2012 Bored of Studies Trial Examinations

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 4

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 17

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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1 A particle is travelling in simple harmonic motion such that its velocity, in metres per second, is given by the equation $v^2 = a^2 - b^2x^2$, where $a, b \neq 0$.

What is the period of motion?

(A) a.

(B) b.

(C) $2\pi/a$.

(D) $2\pi/b$.

2 For what positive values of n does the expansion of $\left(x^2 - \frac{1}{x}\right)^n$ have a constant term?

(A) Odd values of n.

(B) Even values of n.

(C) Multiples of 3.

(D) Multiples of 4.

3 Solve the following expression for x, where a and k are positive real numbers.

$$\frac{a-x}{x} \leq k .$$

(A) $0 < x \leq \frac{a}{k+1}$.

(B) $0 < x < \frac{a}{k+1}$.

(C) $x > \frac{a}{k+1}$ or $x < 0$.

(D) $x \geq \frac{a}{k+1}$ or $x < 0$.

4 Consider the curve $y = \cos^{-1}\left(\frac{1}{x}\right)$. What is the domain and range?

(A) Domain: $x \le -1$ or $x \ge 1$.

Range: $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

(B) Domain: $-1 \le x \le 1$.

Range: $0 < y < \pi$, $y \ne \frac{\pi}{2}$.

(C) Domain: $x < -1$ or $x > 1$.

Range: $0 < y < \pi$, $y \ne \frac{\pi}{2}$.

(D) Domain: $-1 < x < 1$.

Range: $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

5 The line $y = kx + 4$ makes an acute angle θ with the line $y = -2x + 3$.

For what values of k is $\theta \le 45^\circ$?

(A) $-\frac{1}{3} \le k \le 3$.

(B) $\frac{1}{3} \le k \le 3$.

(C) $k \ge 3$ or $k \le -\frac{1}{3}$.

(D) $k \ge 3$ or $k \le \frac{1}{3}$.

6 At their formal, two twin girls Elle and Daisy are to be seated in a circle with five other friends. However, the two of them insist that they be seated next to each other. They are all assigned seats randomly.

What is the probability that such an arrangement is made?

(A) $1/6$.

(B) $1/3$.

(C) $2/7$.

(D) $1/21$.

7 The two curves $y = x$ and $y = e^{-x}$ intersect near $x = 0.5$.

Use Newton's method to find a better approximation, correct to 3 significant figures.

(A) $x = 0.57$.

(B) $x = 0.570$.

(C) $x = 0.56$

(D) $x = 0.566$.

8 Consider two points $A(a_1, a_2)$ and $B(b_1, b_2)$. A point $C(x, y)$ divides the interval AB internally in the ratio $k:1$. If C lies on the line $a_1x + a_2y = 0$, then

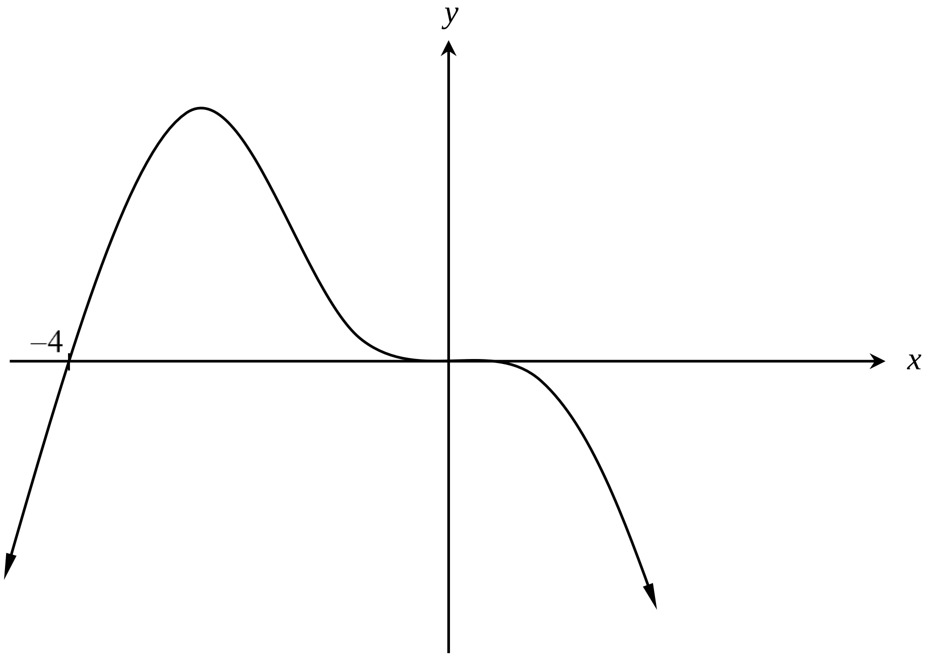
(A) $k = \dfrac{a_1b_1 + a_2b_2}{a_1^{\,2} + a_2^{\,2}}$.

(B) $k = \dfrac{a_1^{\,2} + a_2^{\,2}}{a_1b_1 + a_2b_2}$.

(C) $k = -\dfrac{a_1b_1 + a_2b_2}{a_1^{\,2} + a_2^{\,2}}$.

(D) $k = -\dfrac{a_1^{\,2} + a_2^{\,2}}{a_1b_1 + a_2b_2}$.

9 Which of the following equations best model the curve below?



(A) $y = -x^3(x-4)$.

(B) $y = -x^3(x+4)$.

(C) $y = x^3(x-4)$.

(D) $y = x^3(x+4)$.

10 An n sided 'die' has each of its faces labeled numbers from 1 to n. It is then tossed 3 times, and it is known that the probability of obtaining a particular number exactly 2 times is approximately 2%.

Which of the following is the most likely number of faces that the 'die' has, assuming each number has equal probability?

(A) 6.

(B) 8.

(C) 10.

(D) 12.

Total marks – 60
Attempt Questions 11 – 14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve for $0 \le \theta \le 2\pi$. **3**

$$\sin\theta - \cos\theta = \tan\left(\frac{\theta}{2}\right).$$

(b) How many ways can the letters of the word HOLOMORPHIC be arranged if

(i) There are no restrictions? **1**

(ii) The vowels are to be grouped together? **1**

(iii) No two vowels are next to each other? **2**

(c) Evaluate **3**

$$\int_0^1 \frac{\cos^2\left(\tan^{-1} x\right)}{1+x^2}\,dx\ ,$$

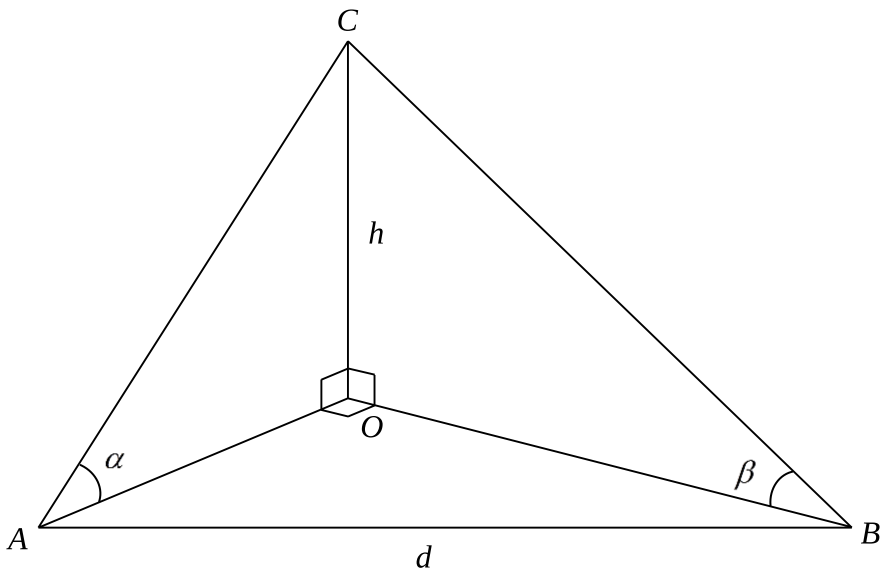
using the substitution $u = \tan^{-1} x$.

(d) A polynomial $P(x)$ has remainder p when divided by $x - p$, and has remainder q when divided by $x - q$, where p and q are real constants such that $p \neq q$. **2**

Show that the remainder when $P(x)$ is divided by $(x-p)(x-q)$, is exactly 1.

Question 11 continues on page 6

(e) Two people A and B observe a tower OC, which is h metres high with angles of elevation being α and β respectively. They are standing d metres apart, such that $\angle AOB = 90^\circ$, where O is the base of the tower. **3**



Prove that

$$h = \frac{d \tan\alpha \tan\beta}{\sqrt{\tan^2\alpha + \tan^2\beta}}.$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) A small aero-plane travels with acceleration $\ddot{x} = \dfrac{k}{x+b}$, for some $b,\ k > 0$. **3**

when the plane initially starts, it has an acceleration of 3ms^{-2}. It is known that once the jet has travelled 10 metres, its acceleration is now 2ms^{-2} and it travels with a velocity of $10\,\text{ms}^{-1}$.

The pilot Jin lands his plane on an island that has a runway exactly 125 metres long, and now wishes to leave. It is known that $V_L = 17$, where V_L is the minimum speed needed for take-off, in metres per second.

Determine if the runway is long enough for the Jin to leave the island.

(b) Define the expression $S(n) = \sum_{p=2}^{n} \dfrac{1}{p^2 - 1}$.

(i) Use Mathematical Induction to prove that **3**

$$S(n) = \frac{(n-1)(3n+2)}{4n(n+1)}.$$

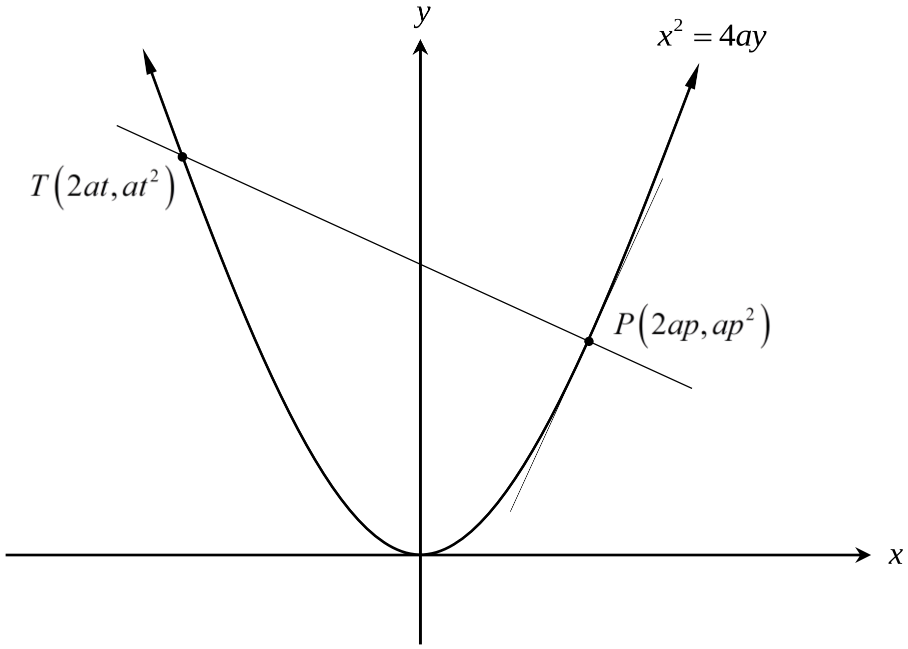
(ii) Hence, or otherwise, find the value of **1**

$$\sum_{p=2}^{\infty} \frac{1}{p^2 - 1}.$$

Question 12 continues on page 8

(c) Two points $P\left(2ap, ap^2\right)$ and $T\left(2at, at^2\right)$ lie on the parabola $x^2 = 4ay$.

A normal is constructed from P such that it meets at T.



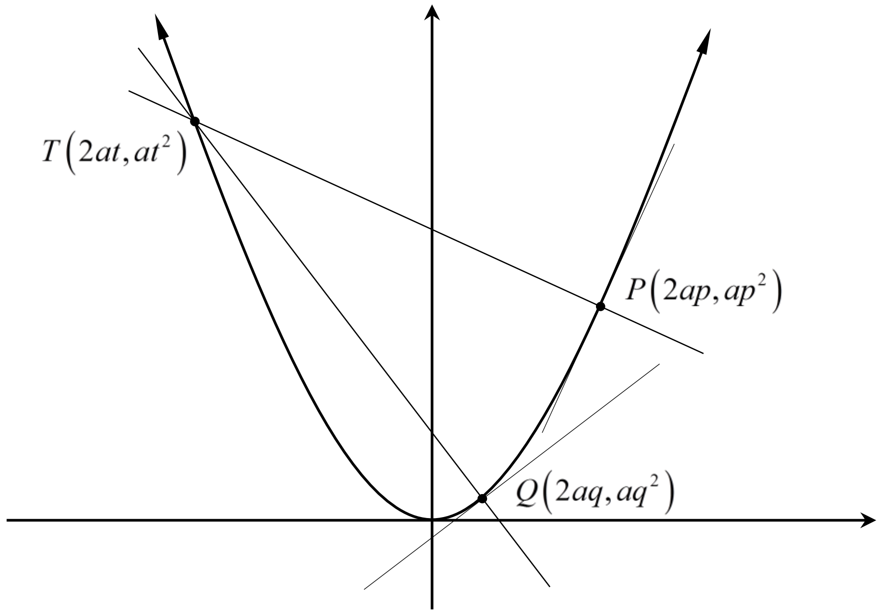
You may assume that the equation of the normal at P is

$$x + py = ap\left(p^2 + 2\right)$$ **(Do NOT prove this).**

(i) Show that $p^2 + pt + 2 = 0$. **2**

Question 12 continues on page 9

(ii) Suppose from another point $Q\left(2aq, aq^2\right)$ on the parabola, a normal is constructed to meet the parabola at T.



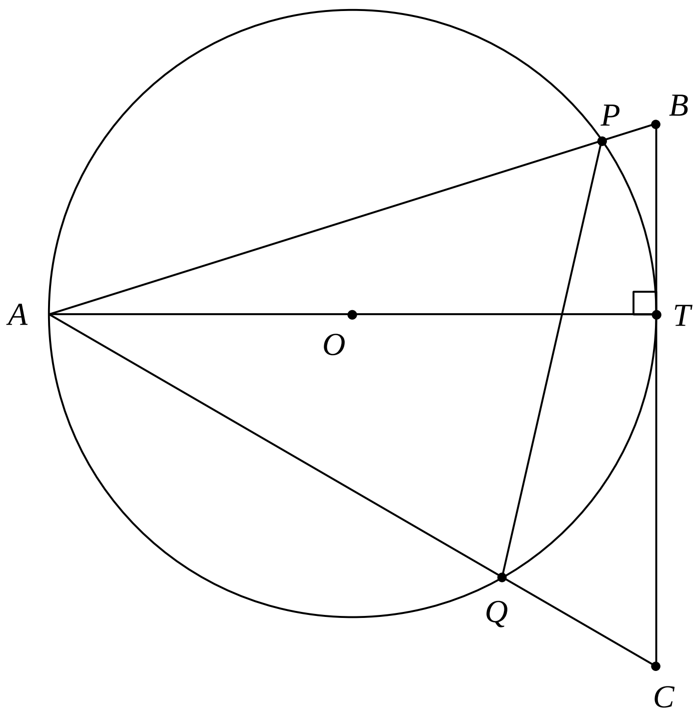
Prove that $p+q+t=0$. **1**

(iii) Deduce that $pq=2$. **1**

Question 12 continues on page 10

Question 12 (continued)

(d) In the diagram below, AT is the diameter of a circle with centre O, and BTC is the tangent to the circle at T. AB and AC intersect the circle at P and Q respectively.



Copy or trace the diagram into your writing booklet.

(i) Prove that $PBCQ$ is a cyclic quadrilateral. **2**

(ii) Hence, or otherwise, prove that $TQ \perp AC$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) An object with initial temperature T_0 is left to cool in a room with some environmental temperature E such that $T_0 > E$. At some time t_n, the temperature is measured to be T_n, where $T_n < T_0$ for all positive integers n. **2**

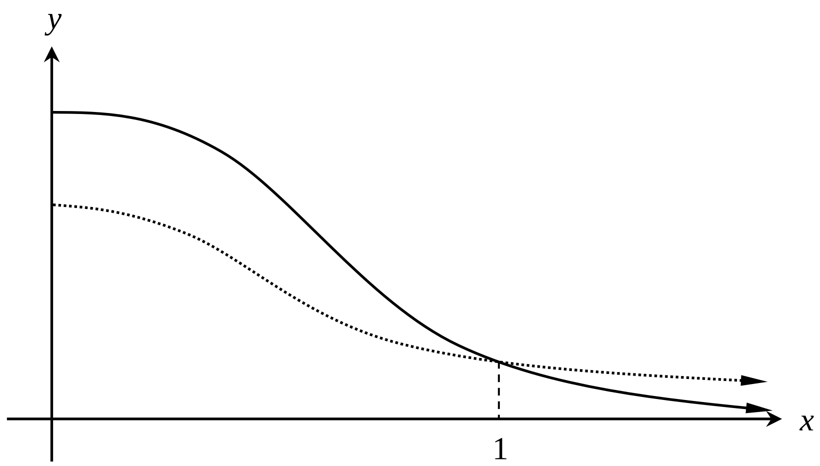
Assuming that the object follows the differential equation

$$\frac{dT}{dt} = k(E - T),$$

where k is some positive real constant and T is the temperature of the object at some time t, prove that

$$k = \frac{1}{t_n}\ln\left(\frac{T_0 - E}{T_n - E}\right).$$

(b) The diagram below shows the graph of two curves $f(x) = \dfrac{1}{\sqrt{a^2 + b^2x^2}}$ and $g(x) = \dfrac{1}{\sqrt{b^2 + a^2x^2}}$, defined for $x \geq 0$, where $a > b > 1$. Let k be some x coordinate such that $k > 1$.



(i) Briefly explain why for $0 \leq x < 1$, $f(x) < g(x)$ and for $x \geq 1$, $f(x) \geq g(x)$. **1**

Question 13 continues on page 12

(ii) The area bounded by the y axis, the two curves and the line $x = 1$, is rotated about the x axis to form a solid. **2**

Prove that the solid has volume

$$V = \frac{\pi}{ab}\tan^{-1}\left(\frac{a^2 - b^2}{2ab}\right).$$

You may assume, without proof, that $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$.

(iii) The area bounded by the two curves and the line $x = k$, is rotated about the x axis to form another solid V_k. **2**

Find V_k and hence prove that $\lim\limits_{k\to\infty} V_k = V$.

(c) A particle satisfies the displacement time equation

$$x = A\cos^2\left(\frac{nt}{2}\right) + B\sin^2\left(\frac{nt}{2}\right),$$

where $A < B$.

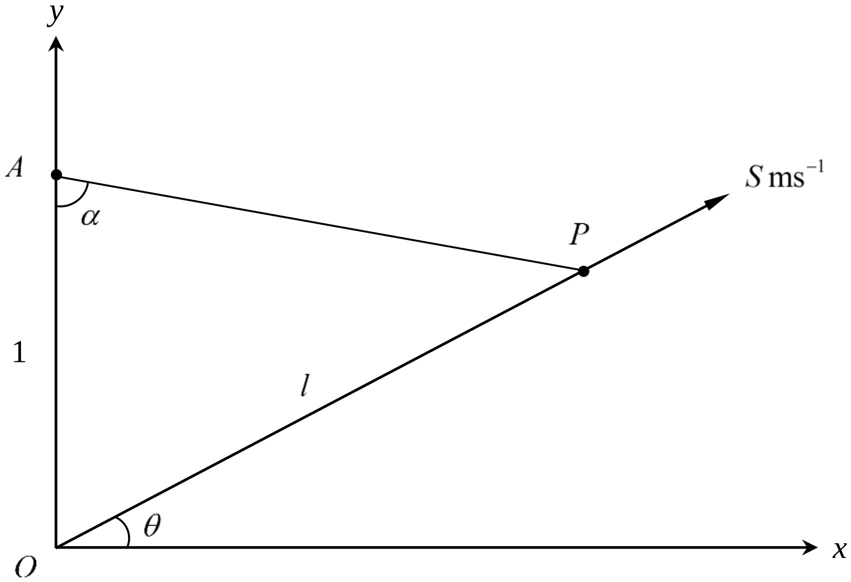
(i) Prove that the motion is simple harmonic. **2**

(ii) Show that **1**

$$A \le x \le B.$$

Question 13 continues on page 13

(d) A point A lies on the y axis, 1 unit away from the origin in the positive direction. Another point P lies on the line $y = x\tan\theta$, where $0^\circ < \theta < 90^\circ$, and travels along the line with some speed $S\ \mathrm{ms}^{-1}$. Let l be the distance from P to the origin, and let $\angle OAP$ be α.



(i) Prove that $l = \dfrac{\sin\alpha}{\cos(\alpha - \theta)}$. **1**

(ii) Let $\dot{\alpha} = \dfrac{d\alpha}{dt}$ and $\ddot{\alpha} = \dfrac{d^2\alpha}{dt^2}$. Show that when $\alpha = 2\theta$,

(1) $\dot{\alpha} = S\cos\theta.$ **2**

(2) $\ddot{\alpha} = -\dot{\alpha} \times S \times l.$ **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let k, m and n be positive integers such that $n > m > k$.

(i) Prove that **2**

$$\binom{m}{0}\binom{n-m}{k}+\binom{m}{1}\binom{n-m}{k-1}+\ldots+\binom{m}{k}\binom{n-m}{0}=\binom{n}{k}.$$

(ii) Hence deduce that **1**

$$\sum_{r=0}^{n}\binom{n}{r}^2=\binom{2n}{n}.$$

(iii) Hence, show that **2**

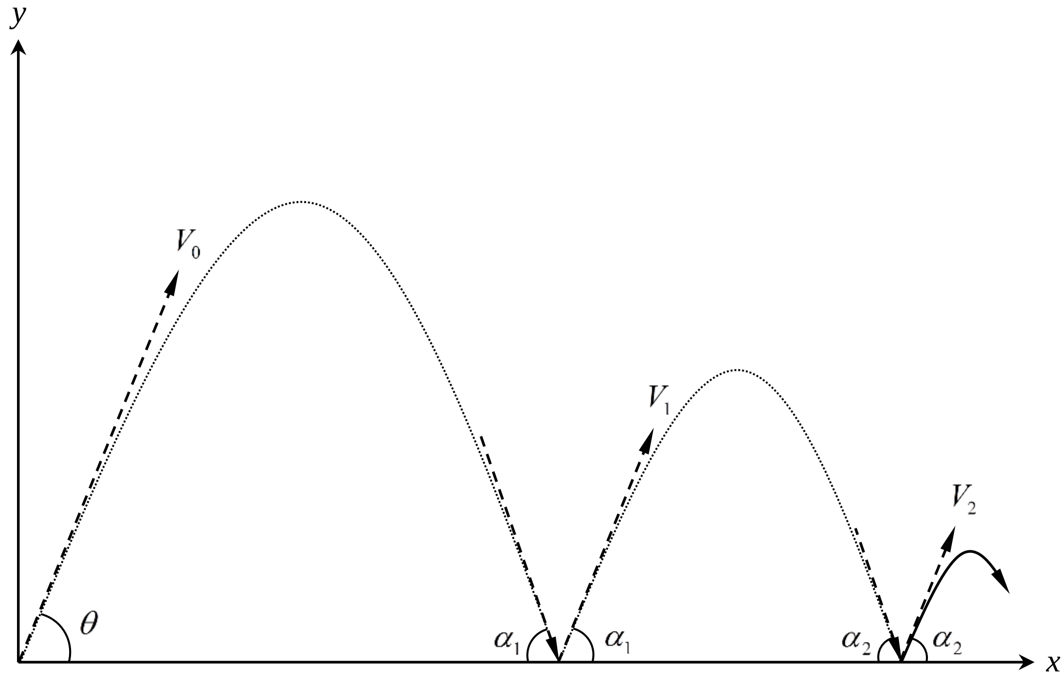
$$\sum_{r=0}^{n}r\binom{n}{r}^2=\frac{n}{2}\binom{2n}{n}.$$

Question 14 continues on page 15

(b) A ball is projected from level ground with some initial velocity V_0 and angle θ. Upon landing, it bounces an infinite number of times.

The n^{th} bounce has some new initial angle α_n being the same acute angle at which it landed, and initial velocity V_n, where $n = 0, 1, 2, \ldots$. However, the velocity V_n is given by the recurrence formula $V_n = kV_{n-1}$, where $0 < k < 1$, and $n \geq 1$.

Let the limiting range be R and let the amount of time it takes to reach that position be T_{V_0} .



You may assume the following equations:

$$x = V_n t \cos\theta ,$$

$$y = -\frac{1}{2} g t^2 + V_n t \sin\theta ,$$

where g is the acceleration due to gravity, and t is time in seconds.

Question 14 continues on page 16

(i) Briefly explain why $\theta = \alpha_n$, for all $n = 1, 2, 3, \ldots$ **1**

(ii) Show that the amount of time it takes to reach the total limiting range R is **2**

$$T_{V_0} = \frac{2V_0 \sin\theta}{g\left(1-k\right)}.$$

(iii) Prove that the total limiting range after infinite bounces is **2**

$$R = \frac{{V_0}^2 \sin 2\theta}{g\left(1-k^2\right)}.$$

Question 14 continues on page 17

(iv) The ball is now set to be projected with the same initial angle θ, but with some velocity $V_R > V_0$ such that when it lands, it lands with horizontal range R in a single projection of the ball. **3**

Let T_R be the time of flight for the ball to have horizontal range R, with initial velocity V_R.

Show that

$$\frac{T_R}{T_{V_0}} = \sqrt{\frac{1-k}{1+k}}\,.$$

(v) Deduce that for any $0 < k < 1$, we have **2**

$$T_R < T_{V_0}\,,$$

and interpret this result physically.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$