2018 Bored of Studies Trial Examinations

Mathematics Extension 1

8^{th} October 2018

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 2 – 4

10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 12

60 marks
- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Section I

10 marks
Attempt Questions 1—10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1—10.

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1 Which of the following best describes the solution set of the inequality below?

$$\left|\frac{x^2+4}{x}\right| < 4$$

(A) $x > 0$

(B) $x < 0$

(C) All real x except $x = 0$

(D) No real solution

2 What is the minimum value of $\sin^2\theta + \cos^2\theta + \tan^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta$?

(A) 1

(B) 3

(C) 5

(D) 7

3 Which of the following values of n does the binomial expansion of $\left(x + \dfrac{1}{x^2}\right)^n$ have a non-zero constant term?

(A) 2015

(B) 2016

(C) 2017

(D) 2018

4 Which of the following conditions ensures that the graphs of $y = \tan^{-1}(ax)$ and $y = bx$ have three points of intersection?

(A) $0 < a < b$

(B) $a < 0 < b$

(C) $a < b < 0$

(D) $b < 0 < a$

5 The normal to the point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ has the equation

$$x + py = 2ap + ap^3.$$

How many possible normals can pass through the point $(0, ka)$, where $k > 2$?

(A) 0

(B) 1

(C) 2

(D) 3

6 Consider the definite integral

$$\int_0^1 \frac{dx}{a^2 + b^2x^2}\, dx.$$

Which of the following conditions will give the largest value of the integral?

(A) $a = \dfrac{b}{2}$

(B) $a = b$

(C) $a = 2b$

(D) $a = 3b$

7 The sum of the solutions to $\sin x = \dfrac{1}{2}$ in the domain $\mathcal{D}$ is 3π. Which of the following is a possible domain $\mathcal{D}$?

(A) $-\pi \leq x \leq \pi$

(B) $-2\pi \leq x \leq 2\pi$

(C) $-3\pi \leq x \leq 3\pi$

(D) $-4\pi \leq x \leq 4\pi$

8 The points of intersection of the chord $x + by + c = 0$ and parabola $y^2 = 4ax$ subtend a right angle at the origin. Which of the following is true?

(A) $4a + c = 0$

(B) $a + c = 0$

(C) $4ab + c = 0$

(D) $4a + 4ab + c = 0$

9 What is the maximum value of the following?

$$\sin(x) + \sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{2\pi}{3}\right)$$

(A) 1

(B) 2

(C) 3

(D) 4

10 A bag contains an equal number of black and white marbles. Three marbles are drawn, with replacement. What is the probability of getting at most one black marble?

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) $\frac{1}{5}$

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Use the substitution $x = \cos 2\theta$ to find $\displaystyle\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$. **3**

(b) The kinetic energy K of an object of mass m and velocity v at time t is given by **1**

$$K = \frac{1}{2}mv^2.$$

Let a be the acceleration of the object. Show that

$$\frac{dK}{dt} = mav.$$

(c) Let $P(x) = ax^3 + bx^2 + cx + d$ represent any cubic polynomial with three real roots x_1, x_2 and x_3. Show that the point of inflexion of $P(x)$ occurs at **2**

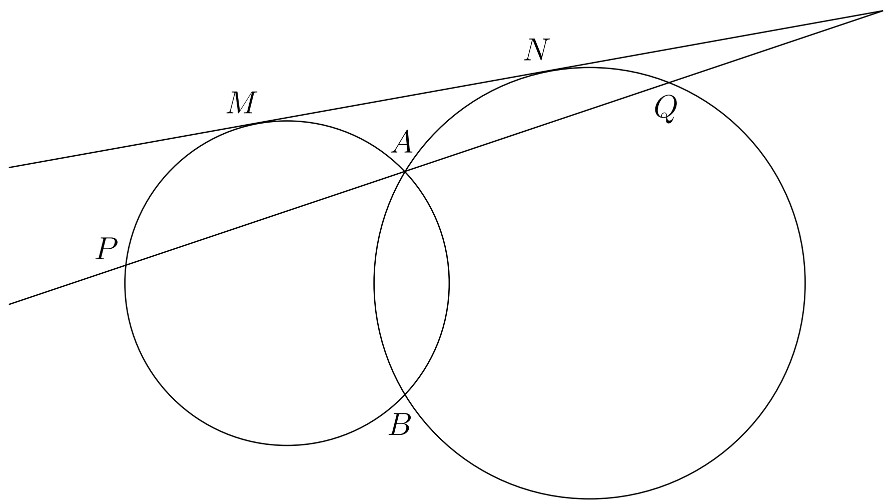
$$x = \frac{x_1 + x_2 + x_3}{3}.$$

(d) In the domain $a \leq x \leq b$, a polynomial $P(x)$ has a double root at $x = \alpha$, and is concave up. Suppose that x_n is the n^{th} application of Newton's method to find an approximation of the root α within the domain. Show that **3**

$$\left| \frac{x_n - \alpha}{x_{n-1} - \alpha} \right| < 1.$$

Question 11 continues on page 6

(e) The diagram below shows two circles intersecting at A and B. **3**



A line drawn through A cuts the circles at P and Q. A common tangent is drawn and touches the circles at M and N. Prove that $\angle PBQ = 2 \times \angle MBN$.

(f) It can be shown that for positive values a and b **3**

$$\tan^{-1} a + \tan^{-1} b = \begin{cases} \tan^{-1}\left(\dfrac{a+b}{1-ab}\right) & \text{if } ab < 1 \\ \tan^{-1}\left(\dfrac{a+b}{1-ab}\right) + \pi & \text{if } ab > 1 \end{cases}$$

(Do NOT prove this)

Use this result to show that for any $x > 0$, $y > 0$ and $z > 0$

$$\tan^{-1}\sqrt{\frac{x(x+y+z)}{yz}} + \tan^{-1}\sqrt{\frac{y(x+y+z)}{xz}} + \tan^{-1}\sqrt{\frac{z(x+y+z)}{xy}} = \pi.$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) Let k be a positive constant. The function

$$f(x) = x + \frac{k^2}{x},$$

has stationary points $(k, 2k)$ and $(-k, -2k)$. **(Do NOT prove this)**

(i) Determine the nature of the stationary points. **1**

(ii) Deduce the range of $f(x)$. **1**

(b) Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be points on the parabola $x^2 = 4ay$ such that pq is a fixed positive constant. It can be shown that the equation of the tangent at P is

$$y = px - ap^2 \qquad \textbf{(Do NOT prove this)}$$

Let R be the point of intersection of the tangents at P and Q.

(i) Show that the coordinates of R are $(a(p+q), apq)$. **2**

(ii) Hence, using the result in part (a) or otherwise, sketch the locus of R. **2**

(c) The letters of the word 'ESPRESSO' are used to create four-letter codes. How many possible distinct four-letter codes can be formed? **3**

(d) By expanding $(a - b)^n$ for appropriate values of a, b and n, show that $7^{100} - 3^{100}$ is divisible by 1000. **2**

Question 12 continues on page 8

(e) A particle moves along the xy-plane with the x and y-components respectively satisfying the equations.

$$\dot{x} = x + 1$$
$$\dot{y} = 3y$$

Initially, the particle is at the point $(0, 1)$ and the initial velocities are $\dot{x} = 1$ and $\dot{y} = 3$.

(i) Show that the displacement equation for x is $x = e^t - 1$. **1**

(ii) By finding an appropriate expression for the displacement equation for y, show that $y = (x + 1)^3$. **2**

(iii) Hence, sketch the path of the particle on the number plane. **1**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) Newton's law of cooling states that the rate of change of the temperature $T(t)$ of an object at time t can be expressed as **1**

$$\frac{dT}{dt} = -k(T - R),$$

where k and R (the room temperature) are positive constants.

Verify that $T(t) = R + (T(0) - R)e^{-kt}$ satisfies the above differential equation.

(b) A student is making a cup of coffee. Assume that the temperature of the coffee can be modelled by Newton's law of cooling as described in part (a) for the same constants k and R. For a given time t, let

- $C_0(t)$ be the temperature of the coffee if no milk added
- $C_1(t)$ be the temperature of the coffee if the student initially adds milk
- $C_2(t)$ be the temperature of the coffee if the student adds milk later at $t = t_0$

(i) When the student adds milk to the coffee initially, then the immediate temperature of the coffee is given by **1**

$$C_1(0) = \alpha C_0(0) + (1 - \alpha)M(0),$$

where $M(t)$ is the temperature of the milk at time t and $0 < \alpha < 1$ is a constant.

Write down the equation of $C_1(t)$ in terms of $R, C_0(0), M(0), k, \alpha$ and t.

(ii) When the student waits and adds milk to the coffee at time $t = t_0$ instead, then the temperature of the coffee at this time is given by **2**

$$C_2(t_0) = \alpha C_0(t_0) + (1 - \alpha)M(t_0).$$

Show that

$$C_1(t_0) - C_2(t_0) = (1 - \alpha)\left[R(1 - e^{-kt_0}) - M(t_0) + M(0)e^{-kt_0}\right].$$

(iii) Suppose that the temperature of the milk also satisfies Newton's law of cooling for $M(t) < R$ with the differential equation **2**

$$\frac{dM}{dt} = -k_M(M - R).$$

Show that if $k_M < k$ then $C_1(t_0) > C_2(t_0)$.

(iv) Explain the physical significance of the result in part (iii). **1**

Question 13 continues on page 10

(c) Prove by mathematical induction that for integers $n \geq 4$, **3**

$$\ln\left[(n+2)!\right] > n+2$$

(d) A function $f(x)$ is defined to have a period of T if **1**

$$f(x) = f(x+T).$$

Suppose that two functions $g(x)$ and $h(x)$ have periods of T and $2T$ respectively. Show that $g(x) + h(x)$ has a period of $2T$.

(e) Two particles A and B interact with each other such that their equations of motion at time t can be written as

$$5\ddot{x}_A + 5x_A + 3x_B = 0,$$

$$5\ddot{x}_B + 5x_B + 3x_A = 0,$$

where x_A and x_B represent the displacement of the particles A and B respectively.

(i) Let $y_1 = x_A + x_B$ and $y_2 = x_A - x_B$. Show that y_1 and y_2 each satisfy the acceleration equation for simple harmonic motion. **2**

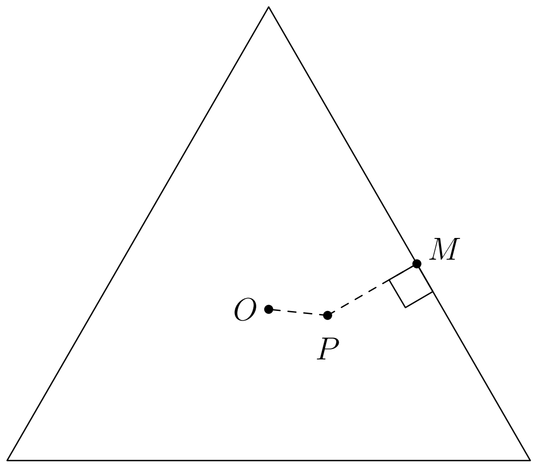
(ii) Use the result in part (d) to find the period of x_A. **2**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) (i) Find the area of the region bounded by the curve $y = x^2\sqrt{3} + \dfrac{1}{4\sqrt{3}}$ and the line $y = \dfrac{x}{\sqrt{3}} + \dfrac{1}{2\sqrt{3}}$ for $x < 0$. **3**

(ii) A dart is randomly thrown at a triangular board with unit side-length. Let O be the centre of the board and P be the point where the dart lands on the board. Let M be the foot of the perpendicular of P to the nearest edge of the board, as shown in the diagram below. **3**



Consider the region $\mathcal{R}$ within the board such that any point P in $\mathcal{R}$ is closer to the centre than the edge.

Using part (i), or otherwise, show that the ratio of the area of $\mathcal{R}$ to the total area of the board is $\dfrac{5}{27}$, and explain the significance of this result.

Question 14 continues on page 12

(b) A particle is launched from the ground with initial speed V and acute angle θ from the horizontal. Let g be the acceleration due to gravity.

You may assume the particle has the following Cartesian equation for its trajectory

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

At the same time, a particle is launched from the same point with the same speed at an angle of projection of $\theta+\alpha$ where $\alpha > 0$. Let P be the point where the *trajectories* of the two particles intersect, but the particles do not necessarily collide.

(i) Find the coordinates of P in terms of V, θ, α, and g. **3**

(ii) Show that as $\alpha \to 0$, P approaches **2**

$$\left(\frac{V^2\cot\theta}{g}, \frac{V^2(1-\cot^2\theta)}{2g}\right).$$

(iii) As θ varies and $\alpha \to 0$, find the locus of P. **1**

(c) Jack has $2n$ identical lollies. He places n lollies in his left pocket and n lollies in his right pocket. Every few minutes, he eats a lolly from a randomly chosen pocket and repeats this process until he reaches in to draw a lolly, only to find that the pocket is empty.

(i) Show that the probability of the remaining pocket having k lollies is **1**

$$\binom{2n-k}{n}\frac{1}{2^{2n-k}}.$$

(ii) Deduce that **2**

$$\sum_{k=0}^{n} 2^{-k}\binom{n+k}{k} = 2^n.$$

End of Question 14

End of paper