EXERCISE 14.1 THE SIGN OF THE DERIVATIVE

2 $f'(x)=2$, so the gradient is constant. This means $f(x)$ must be a straight line.

The equation of a straight line is $y=mx+c$, where m is the gradient and c is the y-intercept.

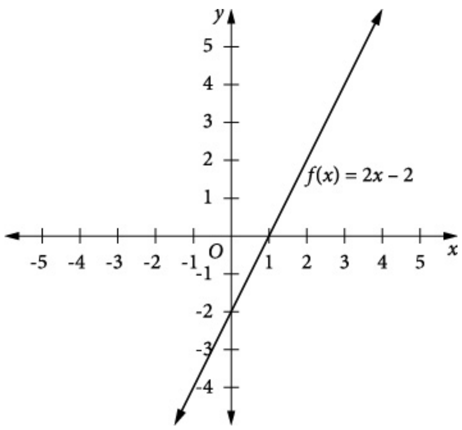
The gradient is 2, so the equation is $y=2x+c$.

$f(1)=0$ means the straight line graph passes through the point $(1,\ 0)$.

$$0=2\times1+c$$
$$c=-2$$

The equation of the line is $y=2x-2$.

$f(x)=2x-2$



4 $f(3)=3,\ f'(3)=0$ and $f(1)=5,\ f'(1)=0$ indicates the graph has gradient 0 at $(3,\ 3)$ and $(1,\ 5)$.

Where $x<1,\ f'(x)>0$ indicates the graph is increasing for $x<1$.

Where $x>1,\ f'(x)>0$ indicates the graph is increasing for $x>3$.

Where $1<x<3,\ f'(x)<0$ indicates the graph is decreasing for $1<x<3$.

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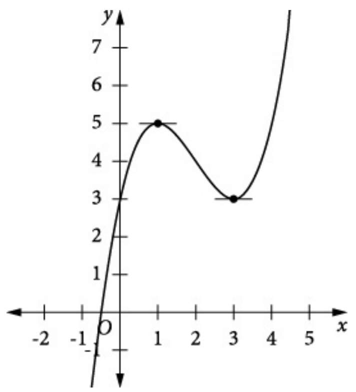

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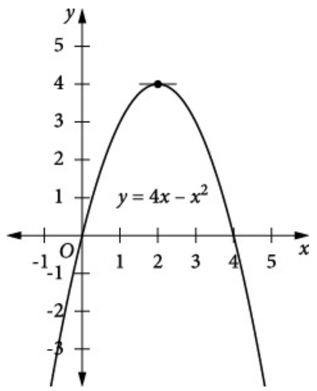

From the graph, $\frac{dy}{dx} = 0$ when $x = 2$.

To the left of $x = 2$, the gradient is positive.

To the right of $x = 2$, the gradient is negative.

8 **(a)** Differentiate $f(x)$ term by term.

$$f(x) = x^3 - 6x^2 + 9x + 2$$
$$f'(x) = 3x^2 - 12x + 9$$

(b) $f'(x) = 0$

$$3x^2 - 12x + 9 = 0$$
$$3(x^2 - 4x + 3) = 0$$
$$3(x-3)(x-1) = 0$$

$f(x)$ is increasing when $(x-3)(x-1) > 0$, i.e., when $x < 1,\ x > 3$.

(c) $f(x)$ is decreasing when $1 < x < 3$.

(d) $f(x)$ changes from increasing to decreasing at $x = 1$.

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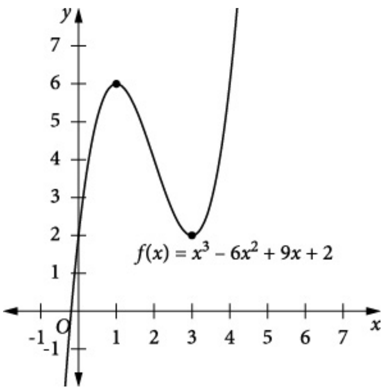
The graph of the function does not have to be drawn, but looking at it below will help students understand the significance of their answers.



10 (a) Use the product rule to differentiate the function.

$$f(x) = (x-1)^2(x+1)$$

$$\begin{aligned} f'(x) &= v\frac{du}{dx} + u\frac{dv}{dx} \\ &= (x+1)\times 2(x-1) + (x-1)^2 \times 1 \\ &= (x-1)(2x+2+x-1) \\ &= (x-1)(3x+1) \end{aligned}$$

$f(x)$ is stationary when $f'(x) = 0$.

$$(x-1)(3x+1) = 0$$

$$x = -\frac{1}{3},\ x = 1$$

(b) $f(x)$ is increasing when $(x-1)(3x+1) > 0 \Rightarrow x < -\frac{1}{3},\ x > 1$.

(c) $f(x)$ is decreasing when $-\frac{1}{3} < x < 1$.

These regions can be seen on the graph below (which does not need to be drawn).

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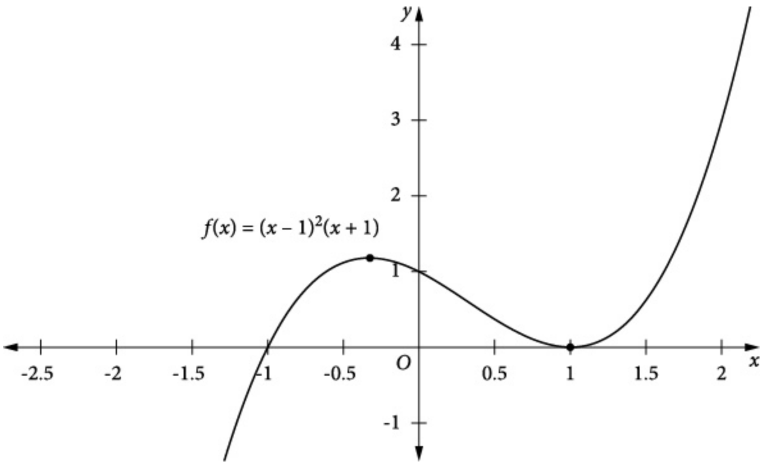


EXERCISE 14.2 THE FIRST DERIVATIVE AND TURNING POINTS

2 C

$$\begin{aligned} f'(x) &= x^2 - 5x - 6 \\ x^2 - 5x - 6 &= 0 \\ (x+1)(x-6) &= 0 \\ x &= -1,\ x = 6 \end{aligned}$$

4 $f(x) = x^3 - 9x^2 + 15x + 16$

$$f'(x) = 3x^2 - 18x + 15$$

Stationary points occur when $f'(x) = 0$.

$$\begin{aligned} 3x^2 - 18x + 15 &= 0 \\ 3(x^2 - 6x + 5) &= 0 \\ 3(x-5)(x-1) &= 0 \\ x &= 1,\ x = 5 \end{aligned}$$

$$\begin{aligned} f(1) &= 1^3 - 9\times1^2 + 15\times1 + 16 \\ &= 23 \end{aligned}$$

$$\begin{aligned} f(5) &= 5^3 - 9\times5^2 + 15\times5 + 16 \\ &= -9 \end{aligned}$$

x	0	1	3	5	6
$f'(x)$	15	0	–12	0	15

At $(1,\ 23)$, the gradient changes from positive to negative, so $(1,\ 23)$ is a maximum turning point.

At $(5,\ -9)$, the gradient changes from negative to positive, so $(5,\ -9)$ is a minimum turning point.

(a) correct

(b) incorrect

(c) correct

(d) correct

6 **(a)** $f(x)=x^3-x^2-x+1$

$f'(x)=3x^2-2x-1$

(b) Stationary points occur when $f'(x)=0$.

$$\begin{aligned} 3x^2-2x-1&=0 \\ (3x+1)(x-1)&=0 \\ x=-\frac{1}{3},\ x&=1 \end{aligned}$$

$$f\left(-\frac{1}{3}\right)=\left(-\frac{1}{3}\right)^3-\left(-\frac{1}{3}\right)^2-\left(-\frac{1}{3}\right)+1=\frac{32}{27}=1\frac{5}{27}$$

$$f(1)=1^3-1^2-1+1=0$$

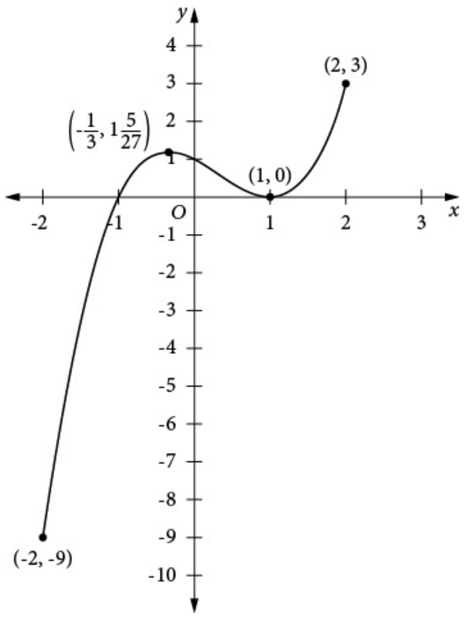
x	-1	$-\frac{1}{3}$	0	1	2
$f'(x)$	4	0	-1	0	7

At $\left(-\frac{1}{3},\ 1\frac{5}{27}\right)$, the gradient changes from positive to negative, so $\left(-\frac{1}{3},\ 1\frac{5}{27}\right)$ is a maximum turning point.

At $(1,\ 0)$, the gradient changes from negative to positive, so $(1,\ 0)$ is a minimum turning point.

(c) $f(-2)=(-2)^3-(-2)^2-(-2)+1=-9$

$f(2)=2^3-2^2-2+1=3$



8 Let $y = x(x-2)+3 = x^2 - 2x + 3$

$$\frac{dy}{dx} = 2x - 2$$

$$2x - 2 = 0$$

$$x = 1$$

$$y = 1\times(1-2)+3 = 2$$

When $x = 0$, $\frac{dy}{dx} = 2\times 0 - 2 = -2$.

When $x = 2$, $\frac{dy}{dx} = 2\times 2 - 2 = 2$.

At $(1,\ 2)$, the gradient changes from negative to positive, so the minimum value of $x(x-2)+3$ is 2.

10 $y = 2x^3 + 3x^2 - 12x + 7$

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

$$6x^2 + 6x - 12 = 0$$

$$6(x^2 + x - 2) = 0$$

$$6(x+2)(x-1) = 0$$

$$x = -2,\ x = 1$$

When $x=-2$,

$$\begin{aligned} y &= 2\times(-2)^3+3\times(-2)^2-12\times(-2)+7 \\ &= 27 \end{aligned}$$

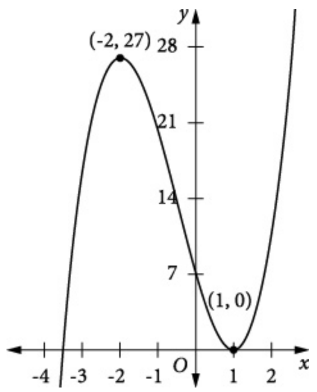
When $x=1$,

$$\begin{aligned} y &= 2\times1^3+3\times1^2-12\times1+7 \\ &= 0 \end{aligned}$$

x	-3	-2	0	1	2
$f'(x)$	24	0	-12	0	24

At $(-2,\ 27)$, the gradient changes from positive to negative, so $(-2,\ 27)$ is a maximum turning point.

At $(1,\ 0)$, the gradient changes from negative to positive, so $(1,\ 0)$ is a minimum turning point.



12 (a) Use the product rule to find the derivative.

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$$f(x) = 9x(x-2)^2$$
$$f'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= (x-2)^2 \times 9 + 9x \times 2(x-2)$$
$$= 9(x-2)(x-2+2x)$$
$$= 9(x-2)(3x-2)$$
$$f'(x) = 0$$
$$9(x-2)(3x-2) = 0$$
$$x = 2,\ x = \frac{2}{3}$$

(b)

x	0	$\frac{2}{3}$	1	2	3
$f'(x)$	36	0	–9	0	63

$$f'(x) > 0$$
$$(x-2)(3x-2) > 0$$
$$x > 2,\ x < \frac{2}{3}$$

(c) $f'(x) < 0$

$$\frac{2}{3} < x < 2$$

(d) $f(-1) = 9 \times (-1) \times ((-1) - 2)^2$

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$$= -81$$

$$f\left(\frac{2}{3}\right) = 9 \times \frac{2}{3} \times \left(\frac{2}{3} - 2\right)^2$$

$$= 10\frac{2}{3}$$

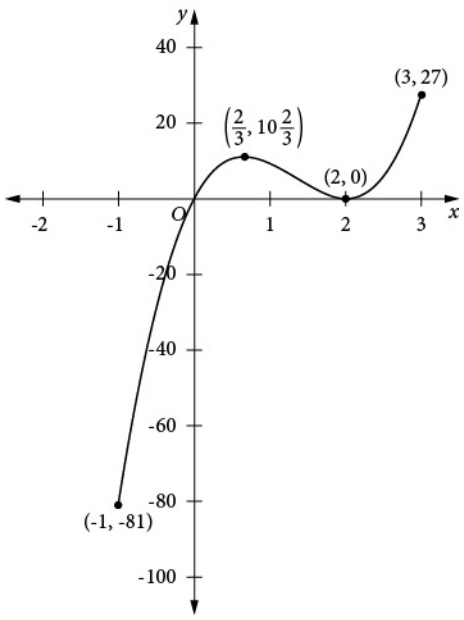
$$f(2) = 9 \times 2 \times (2-2)^2$$

$$= 0$$

$$f(3) = 9 \times 3 \times (3-2)^2$$

$$= 27$$

The range is $27 - (-81) = 108$; its greatest value is 27 and its least value is -81.



14 $y = \frac{1}{x} = x^{-1}$

$$\frac{dy}{dx} = -1 \times x^{-2}$$

$$= -\frac{1}{x^2}$$

$$\neq 0$$

Therefore $y = \frac{1}{x}$ has no turning points.

Since x^2 is always positive for all values of x^2 in the domain, $\frac{dy}{dx}=-\frac{1}{x^2}$ is always negative.

Therefore its gradient is always negative.

EXERCISE 14.3 THE SECOND DERIVATIVE AND CONCAVITY

2 D

$$f(x)=x^6+3x^3-4x+2$$
$$f'(x)=6x^5+9x^2-4$$
$$f''(x)=30x^4+18x$$

4 (a) $y=\sqrt{x}=x^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2}=-\frac{1}{2}\times\frac{1}{2}x^{-\frac{3}{2}}$$
$$=-\frac{1}{4x\sqrt{x}}$$

(b) $y=\sqrt{x-2}$

$$\frac{dy}{dx}=\frac{1}{2}(x-2)^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2}=-\frac{1}{2}\times\frac{1}{2}(x-2)^{-\frac{3}{2}}$$
$$=-\frac{1}{4(x-2)\sqrt{(x-2)}}$$

(c) Use the product rule to find the first derivative, but first find the derivative of $\sqrt{x^2+1}$ using the chain rule.

Let $u=x^2+1$ so that $\frac{du}{dx}=2x$ and $\sqrt{x^2+1}=\sqrt{u}=u^{\frac{1}{2}}$.

$$\frac{d\sqrt{x^2+1}}{dx}=\frac{du^{\frac{1}{2}}}{du}\times\frac{du}{dx}$$
$$=\frac{1}{2}u^{-\frac{1}{2}}\times 2x$$
$$=\frac{x}{\sqrt{x^2+1}}$$

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$$
\begin{aligned}
y &= x\sqrt{x^2+1} \\
\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\
&= \sqrt{x^2+1}\times 1 + x\times\frac{x}{\sqrt{x^2+1}} \\
&= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} \\
&= \frac{x^2+1+x^2}{\sqrt{x^2+1}} \\
&= \frac{2x^2+1}{\sqrt{x^2+1}}
\end{aligned}
$$

Use the quotient rule to find the second derivative.

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{\sqrt{x^2+1}\times 4x - \left(2x^2+1\right)\times\frac{x}{\sqrt{x^2+1}}}{x^2+1} \\
&= \frac{4x\sqrt{x^2+1} - \frac{x\left(2x^2+1\right)}{\sqrt{x^2+1}}}{x^2+1}\times\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} \\
&= \frac{4x\left(x^2+1\right) - x\left(2x^2+1\right)}{\left(x^2+1\right)\sqrt{x^2+1}} \\
&= \frac{4x^3+4x-2x^3-x}{\left(x^2+1\right)^{\frac{3}{2}}} \\
&= \frac{2x^3+3x}{\left(x^2+1\right)^{\frac{3}{2}}} \\
&= \frac{x\left(2x^2+3\right)}{\left(x^2+1\right)^{\frac{3}{2}}}
\end{aligned}
$$

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(d) $y=\dfrac{1}{x}=x^{-1}$

$$\frac{dy}{dx}=-1\times x^{-2}$$
$$=-\frac{1}{x^2}$$
$$\frac{d^2y}{dx^2}=-2\times -x^{-3}$$
$$=\frac{2}{x^3}$$

(e) $y=\dfrac{1}{x+1}=\left(x+1\right)^{-1}$

$$\frac{dy}{dx}=-1\times\left(x+1\right)^{-2}$$
$$=-\frac{1}{\left(x+1\right)^2}$$
$$\frac{d^2y}{dx^2}=-2\times-\left(x+1\right)^{-3}$$
$$=\frac{2}{\left(x+1\right)^3}$$

(f) $y=\dfrac{x}{x+3}$

$$\frac{dy}{dx}=\frac{\left(x+3\right)\times1-x\times1}{\left(x+3\right)^2}$$
$$=\frac{x+3-x}{\left(x+3\right)^2}$$
$$=\frac{3}{\left(x+3\right)^2}$$
$$=3\left(x+3\right)^{-2}$$
$$\frac{d^2y}{dx^2}=3\times-2\left(x+3\right)^{-3}$$
$$=-\frac{6}{\left(x+3\right)^3}$$

Alternatively,

$$y=\frac{x}{x+3}=\frac{x+3-3}{x+3}=1-\frac{3}{x+3}=1-3\left(x+3\right)^{-1}$$
$$\frac{dy}{dx}=-3\times-\left(x+3\right)^{-2}=3\left(x+3\right)^{-2}$$
$$\frac{d^2y}{dx^2}=3\times-2\left(x+3\right)^{-3}=-\frac{6}{\left(x+3\right)^3}$$

(g) Note: The denominator can be written as a power and thus the function can be expressed as a sum of powers (similar to **(h)**), making differentiation simpler.

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$$y=\frac{x^2+1}{\sqrt{x}}=\frac{x^2+1}{x^{\frac{1}{2}}}=x^{\frac{3}{2}}+x^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2}\times\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}\times-\frac{3}{2}x^{-\frac{5}{2}}$$

$$=\frac{3}{4}x^{-\frac{1}{2}}+\frac{3}{4}x^{-\frac{5}{2}}$$

Note: Always take the most negative power out as a common factor.

$$\frac{d^2y}{dx^2}=\frac{3}{4}x^{-\frac{5}{2}}\left(x^2+1\right)$$

$$=\frac{3\left(x^2+1\right)}{4x^{2\frac{1}{2}}}$$

$$=\frac{3\left(x^2+1\right)}{4x^2\sqrt{x}}$$

(h) $y=\left(x^2-1\right)\sqrt{x}=\left(x^2-1\right)x^{\frac{1}{2}}=x^{\frac{5}{2}}-x^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{5}{2}x^{\frac{3}{2}}-\frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{5}{2}\times\frac{3}{2}\times x^{\frac{1}{2}}-\frac{1}{2}\times-\frac{1}{2}x^{-\frac{3}{2}}$$

$$=\frac{15\sqrt{x}}{4}+\frac{1}{4x\sqrt{x}}$$

$$=\frac{15x^2+1}{4x\sqrt{x}}$$

(i) $y=\dfrac{\sqrt{x-1}}{x+1}$

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$$\frac{dy}{dx}=\frac{(x+1)\times\frac{1}{2}(x-1)^{-\frac{1}{2}}-\sqrt{x-1}\times 1}{(x+1)^2}$$

$$=\frac{\frac{(x+1)}{2\sqrt{x-1}}-\sqrt{x-1}}{(x+1)^2}$$

$$=\frac{(x+1)-2(x-1)}{2\sqrt{x-1}(x+1)^2}$$

$$=\frac{3-x}{2\sqrt{x-1}(x+1)^2}$$

$$\frac{d^2y}{dx^2}=\frac{2\sqrt{x-1}(x+1)^2\times -1-(3-x)\left[2\times\frac{1}{2}(x-1)^{-\frac{1}{2}}\times(x+1)^2+2\sqrt{x-1}\times 2(x+1)\right]}{4(x-1)(x+1)^4}$$

$$=\frac{-2(x-1)(x+1)-(3-x)\left[(x+1)+4(x-1)\right]}{4(x-1)^{\frac{3}{2}}(x+1)^3}$$

$$=\frac{-2(x^2-1)-(3-x)(5x-3)}{4(x-1)^{\frac{3}{2}}(x+1)^3}$$

$$=\frac{-2x^2+2-15x+9+5x^2-3x}{4(x-1)^{\frac{3}{2}}(x+1)^3}$$

$$=\frac{3x^2-18x+11}{4(x-1)^{\frac{3}{2}}(x+1)^3}$$

6 $y = 6 - 3x^2$

$$\frac{dy}{dx} = -6x$$
$$\frac{d^2y}{dx^2} = -6$$

Since $\frac{d^2y}{dx^2} < 0$ for all values of x, the curve is concave down. This can also be deduced by the fact that it is a parabola with a negative coefficient of x^2.

8 This function is only defined for $x \geq -1$.

$$y = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x+1}}$$
$$\frac{d^2y}{dx^2} = -\frac{1}{2} \times \frac{1}{2}(x+1)^{-\frac{3}{2}}$$
$$= -\frac{1}{4(x+1)^{\frac{3}{2}}}$$

The graph is concave down when $\frac{d^2y}{dx^2} < 0$

$$-\frac{1}{4(x+1)^{\frac{3}{2}}} < 0$$
$$x + 1 > 0$$
$$x > -1$$

The graph concave down for $x > -1$.

Since $\frac{d^2y}{dx^2} \neq 0$ for x in the domain, there is no point of inflexion.

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10 $y=\frac{1}{x^2}=x^{-2}$

$$\frac{dy}{dx}=-2\times x^{-3}$$
$$\frac{d^2y}{dx^2}=-2\times -3x^{-4}=\frac{6}{x^4}$$

Since x^4 is always positive, then $\frac{d^2y}{dx^2}>0$ for all x in the domain, therefore $y=\frac{1}{x^2}$ is concave up over its domain.

EXERCISE 14.4 THE SECOND DERIVATIVE AND TURNING POINTS

2 **A**

$$f(x)=x^3-3x^2+1$$
$$f'(x)=3x^2-6x$$
$$f''(x)=6x-6$$

For the curve to be concave up, $f(x)>0$.

$$6x-6>0$$
$$x>1$$

4 $8y=8+8x^2-x^4$

$$y=1+x^2-\frac{x^4}{8}$$
$$\frac{dy}{dx}=2x-\frac{4x^3}{8}=2x-\frac{x^3}{2}=\frac{x\left(4-x^2\right)}{2}$$
$$\frac{d^2y}{dx^2}=2-\frac{3x^2}{2}$$

Stationary points occur when $\frac{dy}{dx}=0$.

$$\frac{x\left(4-x^2\right)}{2}=0$$
$$x=0,\ x=\pm 2$$

At $x=0$,

$$\begin{aligned} y &= 1+0^2-\frac{0^4}{8} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2-\frac{3\times 1^2}{2} \\ &= \frac{1}{2} \\ &> 0 \end{aligned}$$

$(0,\ 1)$ is a minimum turning point.

At $x=2$

$$\begin{aligned} y &= 1+2^2-\frac{2^4}{8} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2-\frac{3\times 2^2}{2} \\ &= -4 \\ &> 0 \end{aligned}$$

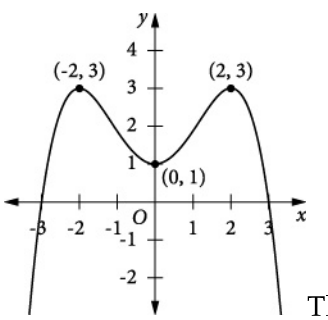
$(2,\ 3)$ is a maximum turning point.

At $x=-2$

$$\begin{aligned} y &= 1+(-2)^2-\frac{(-2)^4}{8} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2-\frac{3\times (-2)^2}{2} \\ &= -4 \\ &> 0 \end{aligned}$$

$(-2,\ 3)$ is a maximum turning point.



The greatest value of the function is the greatest maximum, which is 3.

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6 **(a)** $f(x) = x^3 - 6x^2 + 2$

$$f'(x) = 3x^2 - 12x$$
$$f''(x) = 6x - 12$$
$$6x - 12 = 0$$
$$x = 2$$

(b) $f''(x) > 0$

$$6x - 12 > 0$$
$$x > 2$$

(c) $f''(x) < 0$

$$6x - 12 < 0$$
$$x < 2$$

8 **(a)** Let $x = 0$.

$$f(x) = x^4 - x^2$$
$$f(0) = 0$$

$f(x)$ crosses the y - axis at the origin $(0,\ 0)$.

Let $f(x) = 0$.

$$x^4 - x^2 = 0$$
$$x^2(x^2 - 1) = 0$$
$$x = 0,\ x = \pm 1$$

$f(x)$ Crosses the x - axis at the origin $(0,\ 0)$, $(1,\ 0)$ and $(-1,\ 0)$.

(b) $f(x) = x^4 - x^2$

$$f'(x) = 4x^3 - 2x$$
$$f''(x) = 12x^2 - 2$$

Stationary points occur when $f'(x) = 0$

$$4x^3 - 2x = 0$$
$$2x(2x^2 - 1) = 0$$
$$x = 0,\ x = \pm\frac{1}{\sqrt{2}}$$

At $x = 0$

$$f''(x) = 12 \times 0 - 2$$
$$= -2$$
$$< 0$$

$(0,\ 0)$ is a maximum turning point.

At $x = -\frac{1}{\sqrt{2}}$

$$f(x) = \left(-\frac{1}{\sqrt{2}}\right)^4 - \left(-\frac{1}{\sqrt{2}}\right)^2$$
$$= -\frac{1}{4}$$

$$f''(x) = 12 \times \left(-\frac{1}{\sqrt{2}}\right)^2 - 2$$
$$= 4$$
$$> 0$$

$\left(-\frac{1}{\sqrt{2}},\ -\frac{1}{4}\right)$ is a minimum turning point.

At $x = \frac{1}{\sqrt{2}}$

$$f(x) = \left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{\sqrt{2}}\right)^2$$
$$= -\frac{1}{4}$$

$$f''(x) = 12 \times \left(\frac{1}{\sqrt{2}}\right)^2 - 2$$
$$= 4$$
$$> 0$$

$\left(\frac{1}{\sqrt{2}},\ -\frac{1}{4}\right)$ is a minimum turning point.

(c) $f''(x) = 0$

$$12x^2 - 2 = 0$$
$$2(6x^2 - 1) = 0$$
$$x = \pm\frac{1}{\sqrt{6}}$$

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$$f\left(-\frac{1}{\sqrt{6}}\right)=\left(-\frac{1}{\sqrt{6}}\right)^4-\left(-\frac{1}{\sqrt{6}}\right)^2$$
$$=-\frac{5}{36}$$
$$f\left(\frac{1}{\sqrt{6}}\right)=\left(\frac{1}{\sqrt{6}}\right)^4-\left(\frac{1}{\sqrt{6}}\right)^2$$
$$=-\frac{5}{36}$$

x	-1	$-\frac{1}{\sqrt{6}}$	0	$\frac{1}{\sqrt{6}}$	1
$f''(x)$	10	0	-2	0	10

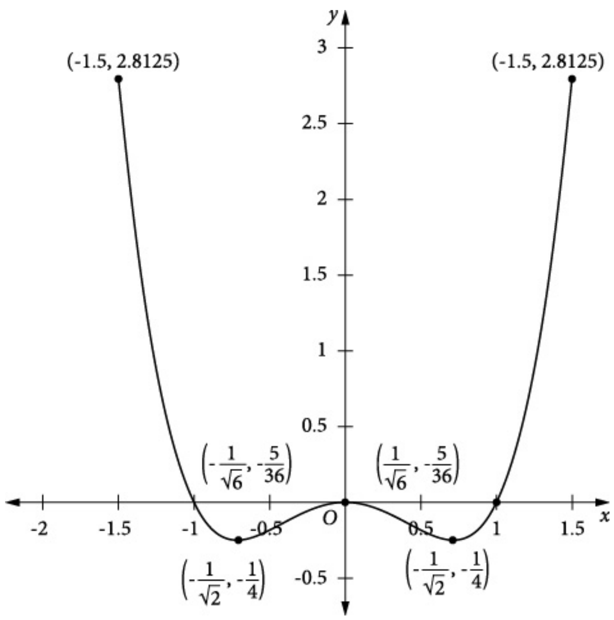
The concavity changes, therefore $\left(-\frac{1}{\sqrt{6}},\ -\frac{5}{36}\right)$ and $\left(\frac{1}{\sqrt{6}},\ -\frac{5}{36}\right)$ are points of inflections.

(d) At $x=-1.5$

$$f(x)=(-1.5)^4-(-1.5)^2$$
$$=2.8125$$

At $x=1.5$

$$f(x)=1.5^4-1.5^2$$
$$=2.8125$$



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(e) The curve is concave down when $f''(x) < 0$.

$$6x^2 - 1 < 0 \text{ or } 6x^2 < 1 \Rightarrow x^2 < \frac{1}{6}$$
$$-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$$
$$-\frac{\sqrt{6}}{6} < x < \frac{\sqrt{6}}{6}$$

10 $y = -x^3 + 3x^2 - 3x$

$$\frac{dy}{dx} = -3x^2 + 6x - 3$$
$$\frac{d^2y}{dx^2} = -6x + 6$$

Stationary points occur when $\frac{dy}{dx} = 0$.

$$-3x^2 + 6x - 3 = 0$$
$$-3(x^2 - 2x + 1) = 0$$
$$-3(x-1)^2 = 0$$
$$x = 1$$

Where $x = 1$, $y = -1^3 + 3\times1^2 - 3\times1 = -1$

$$\frac{d^2y}{dx^2} = -6\times1 + 6 = 0$$

We must investigate further.

Where $x = 0$, $\frac{d^2y}{dx^2} = -6\times0 + 6 = 6$

Where $x = 2$, $\frac{d^2y}{dx^2} = -6\times2 + 6 = -6$

Concavity changes, so $(1, -1)$ is a horizontal point of inflection.

$$\frac{dy}{dx} = -3(x-1)^2$$

Since $(x-1)^2 > 0$ for all real x except $x = 1$, then $-3(x-1)^2 < 0$.

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Hence, $\frac{dy}{dx} < 0$ for all real x except $x = 1$.

$$\begin{aligned} y &= -x^3 + 3x^2 - 3x \\ &= -x\left(x^2 - 3x + 3\right) \end{aligned}$$

Curve cuts the x-axis when $y = 0$.

$$-x\left(x^2 - 3x + 3\right) = 0$$

$$x = 0 \text{ or } x^2 - 3x + 3 = 0$$

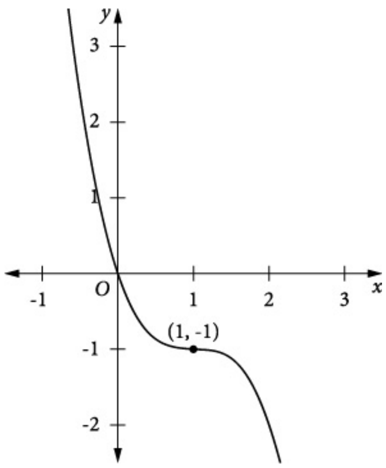
For $x^2 - 3x + 3 = 0$,

$$\begin{aligned} \Delta &= (-3)^2 - 4 \times 1 \times 3 \\ &= -3 \\ &< 0 \end{aligned}$$

$x^2 - 3x + 3 = 0$ has no real roots.

Where $x = 0,\ y = 0$

Therefore the curve only cuts the x-axis at the origin $(0,\ 0)$.



12 (a) $y = x^3 - 6x^2 + 6x$

$$\frac{dy}{dx} = 3x^2 - 12x + 6 = 3\left(x^2 - 4x + 2\right)$$

$x^2 - 4x + 2 = 0$

Use the quadratic formula or complete the square.

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$$x=\frac{4\pm\sqrt{8}}{2}=2\pm\sqrt{2}$$

Where $x=2+\sqrt{2}$,

$$\begin{aligned}y&=\left(2+\sqrt{2}\right)^3-6\left(2+\sqrt{2}\right)^2+6\left(2+\sqrt{2}\right)\\&=8+12\sqrt{2}+12+2\sqrt{2}-6\left(4+4\sqrt{2}+2\right)+12+6\sqrt{2}\\&=32+20\sqrt{2}-24-24\sqrt{2}-12\\&=-4-4\sqrt{2}\\&\approx-9.66\end{aligned}$$

Where $x=2-\sqrt{2}$,

$$\begin{aligned}y&=\left(2-\sqrt{2}\right)^3-6\left(2-\sqrt{2}\right)^2+6\left(2-\sqrt{2}\right)\\&=8-12\sqrt{2}+12-2\sqrt{2}-6\left(4-4\sqrt{2}+2\right)+12-6\sqrt{2}\\&=32-20\sqrt{2}-24+24\sqrt{2}-12\\&=-4+4\sqrt{2}\\&\approx1.66\end{aligned}$$

The two stationary points are $\left(2+\sqrt{2},-4-4\sqrt{2}\right)$, $\left(2-\sqrt{2},-4+4\sqrt{2}\right)$.

Where $x=2+\sqrt{2}$, $\dfrac{d^2y}{dx^2}=6x-12=12+12\sqrt{2}-12=12\sqrt{2}>0$.

$\left(2+\sqrt{2},-4-4\sqrt{2}\right)$ is a minimum turning point.

Where $x=2-\sqrt{2}$, $\dfrac{d^2y}{dx^2}=6x-12=12-12\sqrt{2}-12=-12\sqrt{2}<0$.

$\left(2-\sqrt{2},-4+4\sqrt{2}\right)$ is a minimum turning point.

Find the end points.

$y=x^3-6x^2+6x$

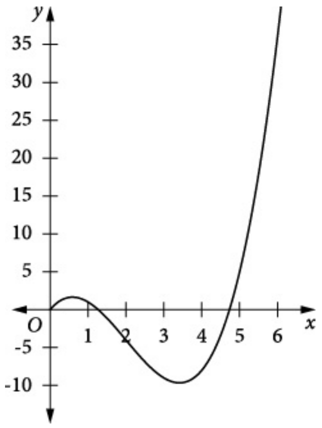
Where $x=0$, $y=0$

Where $x=6$, $y=6^3-6\times6^2+6\times6=36$

The greatest value of the function is 36 when x = 6.

The least value of the function is $-4\left(1+\sqrt{2}\right)$ when $x = 2+\sqrt{2}$.

(b)



14 $R = \dfrac{80x - x^2}{4}$

$$\begin{aligned}\frac{dR}{dx} &= \frac{80-2x}{4}\\ &= \frac{40-x}{2}\end{aligned}$$

The only stationary points occurs when $\dfrac{dR}{dx} = 0$.

$$\begin{aligned}\frac{40-x}{2} &= 0\\ x &= 40\end{aligned}$$

$$\begin{aligned}\frac{d^2R}{dx^2} &= -\frac{1}{2}\\ &< 0\end{aligned}$$

The maximum revenue occurs when $x = 40$.

40 units must be sold to maximise revenue.

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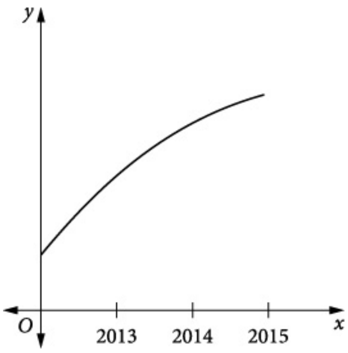
16 (a) From the table, the derivative is clearly decreasing

(b) It decreases first by 0.7, then by 0.1, so it is decreasing at a decreasing rate.

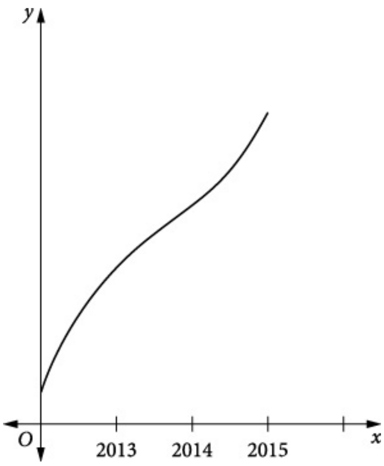
(c) If the derivative is decreasing, the second derivative is negative, and so the graph of $f(t)$ is concave down.

(d) Plot the extra point and join in a smooth curve.

$f'(t) = 3.0$



$f'(t) = 3.2$



(e) For 3.0, the derivative has not changed, or is zero, so concavity is neither up nor down. For 3.2, the derivative has increased, so the second derivative is now increasing, and has changed, so there must be a point of inflection.

(f) It is still increasing at a decreasing rate.

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EXERCISE 14.5 PROBLEM SOLVING WITH DERIVATIVES

2 B

Let the numbers be x and y.

$$\begin{aligned} x+y=12 \Rightarrow y &= 12-x \\ P &= xy \\ &= x(12-x) \\ &= 12x-x^2 \\ P' &= 12-2x \\ 12-2x &= 0 \\ x &= 6 \\ P'' &= -2<0 \end{aligned}$$

$x=6$ makes the product of the two numbers a maximum.

4 (a) From the diagram, there are the equivalent of three lengths and four widths used, making a total 120 m.

$$\begin{aligned} 4x+3y &= 120 \\ 3y &= 120-4x \\ y &= \frac{120-4x}{3} \end{aligned}$$

(b) $$\begin{aligned} A(x) &= x\left(\frac{120-4x}{3}\right) \\ &= \frac{120x-4x^2}{3} \end{aligned}$$

(c) $$A'(x)=\frac{120-8x}{3}$$

$$\begin{aligned} \frac{120-8x}{3} &= 0 \\ x &= 15 \\ A''(x) &= -\frac{8}{3}<0 \end{aligned}$$

$\therefore x=15$ m produces the maximum possible area.

$$\begin{aligned} A &= \frac{15\times(120-4\times15)}{3} \\ &= 300 \end{aligned}$$

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$\therefore 300\ \text{m}^2$ is the maximum possible area.

(d) The total length of fencing now includes the six gates $= 120 + 6 \times 3 = 138$ m.

$$\begin{aligned} 4x + 3y &= 138 \\ 3y &= 138 - 4x \\ y &= \frac{138 - 4x}{3} \end{aligned}$$

$$\begin{aligned} A(x) &= x\left(\frac{138 - 4x}{3}\right) \\ &= \frac{x(138 - 4x)}{3} \\ &= \frac{138x - 4x^2}{3} \end{aligned}$$

$$\begin{aligned} A'(x) &= \frac{138 - 8x}{3} \\ \frac{138 - 8x}{3} &= 0 \\ x &= 17.25 \\ A''(x) &= -\frac{8}{3} < 0 \end{aligned}$$

$\therefore x = 17.25$ m produces the maximum possible area.

$$\begin{aligned} A &= \frac{17.25 \times (138 - 4 \times 17.25)}{3} \\ &= 396.75 \end{aligned}$$

$\therefore 396.75\ \text{m}^2$ is the maximum possible area.

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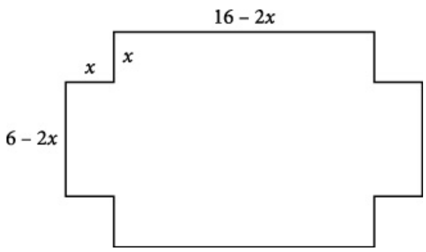
6 Let the side of the square be x. Draw a diagram (if necessary). Imagine the edges folded up.



$$\begin{aligned} V &= x(6-2x)(16-2x) \\ &= x(96-44x+4x^2) \\ &= 96x-44x^2+4x^3 \end{aligned}$$

$$\begin{aligned} V' &= 96-88x+12x^2 \\ &= 4(24-22x+3x^2) \\ &= 4(3x-4)(x-6) \end{aligned}$$

$$V' = 0 \Rightarrow x = \frac{4}{3},\ x = 6$$

$x \neq 6$ due to the original dimension of the rectangular sheet.

$$V'' = -88+6x$$

When $x = \frac{4}{3}$, $V'' = -88 + 6\times\frac{4}{3} = -80 < 0$

$\therefore x = \frac{4}{3}$ produces the maximum volume.

$$\begin{aligned} V &= \frac{4}{3}\left(6-2\times\frac{4}{3}\right)\left(16-2\times\frac{4}{3}\right) \\ &= \frac{1600}{27} \end{aligned}$$

$\therefore V = \frac{1600}{27}$ cm^3 is the maximum volume.

8 Let the sides of the square base be x, and the height of the solid be y.

$$x+x+y=20$$
$$y=20-2x$$

$$\begin{aligned} V &= x\times x\times y \\ &= x^2(20-2x) \\ &= 20x^2-2x^3 \\ V'' &= 40x-6x^2 \\ &= 2x(20-3x) \end{aligned}$$

$$40x-6x^2=0$$
$$2x(20-3x)=0$$
$$V'=0\Rightarrow x=0,\ x=\frac{20}{3}$$

Since $x\neq 0$, then $x=\frac{20}{3}$

When $x=\frac{20}{3}$, $V''=40-12x=40-12\times\frac{20}{3}=-40<0$

$\therefore x=\frac{20}{3}$ cm produces the maximum volume.

$$\begin{aligned} V &= \left(\frac{20}{3}\right)^2\times\left(20-2\times\frac{20}{3}\right) \\ &= \frac{8000}{27} \end{aligned}$$

$\therefore V=\frac{8000}{27}\ \text{cm}^3$ is the maximum volume.

10 (a) The width is x, the length is $2x$ and the height is y.

$$\begin{aligned} 2x\times x\times 2+2xy+2\times 2x\times y &= 216 \\ 4x^2+6xy &= 216 \\ 6xy &= 216-4x^2 \\ y &= \frac{216-4x^2}{6x} \\ y &= \frac{108-2x^2}{3x} \end{aligned}$$

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(b) $V = x \times 2x \times y$

$$= 2x^2 \times \frac{108 - 2x^2}{3x}$$
$$= \frac{2x\left(108 - 2x^2\right)}{3}$$
$$= \frac{216x - 4x^3}{3}$$

(c) $V' = \dfrac{216 - 12x^2}{3}$

$$\frac{216 - 12x^2}{3} = 0$$
$$12\left(18 - x^2\right) = 0$$
$$x = \pm\sqrt{18}$$

$x = -\sqrt{18}$ is not possible since x is the width.

$V'' = -8x$

When $x = \sqrt{18}$, $V'' = -8x = -8\sqrt{18} < 0$

$\therefore x = \sqrt{18}$ cm produces the maximum volume.

$$V = \frac{216 \times \sqrt{18} - 4 \times \left(\sqrt{18}\right)^3}{3}$$
$$= \frac{216\sqrt{18} - 72\sqrt{18}}{3}$$
$$= 48\sqrt{18}$$
$$= 144\sqrt{2}$$

$\therefore V = 144\sqrt{2}$ cm^3 is the maximum volume.

12 (a) Let the length of the other edge of the base be y.

$$x^2 + y^2 = 10^2$$
$$y^2 = 100 - x^2$$
$$y = \pm\sqrt{100 - x^2}$$

y is positive as it is a length, so $y = \sqrt{100 - x^2}$.

(b) $V = x \times \sqrt{100 - x^2} \times \sqrt{100 - x^2}$

$$= x\left(100 - x^2\right)$$
$$= 100x - x^3$$

(c) $V' = 100 - 3x^2$

$$100 - 3x^2 = 0$$
$$x^2 = \frac{100}{3}$$
$$x = \pm\frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$$

x is positive as it is a length, so $x = \dfrac{10\sqrt{3}}{3}$.

$$V'' = -6x$$

Where $x = \dfrac{10\sqrt{3}}{3}$, $V'' = -6 \times \dfrac{10\sqrt{3}}{3} < 0$

$\therefore x = \dfrac{10}{\sqrt{3}}$ cm produces the maximum volume.

$$V = 100 \times \frac{10\sqrt{3}}{3} - \left(\frac{10\sqrt{3}}{3}\right)^3$$
$$= \frac{1000\sqrt{3}}{3} - \frac{1000 \times 3\sqrt{3}}{27}$$
$$= \frac{3000\sqrt{3} - 1000\sqrt{3}}{9}$$
$$= \frac{2000\sqrt{3}}{9}$$

$\therefore V = \dfrac{2000\sqrt{3}}{9}$ cm^3 is the maximum volume.

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14 Let one section of the wire be x; the other section is $30-x$.

$$\begin{aligned} A &= \left(\frac{x}{4}\right)^2 + \left(\frac{30-x}{4}\right)^2 \\ &= \frac{x^2}{16} + \frac{900-60x+x^2}{16} \\ &= \frac{2x^2-60x+900}{16} \\ &= \frac{x^2-30x+450}{8} \end{aligned}$$

$$A' = \frac{2x-30}{8}$$

$$\begin{aligned} \frac{2x-30}{8} &= 0 \\ 2x &= 30 \\ x &= 15 \end{aligned}$$

$$A'' = \frac{1}{4} > 0$$

$\therefore x = 15$ cm produces the smallest area.

$$\begin{aligned} A &= \frac{15^2 - 30\times 15 + 450}{8} \\ &= \frac{225}{8} \\ &= 28\frac{1}{8} \end{aligned}$$

$\therefore A = 28\frac{1}{8}$ cm^2 is the smallest possible area.

16 Let the shorter side of the rectangle be x, then the longer side is $3x$.

Note: It would be equally valid to start with x as the side of the square, although it is simpler to use x as the shorter side of the rectangle.

The wire left for the square is $50-2(x+3x) = 50-8x$.

The side length of the square is $\dfrac{50-8x}{4} = \dfrac{25-4x}{2}$.

$$A = x \times 3x + \left(\frac{25-4x}{2}\right)^2$$
$$= 3x^2 + \frac{625-200x+16x^2}{4}$$
$$= \frac{28x^2-200x+625}{4}$$

$$A' = \frac{56x-200}{4}$$
$$= 14x-50$$

$$14x-50=0$$
$$x = \frac{25}{7}$$

$$A'' = \frac{56}{4} = 14 > 0$$

$\therefore x = \frac{25}{7}$ cm produces the smallest area.

The dimensions of the rectangle are $\frac{25}{7}$ cm or $3\frac{4}{7}$ cm and $3 \times \frac{25}{7} = \frac{75}{7}$ cm or $9\frac{5}{7}$ cm.

The side of the square is $\dfrac{25-4\times\frac{25}{7}}{2} = 5\frac{5}{14}$ cm.

18 (a) $f(x) = \frac{x^3}{3} - \frac{45x^2}{2} + 450x$

$$f'(x) = x^2 - 45x + 450$$

$$x^2 - 45x + 450 = 0$$
$$(x-30)(x-15) = 0$$
$$x = 15,\ x = 30$$

$$f''(x) = 2x - 45$$
$$f''(15) = 2\times 15 - 45 = -15 < 0$$
$$f''(30) = 2\times 30 - 15 = 45 > 0$$

$\therefore x = 30$ produces a local minimum and $x = 15$ produces a local maximum sales.

Since x is the number of thousands of dollars spent on advertising, then $\$15\ 000$ can be expected to produce a maximum number of sales.

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(b) $f\left(15\right)=\frac{15^3}{3}-\frac{45\times15^2}{2}+450\times15=2812\frac{1}{2}$ (from part **(a)**)

About 2812 sales would be expected.

20 (a) $h^2+r^2=a^2$

$$h=\sqrt{a^2-r^2}$$

$$\begin{aligned}V&=\pi r^2\times2\sqrt{a^2-r^2}\\&=2\pi r^2\sqrt{a^2-r^2}\end{aligned}$$

(b) $V'\left(r\right)=4\pi r\sqrt{a^2-r^2}+2\pi r^2\times\frac{1}{2}\times-2r\times\left(a^2-r^2\right)^{-\frac{1}{2}}$

$$\begin{aligned}&=4\pi r\sqrt{a^2-r^2}-\frac{2\pi r^3}{\sqrt{a^2-r^2}}\\&=2\pi r\left(\frac{2\left(a^2-r^2\right)-r^2}{\sqrt{a^2-r^2}}\right)\\&=2\pi r\left(\frac{2a^2-3r^2}{\sqrt{a^2-r^2}}\right)\end{aligned}$$

$$2\pi r\left(\frac{2a^2-3r^2}{\sqrt{a^2-r^2}}\right)=0$$

$r=0$ or $3r^2=2a^2$

$r\neq0$, so $r=\frac{\sqrt{2}a}{\sqrt{3}}=\frac{\sqrt{6}a}{3}$

Finding the second derivative is very complicated, so in this case we should consider the values of the derivative on either side of the stationary point.

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$$\text{If } r = \frac{a}{\sqrt{3}} < \frac{\sqrt{2}a}{\sqrt{3}},$$

$$V(r) = 2\pi r\left(\frac{2a^2 - 3r^2}{\sqrt{a^2 - r^2}}\right)$$

$$= 2\pi r\left(\frac{2a^2 - \frac{3a^2}{3}}{\sqrt{a^2 - r^2}}\right)$$

$$= 2\pi r\left(\frac{a^2}{\sqrt{a^2 - r^2}}\right) > 0$$

There is no need to substitute for r in other parts of the formula as $r > 0$ and the denominator must be positive.

$$\text{If } r = \frac{2a}{\sqrt{3}} > \frac{\sqrt{2}a}{\sqrt{3}},$$

$$V(r) = 2\pi r\left(\frac{2a^2 - 3r^2}{\sqrt{a^2 - r^2}}\right)$$

$$= 2\pi r\left(\frac{2a^2 - \frac{3 \times 4a^2}{3}}{\sqrt{a^2 - r^2}}\right)$$

$$= 2\pi r\left(\frac{-2a^2}{\sqrt{a^2 - r^2}}\right) < 0$$

$\therefore r = \dfrac{\sqrt{6}a}{3}$ gives the maximum volume.

Sample answer: In this case, the derivative is positive, and so the volume is increasing up to $\therefore r = \dfrac{\sqrt{6}a}{3}$, after which the volume is decreasing as the derivative is negative.

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EXERCISE 14.6
APPLICATIONS OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

2 $y = xe^{-0.5x}$

$$f'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\begin{aligned}\frac{dy}{dx} &= e^{-0.5x} \times 1 + x \times -0.5e^{-0.5x} \\ &= e^{-0.5x}\left(1 - \frac{x}{2}\right) \\ &= \frac{(2-x)e^{-0.5x}}{2}\end{aligned}$$

A stationary point occurs when $\frac{dy}{dx} = 0$.

$$\begin{aligned}\frac{(2-x)e^{-0.05x}}{2} &= 0 \\ 2 - x &= 0 \\ x &= 2\end{aligned}$$

$$\begin{aligned}y &= 2 \times e^{-0.5\times 2} \\ &= 2e^{-1} \\ &= \frac{2}{e}\end{aligned}$$

x	1	2	3
$\frac{dy}{dx}$	$\frac{1}{2e^{0.5}}$	0	$-\frac{1}{2e^{0.5}}$

$\therefore \left(2, \frac{2}{e}\right)$ is a maximum turning point.

(a) $e^{-0.5x} > 0$ for all values of x.

$y > 0 \Rightarrow xe^{-0.5x} > 0 \Rightarrow x > 0$

(b) $\frac{dy}{dx} > 0$

$$\frac{(2-x)e^{-0.5x}}{2} > 0$$
$$2-x > 0$$
$$x < 2$$

4 **(a)** Use the chain rule.

$$f(x) = e^{-x^2}$$
$$f'(x) = \frac{de^{-x^2}}{dx^2} \times \frac{dx^2}{dx}$$
$$= -e^{-x^2} \times 2x$$
$$= -2xe^{-x^2}$$

(b) (i) $f'(x) = 0$

$$-2xe^{-x^2} = 0$$
$$x = 0$$

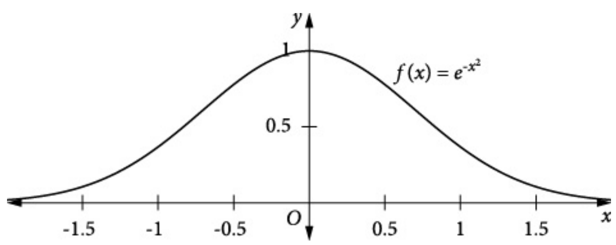
(ii) $f'(x) > 0$

$$-2xe^{-x^2} > 0$$
$$-2x > 0$$
$$x < 0$$

(iii) $f'(x) < 0$

$$-2xe^{-x^2} < 0$$
$$-2x < 0$$
$$x > 0$$

(c)



6 $y = e^t + 4e^{-t}$

$$y' = e^t - 4e^{-t}$$
$$y'' = e^t + 4e^{-t}$$

Stationary points occur when $y' = 0$.

$$e^t - 4e^{-t} = 0$$
$$\frac{e^{2t} - 4}{e^t} = 0$$
$$e^{2t} = 4$$
$$\left(e^t\right)^2 = 4$$

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Since $e^t > 0$, then $e^t = 2$.

When $e^t = 2$,

$$\begin{aligned} y &= 2 + \frac{4}{2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} y'' &= 2 + \frac{4}{2} \\ &= 4 \\ &> 0 \end{aligned}$$

$\therefore$ the minimum value is 4.

(a) correct **(b)** incorrect

(c) correct **(d)** correct

8 **(a)** $A = 2xy$

$= 2xe^{-x^2}$

Let $u = 2x,\ v = e^{-x^2}$

$$\frac{du}{dx} = 2,\ \frac{dv}{dx} = -2xe^{-0.5x}$$

$$\begin{aligned} A &= 2xe^{-x^2} \\ \frac{dA}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\ &= e^{-x^2} \times 2 + 2x \times -2xe^{-x^2} \\ &= e^{-x^2}\left(2 - 4x^2\right) \\ &= 2\left(1 - 2x^2\right)e^{-x^2} \end{aligned}$$

Let $u = 2\left(1 - 2x^2\right),\ v = e^{-x^2}$

$$\frac{du}{dx} = -8x,\ \frac{dv}{dx} = -2xe^{-x^2}$$

$$\begin{aligned}\frac{d^2A}{dx^2} &= v\frac{du}{dx}+u\frac{dv}{dx}\\ &= e^{-x^2}\times -8x+2\left(1-2x^2\right)\times -2xe^{-x^2}\\ &= -8xe^{-x^2}-4x\left(1-2x^2\right)e^{-x^2}\\ &= 4xe^{-x^2}\left(-2-1+2x^2\right)\\ &= 4xe^{-x^2}\left(2x^2-3\right)\end{aligned}$$

$$\begin{aligned}\frac{dA}{dx} &= 0\\ 2e^{-x^2}\left(1-2x^2\right) &= 0\\ 2x^2 &= 1\\ x &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

Where $x=\frac{1}{\sqrt{2}}$

$$\begin{aligned}\frac{d^2A}{dx^2} &= 4\times\frac{1}{\sqrt{2}}\times e^{-\frac{1}{2}}\left(2\times\frac{1}{2}-3\right)\\ &= -\frac{8}{\sqrt{2e}}\\ &< 0\end{aligned}$$

$\therefore x=\frac{1}{\sqrt{2}}$ gives the maximum area.

From the diagram, $x=-\frac{1}{\sqrt{2}}$ will just be a reflection in the y-axis and give the same area as where $x=\frac{1}{\sqrt{2}}$.

(b) $A=2xe^{-x^2}$

$$\begin{aligned}&= 2\times\frac{1}{\sqrt{2}}\times e^{-\frac{1}{2}}\\ &= \frac{\sqrt{2}}{\sqrt{e}}\\ &\approx 0.86 \text{ units}^2\end{aligned}$$

10 $y = e^{2x} + 4e^{-2x}$

$$\frac{dy}{dx} = 2e^{2x} - 8e^{-2x}$$

$$\frac{d^2y}{dx^2} = 4e^{2x} + 16e^{-2x}$$

Stationary points occur when $\frac{dy}{dx} = 0$.

$$\begin{aligned} 2e^{2x} - 8e^{-2x} &= 0 \\ e^{2x} &= 4e^{-2x} \\ e^{4x} &= 4 \\ 4x &= \ln 4 \\ x &= \frac{\ln 4}{4} \\ &= \frac{2\ln 2}{4} \\ &= \frac{\ln 2}{2} \\ y &= e^{2\times\frac{\ln 2}{2}} + 4e^{-2\times\frac{\ln 2}{2}} \\ &= 2 + \frac{4}{2} \\ &= 4 \end{aligned}$$

Where $x = \frac{\ln 2}{2}$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= 4e^{2\times\frac{\ln 2}{2}} + 16e^{-2\times\frac{\ln 2}{2}} \\ &= 8 + \frac{16}{2} \\ &= 16 \\ &> 0 \end{aligned}$$

$\therefore \left(\frac{\ln 2}{2},\ 4\right)$ is a minimum turning point and the minimum value is 4.

12 (a) $\log_e(\sin x)$ is defined when $\sin x > 0$, so $0 < x < \pi$.

(b) Let $y = \log_e(\sin x)$ for $0 < x < \pi$.

Use the chain rule.

$$\begin{aligned}\frac{d\log_e(\sin x)}{dx} &= \frac{d\log_e(\sin x)}{d\sin x}\times\frac{d\sin x}{dx}\\ &= \frac{1}{\sin x}\times\cos x\\ &= \frac{\cos x}{\sin x}\\ &= \cot x\end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{d\cot x}{dx} = -\text{cosec}^2 x$$

Stationary points occurs when $\frac{dy}{dx} = 0$.

$$\begin{aligned}\cot x &= 0\\ x &= \frac{\pi}{2}\\ y &= \log_e\left(\sin\frac{\pi}{2}\right) = \log_e 1 = 0\end{aligned}$$

Where $x = \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} = -\text{cosec}^2\frac{\pi}{2} = -1 < 0$$

$\therefore$ the maximum value of $\log_e(\sin x)$ is 0 when $x = \frac{\pi}{2}$.

14 (a) Where $a = 0.5$, $y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$

Where $x = 0$, $y = e^0 + e^0 = 1 + 1 = 2$

The domain will be from -4 to 4.

Where $x = \pm 4$, $y = e^2 + e^{-2} \approx 7.52$ (2 d.p.)

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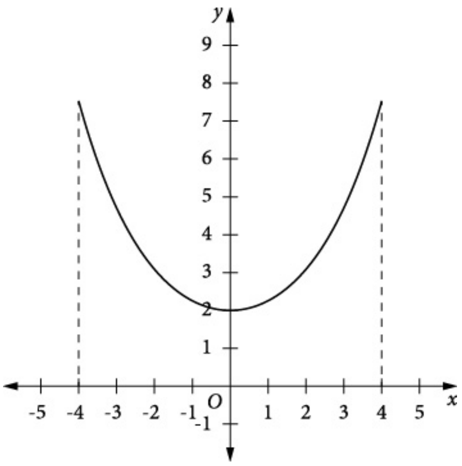

(b) $\frac{dy}{dx} = 0.5e^{0.5x} - 0.5e^{-0.5x} = 0.5\left(e^{0.5x} - e^{-0.5x}\right)$

$$e^{0.5x} - e^{-0.5x} = 0$$
$$e^{-0.5x}\left(e^{0.5x} - 1\right) = 0$$
$$e^{0.5x} = 1$$
$$x = 0$$

$$\frac{d^2y}{dx^2} = 0.5\left(0.5e^{0.5x} + 0.5e^{-0.5x}\right) > 0$$

This value is a minimum.

The least height of the function is 2 units, so the sag is $7.52 - 2 = 5.52$ units (2 d.p.).

(c) At the right end, $x = 4$,

$$\tan\theta = \frac{dy}{dx} = 0.5\left(e^{0.5x} + e^{-0.5x}\right) = 0.5\left(e^{2} + e^{-2}\right)$$
$$\theta \approx 74°35'$$

The curve is symmetrical so the angle will be the same at the other end.

The angle of inclination at the ends is $74°35'$.

EXERCISE 14.7 FURTHER APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

2 $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

At $x = \frac{\pi}{4}$,

$$m_T = \sec^2 \frac{\pi}{4} = 2$$

$$y = \tan \frac{\pi}{4} = 1$$

Equation of tangent:

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y - 1 = 2x - \frac{\pi}{2}$$

$$4x - 2y + 2 - \pi = 0$$

4 Use the chain rule.

$$y = e^{\sin x}$$

$$\frac{dy}{dx} = \frac{de^{\sin x}}{d\sin x} \times \frac{d\sin x}{dx}$$

$$= e^{\sin x} \times \cos x$$

$$= \cos x e^{\sin x}$$

Where $x = 0$,

$$m_T = \cos x e^{\sin x} = \cos 0 e^{\sin 0} = 1 \times e^0 = 1$$

$$m_N = -1$$

$$y = e^{\sin 0} = 1$$

Equation of normal:

$$y - 1 = -1(x - 0)$$

$$y - 1 = -x$$

$$y = 1 - x$$

$$x + y - 1 = 0$$

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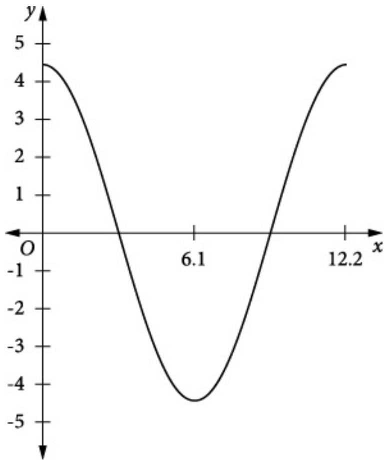
6 (a) The amplitude is the distance from the median to a maximum or minimum.

It is half the difference between maximum or minimum $=\frac{9}{2}=4.5$ m.

(b) One period is the difference between successive high tides = $6.1\times 2=12.2$ hours.

$$T=\frac{2\pi}{n}$$
$$n=\frac{2\pi}{T}$$
$$=\frac{2\pi}{12.2}$$
$$=\frac{10\pi}{61}$$

(c) $y=4.5\cos\frac{10\pi}{61}t$



(d) Low tide occurs at $t=6.1$, one hour after will be $t=7.1$.

$$y=4.5\cos\frac{10\pi}{61}\times 7.1$$
$$y=-3.916...$$
$$y=-3.92 \text{ m } (2 \text{ d.p.})$$

Depth of water is $0.5+(4.5-3.92)=1.08$ m

8 $y=\cot x$

$$\frac{dy}{dx}=-\operatorname{cosec}^2 x$$

At $x = \frac{\pi}{4}$,

$$\begin{aligned} m_T &= -\operatorname{cosec}^2 x \\ &= -2 \end{aligned}$$

$$m_N = \frac{1}{2}$$

Equation of normal:

$$\begin{aligned} y - 1 &= \frac{1}{2}\left(x - \frac{\pi}{4}\right) \\ 2y - 2 &= x - \frac{\pi}{4} \\ x - 2y + 2 - \frac{\pi}{4} &= 0 \\ 4x - 8y + 8 - \pi &= 0 \end{aligned}$$

10 $y = 3\cos 4x$

$$\frac{dy}{dx} = -12\sin 4x$$
$$\frac{d^2y}{dx^2} = -48\cos 4x$$

$$\begin{aligned} LHS &= \frac{d^2y}{dx^2} + 16y \\ &= -48\cos 4x + 16(3\cos 4x) \\ &= -48\cos 4x + 48\cos 4x \\ &= 0 \\ &= RHS \end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} + 16y = 0$$

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EXERCISE 14.8 USING DERIVATIVES IN MOTION IN A STRAIGHT LINE

2 C

$$x = 4t^3 - 3t^2 + 5t - 1$$

$$v = \frac{dx}{dt} = 12t^2 - 6t + 5$$

$$a = \frac{d^2x}{dt^2} = 24t - 6$$

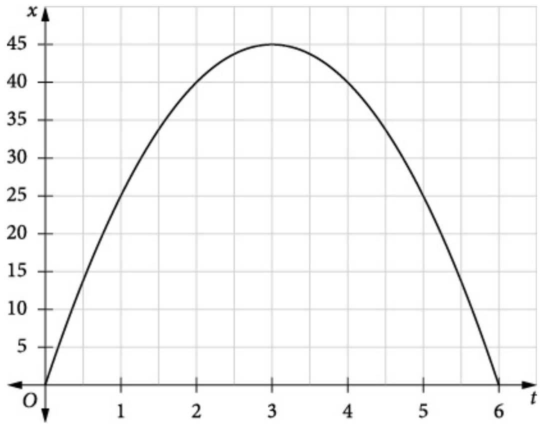
4 (a) $x = 30t - 5t^2$

$= 5t(6-t)$

This is an inverted parabola with horizontal axis intercepts at $t = 0$ and $t = 6$.

The maximum will be at the mid-point of $t = 0$ and $t = 6$, which is $t = 3$.

Where $t = 3$, $x = 5t(6-t) = 5\times3\times3 = 45$



(b) $v = \frac{dx}{dt} = 30 - 10t$

(c) Where $t = 0$, $v = 30 - 10(0) = 30$

The initial velocity is 30 m s^{-1}.

(d) The turning point on the graph is $(3, 45)$.

The particle reaches its greatest height of 45 m when $t = 3$ s.

(e) From the graph, the particle returns to the ground after another 3 seconds.

Algebraically, it is the second time the height is zero, when $t = 6$, which is another 3 seconds.

(f) $t = 6$

$$\begin{aligned}\frac{dx}{dt} &= 30 - 10(6)\\ &= 30 - 60\\ &= -30\end{aligned}$$

Since speed is always positive, its speed is 30 m s^{-1}.

(g) $v = \frac{dx}{dt} = 30 - 10t$

$$a = \frac{d^2x}{dt^2} = -10$$

The particle has a uniform acceleration of $-10\,\text{ms}^{-2}$.

6 (a) $t = 1$

$$\begin{aligned}v &= 20 + (2t-1)e^{-0.5t}\\ &= 20 + \left(2\times1-1\right)e^{-0.5\times1}\\ &= 20.61\text{ km h}^{-1} \quad \text{(2 d.p.)}\end{aligned}$$

(b) Use the product rule.

$$v = 20 + (2t-1)e^{-0.5t}$$
$$v' = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^{-0.5t}\times 2 + \left(2t-1\right)\times -0.5e^{-0.5t}$$
$$= e^{-0.5t}\left(2 - 0.5\left(2t-1\right)\right)e^{-0.5t}$$
$$= e^{-0.5t}\left(2 - t + 0.5\right)e^{-0.5t}$$
$$= \left(2.5 - t\right)e^{-0.5t}$$
$$0 = \left(2.5 - t\right)e^{-0.5t}$$
$$t = 2.5$$

Maximum velocity occurs after 2.5 hours

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CHAPTER REVIEW 14

2 **(a)** $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12$$
$$f'(x) = 0$$
$$6x^2 + 6x - 12 = 0$$
$$6(x^2 + x - 2) = 0$$
$$6(x+2)(x-1) = 0$$
$$x = -2,\ x = 1$$

(b) $f'(x) > 0$

$$6(x+2)(x-1) > 0$$
$$x < -2,\ x > 1$$

(c) $f'(x) < 0$

$$6(x+2)(x-1) < 0$$
$$-2 < x < 1$$

4 **(a)** $y = 3x^3 - 2x^2$

$$\frac{dy}{dx} = 9x^2 - 4x$$
$$\frac{d^2y}{dx^2} = 18x - 4$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$9x^2 - 4x = 0$$
$$x(9x - 4) = 0$$
$$x = 0,\ x = \frac{4}{9}$$

At $x = 0$, $y = 0$

$$\frac{d^2y}{dx^2} = 18 \times 0 - 4 = -4 < 0$$

$\therefore (0, 0)$ is a maximum turning point.

At $x = \frac{4}{9}$, $y = 3 \times \left(\frac{4}{9}\right)^3 - 2 \times \left(\frac{4}{9}\right)^2 = -\frac{32}{243}$

$$\frac{d^2y}{dx^2} = 18 \times \frac{4}{9} - 4 = 4 > 0$$

$\therefore \left(\frac{4}{9}, -\frac{32}{243}\right)$ is a minimum turning point.

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(b) $y = x^3 - 3x^2 - 9x$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$3x^2 - 6x - 9 = 0$$
$$3(x^2 - 2x - 3) = 0$$
$$3(x+1)(x-3) = 0$$
$$x = -1, x = 3$$

At $x = -1$, $y = (-1)^3 - 3\times(-1)^2 - 9\times(-1) = 5$

$$\frac{d^2y}{dx^2} = 6\times(-1) - 6 = -12 < 0$$

$\therefore (-1, 5)$ is a maximum turning point.

At $x = 3$, $y = 3^3 - 3\times 3^2 - 9\times 3 = -27$

$$\frac{d^2y}{dx^2} = 6\times 3 - 6 = 12 > 0$$

$\therefore (3, -27)$ is a minimum turning point.

6 Let the side of the square be x and the dimensions of the rectangle be $y \times 2y$.

$$4x + 6y = 8$$
$$y = \frac{8-4x}{6}$$
$$y = \frac{4-2x}{3}$$

$$\begin{aligned}A &= x^2+2y^2\\&= x^2+2\times\left(\frac{4-2x}{3}\right)^2\\&= x^2+\frac{2(4-2x)^2}{9}\end{aligned}$$

$$\begin{aligned}A' &= 2x+\frac{2\times2\times-2(4-2x)}{9}\\&= 2x+\frac{-8(4-2x)}{9}\\&= \frac{18x-32+16x}{9}\\&= \frac{34x-32}{9}\end{aligned}$$

$$\frac{34x-32}{9}=0$$

$$x=\frac{32}{34}=\frac{16}{17}$$

$$A''=\frac{34}{9}>0$$

$\therefore\ x=\dfrac{16}{17}$ gives the least area.

The length of the square part of the wire is $4\times\dfrac{16}{17}=3\dfrac{13}{17}$ cm.

The length of the rectangular part of the wire is $8-4\times\dfrac{16}{17}=4\dfrac{4}{17}$ cm.

8 **(a)** $f(x)=\dfrac{x^3}{3}-4x+3$

$$f'(x)=x^2-4$$
$$f''(x)=2x$$

$$f'(x)=0$$
$$x^2-4=0$$
$$x=\pm2$$

(b) $f'(x)<0$

$x^2 - 4 < 0$
$-2 < x < 2$

(c) Consider the values of x where the derivative is zero.

Where $x = -2$:

$$f(-2) = \frac{(-2)^3}{3} - 4(-2) + 3 = 8\frac{1}{3}$$
$$f''(-2) = 2 \times -2 = -4 < 0$$

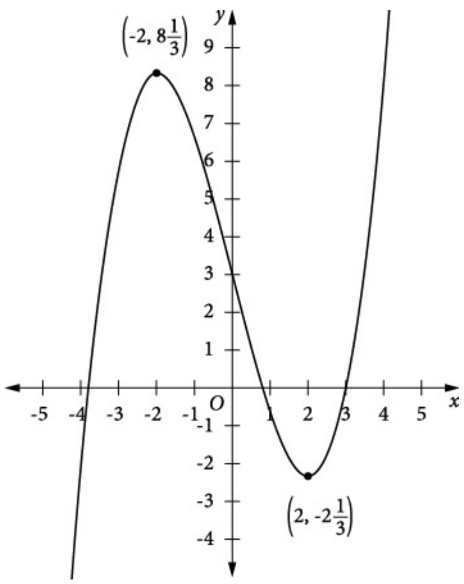
$\therefore \left(-2, 8\frac{1}{3}\right)$ is a maximum turning point.

Where $x = 2$:

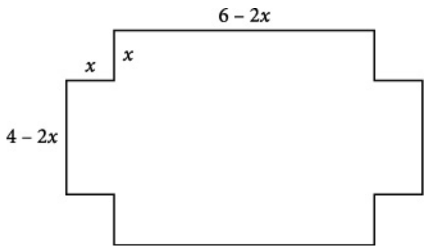
$$f(2) = \frac{2^3}{3} - 4 \times 2 + 3 = -2\frac{1}{3}$$
$$f''(2) = 2 \times 2 = 4 > 0$$

$\therefore \left(2, -2\frac{1}{3}\right)$ is a minimum turning point.

(d) Note that the y-intercept is 3.



10 Let the side of the square be x cm. Draw a diagram and imagine the sides folded up.



$$
\begin{aligned}
V &= x(6-2x)(4-2x) \\
&= x\left(24-20x+4x^2\right) \\
&= 24x-20x^2+4x^3
\end{aligned}
$$

$$V' = 24-40x+12x^2$$

$$V' = -40+24x$$

$$
\begin{aligned}
V' &= 0 \\
24-40x+12x^2 &= 0 \\
4\left(3x^2-10x+6\right) &= 0 \\
x &= \frac{10 \pm \sqrt{(-10)^2-4\times 3\times 6}}{2\times 3} \\
x &= \frac{10 \pm \sqrt{28}}{6} \\
x &= \frac{5 \pm \sqrt{7}}{3}
\end{aligned}
$$

Where $x = \dfrac{5+\sqrt{7}}{3} > \dfrac{5+\sqrt{4}}{3} > 2,$ the shape cannot be made as x is longer than the half the width of the sheet, and is not possible.

Where $x = \dfrac{5+\sqrt{7}}{3}$,

$$V'' = -40+24\times\left(\frac{5-\sqrt{7}}{3}\right) = -8\sqrt{7} < 0$$

$x = \dfrac{5-\sqrt{7}}{3}$ gives the maximum volume.

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12 x-intercepts occur where $y = x^3(3-x) = 0$.

$x = 0$ and $x = 3$.

The x-intercepts are $(0,\ 0)$ and $(3,\ 0)$.

The y-intercepts is $(0,\ 0)$.

$$\begin{aligned} y &= x^3(3-x) = 3x^3 - x^4 \\ y' &= 9x^2 - 4x^3 \\ &= x^2(9-4x) \\ y'' &= 18x - 12x^2 = 6x(3-2x) \end{aligned}$$

Stationary points occur when $y' = 0$.

$$x^2(9-4x) = 0$$
$$x = 0,\ x = 2\frac{1}{4}$$

Where $x = 0$, $y' = 0$, $y'' = 0$.

We must investigate further.

Where $x = -1$, $y' = (-1)^2(9 - 4\times -1) > 0$

Where $x = 1$, $y' = 1^2 \times (9 - 4\times 1) > 0$

The concavity does not change, so $(0,\ 0)$ is a horizontal point of inflexion.

Where $x = 2\frac{1}{4}$, $y = \left(2\frac{1}{4}\right)^3 \times \left(3 - 2\frac{1}{4}\right) = 8\frac{139}{256}$

$$y'' = 6 \times 2\frac{1}{4} \times \left(3 - 2\times 2\frac{1}{4}\right) < 0$$

$\therefore \left(2\frac{1}{4}, 8\frac{139}{256}\right)$ is a maximum turning point.

$$\begin{aligned} y'' &= 0 \\ 6x(3-2x) &= 0 \\ x &= 0,\ x = 1\frac{1}{2} \end{aligned}$$

Where $x = 1\frac{1}{2}$, $y = \left(1\frac{1}{2}\right)^3 \times \left(3 - 1\frac{1}{2}\right) = 5\frac{1}{16}$

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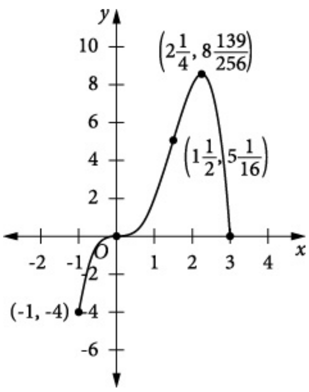
x	1	$1\frac{1}{2}$	2
y''	6	0	-12

Concavity changes, therefore $\left(1\frac{1}{2}, 5\frac{1}{16}\right)$ is a point of inflexion.

Where $x=-1$, $y=(-1)^3\times(3+1)=-4$.

Where $x=3$, $y=3^3\times(3-3)=0$.



14 $C=v+\dfrac{3600}{v}$

$C'=1-\dfrac{3600}{v^2}$

$C''=\dfrac{7200}{v^3}$

$$C'=0$$

$$1-\frac{3600}{v^2}=0$$

$$\frac{3600}{v^2}=1$$

$$v^2=3600$$

$$v=+60$$

The average speed cannot be negative.

When $v=-60$,

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$$C'' = \frac{7200}{60^3} > 0$$

$\therefore v = -60$ minimises the overhead cost.

The average delivery speed to minimise the overhead cost is 60 km/h.

16 $\theta = \theta_0 e^{-kt}$

$$\begin{aligned}\frac{d\theta}{dt} &= \theta_0 \times -ke^{-kt}\\ &= -k \times \theta_0 e^{-kt}\\ &= -k\theta\end{aligned}$$

18 $-\frac{\pi}{4} < x < \frac{3\pi}{4}$

$$-\frac{\pi}{2} < 2x < \frac{3\pi}{2}$$

$$\begin{aligned}y &= 3\sec 2x\\ &= 3(\cos 2x)^{-1}\\ \frac{dy}{dx} &= -3(\cos 2x)^{-2} \times (-2\sin 2x)\\ &= \frac{6\sin 2x}{\cos^2 2x}\\ &= 6\sec 2x \tan 2x\\ &= \frac{6\sin 2x}{\cos^2 2x}\\ \frac{d^2y}{dx^2} &= \frac{\cos^2 2x \times 12\cos 2x + 4\sin 2x \cos 2x \times 6\sin 2x}{\cos^4 2x}\\ &= \frac{12\cos^2 2x + 24\sin^2 2x}{\cos^3 2x}\\ &= \frac{12 + 12\sin^2 2x}{\cos^3 2x}\end{aligned}$$

Stationary points occur when $\frac{dy}{dx} = 0$.

$$y = 3\sec 2x$$

$$\frac{dy}{dx} = 3 \times 2\sec 2x \tan 2x = 6\sec 2x \tan 2x$$

$\sec 2x \neq 0$, so $\dfrac{dy}{dx} = 0$ where $\tan 2x = 0$.

$$-\frac{\pi}{4} < x < \frac{3\pi}{4}$$
$$-\frac{\pi}{2} < 2x < \frac{3\pi}{2}$$

$$\tan 2x = 0 \Rightarrow 2x = 0,\ \pi \Rightarrow x = 0,\ \frac{\pi}{2}$$

Where $x = 0$, $y = 3\sec 0 = 3$

Finding the second derivative is more complicated than considering the behaviour of the derivative on either side of the stationary points. We don't need to evaluate the derivatives. All we need to know is whether they are positive or negative.

Where $x = -\dfrac{\pi}{6}$, $\dfrac{dy}{dx} = 6\sec 2x \tan 2x = 6\sec\left(-\dfrac{\pi}{3}\right)\tan\left(-\dfrac{\pi}{3}\right) < 0$

Where $x = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = 6\sec 2x \tan 2x = 6\sec\left(\dfrac{\pi}{3}\right)\tan\left(\dfrac{\pi}{3}\right) > 0$

The slope changes from negative to positive, so $(0,\ 3)$ is a minimum turning point.

Where $x = \dfrac{\pi}{2}$, $y = 3\sec\left(2\times\dfrac{\pi}{2}\right) = -3$.

Where $x = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = 6\sec 2x \tan 2x = 6\sec\left(\dfrac{2\pi}{3}\right)\tan\left(\dfrac{2\pi}{3}\right) = (-)\times(-) > 0$

Where $x = \dfrac{2\pi}{3}$, $\dfrac{dy}{dx} = 6\sec 2x \tan 2x = 6\sec\left(\dfrac{4\pi}{3}\right)\tan\left(\dfrac{4\pi}{3}\right) = (-)\times(+) < 0$

The slope changes from negative to positive, so $\left(\dfrac{\pi}{2},\ -3\right)$ is a maximum turning point.

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