Module 5: Advanced Mechanics

Part 1: Projectile Motion

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Syllabus content: Advanced Mechanics

Projectile motion

Inquiry question: How can models that are used to explain projectile motion be used to analyse and make predictions?

Students:

- analyse the motion of projectiles by resolving the motion into horizontal and vertical components, making the following assumptions:
  - a constant vertical acceleration due to gravity
  - zero air resistance
- apply the modelling of projectile motion to quantitatively derive the relationships between the following variables:
  - initial velocity
  - launch angle
  - maximum height
  - time of flight
  - final velocity
  - launch height
  - horizontal range of the projectile (ACSPH099)
- conduct a practical investigation to collect primary data in order to validate the relationships derived above.
- solve problems, create models and make quantitative predictions by applying the equations of motion relationships for uniformly accelerated and constant rectilinear motion

Overview - Modelling special cases for motion

In this section (on projectile motion) and the next section (on circular motion) we model two special cases of motion.

Modelling

A *model* can mean different things in different contexts in science (and in physics). The syllabus definition is

> In general, a **model** is a representation that describes, simplifies, clarifies or provides an explanation of the workings, structure or relationships within an object, system or idea.

A definition that is more specific to the context of describing motion might be:

> A **model** (in the context of motion) is a mathematical representation of a phenomena which captures or describes its most significant or relevant features.

A model may not capture the complete behaviour of the system. It may involve simplifying assumptions. In the case of projectile motion these will be that drag is negligible and that the acceleration due to gravity is constant. In the case of uniform circular motion it is that the motion is uniform (constant speed) and exactly circular. Before applying one of these models to a real physical system, such as a basketball thrown through the air or a car on a roundabout, it is necessary to think about whether the assumptions made in this model are reasonable for that particular system.

Projectile motion

Projectile motion occurs when a net force with a constant magnitude and direction resulting in a trajectory that is parabolic. Examples of forces which can produce projectile motion are the force of gravity near the surface of the earth, where the gravitational field is uniform, and the electric force acting on a charge between two charged parallel plates, where the electric field is uniform.

Uniform Circular motion

In uniform circular motion, a net force with a constant magnitude acts on an object, but with a direction that is always perpendicular to the velocity of an object. Such a force produces a trajectory that is circular, along which the object moves with a constant speed. Examples of forces which can produce uniform circular motion are the force of gravity acting on a satellite in orbit around earth, or the magnetic force acting on a particle moving perpendicular to a magnetic field.

Simple Harmonic Motion (extension)

A third special case of motion is simple harmonic motion. Students taking Extension 1 mathematics will encounter it in that course. This is the type of motion is caused by a force that acts towards an equilibrium position, with a magnitude that is proportional to the displacement from the equilibrium position. It is very important, occurring in many fields of physics in which oscillations occur. You have encountered this already when you studied waves - it is the type of motion that particles undergo as a mechanical wave passes through a medium.

The reason to mention this here is to emphasise that there is actually a range of "basic" types of motion that one should be familiar with, and can be used to approximately describe (that is, model) the motion of a real system.

Motion - two big ideas

There are two "big ideas" that we will discuss before we get to the 'nitty gritty' of the two special cases of motion that this module focuses on.

1. Galilean relativity
2. Thinking iterative about motion

Galilean relativity - independence of horizontal and vertical motion

Figure 1: Traveller eating on a plane travelling at constant speed.

Figure 2: Observer on the ground (here we imagine they can watch the traveller's fork as it falls into their lap...!

The first "big idea" is that all observers that are moving at a constant velocity (called "inertial" observers) agree that Newton's laws hold (observers that are accelerating must take their own acceleration into account to use Newton's laws). This means that all observers, such as a traveller on a plane moving at constant speed and an observer on the ground will agree about accelerations and forces, and find that momentum is conserved for isolated systems. As we study motion, we will see that this means that motion in one direction is not affected by motion in another direction, so we can analyse motion in different directions independently.

For the case of a traveller on a plane who drops their fork, both the plane traveller and the observer on the ground (if they could see!) would agree that the fork drops into the travellers lap with a constant vertical acceleration, and that it has no horizontal acceleration. They would disagree about the horizontal velocity and horizontal distance covered by the fork during the fall.

Independence of horizontal and vertical motion - prac

In this module, we focus on mechanics in two dimensions. An important idea that we need to equip ourselves with before continuing if the idea that motion in one dimension does not affect motion in another dimension. We will consider the experiment below, in which two ball bearings are released at the same time, one dropped (almost) straight down, and one fired sideways.

Will they hit the ground at the same time?

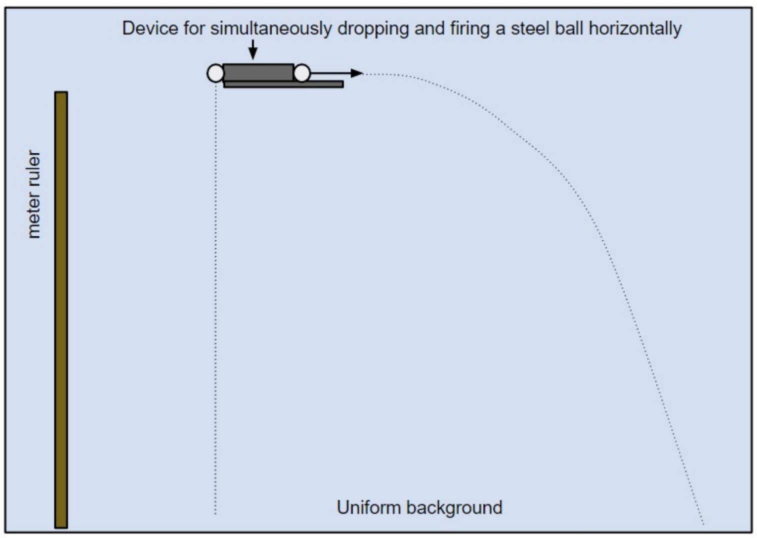


Figure 3: Link to a video of this experiment: `https://youtu.be/Ty7h0cF5uiM`. Here's another video of an equivalent (and higher production quality!) experiment (listen for the sound of the balls hitting the floor) `https://www.youtube.com/watch?v=zMF4CD7i3hg`.

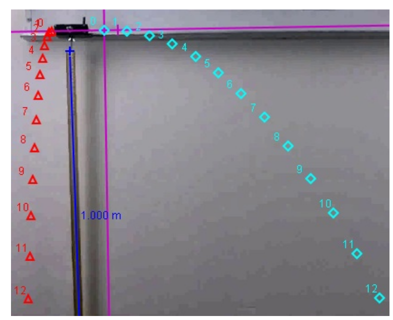


Figure 4: Positions of two steel balls, marked at equal time intervals (0.02s apart). One dropped (almost) directly down, and one fired sideways. Data taken using Tracker by Tammy Humphrey.

We will watch a segment from the documentary on Galileo, "The battle of the heavens" showing a thought experiment with a horse rider and ball: `https://www.youtube.com/watch?v=XCxkdR092c4` 53:30min to 56:10min.

What if we 'flip' the situation? Picture an observer on the ground dropping a ball.

What trajectory does he see the ball take?

What trajectory does *the rider on the horse* see the ball take?

Thinking iteratively about motion

Thinking *iteratively* about changes in motion is very helpful in order to continue to build a strong understanding of kinematics - this is our 2nd "big idea". The goal is to help you see clearly how each of the types of motion we will study are a "special case" of a more general way of thinking about changes in motion.

Velocity changes when an acceleration occurs over time

In the figure on the right, an object moving with some initial velocity $\vec{u}$ experiences a displacement of $\vec{u}\Delta t$ over some (very short) time Δt. If it is subject to a net force which produces an acceleration $\vec{a}$, then its new (final) velocity will be:

$$\vec{v} = \vec{u} + \Delta\vec{a}\Delta t$$

Even if the acceleration varied with time, it would be possible to calculate the new velocity (and position) after some short time period via this process.

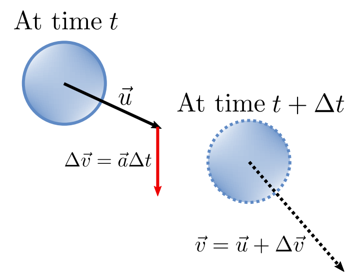
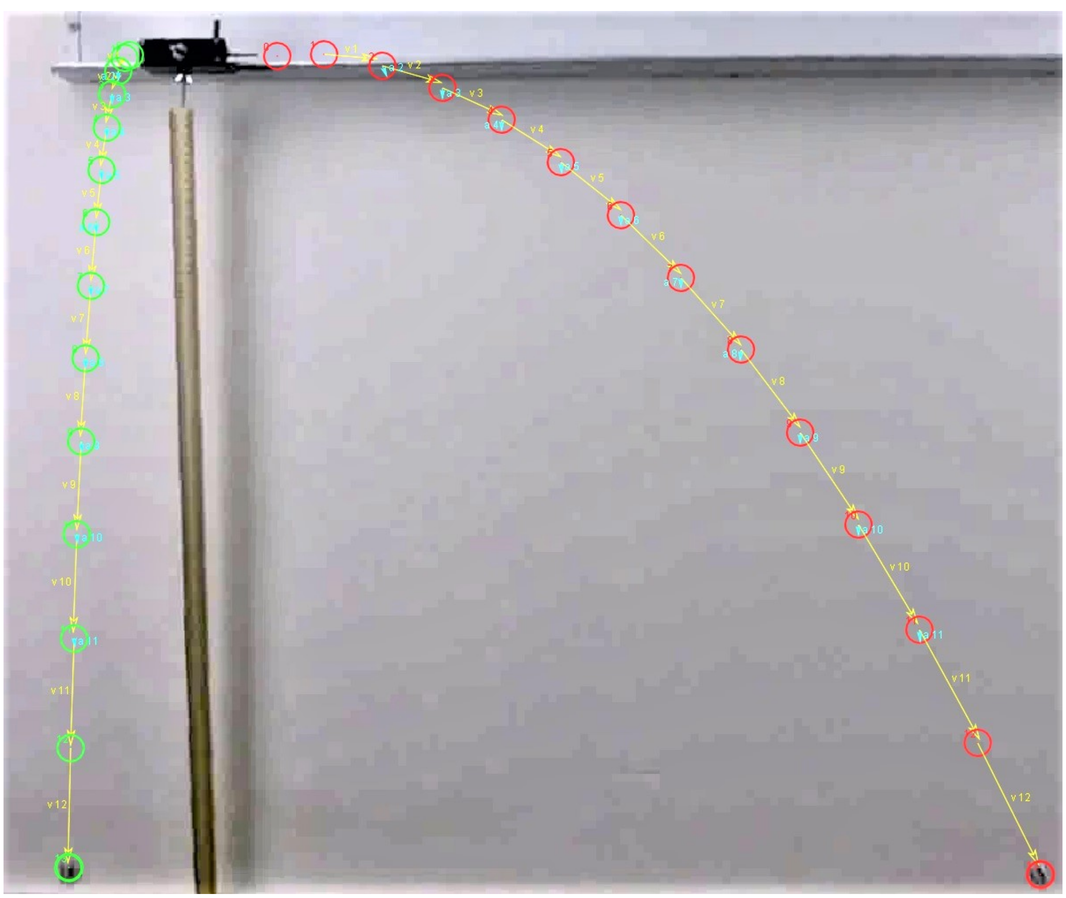


Figure 5: If an object moving with an initial velocity $\vec{u}$ is subject to an acceleration a over a small time period Δt then over this time period its velocity changes to a new velocity which is the vector sum of $\vec{u} + \Delta\vec{v}$.



Projectile motion

The syllabus asks us to:

Analyse the motion of projectiles by resolving the motion into horizontal and vertical components, making the following assumptions:

- a constant vertical acceleration due to gravity
- zero air resistance

The first assumption is an excellent assumption for projectiles that do not travel too high (i.e. less than a few kilometers!) or too far horizontally (i.e. less than hundreds of kilometers!) so that both the magnitude and direction of the acceleration due to gravity remain close to constant.

The second assumption is a good assumption for objects that are dense (i.e. a surface area that is very large compared to the mass of the object) and are not moving too fast (as drag increases with increasing velocity).

Derivation of the equations of constant acceleration

In deriving the equations of projectile motion is is not necessary for us to know what particular force is acting on a particle, only that it is **constant in both magnitude and direction**.

If a constant force acts on a particle in the vertical direction, then is must undergo a **constant acceleration** in the vertical direction, as

$$\Sigma F_y = ma_y$$

From our previous discussion (and from year 11!) we know that we can determine the increase in velocity over a short time period Δt as

$$\vec{v} = \vec{u} + \vec{a}\Delta t$$

If the acceleration has constant magnitude and direction (here, we take this to be in the y-direction), then the acceleration doesn't change over time, so the time step we consider does *not have to be very small*, and we can calculate the velocity at any point in the future using:

$$v_y = u_y + a_y t \tag{1}$$

We can also note that **as long as the acceleration is constant** the average velocity is equal to the displacement in the vertical direction, Δy, over time

$$\Delta y = \frac{u_y + v_y}{2} t$$

substituting equation 1 for the final velocity $v_y = u_y + a_y t$, we can write

$$\Delta y = \frac{u_y + u_y + at}{2} t$$

so that

$$\Delta y = u_y t + \frac{1}{2} a t^2 \tag{2}$$

We can eliminate time t between equations 1 and 2 to obtain a third equation that is useful in the case that a projectile motion problem doesn't give us information about time, or required us to calculate the time a projectile is in the air.

Rearranging equation 1 to make t the subject, we obtain

$$t = \frac{v - u}{a}$$

which we can substitute into

$$\Delta y = \frac{u_y + v_y}{2} t$$

to eliminate t to obtain:

$$v_y^2 = u_y^2 + 2 a_y \Delta y \tag{3}$$

Below is some space for you to use to work through this yourself:

Motion in the horizontal direction

If there is **no acceleration in the x-direction** (i.e. that there is **no drag force**), then the velocity in the x-direction remains constant,

$$v_x = u_x$$

and the displacement in the x-direction is given by

$$\Delta x = u_x t \tag{4}$$

Projectiles follow a parabolic trajectory

Under the assumptions we have made, that:

- the **acceleration is constant in one dimension** (here the vertical direction) and there is
- **no acceleration in the other (perpendicular) direction** (here the horizontal direction)

we can obtain the trajectory followed by the projectile by eliminating the time t between equations 2 and 4 to obtain the following (somewhat messy!) equation:

$$\Delta y = \Delta x \left(\frac{u_y}{u_x}\right) + \frac{a_y}{2u_x^2}\Delta x^2$$

The point of doing this is simply to show that the function $\Delta y(\Delta x)$ is a quadratic - meaning that the path taken by the projectile is a **parabola**.

Solving projectile motion problems

We will use equations 1, 2, 3 and 4 to solve projectile motion problems relating to:

- initial velocity
- launch angle
- maximum height
- time of flight
- final velocity
- launch height
- horizontal range of the projectile

Some guidance for starting projectile motion problems

Depending upon the requirements of the question, it may be useful to:

- Draw a diagram of the problem, marking any relevant information.
- Write down a list of the variables you are given the values of in the question, and identify any variables you have been asked to find.

- Draw a vector diagram representing the magnitude and direction of the initial velocity
- Resolve the initial velocity into components in the x- and y- directions
- Write down the equations that are relevant to the problem.
- Report intermediate answers that you will use for further calculations to a couple of significant figures more than given in the questions.
- Generally, you should report final answers only to the number of significant figures of the lowest precision variable used in the calculation.
- Always include units when reporting your answer.

Example projectile motion problems

We will do the following problems together:

Example 1. A projectile is fired at a velocity of 50ms^{-1} an angle of $30°$ to the horizontal.
(a) Determine the horizontal and vertical components of the velocity
(b) Calculate the time of flight
(c) Find the maximum height
(d) Calculate the range of the projectile

Example 2. An aid plane flying at an altitude of 500m with a horizontal velocity of 100ms^{-1} drops a food supply parcel when directly over a village. Calculate how far the villagers will have to walk to retrieve the parcel.

Example 3. A batsman in a game of cricket hits the ball and it leaves his bat at a height of 70cm. It travels for 2.3s and is caught by a fielder at a height of 1.2m, a horizontal distance of 46m away from the batsman.

What was the initial velocity of the ball as it left the batsman's bat?

Worked examples

Worked Example 1. A projectile is fired at a velocity of $u = 50\text{ms}^{-1}$ an angle of $\theta = 30°$ to the horizontal.

(a) Determine the horizontal and vertical components of the velocity

The initial horizontal component is $u_x = u\cos\theta = 50\cos(30°) = 43.3\text{ms}^{-1} = 43\text{ms}^{-1}(2s.f.)$

The initial vertical component is $u_y = u\sin\theta = 50\sin(30°) = 25\text{ms}^{-1}(2s.f.)$

(b) Calculate the time of flight

To find the time of flight we solve equation 2 for $\Delta y = 0$. Discarding the $t = 0$ solution, we obtain

$$t = \frac{-2u_y}{a} = \frac{-2 \times 25}{-9.8} = 5.1\text{s}$$

(c) Find the maximum height

The projectile will be at its maximum height when the vertical velocity is zero. We will use equation 3 for this.

$$0 = u_y^2 + 2a\Delta y$$

$$\Delta y = \frac{-u_y^2}{2a} = \frac{-25}{2 \times (-9.8)} = 1.28\text{m} = 1.3\text{m}(2s.f.)$$

(d) Calculate the range of the projectile

To find the range, we use the time of flight and equation 4.

$$\Delta x = u_x t = 43.3 \times 5.1 = 220\text{m}(2s.f.)$$

Worked Example 2. An aid plane flying at an altitude of 500m with a horizontal velocity of 100ms^{-1} drops a food supply parcel when directly over a village.

Calculate how far the villagers will have to walk to retrieve the parcel.

The initial vertical velocity in this situation is zero. So we need to know how long it will take a projectile that is dropped from 500m takes to hit the ground (as the horizontal motion doesn't affect the time taken to fall).

$$\Delta y = u_y t + \frac{1}{2}at^2$$

If $u_y = 0$, then

$$t = \sqrt{\frac{2 \times (-500)}{-9.8}} = 10.1\text{s}$$

so

$$\Delta x = u_x t = 100 \times 10.1 = 1010\text{m} = 1\text{km}(1s.f.)$$

Worked Example 3. A batsman in a game of cricket hits the ball and it leaves his bat at a height of 70cm. It travels for 2.3s and is caught by a fielder at a height of 1.2m, a horizontal distance of 46m away from the batsman.

What was the initial velocity of the ball as it left the batsman's bat?

This question is a little bit harder as you need to 'work backwards'. We know the time of flight as well as the range of the projectile, so from this information we can work out the horizontal component of the initial velocity as

$$u_x = \frac{\Delta x}{t} = \frac{46\text{m}}{2.3\text{s}} = 20\text{ms}^{-1}$$

We can work out the initial vertical component of the flight by noting that the overall displacement of the ball is $\Delta y = +0.5\text{m}$, so rearranging 2 we can obtain:

$$u_y = \frac{\Delta y}{t} - \frac{at}{2} = \frac{0.5\text{m}}{2.3\text{s}} - \frac{-9.8\text{ms}^{-2} \times 2.3\text{s}}{2} = 11.5\text{ms}^{-1}$$

So the magnitude of the initial velocity is

$$u = \sqrt{20^2 + 11.5^2} = 23ms^{-1}$$

The angle above the horizontal is

$$\theta = \tan^{-1}\left(\frac{u_y}{u_x}\right) = 30°$$

References