2024 **BORED OF STUDIES TRIAL EXAMINATION**

6th October

Mathematics Extension 1

General instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the value of $\displaystyle\int_0^{\frac{\pi}{4}} (2\cos^2 x + 1)(4\cos^2 x - 5)\,dx$?

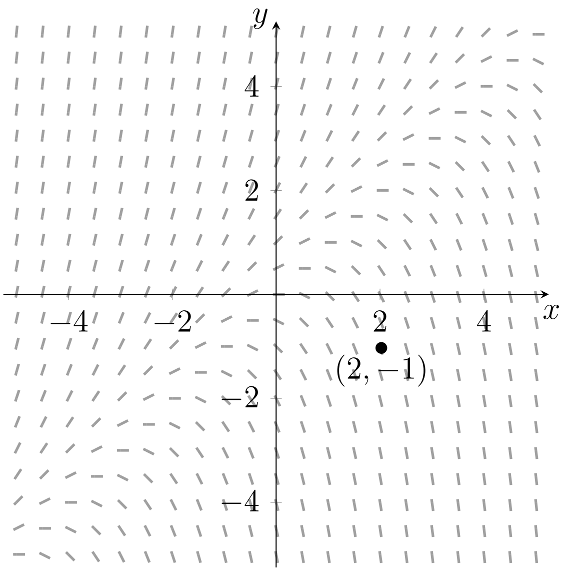
(A) $-\dfrac{5\pi}{4}$ (B) $-\dfrac{3\pi}{4}$ (C) $-\dfrac{1+5\pi}{4}$ (D) $\dfrac{2-5\pi}{4}$

2 Which of the following is NOT equivalent to $\sin(2\cos^{-1} x)$?

(A) $\sin(2\sin^{-1} x)$ (B) $\sin(\pi + 2\sin^{-1}(-x))$

(C) $2x\sqrt{1-x^2}$ (D) $2\sqrt{x^2(1-x^2)}$

3 The following is the direction field of a function $y = f(x)$ for the domain $x \in [-5, 5]$.



Suppose that $f(2) = -1$. Which of the following statements is NOT true about $y = f(x)$?

(A) $f'(2) < 0$ (B) $f''(2) < 0$ (C) $f'(-2) < 0$ (D) $f''(-2) < 0$

4 The displacement vector of a particle at time t is given by $\underset{\sim}{r} = 4\sin 2t\,\underset{\sim}{i} + 3\cos 2t\,\underset{\sim}{j}$.

What is the maximum magnitude of the particle's acceleration?

(A) 8 (B) 16 (C) 20 (D) 28

5 The gradient of the normal to any point P on a particular curve is twice the gradient of the line joining P and the origin.

Which of the following differential equations does this curve satisfy?

(A) $\dfrac{dy}{dx} + \dfrac{x}{2y} = 0$ (B) $\dfrac{dy}{dx} + \dfrac{2x}{y} = 0$

(C) $\dfrac{dy}{dx} - \dfrac{x}{2y} = 0$ (D) $\dfrac{dy}{dx} - \dfrac{2x}{y} = 0$

6 Let $P(x)$ be a quartic polynomial. Suppose that $P'(x) = 0$ has three distinct real solutions x_1, x_2 and x_3, where $x_1 < x_2 < x_3$.

Which of the following inequalities CANNOT be true for $P(x)$?

(A) $P(x_1) < P(x_2) < P(x_3)$ (B) $P(x_1) < P(x_3) < P(x_2)$

(C) $P(x_3) < P(x_1) < P(x_2)$ (D) $P(x_2) < P(x_3) < P(x_1)$

7 Let p be the expected proportion of n students from a school that achieved a band E4 in Mathematics Extension 1. The probability that up to 30% of the school's cohort scores a band E4 in Mathematics Extension 1 is estimated to be 40%.

The following approximate probabilities are provided for $Z \sim N(0, 1)$

$$P(Z < 0.53) \approx 0.7 \quad \text{and} \quad P(Z < 0.26) \approx 0.6.$$

Which of the following values of n and p best support this estimate?

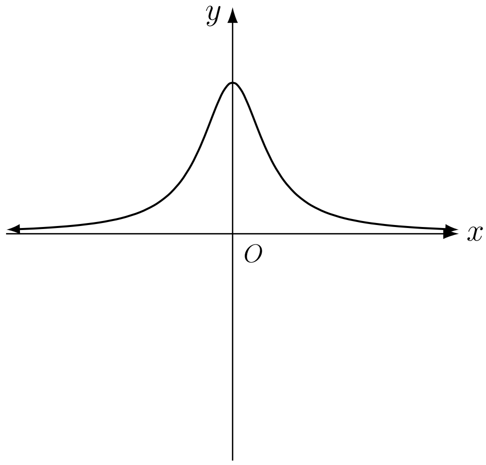
(A) $n = 64$ and $p = 0.315$ (B) $n = 79$ and $p = 0.328$

(C) $n = 70$ and $p = 0.286$ (D) $n = 66$ and $p = 0.271$

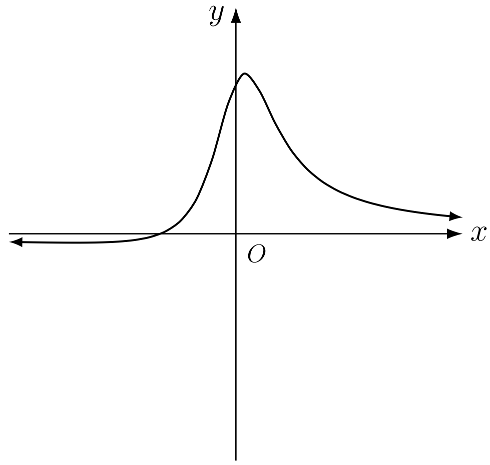
8 A quartic polynomial is divided by $A(x)$ with a remainder of $R(x)$. $A(x)$ is a monic quadratic polynomial with no real roots and $R(x)$ is such that $R(0) = 4$ and $R(1) = 4$.

Which of the following best represents the graph of $y = \dfrac{R(x)}{A(x)}$?

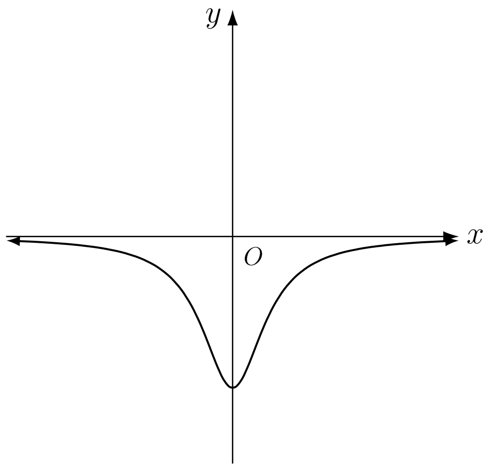
(A)



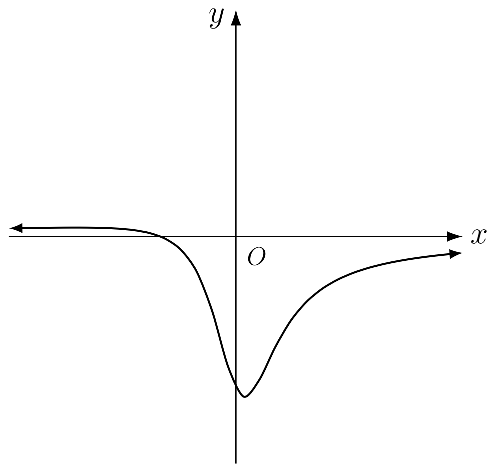
(B)



(C)



(D)



9 A bag contains 4 red, 7 blue, 10 yellow, 12 black and 15 white balls. What is the minimum number of balls that must be taken out of the bag at random, such that there will always be 9 balls of the same colour?

(A) 25 (B) 36 (C) 44 (D) 48

10 What is the range of $y = \sec\left(\tan^{-1}\dfrac{x}{x-1}\right)$?

(A) $y \leq -1$ or $y > \sqrt{2}$

(B) $y \neq \sqrt{2}$

(C) $y \geq 1$

(D) $y \neq 1$

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Let $f(x) = \tan^{-1}\dfrac{x}{x+1} + \tan^{-1}\dfrac{1}{2x+1}$.

(i) Find $f'(x)$. **2**

(ii) Hence, sketch $y = f(x)$. **2**

(b) Suppose that the solutions to the inequality $\dfrac{3x^2-1}{2x^2+x} > k$ are given by $x \in \left(-\frac{1}{2}, 0\right)$. Find the range of values of k. **2**

(c) A monic cubic polynomial $P(x)$ has the roots α, β and γ. Suppose that **3**

$$P'(\alpha) = 0, \quad P'(\beta) = 0, \quad P'(\gamma) = 25,$$
$$\alpha + \beta + \gamma = -1, \text{ and } \alpha\beta\gamma = 12.$$

Find the roots of $P(x)$.

(d) Let A be the area bounded by the curve $y = \sin^{-1} x$, the line $y = \dfrac{\pi}{4}$ and the y-axis.

(i) Show that $A = 1 - \dfrac{1}{\sqrt{2}}$. **1**

(ii) The area bounded by the curves $y = \sin^{-1} x$, $y = \cos^{-1} x$ and the y-axis is rotated about the x-axis to form a solid of revolution. Use the result of A in part (i) to find the volume of this solid. **3**

(e) Suppose that $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ with $\alpha + \beta + \gamma = \dfrac{\pi}{2}$. Show that **2**

$$\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} + \frac{\cos(\beta+\gamma)}{\cos\beta\cos\gamma} + \frac{\cos(\alpha+\gamma)}{\cos\alpha\cos\gamma} = 2.$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) A beaker containing a chemical is partly submerged in a water bath. Over time, the chemical cools down whilst the water bath warms up. The chemical and the water bath have initial temperatures of 160°C and 20°C respectively. **3**

Let T_c and T_w be the temperatures of the chemical and water bath respectively at time t. The rate of change of the temperatures is given by

$$\frac{dT_c}{dt} = -k(T_c - T_w)$$
$$\frac{dT_w}{dt} = \frac{3}{4}k(T_c - T_w)$$

where k is a positive constant.

Show that the chemical will cool down towards a temperature of 80°C over time.

(b) Prove by mathematical induction that $(2n+1)7^n - 1$ is divisible by 4 for positive integer n. **3**

(c) Show that for any constant $0 < \theta < \frac{\pi}{2}$, **3**

$$\int_0^1 \frac{\cos\theta}{x^2 - 2x\sin\theta + 1}\,dx = \frac{\pi}{4} + \frac{\theta}{2}.$$

(d) A particle has the position $P(x_t, y_t)$ on the x-y plane at time t. Let d be the shortest distance between the particle and the line ℓ, which has the Cartesian equation $3x - 4y + 4 = 0$.

(i) Let Y be the y-intercept of the line ℓ and Q be a point on the line ℓ where $\overrightarrow{QY} \cdot \overrightarrow{QP} = 0$. By considering the projection of $\overrightarrow{YP}$ onto $\overrightarrow{YQ}$, show that **3**

$$d^2 = \frac{(3x_t - 4y_t + 4)^2}{25}.$$

(ii) Suppose that the particle itself moves according to the displacement vector **2**

$$\underset{\sim}{r} = t\underset{\sim}{i} + \frac{t^2}{4}\underset{\sim}{j}.$$

Sketch the graph of d over time, showing any intercepts and turning points.

(iii) How many times is the particle exactly 1 unit away (in terms of shortest distance) from the line ℓ, throughout its movement? **1**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

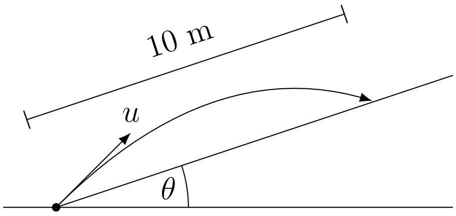
(a) A particle is projected from the ground at an angle of 45° to the horizontal with an initial speed of u ms^{-1}. Let the point of projection be the origin. It can be shown that the velocity vector of the particle relative to the ground is

$$\dot{\underset{\sim}{r}} = \frac{u}{\sqrt{2}}\underset{\sim}{i} + \left(\frac{u}{\sqrt{2}} - gt\right)\underset{\sim}{j} \qquad \textbf{(Do NOT prove this)}$$

where g is the acceleration due to gravity in ms^{-2}.

(i) Show that the Cartesian equation of the particle's path is $y = x - \dfrac{gx^2}{u^2}$. **2**

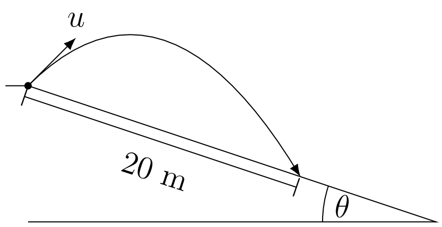
(ii) Suppose that there is an uphill slope with an angle of θ to the horizontal. **1**



When the particle is launched, it lands at a distance of 10 metres up along the slope as shown in the diagram above. Show that

$$\tan\theta = 1 - \frac{10g\cos\theta}{u^2}.$$

(iii) Now suppose instead that there is a downhill slope also with an angle of θ to the horizontal. **1**



When the particle is launched, it lands at a distance of 20 metres down along the slope as shown in the diagram above. Show that

$$\tan\theta = \frac{20g\cos\theta}{u^2} - 1.$$

(iv) Hence, show that $\tan\theta = \dfrac{1}{3}$. **1**

Question 13 continues on page 8

(b) Use the substitution $t = \tan\left(\frac{\pi}{4} - x\right)$ to show (without induction) that for some integer $n \geq 2$ **3**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + \cos x)^{2n}} = \frac{\binom{n-1}{0}}{1 \times 2^{n-1}} + \frac{\binom{n-1}{1}}{3 \times 2^{n-1}} + \frac{\binom{n-1}{2}}{5 \times 2^{n-1}} + \cdots + \frac{\binom{n-1}{n-1}}{(2n-1) \times 2^{n-1}}.$$

(c) Let $P(x) = 8x^3 + 4x^2 - 6x - 2$. It can be shown that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \qquad \text{and} \qquad \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

(Do NOT prove these)

(i) Use the above results to find the solutions to **4**

$$\cos\theta P(\cos\theta) = 0$$

for $0 \leq \theta \leq 2\pi$.

(ii) Find the roots of $P(x)$ in the form $a + \sqrt{b}$ for rational values of a and b, given that $P(-1) = 0$. **2**

(iii) Hence, find the exact value of $\cos\dfrac{2\pi}{5}$. **1**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) On a street there are 20 identical vacant houses in a straight line. There are 16 interested tenants who apply to live in these vacant houses, but only 12 can be successful. The successful tenants are then given one of houses to live in at random.

(i) Suppose that each tenant has an equal chance of being successful in their application. How many ways can the 16 interested tenants be arranged amongst the houses, if they were successful? **1**

(ii) Find the probability that two particular tenants are successful and are also neighbours (i.e. living in adjacent houses). **2**

(b) Water from a contaminated lake is being transported into a treatment plant via a tank. Let t be the number of minutes that have passed since commencing the water transportation,

Suppose that the contaminated lake contains $e^{-0.2t}$ grams of radioactive chemicals per litre after t minutes. The tank initially contains 100 litres of pure water (i.e. initially has zero grams of radioactive chemicals).

Water from the lake flows into the tank at a rate of 200 litres per minute and is uniformly mixed. At the same time, water flows out of the tank into the treatment plant at a rate of 100 litres per minute.

Let m be the mass of radioactive chemicals inside the tank (in grams) after t minutes.

(i) By considering the mass of radioactive chemicals per litre in the tank, explain why **2**

$$\frac{dm}{dt} = 200e^{-0.2t} - \frac{m}{1+t}.$$

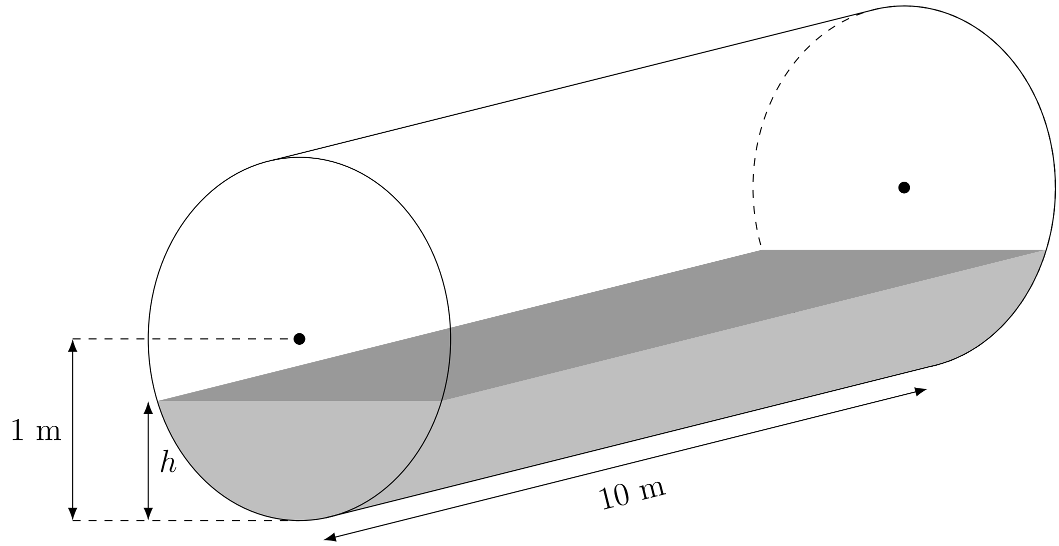
(ii) By considering $\frac{d}{dt}\left[m(1+t)\right]$, find the solution to the differential equation in part (i). **2**

You may assume without proof that $\int xe^{-kx}\,dx = -\frac{1}{k^2}e^{-kx}(1+kx) + C.$

(iii) Calculate the mass of the radioactive chemicals which have flowed *out* of the tank in the first 4 minutes. Provide your answer to the nearest gram. **2**

Question 14 continues on page 10

(c) A horizontal cylindrical enclosed tank has a radius of 1 metre and a length of 10 metres. The tank contains oil at a height of h metres at time t, where $h < 1$ as shown in the diagram below. **4**



Suppose that the oil is evaporating from the tank at a rate of 5 cubic metres per week. Let S be the area of the tank's cylindrical surface that is in contact with the oil after t weeks.

By first finding the volume of the oil after t weeks, show that

$$\frac{dS}{dt} = \frac{5}{h(h-2)} - 1.$$

(d) A fair coin is tossed n times to create a sequence of heads (H) and tails (T). **2**

Define a 'streak' to be a sequence of consecutive coin tosses that have the same outcome. For example, if the coin is tossed 10 times with the sequence of outcomes as $\{HTTTTHHHTH\}$, then the streaks are $\{H\}$, $\{TTTT\}$, $\{HHH\}$, $\{T\}$ and $\{H\}$, In this case, the 10 tosses gave rise to 5 streaks.

Let S be the random variable that represents the total number of streaks from n consecutive coin tosses,

Find $E(S)$ in terms of n.

End of paper

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