Module 5: Advanced Mechanics

Part 2: Uniform Circular Motion

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Contents

Syllabus content: Advanced Mechanics

Circular motion

Inquiry question: Why do objects move in circles?

Students:

- conduct investigations to explain and evaluate, for objects executing uniform circular motion, the relationships that exist between:
  - centripetal force
  - mass
  - speed
  - radius
  - analyse the forces acting on an object executing uniform circular motion in a variety of situations, for example:
  - cars moving around horizontal circular bends
  - a mass on a string
  - objects on banked tracks (ACSPH100)
- solve problems, model and make quantitative predictions about objects executing uniform circular motion in a variety of situations, using the following relationships:
  - $a_c = \frac{v^2}{r}$
  - $v = \frac{2\pi r}{t}$
  - $F_c = \frac{mv^2}{r}$
  - $\omega = \frac{\Delta\theta}{t}$
- investigate the relationship between the total energy and work done on an object executing uniform circular motion
- investigate the relationship between the rotation of mechanical systems and the applied torque
  - $\tau = r_{\perp}F = rF\sin\theta$

Uniform circular motion

Figure 1: A centripetal force!

Uniform circular motion is a "special case" for motion, where an object moves in a circle with constant speed.

The object experiences an acceleration as, although its speed is constant, its velocity is constantly changing as it is continually changing direction.

Real world examples of objects that move in a circular path:

Which of these move with *constant* speed (*uniform* circular motion)?

An accelerometer on a record player turntable

We will focus on a particular case of uniform circular motion - an accelerometer attached to a record player turntable by blue-tac.

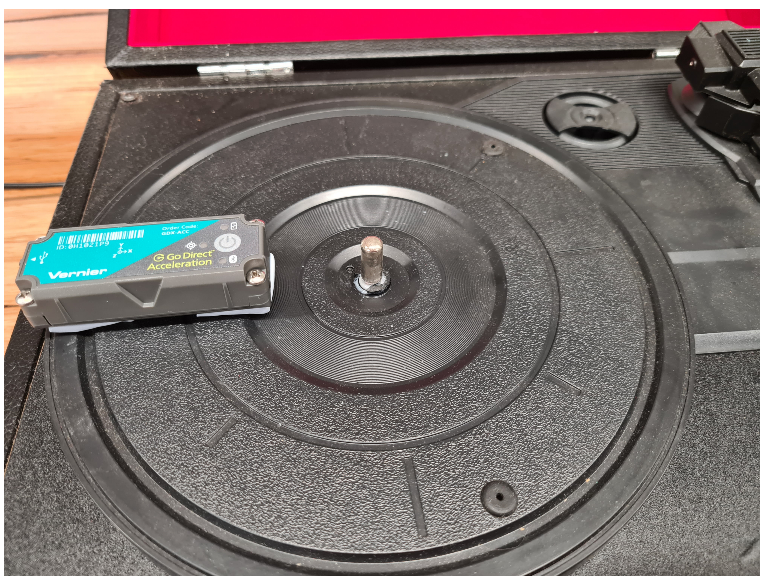

Figure 2: A wireless accelerometer attached to a record player turntable with bluetac.

Our goal is to determine what force/s must act on the accelerometer and in which direction these must act in order to move it at constant speed in a circle.

I will demonstrate the motion of the turntable for you before we actually take some data (if online, use: `https://youtu.be/Q_d7Ji1ishc`).

Predictions

Take the accelerometer as the system. What force/s must act on the accelerometer so that it moves in a circle at constant speed?

Figure 3: A "top down" view of the accelerometer on the record player

In the space on the right, draw a "top-down" and (separately) a "side-on" free body diagram for the accelerometer in the position shown.

Predicting accelerations for uniform circular motion

Using:

- the directions marked on the accelerometer, and
- your free body diagrams (if there is a net force in the $x-$, $y-$ or $z-$ directions then there will be an acceleration in that direction)

Circle your prediction about whether we will measure an acceleration in each direction in the space below

x-**direction**	Zero/Non-zero	+ve/-ve
y-**direction**	Zero/Non-zero	+ve/-ve
z-**direction**	Zero/Non-zero	+ve/-ve

Consider the moment when the turntable is switched on, and the moment when it is switched off. Do you expect the acceleration you have predicted to vary during these times? How?

We will take some data using the accelerometer.
What actually happened?

x**-direction**	Zero/Non-zero	+ve/-ve	Why?
y**-direction**	Zero/Non-zero	+ve/-ve	Why?
z**-direction**	Zero/Non-zero	+ve/-ve	Why?

Forces in uniform circular motion

Draw a "top-down" and "side on" free body diagram for the accelerometer undergoing uniform circular motion in light of our data.

A couple of videos to discuss:

- Playground merry-go-round (AKA ...never trust your brother!): `https://youtu.be/dCUIYW2-RVY`
- Sparks from a grinding wheel: `https://youtu.be/kyXa2F_84bs?si=CDGkY5zUr16fj3xm&t=74`

Figure 4: Girl undergoing uniform circular motion (still from the video)

Figure 5: Sparks from a grinding wheel

Derivation of centripetal acceleration

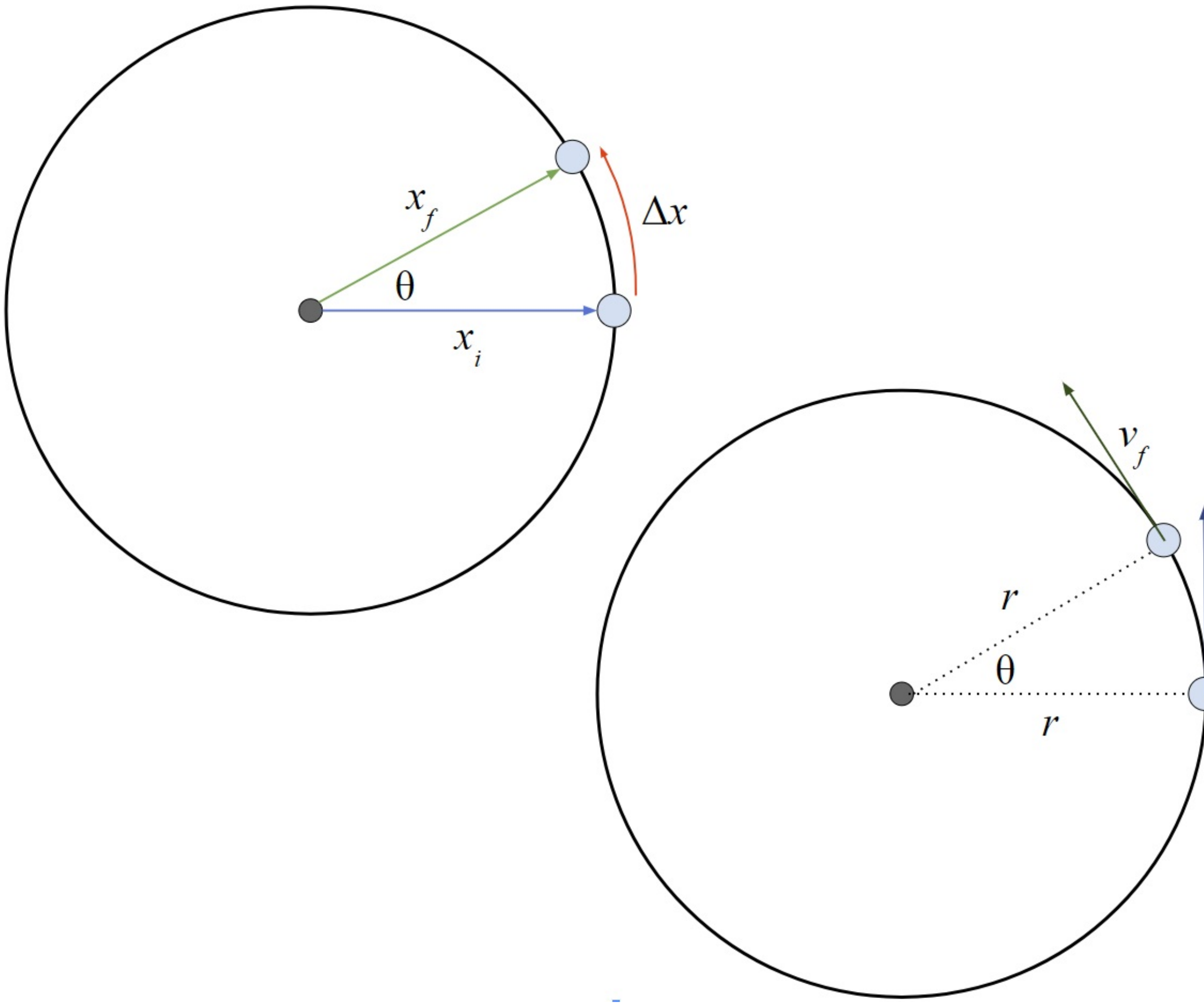

Figure 6: Diagrams for derivation of centripetal acceleration.

Now that we have spent some time thinking about uniform circular motion qualitatively, we wish to derive the magnitude of the acceleration, a, of an object undergoing uniform circular motion with a radius r and a constant speed be v.

We start by considering an object (e.g. a coin)

- moving from an initial displacement $\vec{x}_i$
- to a final displacement $\vec{x}_f$ from the center
- in time Δt

as shown in figure 6 on the left.

If we consider the velocity of the object

- the velocity at its initial position is $\vec{v}_i$
- and its velocity at its final displacement is $\vec{v}_f$
- which both have the same magnitude v

as shown in figure 6 on the right. To find the acceleration of the coin:

1. Write down an expression for the average acceleration of the coin as its velocity changes from $\vec{v}_i$ to $\vec{v}_f$ over a small time Δt.
2. Write down an expression relating the magnitude of the velocity, v, to the magnitude of the change in displacement Δx over a small time Δt.
3. Draw a vector diagram below, representing the change in velocity $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$
4. Draw a vector diagram below, representing the change in displacement, $\Delta\vec{x} = \vec{x}_f - \vec{x}_i$
5. Note that the vector diagrams are you have drawn are similar isosceles triangles!
6. Re-label the equal sides on the vector diagram for change in displacement to be r (as the magnitudes of $\vec{x}_f$ and $\vec{x}_i$ are the radius of the circle).
7. Re-label the equal sides of the diagram for change in velocity to be v (as the magnitudes of $\vec{v}_f$ and $\vec{v}_f$ are the speed, v).
8. Write down a relationship between the corresponding sides of your similar triangles.
9. Using your previous expressions for the relationship between change in displacement and time, and change in velocity and time, obtain an expression for the acceleration which depends only upon the speed v and the radius of the circle r.

We have found the magnitude of the acceleration.

Can you determine the direction as well from your vector diagrams? Note that the acceleration is in the same direction as the vector representing the change in velocity

Direction of the acceleration:

Centripetal acceleration and centripetal force

The acceleration that an object experiences when it undergoes uniform circular motion is known as centripetal acceleration. It has magnitude

$$a_c = \frac{v^2}{r}$$

and always points to the center.

The net force which causes the centripetal acceleration is called a centripetal force.

$$F_c = ma_c = \frac{mv^2}{r}$$

It is very important to note that this is NOT an additional force, it is simply the sum of the actual forces which act on an object in the specific circumstance that it is moving in a circle at a constant speed.

As centripetal forces always act perpendicular to the displacement of the object, the work done by a centripetal force is zero.

Angular velocity

It can be useful in circular motion to work in terms of *angular velocity* rather than linear velocity.

In an analogous way to linear velocity being the time rate of change of the displacement, angular velocity, ω, is the time rate of change of the angular displacement θ, that is,

$$\omega = \frac{\Delta\theta}{t} \tag{1}$$

where the angular displacement θ is measured in radians. The symbol for radian measure is a superscript ^{c} (standing for 'circular measure')

Example 1. What is the angular velocity of the coin on the turntable, if it is rotating at $78rpm$?

Quantitative centripetal acceleration data

We'll take some quantitative data now for our accelerometer on the turntable.

	$a(\mathrm{ms}^{-2})$		
rpm ω	78	45	33
x-direction			
y-direction			
z-direction			

To have some extra practice using the expression for centripetal acceleration, we will calculate the position of the small accelerometer chip inside the wireless accelerometer.

We can use the data we obtained for the acceleration at the position of the chip at each speed of the turntable to determine the distance, r, it must be from the axis of the record player.

The Problem - without hints

If you would enjoy tackling the problem on your own, do it on this page. If you'd prefer a bit more direction, flip to the next page.

The problem - in steps

We will calculate the position of the small accelerometer chip inside the wireless accelerometer using $a_c = \frac{v^2}{r}$.

1. If the turntable is rotating at 78 revolutions per minute, how many revolutions is it making every second?

2. If the accelerometer chip is at a distance r from the center, how far does it travel each time it does one revolution? (You will need the expression for the circumference of a circle of radius r)

3. Write down an expression for the velocity v in terms of r (the number of revolutions per second times the distance it travels in one revolution)

4. Using $a_c = \frac{v^2}{r}$ rearrange to make r the subject, then substitute the value for the centripetal acceleration for 78rpm we measured to obtain a value for r.

5. Repeat this calculation for the other rotational speeds and record your 3 calculated values of r in the table below

	78rpm	45rpm	33rpm
r(m)			

6. Calculate the average value for r and estimate the uncertainty in your answer (Range/2 is probably the way to go for the uncertainty).

7. Check your answer against the image from Vernier on the last page (You might need to use your ruler to measure distances on the accelerometer to check)

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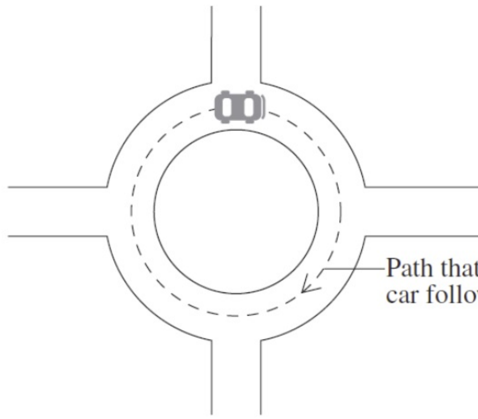
A car with a mass of 800 kg travels at a constant speed of 7.5 m s^{-1} on a roundabout so that it follows a circular path with a radius of 16 m. **4**



A person observing this situation makes the following statement.

> 'There is no net force acting on the car because the speed is constant and the friction between the tyres and the road balances the centripetal force acting on the car.'

Assess this statement. Support your answer with an analysis of the horizontal forces acting on the car, using the numerical data provided above.

Extension question

Calculate bounds (the maximum minimum possible values) for the coefficient of static friction between a coin and the surface of the turntable, given that it slides off for 78rpm and stays on for 45rpm.

Banked Tracks

In the previous section we considered a question about the motion of a car around a roundabout. The friction between the tires of the car and the surface of the road provide the centripetal force required for the car to move in a circle.

What if there is a very low coefficient of friction between the tires of the car and road - if it has been raining? (or even snowing?) In this video `https://www.youtube.com/watch?v=gY0gda12pXg` at 3:27min, a car tries to turn left on an icy road - but there is insufficient friction to provide the centripetal force, producing some undesirable results for the driver. See 4:05min for another one of a taxi attempting to turn right and failing...

It is possible to make gravity provide the centripetal force required for a car to execute a turn (at least if the driver is travelling at the speed limit!) if we design curves in the road so that they are **banked**. Figure 7 shows a very steeply banked track on a velodrome, and figure 8 shows a banked curve on a road.

Figure 7: Cyclists on a steeply banked velodrome track.

Figure 8: Banked curve in California near Los Angeles.

Free body diagram for a car on a banked curve

Figure 9 shows a free body diagram for a car on a banked curve for the case that there is no frictional force between the tyres and the road. In general there might also be a frictional component up or down the road, depending whether the component of the normal force to the centre is larger or smaller than the required centripetal force.

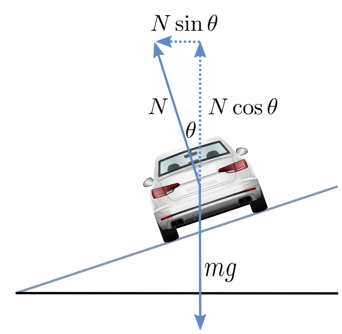


Figure 9: Free body diagram for a "cut-through" of a car on a banked curve

As the road curves *up* in *front* of the car on a banked curve, the normal force must be larger than weight of the car to change the direction of motion of the car (i.e. accelerate it towards the center of the curve), i.e. $|N| > |mg|$.

If the car does not accelerate vertically, then the *vertical component* of the normal force is equal to the weight force (see figure 9)

$$\Sigma F_y = ma_y$$

and $a_y = 0$, so

$$N\cos\theta - mg = 0$$

so

$$N = \frac{mg}{\cos\theta}$$

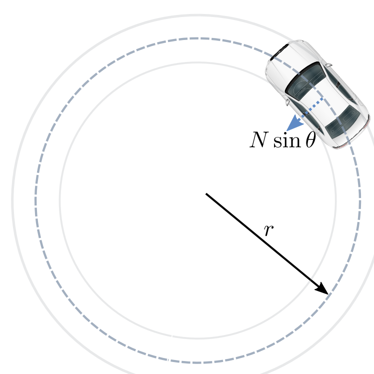


Figure 10: "Top-down" view of a car on a banked curve.

In the horizontal direction, **if** the car is moving with uniform circular motion, then there **must** be a force, or forces which add as vectors to supply the required centripetal force in the radial direction. If there were not a net force equal to the required centripetal force acting towards the centre, then the car would not move in a circle at uniform speed (e.g. when the cars moving on icy roads in the video we played earlier tried to make a turn, but slid instead).

In this case the force is the component of the normal force that is in the horizontal direction, $N \sin\theta$ (see figure 10)

$$\Sigma F_r = ma_r$$

$$N \sin\theta = m\frac{v^2}{r}$$

and

$$mg\frac{\sin\theta}{\cos\theta} = m\frac{v^2}{r}$$

and we can simplify to obtain an expression for the angle at which a curve of radius r should be banked so that cars which are travelling at a speed v do not need to rely on friction in order to make the bend.

$$\tan\theta = \frac{v^2}{rg} \tag{2}$$

Example 2. A car takes a curve of radius $r = 200$m which has been banked at an angle of $10°$ during heavy rain. At what speed should the car travel so that no friction is required in order to make the turn?

Vertical circular motion - a mass on a string

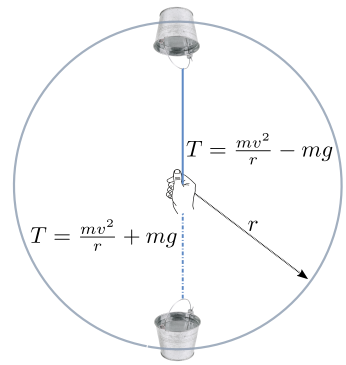


Figure 11: A bucket of water on a string that is swung in a vertical circle at constant speed.

Another scenario of interest is the vertical "loop-the-loop", for example a stunt plane flying in a vertical circle at constant speed, or someone swinging an object on a string in a vertical circle. Note that, in general, it is most likely that such objects will not move at uniform speed, but to give us a place to begin our analysis, we will be assuming a constant speed. Assuming drag is negligible, there are two forces acting on an object on a string that is swung in a vertical circle, the tension in the string and the weight force.

The weight force remains constant in magnitude and in direction throughout the motion. If the object undergoes uniform circular motion, then at all positions around the loop, these two forces must add as vectors to provide the required centripetal force to the centre. This means that the tension force varies in both magnitude and direction so that the net force acting on the object stays constant in magnitude and is always directed towards the centre.

At the top

At the top of the motion, the tension and weight force are both downwards, as is the required centripetal force, so

$$\Sigma F = -\frac{mv^2}{r}$$

$$-T - mg = -\frac{mv^2}{r}$$

and so the tension in the string at the top is

$$T = \frac{mv^2}{r} - mg$$

At the bottom

At the bottom of the motion, the tension and required centripetal force are directed upwards, while the weight force remains directed downwards (as is the case throughout the motion), so

$$\Sigma F = \frac{mv^2}{r}$$

$$T - mg = \frac{mv^2}{r}$$

and so the tension in the string at the bottom is

$$T = \frac{mv^2}{r} + mg$$

Other examples of vertical circular motion

Note that in other cases of vertical circular motion, the force to the center can be provided by forces such as lift (for an entire airplane) or the normal force (when we are considering the pilot of the airplane as our object undergoing uniform circular motion).

In situations where the speed is small, such as ferris wheels, the normal force acting on a person in the ferris wheel acts upwards the entire time, but changes in magnitude so as to always add as a vector with the weight force to give the required (small) centripetal force to the center.

If a person is standing at the equator, then the upward normal force is always slightly smaller than the weight force so that the difference provides the (small) centripetal force required for that person to move in a circle as the earth spins (instead of continuing to move in a straight line off into space).

Example 3. A fighter jet pilot of mass 70kg completes a vertical loop-the-loop at a constant speed of 500km/h, and with a radius of 750m. What normal force does he experience from his seat (a) at the top and (b) at the bottom of the loop-the-loop?

Figure 12: Ferris wheel (long exposure at night, blurry but pretty...)

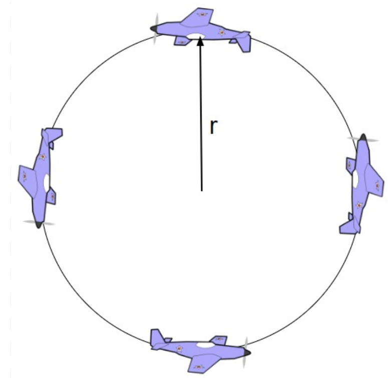


Figure 13: An airplane flying in a "loop-the-loop"

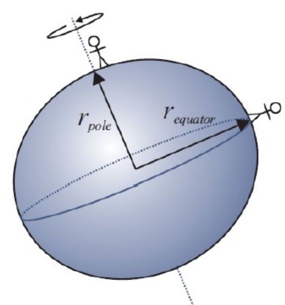


Figure 14: A person standing at the equator undergoes uniform circular motion, with a period of one day and a radius of approximately $r = 6400$km. They are therefore accelerating towards the center of the earth and experience a normal force (very slightly) larger than their weight force.

A person standing at the pole is not accelerating downwards, so the normal force acting on them equals their weight force.

Torque

So far we have focused on *uniform* circular motion. In general, the speed of rotation may not be constant, as when the turntable in our previous example was speeding up or slowing down.

Whenever the speed of rotation is not constant, and there is tangential acceleration, then there must be a net **torque** (the rotational analogue of a force) acting.

Torque: the big ideas:

- To change the rotational motion of an object, the force must be applied at a distance from the axis of rotation. Door handles are located as far as possible from the hinge (door handles themselves are designed so that you apply a force at a distance from the axis of rotation!)
- A bigger force produces a bigger angular acceleration. Push harder and the door changes its motion more rapidly!
- Maximum rotational acceleration is produced when the force is applied perpendicular to the door. If you pull or push on the door if the same plane as the door, it doesn't rotate.

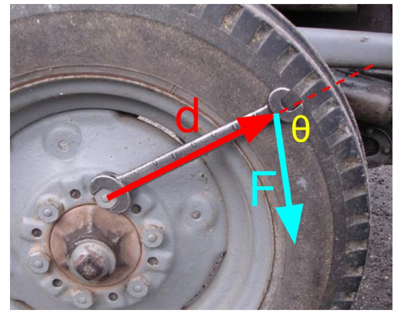


Figure 15: Torque depends on the applied force, how far it is applied from the axis, and the angle at which the force is applied.

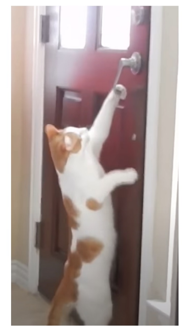
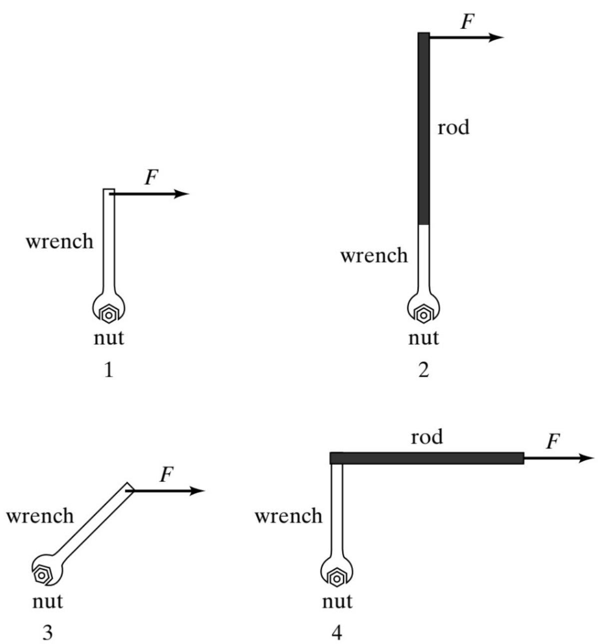


Figure 16: Torque depends on the applied force, how far it is applied from the axis, and the angle at which the force is applied.

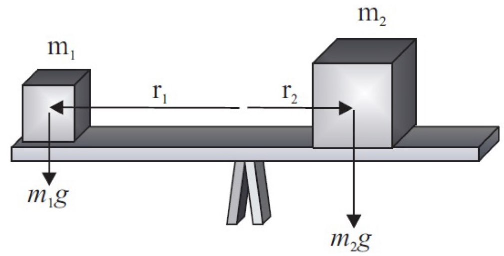
Example 4. Rank the following situations according to torque applied to the wrench to loosen the nut.



Torque is given by

$$\tau = r_{\perp} F = |\vec{r}||\vec{F}| \sin\theta \tag{3}$$

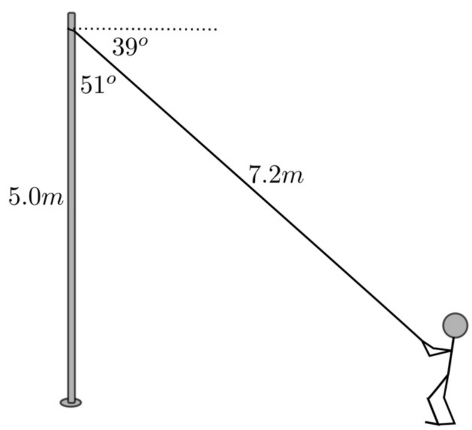
where $\vec{r}$ is a vector pointing from the axis to the point of application of the force $\vec{F}$ and θ is the angle between $\vec{r}$ and $\vec{F}$.



Example 5. Calculate the net torque acting on the plank shown in the figure on the right if $m_1 = 2$kg, $m_2 = 8$kg, $r_1 = 4$m and $r_2 = 1.5$m.

Example 6.

A person pulls on a rope attached to the top of a flagpole with a tension force of $350N$.



What is the magnitude of the torque the person exerts on the flagpole, which is attached to the ground at its base?

(A) $\tau = 7.2 \times 350\, sin(39^o)$Nm
(B) $\tau = 5.0 \times 350\, sin(39^o)$Nm
(C) $\tau = 7.2 \times 350\, sin(51^o)$Nm
(D) $\tau = 5.0 \times 350\, sin(51^o)$Nm

Answers

The following situations could potentially involve *uniform* circular motion, depending on whether the object is travelling with a constant speed as it moves in a circle:

- planetary motion
- cars driving around roundabouts, wheels/hard disks/DVDs turning
- people swinging rocks on strings or water buckets around their heads...
- easter show rides...

Worked Example 1. If the coin is rotating at $78rpm$, then every minute it rotates 78 times. So in one second it rotates $\frac{78}{60} = 1.3$ times, which is $1.3 \times 2\pi = 8.2 \text{rads}^{-1}$.

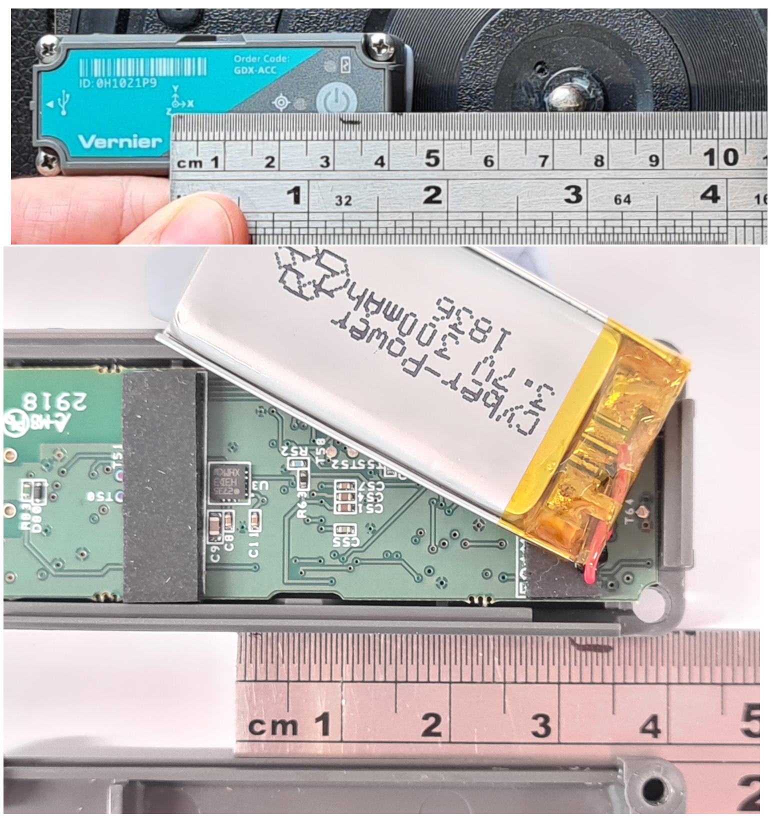


Figure 17: (Top) User manual from Vernier notes that the axes indicate the approximate position of the accelerometer chip inside the accelerometer. (Bottom) Open accelerometer box.

Question 18

The better responses succinctly identified the two clear errors in the statement and used the centripetal force equation to support their judgement. Many candidates incorrectly interpreted the statement as being correct and therefore had difficulty as they tried to justify its elements.

MARKING GUIDELINES

Criteria	Marks
• Makes a correct judgement supported by arguments addressing horizontal forces and backed up by numerical data	4
• Makes a judgement insufficiently supported by argument and/or numerical data	2–3
• Makes correct statement about car moving in circular motion	1

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A suitable answer would be:

- The statement is incorrect.
- Even though the speed is constant the velocity is continuously changing and the car changes direction.
- The centripetal force to make the car move in a circle does not cancel with the force due to friction. Instead, the frictional force supplies the required centripetal force
- The amount of centripetal force required in this situation is $\Sigma F = m\frac{v^2}{r} = \frac{800\times7.5^2}{16} = 2812\text{N}$.

Worked Example 2. We rearrange the formula 2 for v and solve as follows

$$v = \sqrt{(rg\tan\theta)} = \sqrt{200\times9.8\times\tan(10)} = 18.6\text{ms}^{-1} = 67\text{km/h}$$

Worked Example 3. We apply Newton's 2nd law to the fighter pilot. As the question specifies that he is moving in uniform circular motion, we know that his acceleration is always to the center of the loop, and has a magnitude equal to

$$a_c = \frac{v^2}{r}$$

Draw a free body diagram for both (a) and (b).

In (a) the normal force from the pilots seat acts downwards, as does gravity, so

$$-N - mg = -m\frac{v^2}{r}$$

$$N = \frac{mv^2}{r} - mg = 70 \times (\frac{(500/3.6)^2}{750} - 9.8) = 70\text{kg} \times 15.9\text{ms}^{-2} = 1.1 \times 10^3\text{N}$$

For (b) the normal force on the pilot acts upwards, opposing the weight force. The two forces add as vectors to give an upwards centripetal force. By applying Newton's 2nd law we have:

$$N - mg = m\frac{v^2}{r}$$

$$N = \frac{mv^2}{r} + mg = 70 \times (\frac{(500/3.6)^2}{750} + 9.8) = 70\text{kg} \times 34.8\text{ms}^{-2} = 2.5 \times 10^3\text{N}$$

Worked Example 4.

- In situation 2, the distance from the point of application of the force to the nut is largest, and the force is applied perpendicular to the vector points from the nut to the point of application of the force. This situation provides the largest torque.
- Situations 1 and 4 provide identical torques, as in both cases the force is applied to the end of the wrench, at right angles (the rod in situation 4 doesn't change where the force is applied).
- Situation 3 provides the smallest torque as the force is not applied at right angles to the end of the wrench.

Worked Example 5. The torque due to m_1 acts in a counterclockwise direction and is equal to

$$\tau_1 = (r_1) \times (m_1 g) = (4m) \times (2kg \times 9.8ms^{-2}) = 78.4Nm$$

The torque due to m_2 acts in a clockwise direction and is equal to

$$\tau_2 = (r_2) \times (m_2 g) = 117.6Nm$$

. The net torque (taking counterclockwise torques as positive) is then

$$\Sigma\tau = 78.4Nm - 117.6Nm = -39.2Nm$$

That is, $39.2Nm$ in a clockwise direction.

Worked Example 6. This multiple choice requires us to identify the correct value for the vector that points from the axis (point of rotation) to the point of application of the force. Here the point about which the torque is applied is the base of the flagpole, so the correct value for r is $5m$, the length of the flagpole. The angle between the force and this vector is 129^o. As we are only interested in the magnitude of the torque, $\sin 129 = \sin 39$, so the correct answer is B.