EXERCISE 18.1 INVESTMENTS AND LOANS

2 (a) **B**

The key value from the table is $\$1125.51$. Multiplying this by 3 gives $\$3376.53$.

(b) **D**

The key value from the table is $\$1340.09$. Multiplying this by 4.5 gives $\$6030.405$.

(c) **A**

The key value from the table is $\$1216.65$. Multiplying this by 0.5 gives $\$608.325$.

(d) **C**

The key value from the table is $\$1061.21$. Multiplying this by 0.7 gives $\$742.847$.

4 Use $PV = \dfrac{FV}{(1+r)^n}$ where $FV = 50\ 000$, $r = \dfrac{6\%}{12} = 0.005$, $n = 48$

$$PV = \frac{50\ 000}{(1.005)^{48}}$$
$$\approx 39\ 354.92$$

This amount must be rounded up, since rounding down would cause the amount to be just below what is required.

To the next dollar, $\$39\ 355$ would need to be invested.

6 (a) **A**

Every 4 months for 2 years (3 periods per year) gives 6 periods.

6% p.a. equates to 2% per 4 months.

The key value from the table is 6.3081. Multiplying this by $\$1000$ gives $\$6308.1$, or $\$6308.10$.

(b) **D**

Each 6 months for 2 years gives 4 periods.

4% p.a. equates to 2% per 6 months.

The key value from the table is 4.1216. Multiplying this by $\$5000$ gives $\$20\ 608$.

(c) **C**

Each year for 6 years gives 6 periods.

The key value from the table is 6.8019. Multiplying this by $\$500$ gives $\$3400.95$

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8 (a) B

$0.75\% = 0.0075$

The key value is 54.8929. Multiplying this by 200 gives 10978.58

(b) C

$0.8\% = 0.008$

The key value is 55.6845. Multiplying this by 150 gives 8352.675

(c) A

10.8% p.a. is $\dfrac{10.8\%}{12}$ per month, or $0.9\% = 0.009$.

6 years is 72 months

The key value is 52.8212

$$\frac{20\ 000}{52.8212} = 378.635$$

Rounded to the nearest dollar, this is $\$379$ per month.

(d) D

9% p.a. is $\dfrac{9}{12}\%$ per month, or 0.75% or 0.0075

6 years is 72 months.

The key value is 55.4769

$$\frac{8000}{55.4769} = 144.204$$

Rounded to the next dollar, this is $\$145$ per month.

10 (a) The key value is 1.093

$\$3000 \times 1.093 = \$3279$

There is $\$3279$ in the account at the end of three years.

(b) The rate is now 6% and the starting value is $\$3279$.

The key value is 1.191

$\$3279 \times 1.191 = \$3905.289$

There is $\$3905.29$ in the account after six years.

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(c) 4% p.a. compounded six monthly for three years means the rate is 2% and the number of periods is 6.

The key value is 1.126.

$\$3000 \times 1.126 = \$3378$

The rate is now 5%.

The key value is 1.158

$\$3378 \times 1.158 = \$3911.724$

There is $\$3911.72$ in the account after six years.

(d) The second strategy is marginally better.

12 Convert both rates to annual rates using effective annual interest rate $= \left(1+\dfrac{r}{n}\right)^n - 1.$

B: 5.1% p.a. is a rate of $r = 0.051.$

In one year there are $n = 12$ compounding periods.

$$\left(1+\frac{0.051}{12}\right)^{12} - 1 = 0.052\ 209... \approx 5.221\%$$

C: 5% p.a. is a rate of $r = 0.05.$

In one year there are $n = 365$ compounding periods.

$$\left(1+\frac{0.05}{365}\right)^{365} - 1 = 0.051\ 267... \approx 5.127\%$$

B is best.

14 N: 5.4% p.a. is a rate of $r = 0.054.$

In one year there are $n = 12$ compounding periods.

$$\left(1+\frac{0.054}{12}\right)^{12} - 1 = 0.055\ 356... \approx 5.536\%$$

T: 5.35% p.a. is a rate of $r = 0.0535.$

In one year there are $n = 365$ compounding periods.

$$\left(1+\frac{0.0535}{365}\right)^{365} - 1 = 0.054\ 952... \approx 5.495\%$$

T is best.

EXERCISE 18.2 ARITHMETIC SEQUENCES

2 **(a)** The first term is a, so $a = 5$.

(b) $8-5=11-8=14-11=3$

The difference between two terms: $d = 3$

(c) $T_n = a+(n-1)d$

$$\begin{aligned} &= 5+(n-1)3 \\ &= 5+3n-3 \\ &= 3n+2 \end{aligned}$$

(d) Use the formula from part **(c)**.

$T_{13} = 3\times 13+2 = 41$

Alternatively, use the formula.

$$\begin{aligned} T_{13} &= a+12d \\ &= 5+12\times 3 \\ &= 5+36 \\ &= 41 \end{aligned}$$

(e) $T_k = a+(k-1)d$

$$\begin{aligned} 98 &= 5+(k-1)\times 3 \\ &= 5+3k-3 \\ &= 2+3k \\ 96 &= 3k \\ k &= 32 \end{aligned}$$

Alternatively, use the formula from part **(c)**.

$$\begin{aligned} T_k &= 3k+2 \\ 98 &= 3k+2 \\ 96 &= 3k \\ k &= 32 \end{aligned}$$

4 For 8, 14, 20, 26, … the first term is 8.

$a = 8$

$d = 14-8 = 20-14 = 26-20 = 6$

$$\begin{aligned} T_n &= a+(n-1)d \\ &= 8+(n-1)\times 6 \\ &= 6n+2 \end{aligned}$$

$T_8 = 6\times 8+2 = 50$

$T_{14} = 6\times 14+2 = 86$

6 $T_1 = p$, $T_2 = q$

First term: $a = p$

Common difference: $d = q-p$

$$\begin{aligned} T_n &= a+(n-1)d \\ T_{10} &= p+(10-1)(q-p) \\ &= p+9q-9p \\ &= 9q-8p \end{aligned}$$

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8 $T_n = a + (n-1)d$

T_5: $a + 4d = 17 \quad ...(1)$
T_{12}: $a + 11d = 52 \quad ...(2)$

$(2) - (1)$ gives $7d = 35$
$d = 5$

Substitute $d = 5$ into (1).

$a + 4 \times 5 = 17$
$a + 20 = 17$
$a = -3$

The sequence is $-3, 2, 7, 12, \ldots$.

10 Using $T_n = a + (n-1)d$:

$T_5 : a + 4d = m...(1)$
$T_{11} : a + 10d = n...(2)$

$6d = n - m...(2) - (1)$
$d = \dfrac{n-m}{6}$

$$\begin{aligned} T_7 &= T_5 + 2d \\ &= m + 2 \times \frac{n-m}{6} \\ &= m + \frac{n-m}{3} \\ &= \frac{3m + n - m}{3} \\ &= \frac{2m+n}{3} \end{aligned}$$

12 $a = T_1 = 9$

$d = 12 - 9 = 3$

We must find n (in terms of p) when $T_n = 6p + 15$.

$$\begin{aligned} T_n &= a + (n-1)d \\ 6p + 15 &= 9 + (n-1) \times 3 \\ 6p + 15 &= 9 + 3n - 3 \\ 6p + 15 - 9 + 3 &= 3n \\ 3n &= 6p + 9 \\ n &= 2p + 3 \end{aligned}$$

There are $2p + 3$ terms in the sequence.

14 $T_1 : a = 6$

$T_n = a + (n-1)d$

$T_4 = 6 + 3d$

$T_5 = 6 + 4d$.

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$$\begin{aligned} T_5 &= 2T_4 \\ a+(5-1)d &= 2\left(a+(4-1)d\right) \\ 6+4d &= 2(6+3d) \\ 6+4d &= 12+6d \\ 2d &= -6 \\ d &= -3 \end{aligned}$$

The common difference is -3.

16 Let the top row be the first term.

$a = 3$

$d = 5-3 = 2$

$$\begin{aligned} T_n &= a+(n-1)d \\ T_{12} &= 3+11\times 2 \\ &= 25 \end{aligned}$$

There are 25 cans of juice on the bottom row.

18 C

$a = 5\sqrt{2}-\sqrt{3}$

Choose the simplest two terms to find the common difference.

$$\begin{aligned} d &= \left(\sqrt{3}+\sqrt{2}\right)-3\sqrt{2} \\ &= \sqrt{3}-2\sqrt{2} \end{aligned}$$

Alternatively,

$$\begin{aligned} d &= 3\sqrt{2}-\left(5\sqrt{2}-\sqrt{3}\right) \\ &= 3\sqrt{2}-5\sqrt{2}+\sqrt{3} \\ &= \sqrt{3}-2\sqrt{2} \end{aligned}$$

$$\begin{aligned} T_6 &= a+5d \\ &= 5\sqrt{2}-\sqrt{3}+5\left(\sqrt{3}-2\sqrt{2}\right) \\ &= 5\sqrt{2}-\sqrt{3}+5\sqrt{3}-10\sqrt{2} \\ &= 4\sqrt{3}-5\sqrt{2} \end{aligned}$$

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20 (a) $I = \$1000 \times \frac{0.5}{100} = \$5.00$

(b) $\$1000 + \$5 = \$1005$

(c) $\$1000 + \$5 \times 12 = \$1060$

(d) $\$1000 + \$5 \times 24 = \$1120$

EXAMPLE 18.3 SERIES AND SIGMA NOTATION (∑)

2 **(a)** correct

(b) incorrect

(c) correct

(d) incorrect

$$\begin{aligned}\sum_{n=0}^{10} x^n &= x^0 + x^1 + x^2 + \ldots + x^{10} \\ &= 1 + x + x^2 + \ldots + x^{10}\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{10} x^n &= x^1 + x^2 + \ldots + x^{10} \\ &= x + x^2 + x^3 + \ldots + x^{10}\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{11} x^{n-1} &= x^{1-1} + x^{2-1} + x^{3-1} + \ldots + x^{11-1} \\ &= x^0 + x^1 + x^2 + \ldots + x^{10} \\ &= 1 + x + x^2 + \ldots + x^{10}\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{10} x^{n-1} &= x^{1-1} + x^{2-1} + x^{3-1} + \ldots + x^{10-1} \\ &= x^0 + x^1 + x^2 + \ldots + x^9 \\ &= 1 + x + x^2 + \ldots + x^9\end{aligned}$$

4 **(a)** $$\begin{aligned}\sum_{n=1}^{4} n^2 &= 1^2 + 2^2 + 3^2 + 4^2 \\ &= 1 + 4 + 9 + 16 \\ &= 30\end{aligned}$$

(b) $$\begin{aligned}\sum_{n=1}^{6} (2n+1) &= (2\times1+1) + (2\times2+1) + (2\times3+1) + (2\times4+1) + (2\times5+1) + (2\times6+1) \\ &= 3 + 5 + 7 + 9 + 11 + 13 \\ &= 48\end{aligned}$$

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(c) $\sum_{k=1}^{4}(3k-2)=(3\times1-2)+(3\times2-2)+(3\times3-2)+(3\times4-2)$

$$=1+4+7+10$$
$$=22$$

(d) $\sum_{r=1}^{4}2^r=2^1+2^2+2^3+2^4$

$$=2+4+8+16$$
$$=30$$

(e) $\sum_{n=0}^{5}n^2=0^2+1^2+2^2+3^2+4^2+5^2$

$$=0+1+4+9+16+25$$
$$=55$$

(f) $\sum_{n=0}^{5}(2n-1)=(2\times0-1)+(2\times1-1)+(2\times2-1)+(2\times3-1)+(2\times4-1)+(2\times5-1)$

$$=-1+1+3+5+7+9$$
$$=24$$

(g) $\sum_{n=1}^{4}(n^2+n)=(1^2+1)+(2^2+2)+(3^2+3)+(4^2+4)$

$$=2+6+12+20$$
$$=40$$

(h)

$$\sum_{n=1}^{6}(12-3n)=(12-3\times1)+(12-3\times2)+(12-3\times3)+(12-3\times4)+(12-3\times5)+(12-3\times6)$$
$$=9+6+3+0+(-3)+(-6)$$
$$=9$$

(i) $\sum_{r=1}^{4}r^r=1^1+2^2+3^3+4^4$

$$=1+4+27+256$$
$$=288$$

EXERCISE 18.4 ARITHMETIC SERIES

2 $a=8$

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$$\begin{aligned} T_{12} &= 41 \\ 8+11d &= 41 \\ 11d &= 33 \\ d &= 3 \end{aligned}$$

$$\begin{aligned} S_{12} &= \frac{12}{2}(2\times 8+11\times 3) \\ &= 6\times(16+33) \\ &= 294 \end{aligned}$$

The sum of the first 12 terms is 294.

4 **(a)** $a=-2$

$$\begin{aligned} d &= 3-(-2)=5 \\ T_{60} &= -2+59\times 5=293 \end{aligned}$$

The 60th term is 293.

(b) $S_{60}=\frac{n}{2}(a+l)$

$$\begin{aligned} &= \frac{60}{2}(-2+293) \\ &= 30\times 291 \\ &= 8730 \end{aligned}$$

The sum of the first 60 terms is 8730.

6 **(a)** Since $14.7-4.9=24.5-14.7=9.8,$ these distances would appear to form an arithmetic sequence where $d=9.8$ and $a=4.9$.

$$\begin{aligned} S_n &= \frac{n}{2}(2a+(n-1)d) \\ S_6 &= \frac{6}{2}(2\times 4.9+5\times 9.8) \\ &= 3\times 58.8 \\ &= 176.4 \end{aligned}$$

It has fallen 176.4 metres after six seconds.

(b) 'Between the fifth and the sixth second' means 'in the sixth second' so $n=6$.

$T_6=4.9+5\times 9.8=53.9$

It has fallen 53.9 metres between the fifth and the sixth second.

8 $T_8 : a+7d=6\ldots[1]$

$T_{12} : a+11d=9\ldots[2]$

$[2]-[1]$

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$$4d = 3$$
$$d = \frac{3}{4}$$

Substitute into [1]

$$a + 7 \times \frac{3}{4} = 6$$
$$a = \frac{3}{4}$$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$S_{20} = \frac{20}{2}\left(2 \times \frac{3}{4} + 19 \times \frac{3}{4}\right)$$
$$= 10 \times \frac{63}{4}$$
$$= 157.5$$

The sum of the first 20 terms is 157.5.

10 (a) $\sum_{k=1}^{10}(3k-7) = (3\times1-7)+(3\times2-7)+(3\times3-7)+\ldots+(3\times10-7)$

$$= (-4)+(-1)+2+\ldots+23$$

$a = -4$ and $l = 23$.

$$\sum_{k=1}^{10}(3k-7) = S_{10}$$
$$= \frac{10}{2}\times(-4+23)$$
$$= 95$$

(b) $\sum_{k=1}^{8}(4k+1) = (4\times1+1)+(4\times2+1)+(4\times3+1)+\ldots+(4\times8+1)$

$$= 5+9+13+\ldots+33$$

$a = 5$ and $l = 33$.

$$\sum_{k=1}^{8}(4k+1) = S_8$$
$$= \frac{8}{2}\times(5+33)$$
$$= 152$$

(c) $\sum_{k=1}^{n}(4k-1)=(4\times1-1)+(4\times2-1)+(4\times3-1)+\ldots+(4\times n-1)$

$=3+7+11+\ldots+(4n-1)$

$a=3$ and $l=4n-1$.

$$\begin{aligned}\sum_{k=1}^{n}(4k-1)&=S_n\\&=\frac{n}{2}\times(3+(4n-1))\\&=\frac{n}{2}(4n+2)\\&=n(2n+1)\end{aligned}$$

(d) $\sum_{k=1}^{n}(6k+2)=(6\times1+2)+(6\times2+2)+(6\times3+2)+\ldots+(6\times n+2)$

$=8+14+20+\ldots+(6n+2)$

$a=8$ and $l=6n+2$.

$$\begin{aligned}\sum_{k=1}^{n}(6k+2)&=S_n\\&=\frac{n}{2}\times(8+(6n+2))\\&=\frac{n}{2}(6n+10)\\&=n(3n+5)\end{aligned}$$

(e) $\sum_{k=1}^{n}(2k-1)=(2\times1-1)+(2\times2-1)+(2\times3-1)+\ldots+(2\times n-1)$

$=1+3+5+\ldots+(2n-1)$

$a=1$ and $l=2n-1$.

$$\begin{aligned}\sum_{k=1}^{n}(2k-1)&=S_n\\&=\frac{n}{2}\times(1+(2n-1))\\&=\frac{n}{2}(2n)\\&=n^2\end{aligned}$$

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(f) $\sum_{k=1}^{10}(3k+2)=(3\times1+2)+(3\times2+2)+(3\times3+2)+...+(3\times10+2)$

$$=5+8+11+...+32$$

$a=5$ and $l=3n+2$.

$$\begin{aligned}\sum_{k=1}^{10}(3k+2)&=S_{10}\\&=\frac{10}{2}\times(5+32)\\&=185\end{aligned}$$

12 $T_3: a+2d=-2...[1]$

$T_9: a+8d=28...[2]$

$[2]-[1]$

$$\begin{aligned}6d&=30\\d&=5\end{aligned}$$

Substitute into $[1]$

$$\begin{aligned}a+2\times5&=-2\\a+10&=-2\\a&=-12\end{aligned}$$

$$\begin{aligned}S_n&=1092\\\frac{n}{2}(2\times(-12)+(n-1)\times5)&=1092\\n(-24+5n-5)&=2184\\n(5n-29)&=2184\\5n^2-29n-2184&=0\\n&=\frac{29\pm\sqrt{(-29)^2-4\times5\times-2184}}{2\times5}\\n&=\frac{29\pm211}{10}\\n&=-18.2,\ 24\end{aligned}$$

Since the number of terms must be greater than 0, 24 terms are required.

14 (a) $S_n = 2+4+6+...+100$

$n = \frac{100}{2} = 50,\ a = 2,\ l = 100$

$$\begin{aligned} S_{50} &= \frac{50}{2}(2+100) \\ &= 2550 \end{aligned}$$

(b) $5+10+15+...+100$

$n = \frac{100}{5} = 20,\ a = 5,\ l = 100$

$$\begin{aligned} S_{20} &= \frac{20}{2}(5+100) \\ &= 1050 \end{aligned}$$

(c) 'Divisible by 2 and 5' is equivalent to 'divisible by 10'.

$10+20+30+...+100$

$n = \frac{100}{10} = 10,\ a = 10,\ l = 100$

$$\begin{aligned} S_{10} &= \frac{10}{2}(10+100) \\ &= 550 \end{aligned}$$

(d) If we add the numbers divisible by 2 to the numbers divisible by 5, we will have added 'divisible by both' twice.

However, we are not to include 'divisible by both' at all.

Therefore we need 'divisible by 2' plus 'divisible by 5' minus 2 × 'divisible by both'.

Using the answers to the previous parts of this question, the sum of integers divisible by 2 or 5 but not both is $2550+1050-2\times 550 = 2500$

16 $T_5 = 136° \Rightarrow a + 4d = 136°...[1]$

$S_5 = 540° \Rightarrow \frac{5}{2}\times(2a+4d) = 540°$

$$\begin{aligned} 5a + 10d &= 540° \\ a + 2d &= 108°...[2] \end{aligned}$$

$[1]-[2]$

$$\begin{aligned} 2d &= 28° \\ d &= 14° \end{aligned}$$

Substitute into $[1]$

$$\begin{aligned} a + 4\times 14° &= 136° \\ a &= 80° \end{aligned}$$

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$T_2 = 80° + 14° = 94°$
$T_3 = 94° + 14° = 108°$
$T_4 = 108° + 14° = 122°$

The magnitudes of the other four angles are $80°$, $94°$, $108°$, $122°$.

18 $5.25 \text{ m} = 525 \text{ cm}$

$$a = 40$$
$$S_n = 525$$
$$\frac{n}{2}(40 + 30) = 525$$
$$n = 15$$

20 $a = 45$

$$\begin{aligned} d &= 41 - 45 \\ &= -4 \end{aligned}$$

$$\begin{aligned} T_n &= 1 \\ 45 + (n-1) \times (-4) &= 1 \\ 45 - 4n + 4 &= 1 \\ -4n &= -48 \\ n &= 12 \end{aligned}$$

$$\begin{aligned} S_{12} &= \frac{12}{2} \times (45 + 1) \\ &= 276 \end{aligned}$$

22 $S_n = 3n^2 - 11n$

$$\begin{aligned} S_{n-1} &= 3(n-1)^2 - 11(n-1) \\ &= 3(n^2 - 2n + 1) - 11n + 11 \\ &= 3n^2 - 6n + 3 - 11n + 11 \\ &= 3n^2 - 17n + 14 \end{aligned}$$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= (3n^2 - 11n) - (3n^2 - 17n + 14) \\ &= 6n - 14 \end{aligned}$$

$$\therefore T_{n-1} = 6(n-1) - 14 = 6n - 20$$

$$\begin{aligned} T_n - T_{n-1} &= (6n - 14) - (6n - 20) \\ &= 6n - 14 - 6n + 20 \\ &= 6 \end{aligned}$$

Since the difference between any two consecutive terms $n-1$ and n is always 6 and therefore constant, this must be an arithmetic series with $d = 6$.

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24 $T_1 + T_3 = 0$

$$a + a + 2d = 0$$
$$a + d = 0 \ldots [1]$$

$$S_9 = 81$$
$$\frac{9}{2} \times (2a + 8d) = 81$$
$$a + 4d = 9 \ldots [2]$$

$[2] - [1]$

$$3d = 9$$
$$d = 3$$

Substitute into $[1]$

$$a + 3 = 0$$
$$a = -3$$

26 $T_1 = a$

$$T_2 = b$$
$$a + d = b$$
$$d = b - a$$
$$T_n = c$$
$$c = a + (n-1)d$$
$$c = a + (n-1)(b-a)$$
$$c = a + n(b-1) - b + a$$
$$c = 2a - b + n(b-a)$$
$$n = \frac{b + c - 2a}{b - a}$$

$$S_n = \frac{n}{2}(a + l)$$
$$= \frac{n}{2}(a + c)$$
$$= \frac{(b + c - 2a)(a + c)}{2(b - a)}$$

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28 (a) $a = 15$

$d = 1$

$$S_n = 246$$
$$\frac{n}{2}\left(2\times15+(n-1)\times1\right) = 246$$
$$29n + n^2 = 492$$
$$n^2 + 29n - 492 = 0$$
$$n = \frac{-29 \pm \sqrt{29^2 - 4\times1\times(-492)}}{2\times1}$$
$$n = \frac{-29 \pm 53}{2\times1}$$
$$n = -41,\ 12$$

Since n is the number of rows, $n > 0$, so $n = 12$.

There are 12 rows.

(b) $T_{12} = 15 + 11\times1$

$= 26$

There are 26 logs on the bottom row.

30 $a = 6$

$d = 10 - 6 = 4$

$$S_n = \frac{n}{2}\times\left(2\times6+(n-1)\times4\right)$$
$$880 = \frac{n}{2}(12 + 4n - 4)$$
$$880 = 4n + 2n^2$$
$$n^2 + 2n - 440 = 0$$
$$(n-20)(n+22) = 0$$
$$n = -22,\ 20$$

Since n is the number of terms, $n > 0$, so $n = 20$.

20 terms must be taken to give the sum required.

32 (a) $a = 1$

$d = 3 - 1 = 2$

$$\begin{aligned} S_n &= \frac{n}{2}\left(2\times1+\left(n-1\right)\times2\right) \\ &= \frac{n}{2}\left(2+2n-2\right) \\ &= n^2 \end{aligned}$$

(b) $a = 2$

$$\begin{aligned} d &= 4-2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} S_n &= \frac{n}{2}\left(2\times2+\left(n-1\right)\times2\right) \\ &= \frac{n}{2}\left(4+2n-2\right) \\ &= \frac{n}{2}\left(2n+2\right) \\ &= n^2+n \end{aligned}$$

(c) $a = 1$

$d = 2 - 1 = 1$

$$\begin{aligned} S_n &= \frac{n}{2}\left(2\times1+\left(n-1\right)\times1\right) \\ &= \frac{n}{2}\left(2+n-1\right) \\ &= \frac{n\left(n+1\right)}{2} \\ &= \frac{n^2+n}{2} \end{aligned}$$

$$\begin{aligned} 210 &= \frac{n^2+n}{2} \\ n^2+n-420 &= 0 \\ \left(n-20\right)\left(n+21\right) &= 0 \end{aligned}$$

$n = -21,\ n = 20$

Since n is the number of terms, $n > 0$, so $n = 20$.

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34 (a) $a = 3$

$d = 5 - 3 = 2$

$T_{10} = 3 + 9 \times 2 = 21$

(b) $S_{10} = \frac{10}{2} \times (3 + 21) = 120$

36 $a = 95°$

The angle sum of a hexagon is $(6-2) \times 180° = 720°$

$$\begin{aligned} S_6 &= 720° \\ \frac{6}{2} \times (2 \times 95° + 5d) &= 720° \\ 190° + 5d &= 240° \\ 5d &= 50° \\ d &= 10° \end{aligned}$$

$$\begin{aligned} T_1 &= 95° \\ T_2 &= 95° + 10° = 105° \\ T_3 &= 105° + 10° = 115° \\ T_4 &= 115° + 10° = 125° \\ T_5 &= 125° + 10° = 135° \\ T_6 &= 135° + 10° = 145° \end{aligned}$$

The other angles are $105°,\ 115°,\ 125°,\ 135°,\ 145°$

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EXERCISE 18.5 GEOMETRIC SEQUENCE

2 **(a)** For a is the first term, which is 1.

$$a = 1$$

(b) $r = \frac{3}{1} = \frac{9}{3} = \frac{27}{9} = 3$

(c) $T_n = ar^{n-1}$

$$T_n = 1 \times 3^{n-1} = 3^{n-1}$$

(d) $T_{10} = 3^{10-1} = 3^9 = 19\ 683$

(e) $T_k = 6561$

$$\begin{aligned} 3^{k-1} &= 6561 \\ 3^{k-1} &= 3^8 \\ k-1 &= 8 \\ k &= 9 \end{aligned}$$

4 $a = 4$

$$r = \frac{6}{4} = \frac{3}{2}$$

The sequence is geometric, with $a = 4$, $r = \frac{3}{2}$.

$$\begin{aligned} T_6 &= 4 \times \left(\frac{3}{2}\right)^{6-1} \\ &= \frac{243}{8} \end{aligned}$$

6 $r = \frac{T_7}{T_6} = \frac{\frac{64}{27}}{\frac{32}{9}} = \frac{64}{27} \times \frac{9}{32} = \frac{2}{3}$

$$\begin{aligned} T_6 &= ar^5 \\ \frac{32}{9} &= a \times \left(\frac{2}{3}\right)^5 \\ \frac{32}{9} &= a \times \frac{32}{243} \\ a &= \frac{32}{9} \times \frac{243}{32} = 27 \end{aligned}$$

The first term is 27. The common ratio is $\frac{2}{3}$.

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8 There must be a common ratio.

$$\frac{5p}{2p+1}=\frac{12p-4}{5p}$$
$$25p^2=(12p-4)(2p+1)$$
$$25p^2=24p^2+4p-4$$
$$p^2-4p+4=0$$
$$(p-2)^2=0$$
$$p=2$$

10 Arithmetic sequence: $1, x, y$

Common difference:

$$x-1=y-x$$
$$2x-1=y$$
$$y=2x+1...[1]$$

Geometric sequence: $1, y, x$

Common ratio:

$$\frac{y}{1}=\frac{x}{y}$$
$$y^2=x...[2]$$

Substitute [1] into [2]

$$(2x-1)^2=x$$
$$4x^2-4x+1-x=0$$
$$4x^2-5x+1=0$$
$$(4x-1)(x-1)=0$$
$$x=\frac{1}{4},\ x=1$$

Substitute $x=\frac{1}{4}$ into [1]

$$y=2\times\frac{1}{4}-1=-\frac{1}{2}$$

Substitute $x=1$ into [1]

$$y=2\times1-1=1$$

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Answer: $x=\frac{1}{4},\ y=-\frac{1}{2}$ or $x=1,\ y=1$.

Note that when $x=1,\ y=1$, the sequences both become $1,\ 1,\ 1,\ldots$, which, while it is strictly both an arithmetic and a geometric sequence, could be dismissed as a trivial case.

12 Decreasing by 10% is equivalent to taking 90% of the preceding year's population.

$$a=10000$$
$$r=90\%=0.9$$

Taking the initial population as $T_1=10\ 000$, 'after 5 years' time' gives the sixth term of the sequence.

$$T_6=10\ 000\times 0.9^5=5904.9\approx 5905$$

The population will be approximately 5905 in 5 years' time.

14 $\log ar=\log a+\log r$

$$\begin{aligned}\log ar &= \log a+\log r\\ \log ar^2 &= \log a+\log r^2\\ &= \log a+2\log r\\ \log ar^3 &= \log a+\log r^3\\ &= \log a+3\log r\\ &\ .\\ &\ .\\ &\ .\\ \log ar^n &= \log a+\log r^n\\ &= \log a+n\log r\end{aligned}$$

The sequence becomes $\log a,\ \log a+\log r,\ \log a+2\log r,\ \log a+3\log r,\ldots$ which is arithmetic with first term $\log a$ and a common difference of $\log r$.

16 (a) The number present follows a geometric progression with

$$a=200,\ r=2.$$

Taking 'initially' as the first time, then 'after 4 hours' is the fifth term.

$$\begin{aligned}T_5 &= 200\times 2^4\\ &= 3200\end{aligned}$$

There will be 3200 bacteria present after 4 hours.

(b) Taking initially as the first time, then 'after 10 hours' is the eleventh term.

$T_{11} = 200 \times 2^{10} = 204\ 800$

There will be $204\ 800$ bacteria present after 10 hours.

(c) Taking initially as the first time, then 'after t hours' is the $(t+1)\text{th}$ term.

$$\begin{aligned} T_{t+1} &= 200 \times 2^{t} \\ N &= 200 \times 2^{t} \end{aligned}$$

18 (a) The situation can be modelled by a geometric sequence. 'Decreasing by 15% is equivalent to taking 85% of the preceding year's population.

$$\begin{aligned} a &= 3000 \\ r &= 0.85 \end{aligned}$$

Taking initially as the first time then 'after 5 hours' is the sixth term.

$T_6 = 30\ 000 \times 0.85^5 \approx 13\ 311$

After 5 years there will be $13\ 000$ birds, to the nearest thousand.

(b) Let t be the number of years.

$$\begin{aligned} 9000 &= 30\ 000 \times 0.85^{t} \\ 0.85^{t} &= \frac{9000}{30\ 000} \\ 0.85^{t} &= 0.3 \\ \ln 0.85^{t} &= \ln 0.3 \\ t \ln 0.85 &= \ln 0.3 \\ t &= \frac{\ln 0.3}{\ln 0.85} \\ &\approx 7.4 \end{aligned}$$

It will take about 7.4 years for the population to fall to 9000.

(c) Let P be population, and t be the number of years $(t \geq 0)$

$P = 30\ 000 \times 0.85^{t}$

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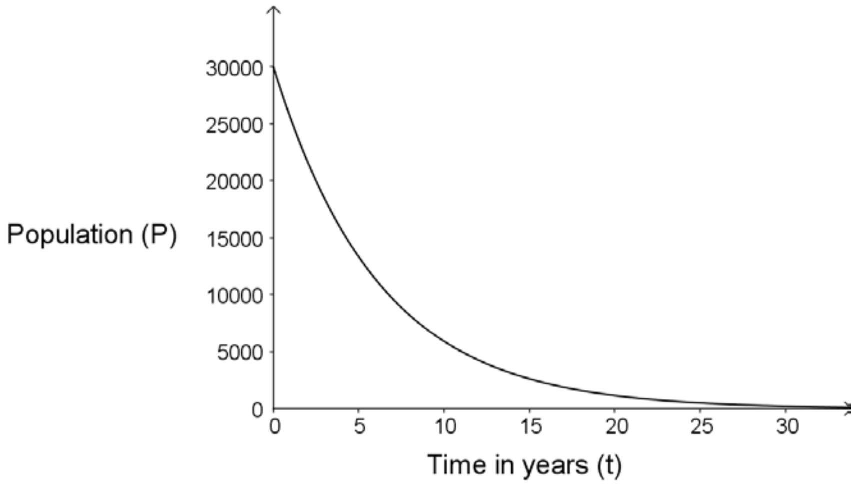

EXERCISE 18.6 FINITE GEOMETRIC SERIES

2 $a = 8,\ r = \frac{-4}{8} = -\frac{1}{2} < 1$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$S_{10} = \frac{8 \times \left[1 - \left(-\frac{1}{2}\right)^{10}\right]}{1 - \left(-\frac{1}{2}\right)}$$

$$= \frac{341}{64}$$

4 $a = 16,\ r = \frac{-8}{16} = -\frac{1}{2} < 1$

Find the number of terms.

$$T_n = ar^{n-1}$$

$$\frac{1}{16} = 16 \times \left(-\frac{1}{2}\right)^{n-1}$$

$$\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{256} = \frac{1}{2^8} = \left(\frac{1}{2}\right)^8 = \left(-\frac{1}{2}\right)^8$$

$$n - 1 = 8$$

$$n = 9$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$S_9 = \frac{16\times\left[1-\left(-\frac{1}{2}\right)^9\right]}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{171}{16}$$

6 $a = x$

$$r = \frac{2x^2}{3} \div x = \frac{2x}{3}$$

If $x = 3,\ r = 2 > 1$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$n = 6,\ x = 3,\ a = x = 3,\ r = 2$

$$S_6 = \frac{3\left(2^6 - 1\right)}{2-1} = 3\left(2^6 - 1\right) = 189$$

8 $a = \log_{10} 3$

$$\begin{aligned} T_2 - T_1 &= \log_{10} 6 - \log_{10} 3 \\ &= \log_{10} \frac{6}{3} \\ &= \log_{10} 2 \end{aligned}$$

$$\begin{aligned} T_3 - T_2 &= \log_{10} 12 - \log_{10} 6 \\ &= \log_{10} \frac{12}{6} \\ &= \log_{10} 2 \end{aligned}$$

$$T_3 - T_2 = T_2 - T_1$$

$$\begin{aligned} S_{10} &= \frac{10}{2}\left(2\times\log_{10} 3 + 9\times\log_{10} 2\right) \\ &= 10\log_{10} 3 + 45\log_{10} 2 \end{aligned}$$

10 (a) Take one A0-sized sheet of paper as the first term. The number of pieces of paper doubles every time it is folded and cut, so this is a geometric sequence where the number of A3-sized pieces of paper is the fourth term.

 ISBN 978 1 4886 1829 1

$a = 1,\ r = 2$

$$\begin{aligned} T_4 &= 1 \times 2^3 \\ &= 8 \end{aligned}$$

8 pieces of A3-sized paper are created.

(b) (i) The number of A10-sized pieces of paper is the eleventh term.

$T_{11} = 1 \times 2^{10} = 1024$

There are 1024 A10-sized sheets created.

(ii) $\text{Sum} = 1 + 2 + 2^2 + ... + 2^{10}$

Note that there are 11 terms.

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{11} = \frac{1\left(2^{11} - 1\right)}{2 - 1} = 2047$$

There are 2047 sheets of paper in the pile.

(iii) One sheet of paper is $\frac{55}{500}$ mm thick

2047 sheets of paper are $2047 \times \frac{55}{500} = 225.17$ mm thick

The stack of sheets is approximately 225 mm 22.5 cm high.

EXERCISE 18.7 INFINITE GEOMETRIC SERIES

2 $a = 1$

$$r = \frac{2x}{1} = 2x$$

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
\frac{3}{4} &= \frac{1}{1-2x} \\
3(1-2x) &= 4 \\
3-6x &= 4 \\
-6x &= 1 \\
x &= -\frac{1}{6}
\end{aligned}$$

4 D

$$a = 6,\ r = \frac{3}{6} = \frac{1}{2}$$

$$S_\infty = \frac{a}{1-r} = \frac{6}{1-\frac{1}{2}} = 12$$

6 $a = 12$

$$r = \frac{8}{12} = \frac{2}{3}$$

$$S_\infty = \frac{a}{1-r} = \frac{12}{1-\frac{2}{3}} = 36$$

8 $a = \sqrt{5} + \sqrt{3}$

$$\begin{aligned}
r &= \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} \times \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} \\
&= \frac{5-2\sqrt{15}+3}{5-3} \\
&= \frac{8-2\sqrt{15}}{2} \\
&= 4-\sqrt{15}
\end{aligned}$$

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$$S_\infty = \frac{\sqrt{5}+\sqrt{3}}{1-\left(4-\sqrt{15}\right)}$$
$$= \frac{\sqrt{5}+\sqrt{3}}{\sqrt{15}-3} \times \frac{\sqrt{15}+3}{\sqrt{15}+3}$$
$$= \frac{5\sqrt{3}+3\sqrt{5}+3\sqrt{5}+3\sqrt{3}}{15-9}$$
$$= \frac{8\sqrt{3}+6\sqrt{5}}{6}$$
$$= \frac{4\sqrt{3}+3\sqrt{5}}{3}$$

10 $0.323232... = 0.32 + 0.0032 + 0.000\ 032...$

$$a = 0.32$$
$$r = \frac{0.0032}{0.32} = 0.01$$
$$S_\infty = \frac{0.32}{1-0.01}$$
$$\frac{p}{q} = \frac{32}{99}$$
$$p = 32,\ q = 99$$

12 $1+2+3+...+10 = \frac{10}{2} \times \left(1+10\right) = 55$

$$T_n = 1 \times \left(\frac{1}{2}\right)^{n-1}$$
$$\frac{1}{512} = \frac{1}{2^{n-1}}$$
$$\frac{1}{2^9} = \frac{1}{2^{n-1}}$$
$$n-1 = 9$$
$$n = 10$$

 ISBN 978 1 4886 1829 1

$$1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{512}=\frac{1\left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\frac{1}{2}}$$

$$=\frac{1023}{512}$$

$$\frac{1+2+3+...+10}{1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{512}}=\frac{55}{\frac{1023}{512}}=55\times\frac{512}{1023}=\frac{2560}{93}=27\frac{49}{93}$$

14 (a) Each subsequent area is $\frac{3}{4}$ the size of the previous area.

Therefore, the area can be modelled by a geometric sequence.

$a=1,\ r=\frac{3}{4}$

$$T_3=ar^2$$

$$=1\times\left(\frac{3}{4}\right)^2$$

$$=\frac{9}{16}$$

The area shaded in the third diagram is $\frac{9}{16}$ of the area of the original triangle.

(b) $T_{10}=ar^9$

$$=1\times\left(\frac{3}{4}\right)^9$$

$$=\frac{19\ 683}{262\ 144}$$

The area shaded in the tenth diagram is $\frac{19\ 683}{262\ 144}\approx 0.075$ square units.

(c) The remaining shaded area becomes less, approaching zero, but never (theoretically) reaching it.

 ISBN 978 1 4886 1829 1

EXERCISE 18.8 COMPOUND INTEREST APPLICATIONS

2 **(a)** Each year the interest will increase by a factor of $1+3.1\% = 1+0.031 = 1.031$.

The initial investment is $\$1000$, so $A_0 = 1000$.

$A_0 = 1000,\ A_n = 1.031 \times A_{n-1}$

(b) The six-monthly interest rate is $\dfrac{6.2\%}{2} = 3.1\%$

Every six months the interest will increase by a factor of $1+3.1\% = 1+0.031 = 1.031$.

The initial investment is $\$5000$, so $A_0 = 5000$.

$A_0 = 5000,\ A_n = 1.031 \times A_{n-1}$

(c) The monthly interest rate is $\dfrac{2.7\%}{12} = 0.225\%$

Every month the interest will increase by a factor of $1+0.225\% = 1+0.002\ 25 = 1.002\ 25$.

The initial investment is $\$10\ 000$, so $A_0 = 10\ 000$.

$A_0 = 10\ 000,\ A_n = 1.002\ 25 \times A_{n-1}$

4 **(a)** $A_0 = 5000$

$$r = 20\%$$
$$R = 1.2$$
$$n = 5$$

$$\begin{aligned} A_5 &= 5000(1.2)^5 \\ &= 12\ 442 \end{aligned}$$

(b) This includes both ends, so there are six years involved and therefore $n = 6$.

$a = 5000,\ R = 1.2$

$$S_n = \frac{a\left(R^n - 1\right)}{R-1}$$
$$S_5 = \frac{5000\left(1.2^6 - 1\right)}{1.2-1} = 49\ 649.6 \approx 49\ 650$$

 ISBN 978 1 4886 1829 1

6 $A_0 = 15000$

$$r = 8\%$$
$$R = 1.08$$
$$n = 8$$

$$\begin{aligned} A_8 &= 15\ 000(1.08)^8 \\ &\approx 27\ 764 \end{aligned}$$

8 **(a)** Megan earns an extra 5% of her salary each year.

$$A_0 = 83\ 000$$
$$r = 5\%$$
$$R = 1.05$$
$$n = 4$$

$$A_1 = A_3 = A_3 = A_4 = 83\ 000 \times 1.05 = 87\ 150$$

$$S_4 = 83\ 000 \times 1.05 \times 4 = 348\ 600$$

Megan has earned $\$348\ 600$ over the four years.

Paul:

$$A_0 = 83\ 000$$
$$r = 3\%$$
$$R = 1.03$$

$$\begin{aligned} A_4 &= 83\ 000\left(1 + 1.03 + 1.03^2 + 1.03^3\right) \\ &= 347\ 241.04 \end{aligned}$$

Paul has earned $\$347\ 241$

(b) $83\ 000(1.03)^n > 83\ 000(1.05)$

$$\begin{aligned} 1.03^n &> 1.05 \\ n\log_{10}(1.03) &> \log_{10}(1.05) \\ n &> \frac{\log_{10}(1.05)}{\log_{10}(1.03)} = 1.65 \end{aligned}$$

If A_0 is the first year, then at the *end* of the second year, Paul's earnings will increase for a second time so they will actually exceed Megan's in the third year.

 ISBN 978 1 4886 1829 1

10 (a) Eleanor:

$$
\begin{aligned}
A_0 &= 40\ 000 \\
r &= 4\% \\
R &= 1.04 \\
n &= 3
\end{aligned}
$$

$$
\begin{aligned}
A_n &= A_0R^n \\
A_3 &= 40\ 000(1.04)^3 \\
&= 44\ 994.56
\end{aligned}
$$

Eleanor's salary will be $\$44\ 994.56$.

Henry:

$$A_3 = 40000 + 2000 \times 3 = 46\ 000$$

Henry's salary will be $\$46000$.

(b) Eleanor:

$$
\begin{aligned}
S_{10} &= \frac{40\ 000\left(1.04^{10} - 1\right)}{1.04 - 1} \\
&= 480\ 244.28
\end{aligned}
$$

Eleanor will have earned $\$480\ 244.28$.

Henry:

$$
\begin{aligned}
S_{10} &= \frac{10}{2}\left(2 \times 40\ 000 + (10 - 1) \times 2000\right) \\
&= 490\ 000
\end{aligned}
$$

Henry will have earned $\$490\ 000$.

(c) $40\ 000(1.04)^n = 40\ 000 + 2000n$

Use technology to solve the above equation, or graph $y = 40\ 000(1.04)^x$ and $y = 40\ 000 + 2000x$, and find the intersection point.

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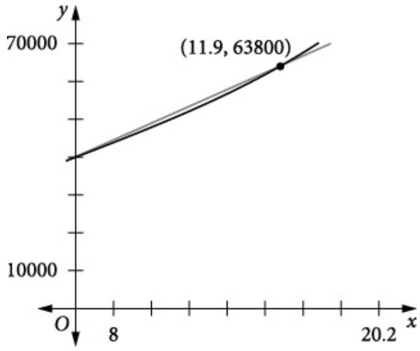

The point of intersection occurs when $x = 11.9$.

Eleanor's salary first exceeds Henry's after 12 years, that is, at the beginning of the 13th year.

12 (a) Interest $r = 8.4\%$ p.a.

Monthly interest rate $= \dfrac{8.4}{12} = 7\%$

$200000 \times 0.007 = 1400$

The interest paid on the first month is $0.7\% \times \$200\ 000 = \$1400$.

(b) Kathryn's payments should be more than $\$1400$ per month to pay off more than the interest rate so that the balance owed is reduced.

(c) $A_0 = 200\ 000$

$$\begin{aligned} A_1 &= 200\ 000 \times 1.007 - 3000 \\ A_2 &= (200\ 000 \times 1.007 - 3000) \times 1.007 - 3000 \\ &= 200\ 000 \times 1.007^2 - 3000 \times 1.007 - 3000 \\ &= 200\ 000 \times 1.007^2 - 3000(1 + 1.007) \end{aligned}$$

(d) $A_0 = 200\ 000$

$$\begin{aligned} A_1 &= 200\ 000 \times 1.007 - 3000 \\ A_2 &= 200\ 000 \times 1.007^2 - 3000(1 + 1.007) \\ A_3 &= 200\ 000 \times 1.007^3 - 3000(1 + 1.007 + 1.007^2) \\ A_n &= 200\ 000 \times 1.007^n - 3000(1 + 1.007 + 1.007^2 + \ldots + 1.007^{n-1}) \end{aligned}$$

 ISBN 978 1 4886 1829 1

$1+1.007+1.007^2+...+1.007^{n-1}$ is a geometric series with $a=1,\ r=1.007.$

$$\begin{aligned}A_n &= 200\ 000\times1.007^n-\frac{3000\left(1.007^n-1\right)}{1.007-1}\\ &= 200\ 000\times1.007^n-\frac{3000\left(1.007^n-1\right)}{0.007}\end{aligned}$$

(e) $200\ 000\times1.007^n-\dfrac{3000\left(1.007^n-1\right)}{0.007}=0$

Multiply both sides by 0.007, then solve for 1.007^n.

$$\begin{aligned}0.007\times200\ 000\times1.007^n-3000\left(1.007^n-1\right)&=0\\ 1400\times1.007^n-3000\times1.007^n+3000&=0\\ -1600\times1.007^n&=-3000\\ 1.007^n&=\frac{-3000}{-1600}=1.875\\ n\log1.007&=\log1.875\\ n&=\frac{\log1.875}{\log1.007}=90.115...\end{aligned}$$

$$200\ 000\times1.007^n-\frac{3000\times1.007^n}{0.007}+\frac{3000}{0.007}=0$$

It will take Kathryn 90 months to pay off the loan. The loan will not be quite paid off in 90 months and her last payment will be small.

EXERCISE 18.9 FURTHER APPLICATIONS OF SERIES

2 $P=20\ 000,\ r=0.0075,\ R=1.0075$, repayment (instalment) is Q per quarter.

Amount owing after first repayment

$$A_1=20\ 000\times1.0075-Q$$

Amount owing after second repayment

$$\begin{aligned}A_2&=\left(20\ 000\times1.0075-Q\right)\times1.0075-Q\\ &=20\ 000\times1.0075^2-Q\left(1+1.0075\right)\end{aligned}$$

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Amount owing after third repayment

$$A_3 = \left(20\ 000 \times 1.0075^2 - Q \times \left(1 + 1.0075\right)\right) \times 1.0075 - Q$$
$$= 20\ 000 \times 1.0075^3 - Q\left(1 + 1.0075 + 1.0075^2\right)$$

Amount owing after the 16th repayment is zero.

$$A_{16} = 20000 \times 1.0075^{16} - Q\left(1 + 1.0075 + 1.0075^2 + ... + 1.0075^{15}\right)$$

$1 + 1.0075 + 1.0075^2 + ... + 1.0075^{15}$ is a geometric series with $a = 1, r = 1.0075, n = 16$

$$\therefore A_{16} = 20000 \times 1.0075^{16} - \frac{Q\left(1.0075^{16} - 1\right)}{1.0075 - 1} = 0$$
$$\frac{Q\left(1.0075^{16} - 1\right)}{0.0075} = 20000 \times 1.0075^{16}$$
$$Q = \frac{20000 \times 1.0075^{16} \times 0.0075}{\left(1.0075^{16} - 1\right)} \approx 1331.18$$

Each instalment is $\$1331.18$.

4 **(a)** $r = \frac{0.06}{12} = 0.005,\ R = 1.005$

$$A_1 = 400\ 000 \times 1.005 - 2578 = 399\ 422$$

They owed $\$399\ 422$ after their first monthly repayment

(b) $A_n = 392\ 870 \times 1.0075^n - 1.0075^{n-1}M - ... - 1.0075M - M$

$$A_{288} = 0 = 392\ 870 \times 1.0075^{288} - \frac{M\left(1.0075^{288} - 1\right)}{1.0075 - 1}$$
$$\frac{M\left(1.0075^{288} - 1\right)}{1.0075 - 1} = 392\ 870 \times 1.0075^{288}$$
$$M = \frac{392\ 870 \times 1.0075^{288} \times 0.0075}{\left(1.0075^{288} - 1\right)} = 3334.15$$

The monthly repayment is $\$3334.15$

 ISBN 978 1 4886 1829 1

(c)

$$A_n = 392\ 870 \times 1.0075^n - \frac{M\left(1.0075^n - 1\right)}{0.0075}$$

$$0 = 392\ 870 \times 1.0075^n - \frac{3500\left(1.0075^n - 1\right)}{0.0075}$$

$$392\ 870 \times 1.0075^n = \frac{3500\left(1.0075^n - 1\right)}{0.0075}$$

$$0.0075 \times 392\ 870 \times 1.0075^n = 3500 \times 1.0075^n - 3500$$

$$2946.525 \times 1.0075^n - 3500 \times 1.0075^n = 3500$$

$$-553.475 \times 1.0075^n = -3500$$

$$1.0075^n = \frac{3500}{553.475}$$

$$n \log_{10} 1.0075 = \log_{10}\left(\frac{3500}{553.475}\right)$$

$$n = \log_{10}\left(\frac{3500}{553.475}\right) \div \log_{10} 1.0075 = 246.8$$

It will take 247 months to repay the loan at the higher rate.

(d) Amount saved: $288 \times \$3334.15 - 247 \times \$3500 = \$95\ 735.20$

They will save approximately $\$95\ 735$

We could assume that the last payment might be about $0.8 \times \$3500 = \$2800$ instead of $\$3500$ so they would save an extra $\$700$ on the last repayment (the above assumes the last payment is $\$3500$), making a more accurate estimate of the savings as $\$95\ 735 + \$700 = \$96\ 435$.

6 **(a)** Value after 11 years:

$$\$90\ 000 \times 1.05^{11} - \frac{\$10\ 000\left(1.05^{11} - 1\right)}{0.05} = \$11\ 862.67$$

This is a good idea because now, after receiving the same payments, you still have over $\$10\ 000$ left, so you get an extra year's payment, and still have quite a bit left.

(b) $\$90\ 000 \times 1.05^{11} - \dfrac{\$10\ 000\left(1.05^{11} - 1\right)}{0.05} = \$11\ 862.67$

(c) $A_{12} = \$90\ 000 \times 1.05^{12} - \dfrac{\$10\ 000\left(1.05^{12} - 1\right)}{0.05} = \$2455.80$

 ISBN 978 1 4886 1829 1

CHAPTER REVIEW 18

2 **(a)** The first term is 22, so $a = 22$.

$d = 15 - 22 = -7$

$T_n = a + (n-1)d$

$$\begin{aligned} T_{10} &= 22 + 9 \times (-7) \\ &= -41 \end{aligned}$$

(b) $T_k = -90$

$$\begin{aligned} 22 + (k-1)(-7) &= -90 \\ 22 - 7k + 7 &= -90 \\ 119 &= 7k \\ k &= 17 \end{aligned}$$

4 **(a)** $T_4 = 600$, $T_{10} = 75$

$a + 3d = 600 \ \ldots(1)$

$a + 9d = 75 \ \ldots(2)$

$(2) - (1)$:

$$\begin{aligned} 6d &= -525 \\ d &= -87.5 \end{aligned}$$

Substitute $d = -87.5$ in (1):

$$\begin{aligned} a + 3(-87.5) &= 600 \\ a &= 862.5 \end{aligned}$$

$T_1 = 862.5$

$T_2 = 862.5 - 87.5 = 775$

$T_3 = 775 - 87.5 = 687.5$

(b) $T_4 = 600$, $T_{10} = 75$

For a geometric sequence,

$ar^3 = 600 \ \ldots(1)$

$ar^9 = 75 \ \ldots(2)$

$(2) \div (1)$

$$\begin{aligned} r^6 &= \frac{75}{600} = \frac{1}{8} \\ r &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Substitute $r = \pm\frac{1}{\sqrt{2}}$ in (1):

$$a\left(\pm\frac{1}{\sqrt{2}}\right)^3 = 600$$
$$a\left(\pm\frac{1}{2\sqrt{2}}\right) = 600$$
$$a = \pm 1200\sqrt{2}$$

There are two possible sequences:

$T_1 = 1200\sqrt{2}$, $T_2 = 1200\sqrt{2}\times\frac{1}{\sqrt{2}} = 1200$, $T_3 = 1200\times\frac{1}{\sqrt{2}} = 600\sqrt{2}$

or $T_1 = -1200\sqrt{2}$, $T_2 = -1200\sqrt{2}\times -\frac{1}{\sqrt{2}} = 1200$, $T_3 = 1200\times -\frac{1}{\sqrt{2}} = -600\sqrt{2}$

6 Compare consecutive terms.

$$\frac{3}{6} = \frac{1.5}{3} = \frac{1}{2}$$

$6+3+1.5+...$ is a geometric sequence with $a = 6,\ r = 0.5.$

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{6}{1-0.5} = 12$$

8 This is an arithmetic sequence.

$$T_1 = a$$
$$T_2 = a + d$$
$$T_3 = a + 2d$$

$$T_1 + T_2 + T_3 = 15$$

$$a + a + d + a + 2d = 15$$
$$3a + 3d = 15$$
$$a + d = 5$$

Geometric series: $a+1, a+d+1, a+2d+4$

$$r = \frac{a+d+1}{a+1} = \frac{a+2d+4}{a+d+1}$$

$$(a+d+1)^2 = (a+1)(a+2d+4)$$
$$a^2 + 2ad + 2a + d^2 + 2d + 1 = a^2 + 2ad + 5a + 2d + 4$$
$$d^2 - 3a - 3 = 0$$
$$a = \frac{d^2 - 3}{3}$$

Substitute $a = \frac{d^2-3}{3}$ into $a + d = 5$.

$$\left(\frac{d^2-3}{3}\right) + d = 5$$
$$d^2 - 3 + 3d = 15$$
$$d^2 + 3d - 18 = 0$$
$$(d+6)(d-3) = 0$$

$\therefore d = -6$ or 3

Substitute $d = -6,\ 3$ into $a + d = 5$

$$a + (-6) = 5$$
$$a = 11$$

$$a + (3) = 5$$
$$a = 2$$

There are two possible solutions:

$a = 2,\ d = 3$ will give the three numbers as $2,\ 5,\ 8.$

$a = 11,\ d = -6$ will give the sequence $11,\ 5,\ -1.$

Check: adding 1, 1 and 4 gives $3,\ 6,\ 12$ and $12,\ 6,\ 3;$ both of which are geometric sequences.

10 $0.2333... = 0.2 + 0.0333...$

$$= \frac{1}{5} + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$$

After the first term, there is an infinite geometric series with $a = \frac{3}{100} = 0.03,\ r = \frac{1}{10} = 0.1.$

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$$\begin{aligned}0.2333... &= \frac{1}{5}+\frac{a}{1-r}\\ &= \frac{1}{5}+\frac{0.03}{1-0.1}\\ &= \frac{1}{5}+\frac{0.03}{0.9}\\ &= \frac{1}{5}+\frac{3}{90}\\ &= \frac{6}{30}+\frac{1}{30}\\ &= \frac{7}{30}\end{aligned}$$

$$\therefore \frac{m}{n}=\frac{7}{30}$$

12 $S_n = n^2 - 3n$

$$\begin{aligned}T_n &= S_n - S_{n-1}\\ &= n^2-3n-\left((n-1)^2-3(n-1)\right)\\ &= n^2-3n-(n-1)^2+3n-3\\ &= n^2-3n-n^2+2n-1+3n-3\\ &= 2n-4\end{aligned}$$

$$\therefore T_n = 2n-4$$

Find the difference between any two terms.

$$T_n = 2n-4 \Rightarrow T_{n+1} = 2(n+1)-4 = 2n-2$$

$$T_{n+1}-T_n = (2n-2)-(2n-4) = 2$$

Since the difference between consecutive terms is constant, this sequence must be an arithmetic progression.

14 An increase of 25% means each new year's population will be 125% of the previous year's population.

The sequence of populations is geometric, with $a = 24\ 000,\ r = 125\% = 1.25$.

If 2010 is T_1, 2030 is T_{31}.

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$$T_n = ar^{n-1}$$
$$T_{21} = 24\ 000 \times 1.25^{20}$$
$$= 2\ 082\ 000$$

The population in 2030 will be $2\ 082\ 000$.

16 $T_r = 2^r + 2r - 1$

$$S_n = \left(2^1 + 2(1) - 1\right) + \left(2^2 + 2(2) - 1\right) + \left(2^3 + 2(3) - 1\right) + ... + \left(2^n + 2n - 1\right)$$
$$= \left(2^1 + 1\right) + \left(2^2 + 3\right) + \left(2^3 + 5\right) + ... + \left(2^n + 2n - 1\right)$$
$$= \left(2^1 + 2^2 + 2^3 + ... + 2^n\right) + \left(1 + 3 + 5 + ... + (2n - 1)\right)$$

The first is a geometric series with $a = 2,\ r = 2$, the second is an arithmetic series with $a = 1,\ d = 2$.

$$S_n = \frac{a_1\left(r^n - 1\right)}{r - 1} + \frac{n}{2}\left(a_2 + l\right)$$
$$\frac{2\left(2^n - 1\right)}{2 - 1} + \frac{n}{2}\left(1 + 2n - 1\right)$$
$$= 2 \times 2^n - 2 + n^2$$
$$= 2^{n+1} + n^2 - 2$$

18 (a) From the table, an annuity of $\$1$ per year for 6 years at 4% per period will have a future value of $\$6.6330$.

This means that an annuity of $\$4500$ under these conditions will be worth $\$6.6330 \times \$4500 = \$29\ 848.50$.

(b) From the table, an annuity of $\$1$ per year for 5 years at 2% per period will have a future value of $\$5.2040$.

This means that an annuity of $\$700$ under these conditions will be worth $\$5.2040 \times \$700 = \$3642.80$.

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20 (a) 74 months at 0.005 gives a multiplication factor of $61.725\ 71$ from the table.

Present value of the annuity $= \$200 \times 61.725\ 71 \approx \$12\ 345.14$

(b) 5 years 11 months $= 5 \times 12 + 11 = 71$ months, $N = 71$.

9% p.a. is $\dfrac{0.09}{12} = 0.0075$ per month so $r = 0.0075$ as a decimal.

From the table, the value to be used is $54.892\ 93$.

Hence $\$19\ 000 = \$M \times 54.892\ 93$

$\$M = \$19\ 000 \div 54.892\ 93 = \$346.128...$

The repayments would be $\$347$ per month, rounded to the next dollar.

22 Working in centimetres:

Let the amount they need be $\$P$.

3.6% p.a. is $\dfrac{0.036}{12} = 0.003$ per month so $r = 0.003$ as a decimal.

$n = 5 \times 12 = 60$

$$\begin{aligned} A &= P(1+R)^n \\ 20\ 000 &= P \times 1.003^{60} \\ P &= 20\ 000 \div 1.003^{60} \\ &= 16709.906... \end{aligned}$$

To the nearest dollar, they must deposit $\$16\ 710$.

24 (a) $A_{12} = \dfrac{100 \times 1.06\left(1.06^{12} - 1\right)}{0.06} = 1788.21$

The accumulated value at the end of 12 years is $\$1788.21$

(b) The \$100 that was not invested at the beginning of the fifth year would become $\$100 \times 1.06^8$

$\$1788.21 - \$100 \times 1.06^8 \approx \$1628.83$

The accumulated value will be $\$1628.83$.

 ISBN 978 1 4886 1829 1

26 (a) $3+\frac{3}{\sqrt{3}+1}+\frac{3}{\left(\sqrt{3}+1\right)^2}+...$ is a geometric series with $a=3, r=\frac{1}{\sqrt{3}+1}$.

$$S_\infty=\frac{3}{1-\left(\frac{1}{\sqrt{3}+1}\right)}$$

Multiply numerator and denominator by $\sqrt{3}+1$.

$$\begin{aligned}S_\infty&=\frac{3}{\sqrt{3}+1-1}\\&=\frac{3\left(\sqrt{3}+1\right)}{\sqrt{3}}\\&=\frac{3\sqrt{3}+3}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{9+3\sqrt{3}}{3}\\S_\infty&=3+\sqrt{3}\end{aligned}$$

(b) $3+\frac{3}{\sqrt{3}-1}+\frac{3}{\left(\sqrt{3}-1\right)^2}+...$ is a geometric series with $a=3, r=\frac{1}{\sqrt{3}-1}$

A geometric series has a limiting sum if the common ratio $|r|<0$

$$\begin{aligned}r&=\frac{1}{\sqrt{3}-1}\\&=\frac{1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\\&=\frac{\sqrt{3}+1}{2}\end{aligned}$$

$$\frac{\sqrt{3}+1}{2}>0,\quad\therefore |r|>0$$

Therefore the series does not have a limiting sum.

 ISBN 978 1 4886 1829 1