg mass?

Orbital motion

Newton's cannonball thought experiment

Newton made the link between projectile motion and orbital motion, in his 1685 book "A treatise of the System of the World", where he describes the following thought experiment:

The effects of centripetal forces.

THAT by means of centripetal forces, the Planets may be retained in certain orbits, we may eaſily underſtand, if we conſider the motions of projectiles. For a ſtone projected is by the preſſure of its own weight forced out of the rectilinear path, which by the projection alone it ſhould have purſued, and made to deſcribe a curve line in the air; and through that crooked way is at laſt brought down to the ground. And the greater the velocity is with which it is projected, the farther it goes before it falls to the Earth. We may therefore ſuppoſe the velocity to be ſo encreaſed, that it would deſcribe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the Earth, till at laſt exceeding the limits of the Earth, it ſhould paſs quite by without touching it.

P. 4, 5, 6. Vol. I.

Let AFB repreſent the ſurface of the Earth, C its center, VD, VE, VF, the curve lines which a body would deſcribe, if projected in an horizontal direction from the top of an high mountain, ſucceſſively with more and more velocity. And, becauſe the celeſtial motions are ſcarcely retarded by the little or no reſiſtance of the ſpaces in which they are performed; to keep up the parity of caſes, let us ſuppoſe either that there is no air about the Earth, or at leaſt that it is endowed with little or no power of reſiſting. And for the ſame reaſon that the body projected with a leſs velocity, deſcribes the leſſer arc VD, and with a greater velocity, the greater arc VE, and augmenting the velocity, it goes farther and farther to F and G; if the velocity was ſtill more and more augmented, it would reach at laſt quite beyond the circumference of the Earth, and return to the mountain from which it was projected.

And ſince the area's, which by this motion it deſcribes by a radius drawn to the center of the Earth, are (by *Prop.* 1. *Book* 1. *Princip. Math.*) proportional to the times in which they are deſcribed; its velocity, when it returns to the mountain, will be no leſs than it was at firſt; and retaining the ſame velocity, it will deſcribe the ſame curve over and over, by the ſame law.

But if we now imagine bodies to be projected in the directions of lines parallel to the horizon from greater heights, as of 5, 10, 100, 1000 or more miles, or rather as many ſemi-diameters of the Earth; thoſe bodies, according to their different velocity, and the different force of gravity in different hights, will deſcribe arcs either concentric with the Earth, or variouſly excentric, and go on revolving through the heavens in thoſe trajectories, juſt as the Planets do in their orbs.

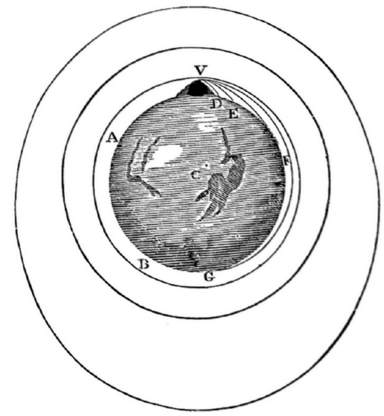

Figure 5: The illustration accoumpanying Newton's cannon thought experiment from "A treatise of the System of the World". An interactive version: `https://physics.weber.edu/schroeder/software/NewtonsCannon.html`.

Newton's ideas were extremely significant in that they suggested that the same physics (the law of universal gravitation) is responsible for both projectile motion and orbital motion. Prior to Newton (and Galileo's) work, Aristotle had proposed that terrestrial and celestial motion was governed by entirely different physical laws.

Physics often progresses through recognising that disparate physical phenomena can be explained by the same underlying theory.

Gravitational field strength

The form of equation 1 suggests that masses exert a force on each other directly through space, implying 'action-at-a-distance".

We can also tease this equation apart, and choose to express this force law in terms of a 'gravitational field' that mass M produces, which points inwards towards M (see figure 6a) and a gravitational force $\vec{F}$ that mass m experiences as it interacts with the gravitational field produced by the first mass.

Gravitational field strength:

$$|\vec{g}| = \frac{GM}{r^2} \tag{2}$$

Gravitational force (weight):

$$\vec{F} = m\vec{g} \tag{3}$$

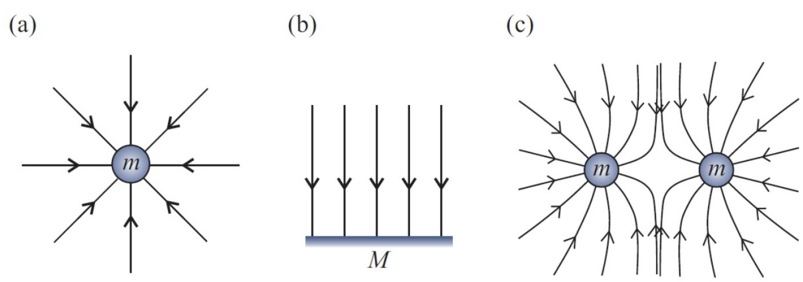

Figure 6: Gravitational field around a point mass (b) Near the surface of the earth (c) between two masses

Although the gravitational field produced by a spherical mass is radially symmetric as shown in figure 6(a), if we "zoom in" close to the surface of a large mass (such as the earth) the gravitational field lines appear almost parallel, as shown in (b). When we looked at projectile motion in Module 5 and made the assumption that the gravitational field was constant, we were making the assumption that gravitational field lines near the surface of the earth are uniform.

Gravitational fields are subject to the principle of superposition. The gravitational field at a point in space is the vector addition of the gravitational fields due to all nearby masses. Figure 6(c) shows the gravitational field around a pair of identical masses.

Example 2.

(a) Calculate the gravitational field strength at the altitude of the International Space Station (ISS) 405km above the surface of the earth.

(b) What is the weight of a 70kg astronaut on the ISS?

(c) Why do astronauts on the ISS appear weightless?

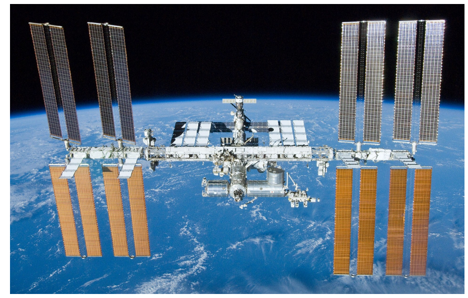

Figure 7: Image: NASA (posted on wikipedia). The International Space Station is featured in this image photographed by an STS-132 crew member on board the Space Shuttle Atlantis after the station and shuttle began their post-undocking relative separation in 2010.

Factors that affect the gravitational field strength

Some factors that affect the gravitational field strength around the surface of the earth:

- The earth is an oblate spheroid (not a perfect sphere)
- The surface is not uniform (there are valleys and mountains)
- The density of the earth is not uniform

Finally, there is a 'pseudo' effect as the apparent weight of an object (the normal force exerted by the ground on the object) varies due to the different centripetal accelerations experienced at different latitudes - near the equator the normal force acting on an object is less than the weight force so that a net force to the center of the earth provides the required centripetal force.

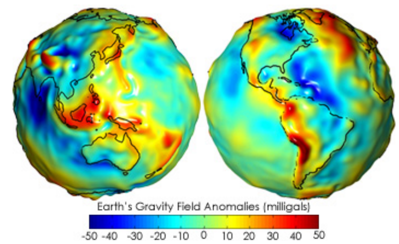

Figure 8: Earth's gravity measured by NASA GRACE mission, showing deviations from the theoretical gravity of an idealized smooth Earth, the so-called Earth ellipsoid. Red shows the areas where gravity is stronger than the smooth, standard value, and blue reveals areas where gravity is weaker. `http://earthobservatory.nasa.gov/Features/GRACE/page3.php`

Example 3.

(a) Calculate the centripetal acceleration of an object located at the equator on earth. (The radius of the earth is 6.4×10^6m)

(b) What is the percentage difference in the normal force exerted by the earth on an object at the equator compared to an object at the poles due to the rotation of the earth (assume the earth is a perfect sphere)?

Orbital velocity

Newton's cannon thought experiment suggests that there is a tangential speed at which an object at a distance r from the center of the earth will move in a circular orbit. To find this speed, we note that the gravitational force between the masses provides the centripetal force required for the orbit.

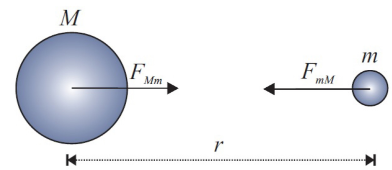

Figure 9: The force due to universal gravitational attraction provides the centripetal force required for two masses to orbit around their center of mass.

$$\Sigma F = ma$$
$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

The mass of the body we are considering the motion of, m, cancels from both sides, so that

Orbital speed (more usually called 'orbital velocity')

$$v = \sqrt{\frac{GM}{r}} \tag{4}$$

Assumptions we are making in our derivation of orbital velocity

In going from $\Sigma F = ma$ to the next line, we are assuming that:

- The only force acting on mass m is due to mass M
- The orbit is circular
- The 'r' in Newton's law of Universal gravitation is the same 'r' in the expression for centripetal acceleration. That is, that the radius of the orbit is the same as the distance between the centers of the masses.

The last assumption is strictly true only if the centre of masses of the two masses is located at the center of the larger mass M, but is an excellent approximation if $M >> m$, as is the case for the planets orbiting the sun, or satellites orbiting earth.

In physics we use simplifying approximations whenever these have negligible impact on our results - but it is important to know the limits of applicability of the formula you are using.

Example 4. Calculate the required orbital velocity for the International Space Station, if it is to maintain a circular orbit with altitude 405km above the surface of the earth.

Kepler's 3rd law revisited

In our previous statement of Kepler's 3rd law we said that $r^3 \propto T^2$, using the expression for orbital velocity we have derived from Newton's law of universal gravitation we can determine the constant of proportionality.

The orbital velocity is equal to the circumference of the orbit ($2\pi r$) divided by the period of its orbit, T (i.e. speed is equal to distance travelled over time!):

$$v = \frac{2\pi r}{T} = \sqrt{\frac{GM}{r}}$$

so, rearranging we have

Kepler's 3rd law:

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2} \tag{5}$$

Assumptions we are making in our derivation of Kepler's 3rd law

We have used our result for orbital velocity - so we are making the same assumptions as we did in that section.

Kepler's 3rd law does also apply to elliptical orbits where the *'r'* in that case is the semimajor axis of its orbit - but we have not not proved that here.

Example 5. Calculate the period of the International space station in its orbit around earth.

Energy

Gravitational potential energy near the surface of the earth

In year 11 we introduced the idea that work done by a force $\vec{F}$ acting on an object that undergoes a displacement of $\vec{x}$ is equal to

$$W = |\vec{F}||\vec{x}| \cos\theta$$

where θ is the angle between the force and displacement vectors. This equation assumes that the force is constant over the entire displacement.

We also considered the change in gravitational potential energy for a mass that is displaced vertically near the surface of the earth, where the force due to gravity is constant (to an excellent approximation).

In physics, when a (conservative) force such as gravity or the electric field produced by an isolated charge does work on an object such as a mass or a charge, this is associated with a change in potential energy of

$$U = -W$$

To convince yourself that this must be the case, consider a mass undergoing a displacement upwards - the force due to gravity, mg, is in the opposite direction as the vertical component of the displacement h, so the work done by gravity is negative, but if the mass moves further from the earth its potential energy must have *increased*.

In this case the change in gravitational potential energy for a small mass moving near the surface of the earth is

$$\Delta U = mgh \tag{6}$$

where h is the displacement and mg the magnitude of the force due to gravity.

Gravitational potential energy

If the displacement of the mass is so large that the force of gravity is no longer approximately constant over the displacement, then we can no longer use equation 6. Instead, we must use the more general equation for the gravitational potential energy associated with two masses, M and m which are separated by a distance r

Gravitational potential energy:

$$U = -\frac{GMm}{r} \tag{7}$$

The maximum value this potential energy takes is zero, when the masses are infinitely far apart ($U \to 0$ as $r \to \infty$). As the masses become closer, the potential energy decreases to increasingly negative values.

Derivation (Requires Calculus)

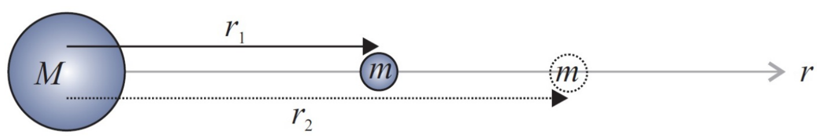

Figure 10: An object of mass m is moved from a distance of r_1 to a distance r_2 from another mass M.

To derive equation 7 we need to calculate the total work done by the gravitational force in a situation such as that shown in figure 10. This requires that we integrate:

$$W = \int_{r_1}^{r_2} |\vec{F}||d\vec{r}| \cos\theta = \int_{r_1}^{r_2} -\frac{GMm}{r^2} dr = GMm \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

The change in gravitational energy if a mass is moved from r_1 to r_2 is therefore:

$$\Delta U = U_2 - U_1 = -W = -GMm \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$$

where r_1 and r_2 are the initial and final separations of the masses, respectively.

Choosing a zero for gravitational potential energy

The work done by the field allows us to calculate *changes* in gravitational energy. To define an actual value for the gravitational potential energy associated with a particular separation of masses, we need to pick an arbitrary zero.

We choose to make the gravitational potential energy for *infinite separation* zero, so the energy of a mass does not depend on objects infinitely far from it.

Substituting $r_1 = \infty$ and $r_2 = r$ in our equation for changes in potential energy, we can define the **gravitational potential energy** of two masses M and m separated by a distance r

$$U = -\frac{GMm}{r}$$

Video from the European space agency showing the launch of the Soyuz: `https://www.youtube.com/watch?v=AVvgpKt5uCA`

Figure 11: The Soyuz.

Example 6.

(a) Calculate the gravitational potential energy of a Soyuz capsule when it is on the launchpad in Baikonur in Kazakhstan before launch. The Soyuz capsule has a mass of 7.1×10^3kg.

(b) Calculate the gravitational potential energy of a Soyuz capsule when it is docked at the international space station.

(c) Calculate the change in gravitational potential energy for the Soyuz capsule as it moves from the surface of the earth to the ISS.

Escape speed (escape velocity)

If you throw an object such as a ball up into the air, it will travel some distance upwards, then slow down and stop, and then fall back down. If you throw the ball faster, it will travel higher before falling back.

The speed that you would need to throw an object with so that it *never* came back is called *escape speed* or *escape velocity* and corresponds to the case where the object would have just enough energy to move infinitely far away before it stopped. In this case (and assuming there are no dissipative forces such as drag) the initial kinetic energy is equal and opposite to the initial gravitational potential energy.

$$U + K = -\frac{GMm}{r} + \frac{1}{2}mv_{esc}^2 = 0$$

so that

Escape velocity:

$$v_{esc} = \sqrt{\frac{2GM}{r}} \tag{8}$$

The escape velocity of an object m does *not* depend on its mass or the direction that the object is fired away from M.

Example 7.

(a) Calculate the escape velocity of an object from earth (in reality, the atmosphere of earth would exert significant drag force on a projectile travelling this fast.)

(b) Calculate the escape velocity of an object from Ceres, a dwarf planet that is the largest object in the asteroid belt in the solar system, which has a mass of $m = 9.4 \times 10^{20}$kg and a radius of 473km.

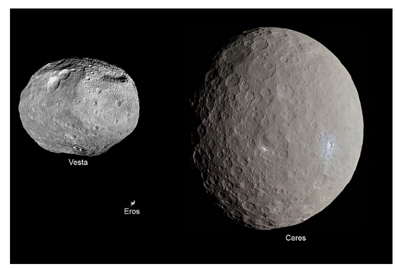

Figure 12: Size comparison of Ceres, Vesta and Eros. Image credit: NASA.

Total (mechanical) energy

The total energy of an object in orbit consists of the sum of its kinetic and gravitational potential energies.

For an object in a circular orbit with a radius r, the orbital velocity is $v_o = \sqrt{\frac{GM}{r}}$, so its kinetic energy is

$$K = \frac{1}{2}mv_o^2 = \frac{GMm}{2r}$$

Its gravitational potential energy is

$$U = -\frac{GMm}{r}$$

so its total mechanical energy is given by

$$U + K = -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r} \tag{9}$$

Example 8.

(a) Calculate the total mechanical energy of the Soyuz capsule just before it docks with the ISS.

(b) Is this the amount of energy we must supply to put the Soyuz in orbit at that altitude?

Satellite orbits around earth

An excellent resource from NASA on this section: `https://earthobservatory.nasa.gov/Features/OrbitsCatalog`.

Figure 13: Satellite orbits around earth. High earth orbits begin at an altitude of about one-tenth of the distance to the moon. Image credit: NASA

Low earth orbits (LEOs)

Altitudes between $180 - 2000$km - just above the atmosphere and below the inner Van Allen radiation belt (which extends down to 1000km above the surface of the earth).

Advantages:

- Smaller amount of energy required to insert satellites into lower orbits
- Close enough to monitor and image earth's surface
- Minimal signal delay when communicating with these satellites
- Polar orbiting low earth satellite can image the polar regions (not imaged well by geostationary satellites).

Disadvantages:

- Suffer from orbital decay
- Satellite dishes must move to track these satellites (period of 90min)

Uses:

- Communications requiring low signal latency (such as voice communications).
- High resolution reconnaissance (spy) satellites

Figure 14: An example of an image from a low earth satellite. The VIIRS instrument on the joint NASA/NOAA Suomi NPP satellite observed Hurricane Florence as it developed in the Atlantic Ocean and made landfall in North Carolina on Sept. 14, 2018. Credit: NASA Worldview. Animated gif: `https://blogs.nasa.gov/disaster-response/wp-content/uploads/sites/275/2018/09/FlorenceNPPsept16.gif`

- High resolution imaging across the EM spectrum from the UV to the visible, infrared and microwave wavlengths that allows monitoring of an array of factors such as soil moisture, sea ice, land ice, snow cover, aerosol levels in the atmosphere, vegetation cover on land, phytoplankton in the oceans and sea and land temperatures, amount of energy absrobed and reflected from the earth's surface [1].

[1] https://aqua.nasa.gov/ and https://ceres.larc.nasa.gov/

Orbital decay

Orbital decay occurs for satellites in low earth orbits due to drag that occurs as a result of collisions between the satellite at molecules in the upper reaches of the atmosphere.

The drag force reduces the satellite's tangential velocity below the required orbital velocity to maintain a stable circular orbit, which causes it to lose altitude. It is important to note that as the satellite loses altitude the potential energy lost is converted to kinetic energy and so it actually *gains* speed as it falls to a lower orbit.

- Some satellites have thrusters to regularly 'lift' the satellite back to the correct altitude
- Becomes more significant during periods of increased solar activity when the atmosphere heats and expands to higher altitudes.

Eccentricity and Orbital inclination

Orbits can also be classified according to their eccentricity (how elliptical the orbit is) and their orbital inclination.

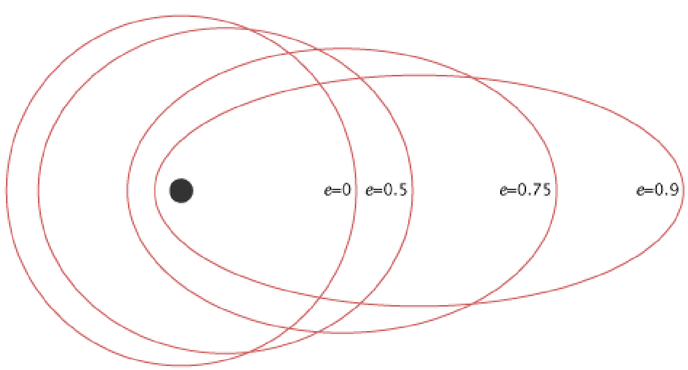

Figure 15: The eccentricity (e) of an orbit indicates the deviation of the orbit from a perfect circle. A circular orbit has an eccentricity of 0, while a highly eccentric orbit is closer to (but always less than) 1. A satellite in an eccentric orbit moves around one of the ellipse's focal points, not the center. Image credit: NASA.

The altitude, eccentricity of the orbit and the inclination of the orbit to the earth's axis are chosen to achieve particular purposes. For low earth orbits, the low altitude allowed the satellites an excellent view of the surface of the earth.

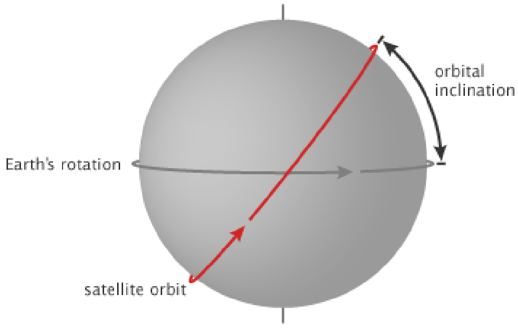

Figure 16: Orbital inclination is the angle between the plane of an orbit and the equator. An orbital inclination of 0° is directly above the equator, 90° crosses right above the pole, and 180° orbits above the equator in the opposite direction of Earth's spin. Image credit: NASA.

Medium earth orbits (MEOs)

These orbits lie between 2000km and 35000km above the surface of the earth (GPS satellites are in MEOs).

While LEOs are highly circular (so they do not collide with earth!) MEOs can be highly eccentric and have large orbital inclinations.

Advantages:

- Highly elliptical orbits such as the "Molniya" orbits commonly used by Russian satellites allow the satellite to spend most time over northern latitudes

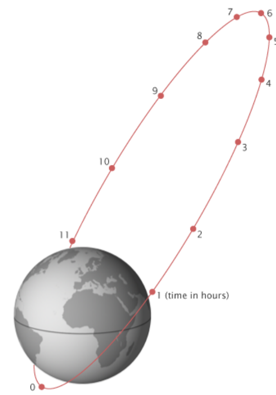

Figure 17: The Molniya orbit combines high inclination (63.4°) with high eccentricity (0.722) to maximize viewing time over high latitudes. Each orbit lasts 12 hours, so the slow, high-altitude portion of the orbit repeats over the same location every day and night. Russian communications satellites and the Sirius radio satellites currently use this type of orbit. Image credit: NASA.

Disadvantages:

- the satellite crosses through the Van Allen radiation belt during its orbit
- the satellite does not remain stationary in the sky so it is necessary for the ground communication station to have dishes that can move to track the satellite across the sky

High earth orbits (HEOs) and Geostationary orbits

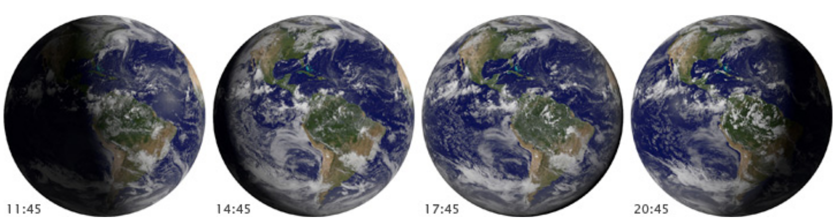

Figure 18: Satellites in geostationary orbit rotate with the Earth directly above the equator, continuously staying above the same spot. This position allows satellites to observe weather and other phenomena that vary on short timescales. Image credit: NASA.

High earth orbits are characterised as those with an altitude above 35000km.

HEOs are "Geosychronous" if at an altitude where the period of the satellite is the same as the earth's rotation, and "Geostationary" if geosynchronous and orbiting around the equator, so that they remain in a constant position in the sky, as viewed from earth.

Advantages:

- Satellites in Geostationary orbits remain located at a fixed position in the sky (satellite dishes do not need to track them). This allows small, cheap satellite dishes to be used to receive signals from these satellites.

Disadvantages:

- Takes more energy to insert satellites into a HEO
- There is a significant signal delay in communication with satellites in HEO (about 240*ms*), which means that there is noticeable lag if geostationary satellites are used for relaying voice communications. These satellites are low in the sky and subject to increased atmospheric refraction for users located in the far northern hemisphere (other orbits, such as Molniya, are used instead).
- Imaging of the Earth from geostationary satellites has a low spatial resolution (of the order of a kilometer[2]).

[2] http://www.bom.gov.au/australia/satellite/about_satellites.shtml

Uses:

- Geostationary satellites are used for telecommunications, particularly applications where low signal latency (minimal delay) is not essential, for example entertainment streaming services.
- Networks of geostationary satellites are used for meteorological monitoring for applications where continuous monitoring with a wide field of view is required, in the infrared and visible wavelengths. This includes application such as weather forecasting, measuring temperatures, cloud and vegetation coverage, tracking cyclones or ash clouds from volcanic events, and oceanography.

Orbital manoeuvres

Very commonly, it is necessary to manoever a spacecraft from an orbit around earth to another destination such as a higher altitude orbit, or an inner or outer planet. In this case an important consideration is minimising the weight of propellant required for the mission.

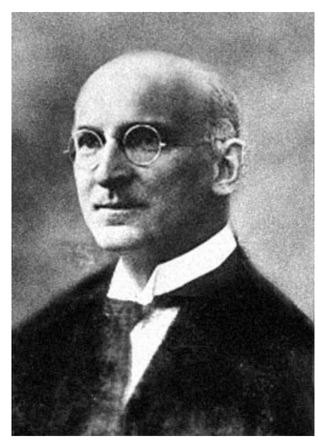

Figure 19: Walter Hohmann. Image credit: Smithsonian Institution.

Hohmann transfer orbits

The most energy efficient way to transfer a spacecraft from one orbit to another was calculated by Walter Hohmann, a German engineer, and published in 1925. If a spacecraft is to be sent to a higher orbit, then it undergoes an engine burn which places it into an elliptical orbit which intersects with the higher orbit at apoapsis (the point where it is travelling slowest and is furtherest from the body it is orbiting). It then undergoes another engine burn to speed it up to the correct orbital velocity for the higher orbit (see figure 20).

If the spacecraft is to be transferred to a lower orbit, for example an orbit closer to the sun than earth's orbit, then the first engine burn is made in a direction *opposite* to the motion of the earth around the sun, to slow the spacecraft down so it no longer had sufficient orbital velocity to maintain a circular orbit around the sun. The spacecraft "falls" inwwards in an elliptical orbit. At the periapsis (the fastest part of the orbit, and closest point to the body it is orbiting around), a second engine burn is performed to slow it down further so that it has the correct orbital velocity for an orbit at that altitude.

More detailed information on Hohmann transfer orbits, as well as the use of gravity assist maneuvers is available from NASA here: `https://solarsystem.nasa.gov/basics/chapter4-1/`.

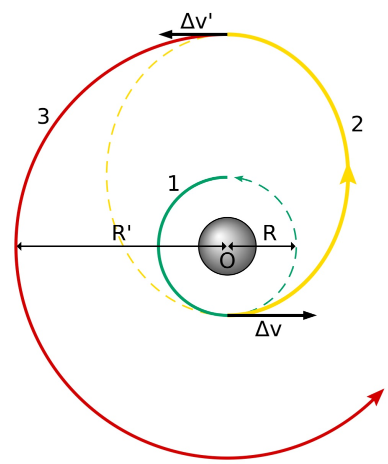

Figure 20: Hohmann transfer orbit for increasing orbital altitude.

Image credit: By Leafnode - Own work based on image by Hubert Bartkowiak, CC BY-SA 2.5, https://commons.wikimedia.org/w/index.php?curid=18

See `https://www.jpl.nasa.gov/edu/teach/activity/lets-go-to-mars-calculating-launch-windows/` for an animation.

Answers

Worked Example 1.

a) The force on a 1kg mass due to it's gravitational attraction towards earth (i.e. its weight) is given by

$$\frac{GMm}{r^2} = \frac{6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2} \times 6.0 \times 10^{24}\mathrm{kg} \times 1\mathrm{kg}}{(6.4 \times 10^6\mathrm{m})^2} = 9.8\mathrm{N}$$

(b) The force on earth is the same, 9.8N

Worked Example 2.

(a) The gravitational field strength at the altitude of the ISS is given by

$$g = \frac{GM}{r^2} = \frac{6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2} \times 6.0 \times 10^{24}}{(0.405 + 6.4) \times 10^6\mathrm{m})^2} = 8.64\mathrm{ms}^{-2}$$

(b) The weight of a 70kg astronaut is $mg = 70\mathrm{kg} \times 8.64\mathrm{ms}^{-2} = 605\mathrm{N}$

(c) The astronauts are in free-fall!

Worked Example 3.

(a) Centripetal acceleration of person on the equator.

The tangential velocity of a person on the surface of the earth at the equator is:

$$v = \frac{2\pi r}{T} = \frac{2\pi.4 \times 10^6}{24 \times 3600} = 465\mathrm{ms}^{-1}$$

$$a_c = \frac{v^2}{r} = \frac{465}{26.4 \times 10^6 = 3.4 \times 10^{-2}\mathrm{ms}^{-2}}$$

(b) Percentage difference in normal force

$$\Sigma F = ma$$

$$N - mg = -m\frac{v^2}{r}$$

(the centripetal force is negative as it is directed downwards)

$$N = m(g - a_c) = m(9.8 - 0.034)$$

$$\frac{N_{eq}}{N_{pole}} = \frac{9.8 - 0.034}{9.8} = 0.997$$

Worked Example 4.

$$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67^{-11} \times 6.0 \times 10^{24}}{(6.4 + 0.405) \times 10^6}} = 7.7 \times 10^3 \mathrm{ms}^{-1}$$

Worked Example 5.
Period of ISS

$$T = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(6.4 + 0.405) \times 10^6)^3}{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}} = 5.6 \times 10^3 \mathrm{s} = 1.5 \mathrm{hr}$$

Worked Example 6.
(a) GPE of Soyuz on the ground

$$U_{ground} = -\frac{GMm}{r} = -\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 7.1 \times 10^3}{6.4 \times 10^6} = -4.44 \times 10^{11} \mathrm{J}$$

(b) GPE of Soyuz docked

$$U_{docked} = -\frac{GMm}{r} = -\frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 7.1 \times 10^3}{(6.4 + 0.405) \times 10^6} = -4.18 \times 10^{11} \mathrm{J}$$

(c) Change in GPE

$$\Delta U = U_f - U_i = U_{docked} - U_{ground} = 2.6 \times 10^{11} J$$

Worked Example 7.
(a) Escape velocity of object from earth.

$$v_{esc} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{6.4 \times 10^6}} = 1.1 \times 10^5 \mathrm{ms}^{-1} = 11 \mathrm{km/s}$$

(b) Escape velocity from Ceres.

$$v_{esc} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 9.4 \times 10^{20}}{473 \times 10^3}} = 515 \mathrm{ms}^{-1}$$

Worked Example 8.
(a) Total mechanical energy of Soyuz just before docking.

$$U_{tot} = -\frac{GMm}{2r} = -2.1 \times 10^{11} \mathrm{J}$$

(b) Is this the energy we must supply to put the Soyuz in orbit at that altitude?

Extra energy is required due to the work done against the drag force exerted by the atmosphere during ascent. The correct calculation would determine the difference between the initial gravitational potential energy and initial kinetic energy the saletellite has before launch (KE due to the rotation of the earth and GPE due to its initial distance from the center of the earth).