2020 **BORED OF STUDIES TRIAL EXAMINATION**

9th October

Mathematics Extension 1

General instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–11)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

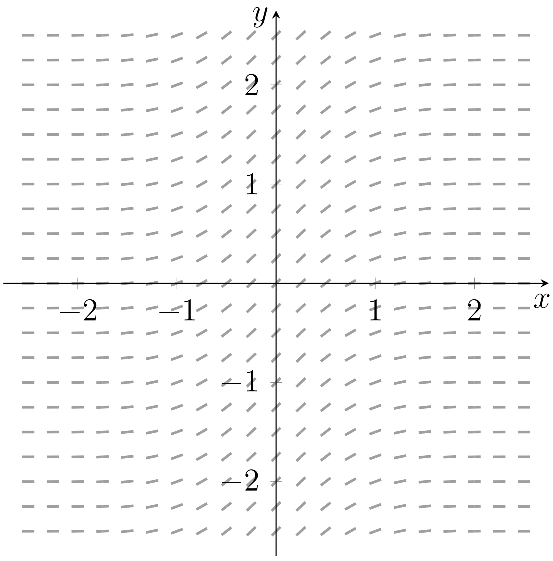
Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

---

1 Which of the following differential equations best represents the following direction field?



(A) $\dfrac{dy}{dx} = \cos x$ (B) $\dfrac{dy}{dx} = 1 - x^2$ (C) $\dfrac{dy}{dx} = e^{-x^2}$ (D) $\dfrac{dy}{dx} = \ln(x^2 + e)$

2 What is the angle between the vectors $\begin{pmatrix} 9 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$?

(A) $\sin^{-1}(0.08)$ (B) $\cos^{-1}(0.08)$ (C) $\sin^{-1}(0.8)$ (D) $\cos^{-1}(0.8)$

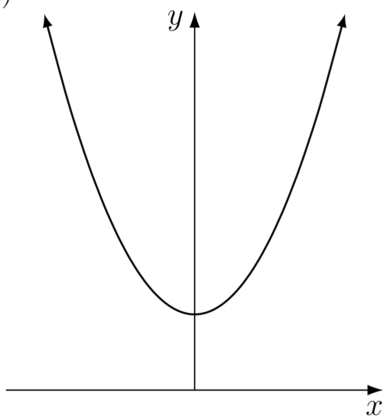
3 A monic cubic polynomial $P(x)$ has a maximum turning point located at the origin. Which quadrant does the other turning point lie in?

(A) 1st quadrant (B) 2nd quadrant (C) 3rd quadrant (D) 4th quadrant

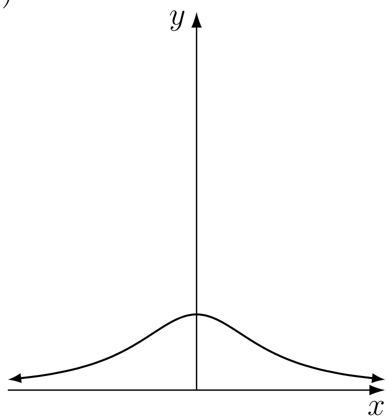
4 Which of the following graphs represents a solution to the differential equation below?

$$\frac{dy}{dx} = 1 + y^2$$

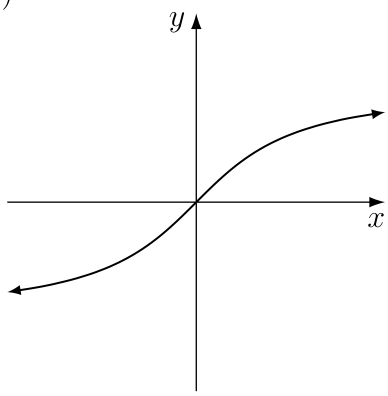
(A)



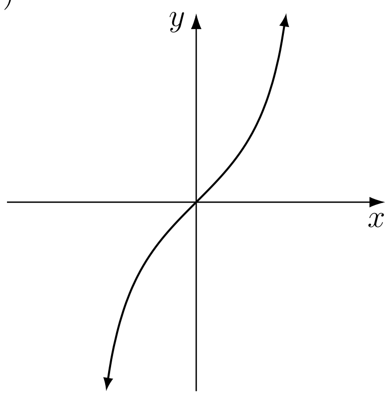
(B)



(C)



(D)



5 Let $P(x) = x^2 + ax + b$ for some non-zero real values a and b. Suppose that

$$\int \frac{dx}{P(x)} = K \tan^{-1}(Ax + B) + C$$

for some real constants A, B, C and K.

Which of the following must always be true?

(A) $a > 0$ (B) $a < 0$ (C) $b > 0$ (D) $b < 0$

6 An object is subject to two forces. One force acts in the direction of $\underset{\sim}{j}$ with a magnitude of 1 newton. The other force acts in the direction of $\sqrt{3}\underset{\sim}{i}-\underset{\sim}{j}$ with a magnitude of 4 newtons.

What is the magnitude of the total force on the object, in newtons?

(A) $\sqrt{3}$ (B) $\sqrt{13}$ (C) 3 (D) 5

7 A curve in the x-y plane is represented by the graph of $y = \dfrac{P(x)}{Q(x)}$ where $P(x)$ is a cubic polynomial and $Q(x)$ is a quadratic polynomial. $P(x)$ and $Q(x)$ have no common factors. What is the minimum number of asymptotes that this curve can have?

(A) 0 (B) 1 (C) 2 (D) 3

8 What is the solution set to $|x-3| \geq \dfrac{1}{x-1}$?

(A) $\{x < -2, x = \sqrt{2}-2, x > 1\}$ (B) $\{x > 1, x = -2, x \leq -2-\sqrt{2}\}$

(C) $\{x \leq -1, x = 2-\sqrt{2}, x \geq 2\}$ (D) $\{x < 1, x = 2, x \geq 2+\sqrt{2}\}$

9 A particle moves along a number plane at time t according to the displacement vector

$$\underset{\sim}{r} = (t\sin t)\underset{\sim}{i} - (t\cos t)\underset{\sim}{j}.$$

The particle is initially at the origin. Let θ be the acute angle at which the path of the particle first crosses the x-axis after leaving the origin. What is the value of θ?

(A) $\dfrac{\pi}{2}$ (B) π (C) $\tan^{-1}\dfrac{\pi}{2}$ (D) $\tan^{-1}\pi$

10 A student tosses a fair coin 100 times. Which of the following is the best estimate for the probability of getting either:

- 60 or more tosses showing heads; or
- 60 or more tosses showing tails.

(A) 0.05 (B) 0.32 (C) 0.68 (D) 0.95

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Evaluate $\displaystyle\int \cos^2 4x \cos^2 x \, dx$. **2**

(b) A particle is launched from the ground with speed V at an angle of projection of θ. Let g be the acceleration due to gravity. The equations of motion at time t are **3**

$$x = Vt\cos\theta, \quad y = Vt\sin\theta - \frac{gt^2}{2} \quad \text{(Do NOT prove this)}$$

Let D be the distance between the particle and its initial launch point.

Show that if D increases throughout the particle's entire trajectory then $\sin^2\theta < \dfrac{8}{9}$.

(c) Using the substitution $u = x\sqrt{x}$, show that **3**

$$\int_0^\infty \sqrt{\frac{x}{e^{x^3}}}\, dx = \frac{\sqrt{2\pi}}{3}.$$

Question 11 continues on page 6

(d) A model which approximates the spread of a virus in a population over time can be described by the differential equation

$$\frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I$$

where

- I is the number of infected people which varies over time t
- S is the number of non-infected people which varies over time t
- N is the total constant population where $N = S + I$
- β and γ are positive constants and $\beta \neq \gamma$.

(i) Show that **1**

$$\frac{dt}{dI} = \frac{1}{\beta - \gamma}\left(\frac{1}{I} + \frac{\beta}{N(\beta - \gamma) - \beta I}\right).$$

(ii) Hence, use integration to show that the general solution is given by **3**

$$I = \frac{N(\beta - \gamma)}{\beta + Ae^{-(\beta-\gamma)t}}.$$

where A is some constant.

(iii) On two separate sets of axes, sketch the graphs of the number of infected people I over time for the two separate cases when $\beta > \gamma$ and $\beta < \gamma$. Indicate any intercepts and asymptotes in terms of A, N, β and γ. **2**

(iv) Hence, explain the physical significance of the ratio $\frac{\beta}{\gamma}$ with regards to how it affects the number of infected people in the population over time. **1**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Two resistors A and B in circuit are connected in parallel such that their effective resistance R_E is given by **2**

$$\frac{1}{R_E} = \frac{1}{R_A} + \frac{1}{R_B}$$

where R_A and R_B are the resistances of A and B respectively.

Suppose that the resistance of A is increasing at a constant rate of 1 ohm per minute and the resistance of B is decreasing at a constant rate of 1 ohm per minute.

Show that the rate of change of the effective resistance (in ohms per minute) is given by

$$\frac{dR_E}{dt} = R_E\left(\frac{1}{R_A} - \frac{1}{R_B}\right).$$

(b) Suppose that X is a continuous random variable with a cumulative distribution function $F(x)$. Let $Y = F(X)$ be another random variable in terms of X. **2**

Show that the probability density function of Y represents a uniform distribution.

(c) Recall that a polyhedron is a solid composed of polygonal faces and straight edges. A cube is an example of a polyhedron. **2**

Let N be the largest number of edges of any given face of a given polyhedron.

By using the pigeonhole principle, or otherwise, prove that there exists at least two faces of the given polyhedron with the same number of edges.

(d) Show that **3**

$$\frac{\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + ... + \sin 44^\circ}{\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + ... + \cos 44^\circ} = \sqrt{2} - 1.$$

Question 12 continues on page 8

(e) Consider the distinct points A and B on the x-y plane with position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Let P be a point that lies strictly within the interval AB such that

$$\frac{AP}{BP} = \frac{1-\mu}{\mu},$$

for some $0 < \mu < 1$.

(i) Show that the position vector of the point P is represented by **1**

$$\underset{\sim}{p} = \mu\underset{\sim}{a} + (1-\mu)\underset{\sim}{b}.$$

(ii) Let two other distinct points C and D lie in the plane with position vectors $\underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively. Suppose that no three points of A, B, C and D are collinear and the intervals AB and CD have a point of intersection. **2**

Using the result in (i), show that there exists some non-zero real numbers $\lambda_1, \lambda_2, \lambda_3$ and λ_4 such that

$$\lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b} + \lambda_3\underset{\sim}{c} + \lambda_4\underset{\sim}{d} = \underset{\sim}{0},$$

where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ and $\underset{\sim}{0} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$.

(f) Suppose that the trigonometric equation $a\sin 4\theta + b\cos 4\theta = c$ has distinct solutions $\theta_1, \theta_2, \theta_3$ and θ_4 for non-zero constants a, b and c. Define **3**

$$S = \tan\theta_1 + \tan\theta_2 + \tan\theta_3 + \tan\theta_4$$
$$T = \tan\theta_2\tan\theta_3\tan\theta_4 + \tan\theta_1\tan\theta_3\tan\theta_4 + \tan\theta_1\tan\theta_2\tan\theta_4 + \tan\theta_1\tan\theta_2\tan\theta_3.$$

Show that

$$S + T = 0.$$

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Let $f(x) = \tan^{-1} x$ and $g(x) = \dfrac{x}{x^2+1}$.

(i) Show that $f'(x) > g'(x)$ for the domain $x > 0$. **2**

(ii) Hence, show that $\dfrac{\sqrt{\tan^{-1} x}}{x} > \dfrac{1}{\sqrt{x(x^2+1)}}$ for the domain $x > 0$. **2**

(iii) Using calculus to investigate the behaviour of each of the curves, sketch **4**

$$y = \frac{1}{\sqrt{x(x^2+1)}} \quad \text{and} \quad y = \frac{\sqrt{\tan^{-1} x}}{x}$$

on the same set of axes.

(iv) The region bounded by the curves $y = \dfrac{1}{\sqrt{x(x^2+1)}}$ and $y = \dfrac{\sqrt{\tan^{-1} x}}{x}$ over the domain $[1, \sqrt{3}]$ is rotated about the x-axis to form a solid of revolution. **3**

Find the volume of this solid.

(b) Let a, b, p and q be real constants and let x_n be a sequence defined by the following relation for integers $n \geq 2$.

$$x_n = a x_{n-1} + b x_{n-2},$$

where $x_1 = p$ and $x_0 = q$.

(i) Prove by mathematical induction for integers $n \geq 0$ **3**

$$\alpha^n (p - \beta q) = x_{n+1} - \beta x_n$$

where α and β are the real roots of the equation $x^2 = ax + b$ and $\alpha \neq \beta$.

(ii) Hence, show that **1**

$$x_n = \frac{(\alpha^n - \beta^n)p + (\alpha^{n-1} - \beta^{n-1})bq}{\alpha - \beta}.$$

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Suppose that X is a non-negative discrete random variable defined on some interval $[a, b]$ with expected value $E(X)$ and variance $\text{Var}(X)$.

(i) Show that for any $r > 0$ **3**

$$\text{Var}(X) \geq rP\left(|X - E(X)| \geq \sqrt{r}\right).$$

(ii) In a large country, a survey was conducted on a random sample of n people. It was found that $100\hat{p}$ percent of them own a bike. Let p be the true proportion of the country's population that own a bike. For any $\varepsilon > 0$, it can be shown that **1**

$$P(|\hat{p} - p| < \varepsilon) \geq L.$$

Use part (i) to find an appropriate value of L in terms of n, p and ε.

(iii) Deduce that **2**

$$\lim_{n \to \infty} P(|\hat{p} - p| < \varepsilon) = 1$$

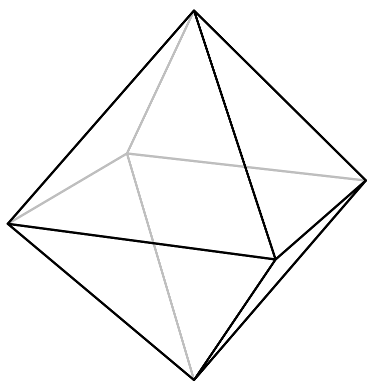
and explain the significance of this result for the survey.

(b) Consider a polyhedron which has n faces and is assigned a distinct colour to each face. Define a "colouring" as a specific arrangement of colours on the faces of the solid. Any rotations of a solid with this specific arrangement are considered the same "colouring".

(i) Find the total number of colourings for a cube. **2**

(ii) Find the total number of colourings for a regular octahedron. **2**

For reference, a diagram of a regular octahedron is provided below.
All 8 faces are equilateral triangles, and all 12 edges are equal in length.



Question 14 continues on page 11

(c) Leonardo is in St. Peter's Basilica and encounters a staircase consisting of n stairs. He is able to "step" up one stair at a time, or "lunge" up two stairs at a time. Let $\psi(n)$ be the number of ways Leonardo can ascend an n-stair staircase.

(i) Explain why $\psi(n) = \psi(n-1) + \psi(n-2)$ for $n \geq 3$. **1**

(ii) The *Fibonacci sequence* F_k is defined by the following relation **1**

$$F_k = F_{k-1} + F_{k-2},$$

where $F_0 = 0$ and $F_1 = 1$.

Show that $\psi(n) = F_{n+1}$.

(iii) Leonardo wishes to lunge exactly k times, where k is an integer satisfying $0 \leq k \leq \dfrac{n}{2}$ for even n, and $0 \leq k \leq \dfrac{n-1}{2}$ for odd n. **1**

Find the number of ways he can do this for a fixed value of k.

(iv) Deduce that **2**

$$F_{n+1} = \begin{cases} \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n}{2}}{\frac{n}{2}} & \text{if } n \text{ is even.} \\ \\ \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots + \binom{\frac{n+1}{2}}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

End of paper