Part 4: Light and Relativity

Tammy Humphrey

Contents

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Syllabus content: The Nature of Light

Light and Special Relativity

Inquiry question: How does the behaviour of light affect concepts of time, space and matter?

Students:

- analyse and evaluate the evidence confirming or denying Einstein's two postulates:
  - the speed of light in a vacuum is an absolute constant
  - all inertial frames of reference are equivalent
- investigate the evidence, from Einstein's thought experiments and subsequent experimental validation, for time dilation $t_v = \frac{t_0}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$ and length contraction $l_v = l_0\sqrt{\left(1-\frac{v^2}{c^2}\right)}$, and analyse quantitatively situations in which these are observed, for example:
  - observations of cosmic-origin muons at the Earth's surface
  - atomic clocks (Hafele-Keating experiment)
  - evidence from particle accelerators
  - evidence from cosmological studies
- describe the consequences and applications of relativistic momentum with reference to:
  - $p_v = \frac{m_0 v}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$
  - the limitation on the maximum velocity of a particle imposed by special relativity
- Use Einstein's mass-energy equivalence relationship $E = mc^2$ to calculate the energy released by processes in which mass is converted to energy, for example:
  - production of energy by the sun
  - particle-antiparticle interactions, eg positron-electron annihilation
  - combustion of conventional fuel

Evidence for the equivalence of inertial reference frames

Galilean relativity

One of Galileo's important contributions to our understanding of motion was to suggest that there is no privileged reference frame in which we can determine an object's absolute motion. He proposed the following thought experiment in his 1632 book "Dialogue Concerning the Two Chief World Systems":

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals by with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath... When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still...[1]

Figure 1: Galileo Galilei

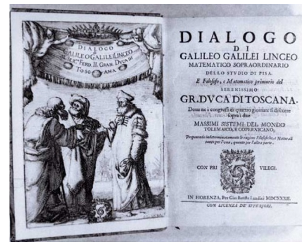

Figure 2: Galileo's Dialogue concerning two Chief World Systems (1632)

[1] A good discussion of Galileo's arguments about inertial reference frames is given in the NOVA documentary on Galileo: Galileo's battle for the heavens `https://www.youtube.com/watch?v=XCxkdR092c4`. Start at 53:29 to 56:49, 142:37 - 1:48:00.
Also, Mythbusters "ball drop" experiment: `https://www.youtube.com/watch?v=BLuI118nhzc`

In an earlier letter to Monsignor Francesco Ingoli in 1624, Galileo describes an experiment in which a cannonball is dropped from the mast of a ship. In the letter he claims to have done the experiment, with the result that the ball lands at the base of the mast.

Inertial frames of reference

An **inertial** frame of reference is **not accelerating**.

A **non-inertial** frame of reference **is accelerating**.

In inertial reference frames, Newton's 1st law holds. Objects with no unbalanced force acting on them continue to move with a constant velocity. The laws of physics (Newton's 2nd law, conservation of momentum and conservation of energy) hold true in all inertial reference frames, and no experiment can distinguish one frame from another. This can be expressed as:

All inertial reference frames are equivalent.

In *non-inertial* reference frames Newton's 1st law does *not* hold. Objects with no net force acting on them can move with a non-constant velocity.

Consider the situation shown in figure 3. The person on the right throws a ball directly towards the person on the left. In the inertial frame of the camera above them, the ball travels in a straight line over the next two frames of the video. In the non-inertial frame of the person on the left on the platform, the ball begins to travel towards them but appears to curve away as it moves, even though there is no horizontal force acting on it in that direction.

By examining the motion of the ball, the observers can tell that they are in a non-inertial reference frame and can measure the acceleration of their frame.

Figure 3: Three frames from a video showing two people seated on a rotating platform.

Equivalence of reference frames in classical mechanics

What we have seen so far is that experiments in classical mechanics, for example throwing a ball, or studying momentum conservation in a collision, yield the same results regardless of the inertial reference frame of the observer.

This is evidence that "all (inertial) reference frames are equivalent" in classical mechanics.

This equivalence is sometimes called "Galilean relativity" to distinguish it from Einstein's special relativity, which we will see is able to explain how inertial reference frames are still equivalent, even if electromagnetic effects are considered.

Equivalence of reference frames and the laws of electromagnetism

With the development of Maxwell's equations in the late 1800's, scientists began to suspect that inertial reference frames were *not* equivalent with regards to the laws of electromagnetism[2].

For example, in year 11 and year 12 you have used the Lorentz force to calculate the force acting on a charge moving in a magnetic

[2] Joe Wolfe's "Einstein Light" website provides a great discussion of this material, as well as the rest of what we will cover in Special Relativity: `http://newt.phys.unsw.edu.au/einsteinlight`

field as

$$F = qvB \sin\theta$$

However the velocity v is determined by the reference frame of the observer. Does this mean that if the observer changes their reference frame to that of the moving charge, the force due to the magnetic field simply disappears? [3]

[3] We will see later that Einstein's theory of special relativity provides a solution to this puzzle.

Scientists in the late 1880s believed that the speed of light that is determined from Maxwell's equations must be the speed of light relative to the medium through which it travelled, the 'aether'. It was expected that if an observer on earth was moving through the aether (as the earth orbits the sun), then the observer would see the speed of light vary, depending on the relative motion of the light and the observer through the aether.

The Michelson- Morley experiment

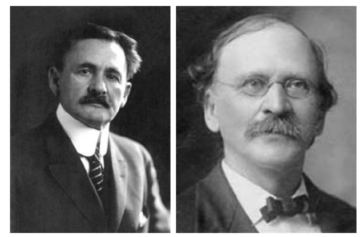

Figure 4: Albert Michelson (left) and Edward Morley (right).

Albert Michelson and Edward Morley [4] designed an experiment to attempt to experimentally determine the velocity of the earth relative to the aether by observing variations in the speed of light .

Michelson and Morley used an interferometer[5] - an apparatus designed to split a coherent light source into two beams which travel along two different paths and are then recombined to form an interference pattern.

[4] Full-text of the Michelson-Morley paper: `https://history.aip.org/exhibits/gap/PDF/michelson.pdf`

[5] See animation at: `https://kcvs.ca/details.html?key=michelsonMorley`. Note: you need to have the flash shockwave player enabled to use this.

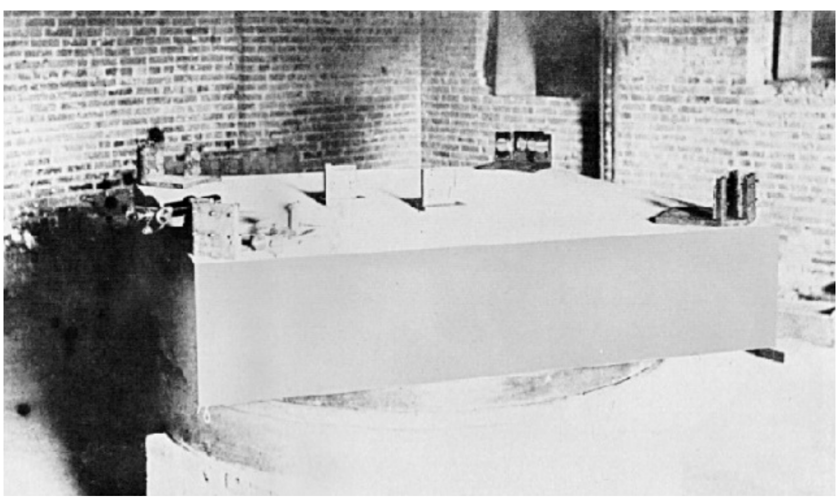

Figure 5: Photo of Michelson and Morley's experiment setup.

Figure 6(a) shows the Michelson-Morley interferometer in the frame of reference of the earth. In this reference frame the light in each beam travels the same distance to each mirror and back, but is expected to travel with different speeds along the two paths. Figure 6(a) shows the Michelson-Morley interferometer in the frame of

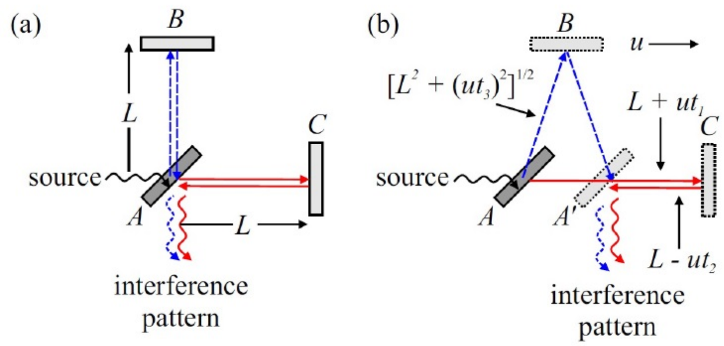

Figure 6: (a) Diagram of the path taken by light in the reference frame of Earth. (b) The path taken by light in the (supposed) reference frame of the aether.

reference of the hypothetical aether [6] if the earth, and thus the interferometer, are moving at a velocity u to the right relative to the aether. In this case the two beams of light do not travel the same distance in the reference frame of the aether. [7] Instead, the distance the light travels along the 'solid' path from the half-silvered mirror A to mirror C is equal to $L + ut_1$ where t_1 is the time light takes to travel that distance. As light travels with speed c in the reference frame of the aether, we can express this as

$$ct_1 = L + ut_1$$

$$t_1 = \frac{L}{c - u}$$

Similarly, the distance that light travels back, from mirror C to the half-silvered mirror A (which has now moved to the position indicated by A′) is

$$L - ut_2$$

where t_2 is the time taken for the trip. This can be expressed as

$$ct_2 = L - ut_2$$

$$t_2 = \frac{L}{c + u}$$

The total time for light to travel along the leg of the interferometer from A to C and back again is thus

$$t_1 + t_2 = \frac{L}{c - u} + \frac{L}{c + u}$$

Which can be rearranged to give

$$t_1 + t_2 = \frac{L\,(c + u) + L\,(c - u)}{c^2 - u^2}$$

$$t_1 + t_2 = \frac{2Lc}{c^2 - u^2}$$

[6] Important: The aether does not exist. However it is convenient for describing the expected results of the Michelson-Morley experiment to temporarily speak as if it did.

[7] Note that they do still travel the same distance in the reference frame of an observer on earth.

$$t_1 + t_2 = \frac{2L/c}{1 - u^2/c^2}$$

The dotted beam travels a distance

$$\sqrt{L^2 + (ut_3)^2}$$

(by Pythagoras' theorem) where t_3 is the time it takes for light to travel from the half-silvered mirror A to the mirror B and ut_3 is thus distance to the right that the apparatus has travelled in time t_3. Light travelling on this branch travels the same distance (by symmetry) from B back to the half-silvered mirror, which has now moved to the right a distance $2ut_3$. Thus the total distance travelled by the light on the dotted path is

$$\sqrt{L^2 + (ut_3)^2}$$

However, as light travels with speed c in the frame of reference of the aether we can express this as

$$2ct_3 = 2\sqrt{L^2 + (ut_3)^2}$$

Which can be rearranged to give

$$c^2 t_3^2 = L^2 + u^2 t^2$$

$$t_3^2 \left(c^2 - u^2\right) = L^2$$

$$t_3^2 = \frac{L^2}{c^2 - u^2}$$

$$t_3^2 = \frac{L^2/c^2}{1 - u^2/c^2}$$

so

$$t_3 = \frac{L/c}{\sqrt{1 - u^2/c^2}}$$

Thus, it can be seen that the time taken for the light to travel along the leg of the interferometer that is parallel to the velocity of the earth through the aether is expected to be longer than the time for it to travel along the leg that is perpendicular to the motion.

$$\frac{2t_3}{\sqrt{1 - u^2/c^2}} = t_1 + t_2$$

This time difference will result in a phase difference when the two beams are recombined to form an interference pattern. In general this will mean that the interference pattern obtained when the apparatus is aligned with neither leg parallel to the motion of the earth through the aether (such that the path lengths are equal) will be different to that obtained when one leg is aligned parallel to the motion of the earth through the aether. As the apparatus is rotated between the two positions, bright and dark bands in the interference pattern should move across the field of view as the relative phase difference between the two beams changes.

It was expected that a shift in the interference pattern of at least 0.4 fringes would be observed as the apparatus rotated through 90°, due to the difference in velocity of the light in the two directions (as a result of the earth's motion through the aether).

No shift was observed in the experiment.

Interpretation of Michelson and Morley's null result

The null result was very puzzling. It was hard to reconcile the existence of the aether and the fact that it could be stationary with respect to the earth (which is rotating and orbiting the sun).

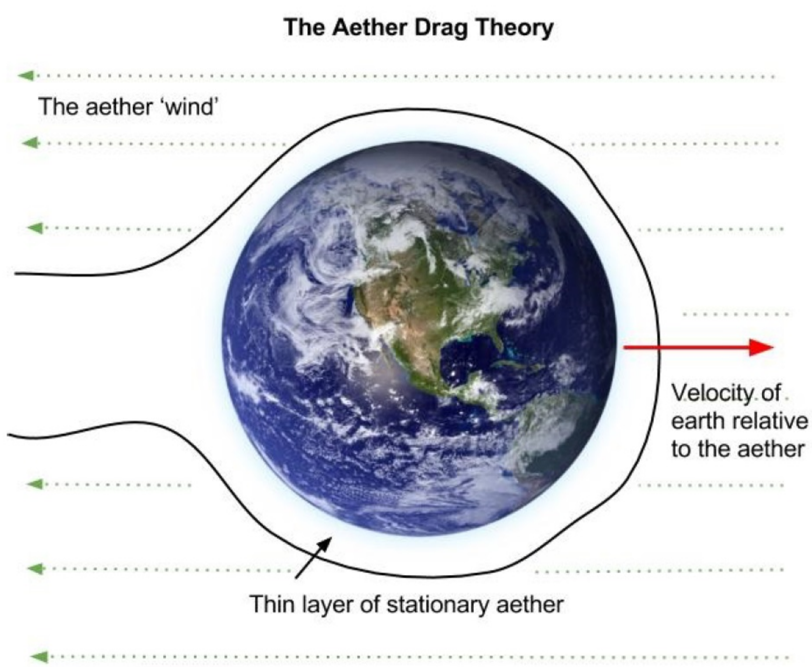

Figure 7: A diagram illustrating the theory of aether drag.

At first, Michelson believed that the result:

> "could still be accounted for by the assumption that the earth drags the aether along at nearly its full speed, so that the relative velocity between the aether and the earth at its surface is zero or very small."

However later experiments by Oliver Lodge and Gustaf Wilhelm Hammar [8] effectively ruled this out.

Lorentz suggested an alternative explanation. Perhaps moving bodies contract in their direction of motion so that their moving

[8] `https://en.wikipedia.org/wiki/Hammar_experiment`

length is related to their rest length by

$$L_v = L_0\sqrt{1 - v^2/c^2}$$

This would shorten the path travelled by the light by exactly the factor required to cause the Michelson-Morley experiment to yield a ‘null’ result. Lorentz also pointed out that this result could be used to make Maxwell’s equations yield the same answer in different reference frames. Nonetheless, Lorentz could not provide a satisfactory explanation for why such a shortening might occur[9].

[9] https://en.wikipedia.org/wiki/Lorentz_ether_theory

Einstein’s theory of Special Relativity

Einstein’s two postulates of Special relativity

In the introduction to his 1905 paper "On the electrodynamics of moving bodies", Einstein makes two points. Firstly, he argues that electromagnetic experiments yield the same result when performed in different reference frames, despite the theoretical explanations for the effects depending upon the velocity of the charges in that frame. Secondly, he notes that attempts to measure the motion of the earth relative to a ‘light medium’ have so far been unsuccessful.

In Einstein’s own words:

> "This suggests that electrodynamics as well as mechanics possess no properties corresponding to the idea of absolute rest. Rather... ...the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We raise this conjecture (...which will hereafter be called ‘The principle of Relativity’) to the status of a postulate , and also introduce another postulate... ...namely that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body."

These ideas can be expressed concisely as follows:

Einstein’s two postulates of special relativity:

- The speed of light in a vacuum is an absolute constant
- All inertial frames of reference are equivalent (i.e. The laws of physics are the same in all inertial reference frames).

Evidence for the equivalence of reference frames in electromagnetism

We have already seen that everyday experiments in mechanics, such as observing that a pendulum hangs straight down in a car or plane (or train..!) moving at constant velocity provides evidence that inertial reference frames are equivalent with regard to mechanics.

In his paper Einstein uses evidence from experiments in electromagnetic induction (ie. moving the coil and leaving the magnet stationary OR keeping the coil stationary and moving the magnet) that electromagnetic experiments yield the same results in different inertial references frames.

Evidence for the constancy of the speed of light from astronomy

Einstein argued that in order that to preserve relativity, it was necessary that all observers measure the same speed for light[10]. At the time Einstein published his paper on special relativity, Michelson-Morley's experiment provided some initial experimental evidence in support of this postulate.

[10] See the Veritasium video: `https://www.youtube.com/watch?v=vVKFBaaL4uM`

In 1913 Willem de Sitter, a Dutch astromomer and mathematician demonstrated that observations of binary stars in astronomy provided additional strong experimental evidence in support of Einstein's postulate.

De Sitter noted that the shifts in the spectrum of binary stars are sharply defined, the spectrum is red and blue shifted alternatively as the stars orbit one another. If the speed of light depended upon the velocity of the emitting body, then light emitted when a star in a binary system was moving towards us could overtake light emitted when the star was moving away from us, which would mean that the signal would be smeared out in time and this periodic shifting would not be observable.[11].

[11] `https://en.wikisource.org/wiki/A_proof_of_the_constancy_of_the_velocity_of_light`

Summary: Evidence "confirming or denying" Einstein's two postulates

All experimental evidence obtained to date has been consistent with Einstein's two postulates of special relativity

1. The speed of light in a vacuum is an absolute constant
2. All inertial frames of reference are equivalent (i.e. The laws of physics are the same in all inertial reference frames).

- Evidence for the first postulate was initially provided by the Michelson-Morley experiment (include a diagram!) which used an interferometer to examine the interference pattern produced by two beams of light that were split and sent on perpendicular paths and then recombined. It was expected that rotating the arms of the interferometer would shift the interference pattern due to a difference in the speed of the earth through the aether in each direction, but no shift was detected. The null results of this experiment were consistent with a constant speed of light.
- Willen de Sitter demonstrated that the clear shift in the spectrum of binary stars as they orbit each other was excellent evidence in support of the first postulate, as any variation in the speed of light emitted by the stars due to their motion would "smear-out" the spectrum as light would arrive at different times due to a difference in speed.
- Evidence for the second postulate is provided by experiments in mechanics, which always yield the same results when performed in different inertial reference frames (for example, a mass on a string will hand straight down whether the refernece frame is "stationary" (relative to the earth) or moving with a constant velocity.
- Evidence for the second postulate is *also* provided by experiments in electromagnetism, which *also* yeild the same results regardless of the inertial reference frame of the observer. For example, Faraday's induction experiment yields the same result whether the magnet is stationary and the coil moves, or the magnet moves and the coil is stationary.

Thought experiments in relativity

The value of thought experiments in special relativity is that they draw out the logical implications of Einstein's two postulates. They therefore serve a different purpose to 'real' experiments, which test whether the predictions of the theory are in accord with reality. Real experiments cannot prove that the theory is correct, they can only yield results that are either consistent or inconsistent with the predictions of special relativity. To date, all relevant experimental results have been consistent with the predictions of special relativity.

Thought experiment - the relativity of simultaneity

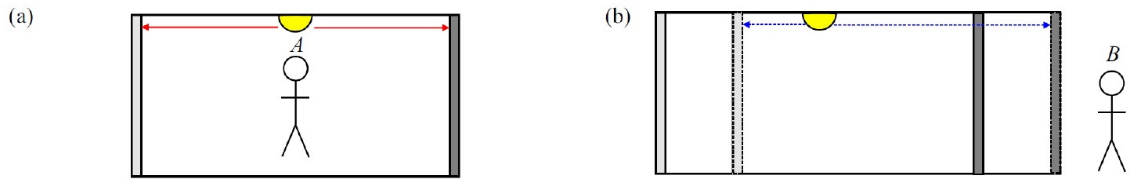

Figure 8: Relativity of Simultaneity "Light operated doors" thought experiment. (a) Observer A inside a train carriage sees two light operated doors which are equidistant from a light in the centre of the carriage open simultaneously when the light is switched on. (b) An observer B with respect to whom the carriage is moving at a constant velocity u to the right, sees the door on the left open before the door on the right, as in his reference frame the light travels a shorter distance to reach this door.

Figure 8 shows a thought experiment which illustrates how Einstein's two postulates imply that events that are simultaneous for one inertial observer will in general not be simultaneous for a different inertial observer.

Figure 8(a) shows observer A inside a train carriage, positioned directly below a light which is in the centre of the carriage. At either end of the carriage are 'light operated' doors. When the light is switched on, observer A sees the doors open simultaneously, as the distance between the light and each of the doors is equal, as is the time taken for light to travel to each.

Figure 8(b) shows observer B who is outside the train carriage, which is moving with a non-zero velocity to the right in this reference frame. In this reference frame light travels further to reach the door on the right (which has moved away from the light source during the time taken for light to travel towards it) than it does to reach the door on the left (which has moved towards the light source during this time). Therefore, in this reference frame of observer B the door on the left-hand side of the train carriage opens before the door on the right.

In general, events that are simultaneous for one inertial observer will not be simultaneous for another.

The thought experiment on the relativity of simultaneity that Einstein himself used in his 1916 book "Relativity" goes along the following lines:

Consider a railway carriage which is struck on each end by lightning. An observer on the ground (who happens to be located in the center of the carriage) sees the strikes to occur at the same time. As light travels the same distance from either end of the carriage to him, he says that the lightning strikes must have been simultaneous.

The observer on the ground will also note that light from the front of the train will arrive at an observer in the center of the train *before* light from the back of the train, as the observer on the train is moving towards the light from the front, and away from the light at the back, so light travels a shorter distance from the front of the train and a longer distance from the back of the train to reach the observer on the train. She will see the front strike before she sees the back strike.

However, if we now shift to the reference frame of the observer on the train, she sees light travel the *same* distance to her from the front and back of the train, and with the same speed as measured by the observer on the ground. This means that if she sees the front strike before the back strike, then in her reference frame *the front strike happened before the back strike*.

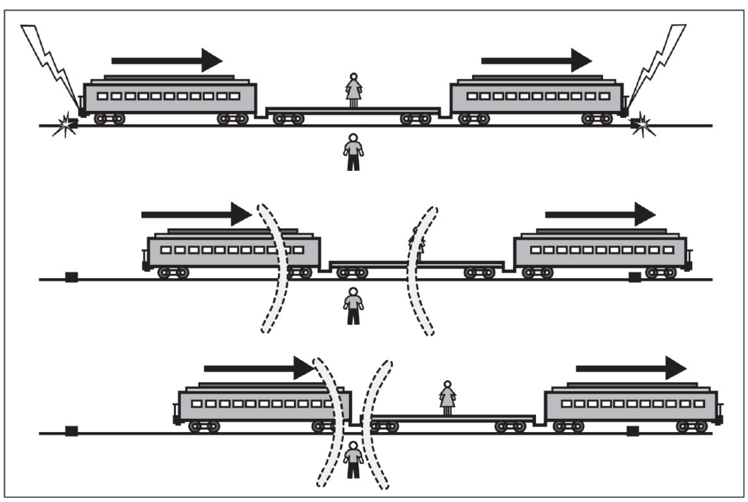

Figure 9: Relativity of Simultaneity "Lightening striking a railway carriage" thought experiment. See `https://www.youtube.com/watch?v=wteiuxyqtoM` for an animated version.

Time dilation

The measurement of time has an important place in special relativity. A 'light clock' is often used in thought experiments (see figure 10) consists of an 'emitter' marked E in figure, which produces a short pulse of light and a mirror M which reflects this pulse back towards a detector D adjacent to the emitter. Each round trip is taken to correspond to one 'tick' of the clock.

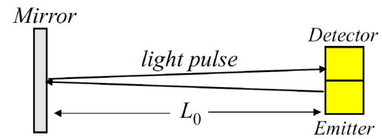

Figure 10: A light clock. A pulse of light is produced by the emitter labelled E and is reflected by the mirror M back to the detector D, and each time light is detected this is counted as one 'tick' of

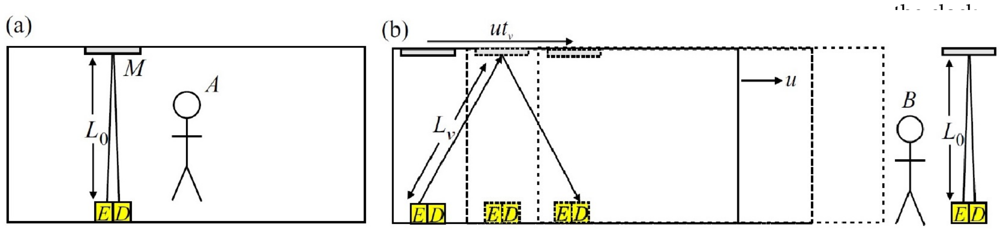

Figure 11: An observer inside a train carriage with their own light clock.

Figure 11(a) shows a light clock inside a train carriage. The mirror is a distance L_0 from the emittor and detector in the rest frame of the light clock, which corresponds to the rest frame of observer A. The time between 'ticks' of the clock for this observer is equal to $t_0 = 2L_0/c$ Figure 11(b) shows the light clock as seen by observer B standing outside the carriage, which is moving with velocity u to the right in this observer's reference frame. Now B measures the distance that the light pulse travels during one 'tick' of the clock to be

$$2L_v = 2\sqrt{L_0^2 + \left(\frac{ut_v}{2}\right)^2}$$

where t_v is the time for one tick as measured by B. As light also travels with speed c in B's reference frame it must be the case that $t_v = 2L_v/c$ so

$$t_v = \frac{2\sqrt{L_0^2 + \left(\frac{ut_v}{2}\right)^2}}{c}$$

whence it can be shown that

$$\frac{t_v}{2} = \frac{\sqrt{\left(\frac{ct_0}{2}\right)^2 + \left(\frac{ut_v}{2}\right)^2}}{c}$$

$$\left(\frac{t_v}{2}\right)^2 = \left(\frac{t_0}{2}\right)^2 + \left(\frac{ut_v}{2c}\right)^2$$

$$t_v^2\left(1 - (u/c)^2\right) = t_0^2$$

$$t_v = \frac{t_0}{\sqrt{1 - u^2/c^2}}$$

Giving the equation for time dilation:

$$t_v = \frac{t_0}{\sqrt{1 - u^2/c^2}} \tag{1}$$

This means that observer B will see the light clock on the carriage tick more slowly than an identical light clock at rest in his reference frame as the light pulse in the clock in the carriage must travel a larger distance (at the same speed c) than the light pulse in his own clock. This 'time-dilation' effect occurs not just for 'light' clocks but for any mechanism for measuring time, atomic (cesium) clocks, the period of a pendulum or your pulse. An observer will measure time to progress more slowly in reference frames that are moving with respect to their own reference frame.

This effect is symmetric, as it must be if all reference frames are equivalent. The observer on the carriage will see a light clock on the train platform to tick more slowly than his own light clock.[12]

[12] See: `http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/Lightclock/home.html`

Length Contraction

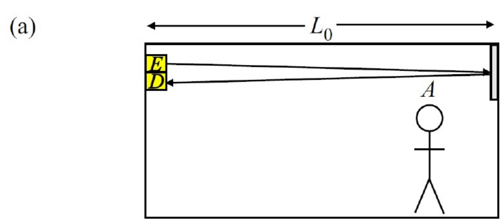

Figure 12: (a) A light clock at rest in the reference frame of observer A in a train carriage of length L_0 as measured by A.

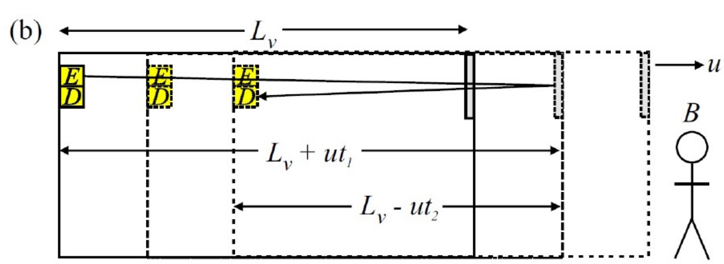

Figure 13: (b) The same light clock as seen by observer B, in whose frame the carriage is moving to the right with velocity u. In B's reference frame the length of the train carriage is contracted to length L_v, given by equation [eq:length contraction].

Figure 12(a) shows a light clock arranged so that the light pulse travels horizontally the whole length of a train carriage and back. Observer A is at rest with respect to this light clock and measures the distance that light travels over a whole 'tick' of the clock to be $ct_0 = 2L_0$

In the reference frame of observer B's the carriage is moving to the right with velocity u. In this reference frame, the light pulse travels a distance $ct_1 = L_v + ut_1$ from the emitter to the mirror in this reference frame, where t_1 is the time taken by the light pulse to travel this distance and L_v is the length of the carriage as measured by B. Then

$$t_1 = \frac{L_v}{c-u}$$

The light pulse travels a distance

$$ct_2 = L_v - ut_2$$

from the mirror back to the detector in this reference frame, where t_2 is the time taken by light to travel this distance. Then

$$t_2 = \frac{L_v}{c+u}$$

From equation 1 we know that

$$t_1 + t_2 = \frac{t_0}{\sqrt{1-u^2/c^2}}$$

It follows that

$$L_v\left(\frac{1}{c-u}+\frac{1}{c+u}\right) = \frac{2L_0}{c\sqrt{1-u^2/c^2}}$$

$$L_v\left(\frac{2c}{c^2-u^2}\right) = \frac{2L_0}{c\sqrt{1-u^2/c^2}}$$

$$L_v\frac{1}{1-u^2/c^2} = \frac{L_0}{\sqrt{1-u^2/c^2}}$$

$$L_v = L_0\sqrt{1-u^2/c^2}L_v = L_0\sqrt{1-u^2/c^2}$$

Which gives the equation for length contraction:

$$L_v = L_0\sqrt{1-u^2/c^2} \qquad (2)$$

Length contraction means that an object which is moving relative to an observer will have a length that is contracted in the direction of motion relative to the length it would have it it was at rest with respect to the observer. As is the case for time dilation, length contraction is a symmetric - observers in different inertial reference frames will measure lengths in the other frame to be contracted.

Note that lengths perpendicular to the direction of motion are not contracted.

Evidence for time dilation and length contraction

Cosmic-origin muons at the earth's surface

Muons are unstable particles that can be created as a result of the interaction of cosmic rays with the upper atmosphere. When at rest (i.e. in their own reference frame) their lifetime is 2.2μs. In 1941 Rossi and Hall measured the flux of muons travelling at $0.994c$ at the top of Mount Evans in Colorado and compared it to the flux of muons at a lower altitude in Denver. Many more muons were detected than would be expected from their short lifetime before they decay. The result could only be explained by assuming that in the earth's frame of reference the muon lifetime was increased due to time dilation, or, in the muon's frame of reference, that the height of the atmosphere is length contracted [13].

Figure 14: Mt Evans and Summit lake, the location of one of Rossi and Hall's experiments to measure the flux of muons at different altitudes.

[13] See the minute physics video for a good discussion: `https://www.youtube.com/watch?v=rVzDP8SMhPo`. A much longer video from 1962 showing details of the experiment: `https://www.youtube.com/watch?v=ivV1FdRXmIk` and a good animation: `http://faraday.physics.utoronto.ca/PVB/Harrison/SpecRel/Flash/LengthContract.html` (needs flash)

The Hafele-Keating Experiment

In 1971 Hafele and Keating flew four atomic clocks around the world on commercial flights both eastwards (so that the velocity of the plane adds to the rotational velocity of the earth) and westwards (so that the velocity of the plane is is in the opposite direction to the rotational velocity of the earth). They compared the time that passed on these clocks compared to identical atomic clocks left behind on the ground at the US Naval observatory. It was found that the moving clocks exhibited time dilation to an extent consistent with the predictions of special relativity (there was also a variation in time passing on the flying clocks due to the predictions of general relativity, and this too was accounted for)[14].

Figure 15: Hafele and Keating with clocks. Time Magazine, October 18, 1971

[14] Full-text of Hafele and Keating's papers: `http://www.personal.psu.edu/rq9/HOW/Atomic_Clocks_Predictions.pdf` and `http://personal.psu.edu/rq9/HOW/Atomic_Clocks_Experiment.pdf`

Evidence from particle accelerators and cosmological studies

Background: The relativistic Doppler effect

The longitudinal relativistic Doppler effect results in a shift in the observed wavelength (and frequency) of light when there is relative motion between the source and the observer[15].

[15] See the discussion at Hyperphysics `http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/reldop2.html#c2`

Time dilation means that there is an additional effect due to time dilation that means that the rate at which crests arrive at the observer (i.e. the observed frequency of the wave) is different to the frequency of the wave measured in the reference frame of the source.

The transverse doppler effect is a purely relativistic effect related to the effect of *transverse* relative motion between the source and observer - the physical situation we considered earlier when we discussed Bradley's measurement of the speed of light using stellar aberration[16].

[16] This is not discussed in depth here as it is not explicitly in the syllabus. The article in Wikipedia provides a good explanation: `https://en.wikipedia.org/wiki/Relativistic_Doppler_effect`

Particle accelerators

In a paper published in Physical Review Letters in 2014[17], physicist Benjamin Botermann reported an experiment to test time-dilation which used the Experimental Storage Ring in Damstadt, Germany (a particle accelerator) to compare the frequency of electron transitions in lithium ions moving at one third the speed of light to the frequency of these transitions at rest. The difference they observed agreed with that predicted by time dilation (via the relativistic transverse Doppler effect).

[17] Summaries of the research for the general public: `https://physics.aps.org/articles/v7/s107` and `https://www.scientificamerican.com/article/einsteins-time-dilation-prediction-verified/`, the article in Physical Review Letters: https://journals.aps.org/prl/abstract/10.1103/PhysRevL and the fulltext: `https://arxiv.org/abs/1409.7951`

Evidence from cosmological studies

Cosmological evidence for the constancy of the speed of light is provided by the distinct shifts in binary star spectral described by de Sitter in 1913 (described in an earlier section).

The longitudinal and transverse Doppler effects are predicted by time-dilation, so cosmological observations consistent with these can be considered to be tests of time-dilation.

One such test is the observation of relativistic beaming of light from very high speed jets of matter in objects such as the supergiant elliptical galaxy Virgo A (known as "M87"). This effect is caused by a combination of two relativistic phenomena.

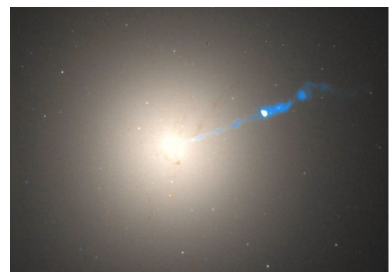

Figure 16: The galaxy M87 showing the blue relativistic jet.

Firstly, time-dilation predicts the relativistic Doppler effect, which produces a shift in the wavelengths of the light from the jet moving towards us to the blue end of the spectrum (as can be seen in figure 16).

The second effect is due to the relativistic aberration (a relativistic form of the aberration of light we discussed as the basis for Bradley's determination of the speed of light), which results in a shift in the direction of light emitted by particles moving at relativitic speeds so that the energy is directed ("beamed") towards the direction of motion of the particle [18]. This effect accounts for why only one beam is seen in M87 - light from the beam in the opposite direction is directed ("beamed") away from us and towards the direction that the particles in the opposite jet are travelling.

[18] `https://www.fourmilab.ch/cship/aberration.html` and `https://en.wikipedia.org/wiki/Relativistic_beaming`

Evidence from particle accelerators for relativistic momentum (inertial mass dilation)

Early confirmation of mass dilation (i.e. an increase in *inertial mass*) was achieved in experiments in 1909 which measured the charge to mass ratio of high speed electrons and found that the ratio depended upon the speed. When mass dilation was accounted for, the charge to mass ratio for all the electrons was found to be constant. A further confirmation of mass dilation is the fact that particle accelerators cannot accelerate particles to speeds equal to or greater than the speed of light. As speed increases, larger and larger forces

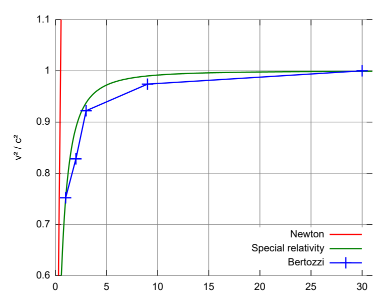

Figure 17: Bertozzi experiment to measure relativistic energy. [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0)]

are required to achieve the same acceleration, implying that the mass of the particles increases with their speed. In the 1960's Bertozzi performed tests using a particle accelerator to compare the speed of electrons to their kinetic energy (as determined by heating effects in an absorber material), with results that agreed with the predictions of special relativity[19].

[19] An original video by Bertozzi demonstrating his experiment at MIT (really good, but about half an hour long...) https://www.youtube.com/watch?v=B0BOpiMQXQA

Relativistic momentum

In order that the momentum of an isolated system is conserved in special relativity, as it is in Newtonian mechanics, it is necessary to change our definition of inertial mass and so our definition of momentum.

Figure 18: Two objects (e.g. spacecraft) approaching each other on either side of an imaginary (or real!) string running along the $x-$axis. Figure from Michael Fowler's lectures on Special Relativity: http://galileo.phys.virginia.edu/classes/109.mf1i.fall03/lectures09.pdf

Conservation of momentum thought experiment

Imagine we have two identical spacecraft approaching each other on either side of an imaginary (or real!) string running along the $x-$axis with a relative speed of $0.6c$. As they pass they experience a glancing collision with each other so that they each acquire a small velocity in the y-direction. The symmetry of the situation requires that the velocity (and so the momentum) they gain in the y-direction, away from the string-line, is equal and opposite, such that momentum is conserved[20].

Figure 19: Two objects (e.g. spacecraft) after a glancing collision. As the situation is symmetric in this reference frame, the momentum gained by each object in the $y-$direction must be equal and opposite.

[20] This thought experiment follows that given in Michael Fowler's lectures: http://galileo.phys.virginia.edu/classes/109.mf1i.fall03/lectures09.pdf

If we now consider the situation as seen by an observer in one of the spacecraft, then they will measure that they are moving away from the string at some rate, for example 10m per second. By symmetry, an observer on the other spacecraft must also measure themselves to be moving away from the string at 10m per second.

The spacecraft have a velocity of $0.6c$ relative to each other in the $x-$direction, however the relative velocity in the y-direction is non-relativistic. This means that distances in the x-direction are length contracted, but distances in the y-direction are not. Both observers will agree about distances in the $y-$direction, *but* each sees the clock on the other spacecraft running slow by a factor $(1 - v^2/c^2)^{-1} = (1 - 0.6^2)^{-1} = 1/0.8 = 1.25$, due to their relative velocity in the x-direction.

Each observer therefore sees themselves moving away from the string ling at 10m every second, but observes 1.25s to pass on their clock before the other spacecraft has moved 10m, so each observer sees the other spacecraft travelling away from the string-line *more slowly* than they themselves are travelling away from the string-line.

If each observer sees the *velocity* of the other spacecraft to be smaller than their own, then in order to conserve momentum in the reference frames of the observers on the spacecraft they must measure the *mass* of the other spacecraft to be larger than their own. In that way the lower velocity of the other spacecraft is exactly balanced by an increase in mass.

We therefore define:

Relativistic momentum:

$$p = \frac{m_0 v}{\sqrt{(1 - \frac{v^2}{c^2})}} \tag{3}$$

The fact that the initial mass of an object depends upon its speed places a *limitation on the maximum velocity of a a particle* - it is not possible to accelerate an object so that it travels with the speed of light (as infinite force would be required).

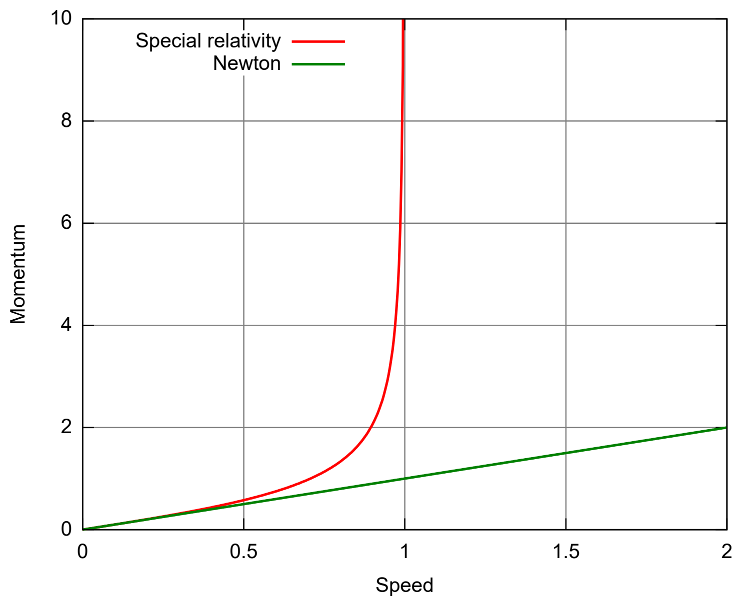

Figure 20: Momentum versus speed for special relativity and classical mechanics.

Energy in special relativity

The relationship between total energy of a particle and its kinetic and 'rest' energies is given by [21]:

> Relativistic energy:
>
> $$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{4}$$

In any process in which energy is released, such as nuclear fission or fusion, or in chemical reactions, or in the emission of light, the mass before the release is greater than the mass after the release of energy. The difference in mass, known as the **mass defect** is related to the energy released by:

> The energy equivalent of mass (for particles with negligible momentum):
>
> $$E = m_0 c^2 \tag{5}$$

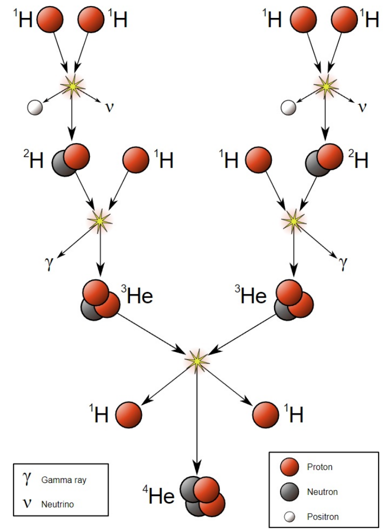

Figure 21: The proton-proton chain fusion cycle for the production of Helium nuclei from protons in stars. Image credit: Wikipedia [Public domain].

Example: Production of energy by the sun

Energy is released in our sun (and in other starts) as a result of the fusion of hydrogen.

There are two pathways taken, the proton-proton chain (see figure 21) and the CNO cycle[22] (see figure 22). The proton-proton chain tends to dominate in low mass stars such as our sun, and the CNO cycle in larger mass stars.

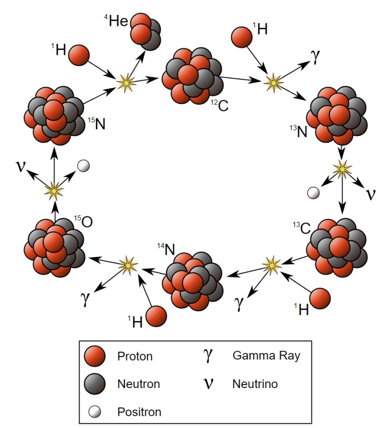

Figure 22: The CNO-cycle mechanism for fusion of hydrogen to helium in which carbon 12 acts as a catalyst. Image credit: Wikipedia, By Borb, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=691758`.

Question 1. Using the table on the next page, calculate (a) the mass defect in kilograms and (b) the energy released in the first part of the proton-proton chain, which involves the reaction:

$^1_1H +^1_1 H \rightarrow^2_1 H + \nu + e^+$

Question 2. Using the table on the next page, calculate the energy released in the last stage of the CNO cycle, which involves the reaction:

$$^{15}_{7}N +^1_1 H \rightarrow^4_2 He +^{12}_{6} C$$

Express your answer in MeV.

[21] For a derivation, see `http://galileo.phys.virginia.edu/classes/109.mf1i.fall03/lectures09.pdf`

[22] See, for example: `http://hyperphysics.phy-astr.gsu.edu/hbase/Astro/solarpp.html#c1` and `http://astronomy.swin.edu.au/cosmos/c/cno+cycle` or `https://www.atnf.csiro.au/outreach//education/senior/astrophysics/stellarevolution_mainsequence.html`

Isotope	Mass (u)
${}^{1}_{1}H$	1.007276466879
${}^{2}_{1}H$	2.01410177811
${}^{3}_{2}He$	3.01602932265
${}^{4}_{2}He$	4.001506179127
${}^{12}_{6}C$	12
${}^{12}_{7}N$	13.00573861
${}^{13}_{6}C$	13.003355
${}^{14}_{7}N$	14.00307400446
${}^{15}_{8}O$	15.0030656
${}^{15}_{7}N$	15.0001088989
e^{+}	$5.48579909 \times 10^{-4}$

Example: Antimatter-matter annihilation

See video by minute physics: `https://www.youtube.com/watch?v=Lo8NmoDL9T8`

Question 3. Calculate the energy released by the annihilation of an electron and a positron.

$$e^{-} + e^{+} \rightarrow 2\gamma$$

Example: Combustion of conventional fuel

Question 4. If the energy released when methane gas combusts with oxygen is 50.1kJ per gram of methane, what is the mass defect between the reactants and the products due to the combustion of 1g of methane?

Extension material for olympiad students (not in HSC syllabus)

Minkowski space-time diagrams

Space-time diagrams are very useful tools for visualising how length and time measurements are affected by relative motion.

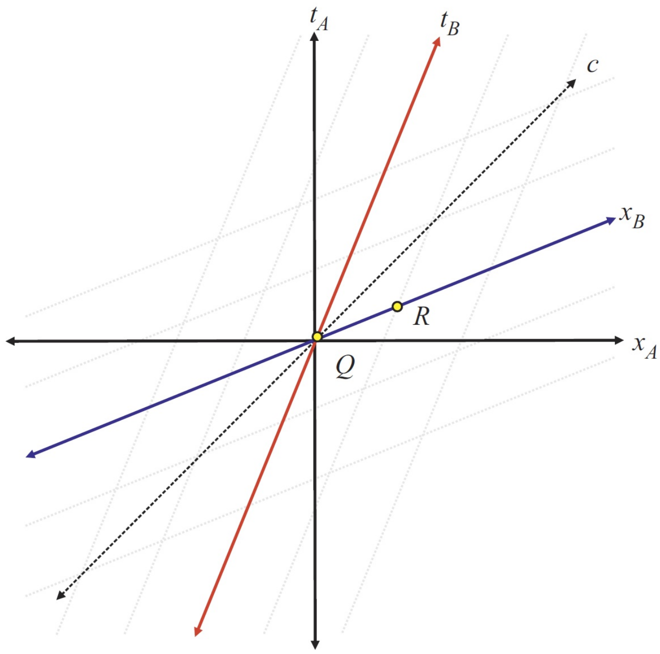

Figure 23: Events that are simultaneous for observer A can be joined by a horizontal line, that is, a line that is parallel to the ct_A axis. Events that are simultaneous for *observer B* can be joined by a line that is parallel to the x_B axis. Thus the events marked Q and R are simultaneous for observer B but not for observer A. Similarly, events which occur at the same location for observer B, separated only by time, lie on the lines which are parallel to the ct_B axis. For observer A, events that occur at the same place, separated only by time, have the same value on the x_A axis (i.e. can be joined by a vertical line).

The 'world line' of an object is its x-position as a function of time. A's world line corresponds to the t axis, as this observer is always located at $x = 0$ in his own reference frame. The world line of a light pulse is a straight line with gradient 1 as indicated in figure 23, as it always moves 1 light-second in a second, or 1 light-year in a year.

To begin, consider two observers A and B, where B is moving with speed 0.5c relative to A in the positive x direction. Let us assume that at the moment their positions coincide they both set their clocks to zero, i.e. $t_A = t_B = 0$ and take that position to be $x_A = x_B = 0$. In figure 23), called a 'Minkowski space-time diagram', we will make the vertical axis the time, and the horizontal axis displacement in the x-direction in A's frame. Displacement in the y and z-directions are not shown. For convenience, if time is measured in seconds, displacements in the x direction are measured in light seconds, or if time is measured in years, displacements are measured in light years.

It follows that the world line of observer B will be a straight line with a gradient of 2 (in units of years per light year), as if this observer is moving with speed 0.5c, they will take two seconds to move one light-second in the x direction, or two years to move one light year.

Now that we have located how B moves through space and time in A's reference frame, we have effectively located the position of B's time axis in A's reference frame. In order to locate B's x-axis, we note that as light must still move with a speed of 1 light-year per year in B's reference frame. This means that B's x-axis must have a gradient of 0.5, as shown in figure 23, in order that light still moves through one unit of space in one unit of time as measured along B's time and space axes.

The very first thing we should notice is that events that are simultaneous in B are not simultaneous in A and vice-versa. This can be seen noting that the 'events' in space-time labelled Q and R in the diagram are clearly not simultaneous for observer A. They are simultaneous for observer B, as they both occur at $t_B = 0$, at different positions along the x_B axis. In general, if we draw lines in parallel to the t_B axis (which correspond to simultaneous events in B's frame), the events lying on these lines do not also lie on lines parallel to the t_A axis.

Ladder in a barn problem

We can use our diagram qualitatively to explore a problem known as the ladder-in-a-barn problem. The problem usually goes something like this. A farmer has a long ladder which he wishes to store in his barn. The difficulty is that the ladder is longer than the barn. Having heard of special relativity the farmer has an idea. If he can get his son to run at relativistic speeds with the ladder, it will be length contracted and he will be able to fit it in the barn (at least temporarily). The son however, while running at relativistic speeds will see the barn to be length contracted, and so to him the problem appears to be exacerbated. Who is right? Will the ladder fit in the barn or won't it?

Figure 24: A long ladder...

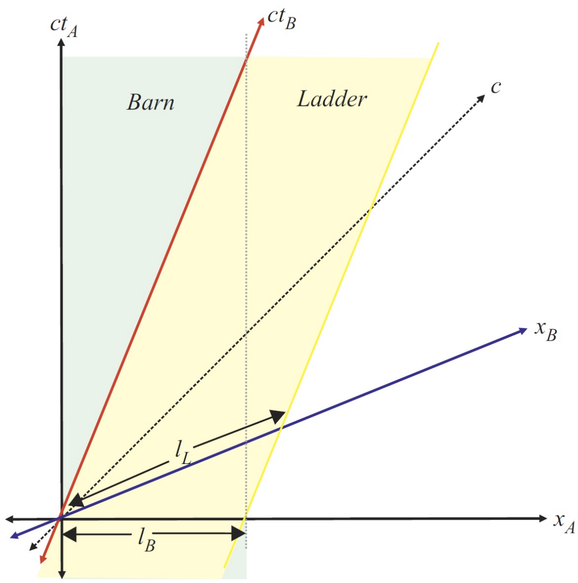

Figure 25: Space-time diagram representing the ladder in the barn problem. The yellow line represents the 'world line' of the right-hand end of the ladder, located at $x_B = l_L$. In this diagram the left-hand side of the ladder is located at the origin of B's reference frame. The ladder therefore traces out the light yellow shaded area on the space-time diagram as time passes. The left hand side of the barn is located (in this diagram) at the origin of A's reference frame, and its right-hand end at position $x_A = l_B$. The barn traces out the light blue shaded area on the diagram with time. Only in the A's frame of reference are the left-hand and right hand ends of the barn simultaneously aligned with those of the ladder.

The answer is that they are both correct in their own frames of reference. The explanation lies in the fact that, in the farmer's reference frame, he sees the event where the left hand end of the ladder lines up with the left hand end of the barn and the event where the right hand side of the ladder lines up with the right hand side of the barn to be simultaneous events, whereas from the perspective of the son, the right hand end of the ladder lining up with the right hand end of the barn occurs well before the left hand end of the ladder lines up with the left hand end of the barn (we will assume that the doors at either end of the barn are open, in order to avoid collisions at relativistic speeds!). This can be illustrated using a Minkowski space-time diagram as shown in figure 25.

The Lorentz transformations

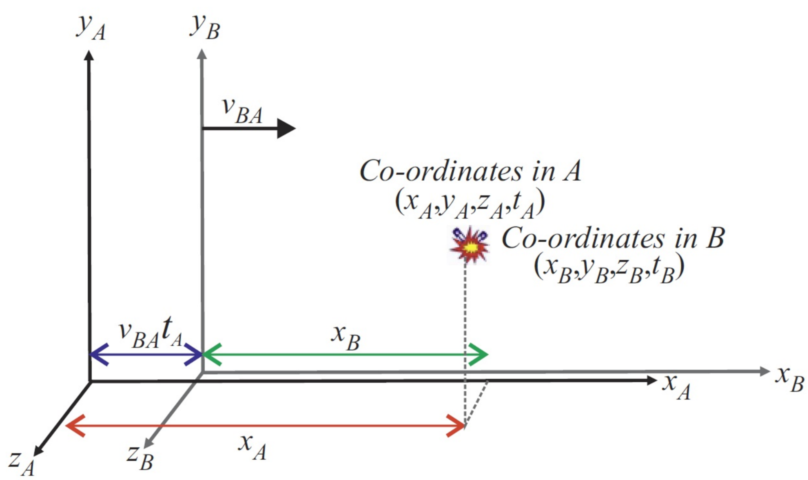

Figure 26: Diagram showing that in the reference frame of observer A, the reference frame of observer B is moving with velocity v to the right. An event (such as the explosion of a firecracker) has different coordinates in time and space in the two reference frames. The Lorentz transformations relate the coordinates of an event in one reference frame to its coordinates in the other frame.

The Lorentz transforms are a list or 'dictionary' of relationships between the quantities x_A, y_A, z_A and t_A for a particular event (for example, the explosion of a firecracker) in one reference frame and the quantities x_B, y_B, z_B and t_B in another reference frame B moving at some velocity v_{BA} with respect to A. As we can freely choose the direction of one of our axes (with the other two then perpendicular to the first and to each other), we will choose to align the x-axis with the direction of the relative velocity of the reference frames, as this will greatly simplify our analysis (without losing any generality).

If we were to analyse the situation from the perspective of classical physics, our task is fairly straightforward. In that case, the y, z and t co-ordinates of the event are same in both reference frames and the x position of the event in A's reference frame is simply the x position of the event in B's reference frame plus the distance that B has moved since the two frames were aligned at t=0.

Thus the so-called 'Galilean' transformations are

$$x_B = x_A - v_{BA}t_A$$

$$y_B = y_A$$

$$z_B = z_A$$

$$t_B = t_A$$

However, when we analyse the situation from the perspective of special relativity we need to be clear that the distance from the origin of B's reference frame to the x-coordinate of the event is not the same in both reference frames due to the effect of length dilation.

In reality,

$$x_A = v_{BA}t_A + x_B\sqrt{1-\left(\frac{v_{BA}}{c}\right)^2}$$

as the distance x_B is length contracted in A's reference frame.

Note that this is often written using the abbreviation

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

where

$$\beta = \frac{v}{c}$$

giving

$$x_B = \gamma\left(x_A - v_{BA}t_A\right)$$

In addition, given what we have learned about special relativity so far, we should be expecting that the two observers will not agree on the time at which the event occured. To determine the relationship between their time co-ordinates for the event we first note that if we switch to B's reference frame then A appears to be receding at a velocity of $v_{AB} = -v_{BA}$. This means that

$$x_B = -v_{BA}t_B + \frac{x_A}{\gamma}$$

and so

$$x_A = \gamma\left(x_B + v_{BA}t_B\right)$$

You can see that this is the same as equation apart from the opposite sign of the velocity. By rearranging for t_B and substituting for x_B we can obtain

$$t_B = \gamma\left(t_A - \frac{v_{BA}}{c^2}x_A\right)$$

Thus the Lorentz transformations to convert co-ordinates in A's frame of reference to co-ordinates in B's frame are:

$$x_B = \gamma\left(x_A - v_{BA}t_A\right)$$

$$y_B = y_A$$

$$z_B = z_A$$

$$t_B = \gamma\left(t_A - \frac{v_{BA}}{c^2}x_A\right)$$

The co-ordinates to convert from B's frame of reference to co-ordinates in A's frame are identical, except that the relevant velocity is v_{AB}, the velocity of A relative to B (where $v_{AB} = -v_{BA}$):

$$x_A = \gamma\left(x_B - v_{AB}t_B\right)$$

$$y_A = y_B$$

$$z_A = z_B$$
$$t_A = \gamma\left(t_B - \frac{v_{AB}}{c^2}x_B\right)$$

We will use these transformations for two purposes here. Firstly, to derive the Einstein velocity addition rule and secondly, to place scales on our axes in our space-time diagrams.

Relativistic addition of velocities

Derivation of the velocity addition rule using the Lorentz transforms.

In Galilean relativity, if an object C is moving with velocity v_{CA} in A's reference frame, then in B's reference frame where A is moving with a velocity v_{AB} the object will be moving with the sum of these velocities,

$$v_{CB} = v_{CA} + v_{AB}$$

To convince yourself of this, imagine that v_{CA} is the velocity of a cricket ball relative to the bowler (observer A). Then if v_{CB} is the velocity of cricket ball relative to a stationary batsman (observer B), this must be the sum of the velocity of the ball relative to the bowler (v_{CA}) and the velocity of the bowler relative to the ground (v_{AB}).

In special relativity, the situation is not so simple - indeed it can't be if light is to travel with a constant speed c in all reference frames.

The velocity v_{CA} of an object C measured in A's reference frame is the ratio of the distance dx_A that it moves in time dt_A, so

$$v_{CA} = \frac{dx_A}{dt_A}$$

In B's reference frame however, the distance that the object moves is

$$dx_B = \gamma\left(dx_A - v_{BA}dt_A\right)$$

during the time interval

$$dt_B = \gamma\left(dt_A - \frac{v_{BA}}{c^2}dx_A\right)$$

The velocity that B measures for the object, v_{CB} is therefore

$$v_{CB} = \frac{dx_B}{dt_B} = \frac{(dx_A - v_{BA}dt_A)}{\left(dt_A - \frac{v}{c^2}dx_A\right)} = \frac{\frac{dx_A}{dt_A} - v_{BA}}{1 - \frac{v_{BA}}{c^2}\frac{dx_A}{dt_A}} = \frac{v_{CA} + v_{AB}}{1 + \frac{v_{AB}v_{CA}}{c^2}}$$

For relative velocities that are small compared to the speed of light the denominator becomes unity and we regain the 'Galilean' velocity addition rule.

Answers

Answer 1. Using the table on the next page, calculate (a) the mass defect in kilograms and (b) the energy released in the first part of the proton-proton chain, which involves the reaction:
${}_1^1H +_1^1 H \rightarrow_1^2 H + \nu + e^+$ The total mass of the reactants is:

$$m_r = 2 \times 1.007276466879 = 2.014552934$$

The total mass of the products is:

$$m_p = 2.01410177811 + 5.48579909 \times 10^4 = 2.014650358$$

The mass defect is:

$$9.7 \times 10^{-5}\text{u}$$

Which is equivalent to

$$m = 1.661 \times 10^{-27}\text{kg/u} \times 9.7 \times 10^{-5}\text{u} = 1.62 \times 10^{-31}\text{kg}$$

(which is about the mass of an electron). This is equivalent to

$$E = 1.62 \times 10^{-31}\text{kg} \times (3^8)^2 = 1.5 \times 10^{-14}\text{J} = 0.1\text{MeV}$$

Answer 2. Using the table on the next page, calculate the energy released in the last stage of the CNO cycle, which involves the reaction:

$${}_7^{15}N +_1^1 H \rightarrow_2^4 He +_6^{12} C$$

Express your answer in MeV.

The total mass of the reactants is:

$$m_r = 15.0001088989 + 1.007276466879 = 16.00738537$$

The total mass of the products is:

$$m_p = 4.001506179127 + 12 = 16.001506179127$$

The mass defect is:

$$5.88 \times 10^{-3}\text{u}$$

Which is equivalent to

$$E = 931.5 \times 5.88 \times 10^{-3} = 5.5\text{MeV}$$

Answer 3. Calculate the energy released by the annihilation of an electron and a positron.

$$e^- + e^+ \rightarrow 2\gamma$$

Electrons and positrons have the same mass, so the energy released is

$$E = 2 \times 9.109 \times 10^{-31} \times (3 \times 10^8)^2 = 1.64 \times 10^{-13}\text{J} = 1.0\text{MeV}$$

Answer 4. If the energy released when methane gas combusts with oxygen is 50.1kJ per gram of methane, what is the mass defect between the reactants and the products due to the combustion of 1g of methane?

$$m = \frac{50.1 \times 10^3}{(3 \times 10^8)^2} = 5.6 \times 10^{-13}\text{kg}$$