2013 Bored of Studies Trial Examinations

# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.

**Total Marks – 70**

**Section I** Pages 1 – 4

**10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

**Section II** Pages 5 – 13

**60 marks**

- Attempt Questions 11 – 13
- Allow about 1 hour 45 minutes for this section.

**Total marks – 10**
**Attempt Questions 1 – 10**
**All questions are of equal value**

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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**1** The point $P(1,4)$ divides the line segment joining $A(-1,8)$ and $B(x,y)$ internally in the ratio $2:3$. What are the coordinates of *B*?

(A) $(4,2)$.

(B) $(4,-2)$.

(C) $(-4,2)$.

(D) $(-4,-2)$.

**2** What is $\lim_{x\to 0}\frac{1-\cos ax}{x^2}$, where $a$ is a real number?

(A) $\frac{a^2}{2}$.

(B) $\frac{2}{a^2}$.

(C) $-\frac{a^2}{2}$.

(D) $-\frac{2}{a^2}$.

**3** Which of the following is NOT equivalent to $\frac{1+\sin\theta}{\cos\theta}$?

(A) $\sec\theta+\tan\theta$.

(B) $\frac{t+1}{t-1}$, where $t=\tan\left(\frac{\theta}{2}\right)$.

(C) $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$.

(D) $\frac{1}{\sec\theta-\tan\theta}$.

**4** Consider the following statements.

Let $C$ be some curve with an inverse relation $C^{-1}$.

Which of the following statements is always true for $C$?

(A) If $C$ is a function, then $C^{-1}$ is a function.

(B) If $C$ is a function, then $C^{-1}$ is not a function.

(C) If $C$ is not a function, then $C^{-1}$ is a function.

(D) None of the above.

**5** Which of the following is the range of $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$?

(A) $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(B) $-\frac{\pi}{2} \le y < \frac{\pi}{2}$.

(C) $-\frac{\pi}{2} < y \le \frac{\pi}{2}$.

(D) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

**6** Use Newton's Method on $f(x) = x^2 - a$ to determine which of the following recursions approaches $x = \sqrt{a}$, given $x_0 > 0$?

(A) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$.

(B) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{a}{x_n}\right)$.

(C) $x_{n+1} = x_n\left(1 - \frac{2}{x_n^2 - a}\right)$.

(D) $x_{n+1} = x_n\left(1 + \frac{2}{x_n^2 - a}\right)$.

**7** Which of the following is the value of $\int_0^a \frac{x^2}{a^2+x^2}\,dx$?

(A) $a\left(1-\frac{\pi}{4}\right)$.

(B) $a\left(\frac{\pi}{4}-1\right)$.

(C) $a^2\left(1-\frac{\pi}{4}\right)$.

(D) $a^2\left(\frac{\pi}{4}-1\right)$.

**8** Cham and Constant are playing a badminton match, where five games must be won in order to win the match. The probability that Cham wins is $0.9$ and the probability that Constant wins is $0.1$.

Which of the following is the approximate probability of Cham winning after 7 games.

(A) $8.9\%$.

(B) $9.8\%$.

(C) $47.8\%$.

(D) $12.4\%$.

**9** Suppose that $f(x)$ and $g(x)$ are increasing functions, and that $\frac{f(x)}{g(x)}$ is also an increasing function for all real *x*. Which of the following statements is always true?

(A) $\frac{f(x)}{f'(x)} > \frac{g(x)}{g'(x)}$.

(B) $\frac{f(x)}{f'(x)} < \frac{g(x)}{g'(x)}$.

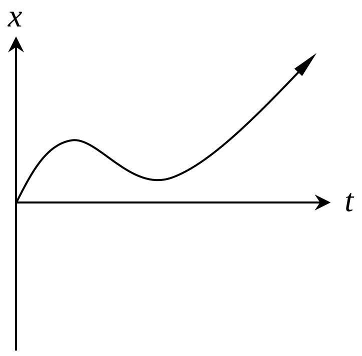
(C) $\frac{f'(x)}{f(x)} > \frac{g'(x)}{g(x)}$.

(D) $\frac{f'(x)}{f(x)} < \frac{g'(x)}{g(x)}$.

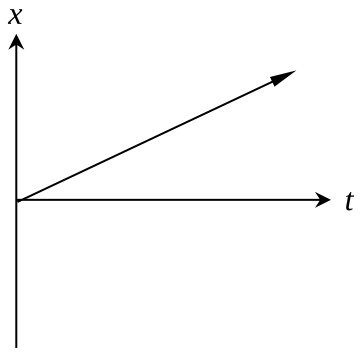
**10** A particle's acceleration is given by $\ddot{x} = kx$, where $k > 0$.

Which of the following graphs best describes the behavior of the particle's displacement with respect to time?

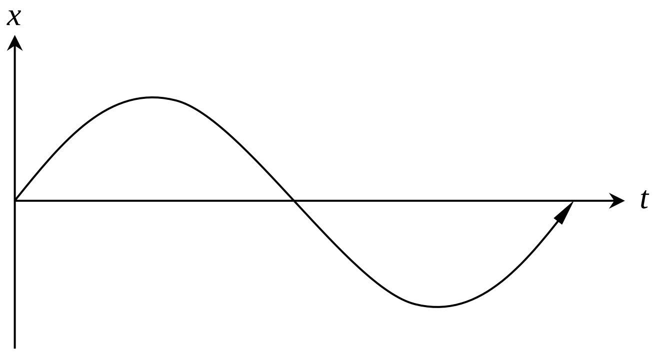
(A)




(B)




(C)

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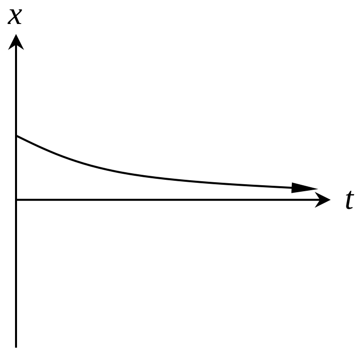

**Total marks – 60**
**Attempt Questions 11 – 14**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

(a)

(i) Find the exact value of the acute angle between the curves $y = \ln x$ and $y = x - 1$. **2**

(ii) Explain the significance of your result. **1**

(b) Use the substitution $u = 1 - \sqrt{x}$ to evaluate $\int_0^1 \sqrt{1 - \sqrt{x}}\, dx$. **3**

(c) Let $f(x) = \ln\left(\dfrac{1 + \sin x}{1 + \cos x}\right)$. **3**

Show that $f(x)$ has no stationary points.

(d) Show that the area between the curve $y = \sin^2(kx)$ and the $x$ axis in the domain $0 \le x \le \dfrac{\pi}{2}$, where $k$ is an integer, is independent of $k$. **2**

(e) A monic polynomial $P(x)$ has roots $\alpha, \beta, \gamma$. The product of its roots is equal to one and the sum of its roots equal to the sum of its reciprocals. Let the sum of its roots be $S$.

(i) Prove that $x = 1$ is a root of the polynomial. **2**

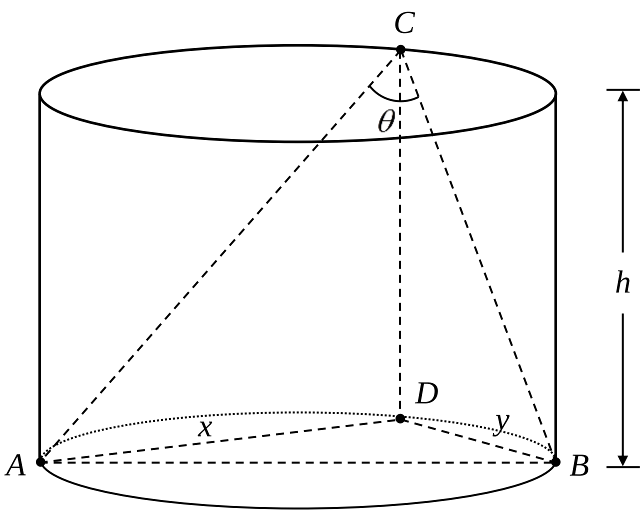
(ii) Hence, find all values of $S$ such that the polynomial has 3 real roots. **2**

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

(a) Consider a cylinder of height $h$ and radius $r$. From the point $C$ on the circumference of the top circle, lines are subtended down to the diameter $AB$. From $C$, a perpendicular is dropped down to the circumference of the base circle at $D$.

Let the angle between $AC$ and $BC$ be $\theta$. Let $DA$ and $DB$ be $x$ and $y$ respectively.




(i) Prove that **2**

$$\cos\theta = \frac{h^2}{\sqrt{h^2+x^2}\sqrt{h^2+y^2}}.$$

(ii) Hence, find the limiting value of $\theta$ as $h \to \infty$. **1**

(b)

(i) Prove algebraically that $n(n+1)$ is even for all integers $n \geq 1$. **1**

(ii) Hence use Mathematical Induction to prove that $n^3+3n^2+2n$ is divisible by 6 for all integers $n \geq 1$. **3**

**Question 12 continues on page 7**

(c) An island is currently dominated by a breed of pigeons, which has population $P_1$. A new breed of pigeons with population $P_2$ is introduced to the island, which competes against $P_1$ for food.

The rate at which $P_1$ and $P_2$ are changing are given respectively by

$$\frac{dP_1}{dt} = -\alpha\left(P_1 - P_2\right),$$

$$\frac{dP_2}{dt} = \beta\left(P_1 - P_2\right),$$

where $\alpha$ and $\beta$ are positive constants.

Let the original population of $P_1$ and $P_2$ be $A$ and $B$ respectively.

(i) By differentiating $\alpha P_2 + \beta P_1$ with respect to $t$, show that **1**

$$\alpha P_2 + \beta P_1 = k,$$

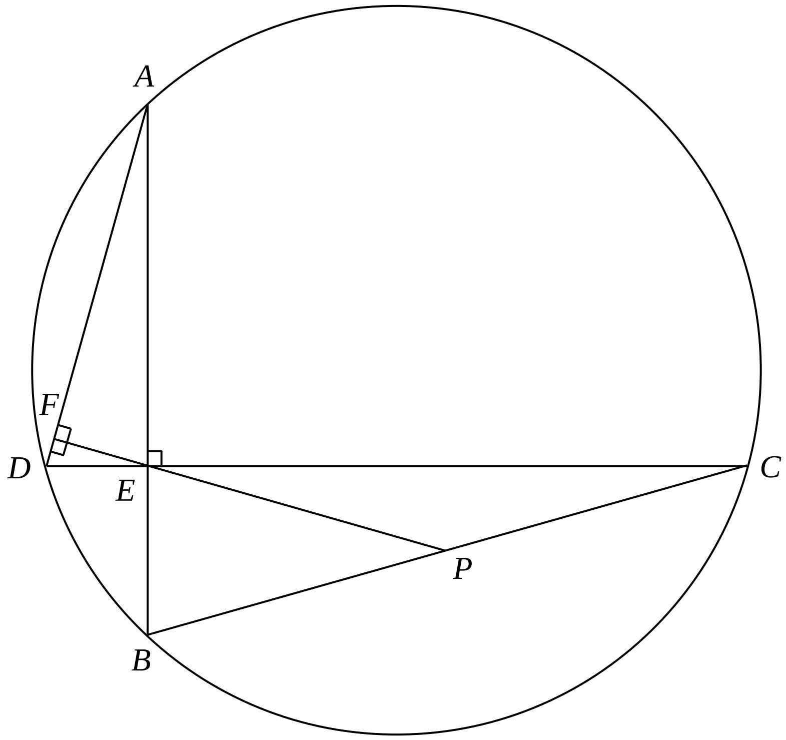
where $k$ is a constant.

(ii) Find the limiting population of both pigeon breeds in terms of $\alpha$, $\beta$ and $k$. **2**

**Question 12 continues on page 8**

(d) Let $AB$ and $CD$ be perpendicular chords in a circle, that intersect at $E$. **2**

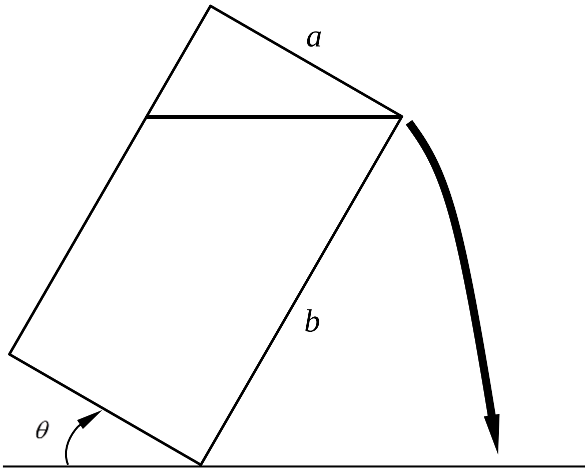
From $E$, a perpendicular is dropped to $AD$ at $F$. The line $EF$ is extended to meet $BC$ at $P$.




Prove that $P$ bisects $BC$.

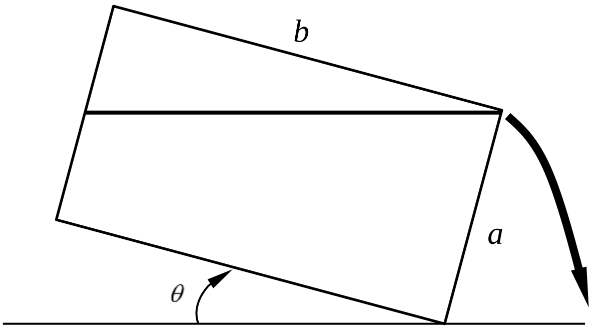
**Question 12 continues on page 9**

(e) A container in the shape of a rectangular prism has length $b$, height $a$ and width $c$, where $0 < b < a$. **3**




It is initially full of water, but the angle it makes with the horizontal slowly increases at a constant rate $\omega$ radians per second until the container is empty.

A similar container is also pouring water out in a similar fashion.




Let the rate at which the first and second container loses water, in terms of $\theta$, be $R_1$ and $R_2$ respectively.

Prove that $\dfrac{R_1}{R_2} = \dfrac{a^2}{b^2}$.

**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet.

(a) A particle moves in simple harmonic motion with amplitude $A$, period $\frac{2\pi}{n}$ and centre of motion at the origin. Initially, the particle starts from the origin and moves towards the positive extremity.

(i) Francis claims that when the particle has displacement $\frac{1}{k}$ of the positive extremity, the velocity is also $\frac{1}{k}$ of the maximum velocity, where $k > 1$. **3**

Prove that if the claim is true, then $k = \sqrt{2}$.

(ii) Let the period of motion be $T$. **1**

When do the conditions of (i) first occur, in terms of $T$?

(b)

(i) By considering $\sin\left(\sin^{-1} x + \cos^{-1} x\right)$, prove that **2**

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

(ii) Sketch the curve **2**

$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x - \frac{\pi}{2},$$

indicating any restrictions.

(iii) Hence, or otherwise, evaluate **1**

$$\int_{-1}^{1} \sin^{-1} x + \cos^{-1} x + \tan^{-1} x \, dx.$$

**Question 13 continues on page 11**

(c) The diagram shows a point $P\left(2ap, ap^2\right)$ on the parabola $x^2 = 4ay$.

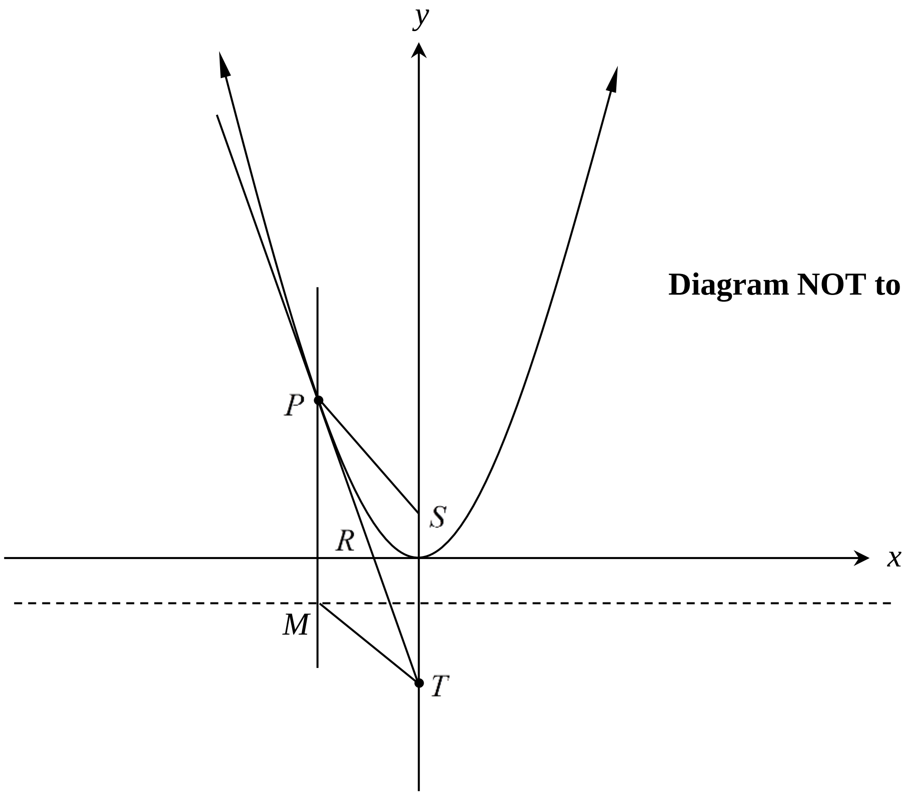



**Diagram NOT to scale.**

From $P$, a vertical line is constructed to meet the directrix at $M$. Also from $P$, a tangent is constructed that meets the $x$ axis at $R$ and the $y$ axis at $T$. Let $S$ be the focus.

You may assume that the equation of the tangent is $y = px - ap^2$.

(i) Find the coordinates of $R$. **1**

(ii) Prove that $PSTM$ is a rhombus. **2**

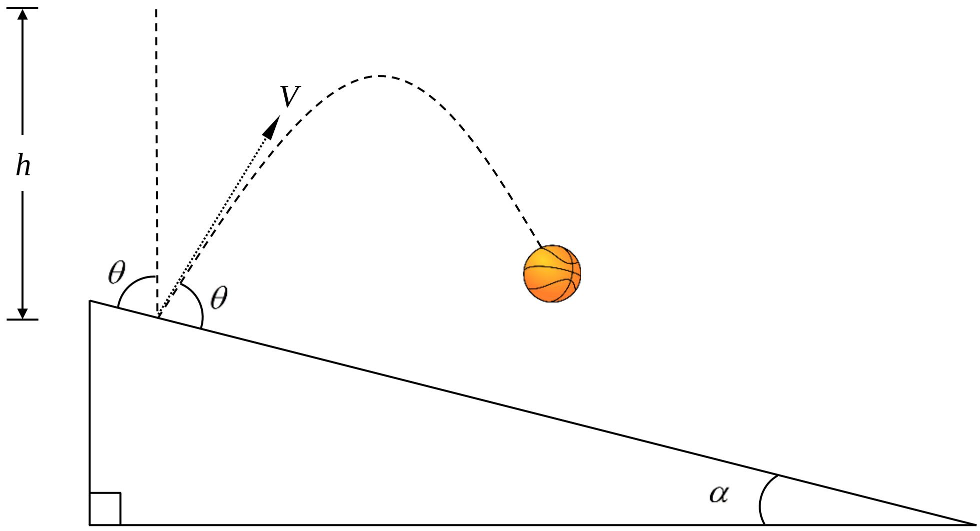
(iii) Prove that $\Delta ORS \,|||\, \Delta RPS$. **2**

(iv) Deduce that $SR^2 = SO \times SP$. **1**

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

(a) A ball is dropped from rest at height $h$ onto a ramp inclined at an angle $\alpha$. It rebounds with the same angle $\theta$ at which it landed from the ramp, and with the same velocity $V$ at which it landed. Let $g$ be the acceleration due to gravity.




(i) Show that the ball initially lands on the ramp with speed $V = \sqrt{2gh}$. **1**

(ii) Prove the following equations of motion.

(1) $x = Vt\sin 2\alpha$. **1**

(2) $y = -\frac{1}{2}gt^2 + Vt\cos 2\alpha$. **1**

(iii) The ball lands on the ramp at some point $P$. Prove that when $\alpha = \frac{\pi}{4}$, the horizontal distance of $P$ from the point where the ball was dropped is maximised. **3**

(iv) Let the ball land on the ramp again with velocity $V_0$. **2**

Prove that when the horizontal range is maximised, $V_0 = V\sqrt{5}$.

**Question 14 continues on page 13**

(b) A class has $r$ boys and $s$ girls. A group of $k$ prefects are to be selected.

(i) Explain why the number of ways of choosing $k$ prefects is $\binom{r+s}{k}$. **1**

(ii) By finding another way of choosing $k$ prefects, explain why **1**

$$\sum_{j=0}^{k}\binom{r}{j}\binom{s}{k-j}=\binom{r+s}{k}$$

(c) A basket contain $N$ pieces of fruit, $M$ of which are old.

(i) From the basket, $n$ pieces of fruit are taken. Let $P(k)$ be the probability of $k$ of those pieces being old. Explain why **1**

$$P(k)=\frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

(ii) Prove that $k\binom{M}{k}=M\binom{M-1}{k-1}$. **1**

(iii) Using (b), or otherwise, prove that **3**

$$\frac{1}{n}\sum_{k=0}^{n}k\times P(k)=\frac{M}{N}.$$

**End of Exam**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$