2019 Bored of Studies Trial Examinations

# Mathematics Extension 1

10th October 2019

**General Instructions**

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- NESA-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 14.

**Total Marks – 70**

**Section I** Pages 2 – 3

**10 marks**
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

**Section II** Pages 4 – 9

**60 marks**
- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

## Section I

**10 marks**
**Attempt Questions 1—10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1—10.

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**1** Consider three points on a line from left to right $A$, $B$ and $C$ where $AB$ is $m$ units in length and $BC$ is $n$ units in length. Which of the following statements is correct?

(A) $B$ divides $AC$ internally in the ratio $n : m$

(B) $B$ divides $CA$ internally in the ratio $m : n$

(C) $A$ divides $BC$ externally in the ratio $(m + n) : n$

(D) $C$ divides $AB$ externally in the ratio $(m + n) : n$

**2** A population is modelled based on the differential equation

$$\frac{dN}{dt} = k(N - b).$$

where $k$ and $b$ are constants, and $N$ is the size of the population at time $t$. Which of the following is NOT a possible solution to the differential equation, given some constant $a$?

(A) $N = ae^{kt} + b$ (B) $N = ae^{kt} - b$ (C) $N = b - ae^{kt}$ (D) $N = b$

**3** Consider the trigonometric equation

$$\frac{2\tan x}{1 + \tan^2 x} = \cos 2x.$$

Which of the following best represents the general solution to the equation for some integer $n$?

(A) $\dfrac{n\pi}{2} + \dfrac{\pi}{4}$ (B) $\dfrac{n\pi}{4} + \dfrac{\pi}{4}$ (C) $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$ (D) $\dfrac{n\pi}{4} + \dfrac{\pi}{8}$

**4** Which of the following best describes the set of solutions to the inequality below?

$$|x - 1| < \frac{1}{x - 1}$$

(A) $1 < x < 2$ (B) $0 < x < 2$ (C) $x < 0$ or $x > 1$ (D) $x < 0$ or $1 < x < 2$

**5** A polynomial $P(x)$ has remainder $x+3$ when divided by $x^2+2x+1$ and satisfies $P(-1) = 2$. Which of the following is NOT a possible remainder when $P(x)$ is divided by $(x+1)^3$?

(A) $5x+2$ (B) $x+3$ (C) $3x^2+7x+6$ (D) $4x^2+9x+7$

**6** Given that $x^2-2x+2$ is a factor of $x^4+4$, what is $\displaystyle\int_{-1}^{1} \frac{(x+1)^2+1}{x^4+4}\,dx$?

(A) $\tan^{-1}\left(\frac{1}{2}\right)$ (B) $\tan^{-1}(2)$ (C) $\dfrac{\pi}{4}$ (D) $\dfrac{\pi}{2}$

**7** Which of the following integrals is equivalent to

$$\int_{-2}^{-1} \frac{(x^2-1)^{\frac{3}{2}}}{x}\,dx$$

after applying the substitution $x = \sec\theta$?

(A) $\displaystyle\int_{\frac{\pi}{3}}^{\pi} \tan^4\theta\,d\theta$ (B) $\displaystyle\int_{\frac{2\pi}{3}}^{\pi} \tan^4\theta\,d\theta$ (C) $\displaystyle\int_{\frac{4\pi}{3}}^{\pi} \tan^4\theta\,d\theta$ (D) $\displaystyle\int_{\frac{5\pi}{3}}^{\pi} \tan^4\theta\,d\theta$

**8** The coefficient of $x^k$ in the binomial expansion of $(1-x)^n$ for positive odd integers $n$ is

$$1+2+3+\ldots+(n-1).$$

What is the value of $k$?

(A) $1$ (B) $2$ (C) $n-1$ (D) $n-2$

**9** Suppose that $2n$ students sitting the BoS Trial exam can completely fill two rows in the exam room. Each row contains $n$ students. There are two different versions of the exam. The exam papers are to be distributed such that no adjacent student (left/right or front/back) has the same version of the exam.

How many ways can be the students and exams be arranged in this way?

(A) $(n!)^2$ (B) $2\times(n!)^2$ (C) $(2n)!$ (D) $2\times(2n)!$

**10** The graph of $y = \sin^{-1}\sqrt{x}$ is increasing in the domain $0 < x < 1$. What is the value of the following definite integral?

$$\int_0^1 \sin^{-1}\sqrt{x}\,dx$$

(A) $\dfrac{\pi}{8}$ (B) $\dfrac{\pi}{4}$ (C) $\dfrac{\pi}{2}$ (D) $\pi$

## Section II

**60 marks**
**Attempt Questions 11—14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use the Question 11 Writing Booklet

(a) Suppose that $\alpha$, $\beta$ and $\gamma$ are angles such that **2**

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{\sin(\alpha+\beta+\gamma)}=\frac{\cos\alpha+\cos\beta+\cos\gamma}{\cos(\alpha+\beta+\gamma)}.$$

Show that

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\alpha+\gamma)=0.$$

(b) Two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ where $a > 0$. The tangents at $P$ and $Q$ intersect at the point $T(a(p+q), apq)$ (Do NOT prove this).

Suppose that $R(2ar, ar^2)$ is a point between $P$ and $Q$ which also lies on the parabola. The tangent at $R$ intersects the tangents at $P$ and $Q$ at the points $P'$ and $Q'$ respectively.

(i) Show that $PT = a|p-q|\sqrt{1+p^2}$. **1**

(ii) Show that $P'T = a|q-r|\sqrt{1+p^2}$. **1**

(iii) Hence show that for any position of $P$ and $Q$ on the parabola **3**

$$\frac{P'T}{PT}+\frac{Q'T}{QT}=1.$$

(c) The polynomial $P(x) = x^3 + 2x^2 - 3x - 1$ has roots $\alpha$, $\beta$ and $\gamma$. Find the value of **2**

$$\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta).$$

**Question 11 continues on page 5**

(d) Newton's method is used to find an approximation of a root of $f(x)$. Let $x_0$ be the initial approximation and $x_k$ be the approximation of the root after $k$ applications of Newton's method. After $(n+1)$ applications, $x_{n+1} = x_0$. Assume that none of the approximations are stationary points. Show that **1**

$$\sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)} = 0.$$

(e) Matthew tosses a fair coin $n$ times. Jonathan independently tosses another fair coin $n$ times.

(i) Explain why the probability that Matthew and Jonathan each toss exactly $k$ heads is **1**

$$\binom{n}{k}^2 \frac{1}{2^{2n}}.$$

(ii) Suppose that Matthew tosses $k$ heads and Jonathan tosses $n-k$ heads. Explain why the probability of this occurring is equal to the result in part (i). **1**

(iii) Hence, show that the probability that Matthew and Jonathan will toss exactly the same number of heads is **2**

$$\binom{2n}{n} \frac{1}{2^{2n}}.$$

(iv) Deduce that **1**

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

**End of Question 11**

**Question 12** (15 marks) Use the Question 12 Writing Booklet

(a) It can be shown that the curve $y = \dfrac{1}{\sqrt{1-x^2}}$ has a minimum turning point at $(0, 1)$ (Do NOT prove this). **2**

Use this information to sketch the graph of $y = \dfrac{1}{\sqrt{1-x^2}}$.

(b) Consider the function $f(x) = \sin^{-1}(2x^2 - 1)$.

(i) Find the domain of $f(x)$. **1**

(ii) Using the result in part (a), or otherwise, sketch the graph of $y = f'(x)$. **3**

(iii) Find the range of $f(x)$. **1**

(iv) Hence sketch the graph of $y = f(x)$. **2**

(c) A particle in a system of springs has the following acceleration equation

$$\frac{d^2x}{dt^2} = \begin{cases} -x\,, & \text{for } x < 2 \\ 4 - x\,, & \text{for } x \geq 2 \end{cases}$$

where $x$ is the particle's displacement at time $t$. The particle is initially at the origin with velocity $v_0$.

(i) Show that if $|v_0| < 2$ then the particle will exhibit simple harmonic motion. **2**

(ii) If $|v_0| < 2$, find the amplitude of the particle's motion in terms of $v_0$. **1**

(iii) Now suppose that the particle's initial velocity is given by $v_0 = 2$. Find the maximum displacement of the particle. **2**

(iv) Hence, describe the particle's displacement over time if $v_0 = 2$. **1**

**End of Question 12**

**Question 13** (15 marks) Use the Question 13 Writing Booklet

(a) Suppose that the graphs of $y = a^x$ and its inverse $y = \log_a x$ are tangent to each other at the point $(p, p)$ for some constant $0 < a < 1$. Let $m$ be the slope of the tangent of the graphs at $(p, p)$.

(i) Explain why $m^2 = 1$. **1**

(ii) Hence, find the values of $a$ and $p$. **3**

(iii) On the same set of axes, sketch the graphs of $y = a^x$ and its inverse $y = \log_a x$ for the value of $a$ found in part (ii). **2**

(b) A particle is projected at an initial speed of $V$ at an angle of $\theta$ to the horizontal. Let $g$ be the acceleration due to gravity. The displacement equations of motion at time $t$ are

$$x = Vt\cos\theta \qquad y = Vt\sin\theta - \frac{gt^2}{2} \qquad \text{(Do NOT prove this.)}$$

The particle has to clear two walls. One wall has a horizontal displacement of $d_1$ from the point of projection and vertical height of $d_2$ where $d_1 \neq d_2$. Another wall has a horizontal displacement of $d_2$ from the point of projection and vertical height of $d_1$.

(i) Show that the angle of projection $\theta$ needed for the particle to just clear the two walls satisfies **3**

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}.$$

(ii) Suppose that the angle of projection is chosen such that $\tan\theta \leq 3$. Show that the particle cannot clear the two walls for any values of $d_1$ and $d_2$. **2**

(c) Let $n$ be a positive integer.

(i) Show that $\left(1 + \dfrac{1}{n}\right)^n \geq 2$. **1**

(ii) Hence, or otherwise, prove by mathematical induction **3**

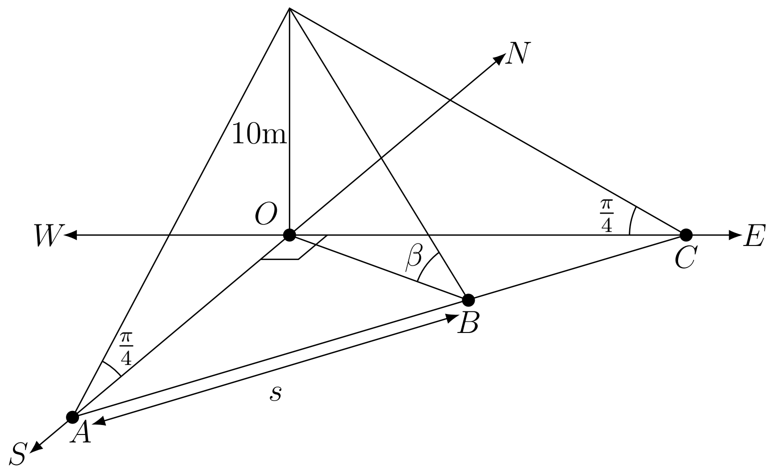
$$(2n-1)! \leq n^{2n-1}.$$

**End of Question 13**

**Question 14** (15 marks) Use the Question 14 Writing Booklet

(a) A monument of height 10 metres stands on point $O$ on the ground. A visitor's centre $A$ is located due south of the monument with an angle of elevation of $\frac{\pi}{4}$ to the top of the monument. Another visitor's centre $C$ is located due east of the monument also with an angle of elevation of $\frac{\pi}{4}$ to the top of the monument.

Visitor $B$ cycles directly from centre $A$ to centre $C$ in a straight line. Let $\beta$ be the angle of elevation of the cycling visitor to the top of the monument at time $t$.




Let $s$ be the displacement of the visitor from centre $A$ at time $t$ in metres. Let $\angle ABO = \theta$ where $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

(i) Show that $s = \dfrac{10\sin\left(\frac{3\pi}{4} - \theta\right)}{\sin\theta}$. **2**

(ii) Suppose that the visitor cycles from $A$ to $C$ at a fixed speed of 5 metres per second. Show that **2**

$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{\sqrt{2}}.$$

(iii) Hence, show that the rate of change of the visitor's angle of elevation is given by **3**

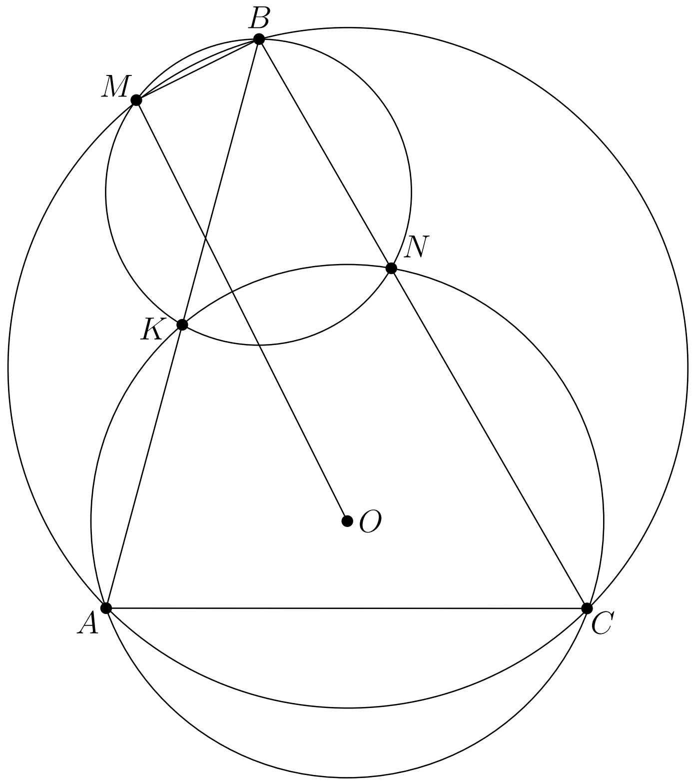
$$\frac{d\beta}{dt} = -\frac{\cos\theta}{2 + \operatorname{cosec}^2\theta}.$$

(iv) What is the largest possible angle of elevation that the visitor can have cycling from $A$ to $C$? Explain your answer. **1**

**Question 14 continues on page 9**

(b) Let $O$ be the centre of a circle $\mathscr{C}_1$ which passes through the vertices $A$ and $C$ of $\triangle ABC$. Let $K$ and $N$ be the points where this circle intersects $AB$ and $BC$ respectively.

Another circle $\mathscr{C}_2$ is drawn to pass through the points $B$, $K$ and $N$. A third circle $\mathscr{C}_3$ is also drawn to pass through the points $A$, $B$ and $C$. Let $M$ be the point of intersection of circles $\mathscr{C}_2$ and $\mathscr{C}_3$.




(i) Prove that $\triangle MKA$ and $\triangle MNC$ are similar. **2**

(ii) Let $S$ and $T$ be the midpoints of $AK$ and $CN$ respectively. Prove that $MSTB$ is a cyclic quadrilateral. **3**

(iii) Explain why $SBTO$ is a cyclic quadrilateral. **1**

(iv) Hence, deduce that $OM \perp BM$. **1**

**End of Question 14**

**End of paper**