EXERCISE 13.1 APPROXIMATIONS OF TRIGONOMETRIC FUNCTIONS WHEN *X* IS SMALL

2 **(a)** $\sin(\pi - x) = -\sin x$ is **incorrect** as $\sin(\pi - x)$ has the same sign as $\sin x$.

(b) $\sin(\pi - x) = \sin x$ is **correct**. This can be verified using a unit circle.

(c) This is **incorrect** as $\lim\limits_{x\to 0}\dfrac{\sin(\pi - x)}{x} = \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

(d) $\lim\limits_{x\to 0}\dfrac{\sin(\pi - x)}{x} = 1$ is **correct**. See part **(c)**.

EXERCISE 13.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

2 **A**

Use the chain rule.

Let $u = \cos 5t$ so $\dfrac{du}{dt} = -5\sin 5t$.

$$\frac{d}{dt} = \frac{d}{du} \times \frac{du}{dt}$$

$$\begin{aligned}\frac{d}{dt}\left(\cos^2 5t\right) &= \frac{d}{du}\left(u^2\right) \times \frac{du}{dt} \\ &= 2u \times -5\sin 5t \\ &= 2\cos 5t \times -5\sin 5t \\ &= -10\sin 5t\cos 5t\end{aligned}$$

4 **(a)** $\dfrac{d}{dt}\left(\sin\dfrac{t}{2} + \dfrac{1}{2}\cos t\right) = \dfrac{1}{2}\cos\dfrac{t}{2} + \dfrac{1}{2} \times -\sin t$

$$= \frac{1}{2}\left(\cos\frac{t}{2} - \sin t\right)$$

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(b) Use the chain rule.

Let $u = \cos t$ so $\frac{du}{dt} = -\sin t$.

$$\frac{d}{dt} = \frac{d}{du} \times \frac{du}{dt}$$

$$\begin{aligned}\frac{d\left(\cos^3 t\right)}{dt} &= \frac{d\left(\cos^3 t\right)}{d\cos t} \times \frac{d\cos t}{dt} \\ &= 3\cos^2 t \times -\sin t \\ &= -3\sin t\cos^2 t\end{aligned}$$

(c) Let $f(t) = t^2 + 1$ so $f'(t) = 2t$.

$$\frac{d}{dt}\cos\left(f(t)\right) = -f'(t)\sin\left(f(t)\right)$$

$$\frac{d}{dt}\cos\left(t^2+1\right) = -2t \times \sin\left(t^2+1\right) = -2t\sin\left(t^2 +\right.$$

(d) $\frac{d}{dt}\sin\left(2t + \frac{\pi}{2}\right) = 2\cos\left(2t + \frac{\pi}{2}\right)$

$$\begin{aligned}&= 2\sin\left[\frac{\pi}{2} - \left(2t + \frac{\pi}{2}\right)\right] \\ &= 2\sin(-2t) \\ &= -2\sin 2t\end{aligned}$$

(e)

$$\frac{d}{dt}\left(t^2 + \tan\frac{t}{2}\right) = 2t + \frac{1}{2}\sec^2\frac{t}{2}$$

(f) Use the product rule, where $u = t^2 - 1$ and $v = \cos 3t$.

$$\begin{aligned}\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{du}{dx} \\ &= \cos 3t \times 2t + \left(t^2 - 1\right) \times -3\sin 3t \\ &= 2t\cos 3t - 3\left(t^2 - 1\right)\sin 3t\end{aligned}$$

(g) $\frac{d\cos\left(2t + \frac{\pi}{3}\right)}{dt} = -2\sin\left(2t + \frac{\pi}{2}\right)$

(h) $\frac{d\cos(3t-2)}{dt} = -3\sin(3t-2)$

6 $f(x)=3\sin\frac{x}{2}-4\cos\frac{3x}{2}-x^3$

$$f'(x)=\frac{d}{dx}\left(3\sin\frac{x}{2}\right)-\frac{d}{dx}\left(4\cos\frac{3x}{2}\right)-\frac{dx^3}{dx}$$

$$\frac{d}{dx}\left(3\sin\frac{x}{2}\right)=3\times\frac{1}{2}\cos\frac{x}{2}=\frac{3}{2}\cos\frac{x}{2}$$

$$\frac{d}{dx}\left(4\cos\frac{3x}{2}\right)=4\times-\frac{3}{2}\cos\frac{3x}{2}=-6\cos\frac{3x}{2}$$

$$\frac{dx^3}{dx}=3x^2$$

$$\begin{aligned}f'(x)&=\frac{d}{dx}\left(3\sin\frac{x}{2}\right)-\frac{d}{dx}\left(4\cos\frac{3x}{2}\right)-\frac{dx^3}{dx}\\&=\frac{3}{2}\cos\frac{x}{2}+6\sin\frac{3x}{2}-3x^2\end{aligned}$$

(a) correct

(b) incorrect

(c) incorrect

(d) correct

8 **(a)** Use the product rule, where $u=e^x$ and $v=\sin x$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{du}{dx}$$

$$\begin{aligned}\frac{d\left(e^x\sin x\right)}{dt}&=\sin x\times e^x+e^x\times\cos x\\&=e^x\left(\sin x+\cos x\right)\end{aligned}$$

(b) Use the product rule, where $u=e^{2x}$ and $v=\cos\frac{x}{2}$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{du}{dx}$$

$$\begin{aligned}\frac{d\left(e^{2x}\cos\frac{x}{2}\right)}{dt}&=\cos\frac{x}{2}\times 2e^{2x}+e^{2x}\times-\frac{1}{2}\sin\frac{x}{2}\\&=2e^{2x}\cos\frac{x}{2}-\frac{1}{2}e^{2x}\sin\frac{x}{2}\\&=e^{2x}\left(2\cos\frac{x}{2}-\frac{1}{2}\sin\frac{x}{2}\right)\end{aligned}$$

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or $\dfrac{e^{2x}}{2}\left(4\cos\dfrac{x}{2}-\sin\dfrac{x}{2}\right)$

(c) Use the product rule, where $u=e^{-x}$ and $v=\sin 3x$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{du}{dx}$$
$$\frac{d\left(e^{-x}\sin 3x\right)}{dt}=\sin 3x\times -e^{-x}+e^{-x}\times 3\cos 3x$$
$$=e^{-x}\left(3\cos 3x-\sin 3x\right)$$

(d) Use the product rule, where $u=e^{x}$ and $v=\cos 4x$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{du}{dx}$$
$$\frac{d\left(e^{x}\cos 4x\right)}{dt}=\cos 4x\times e^{x}+e^{x}\times -4\sin 4x$$
$$=e^{x}\left(\cos 4x-4\sin 4x\right)$$

(e) Use the product rule, where $u=\cos x+\sin x$ and $v=e^{-x}$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{du}{dx}$$
$$\frac{d\left(\left(\cos x+\sin x\right)e^{-x}\right)}{dt}=e^{-x}\times\left(-\sin x+\cos x\right)+\left(\cos x+\sin x\right)\times -e^{-x}$$
$$=e^{-x}\left(-\sin x+\cos x-\cos x-\sin x\right)$$
$$=e^{-x}\left(-2\sin x\right)$$
$$=-2e^{-x}\sin x$$

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(f) Use the chain rule.

Let $u = \sin 2x$ so $\dfrac{du}{dx} = 2\cos 2x$.

$$\frac{d}{dx} = \frac{d}{du} \times \frac{du}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(e^{\sin 2x}\right) &= \frac{d}{du}\left(e^{u}\right) \times \frac{du}{dx} \\ &= e^{u} \times 2\cos 2x \\ &= e^{\sin 2x} \times 2\cos 2x \\ &= 2e^{\sin 2x} \cos 2x\end{aligned}$$

(g) Use the chain rule.

Let $u = \cos x$ so $\dfrac{du}{dx} = -\sin x$.

$$\frac{d}{dx} = \frac{d}{du} \times \frac{du}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(e^{\cos x}\right) &= \frac{d}{du}\left(e^{u}\right) \times \frac{du}{dx} \\ &= e^{u} \times -\sin x \\ &= e^{\cos x} \times -\sin x \\ &= -e^{\cos x} \sin x\end{aligned}$$

(h) Use the chain rule.

Let $u = \sin x + \cos x$ so $\dfrac{du}{dx} = \cos x - \sin x$.

$$\frac{d}{dx} = \frac{d}{du} \times \frac{du}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(e^{\sin x + \cos x}\right) &= \frac{d}{du}\left(e^{u}\right) \times \frac{du}{dx} \\ &= e^{u} \times \left(\cos x - \sin x\right) \\ &= e^{\sin x + \cos x} \times \left(\cos x - \sin x\right) \\ &= \left(\cos x - \sin x\right) e^{\sin x + \cos x}\end{aligned}$$

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EXAMPLE 13.3 DERIVATIVE OF THE LOGARITHM FUNCTION

2 (a) $f(x) = \log_e(3x+2)$

Domain:

$3x + 2 > 0$

$x > -\dfrac{2}{3}$

If $f(x) = \log_e(ax+b)$,

$f'(x) = \dfrac{a}{ax+b}$

$f(x) = \log_e(3x+2)$

$f'(x) = \dfrac{3}{3x+2}$

(b) Domain:

Since $x^2 + 1$ is always positive, the domain is all real x.

Let $g(x) = x^2 + 1 \Rightarrow g'(x) = 2x$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\log_e(g(x)) \\ &= \frac{g'(x)}{g(x)} \\ &= \frac{2x}{x^2+1} \end{aligned}$$

(c) $f(x) = \log_e(x^2 - 4x + 4)$

Domain:

$x^2 - 4x + 4 = (x-2)^2 \geq 0$

$x^2 - 4x + 4$ is only undefined when $x^2 - 4x + 4 = 0$, i.e. when $x \neq 2$

The domain is all x such that $x \neq 2$.

Let $g(x) = x^2 - 4x + 4 \Rightarrow g'(x) = 2x - 4$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\log_e(g(x)) \\ &= \frac{g'(x)}{g(x)} \\ &= \frac{2x-4}{x^2-4x+4} \\ &= \frac{2(x-2)}{(x-2)^2} \\ &= \frac{2}{x-2},\ x \neq 2 \end{aligned}$$

(d) $f(x) = \log_e(4x+3)$

Domain:

$$4x+3>0$$
$$x>-\frac{3}{4}$$

If $f(x)=\log_e(ax+b)$,

$$f'(x)=\frac{a}{ax+b}$$
$$f(x)=\log_e(4x+3)$$
$$f'(x)=\frac{4}{4x+3}$$

$$f'(x)=\frac{4}{4x+3}$$

(e) $\ln\sqrt{x}$

$\sqrt{x}$ is only defined for $x>0$ and in that case $\sqrt{x}>0$ and $\ln\sqrt{x}$ is defined.

Domain: $x>0$

Let $g(x)=\sqrt{x}=x^{\frac{1}{2}}\Rightarrow g'(x)=\frac{1}{2}2x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

$$\begin{aligned}f'(x)&=\frac{d}{dx}\log_e(g(x))\\&=\frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}}\\&=\frac{1}{2\sqrt{x}}\times\frac{1}{\sqrt{x}}\\&=\frac{1}{2x},\ x>0\\&=\frac{2}{x-2}\end{aligned}$$

Alternatively,

$\ln\sqrt{x}=\ln x^{\frac{1}{2}}=\frac{1}{2}\ln x$, so

$$\begin{aligned}\frac{d\ln\sqrt{x}}{dx}&=\frac{d}{dx}\left(\frac{1}{2}\ln x\right)\\&=\frac{1}{2}\frac{d\ln x}{dx}\\&=\frac{1}{2x}\end{aligned}$$

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(f) $\sqrt{x}$ is only defined for $x > 0$ and then $\sqrt{x} > 0$ and $x + \sqrt{x} > 0$, so $\ln \sqrt{x}$ is defined for $x > 0$. Domain: $x > 0$

Let $g(x) = x + \sqrt{x} \Rightarrow g'(x) = 1 + \dfrac{1}{2\sqrt{x}} = \dfrac{2\sqrt{x}+1}{2\sqrt{x}}$

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\log_e\left(g(x)\right) \\
&= \frac{g'(x)}{g(x)} \\
&= \frac{2\sqrt{x}+1}{2\sqrt{x}} \times \frac{1}{x+\sqrt{x}} \\
&= \frac{2\sqrt{x}+1}{2x\sqrt{x}+2x} \\
&= \frac{2\sqrt{x}+1}{2x\left(\sqrt{x}+1\right)},\ x > 0
\end{aligned}$$

4 **(a)** Use the product rule, where $u = x$ and $v = \ln x$.

$$\begin{aligned}
\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\
&= \ln x \times 1 + x \times \frac{1}{x} \\
&= \ln x + 1
\end{aligned}$$

(b) Use the product rule, where $u = x^3$ and $v = \ln x$.

$$\begin{aligned}
\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\
&= \ln x \times 3x^2 + x^3 \times \frac{1}{x} \\
&= 3x^2 \ln x + x^2 \\
&= x^2\left(3\ln x + 1\right)
\end{aligned}$$

(c) Use the product rule, where $u = x + 2$ and $v = \ln\left(x+2\right)$.

$$\begin{aligned}
\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\
&= \ln\left(x+2\right) \times 1 + \left(x+2\right) \times \frac{1}{x+2} \\
&= \ln\left(x+2\right) + 1
\end{aligned}$$

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(d) Use the product rule, where $u = x^2 + 1$ and $v = \ln 2x$.

$$\begin{aligned}\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= \ln 2x \times 2x + (x^2+1)\times\frac{1}{x}\\ &= 2x\ln 2x + \frac{x^2+1}{x}\end{aligned}$$

or $2x\ln 2x + x + \dfrac{1}{x}$

(e) Use the product rule, where $u = 2x - 5$ and $v = \ln x$.

$$\begin{aligned}\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= \ln x \times 2 + (2x-5)\times\frac{1}{2}\\ &= \ln(x+2) + \frac{2x-5}{2}\end{aligned}$$

or $2\ln x + 2 - \dfrac{5}{x}$

(f) Use the product rule, where $u = e^x$ and $v = \ln x$.

$$\begin{aligned}\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= \ln x \times e^x + e^x \times\frac{1}{x}\\ &= e^x\left(\ln x + \frac{1}{x}\right)\end{aligned}$$

or $\dfrac{e^x}{x}(x\ln x + 1)$

(g) Use the product rule, where $u = e^{2x}$ and $v = \ln 2x$.

$$\begin{aligned}\frac{d(uv)}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= \ln 2x \times 2e^{2x} + e^{2x} \times\frac{1}{x}\\ &= e^{2x}\left(2\ln 2x + \frac{1}{x}\right)\end{aligned}$$

or $\dfrac{e^x}{x}\left(2x\ln 2x+1\right)$

(h) Use the quotient rule, where $u = x$ and $v = \log_e x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\log_e x \times 1 - x \times \frac{1}{x}}{\left(\log_e x\right)^2}$$

$$= \frac{\log_e x - 1}{\left(\log_e x\right)^2}$$

(i) Use the quotient rule, where $u = \log_e x$ and $v = x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{x \times \frac{1}{x} - \log_e x \times 1}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

(j) Use the quotient rule, where $u = \log_e x$ and $v = e^x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{e^x \times \frac{1}{x} - \log_e x \times e^x}{\left(e^x\right)^2}$$

$$= \frac{e^x\left(\frac{1}{x} - \log_e x\right)}{e^{2x}}$$

$$= \frac{\frac{1}{x} - \log_e x}{e^x}$$

or $\dfrac{1 - x\log_e x}{xe^x}$

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(k) First find the derivative of $\log_e\left(x^2+1\right)$ using $\frac{d}{dx}\log_e f(x)=\frac{f'(x)}{f(x)}$

$$f(x)=x^2+1 \Rightarrow f'(x)=2x$$
$$\frac{d}{dx}\log_e f\left(x^2+1\right)=\frac{2x}{x^2+1}$$

Use the quotient rule, where $u=\log_e\left(x^2+1\right)$ and $v=x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$
$$\frac{d}{dx}\left(\frac{\log_e\left(x^2+1\right)}{x}\right)=\frac{x\times\frac{2x}{x^2+1}-\log_e\left(x^2+1\right)\times 1}{x^2}$$
$$=\frac{\frac{2x^2}{x^2+1}-\log_e\left(x^2+1\right)}{x^2}$$
$$=\frac{2x^2-\left(x^2+1\right)\log_e\left(x^2+1\right)}{x^2\left(x^2+1\right)}$$

(l) First find the derivative of $\log_e\left(e^x+1\right)$ using $\frac{d}{dx}\log_e f(x)=\frac{f'(x)}{f(x)}$

$$f(x)=e^x+1 \Rightarrow f'(x)=e^x$$
$$\frac{d}{dx}\log_e f\left(e^x+1\right)=\frac{f'(x)}{f(x)}=\frac{e^x}{e^x+1}$$

Use the product rule, where $u=e^x$ and $v=\log_e\left(e^x+1\right)$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$
$$\frac{d}{dx}\left(e^x\log_e\left(e^x+1\right)\right)=\log_e\left(e^x+1\right)\times e^x+e^x\times\frac{e^x}{e^x+1}$$
$$=e^x\log_e\left(e^x+1\right)+\frac{e^{2x}}{e^x+1}$$

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6 Use $\dfrac{d}{dx}\log_e f(x) = \dfrac{f'(x)}{f(x)}$

$$f(x) = x^2 + 1 \Rightarrow f'(x) = 2x$$
$$\frac{d}{dx}\log_e\left(x^2+1\right) = \frac{f'(x)}{f(x)} = \frac{2x}{x^2+1}$$

At $x = 3$,

$$m_T = \frac{2\times 3}{3^2+1} = \frac{6}{10} = \frac{3}{5}$$

8 $y = \log_e x$

When it crosses the x-axis, $y = 0$

$$0 = \log_e x$$
$$x = 1$$

The curve crosses the x-axis at $(0,\ 1)$.

$$y = \log_e x$$
$$\frac{dy}{dx} = \frac{1}{x}$$

At $x = 1$,

$$m_T = \frac{1}{1} = 1$$

The product of perpendicular gradients is -1, so $m_N = -1$.

Equation of tangent at $(1,\ 0)$ is:

$$(y-0) = 1(x-1)$$
$$y = x-1$$

Equation of normal at $(1,\ 0)$ is:

$$(y-0) = -1(x-1)$$
$$y = -x+1$$
$$y = 1-x$$

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10 $y = \log_e\left(e^x\right)$

$$\begin{aligned} &= x\log_e e \\ &= x \times 1 \\ &= x \end{aligned}$$

Therefore, $y = \log_e\left(e^x\right)$ is equivalent to $y = x$ for all values of x for which $\log_e\left(e^x\right)$ is defined.

Since $e^x > 0$ for all x, $\log_e\left(e^x\right)$ is defined for all x, and hence $y = \log_e\left(e^x\right)$ is equivalent to $y = x$ for all values of x.

Hence, the gradient of $y = \log_e\left(e^x\right)$ is equal to the gradient of $y = x$.

The gradient is 1.

12 (a) $e^x = 2$

$$\begin{aligned} \ln\left(e^x\right) &= \ln 2 \\ x &= \ln 2 \approx 0.693 \end{aligned}$$

(b) $e^{3x} = 5$

$$\begin{aligned} \ln\left(e^{3x}\right) &= \ln 5 \\ 3x &= \ln 5 \\ x &= \frac{\ln 5}{3} \approx 0.536 \end{aligned}$$

(c) $e^{2x+3} = 7$

$$\begin{aligned} \ln\left(e^{2x+3}\right) &= \ln 7 \\ 2x+3 &= \ln 7 \\ x &= \frac{\ln 7 - 3}{2} \approx -0.527 \end{aligned}$$

(d) $e^{x^2-1} = 10$

$$\begin{aligned} \ln\left(e^{x^2-1}\right) &= \ln 10 \\ x^2 - 1 &= \ln 10 \\ x^2 &= \ln 10 + 1 \\ x &= \pm\sqrt{\ln 10 + 1} \approx \pm 1.817 \end{aligned}$$

14 (a) $\frac{d}{dx}\left(a^x\right) = a^x \ln a$

$$\frac{d}{dx}\left(2^x\right) = 2^x \ln 2$$

(b) $\frac{d}{dx}\left(a^x\right) = a^x \ln a$

$$\frac{d}{dx}\left(e^x + 3^x\right) = e^x + 3^x \ln 3$$

(c) $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\log_e a}$

$$\frac{d}{dx}\left(\log_2 x\right) = \frac{1}{x\log_e 2}$$

(d) $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\log_e a}$

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$$\frac{d}{dx}\left(x+\log_3 x\right)=1+\frac{1}{x\log_e 3}$$

(e) Use the product rule, where $u=x^2$ and $v=4^x$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$\frac{d}{dx}\left(x^2 4^x\right)=4^x\times 2x+x^2\times 4^x\ln 4$$

$$=4^x\times x\left(2+x\ln 4\right)$$

or $4^x\times 2x\left(1+x\ln 2\right)$

(f) Use the product rule, where $u=x^3$ and $v=\log_5 x$.

$$\frac{d(uv)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$\frac{d}{dx}\left(x^3\log_5 x\right)=\log_5 x\times 3x^2+x^3\times\frac{1}{x\log_e 5}$$

$$=3x^2\log_5 x+\frac{x^2}{\log_e 5}$$

$$=x^2\left(3\log_5 x+\frac{1}{\log_e 5}\right)$$

(g) Use the quotient rule, where $u=2^x$ and $v=x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left(x^3\log_5 x\right)=\frac{x\times 2^x\ln 2-2^x\times 1}{x^2}$$

$$=\frac{2^x\left(x\ln 2-1\right)}{x^2}$$

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(h) Use the quotient rule, where $u = \log_a x$ and $v = x^2$.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left(x^3 \log_a x\right) = \frac{x^2 \times \frac{1}{x\log_e a} - \log_a x \times 2x}{x^4}$$

$$= \frac{\frac{x}{\log_e a} - 2x\log_a x}{x^4}$$

$$= \frac{x - 2x\log_e a\log_a x}{x^4 \log_e a}$$

$$= \frac{x\left(1 - 2\log_e a\log_a x\right)}{x^4 \log_e a}$$

$$= \frac{1 - 2\log_e a\log_a x}{x^3 \log_e a}$$

16 (a) $\frac{d}{dx}\left(a^x\right) = a^x \ln a$

$$\frac{d}{dx}\left(10^x\right) = 10^x \ln 10$$

When $x = 1$, $\frac{dy}{dx} = 10\ln 10.$

Equation of tangent: $y - 10 = 10\ln 10\left(x - 1\right)$

$$y - 10 = 10x\ln 10 - 10\ln 10$$
$$10x\ln 10 - y - 10\ln 10 + 10 = 0$$

(b) $\frac{d}{dx}\left(a^x\right) = a^x \ln a$

$$\frac{d}{dx}\left(5^x\right) = 5^x \ln 5$$

$$m_T \times m_N = -1$$
$$5^x \ln 5 \times m_N = -1$$
$$m_N = -\frac{1}{5^x \ln 5}$$

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$$\frac{dy}{dx}=5^x\ln 5$$

When $x=2,\ m_N=-\frac{1}{5^2\ln 5}=-\frac{1}{25\ln 5}$

Equation of normal: $$\begin{aligned} y-25&=-\frac{1}{25\ln 5}(x-2)\\ -25\ln 5(y-25)&=x-2\\ -25\ln 5y+625\ln 5&=x-2\\ x+25\ln 5y-625\ln 5-2&=0\end{aligned}$$

(c) The tangents are parallel when they have the same gradient.

$$\frac{d}{dx}\left(10^x\right)=10^x\ln 10,\ \frac{d}{dx}\left(5^x\right)=5^x\ln 5$$

The tangents are parallel when

$$\begin{aligned} 10^x\ln 10&=5^x\ln 5\\ \frac{10^x}{5^x}&=\frac{\ln 5}{\ln 10}\\ \left(\frac{10}{5}\right)^x&=\frac{\ln 5}{\ln 10}\\ 2^x&=\frac{\ln 5}{\ln 10}\\ x\ln 2&=\ln\left(\frac{\ln 5}{\ln 10}\right)\\ x&=\frac{\ln\left(\frac{\ln 5}{\ln 10}\right)}{\ln 2}=-0.51669...\approx-0.5167\end{aligned}$$

Check:

$$10^x\ln 10=10^{-0.5167}\times\ln 10=0.3042...\times 2.3025...=0.7006...$$
$$5^x\ln 5=5^{-0.5167}\times\ln 5=0.4353...\times 1.6094...=0.7006...$$

EXERCISE 13.4 DERIVATIVE OF $e^{f(x)}$

2 (a) Use the product rule with $\frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$, where $f(x)=\sin x$.

$$\frac{d}{dx}\left(e^{\sin x}\right)=\cos xe^{\sin x}$$

Use the product rule, where $u = x$ and $v = e^{\sin x}$.

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(xe^{\sin x}\right) &= e^{\sin x} \times 1 + x \times \cos xe^{\sin x} \\ &= e^{\sin x}(x\cos x + 1)\end{aligned}$$

(b) Use the product rule, where $u = e^x$ and $v = \log_e x$.

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(e^x \log_e x\right) &= \log_e x \times e^x + e^x \times \frac{1}{x} \\ &= e^x\left(\log_e x + \frac{1}{x}\right) \text{ or } e^x \log_e x + \frac{e^x}{x}\end{aligned}$$

(c) Use $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$, where $f(x) = \cos(2x+1)$.

$$\begin{aligned}\frac{d}{dx}\left(e^{\cos(2x+1)}\right) &= -\sin(2x+1) \times 2 \times e^{\cos(2x+1)} \\ &= -2\sin(2x+1)e^{\cos(2x+1)}\end{aligned}$$

(d) Differentiate term by term and use the product rule for x^2e^x.

$$\begin{aligned}\frac{d}{dx}\left(1 + x + x^2e^x\right) &= 0 + 1 + 2x \times e^x + e^x \times x^2 \\ &= 1 + 2xe^x + x^2e^x \\ &= 1 + xe^x(x+2)\end{aligned}$$

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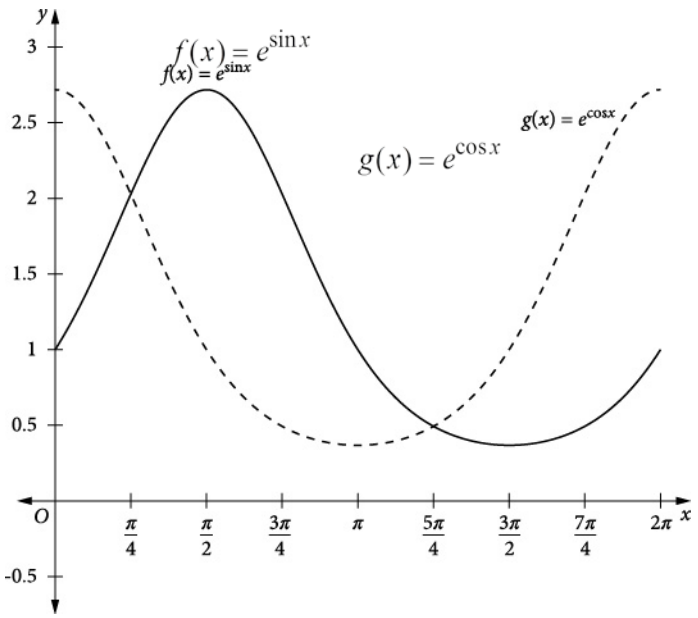
4 (a)



(b) $f(x) = g(x) \Leftrightarrow \sin x = \cos x$

Divide both sides by $\cos x$.

$\tan x = 1 \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ for $0 \le x \le 2\pi$.

$x = \dfrac{\pi}{4} \Rightarrow f(x) = g(x) = e^{\frac{\sqrt{2}}{2}} \approx 2.028$ (3 d.p.)

$x = \dfrac{3\pi}{4} \Rightarrow f(x) = g(x) = e^{-\frac{\sqrt{2}}{2}} \approx 0.493$ (3 d.p.)

The points are $\left(\dfrac{\pi}{4}, 2.028\right)$ and $\left(\dfrac{5\pi}{4}, 0.493\right)$ or $(0.785, 2.028)$ and $(3.927, 0.493)$.

(c) Use $\dfrac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$ to find $f'(x)$ and $g'(x)$.

$f'(x) = \cos x\, e^{\sin x}$

$g'(x) = -\sin x\, e^{\cos x}$

$f'\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}} e^{\frac{1}{\sqrt{2}}}$

$g'\left(\dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}} e^{\frac{1}{\sqrt{2}}}$

$$f'\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}e^{-\frac{1}{\sqrt{2}}}$$

$$g'\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}e^{-\frac{1}{\sqrt{2}}}$$

(d) Where $x = \frac{\pi}{4}$, $f'\left(\frac{\pi}{4}\right) \times g'\left(\frac{\pi}{4}\right) = -\frac{1}{2}e^{\frac{2}{\sqrt{2}}} = -\frac{1}{2}e^{\sqrt{2}} \neq -1$

The tangents are not perpendicular.

Where $x = \frac{5\pi}{4}$, $f'\left(\frac{5\pi}{4}\right) \times g'\left(\frac{5\pi}{4}\right) = -\frac{1}{2}e^{-\frac{2}{\sqrt{2}}} = -\frac{1}{2}e^{-\sqrt{2}} \neq -1$

The tangents are not perpendicular.

CHAPTER REVIEW 13

2 **(a)** Use the product rule, where $u = x^2 + 2x$ and $v = e^x$.

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(x^2+2x\right)e^x &= e^x(2x+2) + \left(x^2+2x\right)e^x \\ &= \left(x^2+4x+2\right)e^x\end{aligned}$$

(b) Use the product rule, where $u = 2e^{-x}$ and $v = \ln x$.

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(2e^{-x}\ln x\right) &= \ln x \times -2e^{-x} + 2e^{-x} \times \frac{1}{x} \\ &= \frac{-2e^{-x}x\ln x + 2e^{-x}}{x} \\ &= \frac{2e^{-x}(1 - x\ln x)}{x}\end{aligned}$$

(c) Use the chain rule.

Let $u = 1 + e^x$ so $\frac{du}{dx} = e^x$.

Let $y = \log_e\left(1+e^x\right)$ so $y = \log_e(u)$ and $\frac{dy}{du} = \frac{1}{u}$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=\frac{1}{u}\times e^x\\&=\frac{e^x}{1+e^x}\end{aligned}$$

(d) Use the chain rule.

Let $u=x^2+2x$ so $\frac{du}{dx}=2x+2$.

Let $y=\log_e\left(x^2+2x\right)$ so $y=\log_e\left(u\right)$ and $\frac{dy}{du}=\frac{1}{u}$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=\frac{1}{u}\times\left(2x+2\right)\\&=\frac{2x+2}{x^2+2x}\end{aligned}$$

(e) Use the product rule, where $u=x^2+3x$ and $v=e^{-3x}$.

$$\frac{d\left(uv\right)}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$\begin{aligned}\frac{d}{dx}\left(\left(x^2+3x\right)e^{-3x}\right)&=e^{-3x}\left(2x+3\right)+\left(x^2+3x\right)\times-3e^{-3x}\\&=e^{-3x}\left(2x+3-3x^2-9x\right)\\&=e^{-3x}\left(3-7x-3x^2\right)\end{aligned}$$

(f) Use the chain rule on both terms

$$\begin{aligned}\frac{d}{dx}\left(e^{\sqrt{x}}+\log_e\sqrt{x}\right)&=\frac{de^{\sqrt{x}}}{dx}+\frac{d\log_e\sqrt{x}}{dx}\\&=\frac{de^{\sqrt{x}}}{d\sqrt{x}}\times\frac{d\sqrt{x}}{dx}+\frac{d\log_e\sqrt{x}}{d\sqrt{x}}\times\frac{d\sqrt{x}}{dx}\\&=e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}+\frac{1}{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\\&=\frac{e^{\sqrt{x}}}{2\sqrt{x}}+\frac{1}{2x}\text{ or }\frac{e^{\sqrt{x}}}{2\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}}+\frac{1}{2x}=\frac{\sqrt{x}e^{\sqrt{x}}+1}{2x}\end{aligned}$$

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4 (a) Use the product rule

$$y = x^2 \sin x$$
$$\frac{dy}{dx} = 2x \sin x + x^2 \cos x$$

(b) Use the quotient rule, where $u = x^2$ and $v = \sin x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{d}{dx}\left(\frac{x^2}{\cos x}\right) = \frac{\cos x \times 2x - x^2 \sin x}{\cos^2 x}$$
$$= \frac{x\left(2\cos x - x\sin x\right)}{\cos^2 x}$$

(c) Use the product rule, where $u = \sin x$ and $v = \cos x$.

$$\frac{d\left(uv\right)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= \cos x \times \cos x + \sin x \times -\sin x$$
$$= \cos^2 x - \sin^2 x$$

(d) Use the chain rule.

Let $u = \tan x$ so $\frac{du}{dx} = \sec^2 x.$

Let $y = \sqrt{\tan x} = \sqrt{u}$ so $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{\tan x}}.$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{2\sqrt{\tan x}} \times \sec^2 x$$
$$= \frac{\sec^2 x}{2\sqrt{\tan x}}$$

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(e) Use the chain rule

$$y = \cos x^2$$

Let $u = x^2$

$$y = \cos u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\sin u \times 2x$$

$$= -2x \sin x^2$$

(f) Use the quotient rule, where $u = 2x+1$ and $v = \sin x$.

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d\left(\frac{\sin x}{2x+1}\right)}{dx} = \frac{(2x+1)\cos x - \sin x \times 2}{(2x+1)^2}$$

$$= \frac{(2x+1)\cos x - 2\sin x}{(2x+1)^2}$$

6 **(a)** Use product rule and chain rule

$$y = \log_e (x \tan x)$$

Let $u = x \tan x$

$$\frac{du}{dx} = \tan x \times 1 + x\sec^2 x$$

$$= \tan x + x\sec^2 x$$

$$y = \log_e u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times \left(\tan x + x\sec^2 x\right)$$

$$= \frac{\tan x + x\sec^2 x}{x \tan x}$$

$$= \frac{\tan x}{x \tan x} + \frac{\cancel{x}\sec^2 x}{\cancel{x}\tan x}$$

$$= \frac{1}{x} + \frac{1}{\cos^2 x} \times \frac{\cos x}{\sin x}$$

$$= \frac{1}{x} + \frac{1}{\sin x \cos x}$$

(b) Use the log laws and $\frac{d}{dx}\log_e\left(f(x)\right)=\frac{f'(x)}{f(x)}$

$$\begin{aligned}
y &= \log_e\left(\frac{x^3-6}{e^{-x}-1}\right)\\
&= \log_e\left(x^3-6\right)-\log_e\left(e^{-x}-1\right)\\
\frac{dy}{dx} &= \frac{3x^2}{x^3-6}-\frac{\left(-e^{-x}\right)}{e^{-x}-1}\\
&= \frac{3x^2}{x^3-6}+\frac{e^{-x}}{e^{-x}-1}
\end{aligned}$$

You can multiply numerator and denominator of the second fraction by e^x to give

$$\frac{dy}{dx}=\frac{3x^2}{x^3-6}+\frac{1}{1-e^x}$$

(c) This expression can be simplified considerably using the log laws and the Pythagorean identity $\sin^2 x+\cos^2 x=1 \Rightarrow \cos^2 x=1-\sin^2 x$.

$$\begin{aligned}
\log_e\left(\frac{\sqrt{x}\cos x}{1-\sin^2 x}\right) &= \log_e\left(\frac{\sqrt{x}\cos x}{\cos^2 x}\right)\\
&= \log_e\left(\frac{\sqrt{x}}{\cos x}\right)\\
&= \log_e\sqrt{x}-\log_e\left(\cos x\right)\\
&= \frac{1}{2}\log_e x-\log_e\left(\cos x\right)
\end{aligned}$$

Now differentiate, using $\frac{d}{dx}\log_e\left(f(x)\right)=\frac{f'(x)}{f(x)}$

$$\frac{d}{dx}\log_e\left(\frac{\sqrt{x}\cos x}{1-\sin^2 x}\right)=\frac{1}{2x}-\frac{-\sin x}{\cos x}=\frac{1}{2x}+\tan x$$

If the Pythagorean identity is not used, we get the following solution.

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$$\log_e\left(\frac{\sqrt{x}\cos x}{1-\sin^2 x}\right)=\log_e\left(\frac{\sqrt{x}\cos x}{(1-\sin x)(1+\sin x)}\right)$$

$$=\log_e\sqrt{x}+\log_e(\cos x)-\log_e(1-\sin x)-\log_e(1+\sin x)$$

$$=\frac{1}{2}\log_e x-\log_e(\cos x)-\log_e(1-\sin x)-\log_e(1+\sin x)$$

$$\frac{d}{dx}\log_e\left(\frac{\sqrt{x}\cos x}{1-\sin^2 x}\right)=\frac{1}{2x}-\frac{-\sin x}{\cos x}-\frac{-\cos x}{1-\sin x}-\frac{\cos x}{1+\sin x}$$

$$=\frac{1}{2x}-\frac{-\sin x\left(1-\sin^2 x\right)+\cos x\times\cos x(1+\sin x)-\cos x\times\cos x(1-\sin x)}{\cos x\left(1-\sin^2 x\right)}$$

$$=\frac{1}{2x}-\frac{-\sin x+\sin^3 x+\cos^2 x+\cos^2 x\sin x-\cos^2 x+\cos^2 x\sin x}{\cos x\left(1-\sin^2 x\right)}$$

$$=\frac{1}{2x}-\frac{-\sin x+\sin^3 x+2\cos^2 x\sin x}{\cos x\left(1-\sin^2 x\right)}$$

$$=\frac{1}{2x}-\frac{\sin x(\sin^2 x+2\cos^2 x-1)}{\cos x\left(1-\sin^2 x\right)}$$

This answer can be shown to be the same as the earlier answer using the Pythagorean identity.

$$-\frac{\sin x(\sin^2 x+2\cos^2 x-1)}{\cos x\left(1-\sin^2 x\right)}=-\frac{\sin x(\sin^2 x+\cos^2 x+\cos^2 x-1)}{\cos x\left(\cos^2 x\right)}$$

$$=-\frac{\sin x(1-\sin^2 x)}{\cos x\left(\cos^2 x\right)}$$

$$=-\frac{\sin x(-\cos^2 x)}{\cos x\left(\cos^2 x\right)}$$

$$=\frac{\sin x}{\cos x}=\tan x$$

$$\therefore\frac{1}{2x}-\frac{\sin x(\sin^2 x+2\cos^2 x-1)}{\cos x\left(1-\sin^2 x\right)}=\frac{1}{2x}+\tan x$$