HSC Mathematics Advanced

Topic 1: Series and Sequences

Arithmetic Sequences

Difference between successive terms is constant (common difference).

- $d = T_n - T_{n-1}$, $n \geq 2$
- $T_n = a + (n-1)d$ where $a = T_1$
  - Three numbers a, m, b form an AP when $m - a = b - m$, and thus when $m = \frac{1}{2}(a + b)$.
  - m is the arithmetic mean of a, b

Geometric Sequences

Ratio between successive terms is non-zero constant (common ratio).

- $\frac{T_n}{T_{n-1}} = r$, $n \geq 2$
- $T_n = ar^{n-1}$ where $a = T_1$
  - Three numbers a, g, b form an GP when $\frac{b}{g} = \frac{g}{a}$, and thus when $g^2 = ab$.
  - g is the geometric mean of a, b
  - 0 cannot be a term in a GP

Summing Sequences

- $S_n = T_1 + T_2 + T_3 + \ldots + T_n$ known as a partial sum.

Sigma can also be used to represent sums (not necessarily AP or GP):

- $\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + T_4 + \ldots T_n = S_n$

Sum of an AP

Let $l = T_n$ be the last term of an AP with $T_1 = a$ and difference d:

$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$

$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a$

Adding the two equations gets $2S_n = n(a + l)$ as there are n terms.

- $S_n = \frac{1}{2}n(a + l)$

Since $l = T_n = a + (n-1)d$, we can derive an alternative formula:

- $S_n = \frac{1}{2}n(2a + (n-1)d)$ and it is evident that:
- $S_n = \sum_{k=1}^{n} (a + (k-1)d)$

Sum of a GP

To find the sum of a geometric sequence:

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

Subtract the second equation from the first:

$\therefore (r-1)S_n = ar^n - a$

- $S_n = \frac{a(r^n - 1)}{r - 1}$ when $|r| > 1$

For when $|r| < 1$, subtract first equation from second:

$\therefore (1 - r)S_n = a - ar^n$

- $S_n = \frac{a(1 - r^n)}{1 - r}$ when $|r| < 1$

Limiting Sum

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, therefore

- $\lim_{n\to\infty} T_n = 0$
- $S_\infty = \lim_{n\to\infty} S_n = \frac{a}{1 - r}$

i.e. $\sum_{n=1}^{\infty} ar^{n-1}$ converges to $\frac{a}{1 - r}$

Recurring Decimals

If we want to express $1.1\dot{0}3\dot{7}$ as a fraction, simply write as GP:

$1.1 + (0.0037 + 0.0000037 + \ldots)$

Inside the brackets we find a limiting sum: $S_\infty = \frac{0.0037}{1 - 0.001}$

$\therefore S_\infty = \frac{1}{270}$ but we need to add 1.1 from before: $= \frac{149}{135}$

Topic 2: Financial Mathematics

Simple and Compound Interest

- Simple interest I is calculated with $I = PRn$, an AP.
  - P is principal, n is units of time, and R is interest rate per time.
  - If calculating total amount, add back the principal in the end.
- Compound interest is found by $A_n = P(1 + R)^n$, a GP.
  - A_n is final amount after n units of time.
  - To find only interest, subtract principal from final amount.
- Depreciation is similarly $A_n = P(1 - R)^n$, a GP
  - R is the rate of depreciation per unit of time.
  - To find only interest, subtract principal from final amount.

Annuities

- Compound interest investments from which equal payments are received on a regular basis for a fixed period of time.

E.g. Minho deposits \$200 per month at the start of each month in an annuity which pays 6%p.a. for 20 years. How much will it be worth after the full 20 years?

After 1 month, the account has $200(1 + 0.005)$ dollars.

After 2 months, $200(1.005)^2 + 200(1.005)$

After n months, we have $200(1.005^n + 1.005^{n-1} + \ldots + 1.005)$

GP in the brackets: $S_{20 \times 12} = \frac{1.005(1.005^{240} - 1)}{1.005 - 1} = 464.3511$

Therefore, $464.3511 \times 200 = \$92870.22$ in total after 20 years.

Future Value and Present Value

- Future value (FV) is the total value of the investment at the end of the term of investment, including all interest.
- Present value (PV) is the single lump of money that could be initially invested to yield a given future value over a given period.

Eg. What is PV of annuity seen above and how much does he save?

$92870.22 = PV(1.005)^{240} \rightarrow PV = \$28055.74$, if he pays in one lump sum he saves $200 \times 240 - 28055.74 = \$19944.26$.

*These notes assume you have mastered everything covered in my Preliminary Maths Notes, download at http://bit.ly/yukimath

Loan Repayments

Loans are usually paid off by regular instalments, with compound interest charged on the balance owing at any time.

- A_n = principal plus interest – instalments plus interest
  - The loan is paid off when A_n becomes zero.

Example: Michael takes out \$10000 to buy a car. He will repay the loan in five years, paying 60 equal monthly instalments, beginning one month after he takes out the loan. Interest is 6%p.a. compounded monthly. How much is the monthly instalment M?

$A_1 = 10000(1.005) - M$

$A_2 = (10000(1.005) - M)(1.005) - M = 10000(1.005)^2 - 1.005M - M$

$A_3 = (10000(1.005)^2 - 1.005M - M)(1.005) - M$

$= 10000(1.005)^3 - M(1 + 1.005 + 1.005^2)$ and then after a while,

$A_{60} = 0 = 10000(1.005)^{60} - M(1 + 1.005 + \ldots + 1.005^{59})$

GP inside brackets: $M = \dfrac{10000(1.005)^{60}}{\dfrac{1.005^{60} - 1}{0.005}} = \$193.33$

- Using future value formula above (much faster):

$A_n = 10000(1.005)^n - M(1 + 1.005 + \ldots + 1.005^{n-1})$

$10000(1.005)^{60} = M(1 + 1.005 + \ldots + 1.005^{59})$

GP in brackets: $S_{60} = \dfrac{1.005^{60} - 1}{0.005} = 69.77$

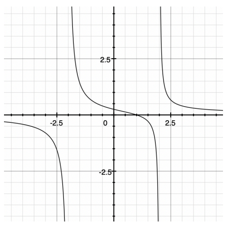
$\therefore M = \$193.33$

Topic 3: Graphs and Equations

Vertical and Horizontal Asymptotes

- If the denominator of a function has a zero at $x = a$, and the numerator is non-zero at $x = a$, then the vertical line $x = a$ is an asymptote.
- For rational functions (num. and den. both polynomials) dividing top and bottom by highest power of x in denominator reveals behaviour for large x:

e.g. for $y = \dfrac{x-1}{x^2-4} \to \dfrac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$



$$\lim_{x\to\pm\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}} = \frac{0}{1} = 0$$

$\therefore y = 0$ is horizontal asymptote.

e.g. for $y = \dfrac{e^x + e^{-x}}{2e^x}$ we wish to find $\displaystyle\lim_{x\to\pm\infty} \frac{e^x + e^{-x}}{2e^x}$

but we know $\displaystyle\lim_{x\to\infty} e^{-x} = 0$ and $\displaystyle\lim_{x\to-\infty} e^x = 0$

For $x \to \infty$ divide by e^x, for $x \to -\infty$ divide by e^{-x}

$\therefore \displaystyle\lim_{x\to\infty} \frac{1 + e^{-2x}}{2}$ and $\displaystyle\lim_{x\to-\infty} \frac{e^{2x} + 1}{2e^{2x}}$

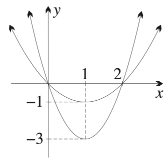
Horizontal asymptote at $y = \dfrac{1}{2}$ (but reaches infinity for $x \to -\infty$)

Shading Regions

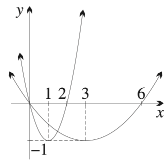
- Draw the curve, dotted line if excluded, solid if included. Then substitute points not on any boundary into inequation to determine where to shade.
- For $\cup$, only if both boundaries at a corner are unbroken is the corner point included.
- If $\cap$, if at least one boundary at a corner is unbroken, then the corner point is included.

Dilations

- To dilate graph vertically by factor of a, replace y with $\dfrac{y}{a}$, i.e.
  - $y = f(x) \to y = af(x)$, where axis of dilation is x-axis.
- To dilate graph horizontally by factor of a, replace x with $\dfrac{x}{a}$, i.e.
  - $y = f(x) \to y = f(\dfrac{x}{a})$, where axis of dilation is y-axis.
- Enlargements are when both dilations are applied, thus, for an enlargement by factor of a, $y = f(x) \to y = af(\dfrac{x}{a})$
- Reflections are dilations with factor -1



Vertical and horizontal dilation by factor of 3, respectively.



Transformations that Commute

- Two translations always commute, i.e. the translations on $y = x^2 \to y = (x+3)^2 - 4$ can be applied either way.
- Two dilations always commute, i.e. the translations on $x^2 + y^2 = 1 \to \dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ can be applied either way.
- Any two transformations with different axis commute (e.g horizontal translation and vertical dilation, or vice versa)
- Any other two transformations do NOT commute.
- Two steps can be taken in any order to convert $y = f(x) \to y = kf(a(x+b)) + c$:
  - A. Dilate horizontally by factor of $\dfrac{1}{a}$, then shift left b
  - B. Dilate vertically by factor k, then shift up c

Transformation of Trig Graphs

- $y = a \sin x$ and $y = a \cos x$ both have amplitude a.
- $y = \sin nx$ and $y = \cos nx$ have period $\dfrac{2\pi}{n}$
- $y = \tan nx$ has period $\dfrac{\pi}{n}$
- The initial phase angle (or phase) of a trig function is the angle when $x = 0$.
  - $y = \sin(x + \dfrac{\pi}{3})$ has phase $\dfrac{\pi}{3}$
- The mean value of a wave is the mean of its maximum and minimum values.
  - $y = \sin x + c$ has mean value c.

Topic 4: Graphing with Calculus

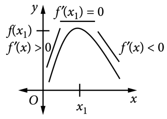
Sign of the Derivative

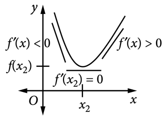
- If $\dfrac{dy}{dx} > 0$ as x increases, the function is increasing.
- If $\dfrac{dy}{dx} < 0$ as x increases, the function is decreasing.
- If $\dfrac{dy}{dx} = 0$, the point is a stationary point, and the reverse is true.
  - At stationary points, the curve is parallel to the x axis.

Local Maxima and Minima

- When function changes from decreasing to increasing ($f'(x)$ changes sign) at a stationary point, it is called a turning point.
- Turning points are classified as maxima or minima depending on whether it is above or below the curve around it, respectively.

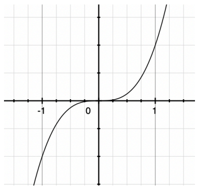
- For $A(a, f(a))$ on $y = f(x)$:
  - A is local maximum if $f(x) \leq f(a)$ for all x in some small interval around a.
  - A is local minimum if $f(x) \geq f(a)$ for all x in some small interval around a.
  - There is not necessarily a tangent at A.
- Discontinuities of y' occur on $y = f(x)$ if there is an asymptote at $x = a$, or if there is a cusp or vertical tangent at $x = a$





Second Derivative and Concavity

- If $f''(a) > 0$, the curve is concave up at $x = a$
- If $f''(a) < 0$, the curve is concave down at $x = a$
- Points of inflection is where the tangent crosses the curve, and also where the graph of $f''(x)$ changes sign. Determine using $f''(x) = 0$
- We can test whether stationary points are maxima or minima by using $f''(a)$ to see if it is concave up/down, revealing min/max.



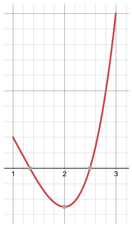
$y = x^3$ has a stationary point of inflection at $(0,0)$

Global Minima and Maxima

- For $A(a, f(a))$ on $y = f(x)$:
  - A is global maximum if $f(x) \leq f(a)$ for all x in the domain.
  - A is global minimum if $f(x) \geq f(a)$ for all x in the domain.
  - Global maxima/minima are also local maxima/minima.
- If the maxima/minima are open or $\pm\infty$, they do not exist.
- To find global maxima/minima, simply examine the turning points, behaviour of large x, and discontinuities of $f'(x)$.

Example 1: Find all min and max in specified domain:
$y = x^4 - 8x^2 + 11,\ 1 \leq x \leq 3$

Examining the graph, we see there is a local maximum at $x = 1$, global minimum at $x = 2$ and global maximum at $x = 3$.



Substituting into equation, we get local maximum of 4, global minimum of -5, and global maximum of 20.

Example 2: Sketch $f(x) = \dfrac{4x}{x^2+9}$

We can find $f'(x) = \dfrac{36-4x^2}{(x^2+9)^2}$ and $f''(x) = \dfrac{8x^3-216x}{(x^2+9)^3}$ and thus

stationary points are $\dfrac{36-4x^2}{(x^2+9)^2} = 0 \rightarrow 9 - x^2 = 0 \rightarrow x = \pm 3$

$f''(3) = -\dfrac{2}{27} < 0$, therefore $x = 3$ is a maximum.

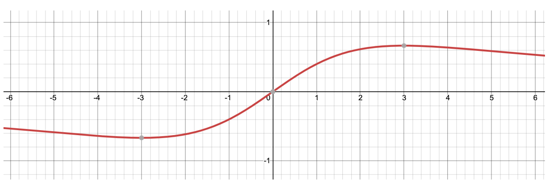
$f''(-3) = \dfrac{2}{27} > 0$, therefore $x = -3$ is a minimum.

Inflections: $\dfrac{8x^3-216x}{(x^2+9)^3} = 0 \rightarrow x^3 - 27x = 0 \rightarrow x = 0, \pm\sqrt{27}$ but since at those x values $f'(x) \neq 0$, they are true points of inflection.

To find horizontal asymptote: $\displaystyle\lim_{x\to\infty} \frac{4x}{x^2+9} = \frac{0}{1} = 0$

$\therefore y = 0$ is asymptote.

- TIP: For an order to approaching curve sketching problems, imagine the devil saying to your friend Safs: Do SIn SAFS!
  - Do for Domain, S for Symmetry, In for Intercepts, S for Sign, A for Asymptotes, F and S for 1st/2nd derivatives.



Minimisation and Maximisation Problems

- Form an equation with two variables, in which one is the quantity to be maximised and the other is the quantity to be varied, then find the global maximum or minimum.

Example: A cylinder has a volume (V) of 250cm^3, find the ratio between r and h that minimises the surface area (S) of the can:

$V = 250 = \pi r^2 h \rightarrow h = \dfrac{250}{\pi r^2}$

$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{500}{r}$

$\dfrac{dS}{dr} = 4\pi r - \dfrac{500}{r^2} = \dfrac{4\pi r^3 - 500}{r^2}$

To find stationary points, $\dfrac{4\pi r^3 - 500}{r^2} = 0 \rightarrow r = \dfrac{5}{\pi^{\frac{1}{3}}}$.

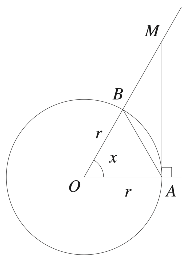
Since $\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{1000}{r^3} > 0$ for $r > 0$, point is global minimum.

Therefore, $h = \dfrac{250}{\pi} \times \dfrac{\pi^{\frac{2}{3}}}{25} = 2r$ and thus the best ratio is $2:1$.

Topic 5: Trigonometry

Trig Limits for Small Angles

Let x be an acute angle as shown, within a circle of centre O and radius r. Let tangent A meet radius OB at M. In $\triangle OAM$, $AM = r\tan x$ and it is evident that area$_{\triangle OAB}$ < area sector$_{OAB}$ < area$_{\triangle OAM}$



$\therefore \frac{1}{2}r^2 \sin x < \frac{1}{2}r^2 x < \frac{1}{2}r^2 \tan x$ and thus

$\sin x < x < \tan x$ for $0 < x < \dfrac{\pi}{2}$. Dividing by $\sin x$,

$1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$ and as $x \rightarrow 0^+$, $\cos x \rightarrow 1$, so

$\displaystyle\lim_{x\to 0^+} \frac{x}{\sin x} = 1$ but since $\dfrac{x}{\sin x}$ is even, $\displaystyle\lim_{x\to 0^-} \frac{x}{\sin x} = 1$ so

$\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\displaystyle\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} = 1$

- We have $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$
  - In other words, for small values of x, we have:
    - $\sin x \approx x$ and $\cos x \approx 1$ and $\tan x \approx x$.

Trigonometric Derivatives

- To find derivative of $\sin x$, we can use first principles:

$\dfrac{d}{dx}\sin x = \displaystyle\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$. Using identity proved in Y11 (3U):

$\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$, we get:

$\displaystyle\lim_{h\to 0} \frac{2\cos(\frac{2x+h}{2})\sin(\frac{h}{2})}{h} = \lim_{h\to 0} \cos(\frac{2x+h}{2}) \times \frac{\sin(\frac{h}{2})}{\frac{h}{2}}$

$= \cos x \times 1$ as $\displaystyle\lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} = 1$ as per small angle theorem.

- $\dfrac{d}{dx}\sin x = \cos x$ and $\dfrac{d}{dx}\sin f(x) = f'(x)\cos f(x)$

$y = \cos x = \sin(\frac{\pi}{2} - x) \rightarrow \dfrac{d}{dx}\sin(\frac{\pi}{2} - x) = -\cos(\frac{\pi}{2} - x)$

- $\dfrac{d}{dx}\cos x = -\sin x$ and $\dfrac{d}{dx}\cos f(x) = -f'(x)\sin f(x)$

$y = \tan x = \dfrac{\sin x}{\cos x} \rightarrow \dfrac{dy}{dx} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$ and simplifying:

- $\dfrac{d}{dx}\tan x = \sec^2 x$ and $\dfrac{d}{dx}\tan f(x) = f'(x)\sec^2 f(x)$

Reciprocal Trig Derivatives

$y = \csc x = \frac{1}{\sin x} \rightarrow \frac{dy}{dx} = -\frac{\cos x}{\sin^2 x}$ and thus we get:

- $\frac{d}{dx}\csc x = -\csc x \cot x$ and $\frac{d}{dx}\csc f(x) = -f'(x)\csc f(x)\cot f(x)$

$y = \sec x = \frac{1}{\cos x} \rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos^2 x}$ and thus we get:

- $\frac{d}{dx}\sec x = \sec x \tan x$ and $\frac{d}{dx}\sec f(x) = f'(x)\sec f(x)\tan f(x)$

$y = \cot x = \frac{1}{\tan x} \rightarrow \frac{dy}{dx} = \frac{-\sec^2 x}{\tan^2 x} = \frac{-1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x}$, thus:

- $\frac{d}{dx}\cot x = -\csc^2 x$ and $\frac{d}{dx}\cot f(x) = -f'(x)\csc^2 f(x)$

Topic 6: Log and Expo Functions

Logarithmic Dominance

- In a graph of $y = x\ln x$ as $x \rightarrow 0^+$, x dominates $\ln x$. This is also true for $y = \frac{\ln x}{x}$ as $x \rightarrow \infty$, with both limits $= 0$.
  - In general, x^k dominates $\ln x$ for $k > 0$.
- $\lim_{x\rightarrow 0^+} x\ln x = 0$ and $\lim_{x\rightarrow\infty}\frac{\ln x}{x} = 0$

Differentiation of Logarithms

Let $y = \ln x$, meaning $x = e^y \rightarrow \frac{dx}{dy} = e^y = x$

- Therefore $\frac{d}{dx}\ln x = \frac{1}{x}$ and in general $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

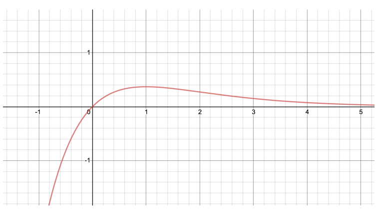
- To differentiate logarithms of non-e base, you can use change-of-base formula to convert to a multiple of a base-e logarithms, or use the standard derivative $\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$:

1. $\frac{d}{dx}\log_3 x = \frac{d}{dx}(\frac{\ln x}{\ln 3}) = \frac{1}{x} \times \frac{1}{\ln 3} = \frac{1}{x\ln 3}$
2. $\frac{d}{dx}\log_3 x = \frac{1}{x\ln 3}$

- Therefore, $\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{f(x)\ln a}$ in general.

Exponential Dominance

- In the graph of $y = xe^{-x}$, e^{-x} gets small as $x \rightarrow \infty$ faster than x gets large, and thus the graph as a whole gets small.
  - i.e. e^{-x} dominates x, and in general e^x dominates x^k for $k > 0$
- $\lim_{x\rightarrow\infty} x^k e^{-x} = 0$ and $\lim_{x\rightarrow-\infty} x^k e^x = 0$



$y = xe^{-x}$

Differentiation of Exponentials

- $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ as established in Y11.

- To differentiate non base-e exponential functions, we know that $e^{\log_e a} = a$ and thus $a^x = (e^{\log_e a})^x = e^{x\log_e a}$.

Differentiating, $\frac{d}{dx}a^x = \log_e a \times e^{x\log_e a} = (\ln a)a^x$

- In general, $\frac{d}{dx}a^{f(x)} = (\ln a)f'(x)a^{f(x)}$

Topic 7: Integration

Primitive Functions

- If $F(x)$ is a function such that $F'(x) = f(x)$, then it is called the anti-derivative or primitive of $f(x)$.
- A function can have an infinite number of primitives, all differing by a constant.

- Primitive of x^n is $\frac{x^{n+1}}{n+1} + C$ where C is a constant and $n \neq -1$
  - This rule is derived from $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$
  - C is usually only added at the final step, to avoid confusion.
- Similarly, primitive of $(ax+b)^n$ is $\frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$
  - This is derived from $\frac{dy}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n$

Indefinite Integrals

- $\int f(x)dx$ denotes the integration of $f(x)$ with respect to x.

For example, $\int (x+6)dx = \frac{x^2}{2} + 6x + C$

Similarly, $\int (2u^3 + u)du = \frac{u^4}{2} + \frac{u^2}{2} + C$

- In general, if $\frac{d}{dx}F(x) = f(x)$ then $\int f(x)dx = F(x) + C$

- We know $\frac{d}{dx}[(f(x)]^{n+1} = (n+1)f'(x)[f(x)]^n$ and rearranging,

$f'(x)[f(x)]^n = \frac{1}{n+1} \times \frac{dy}{dx} \rightarrow \int f'(x)[f(x)]^n dx = \frac{1}{n+1}\int \frac{dy}{dx} \times dx + C$

$\therefore \int f'(x)[f(x)]^n dx = \frac{y}{n+1} + C$ and since $y = [f(x)]^{n+1}$

- We get $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C,\ n \neq -1$
  - This is known as the reverse chain rule.
- No integral may cross an asymptote.

Logarithmic (Reciprocal) Integrals

- Since $\frac{d}{dx}\ln x = \frac{1}{x}$,
  - $\int \frac{1}{x}dx = \ln|x| + C$ as number in the log must be positive.
  - Where before we had $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$ we now have a result for $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$
  - $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$
  - $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

Exponential Integrals

- Since $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$,
  - $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$ and in general,
  - $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ through reverse chain rule.

- Since $a^x = e^{x\ln a}$, integrating both sides:

$\int a^x dx = \int e^{x\ln a}dx = \frac{1}{\ln a}\int (\ln a)e^{x\ln a}dx$ therefore

- $\int a^x dx = \frac{a^x}{\ln a} + C$

Standard Trigonometric Integrals

- Since $\dfrac{d}{dx}\sin x = \cos x$, $\dfrac{d}{dx}\cos x = -\sin x$, $\dfrac{d}{dx}\tan x = \sec^2 x$

we naturally get:

- $\displaystyle\int \cos x\,dx = \sin x + C$
- $\displaystyle\int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$

- If we wish to develop these by substituting x with $ax+b$:

$\dfrac{d}{dx}\sin(ax+b) = a\cos(ax+b) \to \displaystyle\int a\cos(ax+b)dx = \sin(ax+b)$

and then dividing by a (coefficient can be factored out of integrand),

- $\displaystyle\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$ and similarly,
- $\displaystyle\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$
- $\displaystyle\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$

Further Trigonometric Integrals

- In the case of $\displaystyle\int \cot x$, we can write as $\displaystyle\int \frac{\cos x}{\sin x}dx$, noticing that it is of the form $\dfrac{f'(x)}{f(x)}$, thus the integral is as follows:

- $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$ and substituting x with $ax+b$:

$\displaystyle\int \cot(ax+b)dx = \int \frac{\cos(ax+b)}{\sin(ax+b)}dx = \frac{1}{a}\int \frac{a\cos(ax+b)}{\sin(ax+b)}dx$

- $\displaystyle\int \cot(ax+b)dx = \frac{1}{a}\ln|\sin(ax+b)| + C$

- In the case of $\displaystyle\int \tan x$, we use a similar approach to attain:

- $\displaystyle\int \tan(ax+b)dx = -\frac{1}{a}\ln|\cos x| + C$

- Since we found the reciprocal trig derivatives, these are apparent:

- $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
- $\displaystyle\int \sec x\tan x\,dx = \sec x + C$
- $\displaystyle\int \csc x\cot x\,dx = -\csc x + C$ and substituting x with $ax+b$:

$\displaystyle\int a\csc^2(ax+b)dx = -\cot(ax+b) + C$ and dividing by a:

- $\displaystyle\int \csc^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$, similar for the rest,
- $\displaystyle\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$
- $\displaystyle\int \csc(ax+b)\cot(ax+b)dx = -\frac{1}{a}\csc(ax+b) + C$

Example: find $\displaystyle\int \sin x\cos^4 x\,dx$. This resembles reverse chain rule:

$\displaystyle\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$ with $f(x) = \cos x$ and $n = 4$

But $\dfrac{d}{dx}\cos x = -\sin x$, $\displaystyle\int \sin x\cos^4 x\,dx = -\int -\sin x\cos^4 x\,dx$

$\displaystyle = -\frac{1}{5}\int -5\sin x\cos^4 x\,dx = -\frac{1}{5}\cos^5 x + C.$

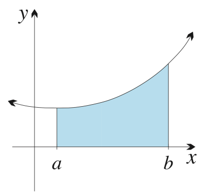
Example 2: find $\displaystyle\int \sin^2 x\,dx$. Since there is no $f'(x)$ anywhere we must alter it using double angle formula (Y11 3U):

$\cos 2x = 1 - 2\sin^2 x \to \sin^2 x = \dfrac{1-\cos 2x}{2}$

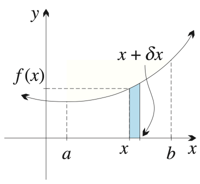
$\displaystyle\therefore \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{4}\int 2\cos 2x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

Definite Integrals

- We see a region contained between a curve $y = f(x)$ and another curve $y = 0$, from $x = a \to b$ where $a \le b$. Both curves should be continuous.



- To find the area of the shaded region, one approach is to cut up into infinite number of rectangles of infinitesimal width, the area of which is $\approx f(x)\delta x$.
- Thus the area of region $\displaystyle\approx \sum_{x=a}^{b} f(x)\delta x$ however as the strips have curved edges, we want to make the widths as thin as possible



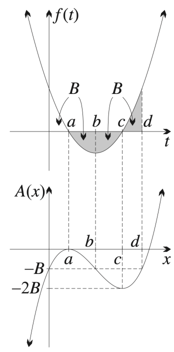
- Area $\displaystyle = \lim_{\delta x\to 0}\sum_{x=a}^{b} f(x)\delta x$ but instead we replace δx with dx, and use Leibnitz's notation:
- Area of shaded region $\displaystyle = \int_a^b f(x)dx$ which is known as the definite integral of the signed area under the curve.
  - $f(x)$ is the integrand, $x = a$ and $x = b$ are called the lower and upper bounds of the integral.
  - Only works if $f(x) \ge 0$ for all x. If the curve ever goes under, you must find where the curve cuts x-axis and consider the regions separately.

Fundamental Theorem of Calculus

- For $f(x)$ that is continuous in interval $[a, b]$, $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$ establishing that taking areas and taking tangents are inverse processes.
- Therefore, when integrating e.g. $\displaystyle\int_{-1}^{5}(2x-3)dx$ we first find the primitive (without C), then substitute the bounds into $F(x)$ as outlined in the fundamental theorem:

$\displaystyle\int_{-1}^{5}(2x-3)dx = \Big[x^2 - 3x\Big]_{-1}^{5}$ as $F(x) = x^2 - 3x$

$\therefore \Big[x^2 - 3x\Big]_{-1}^{5} = (25 - 15) - (1 + 3) = 6$



- $\displaystyle A(x) = \int_a^x f(t)dt$ is defined as the signed area function for $f(t)$, $A(x)$ can be seen on bottom.
- Since $f(t)$ is a parabola with axis of symmetry $x = b$, and the areas marked B are all equal: $A(a) = 0$, at $[a, b]$ $A(x)$ decreases at an increasing rate to $-B$, at $[b, c]$ $A(x)$ decreases at a decreasing rate to $-2B$, etc.
- $A(x)$ in $(-\infty, a]$ is negative as you are finding the integral $\displaystyle\int_{-\infty}^{a} f(t)dt$ which is increasingly negative.

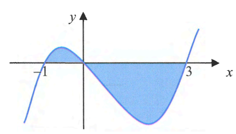
Example: Find area bounded by $y = x(x+1)(x-3)$ and x-axis.

Since $\displaystyle\int_0^3 (x^3 - 2x^2 - 3x)dx < 0$ and we want area.

$\displaystyle\therefore \int_{-1}^{0}(x^3 - 2x^2 - 3x)dx - \int_0^3 (x^3 - 2x^2 - 3x)dx$

$\displaystyle = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2}\right]_0^3$

$= \dfrac{71}{6}$ units^2



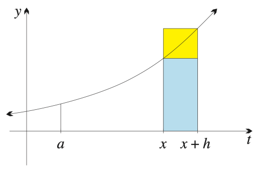
Proving the Fundamental Theorem

Proving Differentially:

We know $A'(x) = \lim_{h \to 0} \dfrac{A(x+h) - A(x)}{h}$ and since

$A(x+h) - A(x) = \displaystyle\int_x^{x+h} f(t)dt$

thus $A'(x) = \displaystyle\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$



As per the diagram, we see $f(t)$ is increasing for $[x, x+h]$ where the lower rectangle has height $f(x)$ and upper rectangle is $f(x+h)$.

$\therefore f(x)h \leq \displaystyle\int_x^{x+h} f(t)dt \leq f(x+h)h$ due to areas of rectangles.

$\therefore f(x) \leq \displaystyle\frac{1}{h}\int_x^{x+h} f(t)dt \leq f(x+h)$ and since $f(x)$ is continuous

$f(x+h) \to f(x)$ as $h \to 0$ and thus $\displaystyle\lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)dt = f(x)$

- Proving that $A'(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)dt = f(x)$

Proving Integrally:

We now know $F(x)$ and $\displaystyle\int_a^x f(t)dt$ are primitives of $f(x)$

and because any two primitives differ only by a constant,

$\displaystyle\int_a^x f(t)dt = F(x) + C$, subs. $x = a$, $\displaystyle\int_a^a f(t)dt = F(a) + C = 0$

$\therefore C = -F(a)$ therefore $\displaystyle\int_a^x f(t)dt = F(x) - F(a)$ and changing

$x = b$ and $t = x$ gives: $\displaystyle\int_a^b f(x)dx = F(b) - F(b)$ as seen last page

Properties of Definite Integrals

- If $f(x)$ is odd, then $\displaystyle\int_{-a}^a f(x)dx = 0$
- If $f(x)$ is even, then $\displaystyle\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$
- $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ no matter where c is.
- $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$ as $F(a) - F(b) = -(F(b) - F(a))$
- $\displaystyle\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$
- If $f(x) \leq g(x)$ in the interval $[a, b]$, $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Example: Find the area of $y^2 = x^2(1 - x^2)$

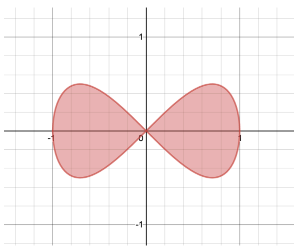
Curve is symmetric around x and y axis, as subbing $-y, -x$ yields the same equation.

$= 4\displaystyle\int_0^1 x\sqrt{1 - x^2}dx$

$= \dfrac{-4}{3}\displaystyle\int_0^1 -3x\sqrt{1 - x^2}dx$

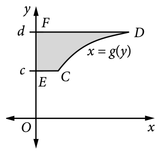
$= -\dfrac{4}{3}\left[(1 - x^2)^{\frac{3}{2}}\right]_0^1$

$= -\dfrac{4}{3}(0 - 1) = \dfrac{4}{3}\text{units}^2$



Areas Bounded by the y-axis

- To find the area bounded by $x = f(x)$ (or any function of y) and the abscissae at $y = c$ and $y = d$ is found by swapping the x and y variables:



- Area $= \displaystyle\int_c^d g(y)dy$

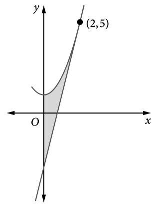
Example: Find the area between $y = x^2 + 1$, $y = 4x - 3$ and y axis.

$4x = y + 3 \to x = \dfrac{y+3}{4}$ and $x^2 = y - 1 \to x = \sqrt{y - 1}$

Total area is $\displaystyle\int_{-3}^5 (\frac{y}{4} + \frac{3}{4})dy - \int_1^5 \sqrt{y-1}dy$

$= \displaystyle\int_{-3}^5 (\frac{y}{4} + \frac{3}{4})dy - \frac{2}{3}\int_1^5 \frac{3}{2}\sqrt{y-1}dy$

$= \left[\dfrac{y^2}{8} + \dfrac{3x}{4}\right]_{-3}^5 - \dfrac{2}{3}\left[(y-1)^{\frac{3}{2}}\right]_1^5 = \dfrac{8}{3}\text{units}^2$



Areas of Compound Regions

- If $f(x) \leq g(x)$ in $[a, b]$, then area between is $\displaystyle\int_a^b (g(x) - f(x))dx$

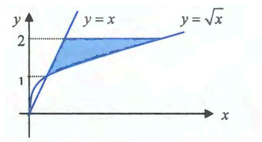
Example: find the shown area.

Converting to functions of y:

$x = y$ and $x = y^2, x > 0$

Area $= \displaystyle\int_1^2 (y^2 - y)dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$

$= \dfrac{7}{3} - \dfrac{3}{2} = \dfrac{5}{6}u^2$



Areas of Trig, Expo and Log Functions

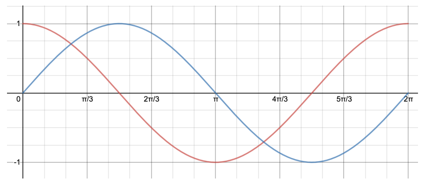
Trig: Find the area between $y = \cos\theta$ and $y = \sin\theta$ for $0 \leq \theta \leq 2\pi$

Finding the intersections:

$\cos\theta = \sin\theta$,

$\therefore \tan\theta = 1$

Intersections at $\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$



$A = \displaystyle\int_0^{\frac{\pi}{4}} (\cos\theta - \sin\theta) + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin\theta - \cos\theta) + \int_{\frac{5\pi}{4}}^{2\pi} (\cos\theta - \sin\theta)$

$= \left[\sin\theta + \cos\theta\right]_0^{\frac{\pi}{4}} - \left[\sin\theta + \cos\theta\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + \left[\sin\theta + \cos\theta\right]_{\frac{5\pi}{4}}^{2\pi}$

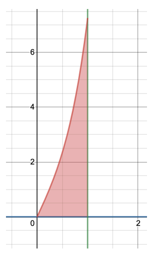
$= \dfrac{2 - \sqrt{2} + 4 + 2 + \sqrt{2}}{\sqrt{2}} = \dfrac{8}{\sqrt{2}} = 4\sqrt{2}u^2$

Expo: Find area between $y = e^{2x} - e^{-2x}$ and $y = 0$ and $x = 1$.

$\displaystyle\int_0^1 (e^{2x} - e^{-2x})dx = \frac{1}{2}\int_0^1 (2e^{2x} - 2e^{-2x})dx$

$= \dfrac{1}{2}\left[e^{2x} + e^{-2x}\right]_0^1 = \dfrac{e^2}{2} + \dfrac{e^{-2}}{2} - 1$

$= 2.76u^2$(2dp)

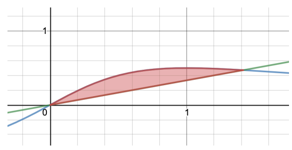


Log/Reciprocal: Find exact area between $y = \dfrac{x}{x^2 + 1}$, $y = \dfrac{x}{3}$ as shown.

$\dfrac{x}{x^2 + 1} = \dfrac{x}{3} \to x = 0, \pm\sqrt{2}$

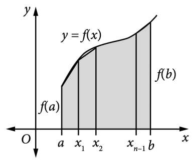
Area $= \displaystyle\int_0^{\sqrt{2}} \left(\frac{x}{x^2 + 1} - \frac{x}{3}\right)$

$= \dfrac{1}{2}\left[\ln|x^2 + 1| - \dfrac{x^2}{3}\right]_0^{\sqrt{2}} = \ln\sqrt{3} - \dfrac{1}{3}$ in exact form.

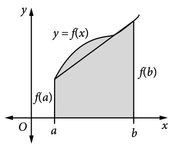


Trapezoidal Rule

- There are many integrals e.g. $\int \ln x \, dx$, $\int \frac{1}{x^4+1} dx$ for which the result is still incalculable. For these we can only approximate.
- If $f(x)$ is a continuous function and $f(x) \geq 0$ for $[a, b]$ then we can approximate $\int_a^b f(x)dx$ by dividing area into a number of trapezia (subintervals) of equal width.



- If there is one subinterval, obviously the approximation would be according to the area of a trapezium: $A = \frac{h}{2}(a+b)$
  - $\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b))$



- With two subintervals (three function points), $h = \frac{b-a}{2}$, and
  - $\int_a^b f(x)dx \approx \frac{h}{2}\left(f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right)$

This trend continues until we can generalise the formula for n subintervals, where $h = \frac{b-a}{n}$, $a = x_0$, and $b = x_n$:

- $\int_a^b f(x)dx \approx \frac{h}{2}(f(a)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(b))$

Example: Use trapezoidal rule with 4 trapezia to find $\int_0^{0.8} xe^{-x} dx$.

$0.8 \div 4 = 0.2$, each subinterval is 0.2. First creating a table of values:

x	0	0.2	0.4	0.6	0.8
$f(x)$	0	0.164	0.268	0.329	0.359

$\int_0^{0.8} xe^{-x}dx \approx 0.1(0+2(0.164)+2(0.268)+2(0.329)+0.359)$

$= 0.1881$

드디어 끝났다!!!!!

너무 피곤했다 x_x

Topic 8: Motion and Rates

Derivatives and Motion

- For x as the displacement of a particle from a fixed position, v its velocity and a as its acceleration,
  - Aver. velocity $= \frac{x_2 - x_1}{t_2 - t_1}$, instant. velocity $v = \frac{dx}{dt} = \dot{x}$
- Acceleration is defined as the rate of change of velocity,
  - Aver. acceleration $= \frac{v_2 - v_1}{t_2 - t_1}$, instant. accel. $a = \frac{dv}{dt} = \dot{v}$

Example: Displacement of particle moving along x-axis is given by $x = 1 + 3t^2 - t^3$, where x is in metres and t is seconds, $t \geq 0$.

a) Find initial position, velocity, accel. and explain the initial motion.

$v = 6t - 3t^2$, $a = 6 - 6t$ when $t = 0$, $x = 1$, $v = 0$, $a = 6$

Initially at rest from point $1m$, about to move further right.,

b) Find a and x when it comes to rest again. What happens after?

$v = 6t - 3t^2 = 0 \to 3t(2-t) = 0 \to t = 2$

$\therefore a = -6$, $x = 5$. Due to negative acceleration, will move to left.

c) What is the maximum velocity and when does it reach it?

When $\dot{v} = 6 - 6t = 0 \to t = 1$ and $\dot{a} = -6$, therefore concave down. $v_{max} = 3\text{m/s}$

Integrating Motion

- When a particle is moving along a straight line,
  - $v = \frac{dx}{dt}$, $\therefore x = \int v\,dt$
  - $a = \frac{dv}{dt}$, $\therefore v = \int a\,dt$

Example: Acceleration of article moving along straight line at t seconds is $a = 5 - 2t\,\text{cms}^{-2}$. We know $t = 0$, $x = 4$, $v = 0$

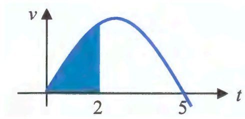
a) Find v and x in terms of t.

$v = \int (5-2t)dt = 5t - t^2 + C$ but $t = 0$, $v = 0$ therefore

$v = 5t - t^2$ and $x = \int (5t - t^2)dt = \frac{5t^2}{2} -$

b) Shade distance travelled in first two seconds in the velocity-time graph, and calculate this distance.

Velocity time graph is $v = t(t-5)$ for $t > 0$, and we shade in up to $x = 2$ as $\int v\,dt = x$:



$\int_0^2 (5t - t^2)dt = \left[\frac{5t^2}{2} - \frac{t^3}{3}\right]_0^2 = \frac{22}{3}\text{cm}$

Integrating Related Rates

- If we know $\frac{dQ}{dt}$ for a situation, the original function Q can be found through integration.

Example: If $\frac{dV}{dt} = 3e^{-0.02t}$ represents the rate of water flowing out during the first t days after time zero.

a) Find V as a function of t.

$V = \int 3e^{-0.02t}dt = -150\int -0.02e^{-0.02t}dt = -150e^{-0.02t} + C$

but when $t = 0, V = 0$ as we know the question only starts at $t = 0$ and so $0 = -150 + C$, thus $C = 150$. $\therefore V = -150e^{-0.02t} + 150$

b) How much water will flow from well during first 100 days?

When $t = 100$, $V = -150e^{-2} + 150 \approx 129.7\text{ML}$

c) Find $\lim_{t\to\infty} V$ and what % of total flow comes in the first 100 days.

$\lim_{t\to\infty} 150(1 - e^{-0.02t}) = 150$ and thus $\frac{129.7}{150} \approx 86.5\,\%$.

Topic 9: Introduction to Data

Different Types of Data

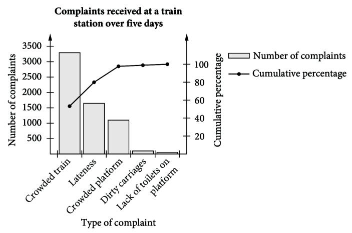
- Categorical data can be grouped into categories.
  - Nominal data has no special order, e.g. hair colour.
  - Ordinal data has some kind of order, e.g. army rank.
- Numerical data can be counted or measured.
  - Discrete data has a countable number of numerical values (not necessarily whole numbers), e.g. ages of students.
  - Continuous data has an infinite number of possibilities within a particular range, e.g. heights of students.
- Categorical data is usually represented by pie graphs or bar graphs, while numerical data is usually illustrated by histograms, stem and leaf plots, box plots or scatterplots.

Representing Data

- Frequency tables and cumulative frequency tables used for organising raw data, the latter of which is only for numeric data.

Mark	0	1	2	3	4	5	6	7	8	9	10	Sum
Tally		‖	‖‖	‖	|	卌 |	卌 |||	卌 ‖	卌 |	‖	‖	—
Frequency	0	2	4	2	1	6	8	7	6	2	2	40
Cumulative frequency	0	2	6	8	9	15	23	30	36	38	40	—

- Categorical data (sometimes discrete) can be represented by Pareto charts, which are usually used to identify the most urgent problems in a business.
  - First arrange categories into descending order of frequency, then add a cumulative frequency column.
  - The Pareto chart contains both a frequency histogram with columns arranged in descending order and a cumulative frequency polygon, with corresponding axes on both sides.



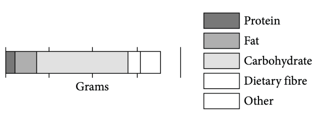
This example shows that the most salient issues are crowded and late trains, and that rectifying those two problems will solve approx. 80% of complaints

- Two-way tables (contingency tables) consist of two or more related frequency tables put together. They are used for conditional probability, and can be somewhat deceptive.
- Two-way tables can identify relations between two variables.

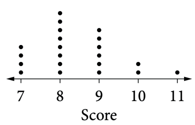
	Japanese	French	German	Total
Girls	6	5	4	15
Boys	2	7	6	15
Total	8	12	10	30

E.g. here we can see that someone studying French is more likely a boy, etc.

- Divided(/segmented) bar graphs are rectangles divided into lengths according to proportion of each category, including a key.



- Dot plots can be used for discrete and categorical data. Values are written along horizontal axis, then dots arranged vertically to represent the number of each category.
  - Good for identifying outliers.
  - Not good for data with > 10 values or frequencies.



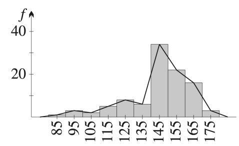
- Stem-and-leaf plots display discrete data, without losing any information.
- Typically the leaf contains the last digits of the values while stem contains all other digits. Stem is usually based on intervals of 10. Include a key to indicate the decimal point at which you are operating, e.g. this is a record of 100m times.

Stem	Leaf
10	9 9
11	0 1 4 5 6 8 9
12	0 1 1 1 2 4 5 6 9
13	0 0 1 1 2 2 2 2 3 4 6 7

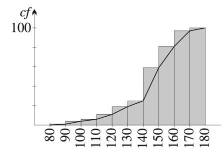
Key: 10|9 = 10.9

Grouping Data

- Grouping data reduces number of rows or columns on tables, and reduces the amount of clutter on graphs.
  - The class centre is the midpoint of the interval used in the grouping.
  - Grouping is a form of rounding, resulting in loss of detail and approximations of the raw data.
- A frequency histogram is a way of displaying frequency of numeric data.
  - Each rectangle is centred on the value or class centre.
  - A frequency polygon is drawn starting from the middle of the column before the first, and joins the centres of the top of each rectangle, ending on the horizontal axis at the middle of next column.

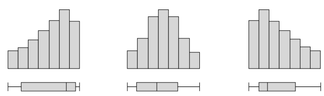


- Cumulative frequency histograms are drawn similarly to frequency histograms.
  - Rectangles pile on top of each other with each cumulative step
  - Cumulative frequency polygon (ogive) starts at zero at bottom left of first rectangle, passing the top right of each rectangle, finishing at the top right corner of the last rectangle.



Measures of Location (Central Tendency)

- Median is a measure of location, represented as Q_2 (2nd quartile).
  - For odd number of scores, median is middle score.
  - For even number, median is average of two middle scores.
- Mode is the most popular score, the simplest measure of location.
  - If 2,3 or more scores have same maximum frequency, we call the data bimodal, trimodal or multimodal.
- Mean is the arithmetic average, 3rd measure of central tendency.
  - For sample data, calculated with $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ for singular values, and $\bar{x} = \dfrac{\sum xf}{\sum f}$ for values with frequencies.
  - For grouped data, $\bar{x} = \dfrac{\sum x_m f}{\sum f}$ where x_m is class centre.
- Here we see a negatively skewed, symmetric, positively skewed distribution respectively.



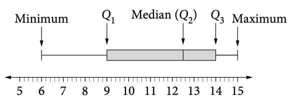
- In a skewed distribution, $\bar{x}$ is further down the tail than m (median), and in the symmetric distribution, $\bar{x}$, m are close together
  - Usually only IQR and median are referenced when talking about shape of distributions, as $\bar{x}$ and s are affected by outliers.
  - If there are no outliers, mean is the better measure of central tendency, otherwise median is better.
- Quartiles subdivide data into four quarters (there are only 3 quartiles; they bridge 4 quarters): Q_1, Q_2(median), Q_3.
- Deciles subdivide into 10 equal parts, and percentiles 100.

Measures of Spread

- Range is the simplest measure of spread, calculated $X_{max} - X_{min}$
    - Only defined for numeric data.
- In Year 11 we found the formula for sample variance:
  $s^2 = \sum (x - \bar{x})^2 f_r = \sum x^2 f_r - \bar{x}^2$ and substituting $f_r = \frac{f}{n}$,
    - $s^2 = \frac{\sum (x - \bar{x})^2 f}{n} = \frac{\sum x^2 f}{n} - \bar{x}^2$
- However, the formula for standard deviation is usually:
    - $\sigma = \sqrt{\frac{\sum (x - \mu)^2 f}{n}}$ but for samples,
    - $s = \sqrt{\frac{\sum (x - \bar{x})^2 f}{n - 1}}$, shown on calculators as σ_{n-1}.
    - This is because $s = \sqrt{\frac{\sum (x - \bar{x})^2 f}{n}}$ usually underestimates the population standard deviation for smaller sample sizes.
- Interquartile Range (IQR) is found by $Q_3 - Q_1$, representing the middle 50% of marks.
    - For odd number of scores, omit the middle score, splitting list into two sublists. Median of left and right list is Q_1, Q_3.
    - For even number of scores, divide through middle. Median of left and right list is Q_1, Q_3, respectively.
    - Outliers are usually defined as a score that is $< Q_1 - 1.5 \times IQR$ OR $> Q_3 + 1.5 \times IQR$
- The five-number summary is: minimum, Q_1, Q_2, Q_3, maximum.

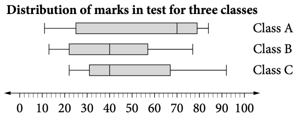
Comparing and Analysing Data

- Five number summary can be used to draw a box&whisker plot.
    - There must be a scale line underneath the box plot.
    - Whiskers should not extend to outliers. Outliers are instead marked with a * or ∘

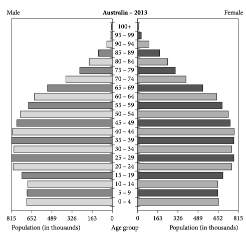


- Parallel box plots compare several box plots on the same scale, and allows us to compare skews of different data groups.

Distribution of marks in test for three classes



- A composite bar graph allows two data sets to be visually compared. For example, the graph on the right compares the same variable across two genders, and is sometimes referred to as a population pyramid.



- Back-to-back stem-and-leaf plots are able to compare two data sets by having a central stem with leaves moving away from the stem in either direction. Both sides should have the lowest leaf values nearest to the stem, thus the ordering is mirrored.

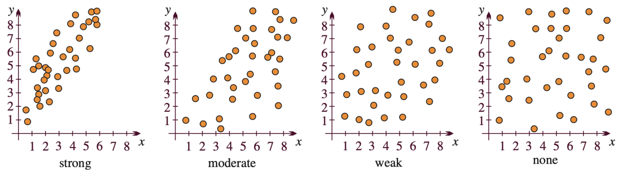
Hockey players' heights (cm)

Team A		Team B
9 8 5	15	1 4 6 7
6 5 4 4 1	16	0 8
9 8 8 6 5	17	2 4 4 6
	18	3 6
	19	0

Key: 17|9 = 179

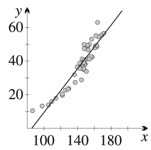
Scatterplots and Correlation

- Where there are two variables, one variable may influence the other, e.g. height has an influence on weight.
- Scatterplots provide a visual representation of any trend in the data, drawn by treating the bivariate data as a series of coordinate pairs and plotting on axes.
    - x-axis for independent var., y-axis for dependent variable.
- Correlation means a statistical relationship, and can be identified by looking at a scatterplot.
- Pearson's correlation coefficient r can be used to quantitatively calculated the correlation of a set of two variables.
    - Data falls between $-1 \leq r \leq 1$, where 1 and -1 are perfect correlation (a single line increasing and decreasing respectively)
    - $1 \leq r < 0$ is negatively correlated, roughly following a line with negative gradient, while $0 < r \leq 1$ is positively correlated, following a line with positive gradient.
    - A horizontal line of points, or clusters in the shape of horizontal lines tend to have 0 correlation.
- Data can sometimes be non-linearly correlated, such as when the trend follows a parabola, or exponential graph.



Line of Best Fit

- The line that correlated data best clusters around is called the line of best fit or regression line.
    - When plotting by eye, aim to have an equal number of points on both sides, with the total distance of points from the line on each side approximately equal.
    - Once an approximate line is produced, find y-intercept and gradient (by choosing two points) to attain the equation.



- Interpolation involves predicting further results within the range of the variables in the data, which can usually be deduced from the line of best fit.
- Extrapolation involves predicting further results outside the range of variables in the data, however the situation outside what is recorded can differ drastically, and thus extrapolation is risky.
- If events A and B are related, one does not necessarily cause the other. In most cases they are influenced by many external causes.

Pearson's Correlation Coefficient

- The correlation coefficient can be calculated by (for data (x, y))
    - $r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}$ or, through use of standard deviation formula, $r = \frac{\sum (x - \bar{x})(y - \bar{y})}{(n - 1)s_x s_y}$.
    - These values are not universal, but r values 0.6-10 represent a strong correlation; 0.4-0.6 is moderate; 0.1-0.4 is weak and 0.0-0.1 is virtually none.
    - Outliers have a major impact on r and must be removed before any calculation is done.

Least Squares Regression Line

- The least squares regression line is the line of best fit that minimises the sum of squares of the vertical distances from each plot to the line.
    - $y = mx + b$ where $m = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}$ and $b = \bar{y} - m\bar{x}$
    - This ensures the line passes through $(\bar{x}, \bar{y})$.
    - m also found by $m = r\frac{s_y}{s_x}$ where r is correlation coefficient.

Topic 10: Continuous Probability Distributions

Continuous Distributions and PDF

- When a variable can take any value in a particular interval, e.g. weight, height etc.
  - A continuous probability distribution is a probability distribution described by a probability density function.
- A continuous random variable is defined by its probability density function (PDF) over $[a, b]$ (usually $(-\infty, \infty)$), which is usually defined as a piecewise function with the following properties:
  - $f(x) \geq 0$ for all x.
  - $\int_a^b f(x)dx = 1$
- Probability is the area under the curve, i.e.

  $P(h \leq X \leq k) = \int_h^k f(x)dx$
  - Probability stays the same regardless of whether $\leq$ or $<$ is used, as $P(X = h) = \int_h^h f(x)dx = 0$ thus making no difference.
- If the density function is constant, the continuous distribution is uniform, and therefore if defined on $[a, b]$ the distribution thus has the equation $y = \frac{1}{b-a}$ as $\int_a^b f(x)dx = 1$.

Mean and Variance

- The mean and variance of a continuous probability density function is similar to that of a discrete one in that instead of summing all the values, we find the area underneath instead.
  - $\mu = \int_a^b xf(x)dx$
  - $\sigma^2 = \left(\int_a^b x^2 f(x)dx\right) - \mu^2$ from $E(X^2) - [E(X)]^2$

    Alternatively, $\sigma^2 = \int_a^b (x-\mu)^2 f(x)dx$ from $E\left((X-\mu)^2\right)$
  - Naturally, $\sigma = \sqrt{\left(\int_a^b x^2 f(x)dx\right) - \mu^2}$

Example: A particular continuous random variable has the PDF $f(x) = \begin{cases} \frac{1}{486}(81 - x^2), & 0 < x < 9 \\ 0, & \text{otherwise} \end{cases}$ hence find μ, σ and Q_2.

To find the mean: $\mu = \int_0^9 \frac{x}{486}(81 - x^2)dx = \left[\frac{81x^2}{972} - \frac{x^4}{1944}\right]_0^9$

$= \frac{27}{8}$ is the mean.

Finding the variance, $\int_a^b x^2 f(x)dx = \int_0^9 \frac{x^3}{486}(81 - x^2)dx$

$= \left[\frac{81}{1458}x^3 - \frac{1}{2430}x^5\right]_0^9 = \frac{81}{5}$ and we know $\mu^2 = \frac{729}{64}$ so:

$\sigma^2 = \frac{81}{5} - \frac{729}{64} = \frac{1539}{320} = 4.809375$

$\therefore \sigma = 2.193$

Median m is found by: $\int_0^m \frac{1}{486}(81 - x^2)dx = \frac{1}{2}$ as in half of all values are before m. Thus we get $\frac{1}{486}\left[81x - \frac{x^3}{3}\right]_0^m = \frac{1}{486}\left(81x - \frac{x^3}{3}\right) = \frac{1}{2}$

By testing values (questions in the exam most likely won't involve cubics and beyond so in the case of quadratics you could simply solve), we get $x = 3.1256$ as the only solution within the boundaries, and thus the median is 3.1256.

Cumulative Distribution Function

- The CDF (represented as $F(x)$) can be thought of as the area function of the PDF, that is, $F(x) = P(X \leq x) = \int_{-\infty}^x f(x)$ or more specifically $\int_a^x f(x)dx$ for a PDF defined in $[a, b]$.
  - Conversely, $f(x) = F'(x)$ in general.

To calculate the corresponding CDF from a PDF, we simply find the antiderivative of the PDF: e.g. $f(x) = \begin{cases} \frac{x}{16}, & 0 < x < 4\sqrt{2} \\ 0, & \text{otherwise} \end{cases}$ can be converted to its CDF counterpart for use for easily calculating probabilities without having to integrate constantly:

$$F(x) = \begin{cases} \frac{x^2}{32}, & 0 < x < 4\sqrt{2} \\ 0, & \text{otherwise} \end{cases}$$

and thus it follows that the median $P(X \leq 0.5)$ can now be found with $F(x) = \frac{x^2}{32} = 0.5$, obtaining $x = 4$ as the median.

The Standard Normal Distribution

- Normal distributions are commonly referred to as bell-shaped curves, and every normal distribution can be obtained from every other normal distribution by shifting and stretching.
- The graph of the standard normal distribution is:

  $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$
  - It is customary to use Z rather than X for the standard normal random variable.
- Most importantly, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz = 1$ and thus it is a PDF.
- When sketching $\phi(z)$, note these particular features:
  - y-intercept is $\phi(0) = \frac{1}{\sqrt{2\pi}} \approx 0.4$
  - $\phi(0)$ is a global maximum and therefore the mode.
  - $\phi(z)$ defined for $x \in \mathbb{R}$ and $\phi(z) > 0$ at all times.
  - Even function, and thus $\mu = 0$.
  - The z-axis is asymptotic in both directions.
  - Points of inflection at $\left(-1, \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}}\right)$ and $\left(1, \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}}\right)$.

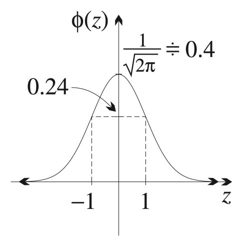
Proof for points of inflection: We know $\frac{d(e^{-\frac{1}{2}z^2})}{dx} = -ze^{-\frac{1}{2}z^2}$

And since $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$ is a constant multiple of $e^{-\frac{1}{2}z^2}$

$\phi'(z) = \frac{-z}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} = -z\phi(z)$ and finding second derivative:

$\phi''(z) = \frac{d}{dz}\left(-z\phi(z)\right) = -\phi(z) - z\phi'(z) = \phi(z)(z^2 - 1)$

Inflection points at $\phi''(z) = \phi(z)(z^2 - 1) = 0$, therefore at $z = \pm 1$.



Mean and Standard Deviation of SND

- To find the mean, we first start with $E(Z) = \int_{-\infty}^{\infty} z\phi(z)dz$ however since $y = z$ is an odd function and $y = \phi(z)$ is even, we know $y = z\phi(z)$ is odd, and therefore $E(Z) = 0$.

We can prove this more rigorously through the following method:
$\mu = \int_{-\infty}^{\infty} z\phi(z)dz = [-\phi(z)]_{-\infty}^{\infty}$ as $\phi'(z) = -z\phi(z)$ proven above.

We cannot substitute bounds of infinity, but we can calculate the limits, and thus we get $\lim_{z\to\infty}(-\phi(z)) - \lim_{z\to-\infty}(-\phi(z))$ but as the z-axis is asymptotic for large values of z, we get $0 - 0 = 0$

- To find the variance, we find $\int_{-\infty}^{\infty} z^2\phi(z)dz$ as $\mu = 0$.

We have proven $z^2\phi(z) = \phi''(z) + \phi(z)$ above, and hence:

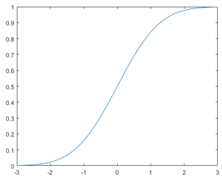
$$\int_{-\infty}^{\infty} z^2\phi(z)dz = \int_{-\infty}^{\infty} \phi''(z)dz + \int_{-\infty}^{\infty} \phi(z)dz$$

$$= [\phi'(z)]_{-\infty}^{\infty} + 1 = 0 + 1 = 1$$

- The variance σ^2 of $\phi(z)$ is 1, and standard deviation σ is also 1.

Estimating Probabilities on Normal Distribution

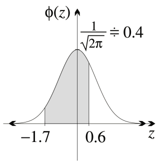
- The CDF of the standard normal distribution is denoted $\Phi(z)$.
- The curve $y = \Phi(z)$ has two asymptotes: $y = 0$ on the left and $y = 1$ on the right, with point symmetry in $(0,\ 0.5)$



	first decimal place									
z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257	0.7580	0.7881	0.8159
1.	0.8413	0.8643	0.8849	0.9032	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713
2.	0.9772	0.9821	0.9861	0.9893	0.9918	0.9938	0.9953	0.9965	0.9974	0.9981
3.	0.9987	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

- This table shows the value of $\Phi(z)$ for the corresponding z values up to 3.9, with the negatives omitted due to $\phi(z)$'s even symmetry.

Example: Find $P(-1.7 \le Z \le 0.6)$.
We wish to find $\Phi(0.6) - \Phi(-1.7)$
But we know $\Phi(-1.7) = 1 - \Phi(1.7)$
$\Phi(0.6) - \Phi(-1.7) = \Phi(0.6) - 1 + \Phi(-1.7)$
≈ 0.6811.



The Empirical Rule

- It is common to need the probabilities that a normally distributed variable is within 1, 2 or 3 standard deviations of the mean. This can be found easily as Z has $\sigma = 1$, and through using the table:
  - $P(-1 \le Z \le 1) \approx 68\%$
  - $P(-2 \le Z \le 2) \approx 95\%$
  - $P(-3 \le Z \le 3) \approx 99.7\%$

General Normal Distributions

- Should we want to dilate or translate the standard normal distribution $y = f(x)$, we can manipulate it similar to any other graph: replace x with $x - h$ when translating h to the right, and divide by b when dilating by b.
- However, in the case of $\phi(z)$, shifting h to the right brings μ to h as well, as it was previously 0, and similarly, dilating by b multiplies the standard deviation by b (as it was previously 1).
- However, in order to keep the function a PDF, we must divide the whole by σ in order to even it out so that $\int_{-\infty}^{\infty} f(x)dx = 1$
- Therefore, the general curve for a normal distribution PDF is:

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ where μ is mean and σ is S.D.

- Naturally, it follows that $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$
- For any normal distribution of this form, the mean, median and mode coincide, and points of inflection are at $x = \mu \pm \sigma$.
- Sometimes a continuous random variable with a normal distribution with mean μ and variance σ^2 may be written as $X \sim N(\mu, \sigma^2)$, in which case the standard normal distribution would be $Z \sim N(0,1)$.

Standardisation with Z-scores

- z-scores are useful for comparing results in different normal distributions, essentially comparing how much each score strays from the mean of their respective distribution in terms of standard deviation.
- To calculate a z-score: $z = \frac{x-\mu}{\sigma}$
- Consequently, to find a value from a z-score: $x = \mu + \sigma z$
- z-scores are negative for scores below the mean.

Example: On one test that followed the distribution $X \sim N(40,4)$ Juan obtained 45, and on another that followed the distribution $Y \sim N(72,25)$ he obtained 85. On which did he do better?

His z-score for the first test is $\frac{X-\mu}{\sigma} = \frac{5}{2} = 2.5$ while for the second test $\frac{Y-\mu}{\sigma} = \frac{10}{5} = 2$ and therefore we can conclude that he did better on his first test, as his z-score is higher.

you've just finished the 2u course! good job!