2014 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

1. B	3. B	5. D	7. A	9. B
2. C	4. B	6. C	8. B	10. D

Working/Justification

Question 1

First find the x-coordinate of the point of intersection.

$$x^2 - (x-2k)^2 = 0$$

$$(x - x + 2k)(x + x - 2k) = 0$$

$$x = k$$

By symmetry it can be observed that the gradient of $y = (x-2k)^2$ at $x = k$ is $-2k$ or it can be observed that

$$\frac{d}{dx}(x^2) = 2x$$

$$= 2k \quad \text{at the intersection point}$$

$$\frac{d}{dx}(x-2k)^2 = 2(x-2k)$$

$$= -2k \quad \text{at the intersection point}$$

The angle between the two tangent lines is 45 degrees hence

$$\tan 45^\circ = \left|\frac{2k-(-2k)}{1+(2k)(-2k)}\right|$$

$$|4k| = |1-4k^2| \quad \text{but } k > 0$$

$$\Rightarrow |1-4k^2| = 4k$$

This leads to solving $1 - 4k^2 = 4k$ and $1 - 4k^2 = -4k$. Either solving directly or plotting $y = 1 - 4x^2$ against $y = 4k$ and $y = -4k$ leads to two possible solutions under the condition of $k > 0$

Question 2

Note that the expression can be written in the form $R\cos(\theta+\alpha)+c=0$ where $R=\sqrt{a^2+b^2}$. This leads to

$$\cos(\theta+\alpha) = -\frac{c}{R}$$

For a solution to exist it must have the condition that

$$\left|\frac{c}{R}\right| \leq 1$$

$$a^2+b^2 \geq c^2$$

$$\Rightarrow \Delta \geq 0$$

When $\Delta > 0$ there are exactly two solutions as there will always exist two solutions for $\alpha \leq \theta+\alpha \leq 2\pi+\alpha$.

When $\Delta < 0$ there are no real solutions as this implies $-\frac{c}{R} > 1$.

When $\Delta = 0$, the equation becomes $\cos(\theta+\alpha) = -1$ or $\cos(\theta+\alpha) = 1$. The latter in particular has potentially two solutions if $b=-c$ and $a=0$ where $\theta = 0, 2\pi$ satisfy the equation and $\Delta = 0$ hence (C) is not always true.

Question 3

First note that $\alpha+\beta+\gamma=0$

$$\begin{aligned}\frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}+\frac{\gamma}{\beta}+\frac{\beta}{\alpha} &= \frac{\beta+\gamma}{\alpha}+\frac{\alpha+\gamma}{\beta}+\frac{\alpha+\beta}{\gamma} \\ &= -\frac{\alpha}{\alpha}-\frac{\beta}{\beta}-\frac{\gamma}{\gamma} \\ &= -3\end{aligned}$$

Question 4

Recall that the velocity displacement equation $v^2 = n^2(a^2-(x-b)^2)$ for amplitude a, period n and centre b. The maximum velocity is an.

If the maximum velocity of particle X is twice the maximum velocity of particle Y then $a_X n_X = 2a_Y n_Y$

If the amplitudes are equal then $n_X = 2n_Y \Rightarrow \frac{2\pi}{n_X} = \frac{1}{2} \times \frac{2\pi}{n_Y}$, hence (I) is true.

If the periods are equal then $a_X = 2a_Y$ hence (IV) is true.

Question 5

$$\sin 2x = \frac{1}{2}$$

$$2x = k\pi + (-1)^k \frac{\pi}{6} \quad \text{for some integer } k$$

$$x = \frac{k\pi}{2} + (-1)^k \frac{\pi}{12}$$

However, since k is any arbitrary integer, so is $-k$ so

$$x = -\frac{k\pi}{2} + (-1)^{-k} \frac{\pi}{12}$$

$$= (-1)^k \frac{\pi}{12} - \frac{k\pi}{2}$$

is also another way to express the general solution. (III) is clearly not correct which can be easily shown by taking $k = 0$ and substituting it into the original equation.

Question 6

When $P(x)$ is divided by $A(x)$ then if the 'remainder' has a larger degree than $A(x)$ then it can be reduced by further division of $A(x)$. Once the degree is less than the degree of $A(x)$ then no further divsion can occur which gives the actual remainder term $R(x)$.

Question 7

Consider the complementary event where no one shares the same birthday. Suppose that there are k people in the room. The first person has 365 days to 'choose' from, after that the second person has 364 days to 'choose' from and so on. The k-th person has $365 - (k - 1)$ days to 'choose' from. Thus, the probability of at least two people sharing the same birthday is

$$1 - \frac{365 \times 364 \times 363 \times .... \times (365 - k + 1)}{365^k}$$

We require this probability to greater than 50% which implies that we require the smallest value of k such that

$$\frac{365 \times 364 \times 363 \times .... \times (365 - k + 1)}{365^k} < 0.5$$

By trial and error of the options we have $k = 23$

Question 8

Search for the value of k such that the solution to the equation $|x-a|+|x-b|=k$ is all $a \leq x \leq b$. Consider the graph of $y=|x-a|+|x-b|$ note that when $a \leq x \leq b$ then $y=(x-a)-(x-b)=b-a$ thus, the only value of k where the line $y=k$ intersects with the graph of $y=|x-a|+|x-b|$ is when $k=b-a$.

Question 9

Note that

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$$

In other words for some point (a, b) and $y=f(x)$ we have

$$f'^{-1}(b)=\frac{1}{f'(a)}$$

From this if $f'(a)=0$ then $f'^{-1}(b)$ is undefined, which shows that (C) is true.

Similarly for $y=f^{-1}(x)$, we have

$$f'(b)=\frac{1}{f'^{-1}(a)}$$

From this if $f'^{-1}(a)$ is zero then $f'(b)$ is undefined, which shows that (D) is true.

(A) is true because the range of $f^{-1}(x)$ is always within the full domain or a subset of the domain of $f(x)$.

However, (B) is not necessarily true because the range of $f(x)$ is not necessarily within the domain of $f^{-1}(x)$, particularly if $f^{-1}(x)$ was found by placing restrictions on the range of $f(x)$.

As an illustrative example, consider $f(x)=x^2$ and a possible inverse $f^{-1}(x)=\sqrt{x}$ (alternatively, another possible inverse $f^{-1}(x)=-\sqrt{x}$). We get

$$\begin{aligned} f^{-1}(f(x)) &= \sqrt{x^2} \\ &= |x| \quad \text{which is not always equal to } x \end{aligned}$$

Question 10

The mechanism behind Newton's method of approximating roots is by solving the x-intercept of the tangent to the point on the curve based on the initial approximation to get the next iteration. Note that by 'approach' we mean by come closer to some extent.

(A) can be true because any point chosen on the side of the curve where $x < a$ sends the tangent to have x-intercepts which are always less than a so the iterations can approach root r_1

(B) can be true because if the tangent is taken at the stationary point which lies approximately on the y-axis it will not approach either r_1 or r_2

(C) can be true because a tangent could taken such there will have an x-intercept which is less than a and hence further iterations come closer to r_1

(D) is never true (i.e. always false) because a tangent taken there will have an x-intercept approach r_2 and never approach r_1

Section II

Question 11

(a)

$$\frac{e^x}{1-e^x} \geq e^x - 1$$

$$\frac{e^x - (1-e^x)(e^x-1)}{1-e^x} \geq 0$$

$$\frac{e^x + e^{2x} - 2e^x + 1}{1-e^x} \geq 0$$

$$(1-e^x)(e^{2x} - e^x + 1) \geq 0$$

However, note that $e^{2x} - e^x + 1$ is a quadratic equation in e^x with disciminant of -3 thus this quadratic is positive definite so $e^{2x} - e^x + 1 \geq 0$ for all real x. This means that the solution comes from solving

$$1 - e^x \geq 0$$

$$e^x \leq 1$$

$$x \leq 0$$

But $e^x \neq 1$ from the denominator of the original inequality which means $x \neq 0$ thus the final solution is $x < 0$.

(b)

$$LHS = \lim_{x\to 0} \frac{\sin ax}{\sin bx}$$

$$= \lim_{x\to 0} \frac{\sin ax}{ax} \times \frac{bx}{\sin bx} \times \frac{a}{b}$$

$$= \frac{a}{b} \times \lim_{x\to 0} \frac{\sin ax}{ax} \times \lim_{x\to 0} \frac{bx}{\sin bx}$$

$$= \frac{a}{b} \times 1 \times 1$$

$$= \frac{a}{b}$$

$$= RHS$$

(c) Let the ratio that P divides the interval AB be $r:1$. Note that this approach minimises the number of variables compared to using the ratio $m:n$.

$$x_0 = \frac{x_1 + rx_2}{r+1}$$

$$y_0 = \frac{y_1 + ry_2}{r+1}$$

$$\frac{x_0}{y_0} = \frac{x_1 + rx_2}{y_1 + ry_2}$$

$$x_0y_1 + rx_0y_2 = x_1y_0 + rx_2y_0$$

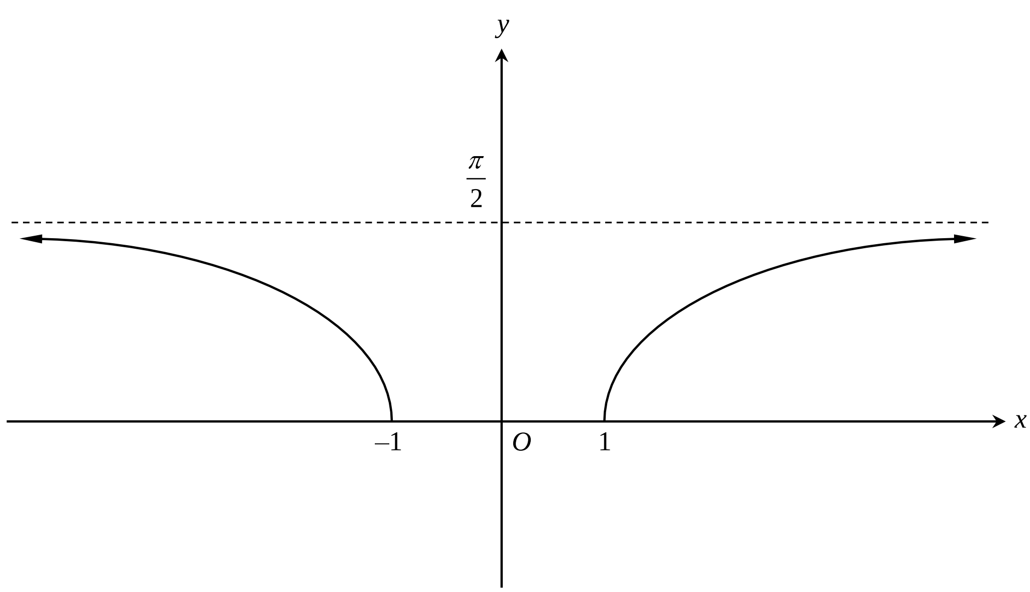
$$r = \frac{x_1y_0 - x_0y_1}{x_0y_2 - x_2y_0} \quad \text{but for external division } r < 0$$

$$\frac{x_1y_0 - x_0y_1}{x_0y_2 - x_2y_0} < 0$$

$$(x_1y_0 - x_0y_1)(x_0y_2 - x_2y_0) < 0$$

$$\therefore (x_0y_1 - x_1y_0)(x_0y_2 - x_2y_0) > 0$$

(d) Note that the function is even and defined for $\frac{1}{x^2} \leq 1$ or equivalently $x \geq 1$ and $x \leq -1$. When $x \to \pm\infty$ then $f(x) \to \frac{\pi}{2}$



(e) (i) $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2\cos^2\theta} \times a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta \quad \text{noting that } a > 0, \cos\theta \geq 0 \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta \quad \text{since } \cos 2\theta = 2\cos^2\theta - 1$$

$$= \frac{a^2}{2} \left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{a^2}{2} [\theta + \sin\theta\cos\theta] + c \quad \text{since } \sin 2\theta = 2\sin\theta\cos\theta$$

But $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$ noting that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Also, $x^2 = a^2(1 - \cos^2\theta)$ so for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ we have $\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$ as $\cos\theta \geq 0$ and $a > 0$. Substituting this all back in:

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2 - x^2}}{a^2}\right] + c$$

(ii) Note that $x^2 = -(a^2 - x^2) + a^2$ so

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\, dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= \int \frac{a^2}{\sqrt{a^2 - x^2}}\, dx - \int \sqrt{a^2 - x^2}\, dx$$

$$= a^2\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2 - x^2}}{a^2}\right] + c \quad \text{from (i)}$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{x\sqrt{a^2 - x^2}}{2} + c$$

Question 12

(a) First we note that

$$\sin^4 x + \cos^4 x = \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x \qquad \text{but } \sin 2x = 2\sin x\cos x$$

$$= (\sin^2 x + \cos^2 x)^2 - \frac{1}{2}\sin^2 2x \qquad \text{but } \sin^2 x + \cos^2 x = 1$$

$$= 1 - \frac{1}{2}\sin^2 2x \qquad \text{but } \cos 4x = 1 - 2\sin^2 2x$$

$$= 1 - \frac{1}{4}(1 - \cos 4x)$$

$$= \frac{3}{4} + \frac{1}{4}\cos 4x$$

Hence

$$V = \pi\int_0^{\frac{\pi}{2}}(\sin^4 x + \cos^4 x)\,dx$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}(3 + \cos 4x)\,dx$$

$$= \frac{\pi}{4}\left[3x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi^2}{8} \qquad \text{cubic units}$$

(b) (i)

$$RHS = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}$$

$$= \frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) + 2\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)}{2\cos^2\left(\frac{\alpha}{2}\right) - 1 + 1 - 2\sin^2\left(\frac{\beta}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)}{\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\beta}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \tan\left(\frac{\alpha+\beta}{2}\right)$$

$$= LHS$$

Alternatively,

$$RHS = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta}$$

$$= \frac{\sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)}$$

$$= \frac{\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \tan\left(\frac{\alpha+\beta}{2}\right)$$

$$= LHS$$

Similarly, replacing β with $-\beta$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = \frac{\sin\alpha + \sin(-\beta)}{\cos\alpha + \cos(-\beta)}$$

$$= \frac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta}$$

since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$

(ii) From the sine rule $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} \Rightarrow a = \dfrac{b\sin\alpha}{\sin\beta}$

$$LHS = \frac{a+b}{a-b}$$

$$= \frac{\frac{b\sin\alpha}{\sin\beta}+b}{\frac{b\sin\alpha}{\sin\beta}-b}$$

$$= \frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

$$= \frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} \times \frac{\cos\alpha+\cos\beta}{\sin\alpha-\sin\beta}$$

$$= \frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)} \quad \text{from part (i)}$$

$$= RHS$$

(c) (i)

$$\ddot{x} = -2k(1-kx)$$

$$\frac{1}{2}\frac{d\dot{x}^2}{dx} = -2k(1-kx)$$

$$\frac{\dot{x}^2}{2} = -2k\int(1-kx)\,dx$$

$$= (1-kx)^2 + c$$

When $x = 0$ then $\dot{x} = \sqrt{2}$ hence $c = 0$

$$\dot{x}^2 = 2(1-kx)^2$$

$$\dot{x} = \sqrt{2}(1-kx) \quad \text{note we take the positive root to satisfy } x = 0, \dot{x} = \sqrt{2}$$

$$\frac{dx}{dt} = \sqrt{2}(1 - kx)$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{2}} \times \frac{1}{1 - kx}$$

$$t = \frac{1}{\sqrt{2}} \int \frac{dx}{1 - kx}$$

$$= -\frac{1}{k\sqrt{2}} \ln(1 - kx) + c$$

When $t = 0$ then $x = 0$ hence $c = 0$

$$t = -\frac{1}{k\sqrt{2}} \ln(1 - kx)$$

$$\ln(1 - kx) = -kt\sqrt{2}$$

$$1 - kx = e^{-kt\sqrt{2}}$$

$$x = \frac{1 - e^{-kt\sqrt{2}}}{k}$$

(ii) When $t \to \infty$ then $e^{-kt\sqrt{2}} \to 0$ noting that $k > 0$. Hence $x \to \dfrac{1}{k}$

(d) (i) The equation of PQ is

$$y - ap^2 = \frac{ap^2 - aq^2}{2ap - 2aq}(x - 2ap) \quad \text{but since } PQ \text{ is a focal chord substitute } (0, a)$$

$$a - ap^2 = -ap(p + q)$$

$$pq = -1$$

The gradient of the tangent at P is $\dfrac{dy}{dx} = \dfrac{dy}{dp}\dfrac{dp}{dx} = p$ and similarly the gradient of the tangent at Q is q.

From the above result, this suggests that $PT \perp QT$. Let A be the area of the ΔPQT

$$A = \frac{1}{2} \times PT \times QT$$

$$PT = \sqrt{(a(p+q) - 2ap)^2 + (apq - ap^2)^2}$$

$$= \sqrt{a^2(p-q)^2 + a^2(pq - p^2)^2} \quad \text{but } pq = -1$$

$$= a\sqrt{\left(p + \frac{1}{p}\right)^2 + (p^2+1)^2} \quad \text{noting that } a > 0$$

$$= a\sqrt{\left(\frac{p^2+1}{p}\right)^2 + (p^2+1)^2}$$

$$= \frac{a}{|p|}\sqrt{(p^2+1)^2 + p^2(p^2+1)^2}$$

$$= \frac{a}{|p|}\sqrt{(p^2+1)^3}$$

$$\text{similarly} \quad QT = \frac{a}{|q|}\sqrt{(q^2+1)^3}$$

$$= a|p|\sqrt{\left(\frac{1}{p^2}+1\right)^3}$$

$$\Rightarrow A = \frac{a^2}{2}\sqrt{(p^2+1)^3\left(\frac{1}{p^2}+1\right)^3}$$

$$= \frac{a^2}{2}\sqrt{\left(p^2+2+\frac{1}{p^2}\right)^3}$$

$$= \frac{a^2}{2}\sqrt{\left(p+\frac{1}{p}\right)^6}$$

$$= \frac{a^2}{2}\left|p+\frac{1}{p}\right|^3$$

Alternatively, find the distance PQ and find the perpendicular distance from T to PQ (let this be d)

$$PQ = \sqrt{(2ap-2aq)^2+(ap^2-aq^2)^2}$$

$$= \sqrt{4a^2(p-q)^2+a^2(p^2-q^2)^2}$$

$$= a\sqrt{4(p-q)^2+(p-q)^2(p+q)^2} \quad \text{noting that } a>0$$

$$= a|p-q|\sqrt{(p+q)^2+4}$$

$$= a|p-q|\sqrt{p^2+2pq+q^2+4} \quad \text{but } pq=-1$$

$$= a\left|p+\frac{1}{p}\right|\sqrt{p^2+2+\frac{1}{p^2}}$$

$$= a\left|p+\frac{1}{p}\right|\sqrt{\left(p+\frac{1}{p}\right)^2}$$

$$= a\left|p+\frac{1}{p}\right|^2$$

Rearrange the equation of the chord PQ

$$y-ap^2=\frac{p+q}{2}(x-2ap)$$

$$2y-2ap^2=(p+q)x-2ap^2-2apq$$

$$(p+q)x-2y-2apq=0$$

$$d = \frac{|(p+q)(a(p+q)) - 2apq - 2apq|}{\sqrt{(p+q)^2+4}}$$

$$= \frac{|a(p+q)^2 - 4apq|}{\sqrt{(p+q)^2+4}}$$

$$= \frac{a\left|\left(p-\frac{1}{p}\right)^2+4\right|}{\sqrt{\left(p-\frac{1}{p}\right)^2+4}} \quad \text{noting that } a>0 \text{ and } pq=-1$$

$$= a\sqrt{\left(p-\frac{1}{p}\right)^2+4}$$

$$= a\sqrt{p^2+2+\frac{1}{p^2}}$$

$$= a\sqrt{\left(p+\frac{1}{p}\right)^2}$$

$$= a\left|p+\frac{1}{p}\right|$$

Hence

$$A = \frac{1}{2} \times d \times PQ$$

$$= \frac{a^2}{2}\left|p+\frac{1}{p}\right|^3$$

(ii) The x-value of P is given by $x_P = 2ap$ and $\dfrac{dx_P}{dt} = 1$

$$\frac{dp}{dt} = \frac{dp}{dx_P}\frac{dx_P}{dt}$$

$$= \frac{1}{2a}$$

If $p > 0$ then $A = \dfrac{a^2}{2}\left(p + \dfrac{1}{p}\right)^3$ which implies that

$$\frac{dA}{dt} = \frac{dA}{dp} \times \frac{dp}{dt}$$

$$= \frac{1}{2a} \times \frac{3a^2}{2}\left(p + \frac{1}{p}\right)^2\left(1 - \frac{1}{p^2}\right)$$

Noting that $a > 0$ then for $\dfrac{dA}{dt} > 0$ we require

$$1 - \frac{1}{p^2} > 0$$

$$p^2 > 1$$

$$p > 1 \quad \text{since } p > 0$$

If $p < 0$ then $A = -\dfrac{a^2}{2}\left(p + \dfrac{1}{p}\right)^3$ which implies that

$$\frac{dA}{dt} = -\frac{1}{2a} \times \frac{3a^2}{2}\left(p + \frac{1}{p}\right)^2\left(1 - \frac{1}{p^2}\right)$$

Noting that $a > 0$ then for $\dfrac{dA}{dt} > 0$ we require

$$1 - \frac{1}{p^2} < 0$$

$$p^2 < 1$$

$$-1 < p < 0 \quad \text{since } p < 0$$

Therefore the solutions are $p > 1$ or $-1 < p < 0$

Question 13

(a) (i) We need to show that $\dfrac{(k+1)(k+2)(k+3).....(k+n)}{n!}$ is an integer. Note that

$$\frac{(k+1)(k+2)(k+3).....(k+n)}{n!} = \frac{k!(k+1)(k+2)(k+3).....(k+n)}{k!n!}$$

$$= \frac{(k+n)!}{k!(k+n-k)!}$$

$$= \binom{k+n}{k}$$

This is the number of ways of choosing k objects from a set $n+k$ objects which must be an integer.

(ii) <u>When $n = 1$</u>:

$$(m! \times 1) = m!$$

$$= 1 \times (m!)^1 \quad \therefore \text{Statement is true for } n = 1:$$

<u>Assume the statement is true for $n = k$</u>:

$$(mk)! = P \times (m!)^k \quad \text{for some integer } P$$

<u>Required to prove the statement is true for $n = k+1$</u>:

$$(m(k+1))! = Q \times (m!)^{k+1} \quad \text{for some integer } Q$$

$$LHS = (m(k+1))!$$

$$= (mk+m)!$$

$$= (mk)!(mk+1)(mk+2)(mk+3)......(mk+m)$$

$$= P \times (m!)^k(mk+1)(mk+2)(mk+3)......(mk+m) \quad \text{by assumption}$$

$$= P \times (m!)^k \times A(m!) \quad \text{using part (i)}$$

$$= Q \times (m!)^{k+1} \quad \text{where } Q = AP$$

$$= RHS$$

The statement is true for $n = k+1$ if it is true for $n = k$. Since the statement is true for $n = 1$ then by induction it is true for all positive integers n.

(b) (i) The probability of obtaining the particular number is $\dfrac{1}{n}$ so the probability of not obtaining this number is $\left(1 - \dfrac{1}{n}\right)$ in a given roll of the die. Thus, the probability of acquiring the number exactly k times out of m rolls of the die is

$$P(k) = \binom{m}{k}\left(\frac{1}{n}\right)^k\left(1 - \frac{1}{n}\right)^{m-k}$$

$$= \binom{m}{k}\frac{1}{n^k} \times \frac{(n-1)^{m-k}}{n^{m-k}}$$

$$= \binom{m}{k}\frac{(n-1)^{m-k}}{n^m}$$

(ii) Since all terms of the binomial expansion of $\left(\frac{1}{n} + (1 - \frac{1}{n})\right)^m$ are positive we find the largest value of k which satisfies the inequality below

$$P(k) \geq P(k-1)$$

$$\binom{m}{k}\frac{(n-1)^{m-k}}{n^m} \geq \binom{m}{k-1}\frac{(n-1)^{m-k+1}}{n^m}$$

$$\frac{m!}{k!(m-k)!} \geq \frac{m!}{(k-1)!(m-k+1)!} \times (n-1)$$

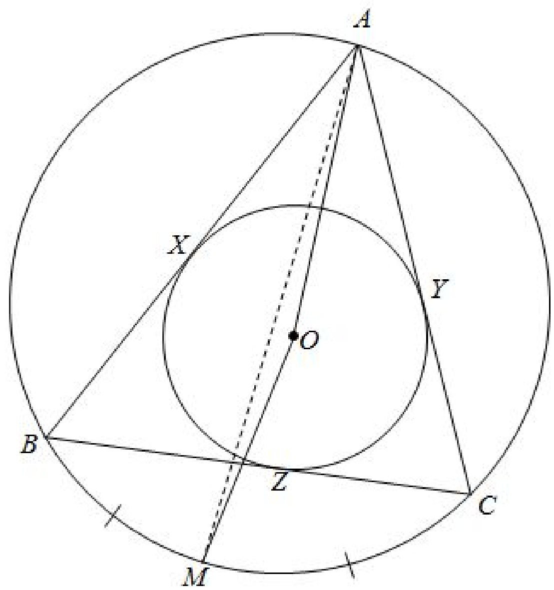
$$m - k + 1 \geq k(n-1)$$

$$k \leq \frac{m+1}{n}$$

Hence the chosen number is most likely to appear approximately $\dfrac{m+1}{n}$ times

(iii) If $m+1$ is divisible by n then $\dfrac{m+1}{n}$ is an integer as is $\dfrac{m+1}{n} - 1 \left(\text{or } \dfrac{m-n+1}{n}\right)$. Note that from part (ii), $k = \dfrac{m+1}{n}$ actually occurs when $P(k) = P(k-1)$ when solving for the largest probability which means the most likely number of times is either k or $k-1$ with equal probability. In other words, the chosen number is most likely to appear exactly $\dfrac{m+1}{n}$ or $\dfrac{m-n+1}{n}$ times.

(c) First we need to prove that AOM is a straight line (note this wasn't specified in the question). Assume that $\angle AOM$ is not a straight line. Let the points X, Y and Z be the points of contact of the tangents to the circles as shown in the diagram below.



$$AX = AY$$

(tangents from an external point)

AO is common

$OX = OY$ (equal radii)

$\therefore \Delta XOA \equiv \Delta YOA \quad (SSS)$

$\Rightarrow \angle XAO = \angle YAO **$

(corresponding angles of congruent triangles)

This suggests that AO bisects $\angle BAC$.

But since arc BM equals arc CM in length then $\angle BAM = \angle CAM$ (equal angles standing on equal arcs)

However, this suggests that AM bisects $\angle BAC$. For both AO and AM to bisect $\angle BAC$ we must have that AM and AO coincide, hence A, O and M are collinear.

From this it can be said that

$\angle MOC = \angle OAC + \angle OCY$ (exterior angle equals sum of opposite interior angles of triangle)

Using a similar argument to prove ** we can say that $\angle OCZ = \angle OCY$. Also

$\angle BAM = \angle BCM$ (angles standing on the same segment)

but as noted previously $\angle BAM = \angle OAC$ (note $\angle OAC$ and $\angle CAM$ are equivalent)

$$\Rightarrow \angle BCM = \angle OAC$$

It can be observed that $\angle OCM = \angle BCM + \angle OCZ$

$$= \angle OAC + \angle OCY$$

$$= \angle MOC$$

$\therefore \Delta OCM$ is isosceles due to two angles being equal

(d) First note that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots.. + \binom{n}{n}x^n$$

Let $x = 1$ and we get the result

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots.. + \binom{n}{n} = 2^n \quad *$$

In the question, first observe that

$$\sum_{k=1}^{n} \binom{k-1}{0} = \binom{0}{0} + \binom{1}{0} + \binom{2}{0} + \ldots.. + \binom{n-1}{0}$$

$$\sum_{k=2}^{n} \binom{k-1}{1} = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \ldots.. + \binom{n-1}{1}$$

$$\sum_{k=3}^{n} \binom{k-1}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots.. + \binom{n-1}{2}$$

....

$$\sum_{k=n-1}^{n} \binom{k-1}{n-2} = \binom{n-2}{n-2} + \binom{n-1}{n-2}$$

$$\sum_{k=n}^{n} \binom{k-1}{n-1} = \binom{n-1}{n-1}$$

Rearranging the sums we get

$$LHS = \sum_{k=1}^{n}\binom{k-1}{0} + \sum_{k=2}^{n}\binom{k-1}{1} + \sum_{k=3}^{n}\binom{k-1}{2} + ... + \sum_{k=n-1}^{n}\binom{k-1}{n-2} + \sum_{k=n}^{n}\binom{k-1}{n-1}$$

$$= \binom{0}{0} + \left[\binom{1}{0} + \binom{1}{1}\right] + \left[\binom{2}{0} + \binom{2}{1} + \binom{2}{2}\right] + ... + \left[\binom{n-1}{0} + \binom{n-1}{1} + ... + \binom{n-1}{n-1}\right]$$

$$= 1 + 2^1 + 2^2 + 2^3 + ... + 2^{n-1} \quad \text{using result above in } *$$

$$= \frac{2^n - 1}{2 - 1}$$

$$= 2^n - 1$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} - 1 \quad \text{using result in } *$$

$$= \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} \quad \text{since } \binom{n}{0} = 1$$

$$= \sum_{k=1}^{n}\binom{n}{k}$$

$$= RHS$$

Question 14

(a) If there are k distinct numbers, each must occur at least once. Consider a set combination of a selection of k numbers to create an n digit password. Construct the problem in terms of inserting 'dividers' ('stars and bars' method) in a combination of numbers where like digits are grouped together. The dividers separate the different groups of the digits.

(insert diagram)

There are $n-1$ 'separating' spaces for each of the dividers available (since the frequency of each digit must be at least one). Since there are k distinct numbers then there $k-1$ dividers to arrange across $n-1$ spaces. Since the dividers are identical there are $\binom{n-1}{k-1}$ possible combinations for a fixed combination of numbers.

However, the k digits can be chosen from 10 digits for which there are $\binom{10}{k}$ combinations.

Hence, the total possible combinations of numbers is $\binom{n-1}{k-1}\binom{10}{k}$

(b) (i) Deriving the equations of motion of the vertical components

$$\ddot{y} = -g$$

$$\dot{y} = -gt + c_1 \quad \text{when } t = T, \dot{y} = 0 \Rightarrow c_1 = gT$$

$$\dot{y} = -g(t-T)$$

$$y = -\frac{g(t-T)^2}{2} + c_2 \text{ when } t = T, y = h \Rightarrow c_2 = h$$

$$y = -\frac{g(t-T)^2}{2} + h$$

Deriving the equations of motion of the horizontal components (noting that the horizontal velocity is constant)

$$x = A\sin(nt) \quad \text{for } t \leq T$$

$$\dot{x} = An\cos(nt)$$

$$= An\cos(nT) \quad \text{at the point of release when particle commences projectile motion when } t = T$$

$$x = Ant\cos(nT) + c_3 \quad \text{for } t \geq T$$

When $t = T$ then $x = x_1 = A\sin(nT)$ hence $c_3 = A\sin(nT) - AnT\cos(nT)$ hence

$$x = An(t - T)\cos(nT) + A\sin(nT)$$

When $t = t_0 + T$ then $x = x_2$

$$x_2 = Ant_0\cos(nT) + A\sin(nT)$$

We wish to choose a time of release T such that x_2 is maximised so first consider

$$\frac{dx_2}{dT} = -An^2t_0\sin(nT) + An\cos(nT)$$

Stationary point occurs when $\dfrac{dx_2}{dT} = 0$

$$An^2t_0\sin(nT) = An\cos(nT)$$

$$\tan(nT) = \frac{1}{nt_0}$$

Note that

$$\frac{d^2x_2}{dT} = -An^3t_0\cos(nT) - An^2\sin(nT) < 0 \quad \text{noting that } 0 \leq nT \leq \frac{\pi}{2} \text{ so } \sin(nT) > 0 \text{ and } \cos(nT) > 0$$

Since the function is continuous with no other turning points in the domain, this confirms that the maximum occurs when $\tan(nT) = \dfrac{1}{nt_0}$

(ii) From part (i), x_2 is maximised when $nt_0 = \dfrac{\cos(nT)}{\sin(nT)}$.

Substitute into our expression for x_2.

$$x_2 = A \times \frac{\cos(nT)}{\sin(nT)} \times \cos(nT) + A\sin(nT)$$

$$= \frac{A}{\sin(nT)}\left(\cos^2(nT) + \sin^2(nT)\right)$$

$$= \frac{A}{\sin(nT)}$$

But since $x_1 = A\sin(nT)$, so $x_1x_2 = A^2$

(c) (i)

$$P = \frac{k}{1+e^{-at}}$$

$$\frac{dP}{dt} = \frac{ake^{-at}}{(1+e^{-at})^2}$$

$$= a \times \frac{k}{1+e^{-at}} \times \frac{1+e^{-at}-1}{1+e^{-at}}$$

$$= aP\left(1 - \frac{1}{1+e^{-at}}\right)$$

$$= aP\left(1 - \frac{P}{k}\right)$$

Hence $P = \dfrac{k}{1+e^{-at}}$ satisfies $\dfrac{dP}{dt} = aP\left(1 - \dfrac{P}{k}\right)$

(ii)

$$N = k - \frac{k}{2}e^{-bt}$$

$$\frac{dN}{dt} = b \times \frac{k}{2}e^{-bt}$$

$$= b(k - N)$$

Hence $N = k - \dfrac{k}{2}e^{-bt}$ satisfies $\dfrac{dN}{dt} = b(k-N)$

(iii) Consider the difference which will be defined as D

$$D = P - N$$

$$= \frac{k}{1+e^{-at}} - k + \frac{k}{2}e^{-bt}$$

$$= \frac{k}{2(1+e^{-at})}\left(2 - 2(1+e^{-at}) + e^{-bt}(1+e^{-at})\right)$$

$$= \frac{ke^{-(a+b)t}}{2(1+e^{-at})}\left(e^{at} - 2e^{bt} + 1\right)$$

Noting that $\dfrac{ke^{-(a+b)t}}{2(1+e^{-at})} \neq 0$ for $k>0$, the aim is prove that if $b<a<2b$ then there exists a value of t such that $D=0$. In other words we need to show that $e^{at}-2e^{bt}+1$ can be zero when $a<b<2b$.

Let $y=e^{at}-2e^{bt}+1$, noting that when $t=0$, $y=0$

$$\frac{dy}{dt} = ae^{at}-2be^{bt}$$

$$= e^{bt}(ae^{(a-b)t}-2b)$$

If $b<a<2b$ then there exists a value of $t>0$ where $\dfrac{dy}{dt}=0$, namely $t=\dfrac{\ln\left(\frac{2b}{a}\right)}{a-b}$ (note that $b<a<2b$ allows $t>0$).

$$\frac{d^2y}{dt^2} = a^2e^{at}-2b^2e^{bt}$$

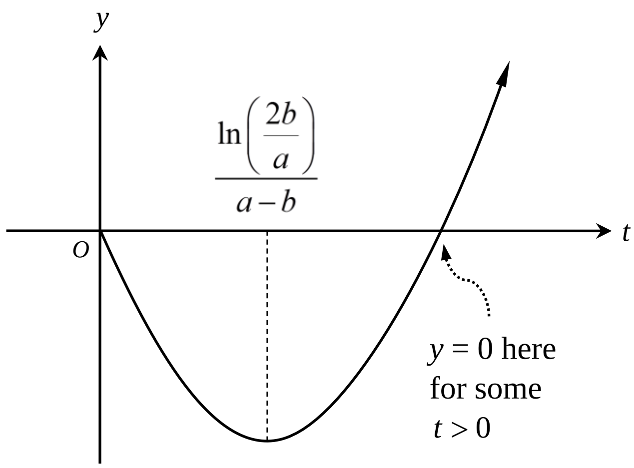
$$= e^{bt}(a^2e^{(a-b)t}-2b^2) \quad \text{substitute } t=\frac{\ln\left(\frac{2b}{a}\right)}{a-b}$$

$$= e^{\frac{b\ln\left(\frac{2b}{a}\right)}{a-b}}\left(a^2\times\frac{2b}{a}-2b^2\right)$$

$$= 2be^{\frac{b\ln\left(\frac{2b}{a}\right)}{a-b}}(a-b)>0 \quad \text{as } a>b>0$$

This shows that $t=\dfrac{\ln\left(\frac{2b}{a}\right)}{a-b}$ gives a minimum value of y (noting that it is the only stationary point of y).

Since a minimum value occurs and $y=0$ at $t=0$ then for $t>0$ there exists some other $t>\dfrac{\ln\left(\frac{2b}{a}\right)}{a-b}$ such that $y=0$ given that y is continuous over all $t>0$.



This means that it is possible for the populations to be equal (i.e. D to be zero) if $b<a<2b$.

2014 Bored Of Studies

Mathematics Extension 1 Trial Examination
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	A
3	B
4	A
5	D
6	D
7	C
8	D
9	B
10	D

Section II

Question 11 (a)

Criteria	Marks
• Correct answer	3
• Obtains $1-e^x \geq 0$	2
• Eliminates denominator correctly	1

Question 11 (b)

Criteria	Marks
• Correct solution	1

Question 11 (c)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Obtains correct expressions for x_0 and y_0	1

Question 11 (d)

Criteria	Marks
• Correct sketch	3
• Identifies both correct asymptote AND correct domain AND $y \geq 0$	2
• Identifies correct asymptote OR correct domain OR equivalent merit	1

Question 11 (e) (i)

Criteria	Marks
• Correct answer	4
• Obtains correct primitive function in terms of θ	3
• Correctly uses double angle formula to manipulate the integral	2
• Substitutes $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$ into integral correctly	1

Question 11 (e) (ii)

Criteria	Marks
• Correct answer	1

Question 12 (a)

Criteria	Marks
• Obtains correct volume	3
• Uses appropriate trigonometric identities to simplify integrand	2
• Writes down the correct integral	1

Question 12 (b) (i)

Criteria	Marks
• Correct proof AND writes down correct expression for $\tan\left(\frac{\alpha-\beta}{2}\right)$	2
• Correct proof OR writes down correct expression for $\tan\left(\frac{\alpha-\beta}{2}\right)$	1

Question 12 (b) (ii)

Criteria	Marks
• Correct proof	1

Question 12 (c) (i)

Criteria	Marks
• Obtains correct displacement-time equation	3
• Makes substantial progress	2
• Obtains correct expression for velocity as a function of displacement	1

Question 12 (c) (ii)

Criteria	Marks
• Correct answer	1

Question 12 (d) (i)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Correctly evaluates an appropriate distance, or equivalent merit	1

Question 12 (d) (ii)

Criteria	Marks
• Correct answer	2
• Obtains correct expression for $\frac{dA}{dp}$, or equivalent merit	1

Question 13 (a) (i)

Criteria	Marks
• Correct solution	1

Question 13 (a) (ii)

Criteria	Marks
• Correct proof	3
• Makes substantial progress for the inductive step	2
• Correctly shows the statement is true for $n=1$	1

Question 13 (b) (i)

Criteria	Marks
• Correct solution	1

Question 13 (b) (ii)

Criteria	Marks
• Correct solution	2
• Makes use of the result $P(k)>P(k-1)$, or equivalent merit	1

Question 13 (b) (iii)

Criteria	Marks
• Correct explanation	1

Question 13 (c)

Criteria	Marks
• Correct proof	4
• Makes substantial progress	3
• Shows that *A*, *O* and *M* are collinear OR proves the result assuming that *A*, *O* and *M* are collinear	2
• Uses a relevant property to make some progress	1

Question 13 (d)

Criteria	Marks
• Correct proof	3
• Makes substantial progress	2
• Identifies that $\sum_{k=0}^{n}\binom{n}{k}=2^n$ OR makes an appropriate rearrangement of the summation, or equivalent merit	1

Question 14 (a)

Criteria	Marks
• Correct answer with appropriate justification	3
• Partially correct answer with appropriate justification OR correct answer with no appropriate justification	2
• Obtains partially correct answer, or equivalent merit	1

Question 14 (b) (i)

Criteria	Marks
• Correct proof	4
• Obtains the result, without showing that a maximum is obtained	3
• Obtains correct equations of motion	2
• Obtains partially correct equations of motion	1

Question 14 (b) (ii)

Criteria	Marks
• Correct solution	2
• Finds the maximum value of x_2, or equivalent merit	1

Question 14 (c) (i)

Criteria	Marks
• Correct solution	1

Question 14 (c) (ii)

Criteria	Marks
• Correct solution	1

Question 14 (c) (iii)

Criteria	Marks
• Correct proof	4
• Makes substantial progress	3
• Makes appropriate use of the constraint $b < a < 2b$ on the properties of the difference between *P* or *N*, or equivalent merit	2
• Obtains an expression for the difference between *P* and *N*, or equivalent merit	1