2022 **BORED OF STUDIES TRIAL EXAMINATION**

Mathematics Extension 1

Solutions

Section I

Answers

1	B	**6**	A
2	D	**7**	A
3	C	**8**	B
4	D	**9**	C
5	B	**10**	D

Brief explanations

1 Since the domain of $\sin^{-1} x$ is $|x| \leq 1$, then the domain of $\sin^{-1}\left(\dfrac{3x-1}{x-1}\right)$ is

$$\begin{aligned} \left|\frac{3x-1}{x-1}\right| &\leq 1 \\ (3x-1)^2 &\leq (x-1)^2 \\ 8x^2 - 4x &\leq 0 \\ 4x(2x-1) &\leq 0 \\ 0 \leq x &\leq \frac{1}{2} \end{aligned}$$

Hence, the answer is (B).

2 First note that (A) has a non-zero discriminant so there is no multiple real root. Also, (C) is always positive due to the even powers so there are no real roots at all. For (B), note that

$$\frac{d}{dx}\left(2x^3 - 6x^2 + 15x + 7\right) = 6x^2 - 12x + 15$$

The discriminant of this quadratic is negative so there is no solution to the first derivative and therefore no multiple root.

For (D), note that

$$\begin{aligned} \frac{d}{dx}\left(3x^5 - 10x^3 + 15x - 8\right) &= 15x^4 - 30x^2 + 15 \\ &= 15(x^4 - 2x^2 + 1) \\ &= 15(x-1)(x+1)(x^2+1) \end{aligned}$$

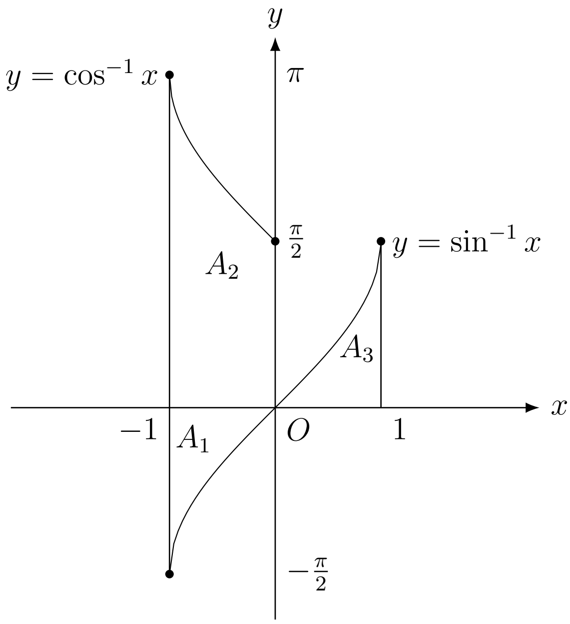
The first derivative has roots of ± 1. Only $x = 1$ makes $3x^5 - 10x^3 + 15x - 8 = 0$, so there exists a multiple real root. Hence, the answer is (D).

3 The acceleration vector is

$$\begin{aligned}
\ddot{\underset{\sim}{r}} &= (-16\sin 2t)\underset{\sim}{i} - (4\cos 2t)\underset{\sim}{j} \\
|\ddot{\underset{\sim}{r}}| &= \sqrt{256\sin^2 2t + 16\cos^2 2t} \\
&= 4\sqrt{16\sin^2 2t + \cos^2 2t} \\
&= 4\sqrt{15\sin^2 2t + 1}
\end{aligned}$$

The maximum occurs when $\sin^2 2t = 1$ so the maximum magnitude of the acceleration vector is 16 units. Hence the answer is (C).

4 Consider the graph of $y = \sin^{-1} x$ and a partial graph of $y = \cos^{-1} x$



From the above graph, define the following areas

$$A_1 = -\int_{-1}^{0} \sin^{-1} x\,dx \quad A_2 = \int_{-1}^{0} \cos^{-1} x\,dx \quad A_3 = \int_{0}^{1} \sin^{-1} x\,dx$$

Note that by the symmetry of the $y = \sin^{-1} x$ curve, $A_1 = A_3$. Going through the options:

- (A) gives $-A_1 - A_2$
- (B) gives $-A_1 + A_2$
- (C) gives $A_2 - A_3$, or equivalently $A_2 - A_1$
- (D) gives $A_2 + A_3$, or equivalently $A_2 + A_1$

Since the area between the curves $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for the domain $x \in [-1, 0]$ is $A_1 + A_2$, the answer is (D).

5 Notice that on the y-axis, the direction field shows a varying negative gradient. This suggests that when $x = 0$, $\dfrac{dy}{dx} < 0$. Options (A) and (C) have $\dfrac{dy}{dx} = 0$ at $x = 0$. Option (D) has $\dfrac{dy}{dx} \geq 0$ at $x = 0$. The only option which has the property $\dfrac{dy}{dx} < 0$ at $x = 0$ is (B).

6 From the double angle formula

$$\begin{aligned} \tan 2x &= \frac{2\tan x}{1-\tan^2 x} \\ \tan 2x(1-\tan^2 x) &= 2\tan x \\ \tan^2 x \tan 2x &= \tan 2x - 2\tan x \end{aligned}$$

This means that

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \tan^2 x \tan 2x \, dx &= \int_0^{\frac{\pi}{6}} (\tan 2x - 2\tan x)\, dx \\ &= \left[-\frac{1}{2}\ln(\cos 2x) + 2\ln(\cos x)\right]_0^{\frac{\pi}{6}} \\ &= -\frac{1}{2}\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{\sqrt{3}}{2}\right) \\ &= \ln\left(\frac{3}{2\sqrt{2}}\right) \end{aligned}$$

Hence, the answer is (A).

7 If $\underset{\sim}{p}$ is the projection of $\underset{\sim}{a}$ onto the direction of $\underset{\sim}{b}$ then $|\underset{\sim}{p}| = |\underset{\sim}{a}\cos\theta|$ where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Since $0 < |\cos\theta| < 1$ when $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-parallel, this suggests that $|\underset{\sim}{p}| < |\underset{\sim}{a}| < |\underset{\sim}{b}|$, which means (C) and (D) are always true.

Also,

$$\begin{aligned} |\underset{\sim}{p}| &= |\underset{\sim}{a}\cos\theta| \\ &= |\underset{\sim}{a}|\left|\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right| \\ &< \frac{|\underset{\sim}{a}\cdot\underset{\sim}{b}|}{|\underset{\sim}{a}|} \\ &= |\hat{\underset{\sim}{a}}\cdot\underset{\sim}{b}| \end{aligned}$$

This suggests that (B) is always true provided $|\underset{\sim}{a}| < |\underset{\sim}{b}|$.

In option (A), note that $|\hat{\underset{\sim}{a}}\cdot\hat{\underset{\sim}{b}}|$ is simply equal to $|\cos\theta|$. The inequality $|\underset{\sim}{p}| < |\cos\theta|$ may not be true if $|\underset{\sim}{a}| \geq 1$. Hence, the answer is (A).

8 The general term of the binomial expansion $(1+0.01)^{10}$ is $\binom{10}{k}(0.01)^k$ for positive integers $0 \le k \le 10$. Consider the first six terms of the expansion where the decimal places increase sequentially with each term.

$$\begin{aligned}
T_1 &= \binom{10}{0}(0.01)^0 = 1 \\
T_2 &= \binom{10}{1}(0.01)^1 = 0.1 \\
T_3 &= \binom{10}{2}(0.01)^2 = 0.0045 \\
T_4 &= \binom{10}{3}(0.01)^3 = 0.00012 \\
T_5 &= \binom{10}{4}(0.01)^4 = 0.0000021 \\
T_6 &= \binom{10}{5}(0.01)^5 = 0.0000000252
\end{aligned}$$

As k increases, the number of decimal places of the terms increases. It can be seen that T_5 (which starts at the 6th decimal place) and T_6 (which starts at the 8th decimal place) make no difference to the 4th decimal place when added on to the prior terms. This can also be argued for the subsequent terms T_7 to T_{11} due to the effect of increasing the powers of 0.01.

Therefore, only the sum of the first 4 terms are needed (i.e. $T_1 + T_2 + T_3 + T_4$) to get an approximation of 1.01^{10} correct to 4 decimal places, hence the answer is (B).

9 The general differential equation for exponential growth is $\dfrac{dP}{dt} = k(P - a)$ for some positive constants a and k.

This can be rewritten as $\dfrac{dP}{dt} = kP - ak$, suggesting $\dfrac{dP}{dt}$ is a linear function of P with positive gradient.

This means that (A) and (B) cannot be correct.

The general solution of the differential equation is $P = a + be^{kt}$ with positive constant b in an exponential growth scenario. Since the population is initially at P_0, then $P_0 = a + b$, so its initial rate of change is

$$\begin{aligned}
\frac{dP}{dt} &= k(P_0 - a) \\
&= kb > 0
\end{aligned}$$

This means that $\dfrac{dP}{dt}$ is initially a positive value before increasing linearly, hence the answer is (C).

10 Since $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, then

$$\begin{aligned}\int_{-1}^{1} \frac{\cos^{-1} x}{x^2+1}\,dx &= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{x^2+1}\,dx - \int_{-1}^{1} \frac{\sin^{-1} x}{x^2+1}\,dx \\ &= \frac{\pi}{2}[\tan^{-1} x]_{-1}^{1} - \int_{-1}^{1} \frac{\sin^{-1} x}{x^2+1}\,dx \\ &= \frac{\pi^2}{4} - \int_{-1}^{1} \frac{\sin^{-1} x}{x^2+1}\,dx \\ &= \frac{\pi^2}{4}\end{aligned}$$

Note that since $\sin^{-1} x$ is an odd function then $\dfrac{\sin^{-1}(-x)}{(-x)^2+1} = -\dfrac{\sin^{-1} x}{x^2+1}$.

This means that the integrand of the above second term is an odd function, which must integrate to zero across a symmetric interval about the y-axis. Hence, the answer is (D).

Question 11

(a) Manipulating the numerator gives

$$\begin{aligned}
\int \frac{\tan^4\theta - 6\tan^2\theta + 1}{\tan^4\theta + 2\tan^2\theta + 1}\,d\theta &= \int \frac{\tan^4\theta - 2\tan^2\theta + 1 - 4\tan^2\theta}{(\tan^2\theta + 1)^2}\,d\theta \\
&= \int \frac{(\tan^2\theta - 1)^2 - (2\tan\theta)^2}{(\tan^2\theta + 1)^2}\,d\theta \\
&= \int \left[\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right)^2 - \left(\frac{2\tan\theta}{\tan^2\theta + 1}\right)^2\right]\,d\theta \\
&= \int (\cos^2 2\theta - \sin^2 2\theta)\,d\theta \qquad \text{by } t\text{-formula in reference sheet} \\
&= \int \cos 4\theta\,d\theta \\
&= \frac{1}{4}\sin 4\theta + c
\end{aligned}$$

(b) (i) Factorising gives

$$\begin{aligned}
27x^3 - 12x &= 69x + 46 \\
3x(9x^2 - 4) &= 23(3x + 2) \\
3x(3x - 2)(3x + 2) - 23(3x + 2) &= 0 \\
(3x + 2)(9x^2 - 6x - 23) &= 0 \\
x &= -\frac{2}{3}, \frac{1 \pm 2\sqrt{6}}{3}
\end{aligned}$$

(ii) Manipulating the trigonometric expressions gives

$$\begin{aligned}
\sin\theta + \cos\theta &= \sqrt{2}\left(\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right) \\
&= \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)
\end{aligned}$$

(iii) Let $u = \sin\theta + \cos\theta$.

$$\begin{aligned}
\sin^3\theta + \cos^3\theta + \frac{23}{27} &= 0\\
(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta) + \frac{23}{27} &= 0\\
(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) + \frac{23}{27} &= 0
\end{aligned}$$

But

$$\begin{aligned}
u^2 &= (\sin\theta + \cos\theta)^2\\
&= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\
\sin\theta\cos\theta &= \frac{u^2-1}{2}
\end{aligned}$$

Substitute u into the equation.

$$\begin{aligned}
u\left(1 - \frac{u^2-1}{2}\right) + \frac{23}{27} &= 0\\
27u^3 - 81u - 46 &= 0\\
27u^3 - 12u &= 69u + 46
\end{aligned}$$

From part (i), this gives the solutions $u = -\frac{2}{3}, \frac{1 \pm 2\sqrt{6}}{3}$.

However, since $u = \sin\theta + \cos\theta$ then from part (ii), $u = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ which has a range of $\left[-\sqrt{2}, \sqrt{2}\right]$. This means that $u \neq \frac{1+2\sqrt{6}}{2}$.

Hence, the solutions are $\sin\theta + \cos\theta = -\frac{2}{3}, \frac{1-2\sqrt{6}}{3}$.

(c) (i) Consider the dot product of $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

$$\begin{aligned}
\overrightarrow{AC} \cdot \overrightarrow{BD} &= [(5\underset{\sim}{i} - 4\underset{\sim}{j}) - (\underset{\sim}{i} - 4\underset{\sim}{j})] \cdot [(2\underset{\sim}{i} + 5\underset{\sim}{j}) - (2\underset{\sim}{i} - 5\underset{\sim}{j})]\\
&= (4\underset{\sim}{i}) \cdot (10\underset{\sim}{j})\\
&= 0
\end{aligned}$$

Hence, $AC \perp BD$.

(ii) Consider the dot product of $\overrightarrow{DA}$ and $\overrightarrow{DC}$.

$$\begin{aligned}
\overrightarrow{DA} \cdot \overrightarrow{DC} &= |\overrightarrow{DA}||\overrightarrow{DC}|\cos\angle ADC\\
[(-\underset{\sim}{i} - 4\underset{\sim}{j}) - (2\underset{\sim}{i} + 5\underset{\sim}{j})] \cdot [(5\underset{\sim}{i} - 4\underset{\sim}{j}) - (2\underset{\sim}{i} + 5\underset{\sim}{j})] &= |\overrightarrow{DA}||\overrightarrow{DC}|\cos\angle ADC\\
(-3\underset{\sim}{i} - 9\underset{\sim}{j}) \cdot (3\underset{\sim}{i} - 9\underset{\sim}{j}) &= \sqrt{3^2+9^2}\sqrt{3^2+9^2}\cos\angle ADC\\
\cos\angle ADC &= \frac{4}{5}
\end{aligned}$$

(iii) Consider the dot product of $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\begin{aligned}
\overrightarrow{BA}\cdot\overrightarrow{BC} &= |\overrightarrow{BA}||\overrightarrow{BC}|\cos\angle ABC \\
[(-\underset{\sim}{i}-4\underset{\sim}{j})-(2\underset{\sim}{i}-5\underset{\sim}{j})]\cdot[(5\underset{\sim}{i}-4\underset{\sim}{j})-(2\underset{\sim}{i}-5\underset{\sim}{j})] &= |\overrightarrow{BA}||\overrightarrow{BC}|\cos\angle ABC \\
(-3\underset{\sim}{i}+\underset{\sim}{j})\cdot(3\underset{\sim}{i}+\underset{\sim}{j}) &= \sqrt{3^2+1^2}\sqrt{3^2+1^2}\cos\angle ABC \\
\cos\angle ABC &= -\frac{4}{5}
\end{aligned}$$

Since $\angle ADC$ and $\angle ADC$ are in $(0,\pi)$ then $\cos\angle ABC = \cos(\pi-\angle ADC) \Rightarrow \angle ABC + \angle ADC = \pi$.

Hence, $\angle ABC$ and $\angle ADC$ are supplementary.

(d) Consider the general term of the sum.

$$\begin{aligned}
\frac{1}{\sin\theta\sin(\theta+1^\circ)} &= \frac{1}{\sin\theta\sin(\theta+1^\circ)}\times\frac{\sin 1^\circ}{\sin 1^\circ} \\
&= \frac{\sin(\theta+1^\circ-\theta)}{\sin 1^\circ\sin\theta\sin(\theta+1^\circ)} \\
&= \frac{\sin(\theta+1^\circ)\cos\theta-\cos(\theta+1^\circ)\sin\theta}{\sin 1^\circ\sin\theta\sin(\theta+1^\circ)} \\
&= \frac{1}{\sin 1^\circ}\left(\cot\theta-\cot(\theta+1^\circ)\right)
\end{aligned}$$

Applying this to the sum

$$\begin{aligned}
&\frac{1}{\sin 45^\circ\sin 46^\circ}+\frac{1}{\sin 47^\circ\sin 48^\circ}+\frac{1}{\sin 49^\circ\sin 50^\circ}+\cdots+\frac{1}{\sin 133^\circ\sin 134^\circ} \\
&= \frac{1}{\sin 1^\circ}\left(\cot 45^\circ-\cot 46^\circ+\cot 47^\circ-\cot 48^\circ+\cdots+\cot 133^\circ-\cot 134^\circ\right) \\
&= \frac{1}{\sin 1^\circ}[(\cot 45^\circ+\cot 47^\circ+\cot 49^\circ+\cdots+\cot 89^\circ+\cot 91^\circ+\cdots+\cot 131^\circ+\cot 133^\circ) \\
&\quad -(\cot 46^\circ+\cot 48^\circ+\cdots+\cot 88^\circ+\cot 90^\circ+\cot 92^\circ+\cdots+\cot 132^\circ+\cot 134^\circ)]
\end{aligned}$$

However, $\cot(\pi-\theta) = -\cot\theta$, which reduces the sum to

$$\begin{aligned}
&\frac{1}{\sin 45^\circ\sin 46^\circ}+\frac{1}{\sin 47^\circ\sin 48^\circ}+\frac{1}{\sin 49^\circ\sin 50^\circ}+\cdots+\frac{1}{\sin 133^\circ\sin 134^\circ} \\
&= \frac{1}{\sin 1^\circ}\left(\cot 45^\circ-\cot 90^\circ\right) \\
&= \frac{1}{\sin 1^\circ}\left(1-\frac{\cos 90^\circ}{\sin 90^\circ}\right) \\
&= \frac{1}{\sin 1^\circ}
\end{aligned}$$

Question 12

(a) Let $\hat{p}$ be a random variable representing the unemployment rate with a mean of p. The distribution of $\hat{p}$ can be approximated by a normal distribution.

From a sample of 196 students surveyed in the past year, the probability of getting less than approximately 4% of students unemployed one year after graduation is 2.5%.

$$P(\hat{p} < 0.04) = 0.025$$

$$P\left(\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{196}}} < \frac{0.04 - p}{\sqrt{\frac{p(1-p)}{196}}}\right) = 0.025$$

$$P\left(Z < \frac{0.04 - p}{\sqrt{\frac{p(1-p)}{196}}}\right) = 0.025 \quad \text{where } Z \sim N(0, 1)$$

From the reference sheet, $P(-2 \leq Z \leq 2) = 0.95$, which implies $P(Z < -2) = 0.025$, by symmetry of the standard normal curve so

$$\begin{aligned}
\frac{0.04 - p}{\sqrt{\frac{p(1-p)}{196}}} &= -2 \quad \text{note that this implies } p > 0.04 \\
196(0.04 - p)^2 &= 4p(1-p) \\
49\left(\frac{1}{625} - \frac{2}{25}p + p^2\right) &= p - p^2 \\
31250p^2 - 3075p + 49 &= 0 \\
p &= \frac{3075 \pm \sqrt{3075^2 - 4(31250)(49)}}{62500} \\
&= \frac{3075 \pm 1825}{62500} \\
&= \frac{49}{625}, \frac{1}{50}
\end{aligned}$$

However, $p > 0.04$, which means the mean of $\hat{p}$ is $p = \dfrac{49}{625}$.

(b) (i) Let V_{ball} and S_{ball} the volume and surface area of a snowball respectively at time t. The snowball has a radius of r at time t. If the rate of decrease of the volume is directly proportional to its surface area then for some positive constant k

$$\begin{aligned}
\frac{dV_{\text{ball}}}{dt} &= -kS_{\text{ball}} \quad \text{from reference sheet } S_{\text{ball}} = 4\pi r^2 \\
\frac{dV_{\text{ball}}}{dr} \times \frac{dr}{dt} &= -4k\pi r^2 \quad \text{from reference sheet } V_{\text{ball}} = \frac{4}{3}\pi r^3 \\
4\pi r^2 \frac{dr}{dt} &= -4k\pi r^2 \\
\frac{dr}{dt} &= -k
\end{aligned}$$

Hence, the radius of each snowball melts at a constant rate.

(ii) Let r_1 and r_2 be the radius of the snowman's head and body respectively at time t. At time t, the height of the snowman as a whole is the sum of the diameters of the spheres, which is $2(r_1 + r_2)$.

From part (i)

$$\begin{aligned} \frac{dr_1}{dt} &= -k \\ r_1 &= -kt + c \qquad \text{when } t = 0, r_1 = R \Rightarrow c = R \\ r_1 &= -kt + R \\ \text{Similarly} \quad r_2 &= -kt + 2R \end{aligned}$$

This means $2(r_1 + r_2) = -4kt + 6R$, so the initial height is $6R$. When the height is half its initial value then

$$\begin{aligned} 3R &= -4kt + 6R \\ -kt &= -\frac{3R}{4} \qquad \text{substitute into } r_1 \text{ and } r_2 \\ r_1 &= \frac{R}{4} \\ r_2 &= \frac{5R}{4} \end{aligned}$$

Let V_0 and V_1 be the volumes of the snowman (which includes head and body) initially and when it reaches half its initial height respectively.

$$\begin{aligned} \frac{V_1}{V_0} &= \frac{\frac{4}{3}\pi\left(\frac{R}{4}\right)^3 + \frac{4}{3}\pi\left(\frac{5R}{4}\right)^3}{\frac{4}{3}\pi R^3 + \frac{4}{3}\pi(2R)^3} \\ &= \frac{7}{32} \end{aligned}$$

Hence, $V_1 : V_0$ is $7 : 32$.

(c) (i) From the horizontal component $t = \dfrac{x}{u\cos\theta}$. Substitute into y to get the Cartesian equation and noting that $u > 0$.

$$\begin{aligned} y &= ut\sin\theta - \frac{gt^2}{2} \\ &= \frac{x\sin\theta}{\cos\theta} - \frac{gx^2}{2u^2\cos^2\theta} \qquad \text{substitute the point at edge of the cliff } (H, H) \\ H &= \frac{H\sin\theta}{\cos\theta} - \frac{gH^2}{2u^2\cos^2\theta} \qquad \text{noting that } H \neq 0 \\ \frac{gH}{2u^2\cos^2\theta} &= \frac{\sin\theta - \cos\theta}{\cos\theta} \\ u^2 &= \frac{gH}{2\cos\theta(\sin\theta - \cos\theta)} \qquad \text{noting that } \theta \neq \frac{\pi}{2} \\ u &= \sqrt{\frac{gH}{2\cos\theta(\sin\theta - \cos\theta)}} \end{aligned}$$

(ii) For acute θ, a necessary condition to hit the cliff edge is for u to exist, so

$$\begin{aligned}\sin\theta &> \cos\theta\\ \tan\theta &> 1\\ \frac{\pi}{4} < \theta &< \frac{\pi}{2}\end{aligned}$$

(iii) When $\theta = \frac{\pi}{4}$, this means that the particle moves in a linear path from the origin to (H, H) on the cliff edge. This is not possible under the influence of gravity so the minimum value of θ must be a small (infinitesimal) value above $\frac{\pi}{4}$.

When $\theta = \frac{\pi}{2}$, this means that the particle is being projected upwards so it will never reach the cliff. This means the maximum value of θ must be a small (infinitesimal) value below $\frac{\pi}{2}$.

(iv) Objective is to minimise u when varying the launch angle θ. Since θ is in the denominator of u^2, then this is equivalent to maximising $\cos\theta(\sin\theta - \cos\theta)$.

$$\begin{aligned}\cos\theta(\sin\theta - \cos\theta) &= \cos\theta\sin\theta - \cos^2\theta\\ &= \frac{1}{2}(\sin 2\theta - \cos 2\theta - 1)\\ &= \frac{1}{2}\left[\sqrt{2}\left(\sin 2\theta\cos\frac{\pi}{4} - \cos 2\theta\sin\frac{\pi}{4}\right) - 1\right]\\ &= \frac{1}{\sqrt{2}}\sin\left(2\theta - \frac{\pi}{4}\right) - \frac{1}{2}\end{aligned}$$

The maximum occurs when $\sin\left(2\theta - \frac{\pi}{4}\right) = 1$, which gives $\theta = \dfrac{3\pi}{8}$.

(d) First note that the sum of all the digits is $0 + 1 + 2 + \cdots + 9 = 45$ which enables the construction of a ten digit number. For an eight digit number to be divisible by 9, two digits are to removed so that the eight digits sum to either $36, 27, 18$ or 9.

However, the sum of any two of the digits can only be between 1 and 17 inclusive. This means that to ensure the eight digit number is divisible by 9, the two digits to be excluded must sum to 9 (making the remaining eight digits sum to 36). These pairs are $(0, 9), (1, 8), (2, 7), (3, 6)$ and $(4, 5)$.

The case of $(0, 9)$ leaves the eight digits with no zeroes to consider. Hence, there are $8!$ possible ways to arrange the digits to construct the eight digit number.

The case of $(1, 8)$ leaves the eight digits with a zero to consider. Since the first digit cannot be zero, then there are $7 \times 7!$ ways (due to the $7!$ ways to arrange the non-zero digits and the 7 ways to position the zero) to arrange the digits to construct the eight digit number. This logic similarly applies to the other three pairs $(2, 7), (3, 6)$ and $(4, 5)$.

Putting this altogether gives $8! + 4(7 \times 7!)$, or equivalently, 181440 ways of making the eight digit number to be divisible by 9.

Question 13

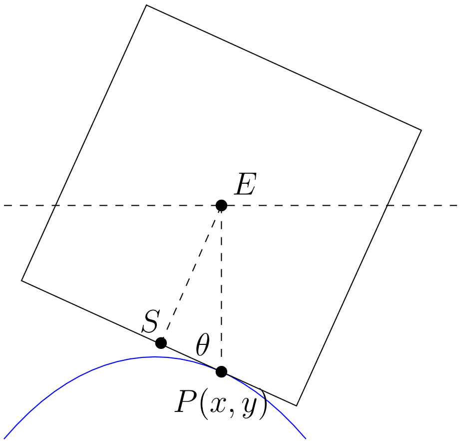
(a) Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$. Note that since $\theta = \tan^{-1} x$ then $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so $\cos\theta > 0$.

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\, d\theta \qquad \text{but } \sec^2\theta - \tan^2\theta = 1 \\
&= \int \sec\theta\, d\theta \\
&= \int \sec\theta \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta \\
&= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta \\
&= \ln(\sec\theta + \tan\theta) + c \\
&= \ln\left(\sqrt{1+\tan^2\theta} + \tan\theta\right) + c \\
&= \ln(x + \sqrt{1+x^2}) + c
\end{aligned}$$

(b) Since $|\underset{\sim}{v}| = \dfrac{dr}{dt}$ then

$$\begin{aligned}
|\underset{\sim}{v}|^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
\left(\frac{dr}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
\left(\frac{dr}{dx} \times \frac{dx}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx} \times \frac{dx}{dt}\right)^2 \\
\left(\frac{dx}{dt}\right)^2 \left(\frac{dr}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right) \\
\left(\frac{dr}{dx}\right)^2 &= 1 + \left(\frac{dy}{dx}\right)^2 \qquad \text{noting that } \frac{dx}{dt} \neq 0
\end{aligned}$$

(c) (i) S is the point on the square that was initially touching the curve as shown on the below diagram. Note that θ is angle between EP and the side of the square touching the curve.



PS is the tangent to the curve at $P(x, y)$. The gradient of this tangent is $\tan\alpha$ where α is angle to the positive horizontal. This implies that

$$\begin{aligned} f'(x) &= \tan\left(\theta + \frac{\pi}{2}\right) \\ &= \frac{\sin\left(\theta + \frac{\pi}{2}\right)}{\cos\left(\theta + \frac{\pi}{2}\right)} \\ &= -\frac{\cos\theta}{\sin\theta} \\ &= -\cot\theta \end{aligned}$$

(ii) The length of PS is equal to the path traced along the curve $y = f(x)$ by the square. Hence, $PS = r$.

Also, note that $ES \perp PS$ which implies $\triangle ESP$ is a right-angled triangle, so

$$\begin{aligned} \tan\theta &= \frac{ES}{PS} \\ PS &= ES\cot\theta \end{aligned}$$

Since the square has a side length of 2 units then $ES = 1$ because E is the centre of the square and S is the midpoint of one of the square's side. This implies that $PS = \cot\theta$ and hence

$$\begin{aligned} f'(x) &= -\cot\theta \\ &= -PS \\ &= -r \\ f''(x) &= -\frac{dr}{dx} \\ &= -\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad \text{noting that } r \text{ is increasing with } x \Rightarrow \frac{dr}{dx} > 0 \\ &= -\sqrt{1 + [f'(x)]^2} \end{aligned}$$

(iii) Let $w = f'(x)$. Using the result in part (a) gives

$$\begin{aligned}
\frac{dw}{dx} &= -\sqrt{1+w^2} \\
\frac{dx}{dw} &= -\frac{1}{\sqrt{1+w^2}} \\
x &= -\ln(w + \sqrt{1+w^2}) + c \qquad \text{when } x = 0, f'(x) = 0 \Rightarrow c = 0 \\
x &= -\ln(w + \sqrt{1+w^2}) \\
e^{-x} - w &= \sqrt{1+w^2} \\
e^{-2x} - 2we^{-x} + w^2 &= 1 + w^2 \\
w &= \frac{e^{-x} - e^{x}}{2}
\end{aligned}$$

(iv) Since $w = f'(x)$ then

$$\begin{aligned}
f'(x) &= \frac{e^{-x} - e^{x}}{2} \\
f(x) &= \frac{-e^{-x} - e^{x}}{2} + c \qquad \text{when } x = 0, f(x) = -1 \Rightarrow c = 0 \\
&= -\frac{e^{-x} + e^{x}}{2}
\end{aligned}$$

(d) (i) Since φ is a root of $x^2 - x - 1 = 0$ then

$$\begin{aligned}
\varphi^2 - \varphi - 1 &= 0 \\
\varphi^2 - 1 &= \varphi \\
(\varphi - 1)(\varphi + 1) &= \varphi \\
\varphi - 1 &= \frac{\varphi}{\varphi + 1} \\
\varphi - a_1 &= \frac{\varphi}{\varphi + a_1}
\end{aligned}$$

The statement is therefore true for $n = 1$.

Assume the statement is true for $n = k$.

$$\varphi - a_k = \frac{\varphi}{(\varphi + a_1)(\varphi + a_2)\cdots(\varphi + a_k)}$$

Required to prove the statement is true for $n = k + 1$

$$\varphi - a_{k+1} = \frac{\varphi}{(\varphi + a_1)(\varphi + a_2)\cdots(\varphi + a_{k+1})}$$

$$
\begin{aligned}
\text{RHS} &= \frac{\varphi}{(\varphi+a_1)(\varphi+a_2)\cdots(\varphi+a_k)(\varphi+a_{k+1})} \\
&= \frac{\varphi-a_k}{\varphi+a_{k+1}} \quad \text{by assumption} \\
&= \frac{\varphi-a_k}{\varphi+a_{k+1}} \times \frac{\varphi-a_{k+1}}{\varphi-a_{k+1}} \quad \text{but } a_{k+1}=\sqrt{1+a_k} \\
&= \frac{\varphi-a_k}{\varphi+\sqrt{1+a_k}} \times \frac{\varphi-a_{k+1}}{\varphi-\sqrt{1+a_k}} \\
&= \frac{(\varphi-a_k)(\varphi-a_{k+1})}{\varphi^2-1-a_k} \quad \text{but } \varphi^2-\varphi-1=0 \Rightarrow \varphi^2-1=\varphi \\
&= \frac{(\varphi-a_k)(\varphi-a_{k+1})}{\varphi-a_k} \\
&= \varphi-a_{k+1} \\
&= \text{LHS}
\end{aligned}
$$

The statement is true for $n=1$, hence by mathematical induction it is true for all integers $n \geq 1$.

(ii) Since φ is a root of quadratic polynomial, there are two possible values of φ. Consequently, there are two possible limits to consider.

Case 1: $a_n \to \varphi = \dfrac{1-\sqrt{5}}{2}$

This is impossible, as $a_n > 0$ for every positive integer n, whereas $\dfrac{1-\sqrt{5}}{2} < 0$, so the sequence cannot possibly approach $\dfrac{1-\sqrt{5}}{2}$.

Case 2: $a_n \to \varphi = \dfrac{1+\sqrt{5}}{2}$

As every term in the sequence is positive, $\varphi + a_k > \varphi$ for any integer k. Hence

$$
\begin{aligned}
\varphi - a_n &= \frac{\varphi}{(\varphi+a_1)(\varphi+a_2)\cdots(\varphi+a_n)} \\
&< \frac{\varphi}{(\varphi)(\varphi)\cdots(\varphi)} \\
&= \frac{1}{\varphi^{n-1}}
\end{aligned}
$$

Since $a_k > 0$ and $\varphi > 0$ then $\dfrac{\varphi}{(\varphi+a_1)(\varphi+a_2)\cdots(\varphi+a_n)} > 0$ so

$$0 < \varphi - a_n < \frac{1}{\varphi^{n-1}}.$$

As $n \to \infty, \varphi - a_n \to 0 \Rightarrow a_n \to \varphi = \dfrac{1+\sqrt{5}}{2}$.

Question 14

(a) Let $x = k\cos\theta$, where $0 \le \theta \le \pi$ for unique values of x, be a root of $x^3 - 3x + 1$ for some unknown positive constant k.

$$\begin{aligned} x^3 - 3x + 1 &= 0 \\ k^3\cos^3\theta - 3k\cos\theta + 1 &= 0 \\ k(k^2\cos^3\theta - 3\cos\theta) + 1 &= 0 \end{aligned}$$

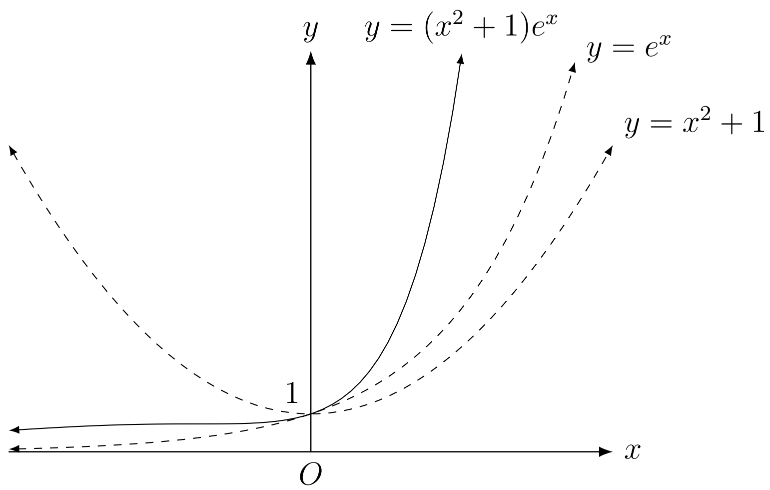
By inspection, $k = 2$ is an appropriate choice to use the result for $\cos 3\theta$.

$$\begin{aligned} 2(4\cos^2\theta - 3\cos\theta) + 1 &= 0 \\ \cos 3\theta &= -\frac{1}{2} \\ 3\theta &= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3} \\ \theta &= \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9} \end{aligned}$$

Hence, the roots are $2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}$. The sum of the square of the roots is

$$\begin{aligned} 4\cos^2\frac{2\pi}{9} + 4\cos^2\frac{4\pi}{9} + 4\cos^2\frac{8\pi}{9} &= \left(2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{8\pi}{9}\right)^2 \\ &\quad - 2\left(4\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + 4\cos\frac{2\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9}\right) \\ &= 0^2 - 2(-3) \\ \therefore \cos^2\frac{2\pi}{9} + \cos^2\frac{4\pi}{9} + \cos^2\frac{8\pi}{9} &= \frac{3}{2} \end{aligned}$$

(b) (i) By multiplication of ordinates, the following sketch is obtained for $y = (x^2+1)e^x$.



(ii) From the graph, the solution to $e^x > \frac{1}{x^2+1}$, or equivalently $(x^2+1)e^x > 1$ is $x > 0$.

(c) (i) Taking the derivative of $f(x)$ gives

$$
\begin{aligned}
f'(x) &= e^x - \frac{k}{1+x^2} \\
&= \frac{(1+x^2)e^x - k}{1+x^2}
\end{aligned}
$$

Stationary points occur when $f'(x) = 0$, which gives $(1+x^2)e^x = k$. For a stationary point to exist, $k > 0$ since $1 + x^2 > 0$ and $e^x > 0$.

Note that the graph of $y = (x^2 + 1)e^x$ is a non-decreasing function. As x increases, y increases (see remark).

Suppose that x_0 is the exact of value of x at the stationary point such that $(1 + x_0^2)e^{x_0} = k$.

For some $x_1 < x_0 < x_2$

$$
\begin{aligned}
(1+x_1^2)e^{x_1} &< (1+x_0^2)e^{x_0} < (1+x_2^2)e^{x_2} \\
(1+x_1^2)e^{x_1} &< k < (1+x_2^2)e^{x_2} \\
f'(x_1) &= \frac{(1+x_1^2)e^{x_1} - k}{1+x_1^2} < 0 \\
f'(x_2) &= \frac{(1+x_2^2)e^{x_2} - k}{1+x_2^2} > 0
\end{aligned}
$$

By the first derivative test, the stationary point is a minimum turning point.

Remark: There is actually a horizontal point of inflexion at $x = -1$ for $y = (x^2 + 1)e^x$. However, since this is the only stationary point then the claim that as x increases, y increases still holds.

(ii) The turning point is the solution to $(1 + x^2)e^x = k$.

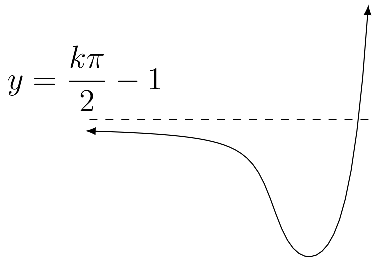
From part (b)(ii), if $k > 1$ then the range of possible x-values of the turning point is $x > 0$, noting the non-decreasing nature of $y = (1 + x^2)e^x$.

(iii) First note that $f(0) = 0$ for **any** value of k.

As $x \to -\infty$, $f(x) \to \dfrac{k\pi}{2} - 1$ and as $x \to \infty$, $f(x) \to \infty$.

This means that there is a horizontal asymptote at at $y = \dfrac{k\pi}{2} - 1$.

The above properties imply that $f(x)$ has the following shape for $k > 0$.



Case 1: $k \leq 0$

There is no real solution to $f'(x) = 0$. Since $(1 + x^2)e^x - k > 0$ then $f'(x) > 0$. This implies that there is only one solution to $f(x) = 0$ since $f(x)$ is continuous and increasing across all real x.

Case 2: $0 < k < 1$

From part (b), this implies that the stationary point occurs when $x < 0$. The point $f(x)$ crosses the origin is on the right side of the turning point.

In order for the right side of the turning point to cross the x-axis, the asymptote must be above the x-axis, which gives the result $\dfrac{2}{\pi} < k < 1$ in this interval to get two distinct solutions.

Case 3: $k = 1$

The solution to $f'(x) = 0$ when $k = 1$ is $x = 0$. This is because from part (b), the point $(0, 1)$ lies on the graph of $y = (x^2 + 1)e^x$. As a result, the turning point coincides with the origin, which only gives one solution to $f(x) = 0$.

Case 4: $k > 1$

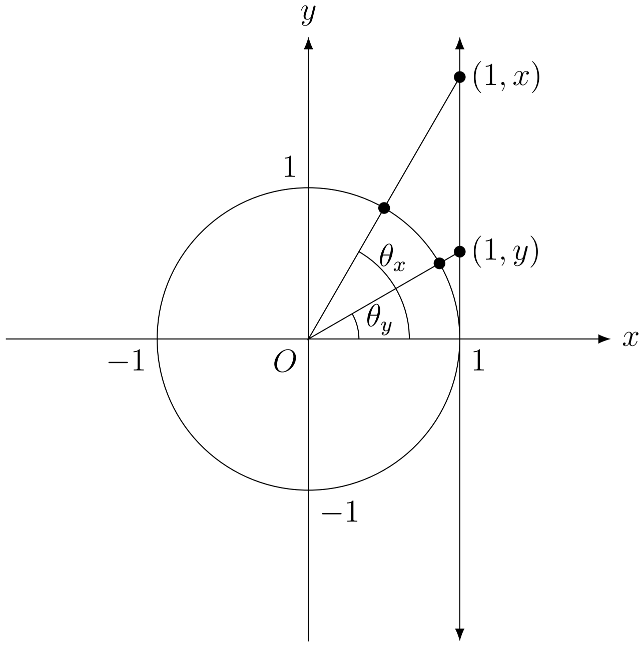
From part (b), this implies that the stationary point occurs when $x > 0$. The origin is on the left side of the turning point. Since $f(x)$ continues to increase without bound for $x > 0$, there will also be a second x-intercept on the right side of the turning point. This means that that there are always two distinct solutions for $k > 1$.

Putting all these cases together, $f(x) = 0$ has exactly two distinct solutions when $k > \dfrac{2}{\pi}$, except for $k = 1$.

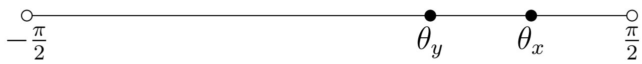
(d) Since $y = \tan^{-1} x$ is an increasing function, its preserves the sign of inequalities. Given the inequality

$$\begin{aligned} -1 &< \frac{x-y}{1+xy} < 1 \\ -\frac{\pi}{4} &< \tan^{-1}\left(\frac{x-y}{1+xy}\right) < \frac{\pi}{4} \\ -\frac{\pi}{4} &< \tan^{-1} x - \tan^{-1} y < \frac{\pi}{4} \end{aligned}$$

Let $\theta_x = \tan^{-1} x$ and $\theta_y = \tan^{-1} y$ and consider the following diagram of the unit circle.



This suggests that instead of considering pairs of real numbers, pairs of numbers from the interval $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ can be considered instead.



Partition the interval into the following four sets of equally spaced sub-intervals:

$$\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right), \left[-\frac{\pi}{4}, 0\right), \left[0, \frac{\pi}{4}\right), \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$$

By the pigeonhole principle, for any 5 numbers from the interval $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$, at least one pair (θ_x, θ_y) must fall into the same sub-interval.

Since the sub-intervals are each of length $\frac{\pi}{4}$, and exclude at least one of their endpoints, then the absolute difference between the angles θ_x and θ_y cannot exceed $\frac{\pi}{4}$, that is

$$-\frac{\pi}{4} < \theta_x - \theta_y < \frac{\pi}{4}.$$

Notice that with the configuration of sets provided, for such a pair, either $x, y < 0$, or $x, y \geq 0$, so $xy > -1$ is always true.