EXERCISE 20.1 CONTINUOUS PROBABILITY DISTRIBUTIONS

2 (a) $P(X \le 60) = \int_{50}^{60} \frac{15\ 000(x-50)}{x^4}\,dx$

$$= \int_{50}^{60} \left(15\ 000x^{-3} - 750\ 000x^{-4}\right) dx$$

$$= \left[\frac{15\ 000}{-2x^2} - \frac{750\ 000}{-3x^3}\right]_{50}^{60}$$

$$= \left[-\frac{7500}{x^2} + \frac{250\ 000}{x^3}\right]_{50}^{60}$$

$$= \left(-\frac{7500}{60^2} + \frac{250\ 000}{60^3}\right) - \left(-\frac{7500}{50^2} + \frac{250\ 000}{50^3}\right)$$

$$= \left(-\frac{25}{27}\right) - (-1)$$

$$= \frac{2}{27}$$

(b) $P(X \le 75) = \int_{50}^{75} \frac{15\ 000(x-50)}{x^4}\,dx$

$$= \int_{50}^{75} \left(15\ 000x^{-3} - 750\ 000x^{-4}\right) dx$$

$$= \left[\frac{15\ 000}{-2x^2} - \frac{750\ 000}{-3x^3}\right]_{50}^{75}$$

$$= \left[-\frac{7500}{x^2} + \frac{250\ 000}{x^3}\right]_{50}^{75}$$

$$= \left(-\frac{7500}{75^2} + \frac{250\ 000}{75^3}\right) - \left(-\frac{7500}{50^2} + \frac{250\ 000}{50^3}\right)$$

$$= \left(-\frac{20}{27}\right) - (-1)$$

$$= \frac{7}{27}$$

(c) $P(X \ge 65) = \int_{65}^{\infty} \frac{15\ 000(x-50)}{x^4}\,dx$

$$= \int_{65}^{\infty} \left(15\ 000x^{-3} - 750\ 000x^{-4}\right) dx$$

$$= \left[\frac{15\ 000}{-2x^2} - \frac{750\ 000}{-3x^3}\right]_{65}^{\infty}$$

$$= \left[-\frac{7500}{x^2} + \frac{250\ 000}{x^3}\right]_{65}^{\infty}$$

$$= 0 - \left(-\frac{7500}{65^2} + \frac{250\ 000}{65^3}\right)$$

$$= 0 - \left(-\frac{1990}{2197}\right)$$

$$= \frac{1990}{2197}$$

(d) $P(X \ge 99) = \int_{99}^{\infty} \frac{15\ 000(x-50)}{x^4}\,dx$

$$= \int_{99}^{\infty} \left(15\ 000x^{-3} - 750\ 000x^{-4}\right) dx$$

$$= \left[\frac{15\ 000}{-2x^2} - \frac{750\ 000}{-3x^3}\right]_{99}^{\infty}$$

$$= \left[-\frac{7500}{x^2} + \frac{250\ 000}{x^3}\right]_{99}^{\infty}$$

$$= 0 - \left(-\frac{7500}{99^2} + \frac{250\ 000}{99^3}\right)$$

$$= 0 - \left(-\frac{492\ 500}{970\ 299}\right)$$

$$= \frac{492\ 500}{970\ 299}$$

4 $\mu = \int_{50}^{\infty} x \frac{15\ 000(x-50)}{x^4} dx$

$$
\begin{aligned}
&= \int_{50}^{\infty} \frac{15\ 000(x-50)}{x^3} dx \\
&= \int_{50}^{\infty} \left(15\ 000x^{-2} - 750\ 000x^{-3}\right) dx \\
&= \left[\frac{15\ 000}{-x} - \frac{750\ 000}{-2x^2}\right]_{50}^{\infty} \\
&= \left[-\frac{15\ 000}{x} + \frac{375\ 000}{x^2}\right]_{50}^{\infty} \\
&= 0 - \left(-\frac{15\ 000}{50} + \frac{375\ 000}{50^2}\right) \\
&= 0 - (-150) \\
&= 150
\end{aligned}
$$

6 $0.5 = \int_{50}^{a} \frac{15\ 000(x-50)}{x^4} dx$

$$
\begin{aligned}
0.5 &= \int_{50}^{a} \left(15\ 000x^{-3} - 750\ 000x^{-4}\right) dx \\
0.5 &= \left[\frac{15\ 000}{-2x^2} - \frac{750\ 000}{-3x^3}\right]_{50}^{a} \\
0.5 &= \left[-\frac{7500}{x^2} + \frac{250\ 000}{x^3}\right]_{50}^{a} \\
0.5 &= \left(-\frac{7500}{a^2} + \frac{250\ 000}{a^3}\right) - \left(-\frac{7500}{50^2} + \frac{250\ 000}{50^3}\right) \\
0.5 &= -\frac{7500}{a^2} + \frac{250\ 000}{a^3} + 1 \\
0.5 &= \frac{7500}{a^2} - \frac{250\ 000}{a^3}
\end{aligned}
$$

Get rid of fractions by multiplying by $2a^3$.

$$
\begin{aligned}
a^3 &= 15\ 000a - 500\ 000 \\
a^3 - 15\ 000a + 500\ 000 &= 0
\end{aligned}
$$

This looks quite a difficult cubic to factorise, but it helps if we examine it carefully. We should also realise that to be the median, a must be greater than 50.

We must use the remainder theorem and consider factors of $500\ 000$.

The last two terms of the cubic are both divisible by 5000, so if the left side is to be zero, the first term must also be divisible by 5000.

For a^3 to be divisible by 5000, a must be divisible by 50.

We must ignore $a = 50$ since it would make the integral
$\int_{50}^{a} \frac{15\ 000(x-50)}{x^4} dx = \int_{50}^{50} \frac{15\ 000(x-50)}{x^4} dx = 0$.

We therefore also ignore any negative solutions to this equation.

Try $a = 100$. $a^3 - 15\ 000a + 500\ 000 = 1\ 000\ 000 - 1\ 500\ 000 + 500\ 000 = 0$

The median is 100.

To make things easier, once we realise that a must be divisible by 50, we can let $b = 50a$.

Our equation becomes $125\ 000b^3 - 750\ 000b + 500\ 000 = 0$, and after dividing each term by $125\ 000$, becomes $b^3 - 6b + 4 = 0$.

Much simpler, and we can also work out that b cannot be odd, and can only be ± 2, ± 4. and for the reasons above, we only try positive values of b.

Try $b = 2$. $b^3 - 6b + 4 = 8 - 12 + 4 = 0$

We can now easily factorise $b^3 - 6b + 4$.

$b^3 - 6b + 4 = (b-2)(b^2 + 2b - 2) = 0$

This gives additional solutions $b = -1 \pm \sqrt{3}$. They are both less than 1, and correspond to values of a less than 50, and so must be ruled out. They will make the integral work, but the resulting value of a cannot be the median.

8 **(a)** $P(1.5 \le X \le 2.5) = \int_{1.5}^{2.5} \frac{2x}{25} dx$

$$= \left[\frac{x^2}{25}\right]_{1.5}^{2.5}$$
$$= \frac{2.5^2}{25} - \frac{1.5^2}{25}$$
$$= \frac{1}{4} - \frac{9}{100}$$
$$= \frac{4}{25}$$
$$= 0.16$$

(b) $P(2 \le X \le 4.5) = \int_{2}^{4.5} \frac{2x}{25} dx$

$$= \left[\frac{x^2}{25}\right]_{2}^{4.5}$$
$$= \frac{4.5^2}{25} - \frac{2^2}{25}$$
$$= \frac{81}{100} - \frac{4}{25}$$
$$= \frac{13}{20}$$
$$= 0.65$$

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(c) $P\left(1.75 \le X \le 3.15\right) = \int_{1.75}^{3.15} \frac{2x}{25} dx$

$$\begin{aligned}
&= \left[\frac{x^2}{25}\right]_{1.75}^{3.15} \\
&= \frac{3.15^2}{25} - \frac{1.75^2}{25} \\
&= \frac{3969}{10000} - \frac{49}{400} \\
&= \frac{343}{1250} \\
&= 0.2744
\end{aligned}$$

10 (a) $\displaystyle\int_0^6 k dx = 1$

$$\begin{aligned}
\left[kx\right]_0^6 &= 1 \\
6k &= 1 \\
k &= \frac{1}{6}
\end{aligned}$$

(b) $P\left(X \le 3\right) = \int_0^3 \frac{1}{6} dx$

$$\begin{aligned}
&= \left[\frac{x}{6}\right]_0^3 \\
&= \frac{3}{6} - 0 \\
&= \frac{1}{2} \\
&= 0.5
\end{aligned}$$

(c) $P\left(X \le 5\right) = \int_0^5 \frac{1}{6} dx$

$$\begin{aligned}
&= \left[\frac{x}{6}\right]_0^5 \\
&= \frac{5}{6} - 0 \\
&= \frac{5}{6}
\end{aligned}$$

12 (a) $\mu = \int_0^9 \frac{s}{486}\left(81 - s^2\right) ds$

$$\begin{aligned}
&= \int_0^9 \left(\frac{s}{6} - \frac{s^3}{486}\right) ds \\
&= \left[\frac{s^2}{12} - \frac{s^4}{1944}\right]_0^9 \\
&= \left(\frac{9^2}{12} - \frac{9^4}{1944}\right) - 0 \\
&= \frac{27}{8} \\
&= 3.375
\end{aligned}$$

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(b) $\sigma^2 = E\left(s^2\right) - \mu^2$

$$= \int_0^9 \frac{s^2}{486}\left(81 - s^2\right) ds - \left(\frac{27}{8}\right)^2$$
$$= \int_0^9 \left(\frac{s^2}{6} - \frac{s^4}{486}\right) ds - \frac{729}{64}$$
$$= \left[\frac{s^3}{18} - \frac{s^5}{2430}\right]_0^9 - \frac{729}{64}$$
$$= \left(\frac{9^3}{18} - \frac{9^5}{2430}\right) - 0 - \frac{729}{64}$$
$$= \frac{81}{5} - \frac{729}{64}$$
$$= \frac{1539}{320}$$

$$\sigma = \sqrt{\frac{1539}{320}}$$
$$\approx 2.19$$

(c) $0.5 = \int_0^a \frac{1}{486}\left(81 - s^2\right) ds$

$$0.5 = \int_0^a \left(\frac{1}{6} - \frac{s^2}{486}\right) ds$$
$$0.5 = \left[\frac{s}{6} - \frac{s^3}{1458}\right]_0^a$$
$$0.5 = \left(\frac{a}{6} - \frac{a^3}{1458}\right) - 0$$
$$729 = 243a - a^3$$
$$0 = a^3 - 243a + 729$$

This equation can be simplified by noting that a^3 must be divisible by 243. This means

Use trial and error to find $a \approx 3.126$

14 (a) $\int_{\frac{1}{4}}^{\frac{5}{4}} k\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2 dt = 1$

$$k\int_{\frac{1}{4}}^{\frac{5}{4}} \left(t-\frac{1}{4}\right)\left(\frac{25}{16}-\frac{5}{2}t+t^2\right)dt = 1$$

$$k\int_{\frac{1}{4}}^{\frac{5}{4}} \left(\frac{25}{16}t-\frac{5}{2}t^2+t^3-\frac{25}{64}+\frac{5}{8}t-\frac{1}{4}t^2\right)dt = 1$$

$$k\int_{\frac{1}{4}}^{\frac{5}{4}} \left(\frac{35}{16}t-\frac{11}{4}t^2+t^3-\frac{25}{64}\right)dt = 1$$

$$k\left[\frac{35}{32}t^2-\frac{11}{12}t^3+\frac{1}{4}t^4-\frac{25}{64}t\right]_{\frac{1}{4}}^{\frac{5}{4}} = 1$$

$$\left[\left(\frac{35}{32}\times\left(\frac{5}{4}\right)^2-\frac{11}{12}\times\left(\frac{5}{4}\right)^3+\frac{1}{4}\times\left(\frac{5}{4}\right)^4-\frac{25}{64}\times\frac{5}{4}\right)-\right.$$
$$\left.\left(\frac{35}{32}\times\left(\frac{1}{4}\right)^2-\frac{11}{12}\times\left(\frac{1}{4}\right)^3+\frac{1}{4}\times\left(\frac{1}{4}\right)^4-\frac{25}{64}\times\frac{1}{4}\right)\right] = 1$$

$$k\left(\frac{125}{3072}-\left(-\frac{131}{3072}\right)\right) = 1$$

$$\frac{257}{3072}k = 1$$

$$k = \frac{3072}{257} = 12$$

(b) $\mu = \int_{\frac{1}{4}}^{\frac{5}{4}} 12t\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2 dt$

$$= 12\int_{\frac{1}{4}}^{\frac{5}{4}} t\left(t-\frac{1}{4}\right)\left(\frac{25}{16}-\frac{5}{2}t+t^2\right)dt$$

$$= 12\int_{\frac{1}{4}}^{\frac{5}{4}} \left(\frac{35}{16}t^2-\frac{11}{4}t^3+t^4-\frac{25}{64}t\right)dt$$

$$= 12\left[\frac{35}{48}t^3-\frac{11}{16}t^4+\frac{1}{5}t^5-\frac{25}{128}t^2\right]_{\frac{1}{4}}^{\frac{5}{4}}$$

$$=12\left[\left(\frac{35}{48}\times\left(\frac{5}{4}\right)^3-\frac{11}{16}\times\left(\frac{5}{4}\right)^4+\frac{1}{5}\times\left(\frac{5}{4}\right)^5-\frac{25}{128}\times\left(\frac{5}{4}\right)^2\right)\right.$$
$$\left.-\left(\frac{35}{48}\times\left(\frac{1}{4}\right)^3-\frac{11}{16}\times\left(\frac{1}{4}\right)^4+\frac{1}{5}\times\left(\frac{1}{4}\right)^5-\frac{25}{128}\times\left(\frac{1}{4}\right)^2\right)\right]$$

$$=12\left(\frac{625}{12288}-\left(-\frac{203}{61440}\right)\right)$$
$$=12\times\frac{13}{240}$$
$$=\frac{13}{20}$$
$$=0.65$$

(c) Solve $\int_{1/4}^{a}12\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2dt=\frac{1}{2}$ for a.

$$\int_{\frac{1}{4}}^{a}12\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2dt=\frac{1}{2}$$
$$12\int_{\frac{1}{4}}^{a}\left(\frac{35}{16}t-\frac{11}{4}t^2+t^3-\frac{25}{64}\right)dt=\frac{1}{2}$$
$$12\left[\frac{35}{32}t^2-\frac{11}{12}t^3+\frac{1}{4}t^4-\frac{25}{64}t\right]_{\frac{1}{4}}^{a}=\frac{1}{2}$$

$$0.5=12\left[\left(\frac{35}{32}\times a^2-\frac{11}{12}\times a^3+\frac{1}{4}\times a^4-\frac{25}{64}\times a\right)-\left(\frac{35}{32}\times\left(\frac{1}{4}\right)^2-\frac{11}{12}\times\left(\frac{1}{4}\right)^3+\frac{1}{4}\times\left(\frac{1}{4}\right)^4-\frac{25}{64}\times\frac{1}{4}\right)\right]$$

$$12\left(\frac{3360a^2-2816a^3+768a^4-1200a}{3072}-\left(-\frac{131}{3072}\right)\right)=\frac{1}{2}$$
$$3360a^2-2816a^3+768a^4-1200a+131=128$$
$$768a^4-2816a^3+3360a^2-1200a+3=0$$

Using trial and error leads to $a\approx 0.636$

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(d) $\int_{\frac{1}{4}}^{1} 12\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2 dt = 12\int_{\frac{1}{4}}^{1}\left(\frac{35}{16}t-\frac{11}{4}t^2+t^3-\frac{25}{64}\right)dt$

$$=12\left[\frac{35}{32}t^2-\frac{11}{12}t^3+\frac{1}{4}t^4-\frac{25}{64}t\right]_{\frac{1}{4}}^{1}$$

$$=12\left[\left(\frac{35}{32}\times1^2-\frac{11}{12}\times1^3+\frac{1}{4}\times1^4-\frac{25}{64}\times1\right)-\left(\frac{35}{32}\times\left(\frac{1}{4}\right)^2-\frac{11}{12}\times\left(\frac{1}{4}\right)^3+\frac{1}{4}\times\left(\frac{1}{4}\right)^4-\frac{25}{64}\times\frac{1}{4}\right)\right]$$

$$=12\left(\frac{7}{192}-\left(-\frac{131}{3072}\right)\right)$$

$$=\frac{243}{256}$$

$$\approx 0.949$$

(e) $\int_{\frac{2}{5}}^{\frac{4}{5}} 12\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2 dt = 12\int_{\frac{2}{5}}^{\frac{4}{5}}\left(\frac{35}{16}t-\frac{11}{4}t^2+t^3-\frac{25}{64}\right)dt$

$$=12\left[\frac{35}{32}t^2-\frac{11}{12}t^3+\frac{1}{4}t^4-\frac{25}{64}t\right]_{\frac{2}{5}}^{\frac{4}{5}}$$

$$=12\left(\frac{35}{32}\times\left(\frac{4}{5}\right)^2-\frac{11}{12}\times\left(\frac{4}{5}\right)^3+\frac{1}{4}\times\left(\frac{4}{5}\right)^4-\frac{25}{64}\times\left(\frac{4}{5}\right)\right)$$
$$-12\left(\frac{35}{32}\times\left(\frac{2}{5}\right)^2-\frac{11}{12}\times\left(\frac{2}{5}\right)^3+\frac{1}{4}\times\left(\frac{2}{5}\right)^4-\frac{25}{64}\times\left(\frac{2}{5}\right)\right)$$

$$=12\left(\frac{617}{30000}-\left(-\frac{2011}{60000}\right)\right)$$

$$=\frac{649}{1000}$$

$$=0.649$$

(f) $P\left(X \geq \frac{3}{4}\right) = \int_{\frac{3}{4}}^{\frac{5}{4}} 12\left(t-\frac{1}{4}\right)\left(\frac{5}{4}-t\right)^2 dt$

$$= 12\int_{\frac{3}{4}}^{\frac{5}{4}} \left(\frac{35}{16}t - \frac{11}{4}t^2 + t^3 - \frac{25}{64}\right)dt$$

$$= 12\left[\frac{35}{32}t^2 - \frac{11}{12}t^3 + \frac{1}{4}t^4 - \frac{25}{64}t\right]_{\frac{3}{4}}^{\frac{5}{4}}$$

$$= 12\left(\frac{35}{32}\times\left(\frac{5}{4}\right)^2 - \frac{11}{12}\times\left(\frac{5}{4}\right)^3 + \frac{1}{4}\times\left(\frac{5}{4}\right)^4 - \frac{25}{64}\times\left(\frac{5}{4}\right)\right)$$

$$-12\left(\frac{35}{32}\times\left(\frac{3}{4}\right)^2 - \frac{11}{12}\times\left(\frac{3}{4}\right)^3 + \frac{1}{4}\times\left(\frac{3}{4}\right)^4 - \frac{25}{64}\times\left(\frac{3}{4}\right)\right)$$

$$= 12\left(\frac{125}{3072} - \left(-\frac{15}{1024}\right)\right)$$

$$= \frac{5}{16}$$

16 (a) By inspection, the graph of $f(x)$ is on or above the x–axis, hence $f(x) \geq 0$.

Check the area under the curve by calculating the area of ΔAPO.

$$A_{\Delta APO} = \frac{1}{2}\times 1\times 2 = 1$$

$$\int_0^1 f(x)dx = A_{\Delta APO}$$

$$= \frac{1}{2}\times 1\times 2$$

$$= 1$$

$f(x)$ is a probability density function because it satisfies the two properties $f(x) \geq 0$ on R and $\int_{-\infty}^{\infty} f(x)dx = 1$

(b) Determine the equations of the line segments OP and AP.

OP passes through the points $(0,\ 0)$ and $\left(\frac{2}{3},\ 2\right)$, hence the gradient is:

$$m_{OP} = \frac{2-0}{\frac{2}{3}-0} = 3$$

The y–intercept, $c = 0$.

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The equation of the line segment OP is $y = 3x$.

AP passes through the points $\left(\frac{2}{3},\ 2\right)$, and $(1,\ 0)$, hence the gradient is

$$m_{AP} = \frac{2-0}{\frac{2}{3}-1} = -6$$

AP passes through $(1,\ 0)$.

$$\begin{aligned} y - 0 &= -6(x-1) \\ y &= -6x + 6 \end{aligned}$$

The equation of line segment AP is $y = -6x + 6$.

For all other values of x, that is $x < 0$ and $x > 1$, $f(x) = 0$

Define $f(x)$ as a piecewise function.

$$f(x) = \begin{cases} 3x, & 0 \le x \le \frac{2}{3} \\ -6x+6, & \frac{2}{3} \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(c) Make two triangles by constructing the altitude PQ of ΔOPA, where Q lies on the x-axis with coordinates $\left(\frac{2}{3},\ 0\right)$.

$P\left(X < \frac{2}{3}\right)$ is the area under the left triangle, which has base $\frac{2}{3}$ and height 2.

$$P\left(X < \frac{2}{3}\right) = \frac{1}{2} \times \frac{2}{3} \times 2 = \frac{2}{3}$$

$P\left(X > \frac{2}{3}\right)$ is the area under the right triangle, which has base $\frac{1}{3}$ and height 2.

$$P\left(X > \frac{2}{3}\right) = \frac{1}{2} \times \frac{1}{3} \times 2 = \frac{1}{3}$$

Note that $P\left(X < \frac{2}{3}\right) + P\left(X > \frac{2}{3}\right) = 1$.

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(d) $P\left(X<\frac{2}{3}\right)=\int_0^{\frac{2}{3}} f(x)dx$

$$=\int_0^{\frac{2}{3}} 3xdx$$

$$=\left[\frac{3x^2}{2}\right]_0^{\frac{2}{3}}$$

$$=\frac{3}{2}\times\left(\frac{2}{3}\right)^2-0$$

$$=\frac{2}{3}$$

$$P\left(X>\frac{2}{3}\right)=\int_{\frac{2}{3}}^1 f(x)dx$$

$$=\int_{\frac{2}{3}}^1(-6x+6)dx$$

$$=\left[-3x^2+6x\right]_{\frac{2}{3}}^1$$

$$=(-3+6)-\left(-\frac{4}{3}+4\right)$$

$$=\frac{1}{3}$$

Regardless of the method used (triangles or integration), the answers are the same.

(e) $P\left(\frac{1}{2}<X<\frac{5}{6}\right)=\int_{\frac{1}{2}}^{\frac{5}{6}} f(x)dx$

$$=\int_{\frac{1}{2}}^{\frac{2}{3}} 3xdx+\int_{\frac{2}{3}}^{\frac{5}{6}}(-6x+6)dx$$

$$=\left[\frac{3x^2}{2}\right]_{\frac{1}{2}}^{\frac{2}{3}}+\left[-\frac{6x^2}{2}+6x\right]_{\frac{2}{3}}^{\frac{5}{6}}$$

$$=\left(\frac{2}{3}-\frac{3}{8}\right)+\left[\left(-\frac{25}{12}+5\right)-\left(-\frac{4}{3}+4\right)\right]$$

$$=\frac{7}{24}+\frac{35}{12}-\frac{8}{3}$$

$$=\frac{13}{24}$$

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EXERCISE 20.2 THE NORMAL DISTRIBUTION

2 **(a)** $m = 10$

$\sigma = \sqrt{4} = 2$

The middle 68% describes the values within one standard deviation of the mean, i.e. $\mu \pm \sigma$.

10 ± 2 gives the range 8 to 12.

(b) The middle 95% describes the values within two standard deviations of the mean, i.e. $\mu \pm 2\sigma$.

10 ± 4 gives the range 6 to 14.

(c) The middle 99.7% describes the values within three standard deviations of the mean, i.e. $\mu \pm 3\sigma$.

10 ± 6 gives the range 4 to 16.

4 C

The mean is $\dfrac{14+26}{2} = 20$

The range 14 to 26 is two standard deviations above and below the mean.

$20 + 2\sigma = 26 \Rightarrow 2\sigma = 6 \Rightarrow \sigma = 3$

So the distribution is $N(20,\ 3)$.

6 Since 95% of the scores are within two standard deviations of the mean, 2.5% are above the mean plus 2 standard deviations.

The minimum score to receive a prize is $85 + 8 = 93$.

8 Since $X \sim N\left(510,\ 25\right)$, a mass less than 500 g will be more than two standard deviations below the mean. Since 95% of masses are less than two standard deviations away from the mean, 5% of masses will be more than two standard deviations away from the mean, of which half, or 2.5%, will be more than two standard deviations below the mean.

This can be expressed algebraically as below.

$$\begin{aligned} P\left(X < 500\right) &< \frac{1}{2} \times \left(1 - 0.95\right) \\ &< 0.025 \\ &< 2.5\% \end{aligned}$$

2.5%.of packets will have a mass less than 500 g.

10 **(a)** Draw a table and count the numbers in each interval. It helps that the heights are listed in order, so the last interval will be $1.95\text{–}<2.0$.

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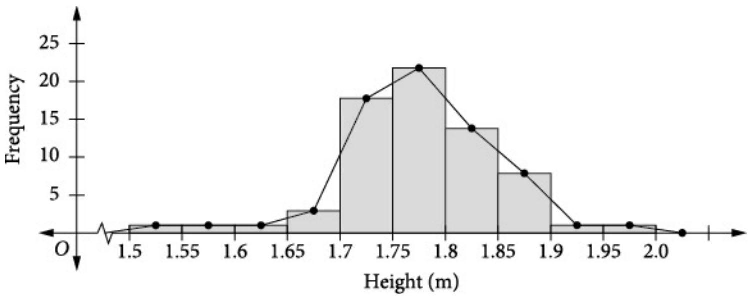
Class interval	Frequency
$1.5-<1.55$	1
$1.55-<1.6$	1
$1.6-<1.65$	1
$1.65-<1.7$	3
$1.7-<1.75$	18
$1.75-<1.8$	22
$1.8-<1.85$	14
$1.85-<1.0$	8
$1.9-<1.95$	1
$1.95-<2.0$	1

Draw the resulting histogram.



(b) The histogram displays a distinct bell shape although it is not perfectly symmetrical (skewed positively). However, it would seem reasonable to at least entertain the notion that the sample follows a normal distribution.

(c) Add the results and divide by 70. The mean is 1.765 m.

Use your calculator to find the standard deviation, which is 0.074 m.

(d) (i) $\mu+\sigma=1.765+0.074=1.839$

$\mu-\sigma=1.765-0.074=1.691$

This corresponds to heights greater than 1.69 m and less than 1.84 m, or from 1.7 m to 1.83 m inclusive.

53 of the results lie between these values.

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$\frac{53}{70} \times 100\% = 75.7\%$ (correct to 1 decimal place)

75.7% of the data lie within one standard deviation of the mean.

(ii) $\mu + 2\sigma = 1.765 + 0.148 = 1.913$

$\mu - 2\sigma = 1.765 - 0.148 = 1.617$

This corresponds to heights greater than 1.61 m and less than 1.92 m, or from 1.62 m to 1.91 m inclusive.

66 of the results lie between these values.

$\frac{66}{70} \times 100\% = 94.3\%$ (correct to 1 decimal place)

94.3% of the data lie within two standard deviations of the mean

(iii) $\mu + 3\sigma = 1.765 + 0.222 = 1.987$

$\mu - 3\sigma = 1.765 - 0.222 = 1.543$

This corresponds to heights greater than 1.54 m and less than 1.99 m, or from 1.55 m to 1.98 m inclusive.

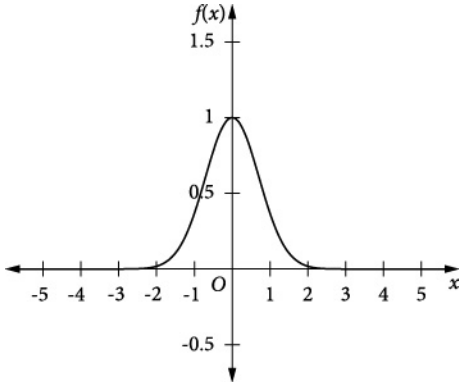
All except one of the results (1.54) lies between these values.

$\frac{69}{70} \times 100\% = 98.6\%$ (correct to 1 decimal place)

100% of the data lies within three standard deviations of the mean.

(e) The expected percentages are 68%, 95% and 99.5% and the actual percentages are 75.7%, 94.3% and 98.6%. One result corresponds to $\frac{1}{70} \times 100\% = 1.4\%$. The result for two standard deviations is as close as possible, and the result for three standard deviations is one measurement less than expected, and since the measurement in question is 0.003 m more than three standard deviations from the mean, this result is reasonable. The result for one standard deviation is 7.7% too high, corresponding to $7.7\% \div 1.4\% = 5.5$ measurements. We would expect $68\% \times 70 = 47.6$ or 48 measurements in this category and we got 53. This could be normal variation, but confirms the conclusion from the histogram that the middle section of this data may be slightly skewed from the normal. Taking the three results together, this data is consistent with an underlying normal distribution.

12 (a)



(b) $\int_{-4}^{4} e^{-x^2}\,dx \approx 1.772$

14 (a) $f(x) = \dfrac{1}{3\sqrt{2\pi}} e^{-\frac{(x-10)^2}{18}}$

x	1	4	7	10	13	16	19
$f(x)$	0.0015	0.0180	0.0807	0.1330	0.0807	0.0180	0.0015

$$\begin{aligned}\int_{1}^{19} \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-10)^2}{18}}\,dx &= \frac{3}{2}\times\big(0.0015 + 2\times 0.0180 + 2\times 0.0807 + 2\times 0.1330 + 2\times 0.0807 + 2\times 0.0180 + 0.0015 \\ &= 0.9957\end{aligned}$$

(b) $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

x	–3	–2	–1	0	1	2	3
$f(x)$	0.0044	0.0540	0.2420	0.3989	0.2420	0.0540	0.0044

$$\begin{aligned}\int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx &= \frac{1}{2}\times\big(0.0044 + 2\times 0.0540 + 2\times 0.2420 + 2\times 0.3989 + 2\times 0.2420 + 2\times 0.0540 + 0.0044\big) \\ &= 0.9953\end{aligned}$$

(c) Both are very close to the answer 0.9973 of **13(b)**. The two answers are identical when rounded to two decimal places.

EXERCISE 20.3

2 (a) $\mu = 40$

$\sigma^2 = 4$

$\sigma = 2$

$$z = \frac{x-\mu}{\sigma} = \frac{42-40}{2} = 1$$

(b) $\mu = 75$

$\sigma^2 = 25$

$\sigma = 5$

$$z = \frac{x-\mu}{\sigma} = \frac{82-75}{5} = \frac{7}{5} = 1.4$$

(c) The z value for the second test was 1.4 which was higher than the z value for the first test.

Therefore, he did better on the second test.

4 (a) $P(X > a) = 1 - P(X < a)$

$$= 1 - 0.214$$
$$= 0.786$$

(b) $P(X > b) = 1 - P(X < b)$

$$= 1 - 0.496$$
$$= 0.504$$

(c) $P(a < X < b) = 1 - \left[P(X < a) + P(X > b)\right]$

$$= 1 - (0.214 + 0.504)$$
$$= 1 - 0.718$$
$$= 0.282$$

6 (a) $P(Z < -1.5) = P(Z > -1.5) = 1 - P(-1.5 < Z < 1.5) = 1 - 0.8864 = 0.1136$

$$P(Z < -1.5) = \frac{0.1136}{2} = 0.0568$$

(b) $P(Z > -1.5) = 1 - P(Z < -1.5)$

$$= 1 - 0.0568$$
$$= 0.9432$$

(c) $P(Z < 1.5) = P(Z > -1.5) = 0.9432$

8 D

The middle 95% of values are two standard deviations either side of the mean.

$a = \mu - 2\sigma = 35 - 2 \times 7 = 21$

$b = \mu + 2\sigma = 35 + 2 \times 7 = 49$

10 (a) $P(M < 49.5) \approx P(Z < -2) = \dfrac{0.05}{2} = 0.025$

$$\begin{aligned} P(M > 49.5) &= 1 - P(M < 49.5) \\ &\approx 1 - 0.025 \\ &= 0.975 \end{aligned}$$

Calculate the z –value using the $m = 50$

$$\begin{aligned} z &= \frac{50-72}{11} \\ &= -\frac{22}{11} \\ &\approx -2 \end{aligned}$$

So 50 is 2 standard deviations below the mean.

$$P(M < 50) = P(Z < -2) = \frac{0.05}{2} = 0.025$$

$$\begin{aligned} P(M > 49.5) &= 1 - P(M < 49.5) \\ &= 1 - 0.025 \\ &= 0.975 \end{aligned}$$

(b) $z = \dfrac{45-72}{11}$

$$\begin{aligned} &= -\frac{27}{11} \\ &\approx -2.4545 \end{aligned}$$

If $z = \dfrac{27}{11}$

$$\begin{aligned} \frac{27}{11} &= \frac{x-72}{11} \\ x &= 72 + 27 \\ &= 99 \end{aligned}$$

Alternatively, students may recognise that the normal curve is symmetrical about the mean.

45% is $72\% - 27\%$ so the other score would be $72\% + 27\% = 99\%$.

(c) The top 2.5% combined with the bottom 2.5% is 5% of the marks, or two standard deviations or more either side of the mean. To be in the top 2.5%, a student must be at least 2 standard deviations above the mean.

$72 + 2 \times 11 = 94$

The minimum mark for A^{++} is 94%.

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(d) $\mu = 70$

$\sigma = \sqrt{144} = 12$

$z = \dfrac{94 - 70}{12} = 2$

Yes, a student who obtained a mark of 94% would be awarded an A^{++} grade as they are (just) in the top 2.5% of the group.

CHAPTER 20 REVIEW

2 No, the hybrid function $f(x)$ does not represent a probability density function, because $f(x)$ is negative for some values of x in the domain. For example,

$$f(0.5) = 0.5^2 + 4 \times 0.5 - \frac{10}{3}$$
$$= -\frac{13}{12}$$

4 **(a)** $\mu = 500,\ \sigma = \sqrt{25} = 5$

A 495 g jar less than one standard deviation below the mean.

68% of jars are less than one standard deviation from the mean.

34% of jars are more than one standard deviation either side of the mean.

$\dfrac{\mathbf{32\%}}{2} = \mathbf{16\%}$ of jars are more than one standard deviation below the mean.

16% of jars will be less than 495 g.

(b) A 490 g jar is 2 standard deviations below the mean.

95% of jars are less than one standard deviation from the mean.

5% of jars are more than one standard deviation either side of the mean.

$\dfrac{5\%}{2} = 2.5\%$ of jars are more than one standard deviation below the mean.

$2.5\% \times 1000 = 25$ jars will be less than 490 g.

In a batch of 1000, you would expect 25 jars to be rejected.

6 **(a)** $\mu = 28,\ \sigma = \sqrt{16} = 4$

$z = \dfrac{24 - 28}{4} = \dfrac{-4}{4} = -1$

(b) $z = \dfrac{40 - 28}{4} = \dfrac{12}{4} = 3$

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8 D

$\mu = 10,\ \sigma = \sqrt{9} = 3$

The middle 95% of observations occur within two standard deviations of the mean.

$$\mu - 2\sigma = 10 - 2\times 3 = 4$$
$$\mu + 2\sigma = 10 + 2\times 3 = 16$$

About 95% of observations would be in the range 4 to 16.

10 C

Graph 2 is to the right of graph 1, so it has the larger mean.

Both graphs have the same shape, (and the same spread) so they both have the same variation.

12 (a) It is reasonable because $f(x) \geq 0$ for the values in the domain and

$$\begin{aligned}\int_0^{\sqrt{3}} \frac{3}{\pi\left(1+x^2\right)}dx &= \frac{3}{\pi}\int_0^{\sqrt{3}} \frac{dx}{1+x^2}\\ &= \frac{3}{\pi}\left[\tan^{-1}x\right]_0^{\sqrt{3}}\\ &= \frac{3}{\pi}\times\left(\frac{\pi}{3}-0\right)\\ &= 1\end{aligned}$$

Hence $f(x)$ could represent a probability density function.

(b) (i) $P(X<1) = \int_0^1 \frac{3}{\pi\left(1+x^2\right)}dx$

$$\begin{aligned}&= \frac{3}{\pi}\left[\tan^{-1}x\right]_0^1\\ &= \frac{3}{\pi}\left[\tan^{-1}1 - \tan^{-1}0\right]\\ &= \frac{3}{\pi}\times\frac{\pi}{4}\\ &= \frac{3}{4}\\ &= 0.75\end{aligned}$$

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(ii) $P(X > 1.25) = 1 - P(X < 1.25)$

$$\begin{aligned}
&= 1 - \int_0^{1.25} \frac{3}{\pi(1+x^2)}\,dx \\
&= 1 - \frac{3}{\pi}\left[\tan^{-1} x\right]_0^{1.25} \\
&= 1 - \frac{3}{\pi}\left[\tan^{-1} 1.25 - \tan^{-1} 0\right] \\
&= 1 - \frac{3}{\pi} \times \tan^{-1} 1.25 \\
&\approx 0.1443
\end{aligned}$$

(iii) $P(0.25 < X < 1.1) = \int_{0.25}^{1.1} \frac{3}{\pi(1+x^2)}\,dx$

$$\begin{aligned}
&= \frac{3}{\pi}\left[\tan^{-1} x\right]_{0.25}^{1.1} \\
&= \frac{3}{\pi}\left[\tan^{-1} 1.1 - \tan^{-1} 0.25\right] \\
&\approx 0.5615
\end{aligned}$$

(c) $E(X) = \displaystyle\int_0^{\sqrt{3}} \frac{3x}{\pi(1+x^2)}\,dx$

Substitute $f(x) = 1 + x^2$ so that $f'(x) = 2x$.

$$\begin{aligned}
E(X) &= \frac{3}{2\pi}\int_0^{\sqrt{3}} \frac{2x}{(1+x^2)}\,dx \\
&= \frac{3}{2\pi}\int_0^{\sqrt{3}} \frac{f'(x)}{f(x)}\,dx \\
&= \frac{3}{2\pi}\left[\log_e(1+x^2)\right]_0^{\sqrt{3}} \\
&= \frac{3}{2\pi}\left[\log_e(1+3) - \log_e 1\right] \\
&= \frac{3}{2\pi}\log_e 4 \\
&= \frac{3}{\pi}\log_e 2 \\
&\approx 0.6619
\end{aligned}$$

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(d) $Var(X) = E\left(X^2\right) - \mu^2$

$$= \int_0^{\sqrt{3}} \frac{3x^2}{\pi\left(1+x^2\right)} dx - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$= \frac{3}{\pi}\int_0^{\sqrt{3}} \frac{1+x^2-1}{\left(1+x^2\right)} dx - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$= \frac{3}{\pi}\int_0^{\sqrt{3}} \left[1 - \frac{1}{\left(1+x^2\right)}\right] dx - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$= \frac{3}{\pi}\left(\int_0^{\sqrt{3}} dx - \int_0^{\sqrt{3}} \frac{1}{1+x^2} dx\right) - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$= \frac{3}{\pi}\left\{[x]_0^{\sqrt{3}} - \left[\tan^{-1} x\right]_0^{\sqrt{3}}\right\} - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$= \frac{3}{\pi}\left(\sqrt{3} - \frac{\pi}{3}\right) - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$= \frac{3\sqrt{3}}{\pi} - 1 - \left(\frac{3}{\pi}\log_e 2\right)^2$$

$$\approx 0.2159$$

14 (a) Xandra:

$$E(X) = \int_0^3 -\frac{x}{9}\left(x^2 - 4x\right) dx$$

$$= -\frac{1}{9}\int_0^3 \left(x^3 - 4x^2\right) dx$$

$$= -\frac{1}{9}\left[\frac{x^4}{4} - \frac{4x^3}{3}\right]_0^3$$

$$= -\frac{1}{9}\left[\left(\frac{3^4}{4} - \frac{4\times 3^3}{3}\right) - 0\right]$$

$$= -\frac{1}{9} \times \left(-\frac{63}{4}\right)$$

$$= \frac{7}{4}$$

$$= 1.75$$

Xandra spends an average of 1.75 hours (1 h 45 min) completing homework on a given day.

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Zack:

$$\begin{aligned}E(Z)&=\int_0^3 \frac{2z}{57}\left(z^2-9z+20\right)dz\\&=\frac{2}{57}\int_0^3\left(z^3-9z^2+20z\right)dz\\&=\frac{2}{57}\left[\frac{z^4}{4}-3z^3+10z^2\right]_0^3\\&=\frac{2}{57}\left[\left(\frac{3^4}{4}-3\times3^3+10\times3^2\right)-0\right]\\&=\frac{2}{57}\times\frac{117}{4}\\&=\frac{39}{38}\\&\approx1.03\end{aligned}$$

Jack spends an average of 1.03 hours (1 h 2 min) completing homework on a given day.

Xandra spends, on average, about 43 more minutes completing homework than Zack.

(b) $P(X<1.5)=\int_0^{1.5}-\frac{1}{9}\left(x^2-4x\right)dx$

$$\begin{aligned}&=-\frac{1}{9}\int_0^{1.5}\left(x^2-4x\right)dx\\&=-\frac{1}{9}\left[\frac{x^3}{3}-2x^2\right]_0^{1.5}\\&=-\frac{1}{9}\left[\left(\frac{1.5^3}{3}-2\times1.5^2\right)-0\right]\\&=-\frac{1}{9}\times\left(-\frac{27}{8}\right)\\&=\frac{3}{8}\\&=0.375\end{aligned}$$

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(c) Xandra:

$$
\begin{aligned}
Var(X) &= E\left(X^2\right) - \mu^2 \\
&= \int_0^3 -\frac{x^2}{9}\left(x^2 - 4x\right)dx - 1.75^2 \\
&= -\frac{1}{9}\int_0^3 \left(x^4 - 4x^3\right)dx - 3.0625 \\
&= -\frac{1}{9}\left[\frac{x^5}{5} - x^4\right]_0^3 - 3.0625 \\
&= -\frac{1}{9}\left[\left(\frac{3^5}{5} - 3^4\right) - 0\right] - 3.0625 \\
&= -\frac{1}{9}\times\left(-\frac{162}{5}\right) - 3.0625 \\
&= \frac{18}{5} - 3.0625 \\
&= 3.6 - 3.0625 \\
&= 0.5375
\end{aligned}
$$

Zack:

$$
\begin{aligned}
Var(Z) &= E\left(X^2\right) - \mu^2 \\
&= \int_0^3 \frac{2z^2}{57}\left(z^2 - 9z + 20\right)dz - \left(\frac{39}{38}\right)^2 \\
&= \frac{2}{57}\int_0^3 \left(z^4 - 9z^3 + 20z^2\right)dz - \left(\frac{39}{38}\right)^2 \\
&= \frac{2}{57}\left[\frac{z^5}{5} - \frac{9z^4}{4} + \frac{20z^3}{3}\right]_0^3 - \left(\frac{39}{38}\right)^2 \\
&= \frac{2}{57}\left[\left(\frac{3^5}{5} - \frac{9\times 3^4}{4} + \frac{20\times 3^3}{3}\right) - 0\right] - \left(\frac{39}{38}\right)^2 \\
&= \frac{2}{57}\times\frac{927}{20} - \left(\frac{39}{38}\right)^2 \\
&= \frac{4137}{7220} \\
&\approx 0.5730
\end{aligned}
$$