EXERCISE 7.1 CONTINUITY AND GRADIENTS OF TANGENTS

2 **(a)** is smooth because the slope of the curve changes continually and not suddenly, even at $(0,\ 0)$.

(b) is discontinuous and undefined when $x=0$, so it is not smooth.

(c) is smooth because the slope of the curve changes continually and not suddenly, even where $y=0$.

(d) has two points where the gradient changes suddenly, so it is not smooth.

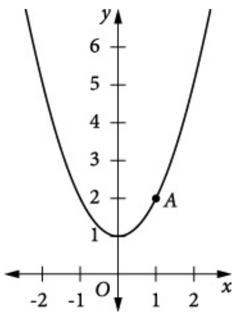
(e) has a discontinuity, and so is not smooth.

(f) is smooth because the slope of the curve changes continually.

(g) has a point where the gradient changes suddenly, so it is not smooth.

(h) will be continuous, but only if the sign changes on both curves at the second x-intercept, and if the line is vertical at the origin.

4 This is the graph of $y=x^2$ raised 1 unit, so its vertex will be $(0,\ 1)$.



(a) $m_{AB}=\dfrac{7.25-2}{2.5-1}=\dfrac{7}{2}=3\dfrac{1}{2}$

Equation of AB:

$$y-2=\frac{7}{2}(x-1)$$
$$y-2=\frac{7}{2}x-\frac{7}{2}$$
$$y=\frac{7}{2}x-\frac{3}{2}$$

$7x-2y-3=0$

$m_{AB_1}=\dfrac{5-2}{2-1}=3$

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Equation of AB_1:

$$\begin{aligned} y-2&=3(x-1)\\ y-2&=3x-3\\ y&=3x-1 \end{aligned}$$
$$3x-y-1=0$$

$$m_{AB_2}=\frac{3.25-2}{1.5-1}=\frac{5}{2}=2\frac{1}{2}$$

Equation of AB_2:

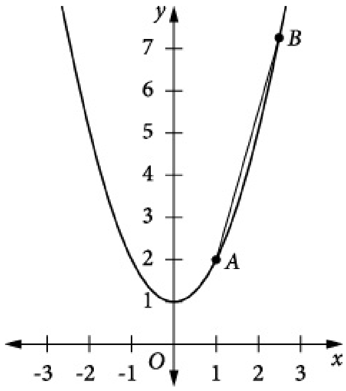
$$\begin{aligned} y-2&=\frac{5}{2}(x-1)\\ y-2&=\frac{5}{2}x-\frac{5}{2}\\ y&=\frac{5}{2}x-\frac{1}{2} \end{aligned}$$
$$5x-2y-1=0$$

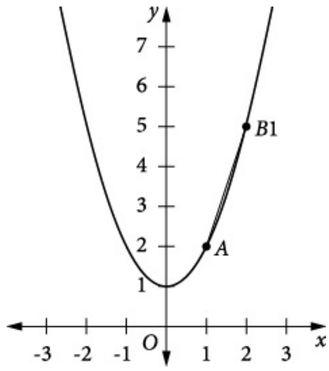
$$m_{AB_3}=\frac{2.21-2}{1.1-1}=\frac{21}{10}=2.1$$

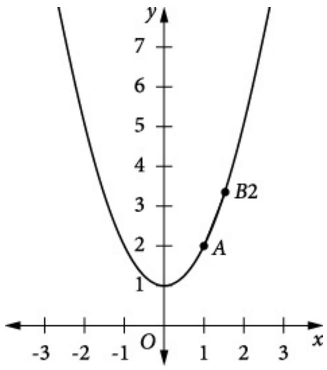
Equation of AB_3:

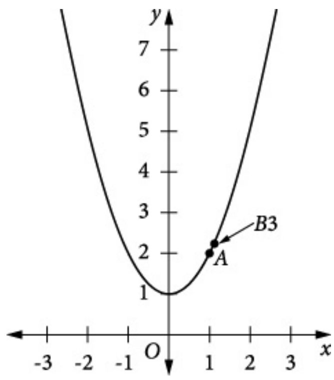
$$\begin{aligned} y-2&=\frac{21}{10}(x-1)\\ y-2&=\frac{21}{10}x-\frac{21}{10}\\ y&=\frac{21}{10}x-\frac{1}{10} \end{aligned}$$
$$21x-10y-1=0$$

(b)







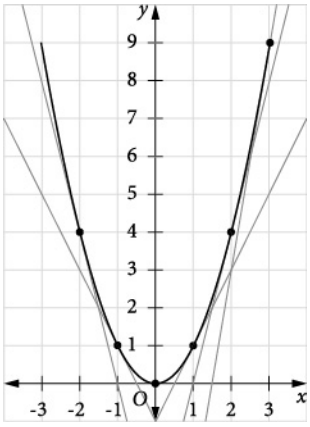


(c) The gradient appears to be approaching 2.

(d) $m_{AB_4} = \dfrac{1.81-2}{0.9-1}$

$= 1.9$

6



(a) When $x=-2,$ the tangent touches the curve at $(-2,\ 4)$ and passes through $(-1,\ 0)$.

$$m_{x=-2}=\frac{0-4}{-1-(-2)}=\frac{-4}{1}=-4$$

When $x=-1,$ the tangent touches the curve at $(-1,\ 1)$ and passes through $(-2,\ 3)$.

$$m_{x=-1}=\frac{1-3}{-1-(-2)}=\frac{-2}{1}=-2$$

When $x=0,$ the tangent touches the curve at $(0,\ 0)$ and is horizontal.

$$m_{x=0}=0$$

When $x=1,$ the tangent touches the curve at $(1,\ 1)$ and passes through $(2,\ 3)$.

$$m_{x=1}=\frac{3-1}{2-1}=\frac{2}{1}=2$$

When $x=2,$ the tangent touches the curve at $(2,\ 4)$ and passes through $(-1,\ 0)$.

$$m_{x=2}=\frac{4-0}{2-1}=\frac{4}{1}=4$$

When $x=3,$ the tangent touches the curve at $(3,\ 9)$ and passes through $(2,\ 3)$.

$$m_{x=3}=\frac{9-3}{2-1}=\frac{6}{1}=6$$

x	-2	-1	0	1	2	3
m	-4	-2	0	2	4	6

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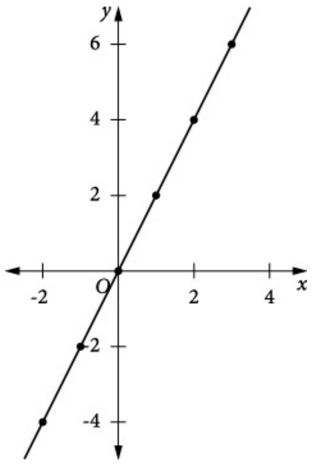
(b)

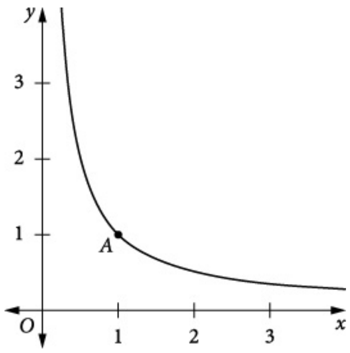


(c) $m = \dfrac{2-0}{1-0}$

$= 2$

It is a straight line through the origin. The equation of the line must be $y = 2x$

8



(a) $m_{AB} = \dfrac{\frac{1}{3}-1}{3-1} = -\dfrac{1}{3}$

Equation of AB:

$$y-1 = -\frac{1}{3}(x-1)$$

$$y-1 = -\frac{1}{3}x + \frac{1}{3}$$

$$y = -\frac{1}{3}x + \frac{4}{3}$$

$$x + 3y - 4 = 0$$

$$m_{AB_1} = \frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2}$$

Equation of AB_1:

$$y-1 = -\frac{1}{2}(x-1)$$
$$y-1 = -\frac{1}{2}x + \frac{1}{2}$$
$$y = -\frac{1}{2}x + \frac{3}{2}$$
$$x + 2y - 3 = 0$$

$$m_{AB_2} = \frac{\frac{2}{3}-1}{\frac{3}{2}-1} = -\frac{2}{3}$$

Equation of AB_2:

$$y-1 = -\frac{2}{3}(x-1)$$
$$y-1 = -\frac{2}{3}x + \frac{2}{3}$$
$$y = -\frac{2}{3}x + \frac{5}{3}$$
$$2x + 3y - 5 = 0$$

$$m_{AB_3} = \frac{\frac{10}{11}-1}{\frac{11}{10}-1} = -\frac{10}{11}$$

Equation of AB_3:

$$y-1 = -\frac{10}{11}(x-1)$$
$$y-1 = -\frac{10}{11}x + \frac{10}{11}$$
$$y = -\frac{10}{11}x + \frac{21}{11}$$
$$10x + 11y - 21 = 0$$

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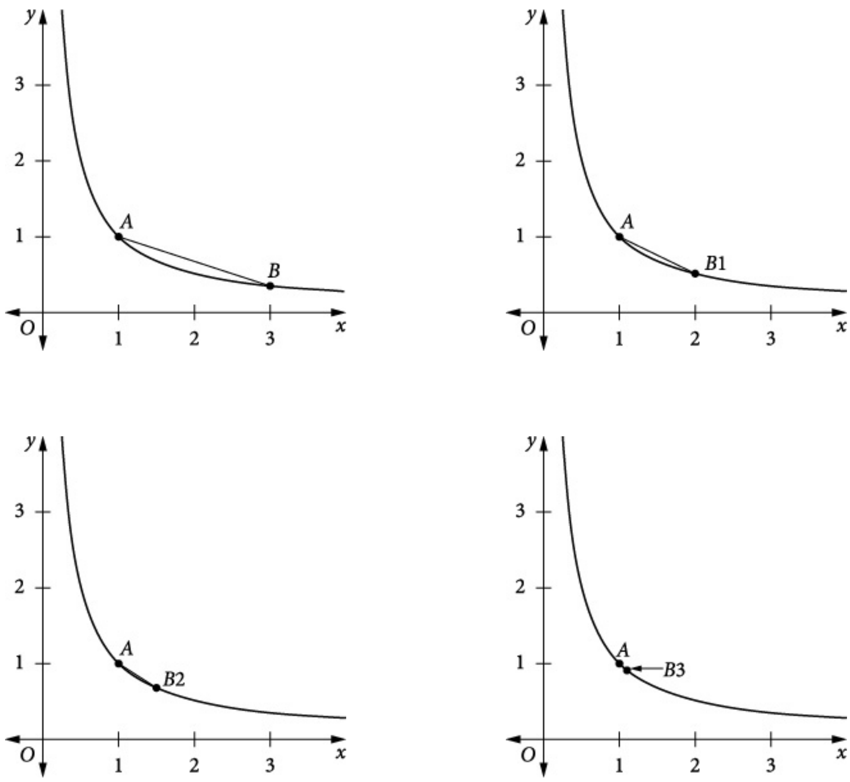
(b)



(c) The gradient appears to be approaching -1.

(d) $m_{AB_4} = \dfrac{\dfrac{10}{9} - 1}{\dfrac{9}{10} - 1} = -\dfrac{10}{9}$

EXERCISE 7.2 LIMIT AND CONTINUITY

2 C

$$\lim_{x \to -3} \frac{(x+5)(x+3)}{x+3} = \lim_{x \to -3}(x+5) \text{ for } x \neq -3.$$
$$= -3 + 5$$
$$= 2$$

4 (a) For $x \geq 0$, $\lim_{x \to 0^+}(x^2 + 1) = 0 + 1 = 1$

For $x < 0$, $\lim_{x \to 0^-}(1) = 1$

$\therefore \lim_{x \to 0} f(x) = 1$

(b) For $x \geq 0$, $\lim\limits_{x \to 1^+} 2x = 2 \times 1 = 2$

For $x < 0$, $\lim\limits_{x \to 1^-} (-2x + 4) = -2 \times 1 + 4 = 2$

$\therefore \lim\limits_{x \to 1} f(x) = 2$

6 (a) As $x < 0$, $\dfrac{1}{x}$ is negative and increases in magnitude (gets more negative).

As $x > 0$, $\dfrac{1}{x}$ is positive and increases in magnitude.

$\therefore \lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist, because it approaches different values when approached from above or from below.

(b) For $x > 0$, $\lim\limits_{x \to 0^+} (0) = 0$

For $x < 0$, $\lim\limits_{x \to 0^-} (1) = 1$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist, because it approaches different values when approached from above or from below.

(c) For $x > 0$, $\lim\limits_{x \to 0^+} f(x) = 0$

For $x < 0$, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 1) = 0 + 1 = 1$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist, because it approaches different values when approached from above or from below.

(d) For $x > 0$, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (x^2 + 1) = 0^2 + 1 = 1$

For $x < 0$, $\lim\limits_{x \to 0^-} (2) = 2$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist, because it approaches different values when approached from above or from below.

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EXERCISE 7.3 GRADIENT OF A CURVE

2 $f(0.9) = 2 \times 0.9 + 3$

$= 4.8$

$$m_1 = \frac{4.8 - 5}{0.9 - 1}$$
$$= 2$$

$$f(0.99) = 2 \times 0.99 + 3$$
$$= 4.98$$

$$m_2 = \frac{4.98 - 5}{0.99 - 1}$$
$$= 2$$

$$f(0.999) = 2 \times 0.999 + 3$$
$$= 4.998$$

$$m_3 = \frac{4.998 - 5}{0.999 - 1}$$
$$= 2$$

$$f(1.001) = 2 \times 1.001 + 3$$
$$= 5.002$$

$$m_4 = \frac{5.002 - 5}{1.001 - 1}$$
$$= 2$$

$$f(1.01) = 2 \times 1.01 + 3$$
$$= 5.02$$

$$m_5 = \frac{5.02 - 5}{1.01 - 1}$$
$$= 2$$

$$f(1.1) = 2 \times 1.1 + 3$$
$$= 5.2$$

$$m_6 = \frac{5.2 - 5}{1.1 - 1}$$
$$= 2$$

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x-coordinate of Q	0.9	0.99	0.999	1.001	1.01	1.1
Gradient of secant PQ	2	2	2	2	2	2

$$\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=2$$

You could predict the answer from the function. It is a straight line, so it has a constant gradient. Using the form $y=mx+b$, the gradient is $m=2$.

4 D

$$\begin{aligned}\frac{g(3.999)-g(4)}{3.999-4}&=\frac{3.999^2-4^2}{3.999-4}\\&\left(=\frac{(3.999-4)(3.999+4)}{3.999-4}\right)\\&=7.999\end{aligned}$$

It is closest to 8.

6 $f(1.9)=\dfrac{1}{1.9}=\dfrac{10}{19}$

$$m_1=\frac{\frac{10}{19}-\frac{1}{2}}{1.9-2}=-\frac{5}{19}\approx -0.2632$$

$$f(1.99)=\frac{1}{1.99}=\frac{100}{199}\approx 0.5025$$

$$m_2=\frac{\frac{100}{199}-\frac{1}{2}}{1.99-2}=-\frac{50}{199}\approx -0.2513$$

$$f(1.999)=\frac{1}{1.999}=\frac{1000}{1999}$$

$$m_3=\frac{\frac{1000}{1999}-\frac{1}{2}}{1.999-2}\approx -0.2501$$

$$f(2.001)=\frac{1}{2.001}=\frac{1000}{2001}$$

$$m_4=\frac{\frac{1000}{2001}-\frac{1}{2}}{2.001-2}\approx -0.2499$$

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$$f(2.01) = \frac{1}{2.01} = \frac{100}{201}$$

$$m_5 = \frac{\frac{100}{201} - \frac{1}{2}}{2.01 - 2} = -\frac{50}{201} \approx -0.2488$$

$$f(2.1) = \frac{1}{2.1} = \frac{10}{21}$$

$$m_6 = \frac{\frac{10}{21} - \frac{1}{2}}{2.1 - 2} = -\frac{5}{21} \approx -0.2381$$

x-coordinate of Q	1.9	1.99	1.999	2.001	2.01	2.1
Gradient of secant PQ	–0.2632	–0.2513	–0.2501	–0.2499	–0.2488	–0.2381

$$\lim_{x \to 1} \frac{f(x) - f(2)}{x - 2} = -0.25$$

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8 (a) $f(x)=2x^2$

$$\begin{aligned}
f(x+h)&=2(x+h)^2\\
&=2\left(x^2+2xh+h^2\right)\\
&=2x^2+4xh+2h^2\\
\frac{f(x+h)-f(x)}{h}&=\frac{2x^2+4xh+2h^2-2x^2}{h}\\
&=\frac{4xh+2h^2}{h}\\
&=\frac{2h(2x+h)}{h}\\
&=2(2x+h)
\end{aligned}$$

(b) $f(x)=2x^2+x$

$$\begin{aligned}
f(x+h)&=2(x+h)^2+(x+h)\\
&=2\left(x^2+2xh+h^2\right)+x+h\\
&=2x^2+4xh+2h^2+x+h\\
\frac{f(x+h)-f(x)}{h}&=\frac{2x^2+4xh+2h^2+x+h-\left(2x^2+x\right)}{h}\\
&=\frac{2x^2+4xh+2h^2+x+h-2x^2-x}{h}\\
&=\frac{4xh+2h^2+h}{h}\\
&=4x+2h+1
\end{aligned}$$

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(c) $f(x) = 4x - x^2$

$$\begin{aligned}
f(x+h) &= 4(x+h) - (x+h)^2 \\
&= 4x + 4h - \left(x^2 + 2xh + h^2\right) \\
&= 4x + 4h - x^2 - 2xh - h^2 \\
\frac{f(x+h) - f(x)}{h} &= \frac{4x + 4h - x^2 - 2xh - h^2 - \left(4x - x^2\right)}{h} \\
&= \frac{4x + 4h - x^2 - 2xh - h^2 - 4x + x^2}{h} \\
&= \frac{4h - 2xh - h^2}{h} \\
&= \frac{h(4 - 2x - h)}{h} \\
&= 4 - 2x - h
\end{aligned}$$

(d) $f(x) = x^3$

$$\begin{aligned}
f(x+h) &= (x+h)^3 \\
&= x^3 + 3x^2h + 3xh^2 + h^3 \\
\frac{f(x+h) - f(x)}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \frac{h\left(3x^2 + 3xh + h^2\right)}{h} \\
&= 3x^2 + 3xh + h^2
\end{aligned}$$

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EXERCISE 7.4 FINDING THE DERIVATIVE FROM FIRST PRINCIPLES

2 **(a)** $f(-2)=(-2)^2=4$

$$f(-2+h)=(h-2)^2=h^2-4h+4$$

$$\begin{aligned}f'(-2)&=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\&=\lim_{h\to 0}\frac{h^2-4h+4-4}{h}\\&=\lim_{h\to 0}\frac{h^2-4h}{h}\\&=\lim_{h\to 0}(h-4)\\&=-4\end{aligned}$$

(b) $f(-1)=(-1)^3=-1$

$$\begin{aligned}(-1+h)^3&=(-1)^3+3(-1)^2h+3(-1)h^2+h^3\\&=-1+3h-3h^2+h^3\end{aligned}$$

$$\begin{aligned}f'(-1)&=\lim_{h\to 0}\frac{(-1+h)^3-x^3}{h}\\&=\lim_{h\to 0}\frac{-1+3h-3h^2+h^3-(-1)}{h}\\&=\lim_{h\to 0}\frac{3h-3h^2+h^3}{h}\\&=\lim_{h\to 0}\left(3-3h+h^2\right)\\&=3\end{aligned}$$

4 **D**

$$\begin{aligned}f(x)&=3-2x+4x^2\\f(1)&=3-2+4=5\\f(1+h)&=3-2(1+h)+4(1+h)^2\\&=3-2-2h+4+8h+h^2\\&=h^2+6h+5\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}&=\lim_{h\to 0}\frac{h^2+6h+5-5}{h}\\&=\lim_{h\to 0}\frac{h^2+6h}{h}\\&=\lim_{h\to 0}(h+6)\\&=0+6\\&=6\end{aligned}$$

6 **(a)** $f(x)=4x^2-1$

$$\begin{aligned}f(x+h)&=4(x+h)^2-1\\&=4\left(x^2+2xh+h^2\right)-1\\&=4x^2+8xh+4h^2-1\end{aligned}$$

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$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{4x^2+8xh+4h^2-1-\left(4x^2-1\right)}{h}\\ &= \lim_{h\to 0}\frac{8xh+4h^2}{h}\\ &= \lim_{h\to 0}(8x+4h)\\ &= 8x+0\\ &= 8x\end{aligned}$$

(b) $f(x)=\dfrac{x^2}{2}-2x-3$

$$\begin{aligned}f(x+h) &= \frac{(x+h)^2}{2}-2(x+h)-3\\ &= \frac{1}{2}\left(x^2+2xh+h^2\right)-2x-2h-3\\ &= \frac{x^2}{2}+xh+\frac{h^2}{2}-2x-2h-3\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{x^2}{2}+xh+\frac{h^2}{2}-2x-2h-3-\left(\frac{x^2}{2}-2x-3\right)}{h}\\ &= \lim_{h\to 0}\frac{xh+\frac{h^2}{2}-2h}{h}\\ &= \lim_{h\to 0}\left(x+\frac{h}{2}-2\right)\\ &= x+0-2\\ &= x-2\end{aligned}$$

(c) $f(x)=x^3-2x^2$

$$\begin{aligned}f(x+h) &= (x+h)^3-2(x+h)^2\\ &= x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2-\left(x^3-2x^2\right)}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-4xh-2h^2}{h}\\ &= \lim_{h\to 0}\left(3x^2+3xh+h^2-4x-2h\right)\\ &= 3x^2+0+0-4x-0\\ &= 3x^2-4x\end{aligned}$$

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EXERCISE 7.5 CONDITIONS FOR DIFFERENTIABILITY

2 For $x \le 1,\ f(x) = x.$

This is a straight line through the origin of gradient 1.

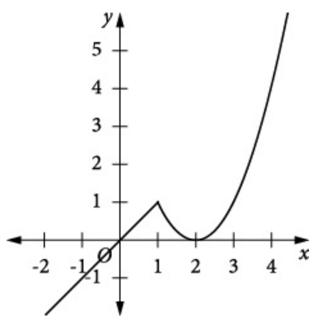
$a = 1,\ x \le 1,\ f(x) = x = 1$

$x > 1,\ f(x) = (x-2)^2.$

This is a parabola, approaching $(1-2)^2 = 1$ as $x \to 1$.

It is concave up, with a vertex at $(2,\ 0)$.

Sketch the graph.



It is continuous when $x = 1$, but the left-hand derivative is 1 and the right-hand derivative is negative, so $f'(a)$ is undefined for $a = 1$.

Where $a = 3$, $f(a) = (a-2)^2$

$$
\begin{aligned}
f'(3) &= \lim_{x\to 3}\frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0}\frac{(3+h-2)^2-(3-2)^2}{h} \\
&= \lim_{h\to 0}\frac{(h+1)^2-1}{h} \\
&= \lim_{h\to 0}\frac{h^2+2h+1-1}{h} \\
&= \lim_{h\to 0}\frac{h^2+2h}{h} \\
&= \lim_{h\to 0}(h+2) \\
&= 2
\end{aligned}
$$

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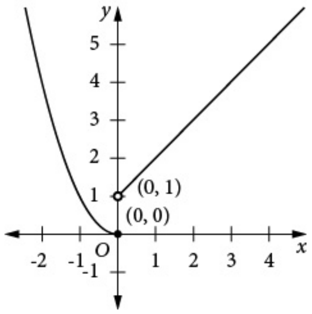
4 Where $x \le 0,$ the graph is the left-hand half of $f(x) = x^2,$ vertex the origin.

Where $x > 0,$ the graph is $y = x + 1,$, a straight line of gradient 1, crossing the y-axis at 1.

Sketch the graph.



$f'(0)$ does not exist. The function is discontinuous at $x = 0$, so it is not differentiable at $x = 0$.

Where $x = -1,$ $f(x) = x^2$.

$$
\begin{aligned}
f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\
&= \lim_{h \to 0} \frac{(-1+h)^2 - (-1)^2}{h} \\
&= \lim_{h \to 0} \frac{(h-1)^2 - 1}{h} \\
&= \lim_{h \to 0} \frac{h^2 - 2h + 1 - 1}{h} \\
&= \lim_{h \to 0} \frac{h^2 - 2h}{h} \\
&= \lim_{h \to 0} (h - 2) \\
&= -2
\end{aligned}
$$

6 For $x > 3,$ the graph is $f(x) = x$, which will start at $(3,\ 3)$, not including $(3,\ 3)$, with a gradient of 1.

For $-3 \le x \le 3,$ the graph is $f(x) = 3$, a horizontal line which will start at $(-3,\ 3)$, including $(-3,\ 3)$, and end at $(3,\ 3)$, including $(3,\ 3)$.

For $x < -3,$ the graph is $f(x) = x + 1$, a straight line of gradient 1. As $x \to -3,$ $f(x) \to -3 + 1 = -2,$ but does not touch it.

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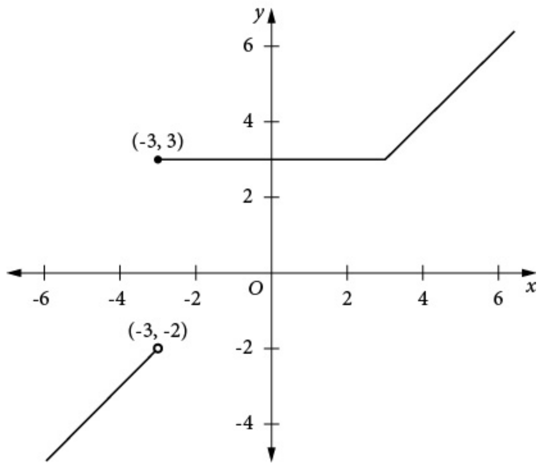

$f'(3)$ does not exist. because the left-hand derivative is 0 and the right-hand derivative is 1.

$f'(-3)$ does not exist as the function is discontinuous at $x=-3$, so it is not differentiable at $x=-3$

EXERCISE 7.6 STANDARD DERIVATIVES

2 (a) $y = x^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1}$$
$$= \frac{3}{2}x^{\frac{1}{2}}$$
$$= \frac{3\sqrt{x}}{2}$$

(b) $y = \frac{2}{x}$

$$= 2x^{-1}$$
$$\frac{dy}{dx} = 2\times(-1)\times x^{-1-1}$$
$$= -\frac{2}{x^2}$$

(c) $y = 2\sqrt{x}$

$$= 2x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2\times\frac{1}{2}\times x^{\frac{1}{2}-1}$$
$$= x^{-\frac{1}{2}}$$
$$= \frac{1}{\sqrt{x}}$$

(d) $v = \sqrt[3]{t^2}$

$$= t^{\frac{2}{3}}$$
$$\frac{dv}{dt} = \frac{2}{3}t^{\frac{2}{3}-1}$$
$$= \frac{2}{3}t^{-\frac{1}{3}}$$
$$= \frac{2}{3\sqrt[3]{t}}$$

(e) $h(m) = \frac{1}{m^3}$

$$= m^{-3}$$
$$h'(m) = -3\times m^{-3-1}$$
$$= -3\times m^{-4}$$
$$= -\frac{3}{m^4}$$

(f) $f(x) = \frac{1}{\sqrt{x}}$

$$= x^{-\frac{1}{2}}$$
$$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1}$$
$$= -\frac{x^{-\frac{3}{2}}}{2}$$
$$= -\frac{1}{2\sqrt{x^3}}$$
$$= -\frac{1}{2\sqrt{x^2\times x}}$$
$$= -\frac{1}{2x\sqrt{x}}$$

4 (a) $y = (x-1)(x+2)$

$$= x^2 + x - 2$$
$$\frac{dy}{dx} = 2x + 1 - 0 = 2x - 1$$

(b) $y = 3x(x^2-2)$

$$= 3x^3 - 6x$$
$$\frac{dy}{dx} = 3\times 3x^2 - 6$$
$$= 3(3x^2-2)$$

(c) $y = (2x-3)^2$

$$= 4x^2 - 12x + 9$$
$$\frac{dy}{dx} = 8x - 12$$
$$= 4(2x-3)$$

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(d) $y=(x-4)(x+4)$

$$=x^2-16$$

$$\frac{dy}{dx}=2x$$

(e) $y=(2x-3)^3$

$$=8x^3+3\times(2x)^2\times(-3)+3\times 2x\times(-3)^2-27$$
$$=8x^3-36x^2+54x-27$$
$$\frac{dy}{dx}=24x^2-72x+54$$
$$=6(4x^2-12x+9)$$
$$=6(2x-3)^2$$

(f) $y=(x-2)(x+1)(3x+1)$

$$=(x^2-x-2)(3x+1)$$
$$=3x^3+x^2-3x^2-x-6x-2$$
$$=3x^3-2x^2-7x-2$$
$$\frac{dy}{dx}=9x^2-4x-7$$

6 **(a)** $f(x)=x+\sqrt{x}$

$$=x+x^{\frac{1}{2}}$$
$$f'(x)=1+\frac{1}{2}x^{-\frac{1}{2}}$$
$$=1+\frac{1}{2\sqrt{x}}$$

(b) $f(x)=x^2+\frac{1}{x}$

$$=x^2+x^{-1}$$
$$f'(x)=2x-x^{-2}$$
$$=2x-\frac{1}{x^2}$$

(c) $f(x)=x^2+x+1+\frac{1}{x}+\frac{1}{x^2}$

$$=x^2+x+1+x^{-1}+x^{-2}$$
$$f'(x)=2x+1+0-x^{-2}-2x^{-3}$$
$$=2x+1-\frac{1}{x^2}-\frac{2}{x^3}$$

(d) $f(x)=x^{\frac{2}{3}}+x^{\frac{1}{3}}$

$$f'(x)=\frac{2}{3}x^{-\frac{1}{3}}+\frac{1}{3}x^{-\frac{2}{3}}$$
$$=\frac{2}{3\sqrt[3]{x}}-\frac{1}{3\sqrt[3]{x^2}}$$

(e) $f(x)=\left(x-\dfrac{1}{x}\right)^2$

$$=x^2-2\times x\times\frac{1}{x}+\frac{1}{x^2}$$
$$=x^2-2+x^{-2}$$
$$f'(x)=2x-2x^{-3}$$
$$=2x-\frac{2}{x^3}$$

(f) $f(x)=x\sqrt{x}$

$$=x^1\times x^{\frac{1}{2}}$$
$$=x^{\frac{3}{2}}$$
$$f'(x)=\frac{3}{2}x^{\frac{1}{2}}$$
$$=\frac{3\sqrt{x}}{2}$$

8 (a) $f(x)=x^2-4$

$$f'(x)=2x$$
$$2x=0$$
$$x=0$$

(b) $f(x)=2x^3-6x$

$$f'(x)=6x^2-6$$
$$6x^2-6=0$$
$$6(x^2-1)=0$$
$$6(x+1)(x-1)=0$$
$$x=\pm1$$

(c) $f(x)=x^3-4x^2$

$$f'(x)=3x^2-8x$$
$$3x^2-8x=0$$
$$x(3x-8)=0$$
$$x=0,\ x=\frac{8}{3}$$

10 $f(x)=x^3-x^2-6x+1$

$$f'(x)=3x^2-2x-6$$
$$3x^2-2x-6=-5$$
$$3x^2-2x-1=0$$
$$(3x+1)(x-1)=0$$
$$x=-\frac{1}{3},\ x=1$$

12 (a) $f(x)=(x-1)^2$

$$(x-1)^2=0$$
$$x=1$$

(b) $f(x)=(x-1)^2$

$$=x^2-2x+1$$
$$f'(x)=2x-2$$
$$2x-2=0$$
$$2x=2$$
$$x=1$$

(c) $f'(x)=2x-2$

$$2x-2=-1$$
$$2x=1$$
$$x=\frac{1}{2}$$

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14 (a) $y=\frac{1}{3}x^3-\frac{3}{2}x^2+2x+1$

$$\frac{dy}{dx}=\frac{1}{3}\times 3x^2-\frac{3}{2}\times 2x+2$$
$$=x^2-3x+2$$

If the tangent is parallel to the x-axis, the gradient is 0.

$$x^2-3x+2=0$$
$$(x-1)(x-2)=0$$
$$x=1,\ x=2$$

(b) $\frac{dy}{dx}=\tan\theta=2$ indicates gradient is 2.

$$x^2-3x+2=2$$
$$x^2-3x=0$$
$$x(x-3)=0$$
$$x=0,\ x=3$$

(c) $y-6x-1=0$

$$y=6x+1$$

If the tangent is parallel to the line, the gradients are the same, so it is $\frac{dy}{dx}=m=6$.

$$x^2-3x+2=6$$
$$x^2-3x-4=0$$
$$(x+1)(x-4)=0$$
$$x=-1,\ x=4$$

16 $c=0.6x^2+4x+650$

$$\frac{dc}{dx}=1.2x+4$$

When $x=40$,

$$\frac{dc}{dx}=1.2\times 40+4$$
$$=52$$

It will cost about $\$52$ to make the next item.

The marginal cost is $52.

18 The rate of the change in value is the derivative.

$$v=40000-4000t$$
$$\frac{dv}{dt}=-4000$$

The depreciation is constant at $4000 per year, no matter what the time, for up to 10 years.

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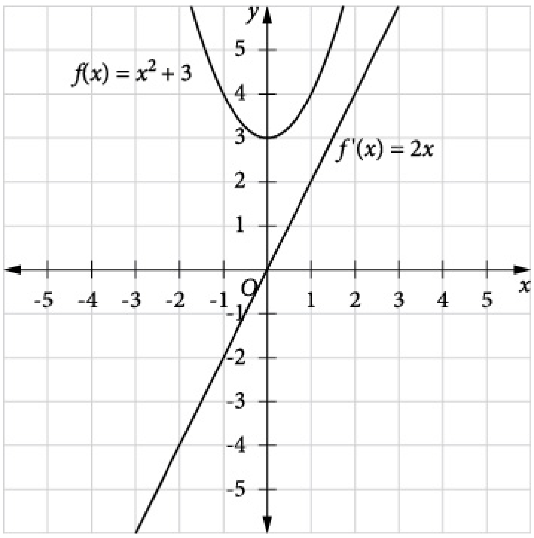
20 (a) $f(x) = x^2 + 3$

$f'(x) = 2x$

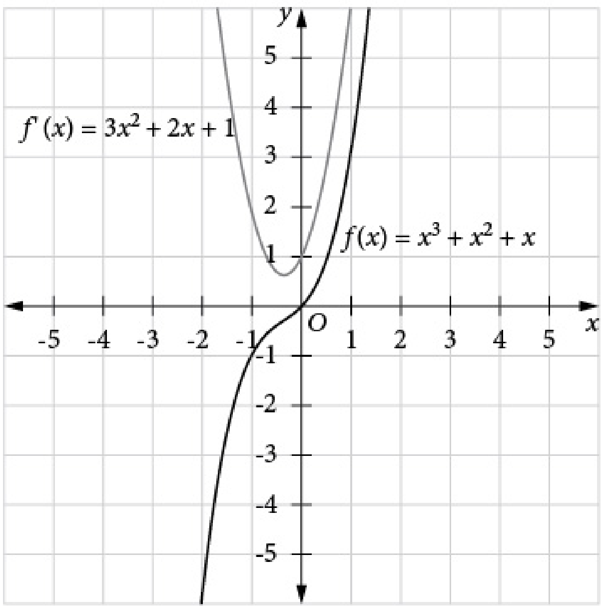
(b)



22 (a) $f(x) = x^3 + x^2 + x$

$f'(x) = 3x^2 + 2x + 1$

(b)



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EXERCISE 7.7 THE PRODUCT RULE

2 **A**

$$g(x) = (3x-1)(3x^2+1)$$
$$u = 3x-1,\ v = 3x^2+1$$
$$\frac{du}{dx} = 3,\ \frac{dv}{dx} = 6x$$
$$\begin{aligned}g'(x) &= \frac{d}{dx}(uv)\\ &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= (3x^2+1)\times 3 + (3x-1)\times 6x\\ &= 9x^2+3+18x^2-6x\\ &= 27x^2-6x+3\end{aligned}$$

4 **(a)** $g(x) = (x^2+5x)(x^3+x^2+1)$

$$u = x^2+5x,\ v = x^3+x^2+1$$
$$\frac{du}{dx} = 2x+5,\ \frac{dv}{dx} = 3x^2+2x$$
$$\begin{aligned}g'(x) &= \frac{d}{dx}(uv)\\ &= v\frac{du}{dx} + u\frac{dv}{dx}\end{aligned}$$
$$\begin{aligned}g'(x) &= (x^3+x^2+1)(2x+5) + (x^2+5x)(3x^2+2x)\\ &= 2x^4+2x^3+2x+5x^3+5x^2+5+3x^4+15x^3+2x^3+10x^2\\ &= 5x^4+24x^3+15x^2+2x+5\end{aligned}$$

(b) $g'(1) = 5\times(1)^4 + 24\times(1)^3 + 15\times(1)^2 + 2(1) + 5$

$$\begin{aligned}&= 5+24+15+2+5\\ &= 51\end{aligned}$$

(c) $g'(-2) = 5\times(-2)^4 + 24\times(-2)^3 + 15\times(-2)^2 + 2(-2) + 5$

$$= -51$$

EXERCISE 7.8 THE CHAIN RULE

2 B

$$f(x)=(x^3-1)^5$$

Let $u=x^3-1$ so $\frac{du}{dx}=3x^2$.

$f(x)=u^5$ so $\frac{d[f(x)]}{du}=5u^4$.

$$\begin{aligned}f'(x)&=\frac{d[f(x)]}{du}\times\frac{du}{dx}\\&=5u^4\times 3x^2\\&=5(x^3-1)^4\times 3x^2\\&=15x^2(x^3-1)^4\end{aligned}$$

4 First find the derivative of $\sqrt[3]{x^2-4}$.

$$f(x)=(x^2-4)^{\frac{1}{3}}$$

Let $u=x^2-4$ so $\frac{du}{dx}=2x$.

$f(x)=u^{\frac{1}{3}}$ so $\frac{d[f(x)]}{du}=\frac{1}{3}u^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{u^2}}$

$$\begin{aligned}f'(x)&=\frac{d[f(x)]}{du}\times\frac{du}{dx}\\&=\frac{1}{3\sqrt[3]{u^2}}\times 2x\\&=\frac{2x}{3\sqrt[3]{(x^2-4)^2}}\end{aligned}$$

$$g(x)=x^2+5x+\sqrt[3]{x^2-4}$$
$$g'(x)=2x+5+\frac{2x}{3\sqrt[3]{(x^2-4)^2}}$$

This can also be written $g'(x)=2x+5+\frac{2x}{3}(x^2-4)^{-\frac{2}{3}}$.

(a) incorrect **(b)** incorrect

(c) correct **(d)** correct

EXERCISE 7.9 THE QUOTIENT RULE

2 **A**

$$y=\frac{x+1}{x^2+1}$$

$$u=x+1,\ v=x^2+1$$

$$\frac{du}{dx}=1,\ \frac{dv}{dx}=2x$$

$$g'(x)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{\left(x^2+1\right)\times 1-(x+1)\times 2x}{\left(x^2+1\right)^2}$$

$$=\frac{x^2+1-2x^2-2x}{\left(x^2+1\right)^2}$$

$$=\frac{1-2x-x^2}{\left(x^2+1\right)^2}$$

4 $y=\dfrac{\sqrt{x}}{x^2+1}$

$$u=\sqrt{x}=x^{\frac{1}{2}},\ v=x^2+1$$

$$\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}},\ \frac{dv}{dx}=2x$$

$$g'(x)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{\left(x^2+1\right)\times\frac{1}{2\sqrt{x}}-\sqrt{x}\times 2x}{\left(x^2+1\right)^2}$$

$$=\frac{\frac{\left(x^2+1\right)}{2\sqrt{x}}-2x\sqrt{x}}{\left(x^2+1\right)^2}\times\frac{2\sqrt{x}}{2\sqrt{x}}$$

$$=\frac{\left(x^2+1\right)-4x^2}{2\sqrt{x}\left(x^2+1\right)^2}$$

$$=\frac{1-3x^2}{2\sqrt{x}\left(x^2+1\right)^2}$$

A correct **B** correct

C correct **D** incorrect

6 $y=\frac{x}{x^2+1}$

$$u=x,\ v=x^2+1$$

$$\frac{du}{dx}=1,\ \frac{dv}{dx}=2x$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{\left(x^2+1\right)\times 1-x\times 2x}{\left(x^2+1\right)^2}$$

$$=\frac{x^2+1-2x^2}{\left(x^2+1\right)^2}$$

$$=\frac{1-x^2}{\left(x^2+1\right)^2}$$

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$$\frac{dy}{dx} = 0$$

$$\frac{1-x^2}{\left(x^2+1\right)^2} = 0$$

$$1-x^2 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Therefore, the gradient of the tangent to this curve is zero twice, at $x = -1$ and $x = 1$.

EXERCISE 7.10 TANGENTS AND NORMALS TO A CURVE

2 $y = 3x - x^2$

$$\frac{dy}{dx} = 3 - 2x$$

At $x = 0$,

$$\frac{dy}{dx} = 3 - 2 \times 0$$

$$m_T = 3$$

$$m_N = -\frac{1}{3}$$

Equation of tangent:

$$(y-0) = 3(x-0)$$

$$y = 3x$$

$$3x - y = 0$$

Equation of normal:

$$(y-0) = -\frac{1}{3}(x-0)$$

$$y = -\frac{1}{3}x$$

$$3y = -x$$

$$x + 3y = 0$$

4 $y = 2x^2 + 3x - 4$

$$\frac{dy}{dx} = 4x + 3$$

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At $x = 0$,

$$\begin{aligned} y &= 2\times 0+3\times 0-4 \\ &= -4 \end{aligned}$$

$$\frac{dy}{dx} = 4\times 0+3$$

$$m_T = 3$$

$$m_N = -\frac{1}{3}$$

Equation of tangent:

$$\begin{aligned} (y-(-4)) &= 3(x-0) \\ y+4 &= 3x \\ y &= 3x-4 \\ 3x-y-4 &= 0 \end{aligned}$$

Equation of normal:

$$\begin{aligned} (y-(-4)) &= -\frac{1}{3}(x-0) \\ y+4 &= -\frac{1}{3}x \\ y &= -\frac{1}{3}x-4 \\ 3y &= -x-12 \\ x+3y+12 &= 0 \end{aligned}$$

6 $y = \frac{1}{x}$

$$= x^{-1}$$

$$\begin{aligned} \frac{dy}{dx} &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

At $x = -2,\ y = -\frac{1}{2}$.

$$\frac{dy}{dx} = -\frac{1}{(-2)^2} = -\frac{1}{4}$$

$$m_T = -\frac{1}{4}$$

$$m_N = 4$$

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Equation of tangent:

$$\begin{aligned}\left(y-\left(-\frac{1}{2}\right)\right)&=-\frac{1}{4}\left(x-(-2)\right)\\ y+\frac{1}{2}&=-\frac{1}{4}(x+2)\\ y&=-\frac{1}{4}x-\frac{1}{2}-\frac{1}{2}\\ y&=-\frac{1}{4}x-1\\ 4y&=-x-4\\ x+4y+4&=0\end{aligned}$$

Equation of normal:

$$\begin{aligned}\left(y-\left(-\frac{1}{2}\right)\right)&=4\left(x-(-2)\right)\\ y+\frac{1}{2}&=4x+8\\ y&=4x+\frac{15}{2}\\ 2y&=8x+15\\ 8x-2y+15&=0\end{aligned}$$

(a) incorrect **(b)** correct

(c) correct **(d)** correct

8 $y=3x^3-7x^2+2x$

$$\frac{dy}{dx}=9x^2-14x+2$$

At $x=2$,

$$\begin{aligned}y&=3\times 2^3-7\times 2^2+2\times 2\\ &=0\end{aligned}$$

$$\frac{dy}{dx}=9\times 2^2-14\times 2+2=10$$

$$m_T=10$$

$$m_N=-\frac{1}{10}$$

Equation of tangent:

$$(y-0)=10(x-2)$$
$$y=10x-20$$
$$10x-y-20=0$$

Equation of normal:

$$(y-0)=-\frac{1}{10}(x-2)$$
$$y=-\frac{1}{10}x+\frac{1}{5}$$
$$10y=-x+2$$
$$x+10y-2=0$$

10 $y=4x-3x^2$

$$4x-3x^2=0$$
$$x(4-3x)=0$$
$$x=0,\ x=\frac{4}{3}$$

$$\frac{dy}{dx}=4-6x$$

At $x=0$,

$$\frac{dy}{dx}=4-6\times 0=4$$
$$m_{T_1}=4$$

Equation of tangent at $(0,\ 0)$:

$$(y-0)=4(x-0)$$
$$y=4x$$
$$4x-y=0$$

At $x=\frac{4}{3}$,

$$\frac{dy}{dx}=4-6\times\frac{4}{3}=-4$$
$$m_{T_1}=-4$$

Equation of tangent at $\left(\frac{4}{3},\ 0\right)$:

$$\begin{aligned}(y-0)&=-4\left(x-\frac{4}{3}\right)\\ y&=-4x+\frac{16}{3}\\ 3y&=-12x+16\end{aligned}$$

$$12x+3y-16=0$$

12 $y=2x^2-6x+5...[1]$

$y=x^2-2x+1...[2]$

Equating $[1]$ and $[2]$,

$$\begin{aligned}2x^2-6x+5&=x^2-2x+1\\ x^2-4x+4&=0\\ (x-2)^2&=0\\ x&=2\end{aligned}$$

Substitute into $[2]$

$$\begin{aligned}y&=2^2-2\times 2+1\\ &=1\end{aligned}$$

The two parabolas intersect at only one point, $(2,\ 1)$.

$$\begin{aligned}y_1&=2x^2-6x+5\\ \frac{dy_1}{dx}&=4x-6\end{aligned}$$

At $x=2$,

$$\begin{aligned}\frac{dy_1}{dx}&=4\times 2-6\\ &=2\end{aligned}$$

$$\begin{aligned}y_2&=x^2-2x+1\\ \frac{dy_2}{dx}&=2x-2\end{aligned}$$

At $x=2$,

$$\begin{aligned}\frac{dy_2}{dx}&=2\times 2-2\\ &=2\end{aligned}$$

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Both curves have the same gradient at their point of intersection.

Equation of tangent at $(2,\ 1)$:

$$\begin{aligned} y-1&=2(x-2) \\ y-1&=2x-4 \\ y&=2x-3 \end{aligned}$$

14 (a) $y=x+1\ldots[1]$

$y=x^2-x-2\ldots[2]$

Equating [1] and [2]

$$\begin{aligned} x^2-x-2&=x+1 \\ x^2-2x-3&=0 \\ (x+1)(x-3)&=0 \end{aligned}$$

$x=-1,\ x=3$

Substitute $x=-1$ into [1]

$$\begin{aligned} y&=-1+1 \\ &=0 \end{aligned}$$

$\therefore P$ is $(-1,\ 0)$.

Substitute $x=3$ into [1]

$$\begin{aligned} y&=3+1 \\ &=4 \end{aligned}$$

$\therefore Q$ is $(3,\ 4)$.

(b) $\dfrac{dy}{dx}=2x-1$

At $P(-1,\ 0)$, $\dfrac{dy}{dx}=2\times(-1)-1$

$m_T=-3$

Equation of tangent at P:

$$\begin{aligned} y-0&=-3(x-(-1)) \\ y&=-3x-3 \\ 3x+y+3&=0 \end{aligned}$$

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At $Q(3,\ 4)$, $\dfrac{dy}{dx} = 2\times3-1$

$m_T = 5$

Equation of tangent at Q:

$$
\begin{aligned}
y-4 &= 5(x-3) \\
y &= 5x-11 \\
5x-y-11 &= 0
\end{aligned}
$$

(c) $3x+y+3=0...[1]$

$5x-y-11=0...[2]$

$[1]+[2]$

$$
\begin{aligned}
8x-8 &= 0 \\
x &= 1
\end{aligned}
$$

Substitute into $[1]$

$$
\begin{aligned}
3\times1+y+3 &= 0 \\
y &= -6
\end{aligned}
$$

The intersection point of these two tangents is $(1,\ -6)$.

16 $y = 2x^2-5x+1$

$$
\begin{aligned}
\frac{dy}{dx} &= 4x-5 \\
4x-5 &= 3 \\
4x &= 8 \\
x &= 2
\end{aligned}
$$

Substitute back into $y = 2x^2-5x+1$

$$
\begin{aligned}
y &= 2\times2^2-5\times2+1 \\
&= -1
\end{aligned}
$$

Equation of tangent at $(2,\ -1)$:

$$
\begin{aligned}
y-(-1) &= 3(x-2) \\
y+1 &= 3x-6 \\
y &= 3x-7 \\
3x-y-7 &= 0
\end{aligned}
$$

Gradient of the normal:

$m_N = -\frac{1}{3}$

Equation of normal at $(2,\ -1)$:

$$\begin{aligned} y-(-1) &= -\frac{1}{3}(x-2) \\ y+1 &= -\frac{1}{3}x+\frac{2}{3} \\ y &= -\frac{1}{3}x-\frac{1}{3} \end{aligned}$$

$x+3y+1=0$

18 $y = x-2 \ldots[1]$

$y = x^3(x-2)\ldots[2]$

Equate [1] and [2]

$$\begin{aligned} x^3(x-2) &= x-2 \\ x^3(x-2)-(x-2) &= 0 \\ (x-2)(x^3-1) &= 0 \\ x &= 1,\ x=2 \end{aligned}$$

Substitute $x=1$ into [1]

$$\begin{aligned} y &= 1-2 \\ &= -1 \end{aligned}$$

$\therefore A$ is $(1,\ -1)$.

Substitute $x=2$ into [1]

$$\begin{aligned} y &= 2-2 \\ &= 0 \end{aligned}$$

$\therefore B$ is $(2,\ 0)$.

$$\begin{aligned} y &= x^3(x-2) \\ y &= x^4-2x^3 \\ \frac{dy}{dx} &= 4x^3-6x^2 \end{aligned}$$

At $A(1,\ -1)$

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$$\frac{dy}{dx} = 4\times 1^3 - 6\times 1^2$$
$$m_{T_A} = -2$$
$$\tan\theta = -2$$
$$\theta \approx 180° - 63°26'$$
$$\theta \approx 116°34'$$

At $B(2,\ 0)$

$$\frac{dy}{dx} = 4\times 2^3 - 6\times 2^2$$
$$m_{T_B} = 8$$
$$\tan\theta = 8$$
$$\theta \approx 82°52'$$

The angle between the two tangents is $116°34' - 82°52' = 33°42'$

20 $y = x^2(2x-3)$

$$= 2x^3 - 3x^2$$
$$\frac{dy}{dx} = 6x^2 - 6x$$

(a) The tangent is parallel to $y = 12x - 1$

$m = 12$

$$6x^2 - 6x = 12$$
$$6x^2 - 6x - 12 = 0$$
$$6(x^2 - x - 2) = 0$$
$$6(x+1)(x-2) = 0$$
$$x = -1,\ x = 2$$

Substitute $x = -1$ into $y = x^2(2x-3)$

$$y = (-1)^2 \times (2\times(-1) - 3)$$
$$= -5$$

Substitute $x = 2$ into $y = x^2(2x-3)$

$$y = 2^2 \times (2\times 2 - 3)$$
$$= 4$$

The points are $(-1,\ -5)$, $(2,\ 4)$.

(b) The tangent is parallel to the x-axis

$m = 0$

$$\begin{aligned}6x^2 - 6x &= 0\\ 6x(x-1) &= 0\\ x &= 0,\ x = 1\end{aligned}$$

Substitute $x = 0$ into $y = x^2(2x-3)$

$$\begin{aligned}y &= (0)^2 \times (2\times 0 - 3)\\ &= 0\end{aligned}$$

Substitute $x = 1$ into $y = x^2(2x-3)$

$$\begin{aligned}y &= 1^2 \times (2\times 1 - 3)\\ &= -1\end{aligned}$$

The points are $(0,\ 0)$, $(1,\ -1)$.

EXERCISE 7.11 THE GRADIENT AS A RATE OF CHANGE

2 D

$$A = 0.6t^2$$

$$\frac{dA}{dt} = 1.2t$$

At $t = 4$

$$\frac{dA}{dt} = 1.2 \times 4 = 4.8 \text{ mm}^2\text{/day}$$

4 (a) $Q(t) = 1000(20-t)^2$

Let $u = 20 - t$ so $\dfrac{du}{dt} = -1$.

$Q(u) = 1000u^2$ so $Q'(u) = 1000 \times 2u = 2000u = 2000(20-t)$.

$$\begin{aligned}Q'(t) &= Q'(u) \times \frac{du}{dt}\\ &= 2000(20-t) \times -1\\ &= -2000(20-t)\end{aligned}$$

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The tank is being emptied at a rate of $2000(20-t)$ litres per minute.

(b) $1000(20-t)^2 = 0 \Rightarrow t = 20$ minutes

(c) $2000(20-t) = 20000$

$20-t = 10$

$t = 10$ minutes

(d) $Q(0) = 1000\times(20-0)^2$

$= 400\ 000$ L

$Q(5) = 1000\times(20-5)^2$
$= 225\ 000$ L

Average rate is $\dfrac{400\ 000-225\ 000}{5} = 35\ 000$ litres per minute.

6 (a) Initially means $t = 0$.

$\dfrac{dQ}{dt} = 2(0)+1 = 1$

1 item per minute

(b) $\dfrac{dQ}{dt} = 2(10)+1 = 21$

21 items per minute

8 (a) $V = \pi r^2 h$

$\dfrac{dV}{dh} = \pi r^2$

(b) $V = \pi r^2 h$

$\dfrac{dV}{dh} = 2\pi rh$

10 (a) $V = 40000-5000t$

$\dfrac{dV}{dt} = -5000$

It is depreciating at a constant rate of $\$5000$ per year.

(b) $V = 40\ 000 - 5000\times 3$

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$V = 25\ 000$

The machine is worth $\$25\ 000$ after 3 years.

EXERCISE 7.12 VELOCITY AND ACCELERATION AS A RATE OF CHANGE

2 C

$$x = 2t^3 - t^2 + 4t + 1$$
$$v = \frac{dx}{dt} = 6t^2 - 2t + 4$$
$$a = \frac{dv}{dt} = 12t - 2$$

4 (a) $x = 2t^3 - 15t^2 + 36t$

$$v = \frac{dx}{dt} = 6t^2 - 30t + 36$$
$$a = \frac{dv}{dt} = 12t - 30$$

(b) Initial means $t = 0$.

$$v(0) = 6(0)^2 - 30(0) + 36 = 36$$

$$a(0) = 12(0) - 30 = -30$$

Initial velocity: $36\ \text{m s}^{-1}$.

Initial acceleration: $-30\ \text{m s}^{-2}$.

(c) $v = 6t^2 - 30t + 36$

$$0 = 6t^2 - 30t + 36$$
$$0 = 6(t^2 - 5t + 6)$$
$$0 = 6(t-2)(t-3)$$

The velocity is zero at $t = 2$ seconds and $t = 3$ seconds.

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(d) $a = 12t - 30$

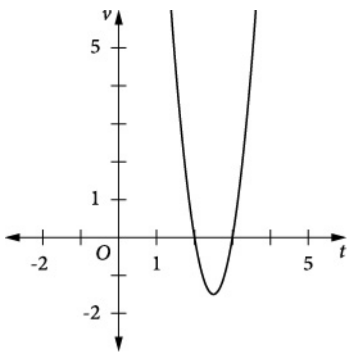
$0 = 12t - 30$
$t = 2.5 \text{ s}$

$v(2.5) = 6(2.5)^2 - 30(2.5) + 36 = -1.5 \text{ m s}^{-1}$

$x(2.5) = 2(2.5)^3 - 15(2.5)^2 + 36(2.5) = 27.5 \text{ m}$

The acceleration is zero at $t = 2.5$ s . At this time the velocity is -1.5 m s^{-1} and the acceleration is 27.5 m s^{-2} .

(e)



The velocity function is negative when $2 < t < 3$, i.e., from 2 s to 3 s.

6 **(a)** $x_1 = t + 6$

$x_2 = t^2 + 4$

They are together when $x_1 = x_2$.

$t + 6 = t^2 + 4$
$0 = t^2 - t - 2$
$0 = (t+1)(t-2)$
$t = -1,\ 2$

Since $t \geq 0$, the bodies are together at $t = 2$ seconds.

(b) $v_1 = \dfrac{dx_1}{dt} = 1$

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$$v_2 = \frac{dx_2}{dt} = 2t$$
$$2t = 1$$
$$t = 0.5$$

The bodies are travelling with the same velocity at $t = 0.5$ seconds.

8 **(a)** $v_A = \frac{dx_A}{dt} = 50 - 40t$

$$s_A = |50 - 40t|$$
$$v_B = \frac{dx_B}{dt} = 160t + 20$$
$$s_B = |160t + 20| = 160t + 20 \quad \text{since } t \geq 0.$$

$$s_A(0) = |50 - 40(0)| = 50$$
$$s_B(0) = 160(0) + 20 = 20$$

Car A speed at point O: 50 km h^{-1}

Car B speed at point O: 20 km h^{-1}

(b) $|50 - 40t| = 160t + 20$

$$50 - 40t = 160t + 20$$
$$t = 0.15$$
$$50 - 40t = -(160t + 20)$$
$$50 - 40t = -160t - 20$$
$$120t = -20 - 50$$
$$t < 0$$

$t = 0.15$ hours, or $t = 0.15 \times 60 = 9$ min.

(c) The cars are at the same point at the same time t if

$$50t - 20t^2 = 80t^2 + 20t$$
$$100t^2 - 30t = 0$$
$$10t(10t - 3) = 0$$
$$t = 0,\ 0.3$$

When $t = 0$, $x_A = x_B = 0$, so they are both at O.

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When $t = 0.3$, $x_A = 50t - 20t^2 = 50 \times 0.3 - 20 \times 0.3^2 = 13.2$

The distance from O to Q is 13.2 km.

(d) Car C is travelling at constant velocity, so $x_C = mt + c$.

When $t = 0$, A and B are at O, and C is 2 km ahead, so $x_C = +2$.

$$\begin{aligned}
x_C &= mt + c \\
2 &= m \times 0 + c \\
c &= 2 \\
x_C &= mt + 2
\end{aligned}$$

C arrives at Q (13.2 km) at $t = 0.3$

$$\begin{aligned}
x_C &= mt + 2 \\
13.2 &= m \times 0.3 + 2 \\
0.3m &= 11.2 \\
m &= \frac{11.2}{0.3} = \frac{112}{3} \\
\therefore x_c &= \frac{112t}{3} + 2
\end{aligned}$$

10 D

If its velocity is positive it must be increasing with time.

If its acceleration is negative, velocity and hence the gradient will be decreasing, in other words the graph will be getting less steep.

These correspond to graph **D**.

CHAPTER REVIEW 7

2 (a) $\lim\limits_{h \to 0} \dfrac{2x^2h + 3h}{h} = \lim\limits_{h \to 0} \left(2x^2 + 3\right)$

$$= 2x^2 + 3$$

(b) $\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}$

$$\begin{aligned}&=\lim_{h\to 0}\frac{4h+h^2}{h}\\&=\lim_{h\to 0}(4+h)\\&=4+0\\&=4\end{aligned}$$

(c) $\lim_{h\to 0}\frac{(1+h)^3-1}{h}=\lim_{h\to 0}\frac{1+3h+3h^2+h^3-1}{h}$

$$\begin{aligned}&=\lim_{h\to 0}\frac{3h+3h^2+h^3}{h}\\&=\lim_{h\to 0}\left(3+3h+h^2\right)\\&=3+0+0\\&=3\end{aligned}$$

4 **(a)** $f(x)=x^2+6x+8$

$$\begin{aligned}f(2)&=2^2+6\times 2+8\\&=24\end{aligned}$$

(b) $f'(x)=2x+6$

$$\begin{aligned}f'(2)&=2\times 2+6\\&=10\end{aligned}$$

(c) $f'(c)=2c+6$

(d) $2c+6=-2$

$$\begin{aligned}2c&=-8\\c&=-4\end{aligned}$$

6 First find the derivative of $f(x)=\left(3x^2-2x+1\right)^5$.

Use w instead of u as we will need u and v for the second part of this question.

Let $w=3x^2-2x+1$ so $\frac{dw}{dx}=6x-2$.

$f(x) = w^5$ so $\dfrac{d}{du} f(x) = 5w^4 = 5\left(3x^2 - 2x + 1\right)^4$

$$\begin{aligned} f'(x) &= \frac{d}{dw} f(x) \times \frac{dw}{dx} \\ &= (6x-2) \times 5\left(3x^2 - 2x + 1\right)^4 \\ &= 2(3x-1) \times 5\left(3x^2 - 2x + 1\right)^4 \\ &= 10(3x-1)\left(3x^2 - 2x + 1\right)^4 \end{aligned}$$

$y = \left(x^2 - 4\right)\left(3x^2 - 2x + 1\right)^5$

$$u = x^2 - 4,\ v = \left(3x^2 - 2x + 1\right)^5$$

$$\frac{du}{dx} = 2x,\ \frac{dv}{dx} = 10(3x-1)\left(3x^2 - 2x + 1\right)^4$$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(uv) \\ &= v\frac{du}{dx} + u\frac{dv}{dx} \\ &= \left(3x^2 - 2x + 1\right)^5 \times 2x + \left(x^2 - 4\right) \times 10(3x-1)\left(3x^2 - 2x + 1\right)^4 \\ &= \left(3x^2 - 2x + 1\right)^4 \left(2x\left(3x^2 - 2x + 1\right) + 10(3x-1)\left(x^2 - 4\right)\right) \\ &= \left(3x^2 - 2x + 1\right)^4 \left(6x^3 - 4x^2 + 2x + 10\left(3x^3 - x^2 - 12x + 4\right)\right) \\ &= \left(3x^2 - 2x + 1\right)^4 \left(6x^3 - 4x^2 + 2x + 30x^3 - 10x^2 - 120x + 40\right) \\ &= \left(3x^2 - 2x + 1\right)^4 \left(36x^3 - 14x^2 + 118x + 40\right) \\ &= 2\left(3x^2 - 2x + 1\right)^4 \left(18x^3 - 7x^2 + 59x + 20\right) \end{aligned}$$

8 **(a)** $\dfrac{d\left(x^4\right)}{dx} = 4x^{4-1} = 4x^3$

(b) $x(x-1) = x^2 - x$

$$\frac{d\left(x(x-1)\right)}{dx} = 2x^{2-1} - 1 = 2x^1 - 1 = 2x - 1$$

(c) $\dfrac{d\left(x^3 - 2x\right)}{dx} = 3x^{3-1} - 2 = 3x^2 - 2$

10 $y = 2x^2(4 - x)$

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$$2x^2(4-x)=0$$
$$x=0,\ x=4$$

$$y=8x^2-2x^3$$
$$\frac{dy}{dx}=16x-6x^2$$

At $x=0$

$$\frac{dy}{dx}=16\times 0-6\times 0$$
$$m_T=0$$

Equation of tangent at $(0,0)$:

$$y=0$$

At $x=4$

$$\frac{dy}{dx}=16\times 4-6\times 4^2$$
$$m_T=-32$$

Equation of tangent at $(4,0)$:

$$y-0=-32(x-4)$$
$$y-0=-32x+128$$
$$y=-32x+128$$

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12 (a) 'Initial' means $t = 0$.

$$v(0) = \frac{2 \times 0}{9 + 0^2} = 0 \text{ m s}^{-1}$$

(b) $v = \dfrac{2t}{9 + t^2}$

$$u = 2t,\ w = 9 + t^2$$

$$\frac{du}{dt} = 2,\ \frac{dw}{dt} = 2t$$

$$\begin{aligned} a = \frac{dv}{dt} &= \frac{w\frac{du}{dt} - u\frac{dw}{dt}}{v^2} \\ &= \frac{(9 + t^2) \times 2 - 2t \times 2t}{(9 + t^2)^2} \\ &= \frac{18 + 2t^2 - 4t^2}{(9 + t^2)^2} \\ &= \frac{18 - 2t^2}{(9 + t^2)^2} \end{aligned}$$

(c) $a = 0 \Rightarrow 18 - 2t^2 = 0$

$$2t^2 = 18$$
$$t^2 = 9$$
$$t = 3$$

Acceleration is zero after 3 seconds.

(d) The acceleration is initially positive, so the velocity is initially increasing. The acceleration is decreasing, and becomes negative after 3 seconds, so after 3 seconds the velocity will decrease.

Therefore, the maximum velocity occurs when $t = 3$ (after 3 seconds).

The maximum velocity is $\dfrac{1}{3}$ m s^{-1}.

$$v(3) = \frac{2 \times 3}{9 + 3^2} = \frac{6}{18} = \frac{1}{3} \text{ m s}^{-1}$$

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