ADVANCED MATHEMATICS

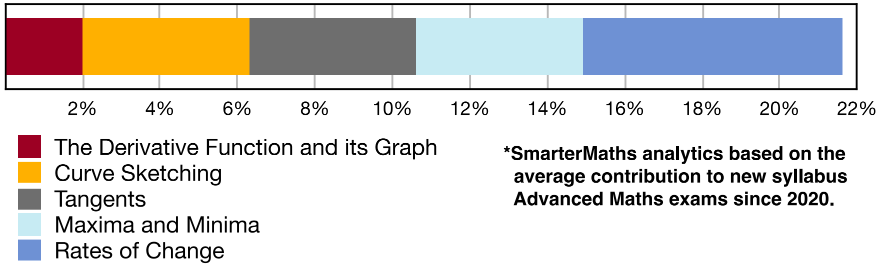
Calculus (Adv), C3 Applications of Calculus (Adv)

The Derivative Function and its Graph (Y12)
Tangents (Y12)
Curve Sketching (Y12)
Maxima and Minima (Y12)

Teacher: Cathyanne Horvat

Exam Equivalent Time: 139.5 minutes *(based on allocation of 1.5 minutes per mark)*

C3 Applications of Calculus



***SmarterMaths analytics based on the average contribution to new syllabus Advanced Maths exams since 2020.**

HISTORICAL CONTRIBUTION

- *C3 Applications of Calculus* is the biggest topic in the Advanced course, contributing a massive 21.6% to new syllabus exams since introduced in 2020.
- This topic has been split into five sub-topics for analysis purposes: *1-The Derivative Function and its Graph (2.0%), 2-Curve Sketching (4.3%), 3-Tangents (4.3%), 4-Maxima and Minima (4.3%) and 5-Rates of Change (6.7%).*
- This analysis looks at *The Derivative Function and its Graph.*

HSC ANALYSIS - What to expect and common pitfalls

- *The Derivative Function and its Graph* is a challenging sub-topic that demands a solid conceptual understanding of the first and second derivative.
- Students are regularly asked to explore the graphical relationship between f(x) and f´(x). This question type has caused problems in the past and deserves attention.
- This sub-topic has been tested in 6 Advanced exams in the last decade (notably absent in 2022), including two questions in 2021 and 2020. With sub-50% mean marks resulting on a majority of occasions, a revision focus is recommended.
- The 2020 exam tested *composite functions* in this context for the first time, producing an 11% mean mark (lowest in the exam). This should be carefully reviewed (see *2020 Adv 10 MC*).

Questions

1. Calculus, 2ADV C3 2013 HSC 12a

The cubic $y = ax^3 + bx^2 + cx + d$ has a point of inflection at $x = p$.

Show that $p = -\dfrac{b}{3a}$. ***(2 marks)***

2. Calculus, 2ADV C3 2005 HSC 2d

Find the equation of the tangent to $y = \log_e x$ at the point $(e, 1)$. ***(2 marks)***

3. Calculus, 2ADV C3 2007 HSC 6b

Let $f(x) = x^4 - 4x^3$.

i. Find the coordinates of the points where the curve crosses the axes. ***(2 marks)***

ii. Find the coordinates of the stationary points and determine their nature. ***(4 marks)***

iii. Find the coordinates of the points of inflection. ***(1 mark)***

iv. Sketch the graph of $y = f(x)$, indicating clearly the intercepts, stationary points and points of inflection. ***(3 marks)***

4. Calculus, 2ADV C3 2020 HSC 16

Sketch the graph of the curve $y = -x^3 + 3x^2 - 1$, labelling the stationary points and point of inflection. Do NOT determine the x-intercepts of the curve. ***(4 marks)***

5. Calculus, 2ADV C3 2021 HSC 16

For what values of x is $f(x) = x^2 - 2x^3$ increasing? ***(3 marks)***

6. Calculus, 2ADV C3 2022 HSC 22

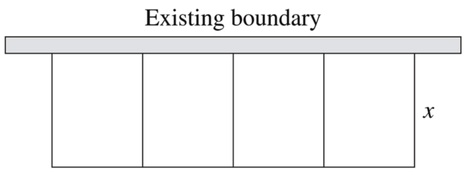
Find the global maximum and minimum values of $y = x^3 - 6x^2 + 8$, where $-1 \leq x \leq 7$. *(4 marks)*

7. Calculus, 2ADV C3 2014 HSC 14a

Find the coordinates of the stationary point on the graph $y = e^x - ex$, and determine its nature. *(3 marks)*

8. Calculus, 2ADV C3 2016 HSC 14c

A farmer wishes to make a rectangular enclosure of area 720 m^2. She uses an existing straight boundary as one side of the enclosure. She uses wire fencing for the remaining three sides and also to divide the enclosure into four equal rectangular areas of width x m as shown.



i. Show that the total length, l m, of the wire fencing is given by

$l = 5x + \dfrac{720}{x}$. *(1 mark)*

ii. Find the minimum length of wire fencing required, showing why this is the minimum length. *(3 marks)*

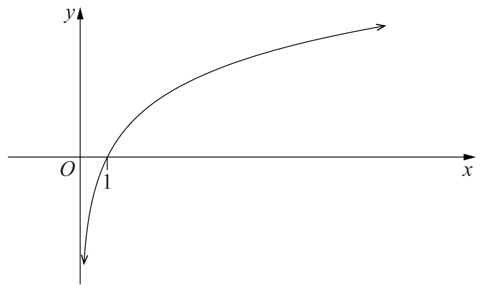
9. Calculus, 2ADV C4 2008 HSC 5a

The gradient of a curve is given by $\dfrac{dy}{dx} = 1 - 6\sin 3x$. The curve passes through the point $(0, 7)$.

What is the equation of the curve? *(3 marks)*

10. Calculus, 2ADV C3 2020 HSC 29

The diagram shows the graph of $y = c\ln x,\ c > 0$.



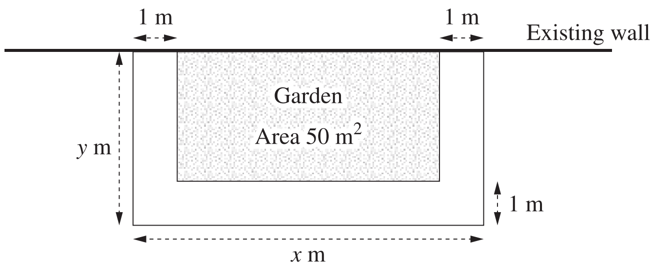
a. Show that the equation of the tangent to $y = c\ln x$ at $x = p$, where $p > 0$, is

$y = \dfrac{c}{p}x - c + c\ln p$. *(2 marks)*

b. Find the value of c such that the tangent from part (a) has a gradient of 1 and passes through the origin.
(2 marks)

11. Calculus, 2ADV C3 2023 HSC 24

A gardener wants to build a rectangular garden of area 50 m² against an existing wall as shown in the diagram. A concrete path of width 1 metre is to be built around the other three sides of the garden.



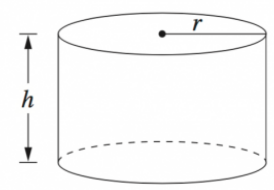
Let x and y be the dimensions, in metres, of the outer rectangle as shown.

a. Show that $y = \dfrac{50}{x-2} + 1$. *(1 mark)*

b. Find the value of x such that the area of the concrete path is a minimum. Show that your answer gives a minimum area. *(4 marks)*

12. Calculus, 2ADV C3 2010 HSC 5a

A rainwater tank is to be designed in the shape of a cylinder with radius r metres and height h metres.



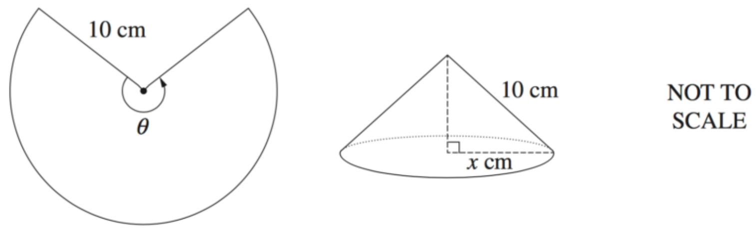
The volume of the tank is to be 10 cubic metres. Let A be the surface area of the tank, including its top and base, in square metres.

i. Given that $A = 2\pi r^2 + 2\pi rh$, show that $A = 2\pi r^2 + \dfrac{20}{r}$. *(2 marks)*

ii. Show that A has a minimum value and find the value of r for which the minimum occurs. *(3 marks)*

13. Calculus, 2ADV C3 2018 HSC 16a

A sector with radius 10 cm and angle θ is used to form the curved surface of a cone with base radius x cm, as shown in the diagram.



The volume of a cone of radius r and height h is given by $V = \dfrac{1}{3}\pi r^2 h$.

i. Show that the volume, V cm³, of the cone described above is given by

$$V = \frac{1}{3}\pi x^2 \sqrt{100 - x^2}.$$ *(1 mark)*

ii. Show that $\dfrac{dV}{dx} = \dfrac{\pi x\left(200 - 3x^2\right)}{3\sqrt{100 - x^2}}$. *(2 marks)*

iii. Find the exact value of θ for which V is a maximum. *(3 marks)*

14. Calculus, 2ADV C3 2010 HSC 8d

Let $f(x) = x^3 - 3x^2 + kx + 8$, where k is a constant.

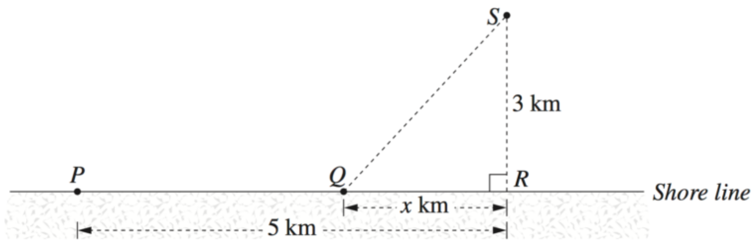
Find the values of k for which $f(x)$ is an increasing function. *(2 marks)*

15. Calculus, 2ADV C3 2009 HSC 9b

An oil rig, S, is 3 km offshore. A power station, P, is on the shore. A cable is to be laid from P to S. It costs \$1000 per kilometre to lay the cable along the shore and \$2600 per kilometre to lay the cable underwater from the shore to S.

The point R is the point on the shore closest to S, and the distance PR is 5 km.

The point Q is on the shore, at a distance of x km from R, as shown in the diagram.



i. Find the total cost of laying the cable in a straight line from P to R and then in a straight line from R to S. *(1 mark)*

ii. Find the cost of laying the cable in a straight line from P to S. *(1 mark)*

iii. Let $\$C$ be the total cost of laying the cable in a straight line from P to Q, and then in a straight line from Q to S.

Show that $C = 1000\left(5 - x + 2.6\sqrt{x^2+9}\right)$. *(2 marks)*

iv. Find the minimum cost of laying the cable. ***(4 marks)***

v. New technology means that the cost of laying the cable underwater can be reduced to $1100 per kilometre.

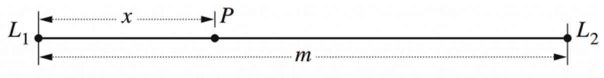
Determine the path for laying the cable in order to minimise the cost in this case. ***(2 marks)***

16. Calculus, 2ADV C3 2007 HSC 10b

The noise level, N, at a distance d metres from a single sound source of loudness L is given by the formula

$$N = \frac{L}{d^2}.$$

Two sound sources, of loudness L_1 and L_2 are placed m metres apart.



The point P lies on the line between the sound sources and is x metres from the sound source with loudness L_1.

i. Write down a formula for the sum of the noise levels at P in terms of x. ***(1 mark)***

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ii. There is a point on the line between the sound sources where the sum of the noise levels is a minimum. Find an expression for x in terms of m, L_1 and L_2 if P is chosen to be this point. ***(4 marks)***

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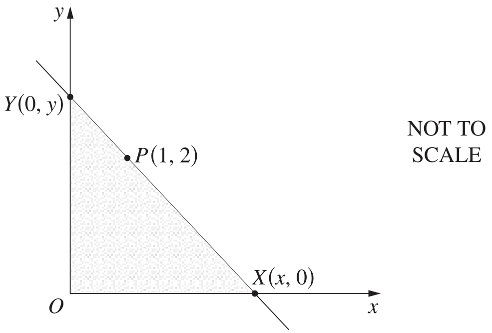
17. Calculus, 2ADV C3 2022 HSC 31

A line passes through the point $P(1, 2)$ and meets the axes at $X(x, 0)$ and $Y(0, y)$, where $x > 1$.

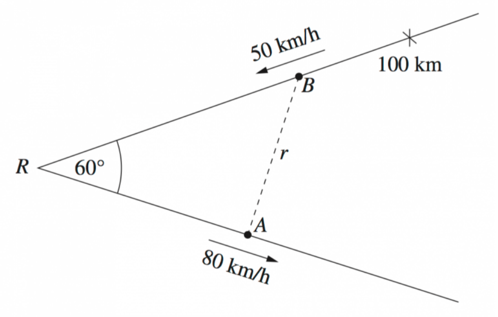


a. Show that $y = \dfrac{2x}{x-1}$. *(2 marks)*

b. Find the minimum value of the area of triangle XOY. *(4 marks)*

18. Calculus, 2ADV C3 2013 HSC 14b

Two straight roads meet at R at an angle of 60°. At time $t = 0$ car A leaves R on one road, and car B is 100km from R on the other road. Car A travels away from R at a speed of 80 km/h, and car B travels towards R at a speed of 50 km/h.

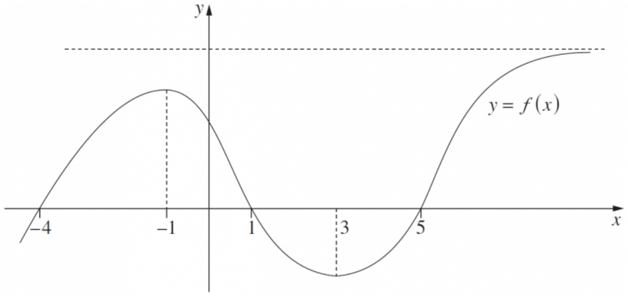


The distance between the cars at time t hours is r km.

i. Show that $r^2 = 12\ 900t^2 - 18\ 000t + 10\ 000$. *(2 marks)*

ii. Find the minimum distance between the cars. *(3 marks)*

19. Calculus, 2ADV C3 2009 HSC 8a



The diagram shows the graph of a function $y = f(x)$.

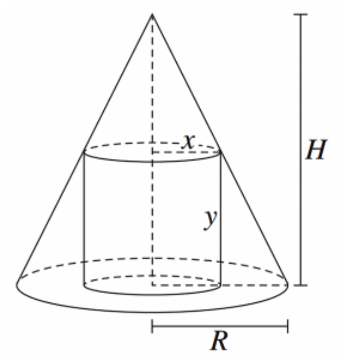
i. For which values of x is the derivative, $f'(x)$, negative? ***(1 mark)***

ii. What happens to $f'(x)$ for large values of x? ***(1 mark)***

iii. Sketch the graph $y = f'(x)$. ***(2 marks)***

20. Calculus, 2ADV C3 2015 HSC 16c

The diagram shows a cylinder of radius x and height y inscribed in a cone of radius R and height H, where R and H are constants.



The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

i. Show that the volume, V, of the cylinder can be written as

$$V = \frac{H}{R}\pi x^2(R - x).$$ *(3 marks)*

ii. By considering the inscribed cylinder of maximum volume, show that the volume of any inscribed cylinder does not exceed $\frac{4}{9}$ of the volume of the cone. *(4 marks)*

Worked Solutions

1. Calculus, 2ADV C3 2013 HSC 12a

Show $p = -\dfrac{b}{3a}$

$$\begin{aligned} y &= ax^3 + bx^2 + cx + d \\ y\prime &= 3ax^2 + 2bx + c \\ y\prime\prime &= 6ax + 2b \end{aligned}$$

Given P.I. occurs when $x = p$

$\Rightarrow y\prime\prime = 0$ when $x = p$

$$\begin{aligned} \therefore\ 6ap + 2b &= 0 \\ 6ap &= -2b \\ p &= -\frac{2b}{6a} \\ &= -\frac{b}{3a} \quad \ldots \text{as required} \end{aligned}$$

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2. Calculus, 2ADV C3 2005 HSC 2d

$y = \log_e x$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

At $(e, 1)$,

$m = \dfrac{1}{e}$

Equation of tangent, $m = \dfrac{1}{e}$, through $(e, 1)$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{e}(x - e) \\ y - 1 &= \frac{x}{e} - 1 \\ y &= \frac{x}{e} \end{aligned}$$

Worked Solutions

3. Calculus, 2ADV C3 2007 HSC 6b

i. $f(x) = x^4 - 4x^3$

Cuts x-axis when $f(x) = 0$

$$\begin{aligned} x^4 - 4x &= 0 \\ x^3(x - 4) &= 0 \\ x &= 0 \text{ or } 4 \end{aligned}$$

$\therefore$ Cuts the x-axis at $(0, 0)$, $(4, 0)$

Cuts the y-axis when $x = 0$

$\therefore$ Cuts the y-axis at $(0, 0)$

ii. $f(x) = x^4 - 4x^3$

$f\prime(x) = 4x^3 - 12x^2$

$f\prime\prime(x) = 12x^2 - 24x$

S.P.'s when $f\prime(x) = 0$

$$\begin{aligned} 4x^3 - 12x^2 &= 0 \\ 4x^2(x - 3) &= 0 \\ x &= 0 \text{ or } 3 \end{aligned}$$

When $x = 0$

$f(0) = 0$

$f\prime\prime(0) = 0$

x	-0.5	0	0.5
$f''(x)$	>0	0	<0

Since concavity changes, a P.I. occurs at $(0, 0)$

When $x = 3$

$$\begin{aligned} f(3) &= 3^4 - 4 \times 3^3 \\ &= -27 \\ f\prime\prime(3) &= 12 \times 3^2 - 24 \times 3 \\ &= 36 > 0 \end{aligned}$$

$\therefore$ Minimum S.P. at $(3, -27)$

iii. P.I. when $f\prime\prime(x) = 0$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0$ or 2

P.I. at $(0, 0)$ (from(ii))

When $x = 2$

x	1.5	2	2.5
$f''(x)$	< 0	0	> 0

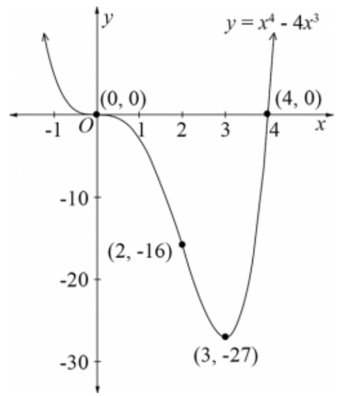
Since concavity changes, a P.I. occurs when $x = 2$

$f(2) = 2^4 - 4 \times 2^3$

$= 16$

$\therefore$ P.I.'s at $(0, 0)$ and $(2, 16)$

iv.



4. Calculus, 2ADV C3 2020 HSC 16

$y = -x^3 + 3x^2 - 1$

$\dfrac{dy}{dx} = -3x^2 + 6x$

$\dfrac{d^2y}{dx^2} = -6x + 6$

SP's when $\dfrac{dy}{dx} = 0$

$-3x^2 + 6x = 0$

$-3x(x-2) = 0$

$x = 0$ or 2

When $x = 0$,

$y = -1$

$\dfrac{d^2y}{dx^2} = 6 > 0$

$\therefore$ MIN at $(0, -1)$

When $x = 2$,

$y = -8 + 12 - 1 = 3$

$\dfrac{d^2y}{dx^2} = -6 \times 2 + 6 = -6 < 0$

$\therefore$ MAX at $(2, 3)$

$\dfrac{d^2y}{dx^2} = 0$ when

$-6x + 6 = 0$

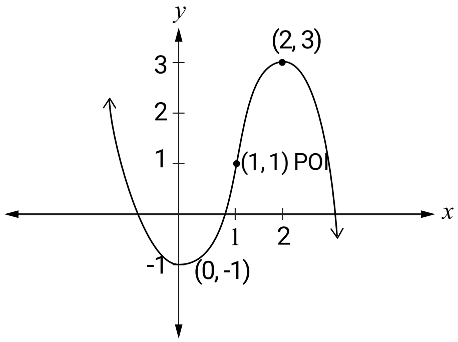
$x = 1$

Checking change of concavity

x	$\frac{1}{2}$	1	$\frac{3}{2}$
$\dfrac{d^2y}{dx^2}$	3	0	−3

Concavity changes either side of $x = 1$

$\therefore$ POI at $(1, 1)$



5. Calculus, 2ADV C3 2021 HSC 16

$$f(x) = x^2 - 2x^3$$
$$f\prime(x) = 2x - 6x^2$$

$f(x)$ is increasing when $f\prime(x) > 0$

$$2x - 6x^2 > 0$$
$$2x(1-3x) > 0$$
$$x \in \left(0, \frac{1}{3}\right)$$

6. Calculus, 2ADV C3 2022 HSC 22

$$y = x^3 - 6x^2 + 8$$
$$\frac{dy}{dx} = 3x^2 - 12x$$
$$\frac{d^2y}{dx^2} = 6x - 12$$

SP's when $\frac{dy}{dx} = 0$:

$$3x^2 - 12x = 0$$
$$3x(x-4) = 0$$

$x = 0$ or 4

When $x = 0,\ \ y = 8,\ \ \frac{d^2y}{dx^2} < 0$

$\Rightarrow$ Local Max at $(0, 8)$

When $x = 4,\ \ y = 4^3 - 6(4^2) + 8 = -24,\ \ \frac{d^2y}{dx^2} > 0$

$\Rightarrow$ Local Min at $(4, -24)$

Check ends of domain:
When $x = -1,\ \ y = -1 - 6 + 8 = 1$
When $x = 7,\ \ y = 7^3 - 6(7^2) + 8 = 57$
$\therefore$ Global max $= 57$
$\therefore$ Global min $= -24$

7. Calculus, 2ADV C3 2014 HSC 14a

$$y = e^x - ex$$
$$\frac{dy}{dx} = e^x - e$$
$$\frac{d^2y}{dx^2} = e^x$$

S.P. when $\dfrac{dy}{dx} = 0$

$$e^x - e = 0$$
$$e^x = e^1$$
$$x = 1$$

At $x = 1$

$$y = e^1 - e = 0$$
$$\frac{d^2y}{dx^2} = e > 0 \quad \Rightarrow \text{MIN}$$

$\therefore$ MINIMUM S.P. at $(1, 0)$

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8. Calculus, 2ADV C3 2016 HSC 14c

i. Area $= xy$

$$720 = xy$$
$$y = \frac{720}{x}$$

$$l = 5x + y$$
$$= 5x + \frac{720}{x} \quad \ldots \text{as required}$$

ii. $\dfrac{dl}{dx} = 5 - \dfrac{720}{x^2}$

$$\frac{d^2l}{dx^2} = \frac{1440}{x^3}$$

Max/Min when $\dfrac{dl}{dx} = 0$,

$$5 = \frac{720}{x^2}$$
$$x^2 = 144$$
$$x = 12, \quad x > 0$$

When $x = 12$, $\dfrac{d^2l}{dx^2} > 0$

$\therefore$ Minimum occurs when $x = 12$

$\therefore$ Minimum fencing

$$= 5 \times 12 + \frac{720}{12}$$
$$= 120 \text{ m}$$

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9. Calculus, 2ADV C4 2008 HSC 5a

$$\frac{dy}{dx} = 1–6\sin 3x$$

$$y = \int 1–6\sin 3x \ dx$$

$$= x + 2\cos 3x + c$$

Passes through $(0, 7)$

$$\Rightarrow 0 + 2\cos 0 + c = 7$$
$$2 + c = 7$$
$$c = 5$$

$\therefore$ Equation is $\quad y = x + 2\cos 3x + 5$

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10. Calculus, 2ADV C3 2020 HSC 29

a. $y = c\ln x$

$$\frac{dy}{dx} = \frac{c}{x}$$

At $x = p$,

$$m_{\text{tang}} = \frac{c}{p}$$

Tangent passes through $(p, c\ln p)$

$\therefore$ Equation of tangent

$$y–c\ln p = \frac{c}{p}(x–p)$$

$$y = \frac{c}{p}x–c + c\ln p$$

b. If $m_{\text{tang}} = 1$,

♦ Mean mark part (b) 40%.

$$\frac{c}{p} = 1$$
$$c = p$$

If tangent passes through $(0, 0)$

$$0 = -c + c\ln c$$
$$0 = c(\ln c–1)$$

$$\ln c = 1 \quad (c > 0)$$
$$\therefore c = e$$

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11. Calculus, 2ADV C3 2023 HSC 24

a. $(y-1)(x-2) = 50$

$$yx - 2y - x + 2 = 50$$
$$y(x-2) = 48 + x$$
$$y = \frac{48+x}{x-2}$$
$$y = \frac{50 + (x-2)}{x-2}$$
$$y = \frac{50}{x-2} + 1$$

♦♦ Mean mark (b) 33%.

b. Let $A =$ area of concrete path

$$A = xy - 50$$
$$= x\left(\frac{50}{x-2} + 1\right) - 50$$
$$= \frac{50x}{x-2} + x - 50$$

$$\frac{dA}{dx} = \frac{50(x-2) - 50x}{(x-2)^2} + 1$$
$$= \frac{-100}{(x-2)^2} + 1$$

Max/min when $\dfrac{dA}{dx} = 0$

$$\frac{-100}{(x-2)^2} + 1 = 0$$
$$(x-2)^2 = 100$$
$$x - 2 = \pm 10$$

$$x = 12 \ (x > 0)$$

Check gradients about $x = 12$:

x	10	12	14
$\dfrac{dA}{dx}$	$-\frac{9}{16}$	0	$\frac{11}{36}$
	\	–	/

$\therefore$ Minimum at $x = 12$.

12. Calculus, 2ADV C3 2010 HSC 5a

i. Show $A = 2\pi r^2 + \dfrac{20}{r}$

Since $V = \pi r^2 h = 10 \quad \Rightarrow h = \dfrac{10}{\pi r^2}$

Substituting into A

$$A = \pi r^2 + 2\pi r\left(\frac{10}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{20}{r} \quad \ldots \text{ as required}$$

MARKER'S COMMENT: Students MUST know the volume formula for a cylinder. Those that did and stated $10 = \pi r^2 h$ most often completed this question efficiently.

ii.

$$A = 2\pi r^2 + \frac{20}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{20}{r^2}$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{40}{r^3} > 0 \quad (r > 0)$$

$$\Rightarrow \text{MIN occurs when } \frac{dA}{dr} = 0$$

$$4\pi r - \frac{20}{r^2} = 0$$

$$4\pi r^3 - 20 = 0$$

$$4\pi r^3 = 20$$

$$r = \sqrt[3]{\frac{5}{\pi}}$$

$$= 1.16754\ldots$$

$$= 1.17 \text{ metres } (2 \text{ d.p.})$$

♦ *Mean mark 44%*
MARKER'S COMMENT: The "table method" or 1st derivative test for proving a minimum (i.e. showing how $\frac{dA}{dr}$ changes sign) was also quite successful.

13. Calculus, 2ADV C3 2018 HSC 16a

i. $V = \dfrac{1}{3}\pi r^2 h$

Using Pythagoras,

$h = \sqrt{100 - x^2}$

$r = x$

$$\therefore \text{ Volume} = \frac{1}{3}\pi x^2\sqrt{100-x^2}$$

ii.

$$V = \frac{1}{3}\pi x^2\sqrt{100-x^2}$$

$$\frac{dV}{dh} = \frac{1}{3}\pi\left[2x \cdot \sqrt{100-x^2} - 2x \cdot \frac{1}{2}\left(100-x^2\right)^{-\frac{1}{2}} \cdot x^2\right]$$

$$= \frac{1}{3}\pi\left[\frac{2x\left(100-x^2\right)-x^3}{\sqrt{100-x^2}}\right]$$

$$= \frac{1}{3}\pi\left[\frac{200x-2x^3-x^3}{\sqrt{100-x^2}}\right]$$

$$= \frac{\pi x\left(200-3x^2\right)}{3\sqrt{100-x^2}} \quad \text{.. as required}$$

♦ *Mean mark 45%.*

iii. Find x when $\dfrac{dV}{dx} = 0$

$$200 - 3x^2 = 0$$

$$x = \sqrt{\frac{200}{3}}$$

When $x < \sqrt{\dfrac{200}{3}}$, $\left(200-3x^2\right) > 0$

$$\Rightarrow \frac{dV}{dx} > 0$$

When $x > \sqrt{\dfrac{200}{3}}$, $\left(200-3x^2\right) < 0$

$$\Rightarrow \frac{dV}{dx} < 0$$

$$\therefore \text{ MAX when } x = \sqrt{\frac{200}{3}}$$

Equating the arc length of the section to the circumference of the cone:

$$2\pi r \cdot \frac{\theta}{2\pi} = 2\pi \cdot x$$

$$10\theta = 2\pi\sqrt{\frac{200}{3}}$$

♦♦ *Mean mark 23%.*

$$\therefore \theta = \frac{2\pi \cdot 10\sqrt{2}}{10 \cdot \sqrt{3}}$$

$$= \frac{2\sqrt{2}\pi}{\sqrt{3}}$$

14. Calculus, 2ADV C3 2010 HSC 8d

$$f(x) = x^3 - 3x^2 + kx + 8$$
$$f\prime(x) = 3x^2 - 6x + k$$

$f(x)$ is increasing when $f\prime(x) > 0$

$\Rightarrow 3x^2 - 6x + k > 0$

♦♦ *Mean mark 28%.*
MARKER'S COMMENT: The arithmetic required to solve $36 - 12k < 0$ proved the undoing of *too many* students in this question. *TAKE CARE!*

$f\prime(x)$ is always positive

$\Rightarrow f\prime(x)$ is a positive definite.

i.e. when $a > 0$ and $\Delta < 0$

$$a = 3 > 0$$
$$\Delta = b^2 - 4ac$$

$$\therefore (-6)^2 - (4 \times 3 \times k) < 0$$
$$36 - 12k < 0$$
$$12k > 36$$
$$k > 3$$

$\therefore f(x)$ is increasing when $k > 3$.

15. Calculus, 2ADV C3 2009 HSC 9b

♦♦ *Although specific data is unavailable for question parts, mean marks were 35% for Q9 in total.*

i. $\text{Cost} = (PR \times 1000) + (SR \times 2600)$

$$= (5 \times 1000) + (3 \times 2600)$$
$$= 12\ 800$$

$\therefore$ Cost is $\$12\ 800$

ii. $\text{Cost} = PS \times 2600$

Using Pythagoras:

$$PS^2 = PR^2 + SR^2$$
$$= 5^2 + 3^2$$
$$= 34$$
$$PS = \sqrt{34}$$

$$\therefore \text{ Cost} = \sqrt{34} \times 2600$$
$$= 15\ 160.474\ldots$$
$$= \$15\ 160 \text{ (nearest dollar)}$$

iii. Show $C = 1000\left(5 - x + 2.6\sqrt{x^2+9}\right)$

$$\text{Cost} = (PQ \times 1000) + (QS \times 2600)$$
$$PQ = 5 - x$$
$$QS^2 = QR^2 + SR^2$$
$$= x^2 + 3^2$$
$$QS = \sqrt{x^2+9}$$
$$\therefore C = (5-x)1000 + \sqrt{x^2+9}\ (2600)$$
$$= 1000\left(5 - x + 2.6\sqrt{x^2+9}\right) \quad \ldots \text{ as required}$$

iv. Find the MIN cost of laying the cable

$$C = 1000\left(5 - x + 2.6\sqrt{x^2+9}\right)$$
$$\frac{dC}{dx} = 1000\left(-1 + 2.6 \times \frac{1}{2} \times 2x\left(x^2+9\right)^{-\frac{1}{2}}\right)$$
$$= 1000\left(-1 + \frac{2.6x}{\sqrt{x^2+9}}\right)$$

MAX/MIN when $\dfrac{dC}{dx} = 0$

$$1000\left(-1 + \frac{2.6x}{\sqrt{x^2+9}}\right) = 0$$

IMPORTANT: Tougher derivative questions often require students to deal with multiple algebraic

$$\frac{2.6x}{\sqrt{x^2+9}} = 1$$

$$2.6x = \sqrt{x^2+9}$$

$$(2.6)^2x^2 = x^2+9$$

$$x^2\left(2.6^2-1\right) = 9$$

$$x^2 = \frac{9}{5.76}$$

$$= 1.5625$$

$$x = 1.25 \quad (x>0)$$

If $x=1$, $\frac{dC}{dx}<0$

If $x=2$, $\frac{dC}{dx}>0$

$\therefore$ MIN when $x = 1.25$

$$C = 1000\left(5-1.25+2.6\sqrt{1.25^2+9}\right)$$

$$= 1000(122)$$

$$= 12\ 200$$

$\therefore$ MIN cost is $\$12\ 200$ when $x = 1.25$

constants. See Worked Solutions in part (iv).

MARKER'S COMMENT: Check the nature of the critical points in these type of questions. If using the first derivative test, make sure some actual values are substituted in.

v. Underwater cable now costs \$1100 per km

$$\Rightarrow C = 1000(5-x)+1100\sqrt{x^2+9}$$

$$= 1000\left(5-x+1.1\sqrt{x^2+9}\right)$$

$$\frac{dC}{dx} = 1000\left(-1+1.1\times\frac{1}{2}\times 2x\left(x^2+9\right)^{-\frac{1}{2}}\right)$$

$$= 1000\left(-1+\frac{1.1x}{\sqrt{x^2+9}}\right)$$

MAX/MIN when $\frac{dC}{dx} = 0$

$$1000\left(-1+\frac{1.1x}{\sqrt{x^2+9}}\right) = 0$$

$$\frac{1.1x}{\sqrt{x^2+9}} = 1$$

$$1.1x = \sqrt{x^2+9}$$

$$1.1^2x^2 = x^2+9$$

$$x^2\left(1.1^2-1\right) = 9$$

$$x^2 = \frac{9}{0.21}$$

$$x \approx 6.5 \text{ km (to 1 d.p.)}$$

$\Rightarrow$ no solution since $x \le 5$

If we lay cable PR then RS

$\Rightarrow$ Cost $= 5\times 1100+3\times 1000 = 8500$

If we lay cable directly underwater via PS

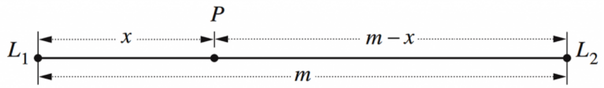
$\Rightarrow$ Cost $= \sqrt{34}\times 1100 = 6414.047\ldots$

$\therefore$ MIN cost is \$6414 by cabling directly from P to S.

MARKER'S COMMENT: Many students failed to interpret a correct calculation of $x>5$ as providing no solution.

16. Calculus, 2ADV C3 2007 HSC 10b

i.



$$N=\frac{L}{d^2}$$

$$\text{Noise from } L_1=\frac{L_1}{x^2}$$

$$\text{Noise from } L_2=\frac{L_2}{(m-x)^2}$$

$$\therefore N=\frac{L_1}{x^2}+\frac{L_2}{(m-x)^2}$$

ii. $N=L_1\,x^{-2}+L_2(m-x)^{-2}$

$$\frac{dN}{dx}=-\,2L_1x^{-3}+-\,2L_2(m-x)^{-3}\times\frac{d}{dx}(m-x)$$

$$=\frac{-\,2L_1}{x^3}+\frac{2L_2}{(m-x)^3}$$

$$\text{Max or min when } \frac{dN}{dx}=0$$

$$\frac{2L_1}{x^3}=\frac{2L_2}{(m-x)^3}$$

$$2L_1(m-x)^3=2L_2\,x^3$$

$$L_1(m-x)^3=L_2\,x^3$$

$$\sqrt[3]{L_1}(m-x)=\sqrt[3]{L_2}\,x$$

$$\sqrt[3]{L_1}\,m-\sqrt[3]{L_1}\,x=\sqrt[3]{L_2}\,x$$

$$\sqrt[3]{L_2}\,x+\sqrt[3]{L_1}\,x=\sqrt[3]{L_1}\,m$$

$$x\left(\sqrt[3]{L_2}+\sqrt[3]{L_1}\right)=\sqrt[3]{L_1}\,m$$

$$x=\frac{\sqrt[3]{L_1}\,m}{\left(\sqrt[3]{L_2}+\sqrt[3]{L_1}\right)}$$

$$\frac{dN}{dx}=-\,2L_1\,x^{-3}+2L_2(m-x)^{-3}$$

$$\frac{d^2N}{dx^2}=6L_1\,x^{-4}-6L_2(m-x)^{-4}\times-1$$

$$=\frac{6L_1}{x^4}+\frac{6L_2}{(m-x)^4}>0$$

$\therefore$ A minimum occurs when

$$x=\frac{\sqrt[3]{L_1}\,m}{\left(\sqrt[3]{L_2}+\sqrt[3]{L_1}\right)}$$

17. Calculus, 2ADV C3 2022 HSC 31

a. Show $y = \dfrac{2x}{x-1}$

Since $m_{YP} = m_{PX}$:

$$\frac{y-2}{0-1} = \frac{2-0}{1-x}$$

$$y - 2 = \frac{-2}{1-x}$$

$$\begin{aligned} y &= 2 - \frac{2}{1-x} \\ &= \frac{2(1-x)-2}{1-x} \\ &= \frac{-2x}{1-x} \\ &= \frac{2x}{x-1} \quad \text{... as required} \end{aligned}$$

♦♦♦ Mean mark part (a) 17%.

COMMENT: $y = \dfrac{2x}{x-1}$ is the expression of a relationship between the intercepts and not the equation of the line.

b.
$$\begin{aligned} A &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2}x\left(\frac{2x}{x-1}\right) \\ &= \frac{x^2}{x-1} \end{aligned}$$

♦♦ Mean mark part (b) 29%.

$$\begin{aligned} \frac{dA}{dx} &= \frac{(x-1)\cdot 2x - x^2(1)}{(x-1)^2} \\ &= \frac{2x^2 - 2x - x^2}{(x-1)^2} \\ &= \frac{x(x-2)}{(x-1)^2} \end{aligned}$$

SP's occur when $\dfrac{dA}{dx} = 0$:

$x = 0 \;$ or $\; 2$

Use 1st derivative test to find max/min:

x	−1	0	1.5	2	3
$\frac{dA}{dx}$	$\frac{3}{4}$	0	−3	0	$\frac{3}{4}$
	+	0	–	0	+

$\Rightarrow$ MIN at $x = 2$

$$\begin{aligned} \therefore A_{\text{min}} &= \frac{1}{2} \times 2 \times \frac{2 \times 2}{2-1} \\ &= 4 \text{ u}^2 \end{aligned}$$

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18. Calculus, 2ADV C3 2013 HSC 14b

i. Need to show $r^2 = 12\ 900t^2 - 18\ 000t + 10\ 000$

$RB = 100 - 50t$

$RA = 80t$

Using the cosine rule

$$\begin{aligned} r^2 &= (RB)^2 + (RA)^2 - 2(RB)(RA)\cos\angle R \\ &= (100-50t)^2 + (80t)^2 - 2(100-50t)(80t)\cos 60 \\ &= 10\ 000 - 10\ 000t + 2500t^2 + 6400t^2 - 8000t + 4000t^2 \\ &= 12\ 900t^2 - 18\ 000t + 10\ 000 \quad \text{... as required} \end{aligned}$$

♦♦ *Mean mark 26%*

ii. Max/min when $\dfrac{dr^2}{dt} = 0$

$$\frac{dr^2}{dt} = 25\ 800t - 18\ 000 = 0$$

$$t = \frac{18\ 000}{25\ 800} = \frac{30}{43}$$

When $t = \dfrac{30}{43}$, $\dfrac{d^2(r^2)}{dt^2} = 25\ 800 > 0 \;\Rightarrow$ MIN

$\therefore$ Minimum distance when $t = \dfrac{30}{43}$ hr

♦♦ *Mean mark 27%*

ALGEBRA TIP: Finding the derivative of r^2 (rather than making r the subject), makes calculations ***much easier***. *ENSURE* you apply the test to confirm a minimum.

Find r when $t = \dfrac{30}{43}$

$$\begin{aligned} r^2 &= 12\ 900\left(\frac{30}{43}\right)^2 - 18\ 000\left(\frac{30}{43}\right) + 10\ 000 \\ &= 3720.9302... \\ \therefore r &= \sqrt{3702.9302...} \\ &\approx 60.99942... \\ &\approx 61 \text{ km (nearest km)} \end{aligned}$$

$\therefore$ MIN distance between the cars is 61 km.

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19. Calculus, 2ADV C3 2009 HSC 8a

i. $f'(x) < 0$ when

$-1 < x < 3$

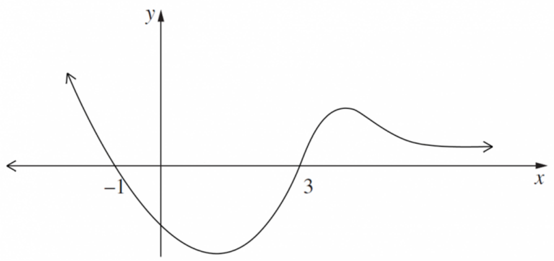
♦♦ *Exact data not available.*

ii. As $x \to \infty$,

$f'(x) \to 0$

iii.

♦♦ *Exact data not available.*
MARKER'S COMMENT: Poorly drawn graphs with axes not labelled and inaccurate scales were common.



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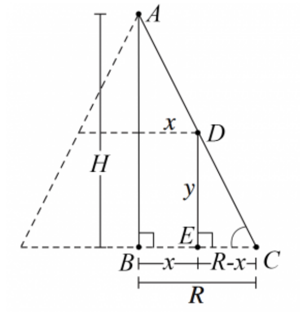
20. Calculus, 2ADV C3 2015 HSC 16c

i. Show $V = \dfrac{H}{R}\pi x^2(R-x)$

♦♦ *Mean mark 16%.*



Consider ΔABC and ΔDEC

$\angle ABC = \angle DEC = 90^\circ$

$\angle BCA$ is common

$\therefore \Delta ABC \ ||| \ \Delta DEC$ (equiangular)

$\therefore \dfrac{DE}{EC} = \dfrac{AB}{BC}$ (corresponding sides of similar triangles)

$$\frac{y}{R-x} = \frac{H}{R}$$

$$y = \frac{H}{R}(R-x)$$

Volume of cylinder

$$= \pi r^2 h$$

$$= \pi x^2 \times \frac{H}{R}(R-x)$$

$$= \frac{H}{R}\pi x^2(R-x) \ \ldots \text{ as required.}$$

ii. $V = \dfrac{H}{R}\pi x^2(R-x)$

♦♦ *Mean mark 21%.*

$$\frac{dV}{dx} = \frac{H}{R}\pi\left[\left(x^2 \times -1\right) + 2x(R-x)\right]$$

$$= \frac{H}{R}\pi\left[-x^2 + 2xR - 2x^2\right]$$

$$= \frac{H}{R}\pi\left[2xR - 3x^2\right]$$

$$\frac{dV^2}{dx^2} = \frac{H}{R}\pi[2R - 6x]$$

Max or min when $\dfrac{dV}{dx} = 0$

$$= 0$$

$$\frac{H}{R}\pi\left[2xR – 3x^2\right]$$

$$2xR – 3x^2 = 0$$

$$x(2R – 3x) = 0$$

$$x = 0 \quad \text{or} \quad 3x = 2R$$

$$x = \frac{2R}{3}$$

When $x = 0$

$$\frac{d^2V}{dx^2} = \frac{H}{R}\pi[2R – 0] > 0$$

$\therefore$ Minimum

When $x = \dfrac{2R}{3}$

$$\frac{d^2V}{dx^2} = \frac{H}{R}\pi\left[2R – 6 \times \frac{2R}{3}\right]$$

$$= \frac{H}{R}\pi[-2R] < 0$$

$\therefore$ Maximum

Maximum Volume of cylinder

$$= \frac{H}{R}\pi\left(\frac{2R}{3}\right)^2\left(R – \frac{2R}{3}\right)$$

$$= \frac{H}{R}\pi \times \frac{4R^2}{9} \times \frac{R}{3}$$

$$= \frac{4\pi HR^2}{27}$$

Volume of cone $= \dfrac{1}{3}\pi R^2 H$

$$\therefore \frac{\text{Max Volume of Cylinder}}{\text{Volume of cone}}$$

$$= \frac{\frac{4\pi HR^2}{27}}{\frac{1}{3}\ \pi R^2 H}$$

$$= \frac{4\pi HR^2}{27} \times \frac{3}{\pi R^2 H}$$

$$= \frac{4}{9}$$

$\therefore$ The volume of the inscribed cylinder does not exceed $\dfrac{4}{9}$ of the cone volume.