2022 **BORED OF STUDIES TRIAL EXAMINATION**

4th October

Mathematics Extension 1

General instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the domain of $\sin^{-1}\left(\dfrac{3x-1}{x-1}\right)$?

(A) $x \leq \dfrac{1}{2}$ or $x > 1$ (B) $0 \leq x \leq \dfrac{1}{2}$

(C) $0 \leq x < 1$ (D) $x \leq 0$ or $x > 1$

2 Which of the following polynomials has a multiple real root?

(A) $x^2 - 6x - 3$ (B) $2x^3 - 6x^2 + 15x + 7$

(C) $3x^4 + x^2 + 1$ (D) $3x^5 - 10x^3 + 15x - 8$

3 At time t, a particle moves on the x-y plane according to the displacement vector

$$\underset{\sim}{r} = (4\sin 2t)\underset{\sim}{i} + (\cos 2t)\underset{\sim}{j}.$$

What is the maximum magnitude of the particle's acceleration vector?

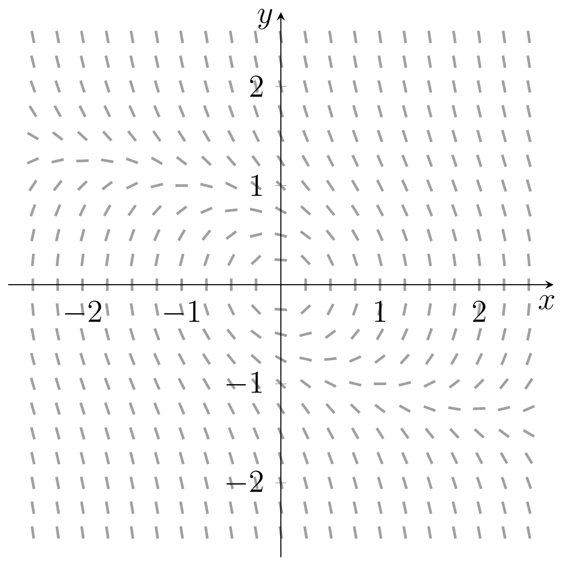
(A) 4 (B) $\sqrt{17}$ (C) 16 (D) 17

4 Which of the following integrals is equivalent to the area between the curves $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for the domain $x \in [-1, 0]$?

(A) $\displaystyle\int_{-1}^{0} \sin^{-1} x\, dx - \int_{-1}^{0} \cos^{-1} x\, dx$ (B) $\displaystyle\int_{-1}^{0} \sin^{-1} x\, dx + \int_{-1}^{0} \cos^{-1} x\, dx$

(C) $\displaystyle\int_{-1}^{0} \cos^{-1} x\, dx - \int_{0}^{1} \sin^{-1} x\, dx$ (D) $\displaystyle\int_{-1}^{0} \cos^{-1} x\, dx + \int_{0}^{1} \sin^{-1} x\, dx$

5 Which of the following differential equations best represents the direction field below?



(A) $\dfrac{dy}{dx} = -\dfrac{x}{y} - x^2$ (B) $\dfrac{dy}{dx} = -\dfrac{x}{y} - y^2$

(C) $\dfrac{dy}{dx} = -\dfrac{x}{y} + x^2$ (D) $\dfrac{dy}{dx} = -\dfrac{x}{y} + y^2$

6 What is the value of $\displaystyle\int_0^{\frac{\pi}{6}} \tan^2 x \tan 2x \, dx$?

(A) $\ln\left(\frac{3}{2\sqrt{2}}\right)$ (B) $\ln\left(\frac{2\sqrt{2}}{3}\right)$

(C) $\dfrac{1}{2}$ (D) $\dfrac{3}{4}$

7 Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be non-zero and non-parallel vectors on the x-y plane such that $|\underset{\sim}{a}| < |\underset{\sim}{b}|$.

Suppose that $\underset{\sim}{a}$ is projected onto the direction of $\underset{\sim}{b}$ to give the vector $\underset{\sim}{p}$.

Which of the following inequalities is *not always* true?

(A) $|\underset{\sim}{p}| < |\hat{\underset{\sim}{a}} \cdot \hat{\underset{\sim}{b}}|$ (B) $|\underset{\sim}{p}| < |\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}|$ (C) $|\underset{\sim}{p}| < |\underset{\sim}{a}|$ (D) $|\underset{\sim}{p}| < |\underset{\sim}{b}|$

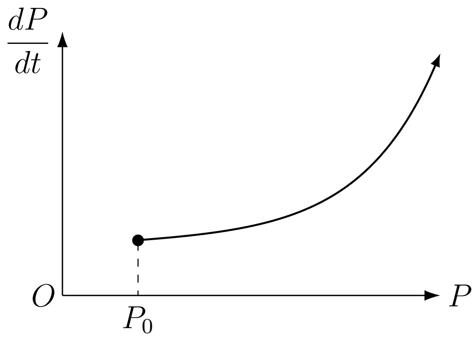
8 The value of 1.01^{10} is to be approximated with the first few terms in the binomial expansion of $(1+0.01)^{10}$. What is the minimum number of terms needed in the binomial expansion so that the approximation is correct to 4 decimal places?

(A) 3 (B) 4

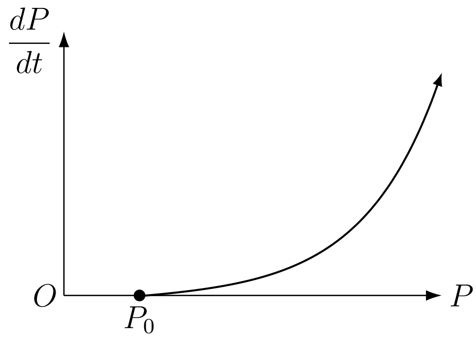
(C) 5 (D) 6

9 A city initially has a population P_0 and then undergoes exponential growth over time. If P is the population of the city at time t, which of the following graphs best shows the behaviour of the city's growth rate as the population increases?

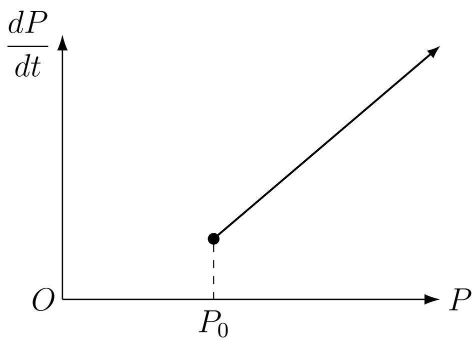
(A)



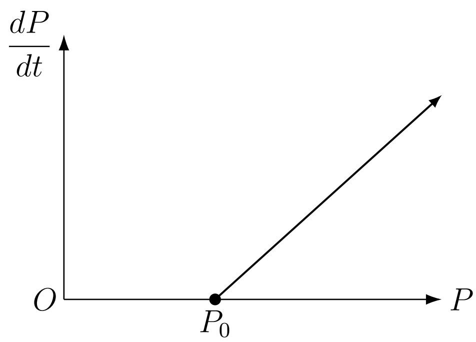
(B)



(C)



(D)



10 Which of the following is equal to $\displaystyle\int_{-1}^{1} \frac{\cos^{-1} x}{x^2+1}\,dx$?

(A) 0 (B) $\dfrac{\pi}{2}$ (C) $\dfrac{\pi^2}{8}$ (D) $\dfrac{\pi^2}{4}$

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Find $\displaystyle\int \frac{\tan^4\theta - 6\tan^2\theta + 1}{\tan^4\theta + 2\tan^2\theta + 1}\,d\theta$. **2**

(b) (i) Find the solutions to $27x^3 - 12x = 69x + 46$. **2**

(ii) Express $\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$ where $0 < \alpha < \dfrac{\pi}{2}$. **1**

(iii) Hence, find the exact value(s) of $\sin\theta + \cos\theta$ given that **3**

$$\sin^3\theta + \cos^3\theta + \frac{23}{27} = 0.$$

(c) Suppose that the position vectors representing the points A, B, C and D on the x-y plane are $-\underset{\sim}{i} - 4\underset{\sim}{j}, 2\underset{\sim}{i} - 5\underset{\sim}{j}, 5\underset{\sim}{i} - 4\underset{\sim}{j}$ and $2\underset{\sim}{i} + 5\underset{\sim}{j}$ respectively.

(i) Show that AC and BD are perpendicular. **1**

(ii) Show that $\cos\angle ADC = \dfrac{4}{5}$. **1**

(iii) Hence, show that $\angle ABC$ and $\angle ADC$ are supplementary. **2**

(d) Show that **3**

$$\frac{1}{\sin 45^\circ \sin 46^\circ} + \frac{1}{\sin 47^\circ \sin 48^\circ} + \frac{1}{\sin 49^\circ \sin 50^\circ} + \cdots + \frac{1}{\sin 133^\circ \sin 134^\circ} = \frac{1}{\sin 1^\circ}.$$

End of Question 11

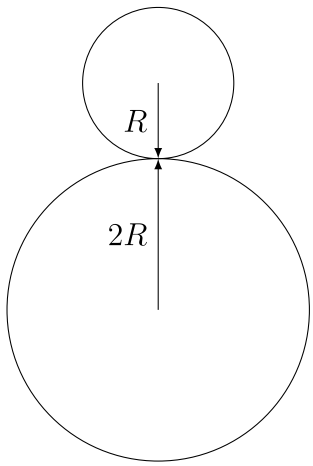
Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) A university wanted to investigate the unemployment rate of its students one year after graduation during different economic conditions. Their historical data shows a distribution of student unemployment rates across different years, which can be approximated with a normal curve. **3**

From a sample of 196 students surveyed in the past year, approximately 4% of students were found to be unemployed one year after graduation. The university claims that based on their historical data, the probability of this level of student unemployment rate or lower is 2.5%.

Find the mean unemployment rate in the university's historical data.

(b) A snowman is made of two spherical snowballs. Initially, the snowball for the head has a radius of R. The head sits on top of a larger snowball for the body, which initially has a radius of $2R$.



As each snowball melts, their volumes each decrease at a rate that is directly proportional to their surface area.

Both snowballs have the same constant of proportionality in the rate of change of their volumes and maintain their spherical shape while melting.

(i) Show that the radius of each snowball melts at a constant rate. **1**

(ii) The snowman melts to half its initial height. Find the ratio of the snowman's volume to its initial volume. **2**

Question 12 continues on page 7

(c) A particle is launched at an acute angle of θ at a speed of u, which lands on the edge of cliff of height H. The particle is also at a distance of H from the base of the cliff.

If the point of launch is defined as the origin then it can be shown that the particle has the following displacement equations at time t.

$$x = ut\cos\theta \qquad\qquad y = ut\sin\theta - \frac{gt^2}{2} \qquad\qquad \textbf{(Do NOT prove these)}$$

where g is the acceleration due to gravity.

(i) Show that $u = \sqrt{\dfrac{gH}{2\cos\theta(\sin\theta - \cos\theta)}}$. **2**

(ii) Hence, deduce that the angle of launch must be such that $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$. **1**

(iii) Explain the physical significance of the minimum and maximum values of θ from the range stated in part (ii). **1**

(iv) Find the value of θ which minimises the initial speed of particle's projection whilst still landing at the edge of the cliff. **2**

(d) An eight digit number is constructed choosing from the numbers 0 to 9 for each digit without replacement. The first digit cannot be 0. **3**

Find the number of ways of making this eight digit number divisible by 9.

You may use, without proof, the fact that if a number is divisible by 9, then the sum of its digits is also divisible by 9.

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Use the substitution $x = \tan\theta$ to show that for some constant c **3**

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln(x + \sqrt{1+x^2}) + c.$$

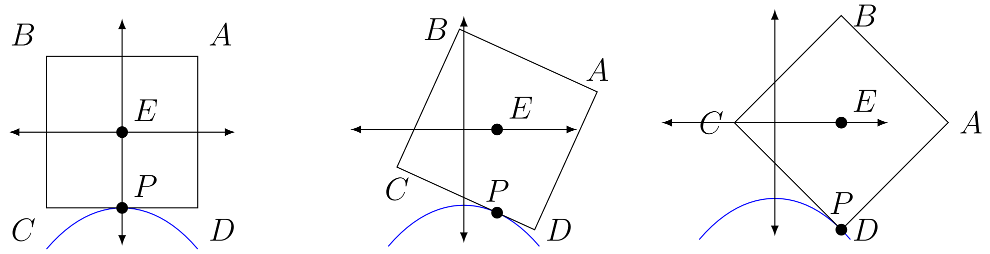
(b) The velocity of a particle moving on the x-y plane at time t is given by the vector **1**

$$\underset{\sim}{v} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j}.$$

Suppose that the horizontal and vertical velocity components of the particle are non-zero and do not change in sign over time. It can be shown that $|\underset{\sim}{v}| = \dfrac{dr}{dt}$, where r is the length of the path traced by the particle **(Do NOT prove this)**.

Using the chain rule on $|\underset{\sim}{v}|$, show that $\left(\dfrac{dr}{dx}\right)^2 = 1 + \left(\dfrac{dy}{dx}\right)^2$.

(c) A square $ABCD$ of side length of 2 units rolls along a curve $y = f(x)$ where its centre E is always vertically above the point of contact P. Suppose that the curve is concave down with maximum turning point at $(0, -1)$. The shape of the curve is such that the square's centre E always remains on the x-axis when it rolls.



Suppose that the point of contact of the square is initially at $(0, -1)$ before rolling to $P(x, y)$ on the curve. Let θ be the angle between EP and the side of the square touching the curve.

(i) Show that $f'(x) = -\cot\theta$. **1**

(ii) Let S be a point on square $ABCD$ that was initially in contact with the curve. By considering the length of PS and the result in (b), show that **2**

$$f''(x) = -\sqrt{1 + [f'(x)]^2}.$$

(iii) Let $w = f'(x)$. Use the result in part (a) to find w in terms of x. **2**

(iv) Hence, find $f(x)$. **1**

Question 13 continues on page 9

(d) Let $\{a_n\}$ be a sequence where $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$ for integers $n \geq 1$.

Let φ be a root of the quadratic polynomial $x^2 - x - 1$.

(i) Use mathematical induction to show that **3**

$$\varphi - a_n = \frac{\varphi}{(\varphi + a_1)(\varphi + a_2)\cdots(\varphi + a_n)}$$

for all integers $n \geq 1$.

(ii) Hence, deduce that $\displaystyle\lim_{n\to\infty} a_n = \frac{1 + \sqrt{5}}{2}$. **2**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) Using a trigonometric substitution to find the roots of $x^3 - 3x + 1 = 0$, show that **3**

$$\cos^2 \frac{2\pi}{9} + \cos^2 \frac{4\pi}{9} + \cos^2 \frac{8\pi}{9} = \frac{3}{2}.$$

You may assume, without proof, that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) (i) By considering the graphs of $y = x^2 + 1$ and $y = e^x$, sketch $y = (x^2+1)e^x$. **2**

You may assume without proof that $\lim_{x \to -\infty} (x^2+1)e^x = 0$.

(ii) Hence, or otherwise, find the solutions to $e^x > \dfrac{1}{1+x^2}$. **1**

(c) Let $f(x) = e^x - k\tan^{-1} x - 1$ for some constant k.

(i) Using the result(s) in part (b), or otherwise, show that $f(x)$ has a minimum turning point when $k > 0$. **2**

(ii) Using the result(s) in part (b), or otherwise, find the range of x-values of the minimum turning point when $k > 1$. **1**

(iii) Hence, find the values of k where $f(x) = 0$ has exactly two distinct solutions. Justify your answer. **3**

(d) Prove that for any set of 5 real numbers, there always exists a pair (x, y) such that **2**

$$-1 < \frac{x-y}{1+xy} < 1,$$

provided that $xy > -1$.

You may assume, without proof, that for $\alpha\beta > -1$,

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right).$$

End of paper