2021 **BORED OF STUDIES TRIAL EXAMINATION**

2nd November

Mathematics Extension 1

General instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–11)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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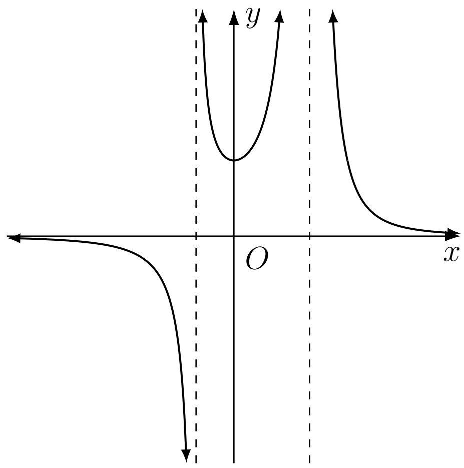
1 Consider the graph below of $y = \dfrac{1}{P(x)}$ where $P(x)$ is a cubic polynomial.



Which of the following is true for $P(x)$?

(A) $P(x)$ has three distinct roots

(B) $P(x)$ has a double root

(C) $P(x)$ has a triple root

(D) $P(x)$ has one real root only

2 Which of the following is equivalent to $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$?

(A) $\dfrac{1}{3}\cos^{-1}\left(\dfrac{3x}{4}\right) + c$

(B) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{3x}{\sqrt{16-9x^2}}\right) + c$

(C) $\dfrac{1}{4}\cos^{-1}\left(\dfrac{3x}{4}\right) + c$

(D) $\dfrac{1}{4}\tan^{-1}\left(\dfrac{3x}{\sqrt{16-9x^2}}\right) + c$

3 Let X be a random variable such that $X \sim \text{Bin}\left(100, \frac{1}{10}\right)$. Which of the following probabilities is closest to 0.95?

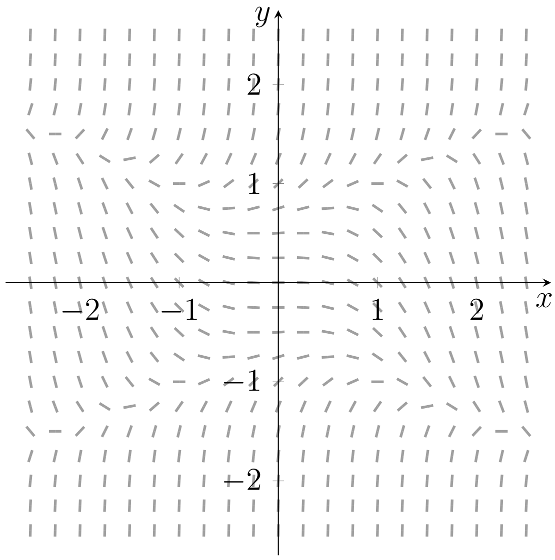
(A) $P(4 \le X \le 16)$

(B) $P(8 \le X \le 28)$

(C) $P(0 \le X \le 95)$

(D) $P(3 \le X \le 98)$

4 Which of the following differential equations best represents the following direction field?



(A) $\dfrac{dy}{dx} = y^2 - 4x^2$ (B) $\dfrac{dy}{dx} = y^4 - x^2$ (C) $\dfrac{dy}{dx} = 4y^2 - x^2$ (D) $\dfrac{dy}{dx} = y^2 - x^4$

5 A particle is moving in a straight line with a velocity v at time t. Its acceleration at time t is given by

$$\frac{dv}{dt} = a + bv$$

where a and b are non-zero constants.

Which of the following conditions are necessary for the velocity to follow an exponential decay over time with a negative limiting value?

(A) $a < 0$ and $b < 0$

(B) $a < 0$ and $b > 0$

(C) $a > 0$ and $b > 0$

(D) $a > 0$ and $b < 0$

6 Suppose that there are 4 students being ranked in a class, where ties are allowed. What is the probability that there will be two pairs of students tied in their rankings?

(A) $\dfrac{2}{24}$ (B) $\dfrac{2}{25}$ (C) $\dfrac{4}{25}$ (D) $\dfrac{4}{27}$

7 The differential equation $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$ is not separable in x and y.

Which of the following substitutions transforms this into a separable differential equation in u and x?

(A) $u = x + y$ (B) $u = x - y$ (C) $u = xy$ (D) $u = \dfrac{y}{x}$

8 Suppose that the numbers 1 to 10 are arranged in a circle. There will always exist n adjacent numbers from this circle with a sum of at least 28. What is the lowest possible value of n?

(A) 3 (B) 4 (C) 5 (D) 6

9 Suppose that $a \sin x + b \cos x$ can be written in the form $R \sin(x + \alpha)$ for non-zero values of a, b, R and α. If $ab < 0$, which of following pairs of quadrants is α most likely to be in?

(A) 1st and 2nd quadrant (B) 3rd and 4th quadrant

(C) 1st and 3rd quadrant (D) 2nd and 4th quadrant

10 A student randomly guesses the answers to 5 multiple choice questions. Each question has four choices and one correct answer.

Having glanced at the student's responses, a teacher tells the student she got at least one question correct.

Given this information, what is the probability that she got exactly 3 questions correct?

(A) $\dfrac{10}{27}$ (B) $\dfrac{45}{512}$ (C) $\dfrac{90}{781}$ (D) $\dfrac{243}{1064}$

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) By considering an appropriate binomial expansion, show that $10^n - 3^n - 7^n$ is divisible by 21. **1**

(b) Solve $\dfrac{\sqrt{2-x}}{x} < 1$. **3**

(c) Let $f(x) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$.

(i) Sketch the graph of $y = f'(x)$. **3**

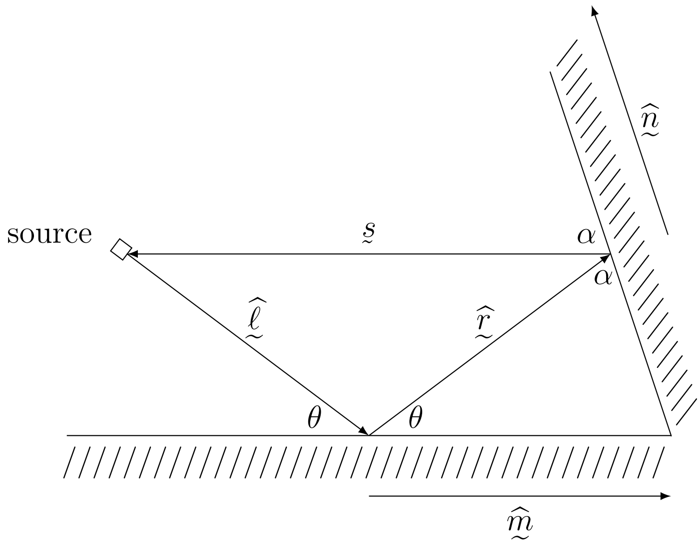
(ii) Hence, sketch the graph of $y = f(x)$ showing any asymptotes. **2**

(iii) The area bounded by the curve $f(x)$ for $x \geq 0$ and the line $y = \frac{\pi}{4}$ is rotated about the y-axis. Find the volume of the solid of revolution. **2**

Question 11 continues on page 6

(d) A laser beam is fired from a source into a mirror system shown in the diagram below.

The beam travels one metre before being reflected at an angle of θ by the first mirror. It then travels another one metre before being reflected at an angle of α relative to the second mirror. The second mirror is positioned in such a way that it reflects the beam back to the initial source.



Let

- $\widehat{\underset{\sim}{\ell}}$ be the unit vector representing the initial beam hitting the mirror;
- $\widehat{\underset{\sim}{r}}$ be the unit vector representing the reflected beam from the first mirror;
- $\underset{\sim}{s}$ be the vector representing the beam reflected back to its initial source;
- $\widehat{\underset{\sim}{m}}$ be a unit vector parallel to first mirror shown on the; diagram; and
- $\widehat{\underset{\sim}{n}}$ be a unit vector parallel to the second mirror shown on the diagram.

(i) Show that $\widehat{\underset{\sim}{r}} = 2(\widehat{\underset{\sim}{\ell}} \cdot \widehat{\underset{\sim}{m}})\widehat{\underset{\sim}{m}} - \widehat{\underset{\sim}{\ell}}$. **2**

(ii) Hence, show that $\underset{\sim}{s} \cdot \widehat{\underset{\sim}{n}} = \sin\dfrac{3\theta}{2} - \sin\dfrac{\theta}{2}$. **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Using trigonometric compound angle identities, show that **2**

$$\sin\left(\frac{\pi}{n}\right)+\sin\left(\frac{2\pi}{n}\right)+\sin\left(\frac{3\pi}{n}\right)+\cdots+\sin\left(\frac{n\pi}{n}\right)=\cot\left(\frac{\pi}{2n}\right).$$

(b) A triangle is formed by the following vectors: **2**

$$\underset{\sim}{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix},\quad \underset{\sim}{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix},\quad\text{and}\quad \underset{\sim}{a}+\underset{\sim}{b}=\begin{pmatrix}a_1+b_1\\a_2+b_2\end{pmatrix}.$$

By considering $\underset{\sim}{a}\cdot\underset{\sim}{b}$, show that the area of the triangle is given by

$$A=\frac{1}{2}|a_1b_2-a_2b_1|.$$

(c) A particle moves in the x-y plane at time t with the displacement vector

$$\underset{\sim}{r}(t)=\left(\frac{2at}{1+t^2}\right)\underset{\sim}{i}+\left(\frac{2at^2}{1+t^2}\right)\underset{\sim}{j}$$

where a is a non-negative constant and $0\leq t\leq T$ for some large value of T.

(i) Let A be the area of the triangle between the vectors $\underset{\sim}{r}(t)$ and $\underset{\sim}{r}(t+h)$ for some positive value h. Using result in part (b), show that **1**

$$A=\frac{h}{2}f(t)f(t+h)$$

where $f(t)=\dfrac{2at}{1+t^2}$.

(ii) Let $A=S(t+h)-S(t)$, for some function $S(t)$. Show that $S'(t)=\dfrac{2a^2t^2}{(1+t^2)^2}$. **1**

(iii) Given that $S(0)=0$, use the substitution $t=\tan\theta$ to find $S(t)$ when $t=T$. **3**

(iv) Hence, find the limiting value of $S(T)$ and explain how it relates to the Cartesian equation of the particle's path. **2**

(d) The polynomial $P(x)=x^3-3x^2+x+1$ has a root at $x=1$.

(i) Find the other two roots of $P(x)$. **2**

(ii) Let $x=\tan\theta$. By solving a trigonometric equation in terms of θ from $P(x)$, show that **2**

$$\tan\frac{\pi}{8}=\sqrt{2}-1.$$

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) A point $P(x, y)$ moves along the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for constants $0 < a < b$. **3**

Suppose that P is initially at the point $(a, 0)$ and then moves into the fourth quadrant of the x-y plane.

Use calculus to show that the length of OP is increasing over time until it reaches a maximum at the point $(0, -b)$.

(b) Let $X_1, X_2, X_3 \ldots, X_n$ be a sequence of independent Bernoulli random variables, each with the same unknown parameter p as the probability of success.

Suppose that after each trial, each Bernoulli random variable takes the particular values $X_1 = x_1, X_2 = x_2, X_3 = x_3, \ldots, X_n = x_n$ where at least one of the values is 0 and at least one of the values is 1.

Let L be the probability of obtaining this particular combination of outcomes.

(i) Show that $L = p^{n\bar{x}}(1-p)^{n(1-\bar{x})}$, where $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$. **1**

(ii) Hence show that for the given combination of outcomes, L is maximised when $p = \bar{x}$. **3**

(iii) Explain the significance of the result in part (ii), with respect to estimating the parameter p for a given combination of outcomes. **1**

(c) Let α, β, γ and δ be the roots of the polynomial **3**

$$P(x) = \binom{n}{0}x^4 - \binom{n}{1}x^3 + \binom{n}{2}x^2 - \binom{n}{3}x + \binom{n}{4},$$

for some integer $n \geq 4$.

Show that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = n$.

Question 13 continues on page 9

(d) At a small enclosure in a reptile zoo, Ms. Drizzle has $(n+1)$ reptiles, where $n > 3$. This enclosure consists of n lizards, each of a different colour, and a chameleon which has the ability to change colour.

Suppose that if the chameleon is kept in the same tank as a group of lizards, then overnight it will change colour to one of them.

For tomorrow, Ms. Drizzle needs to demonstrate a separate pair of reptiles to each of the two schools visiting. She selects 4 reptiles from the enclosure at random and places them in a tank overnight.

On the next day, she splits the reptiles into two different pairs such that no pair has the same colour.

(i) Show, giving reasons, that there are $3\binom{n+1}{4}$ possible pairings of colours that Ms. Drizzle can bring to demonstration tomorrow. **3**

(ii) By considering combinations of multiple pairs of lizards, deduce that **1**

$$3\binom{n+1}{4} = \binom{\binom{n}{2}}{2}.$$

End of Question 13

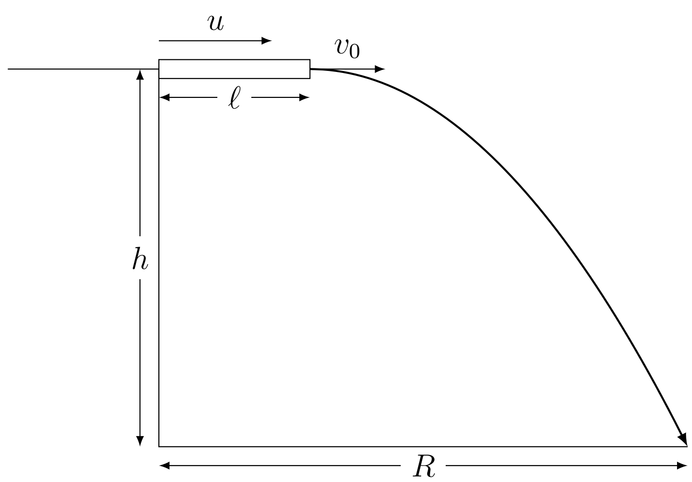
Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) A particle moves in a straight line in the positive direction with an initial velocity of u at the origin. Suppose that the particle is decelerating at a rate of a, where a is a positive constant. Let v and x be the particle's velocity and displacement at time t. **1**

It can be shown that $\dfrac{dv}{dt} = \dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$. (Do NOT prove this)

Show that when the particle has travelled a distance of ℓ then $v = \sqrt{u^2 - 2a\ell}$.

(b) A particle is launched horizontally from a cliff of height h with an initial speed u. At the edge of the cliff, the particle enters a pipe of length ℓ.



The pipe decelerates the particle at a rate of a, where a is a positive constant before it exits the pipe at a speed of v_0 and falls to the bottom of the cliff.

Let g be the acceleration due to gravity. The displacement vector of the particle at time t, from the ground directly below its exit of the pipe, is given by

$$\underset{\sim}{r} = v_0 t\underset{\sim}{i} + \left(h - \frac{gt^2}{2}\right)\underset{\sim}{j}. \qquad \text{(Do NOT prove this)}$$

(i) Let R be the horizontal distance from the base of the cliff to where the particle lands. Using the result in part (a), show that **1**

$$R = \ell + \sqrt{\frac{2h(u^2 - 2a\ell)}{g}}.$$

(ii) Suppose that the deceleration caused by the pipe is such that $a \geq u\sqrt{\dfrac{g}{2h}}$. **3**

Find the length of the pipe which maximises R. Justify your answer.

Question 14 continues on page 11

Question 14 (continued)

(c) It can be shown that for $\alpha\beta > -1$ that **2**

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right) \quad \text{(Do NOT prove this).}$$

Let $S = \tan^{-1}\left(\frac{2}{1^2}\right) + \tan^{-1}\left(\frac{2}{2^2}\right) + \tan^{-1}\left(\frac{2}{3^2}\right) + \cdots + \tan^{-1}\left(\frac{2}{n^2}\right)$.

Use the given result to show that

$$\lim_{n\to\infty} \tan S = -1.$$

(d) Suppose that for some positive integer n

$$a_n = T_{n,0} + T_{n,1} + T_{n,2} + \cdots + T_{n,n}$$

$$b_n = T_{n,1} + 2T_{n,2} + 3T_{n,3} + \cdots + nT_{n,n}$$

where $T_{n,k} = \binom{2k}{k}\binom{2(n-k)}{n-k}$ is the $(k+1)^{\text{th}}$ term of a_n.

(i) Show that **1**

$$b_n = nT_{n,0} + (n-1)T_{n,1} + (n-2)T_{n,2} + \cdots + T_{n,n-1}.$$

(ii) Deduce that $b_n = \dfrac{n}{2}a_n$. **1**

(iii) Show that **4**

$$b_n = 4b_{n-1} + 2a_{n-1}.$$

(iv) Hence, prove by mathematical induction that $a_n = 4^n$. **2**

End of paper