2012 Bored of Studies Trial Examinations

Mathematics Extension 1

SOLUTIONS

Disclaimer: These solutions may contain small errors. If any are found, please feel free to contact either Carrotsticks or Trebla on www.boredofstudies.org, regarding them.

Thanks: To Trebla, for his many hours spent verifying solutions and suggesting alternate methods.

Multiple Choice

1. D
2. C
3. D
4. A
5. C
6. B
7. D
8. D
9. B
10. D

Brief Explanations

Question 1 Re-arrange into standard form $v^2 = n^2\left(A^2 - x^2\right)$.

Question 2 Let the exponent of x in the general term be zero to acquire $2n = 3k$, $k \in \mathbb{N}$.

Question 3 Split numerator into two terms and draw a diagram.

Question 4 Observe limit as $x \to \pm\infty$, and that x cannot lie in $-1 < x < 1$.

Question 5 Angle between two lines formula, and let the expression be ≤ 1.

Question 6 Standard permutations problem. Note that they are in a circle, so it's $(n-1)!$.

Question 7 Standard Newton's method of approximation question.

Question 8 Find the coordinates of C, then substitute into the line.

Question 9 Negative quartic, with a triple root at the origin and a single root at $x = -4$.

Question 10 Binomial probability question. Use guess/check to acquire closest solution.

Written Response

Question 11 (a)

We will use the t formula substitutions.

Let $t = \tan\left(\frac{\theta}{2}\right)$

So our expression is:

$$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = t$$

Re-arrange:

$$\begin{aligned} 2t - 1 + t^2 &= t\left(1+t^2\right) \\ &= t^3 + t \end{aligned}$$

Form a cubic polynomial in t , then solve:

$$\begin{aligned} t^3 - t^2 - t + 1 &= 0 \\ t\left(t^2 - 1\right) - \left(t^2 - 1\right) &= 0 \\ \left(t-1\right)\left(t^2 - 1\right) &= 0 \\ \left(t-1\right)^2\left(t+1\right) &= 0 \\ t &= 1, -1 \\ \tan\left(\frac{\theta}{2}\right) &= 1, -1 \end{aligned}$$

Solve for $0 \le \frac{\theta}{2} \le \pi$:

$$\frac{\theta}{2} = \frac{\pi}{4}, \frac{3\pi}{4}$$

So therefore we have $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Question 11 (b) (i)

There are a total of 11 letters, and we have 2 H's and 3 O's.

So the number of permutations is $\frac{11!}{2!3!}$.

Question 11 (b) (ii)

There are 4 vowels and 7 consonants, and the vowels are to be grouped together.

Example: HCL**(OIOO)**PMHR

Arrange all the consonants and the group (OIOO) to get $\frac{8!}{2!}$. Note we have 2! in the denominator because we have 2 H's.

Arrange the vowels in the group, noting that we have three O', to get $\frac{4!}{3!}$.

So therefore the answer is $\frac{8!}{2!} \times \frac{4!}{3!}$.

Question 11 (b) (iii)

We will count the number of gaps between consonants, and then insert the vowels into these gaps.

We have 7 consonants, so therefore 8 gaps. We insert 4 vowels into these 8 gaps, and thus we have $\binom{8}{4}$.

We must now permute the vowels to acquire $\frac{4!}{3!}$.

Permute the consonants to acquire $\frac{7!}{2!}$.

So therefore the answer is $\binom{8}{4} \times \frac{4!}{3!} \times \frac{7!}{2!}$.

Question 11 (c)

Let $u = \tan^{-1} x$ such that $du = \dfrac{dx}{1+x^2}$.

$u = 1 \Rightarrow x = \dfrac{\pi}{4}$

$u = 0 \Rightarrow x = 0$

$$\begin{aligned}
\int_0^1 \frac{\cos^2\left(\tan^{-1} x\right)}{1+x^2} dx &= \int_0^{\frac{\pi}{4}} \cos^2 u \, du \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\cos 2u + 1\right) \, du \\
&= \frac{1}{2}\left[\frac{1}{2}\sin 2u + u\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right) \\
&= \frac{1}{8}\left(2+\pi\right)
\end{aligned}$$

Question 11 (d)

We know that $P(p) = p$ and $P(q) = q$.

$P(x) = (x-p)(x-p)Q(x) + Ax + B$

Using the above conditions:

$P(p) = Ap + B = p \qquad (1)$
$P(q) = Aq + B = q \qquad (2)$

$(1)-(2)$:

$A(p-q) = p-q \Rightarrow A = 1$

Substitute into (1):

$p + B = p \Rightarrow B = 0$

Hence the remainder is exactly *x*. □

Question 11 (e)

From ΔAOC, $OA=\dfrac{h}{\tan\alpha}$ and similarly in ΔBOC, we have $OB=\dfrac{h}{\tan\beta}$.

Using Pythagoras' Theorem, $OA^2+OB^2=d^2$.

$$\left(\frac{h}{\tan\alpha}\right)^2+\left(\frac{h}{\tan\beta}\right)^2=d^2$$

$$\frac{h^2}{\tan^2\alpha}+\frac{h^2}{\tan^2\beta}=d^2$$

$$h^2\left(\frac{1}{\tan^2\alpha}+\frac{1}{\tan^2\beta}\right)=d^2$$

$$h^2\left(\frac{\tan^2\alpha+\tan^2\beta}{\tan^2\alpha\tan^2\beta}\right)=d^2$$

And therefore:

$$h^2=\frac{d^2\tan^2\alpha\tan^2\beta}{\tan^2\alpha+\tan^2\beta}$$

Since $\alpha<90^\circ$, $\beta<90^\circ$, we have $\tan\alpha>0$, $\tan\beta>0$. Also, we must have $h>0$ and hence:

$$h=\frac{d\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha+\tan^2\beta}} \qquad \square$$

Question 12 (a)

When $x=0,\ \ddot{x}=3$, so we have $3=\dfrac{k}{b}$ and thus $3b=k$.

When $x=10,\ \ddot{x}=2$, so we have $2=\dfrac{k}{10+b}$ and thus $20+2b=k$.

Solving simultaneously yields $b=20$ and thus $k=60$.

So therefore $\ddot{x}=\dfrac{60}{x+20}$ and thus $\dfrac{d}{dx}\left(\dfrac{1}{2}V^2\right)=\dfrac{60}{x+20}$. Integrating both sides with respect to x yields:

$$\frac{1}{2}V^2=60\ln\left(x+20\right)+C$$

We are given that when $x=10,\ v=10$, so:

$$\begin{aligned}50&=60\ln\left(30\right)+C\\ C&=50-60\ln\left(30\right)\end{aligned}$$

So our expression is now:

$$\begin{aligned}\frac{1}{2}V^2&=60\ln\left(x+20\right)+50-60\ln\left(30\right)\\ V^2&=120\ln\left(x+20\right)-120\ln\left(30\right)+100\\ &=120\ln\left(\frac{x+20}{30}\right)+100\end{aligned}$$

Let $V=17$:

$$\begin{aligned}120\ln\left(\frac{x+20}{30}\right)+100&=17^2\\ 120\ln\left(\frac{x+20}{30}\right)&=189\\ \ln\left(\frac{x+20}{30}\right)&=1.575\\ \frac{x+20}{30}&\approx 4.83\\ x&\approx 124.92\text{m}\end{aligned}$$

So Jin JUST makes it out.

Alternatively

From

$$V^2 = 120\ln(x+20) - 120\ln(30) + 100$$
$$= 120\ln\left(\frac{x+20}{30}\right) + 100$$

Substitute $x = 125$:

$$V^2 \approx 289.064$$
$$V \approx 17.001$$

And hence, Jin JUST makes it out.

Question 12 (b) (i)

Base Case: $n=2$.

$$LHS=\sum_{p=2}^{2}\frac{1}{p^2-1}=\frac{1}{2^2-1}=\frac{1}{3} \qquad RHS=\frac{6+2}{8(3)}=\frac{8}{24}=\frac{1}{3}$$

Therefore true for $n=2$.

Inductive Hypothesis: $n=k$.

$$\sum_{p=2}^{k}\frac{1}{p^2-1}=\frac{(k-1)(3k+2)}{4k(k+1)}$$

Inductive Step: $k\Rightarrow k+1$.

Required to prove:

$$\sum_{p=2}^{k+1}\frac{1}{p^2-1}=\frac{k(3k+5)}{4(k+1)(k+2)}$$

$$\begin{aligned}
LHS&=\sum_{p=2}^{k+1}\frac{1}{p^2-1}\\
&=\sum_{p=2}^{k}\frac{1}{p^2-1}+\frac{1}{(k+1)^2-1}\\
&=\frac{(k-1)(3k+2)}{4k(k+1)}+\frac{1}{(k+1)^2-1}\\
&=\frac{(k-1)(3k+2)}{4k(k+1)}+\frac{1}{k(k+2)}\\
&=\frac{(k-1)(3k+2)(k+2)+4(k+1)}{4k(k+1)(k+2)}\\
&=\frac{3k^3+5k^2-4k-4+4k+4}{4k(k+1)(k+2)}\\
&=\frac{3k^3+5k^2}{4k(k+1)(k+2)}\\
&=\frac{k(3k+5)}{4(k+1)(k+2)}\\
&=RHS
\end{aligned}$$

Hence true by induction for all $n\geq 2$.

Question 12 (b) (ii)

$$\begin{aligned}\lim_{n\to\infty} S(n) &= \lim_{n\to\infty}\sum_{p=2}^{n}\frac{1}{p^2-1}\\ &= \lim_{n\to\infty}\frac{(n-1)(3n+2)}{4n(n+1)}\\ &= \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)\left(3+\frac{2}{n}\right)}{4\left(1+\frac{1}{n}\right)}\\ &= \frac{3}{4}\end{aligned}$$

Question 12 (c) (i)

There are a couple of ways to do this question.

Method #1:

The equation of the normal is given to be $x+py=ap\left(p^2+2\right)$. But we know that the point T lies on it, so we will substitute in the point $T\left(2at, at^2\right)$.

$$2at+apt^2=ap\left(p^2+2\right)$$
$$2at+apt^2=ap^3+2ap$$

Re-arrange:

$$\begin{aligned}ap^3-apt^2+2ap-2at&=0\\ ap\left(p^2-t^2\right)+2a(p-t)&=0\\ ap(p-t)(p+t)+2a(p-t)&=0 \qquad \ldots(\text{Noting that } p\neq t)\\ ap(p+t)+2a&=0\\ p(p+t)+2&=0\\ p^2+pt+2&=0 \qquad \square\end{aligned}$$

Method #2:

The equation of the normal intersects the parabola twice, but we know one of the roots is $x = 2ap$. We could easily do it the other way around, by substituting x into $x^2 = 4ay$, but that would be quite tedious.

Substitute the equation of the normal into the parabola:

$$\begin{aligned} x + py &= ap\left(p^2 + 2\right) \\ y &= a\left(p^2 + 2\right) - \frac{x}{p} \end{aligned}$$

Hence we have:

$$\begin{aligned} x^2 &= 4a\left(a\left(p^2 + 2\right) - \frac{x}{p}\right) \\ &= 4a^2\left(p^2 + 2\right) - \frac{4a}{p}x \end{aligned}$$

Re-arranging:

$$x^2 + \frac{4a}{p}x - 4a^2\left(p^2 + 2\right) = 0$$

Sum of roots is $x_1 + x_2 = -\frac{4a}{p}$. But we already know that one of the roots is $x = 2ap$ and the other is $x = 2at$, so therefore we have:

$$\begin{aligned} 2ap + 2at &= -\frac{4a}{p} \\ p + t &= -\frac{2}{p} \end{aligned}$$

And hence the result $p^2 + pt + 2 = 0$. □

Method #3:

The chord PT must be perpendicular to the tangent at P.

$$\begin{aligned}\nabla PT &= \frac{ap^2 - at^2}{2ap - 2at}\\ &= \frac{a(p-t)(p+t)}{2a(p-t)}\\ &= \frac{p+t}{2}\end{aligned}$$

The gradient of the tangent at P is

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dp}{dx/dp}\\ &= \frac{2ap}{2a}\\ &= p\end{aligned}$$

Hence

$$\begin{aligned}p \times \frac{p+t}{2} &= -1\\ p^2 + pt &= -2\\ p^2 + pt + 2 &= 0 \qquad \square\end{aligned}$$

Question 12 (c) (ii)

Similarly to (i), we can deduce the same expression, except with q.

So we have:

$$p^2 + pt + 2 = 0$$

$$q^2 + qt + 2 = 0$$

Subtract the two equations:

$$\begin{aligned}p^2 - q^2 + t(p-q) &= 0\\ (p-q)(p+q) + t(p-q) &= 0 \quad \text{...(note that } p \neq q)\\ p + q + t &= 0 \quad \square\end{aligned}$$

Question 12 (c) (iii)

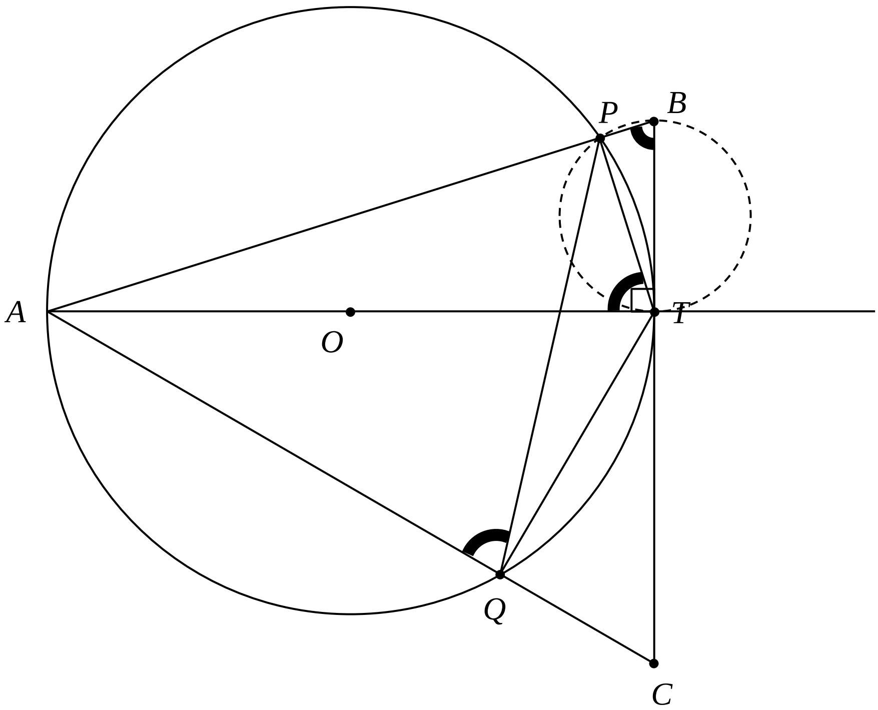
So we now have $p+q+t=0$ and $p^2+pt+2=0$.

Make t the subject to acquire $t=-\left(p+q\right)$, then substitute into $p^2+pt+2=0$:

$$\begin{aligned} p^2-p\left(p+q\right)+2&=0 \\ p^2-p^2-pq+2&=0 \\ pq&=2 \qquad \square \end{aligned}$$

Question 12 (d) (i)

First, we construct *PT* and *TQ*.



Let $\angle PBT=\theta$

$\angle APT=90^\circ$ (Angle subtended from diameter)

Therefore a circle can be constructed through points *B*, *P* and *T* such that *BT* is a diameter (converse of Thale's Theorem). This implies that *AT* is tangential to the circle (also since $\angle ATB=90^\circ$).

Hence, $\angle PBT=\theta=\angle PTA$ (Alternate Segment Theorem)

But $\angle PTA=\angle PQA$ (Angle subtended by common chord)

Therefore $\angle PBT = \theta = \angle PQA$.

Hence $PBCQ$ is a cyclic quadrilateral (converse of exterior angle from cyclic quadrilateral theorem). □

Alternatively

Let $\angle PTA = \theta$

$\angle PQA = \theta$ (Angle subtended by a common chord)

$\angle APT = 90^{\circ}$ (Angle subtended from diameter)

$\therefore \angle PAT = 90 - \theta$ (Angle sum of ΔPAT)

But ΔBTA is also right-angled, so

$$\begin{aligned} \angle PBT &= 90 - (90 - \theta) \quad \text{(Angle sum of } \Delta BTA\text{)} \\ &= \theta \end{aligned}$$

Hence, by the converse of Exterior Angle = Opposite Interior Angle Theorem:

$PBCQ$ is a cyclic quadrilateral □

Question 12 (d) (ii)

A basic angle chase yields the result immediately.

$\angle BCQ = \angle APQ$ (Exterior angle opposite interior angle of a cyclic quadrilateral)

$\angle APQ = \angle ATQ$ (Angle subtended by common chord)

Hence by the converse of the Alternate Segment Theorem, we have the result. □

Alternatively

We can simply observe that $\angle AQT = 90^{\circ}$, since it is an angle subtended from a diameter. It follows, by supplementary angles, that $\angle TQC = 90^{\circ}$ and hence the result by the converse of Thale's Theorem (Angle subtended from diameter is 90°). □

Question 13 (a)

We begin with the differential equation $\dfrac{dT}{dt} = k\left(E - T\right)$.

Separating the terms and grouping them appropriately, we have:

$$-\frac{dT}{T-E} = k\,dt$$

Note that we make the arrangement from $E - T$ to $T - E$ since $E < T$.

Integrate both sides with respect to the appropriate variable:

$$-\int_{T_0}^{T_n} \frac{dT}{T-E} = \int_{t_0}^{t_n} k\,dt$$

$$-\ln\left(T-E\right)\Big|_{T_0}^{T_n} = kt\Big|_{t_0}^{t_n}$$

Substituting and re-arranging, we have:

$$-\ln\left(T_n - E\right) + \ln\left(T_0 - E\right) = k\left(t_n - t_0\right)$$

$$\ln\left(\frac{T_0 - E}{T_n - E}\right) = k\left(t_n - t_0\right)$$

And hence:

$$k = \frac{\ln\left(\dfrac{T_0 - E}{T_n - E}\right)}{\left(t_n - t_0\right)}$$

But recall that $t_0 = 0$, hence:

$$k = \frac{1}{t_n}\ln\left(\frac{T_0 - E}{T_n - E}\right) \qquad \square$$

Question 13 (b) (i)

We are given the domain $0 \le x < 1$, from which we observe that $x^2 < 1$.

Multiply both sides by $a^2 - b^2$:

$$\left(a^2 - b^2\right) x^2 < a^2 - b^2$$

This is allowed since $a > b > 0$.

Expand and re-arrange:

$$\begin{aligned} a^2x^2 - b^2x^2 &< a^2 - b^2 \\ b^2 + a^2x^2 &< a^2 + b^2x^2 \end{aligned}$$

We carefully square root both sides, knowing that the inequality is still preserved.

$$\sqrt{b^2 + a^2x^2} < \sqrt{a^2 + b^2x^2}$$

Flip both sides, and thus the inequality:

$$\frac{1}{\sqrt{a^2 + b^2x^2}} < \frac{1}{\sqrt{b^2 + a^2x^2}}$$

Hence $f(x) < g(x)$.

And so the other inequality follows.

Alternatively

Let $f(x) < g(x)$:

$$\frac{1}{\sqrt{a^2 + b^2x^2}} < \frac{1}{\sqrt{b^2 + a^2x^2}}$$

$$\sqrt{a^2 + b^2x^2} > \sqrt{b^2 + a^2x^2}$$

Square both sides carefully, noting that the inequality is preserved.

$$\begin{aligned} a^2 + b^2x^2 &> b^2 + a^2x^2 \\ x^2\left(a^2 - b^2\right) &< a^2 - b^2 \end{aligned}$$

Hence $x^2 < 1$ and thus $0 \le x < 1$, since $x \ge 0$. The other direction of the inequality follows.

Question 13 (b) (ii)

This is a normal volumes problem now.

Since for $0 \le x < 1$, we have $f(x) < g(x)$, we can compute *V*.

$$\begin{aligned}
V &= \pi \int_0^1 \left(\frac{1}{b^2 + a^2x^2} - \frac{1}{a^2 + b^2x^2} \right) dx \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{ax}{b}\right) - \tan^{-1}\left(\frac{bx}{a}\right) \right]_0^1 \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) \right] \\
&= \frac{\pi}{ab} \tan^{-1}\left(\frac{\frac{a}{b} - \frac{b}{a}}{1 + \frac{a}{b} \times \frac{b}{a}} \right) \\
&= \frac{\pi}{ab} \tan^{-1}\left(\frac{a^2 - b^2}{2ab} \right) \quad \square
\end{aligned}$$

Question 13 (b) (iii)

This is essentially the same thing, with different limits and $f(x) \ge g(x)$.

$$\begin{aligned}
V_k &= \pi \int_1^k \frac{1}{a^2 + b^2x^2} - \frac{1}{b^2 + a^2x^2} dx \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{bx}{a}\right) - \tan^{-1}\left(\frac{ax}{b}\right) \right]_1^k \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{bk}{a}\right) - \tan^{-1}\left(\frac{ak}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{a}{b}\right) \right]
\end{aligned}$$

Note that as $k \to \infty$, $\tan^{-1}\left(\frac{bk}{a}\right) \to \frac{\pi}{2}$ and $\tan^{-1}\left(\frac{ak}{b}\right) \to \frac{\pi}{2}$.

Hence:

$$\begin{aligned}
V_k &\to \frac{\pi}{ab} \left[-\tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{a}{b}\right) \right] \\
&= \frac{\pi}{ab} \left[\tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}\left(\frac{b}{a}\right) \right]
\end{aligned}$$

And this is the same expression as (i). $\square$

Question 13 (c) (i)

We will use the identity $\sin^2\theta + \cos^2\theta = 1$.

Since $A \le B$, we have:

$$
\begin{aligned}
x &= A\left[\cos^2\left(\frac{nt}{2}\right) + \sin^2\left(\frac{nt}{2}\right)\right] + (B-A)\sin^2\left(\frac{nt}{2}\right) \\
&= A + (B-A)\sin^2\left(\frac{nt}{2}\right) \\
&= A + \frac{B-A}{2}(1-\cos nt) \\
&= \frac{A+B}{2} - \left(\frac{B-A}{2}\right)\cos nt
\end{aligned}
$$

Differentiate once with respect to t :

$$\dot{x} = n\left(\frac{B-A}{2}\right)\sin nt$$

Differentate again with respect to t :

$$
\begin{aligned}
\ddot{x} &= n^2\left(\frac{B-A}{2}\right)\cos nt \\
&= -n^2\left[x - \frac{A+B}{2}\right]
\end{aligned}
$$

Hence, the particle moves in Simple Harmonic Motion, with centre of motion being $x = \dfrac{A+B}{2}$. $\square$

Alternatively

Using the results $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, we have:

$$
\begin{aligned}
x &= A\cos^2\left(\frac{nt}{2}\right) + B\sin^2\left(\frac{nt}{2}\right) \\
&= \frac{A}{2}(1+\cos nt) + \frac{B}{2}(1-\cos nt) \\
&= \frac{1}{2}(A+B) + \frac{1}{2}(A-B)\cos nt
\end{aligned}
$$

Differentiate once with respect to t :

$$\dot{x} = -\frac{n}{2}(A-B)\sin nt$$

Differentiate again with respect to t :

$$
\begin{aligned}
\ddot{x} &= -\frac{n^2}{2}(A-B)\cos nt \\
&= -n^2\left[x - \frac{A+B}{2}\right]
\end{aligned}
$$

Question 13 (c) (ii)

Observe that the centre of motion is $x = \dfrac{A+B}{2}$ and amplitude is $\left(\dfrac{B-A}{2}\right)$.

One endpoint is $x_1 = \dfrac{A+B}{2} + \dfrac{B-A}{2} = B$.

The other endpoint is $x_2 = \dfrac{A+B}{2} - \dfrac{B-A}{2} = A$.

Hence $A \le x \le B$.

Question 13 (d) (i)

Using the Sine Rule in ΔAOP, we have:

$$\frac{l}{\sin\alpha} = \frac{1}{\sin\angle APO}$$

But we also have:

$$\begin{aligned}\angle APO &= 180^\circ - \alpha - \angle AOP \\ &= 180^\circ - \alpha - \left(90^\circ - \theta\right) \\ &= 180^\circ - \alpha - 90^\circ + \theta \\ &= 90^\circ - \left(\alpha - \theta\right)\end{aligned}$$

So:

$$\begin{aligned}\frac{l}{\sin\alpha} &= \frac{1}{\sin\left(90^\circ - \left(\alpha-\theta\right)\right)} \\ &= \frac{1}{\cos\left(\alpha-\theta\right)} \\ l &= \frac{\sin\alpha}{\cos\left(\alpha-\theta\right)} \qquad \square\end{aligned}$$

Question 13 (d) (ii) (1)

Using the Chain Rule, we have $\frac{d\alpha}{dt} = \frac{d\alpha}{dl} \times \frac{dl}{dt} = \frac{d\alpha}{dl} \times S$.

$$\frac{dl}{d\alpha} = \frac{\cos\alpha\cos(\alpha-\theta) + \sin(\alpha-\theta)\sin\alpha}{\cos^2(\alpha-\theta)}$$
$$= \frac{\cos\theta}{\cos^2(\alpha-\theta)}$$
$$\frac{d\alpha}{dl} = \frac{\cos^2(\alpha-\theta)}{\cos\theta}$$

$$\begin{aligned}\dot{\alpha} &= \frac{d\alpha}{dt} \\ &= \frac{d\alpha}{dl} \times \frac{dl}{dt} \\ &= \frac{\cos^2(\alpha-\theta)}{\cos\theta} \times S\end{aligned}$$

Let $\alpha = 2\theta$:

$$\begin{aligned}\dot{\alpha} &= \frac{\cos^2(2\theta-\theta)}{\cos\theta} \times S \\ &= \frac{\cos^2\theta}{\cos\theta} \times S \\ &= S\cos\theta \qquad \square\end{aligned}$$

Question 13 (d) (ii) (2)

We will use the formula $\ddot{\alpha} = \dot{\alpha} \times \frac{d\dot{\alpha}}{d\alpha}$.

It may seem unrecognisable now, but it is actually more commonly known as $a = v \times \frac{dv}{dx}$, which is much more well-known (as it is taught that way).

$$\begin{aligned}
\ddot{\alpha} &= \dot{\alpha} \times \frac{d\dot{\alpha}}{d\alpha} \\
&= \dot{\alpha} \times \frac{d}{d\alpha}\left(\frac{\cos^2(\alpha-\theta)}{\cos\theta} \times S\right) \\
&= \dot{\alpha} \times \frac{-2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \times S \\
&= -\dot{\alpha} \times S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta}
\end{aligned}$$

Let $\alpha = 2\theta$:

$$\begin{aligned}
\ddot{\alpha} &= -\dot{\alpha} \times S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \\
&= -\dot{\alpha} \times S \times \frac{2\cos\theta\sin\theta}{\cos\theta} \\
&= -\dot{\alpha} \times S \times 2\sin\theta
\end{aligned}$$

But $l = \frac{\sin\alpha}{\cos(\alpha-\theta)}$ and when $\alpha = 2\theta$,

$$\begin{aligned}
l &= \frac{\sin 2\theta}{\cos\theta} \\
&= \frac{2\sin\theta\cos\theta}{\cos\theta} \\
&= 2\sin\theta
\end{aligned}$$

Hence:

$$\begin{aligned}
\ddot{\alpha} &= -\dot{\alpha} \times S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \\
&= -\dot{\alpha} \times S \times \frac{2\cos\theta\sin\theta}{\cos\theta} \\
&= -\dot{\alpha} \times S \times l \quad \square
\end{aligned}$$

Alternatively

$$\begin{aligned}\ddot{\alpha} &= \frac{d\dot{\alpha}}{dt} \\ &= \frac{d\dot{\alpha}}{d\alpha} \times \frac{d\alpha}{dt} \\ &= \frac{d\dot{\alpha}}{d\alpha} \times S\cos\theta\end{aligned}$$

But recall that $\dot{\alpha} = \frac{\cos^2(\alpha-\theta)}{\cos\theta} \times S$

$$\begin{aligned}\frac{d\dot{\alpha}}{d\alpha} &= -S \times \frac{2\cos(\alpha-\theta)\sin(\alpha-\theta)}{\cos\theta} \\ &= -\frac{S}{\cos\theta}\sin(2\alpha-2\theta)\end{aligned}$$

Hence :

$$\ddot{\alpha} = -S^2 \times \sin(2\alpha-2\theta)$$

Substitute $\alpha = 2\theta$:

$$\begin{aligned}\ddot{\alpha} &= -S^2 \times \sin 2\theta \\ &= -2S^2 \sin\theta\cos\theta\end{aligned}$$

But recall that $\dot{\alpha} = S\cos\theta$. Also, similarly to the alternative solution above, $l = 2\sin\theta$.

Hence $\ddot{\alpha} = -\dot{\alpha} \times S \times l$ □

Question 14 (a) (i)

Consider the expansion $(1+x)^m (1+x)^{n-m} = (1+x)^n$.

Coefficient of x^k from RHS: $\binom{n}{k}$

Coefficient of x^k from LHS:

$$x^0 \times x^k \Rightarrow \binom{m}{0}\binom{n-m}{k}$$
$$x^1 \times x^{k-1} \Rightarrow \binom{m}{1}\binom{n-m}{k-1}$$
$$x^2 \times x^{k-2} \Rightarrow \binom{m}{2}\binom{n-m}{k-2}$$
...
$$x^k \times x^0 \Rightarrow \binom{m}{k}\binom{n-m}{0}$$

Hence coefficient of x^k from LHS is:

$$\binom{n-m}{0}\binom{n}{k} + \binom{n-m}{1}\binom{n}{k-1} + \binom{n-m}{2}\binom{n}{k-2} + \ldots + \binom{n-m}{k}\binom{n}{0}$$

And hence the result. $\square$

Question 14 (a) (ii)

We make the following substitutions, $n \to 2n$, $m \to n$, $k \to n$.

Then the identity from (i) now becomes:

$$\binom{2n-n}{0}\binom{n}{n}+\binom{2n-n}{1}\binom{n}{n-1}+\binom{2n-n}{2}\binom{n}{n-2}+\ldots+\binom{2n-n}{n}\binom{n}{0}=\binom{2n}{n}$$

Simplifying this:

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\ldots+\binom{n}{n}\binom{n}{0}=\binom{2n}{n}$$

But recall the identity $\binom{n}{k}=\binom{n}{n-k}$:

Hence $\binom{n}{0}=\binom{n}{n}$, $\binom{n}{1}=\binom{n}{n-1}$, … , $\binom{n}{n}=\binom{n}{0}$.

Therefore:

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n} \qquad \square$$

Question 14 (a) (iii)

We now make the substitution $r \to n-r$ in the summation on the right.

This will mean that in a similar fashion to Integration by Substitution, $r=0 \Rightarrow n=r$ and $r=n \Rightarrow n=0$. But also note that if we sum from 0 to *n*, or from *n* to 0, it makes no difference so we can just write the bottom limit as 0 and the top limit as *n*, to follow general convention.

Hence, our new expression is:

$$\sum_{r=0}^{n} r\binom{n}{r}^2 = \sum_{r=1}^{n}(n-r)\binom{n}{n-r}^2$$

Expanding the RHS, we get:

$$\sum_{r=0}^{n} r\binom{n}{r}^2 = \sum_{r=1}^{n} n\binom{n}{n-r}^2 - \sum_{r=1}^{n} r\binom{n}{n-r}^2$$

But recall that $\binom{n}{k} = \binom{n}{n-k}$. Applying it, we have:

$$\begin{aligned}\sum_{r=0}^{n} r\binom{n}{r}^2 &= \sum_{r=1}^{n} n\binom{n}{n-(n-r)}^2 - \sum_{r=1}^{n} r\binom{n}{n-(n-r)}^2 \\ &= \sum_{r=1}^{n} n\binom{n}{r}^2 - \sum_{r=1}^{n} r\binom{n}{r}^2\end{aligned}$$

Re-arranging the terms, we have:

$$2\sum_{r=0}^{n} r\binom{n}{r}^2 = n\sum_{r=1}^{n}\binom{n}{r}^2$$

And this is possible, since $\sum_{r=1}^{n} r\binom{n}{r}^2 = \sum_{r=0}^{n} r\binom{n}{r}^2$.

Substitute the result in (ii):

$$\begin{aligned}2\sum_{r=0}^{n} r\binom{n}{r}^2 &= n\sum_{r=1}^{n}\binom{n}{r}^2 \\ &= n\binom{2n}{n}\end{aligned}$$

Dividing both sides by 2 yields the required result. □

Alternatively

$$\binom{n}{1}^2+2\binom{n}{2}^2+\ldots+n\binom{n}{n}^2=\binom{n}{n-1}^2+2\binom{n}{n-2}^2+\ldots+n\binom{n}{n-n}^2$$

$$=n\binom{n}{0}^2+(n-1)\binom{n}{1}^2+(n-2)\binom{n}{2}^2+\ldots+\binom{n}{n-1}^2$$

$$=n\binom{n}{0}^2+\left[n\binom{n}{1}^2-\binom{n}{1}^2\right]+\left[n\binom{n}{2}^2-2\binom{n}{2}^2\right]+\ldots+\binom{n}{n-1}^2$$

$$=n\left[\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n-1}^2\right]-\left[\binom{n}{1}^2+2\binom{n}{2}^2+\ldots+(n-1)\binom{n}{n-1}^2\right]$$

Or in a more compact form

$$\sum_{k=1}^{n}k\binom{n}{k}^2=n\times\left[\sum_{k=0}^{n}\binom{n}{k}^2\right]-\left[\sum_{k=1}^{n}k\binom{n}{k}^2\right]$$

$$2\times\sum_{k=1}^{n}k\binom{n}{k}^2=n\times\left[\sum_{k=0}^{n}\binom{n}{k}^2\right]$$

$$\sum_{k=1}^{n}k\binom{n}{k}^2=\frac{n}{2}\binom{2n}{n}\qquad\square$$

Question 14 (b) (i)

By symmetry of the parabola, we have $\theta = \alpha_1$. Similarly for other trajectories, $\alpha_1 = \alpha_2$, $\alpha_2 = \alpha_3$, …. and inductively, we have $\theta = \alpha_1 = \alpha_2 = \alpha_3 = ...$ Hence $\theta = \alpha_n$, for all $n \in \mathbb{N}$.

Question 14 (b) (ii)

We will first find the time for the first trajectory.

Let $y = 0$:

$$-\frac{1}{2}gt^2 + V_n t \sin\theta = 0$$

$$-\frac{1}{2}gt + V_n \sin\theta = 0 \quad ...(\text{since } t = 0 \text{ is when it is at the origin})$$

$$t = \frac{2V_n \sin\theta}{g}$$

So therefore, we have $t_0 = \dfrac{2V_0 \sin\theta}{g}$.

Similarly:

$t_1 = \dfrac{2V_1 \sin\theta}{g} = \dfrac{2kV_0 \sin\theta}{g}$, using the recurrence $V_n = kV_{n-1}$.

$$t_2 = \frac{2V_2 \sin\theta}{g} = \frac{2kV_1 \sin\theta}{g} = \frac{2k^2V_0 \sin\theta}{g} \ \dots$$

$$t_n = \frac{2k^nV_0 \sin\theta}{g}$$

So summing up an infinite number of these, we have:

$$
\begin{aligned}
T_{V_0} &= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{2k^r V_0\sin\theta}{g} \\
&= \frac{2V_0\sin\theta}{g}\times\lim_{n\to\infty}\sum_{r=0}^{n}k^r \\
&= \frac{2V_0\sin\theta}{g}\times\frac{1}{1-k} \qquad \ldots\left(\text{since } |k|<1\right) \\
&= \frac{2V_0\sin\theta}{g\left(1-k\right)} \qquad \square
\end{aligned}
$$

Question 14 (b) (iii)

We will first find the distance of the first trajectory.

$$
\begin{aligned}
x_0 &= V_0\, t_0 \cos\theta \\
&= V_0 \times \frac{2V_0\sin\theta}{g}\times\cos\theta \\
&= \frac{2V_0{}^2\sin\theta\cos\theta}{g} \\
&= \frac{V_0{}^2\sin 2\theta}{g}
\end{aligned}
$$

Similarly,

$$
x_1 = \frac{V_1^2\sin 2\theta}{g} = \frac{k^2V_0{}^2\sin 2\theta}{g}
$$

$$
x_2 = \frac{V_2{}^2\sin 2\theta}{g} = \frac{k^2V_1^2\sin 2\theta}{g} = \frac{k^4V_0{}^2\sin 2\theta}{g}
$$

…

$$
x_n = \frac{k^{2n}\,V_0{}^2\sin 2\theta}{g}
$$

Hence, the total distance is:

$$\begin{aligned}
R &= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{k^{2r}V_0^{\ 2}\sin 2\theta}{g}\\
&= \frac{V_0^{\ 2}\sin 2\theta}{g}\lim_{n\to\infty}\sum_{r=0}^{n}k^{2r}\\
&= \frac{V_0^{\ 2}\sin 2\theta}{g}\times\frac{1}{1-k^2} \qquad \ldots\left(\text{since } |k|<1\right)\\
&= \frac{V_0^{\ 2}\sin 2\theta}{g\left(1-k^2\right)} \qquad \square
\end{aligned}$$

Question 14 (b) (iv)

We know that $T_R = \dfrac{2V_R\sin\theta}{g}$.

So placing it as a ratio:

$$\begin{aligned}
\frac{T_R}{T_{V_0}} &= \frac{\dfrac{2V_R\sin\theta}{g}}{\dfrac{2V_0\sin\theta}{g\left(1-k\right)}}\\
&= \frac{2V_R\sin\theta}{g}\times\frac{g\left(1-k\right)}{2V_0\sin\theta}\\
&= \frac{V_R}{V_0}\times\left(1-k\right)
\end{aligned}$$

So we must now find $\dfrac{V_R}{V_0}$.

Equate R with the range of any normal trajectory:

$$\frac{V_R^{\ 2}\sin 2\theta}{g} = \frac{V_0^{\ 2}\sin 2\theta}{g\left(1-k^2\right)}$$

$$V_R^{\ 2} = \frac{V_0^{\ 2}}{1-k^2}$$

$$\frac{V_R^{\ 2}}{V_0^{\ 2}} = \frac{1}{1-k^2}$$

$$\frac{V_R}{V_0} = \frac{1}{\sqrt{(1-k)(1+k)}}$$

Hence, substituting it in:

$$\frac{T_R}{T_{V_0}} = \frac{V_R}{V_0}\times(1-k)$$

$$= \frac{1-k}{\sqrt{(1-k)(1+k)}}$$

$$= \frac{\sqrt{1-k}}{\sqrt{1+k}}$$

$$= \sqrt{\frac{1-k}{1+k}} \qquad \square$$

Question 14 (b) (v)

We know that $0 < k < 1$.

Hence $1+k > 1$, and $1-k < 1$ and therefore $0 < \frac{1-k}{1+k} < 1$.

Square rooting both sides yields the same bounds, so $0 < \sqrt{\frac{1-k}{1+k}} < 1$.

But recall that $\frac{T_R}{T_{V_0}} = \sqrt{\frac{1-k}{1+k}}$, so therefore $0 < \frac{T_R}{T_{V_0}} < 1$, and hence $0 < T_R < T_{V_0}$. $\square$

Physically, this means that it will always take less time to get the ball to a location via landing it there in a single larger trajectory, as opposed to bouncing it there with a smaller one.