2024 **BORED OF STUDIES TRIAL EXAMINATION**

Mathematics Extension 1

Solutions

Section I

Answers

1	D	**6**	A
2	D	**7**	A
3	C	**8**	A
4	B	**9**	B
5	A	**10**	C

Brief explanations

1 Given $\cos 2x = 2\cos^2 x - 1$ and $\cos 4x = 2\cos^2 2x - 1$

$$\begin{aligned}\int_0^{\frac{\pi}{4}} (2\cos^2 x + 1)(4\cos^2 x - 5)\,dx &= \int_0^{\frac{\pi}{4}} (\cos 2x + 2)(2\cos 2x - 3)\,dx \\ &= \int_0^{\frac{\pi}{4}} (2\cos^2 2x + \cos 2x - 6)\,dx \\ &= \int_0^{\frac{\pi}{4}} (\cos 4x + \cos 2x - 5)\,dx \\ &= \left[\frac{\sin 4x}{4} + \frac{\sin 2x}{2} - 5x\right]_0^{\frac{\pi}{4}} \\ &= \frac{1}{2} - \frac{5\pi}{4} \\ &= \frac{2 - 5\pi}{4}\end{aligned}$$

Hence, the answer is (D).

2 Assessing each of the options

$$\text{For option (A)} \quad \begin{aligned}\sin(2\sin^{-1} x) &= \sin\left(2\left(\frac{\pi}{2} - \cos^{-1} x\right)\right) \\ &= \sin(\pi - 2\cos^{-1} x) \\ &= \sin(2\cos^{-1} x)\end{aligned}$$

$$\text{For option (B)} \quad \begin{aligned}\sin(\pi + 2\sin^{-1}(-x)) &= \sin(\pi - 2\sin^{-1} x) \\ &= \sin(\pi - 2\sin^{-1} x) \\ &= \sin(2\sin^{-1} x) \\ &= \sin(2\cos^{-1} x) \quad \text{from option (A)}\end{aligned}$$

For option (C), let $u = \cos^{-1} x \Rightarrow x = \cos u$ noting that $0 \leq u \leq \pi$.

$$\begin{aligned}\sin 2u &= 2\sin u \cos u \\ &= 2x\sqrt{1-x^2}\end{aligned}$$

For option (D), note that

$$\begin{aligned}2\sqrt{x^2(1-x^2)} &= 2|x|\sqrt{1-x^2} \\ &\neq 2x\sqrt{1-x^2}\end{aligned}$$

Hence, the answer is (D).

3 The curve looks concave down with a maximum turning point in $-2 < x < 2$. This means that $f'(2) < 0, f''(2) < 0$ and $f'(-2) > 0, f''(-2) < 0$.

Hence, the answer is (C).

4 Deriving the acceleration vector

$$\begin{aligned}\underset{\sim}{r} &= 4\sin 2t\,\underset{\sim}{i} + 3\cos 2t\,\underset{\sim}{j} \\ \dot{\underset{\sim}{r}} &= 8\cos 2t\,\underset{\sim}{i} - 6\sin 2t\,\underset{\sim}{j} \\ \ddot{\underset{\sim}{r}} &= -16\sin 2t\,\underset{\sim}{i} - 12\cos 2t\,\underset{\sim}{j} \\ |\ddot{\underset{\sim}{r}}| &= \sqrt{(-16\sin 2t)^2 + (-12\cos 2t)^2} \\ &= \sqrt{256\sin^2 2t + 144\cos^2 2t} \\ &= \sqrt{144 + 112\sin^2 2t}\end{aligned}$$

This is maximised when $\sin^2 2t = 1$, which gives $|\ddot{\underset{\sim}{r}}| = 16$.

Hence, the answer is (B).

5 For a point $P(x, y)$, the gradient of the normal is $-\dfrac{dx}{dy}$.

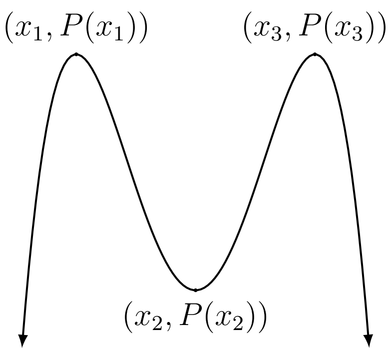
The gradient of the line between $P(x, y)$ and the origin is $\dfrac{y}{x}$.

The gradient of the normal to any point P on a particular curve is twice the gradient of the line joining P and the origin so

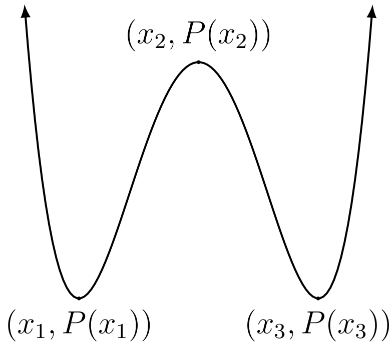
$$\begin{aligned}-\frac{dx}{dy} &= 2 \times \frac{y}{x} \\ \frac{dy}{dx} + \frac{x}{2y} &= 0\end{aligned}$$

Hence, the answer is (A).

6 Given there are distinct stationary points, the shape of $P(x)$ could be



It is possible for $P(x_2) < P(x_3) < P(x_1)$, which is option (D).



It is possible for $P(x_1) < P(x_3) < P(x_2)$ or $P(x_3) < P(x_1) < P(x_2)$ which are options (B) and (C) respectively.

However, it is not possible for $P(x_1) < P(x_2) < P(x_3)$. Hence, the answer is (A).

7 Let $\hat{p}$ be the sample proportion of students in the school that score band E4 in Mathematics Extension 1. The estimate is that

$$P\left(\hat{p} \leq 0.3\right) = 0.4$$

$$P\left(\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \leq \frac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}}\right) = 0.4 \quad \text{for large } n \text{ note that } \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$$P\left(Z \leq \frac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}}\right) \approx 0.4 \quad \text{where } Z \sim N(0,1)$$

Given $P(Z < 0.26) \approx 0.6$ then by symmetry of the normal curve

$$P(Z \geq 0.26) \approx 0.4$$
$$\Rightarrow P(Z \leq -0.26) \approx 0.4$$

Hence, the choices of n and p should have $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} \approx -0.26$.

For option (B), when $p = 0.328$ and $n = 79$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = -0.53$

For option (C), when $p = 0.286$ and $n = 70$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = 0.26$

For option (D), when $p = 0.271$ and $n = 66$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = 0.53$

For option (A), when $p = 0.315$ and $n = 64$, then $\dfrac{0.3-p}{\sqrt{\frac{p(1-p)}{n}}} = -0.26$

Hence, the answer is (A).

8 Since $A(x)$ is monic quadratic with no real roots then $A(x)$ is positive definite.

Also, $R(x)$ is a remainder term it must be a linear polynomial or a constant. Since, $R(0) = R(1)$ then it must be a constant (namely equal to 4).

Since $R(x) = 4$ and $A(x) > 0$ for all real x then $\dfrac{A(x)}{R(x)} > 0$ for all real x.

Hence, the answer is (A).

9 First note that it is not possible to get 9 balls of the same colour that are red or blue.

By the pigeonhole principle, to guarantee 9 balls are the same colour, the red and blue need to be exhausted first and then 8 balls of yellow, black and white need to be drawn out.

This means that $4+7+8\times 3$, or equivalently, 35 balls need to be drawn first and the 36th ball will guarantee there will 9 balls of the same colour.

Hence, the answer is (B).

10 First derive the range of $\dfrac{x}{x-1}$. Noting that

$$\begin{aligned}\frac{x}{x-1} &= \frac{x-1+1}{x-1}\\ &= 1+\frac{1}{x-1}\end{aligned}$$

The range of $\dfrac{x}{x-1}$ is simply $\dfrac{x}{x-1} \neq 1$. This implies that the range of $\tan^{-1}\dfrac{x}{x-1}$ is

$$-\frac{\pi}{2} < \tan^{-1}\frac{x}{x-1} < \frac{\pi}{2} \quad \text{but} \;\; \tan^{-1}\frac{x}{x-1} \neq \frac{\pi}{4}$$

This means that

$$\begin{aligned} 0 < \cos\left(\tan^{-1}\frac{x}{x-1}\right) &\leq 1\\ \sec\left(\tan^{-1}\frac{x}{x-1}\right) &\geq 1\end{aligned}$$

Hence, the answer is (C).

Remark: Although $\tan^{-1}\dfrac{x}{x-1} \neq \dfrac{\pi}{4}$, it is possible for $\tan^{-1}\dfrac{x}{x-1} = -\dfrac{\pi}{4}$. Since $\sec x$ is an even function then it possible to get $\sec\left(\dfrac{\pi}{4}\right)$ because

$$\sec\left(-\frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right)$$

This means that this restriction has no impact on the final range.

Question 11

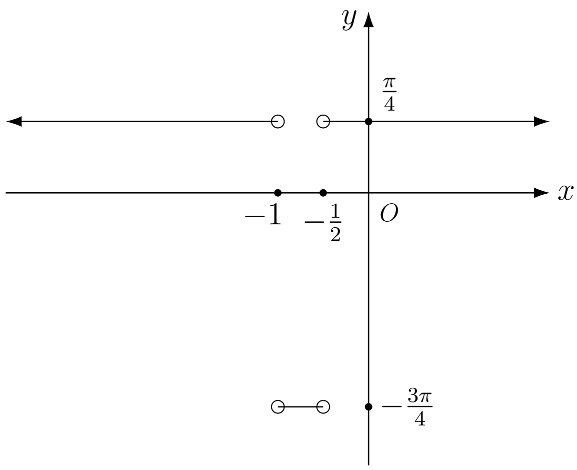
(a) (i) Using the quotient rule

$$
\begin{aligned}
f(x) &= \tan^{-1}\frac{x}{x+1} + \tan^{-1}\frac{1}{2x+1} \\
&= \frac{1}{1+\frac{x^2}{(x+1)^2}} \times \frac{(1)(x+1)-x(1)}{(x+1)^2} + \frac{1}{1+\frac{1}{(2x+1)^2}} \times -\frac{2}{(2x+1)^2} \\
&= \frac{1}{(x+1)^2+x^2} - \frac{2}{(2x+1)^2+1} \\
&= \frac{(2x+1)^2+1-2(x+1)^2-2x^2}{[(x+1)^2+x^2][(2x+1)^2+1]} \\
&= \frac{4x^2+4x+1+1-2x^2-4x-2-2x^2}{[(x+1)^2+x^2][(2x+1)^2+1]} \\
&= 0
\end{aligned}
$$

(ii) From part (i), $f'(x) = 0$ for all defined values of x. However, $f(x)$ is undefined at $x = -1, -\frac{1}{2}$. Consider the regions $x < -1, -1 < x < -\frac{1}{2}$ and $x > -\frac{1}{2}$. Testing different values by calculator in those regions, say $x = -2, -\frac{3}{4}$ and 0 gives

$$
\begin{aligned}
f(-2) &= \frac{\pi}{4} \\
f\left(-\frac{3}{4}\right) &= -\frac{3\pi}{4} \\
f(0) &= \frac{\pi}{4}
\end{aligned}
$$

This gives the following sketch.



(b) From the inequality

$$\begin{aligned}
\frac{3x^2-1}{2x^2+x} &> k \\
\frac{3x^2-1-2kx^2-kx}{x(2x+1)} &> 0 \\
x(2x+1)((3-2k)x^2-kx-1) &> 0
\end{aligned}$$

It is given that the solution to this inequality is $x \in \left(-\frac{1}{2}, 0\right)$.

This implies that $x < 0$ and $2x+1 > 0$, hence $x(2x+1) < 0$. This means that the quadratic factor is required to be negative definite in order for the inequality to hold. A necessary condition for this is

$$\begin{aligned}
\Delta &< 0 \\
k^2 - 4(3-2k)(-1) &< 0 \\
k^2 - 8k + 12 &< 0 \\
(k-2)(k-6) &< 0 \\
2 < k &< 6
\end{aligned}$$

The other condition required for the quadratic to be negative definition is $3-2k < 0$, or equivalently, $k > \frac{3}{2}$.

This is a superset of the above solution, so the range of values of k is just $2 < k < 6$.

(c) Since $P'(\alpha) = 0$ and $P'(\beta) = 0$ then there is a double root where $\alpha = \beta$. Since $P(x)$ is a monic cubic polynomial then

$$\begin{aligned}
P(x) &= (x-\alpha)^2(x-\gamma) \\
P'(x) &= 2(x-\alpha)(x-\gamma) + (x-\alpha)^2 \\
P'(\gamma) &= (\gamma-\alpha)^2 \\
(\gamma-\alpha)^2 &= 25 \\
\gamma &= \alpha \pm 5
\end{aligned}$$

Consider the case where $\gamma = \alpha - 5$, then from the sum of the roots

$$\begin{aligned}
\alpha + \beta + \gamma &= -1 \\
3\alpha - 5 &= -1 \\
\alpha &= \frac{4}{3} \Rightarrow \beta = \frac{4}{3}, \gamma = -\frac{11}{3}
\end{aligned}$$

However, it is given that $\alpha\beta\gamma = 12$ which does not hold from the above solution set.

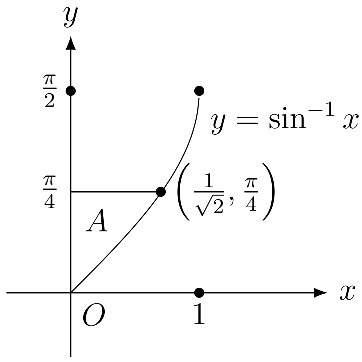
Now consider the case where $\gamma = \alpha + 5$, then from the sum of the roots

$$\begin{aligned}
\alpha + \beta + \gamma &= -1 \\
3\alpha + 5 &= -1 \\
\alpha &= -2 \Rightarrow \beta = -2, \gamma = 3
\end{aligned}$$

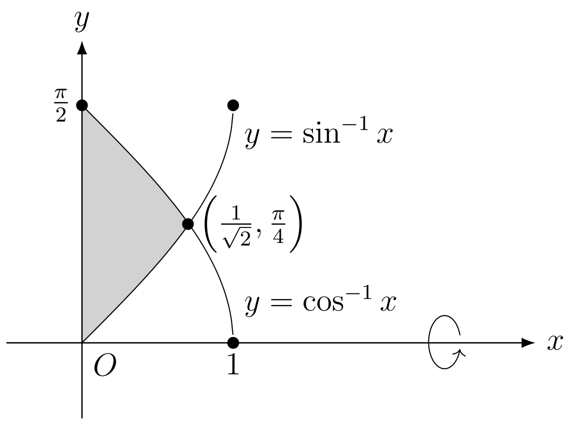
This satisfies the given result $\alpha\beta\gamma = 12$. Hence, the roots are -2 as a double root and 3.

(d) (i) Given $y = \sin^{-1} x$, then $x = \sin y$ so

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} x\,dy \\ &= \int_0^{\frac{\pi}{4}} \sin y\,dy \\ &= [-\cos y]_0^{\frac{\pi}{4}} \\ &= 1 - \frac{1}{\sqrt{2}} \end{aligned}$$



(ii) Note the point of intersection between $y = \sin^{-1} x$ and $y = \cos^{-1} x$ is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$.



The area between the curves is rotated about the x-axis so the volume of the solid of revolution is

$$\begin{aligned} V &= \pi \int_0^{\frac{1}{\sqrt{2}}} \left(\cos^{-1} x\right)^2\,dx - \pi \int_0^{\frac{1}{\sqrt{2}}} \left(\sin^{-1} x\right)^2\,dx \\ &= \pi \int_0^{\frac{1}{\sqrt{2}}} \left(\cos^{-1} x - \sin^{-1} x\right)\left(\cos^{-1} x + \sin^{-1} x\right)\,dx \quad \text{but } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \\ &= \frac{\pi^2}{2} \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{\pi}{2} - 2\sin^{-1} x\right)\,dx \\ &= \frac{\pi^3}{4} \int_0^{\frac{1}{\sqrt{2}}} dx - \pi^2 \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx \\ &= \frac{\pi^3}{4}[x]_0^{\frac{1}{\sqrt{2}}} - \pi^2 \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx \quad \text{but from part (i)} \quad A + \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dx = \frac{\pi}{4\sqrt{2}} \\ &= \frac{\pi^3}{4\sqrt{2}} - \pi^2 \left(\frac{\pi}{4\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}}\right)\right) \\ &= \pi^2 \left(1 - \frac{1}{\sqrt{2}}\right) \end{aligned}$$

(e) Given $\alpha+\beta+\gamma=\dfrac{\pi}{2}$, note that

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\
\tan\left(\frac{\pi}{2}-\gamma\right) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\
\frac{1}{\tan\gamma} &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\
1-\tan\alpha\tan\beta &= \tan\alpha\tan\gamma+\tan\beta\tan\gamma\\
\tan\alpha\tan\beta+\tan\alpha\tan\gamma+\tan\beta\tan\gamma &= 1
\end{aligned}$$

Using this result

$$\begin{aligned}
\text{LHS} &= \frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta}+\frac{\cos(\beta+\gamma)}{\cos\beta\cos\gamma}+\frac{\cos(\alpha+\gamma)}{\cos\alpha\cos\gamma}\\
&= \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}+\frac{\cos\beta\cos\gamma-\sin\beta\sin\gamma}{\cos\beta\cos\gamma}+\frac{\cos\alpha\cos\gamma-\sin\alpha\sin\gamma}{\cos\alpha\cos\gamma}\\
&= 1-\tan\alpha\tan\beta+1-\tan\beta\tan\gamma+1-\tan\alpha\tan\gamma\\
&= 3-(\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\alpha\tan\gamma)\\
&= 2\\
&= \text{RHS}
\end{aligned}$$

Question 12

(a) Given the differential equations and for some constant C

$$\begin{aligned}\frac{3}{4}\cdot\frac{dT_c}{dt}+\frac{dT_w}{dt}&=\frac{3}{4}\times -k(T_c-T_w)+\frac{3}{4}k(T_c-T_w)\\&=0\\ \Rightarrow \frac{3}{4}T_c+T_w&=C\end{aligned}$$

When $t=0, T_c=160$ and $T_w=20 \Rightarrow C=140$. This implies that $\frac{3}{4}T_c+T_w=140$, so

$$\begin{aligned}\frac{dT_c}{dt}&=-k(T_c-T_w)\\&=-k\left(T_c-140+\frac{3}{4}T_c\right)\\&=-k\left(\frac{7}{4}T_c-140\right)\\&=-\frac{7}{4}k\left(T_c-80\right)\end{aligned}$$

The general solution is $T_c = 80 + Ae^{-\frac{7}{4}kt}$ for some constant A. When $t\to\infty$, then $T_c \to 80$. Hence, the chemical will cool down towards a temperature of 80°C over time.

(b) When $n=1$

$$\begin{aligned}(2n+1)7^n-1&=(2(1)+1)7^1-1\\&=20\\&=4\times 5\end{aligned}$$

The statement is true for $n=1$.

Assume the statement is true for $n=k$, that is for some integer M

$$(2k+1)7^k-1=4M.$$

Required to prove the statement is true for $n=k+1$, that is for some integer N

$$(2k+3)7^{k+1}-1=4N.$$

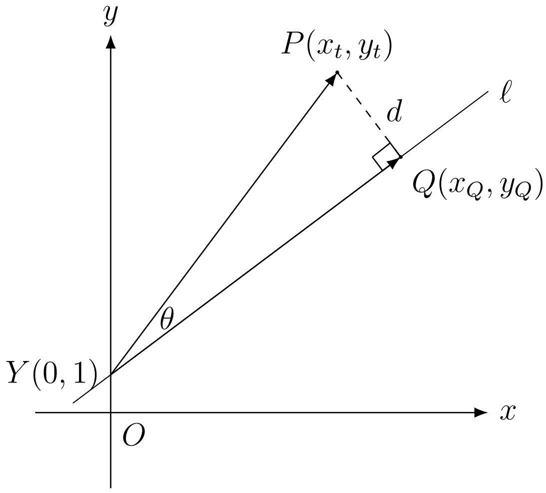
$$\begin{aligned}\text{LHS}&=(2k+3)7^{k+1}-1\\&=(2k+1+2)7^{k+1}-1\\&=(2k+1)7^{k+1}+14\cdot 7^k-1\\&=7(4M+1)+14\cdot(4M+1-2k\cdot 7^k)-1 \quad \text{by assumption}\\&=84M-28k\cdot 7^k+20\\&=4(21M-7k\cdot 7^k+5)\\&=4N \quad \text{where } N=21M-7k\cdot 7^k+5\\&=\text{RHS}\end{aligned}$$

Since statement is true for $n=1$, then by induction it is true for all positive integers n.

(c)

$$
\begin{aligned}
\text{LHS} &= \int_0^1 \frac{\cos\theta}{x^2 - 2x\sin\theta + 1}\,dx \\
&= \int_0^1 \frac{\cos\theta}{x^2 - 2x\sin\theta + \sin^2\theta + \cos^2\theta}\,dx \\
&= \int_0^1 \frac{\cos\theta}{(x-\sin\theta)^2 + \cos^2\theta}\,dx \\
&= \left[\tan^{-1}\left(\frac{x-\sin\theta}{\cos\theta}\right)\right]_0^1 \\
&= \tan^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right) - \tan^{-1}\left(\frac{-\sin\theta}{\cos\theta}\right) \\
&= \tan^{-1}\left(\frac{1-\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}{\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}\right) - \tan^{-1}(-\tan\theta) \\
&= \tan^{-1}\left(\frac{1-2\tan\frac{\theta}{2}+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right) + \theta \\
&= \tan^{-1}\left(\frac{\left(1-\tan\frac{\theta}{2}\right)^2}{\left(1-\tan\frac{\theta}{2}\right)\left(1+\tan\frac{\theta}{2}\right)}\right) + \theta \\
&= \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right) + \theta \quad \text{noting } 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \tan\frac{\theta}{2} < 1 \\
&= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right) + \theta \\
&= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right) + \theta \\
&= \frac{\pi}{4} - \frac{\theta}{2} + \theta \\
&= \frac{\pi}{4} + \frac{\theta}{2} \\
&= \text{RHS}
\end{aligned}
$$

(d) (i) Note that the coordinates of Y are $(0,1)$. Let θ be the angle between $\overrightarrow{YP}$ and $\overrightarrow{YQ}$ as shown in the diagram below. Let (x_Q, y_Q) be the coordinates of Q.



Consider the projection of $\overrightarrow{YP}$ onto $\overrightarrow{YQ}$ in terms of magnitude

$$\begin{aligned}
\left|\overrightarrow{YQ}\right| &= \left|\overrightarrow{YP}\right| |\cos\theta| \\
&= \left|\overrightarrow{YP}\right| \frac{\left|\overrightarrow{YP} \cdot \overrightarrow{YQ}\right|}{\left|\overrightarrow{YP}\right| \left|\overrightarrow{YQ}\right|} \\
&= \frac{\left|\overrightarrow{YP} \cdot \overrightarrow{YQ}\right|}{\left|\overrightarrow{YQ}\right|} \\
&= \frac{|(x_t\underset{\sim}{i} + (y_t - 1)\underset{\sim}{j}) \cdot (x_Q\underset{\sim}{i} + (y_Q - 1)\underset{\sim}{j})|}{\sqrt{x_Q^2 + (y_Q - 1)^2}}
\end{aligned}$$

But Q lies on the line $3x - 4y + 4 = 0$ so

$$\begin{aligned}
3x_Q - 4y_Q + 4 &= 0 \\
y_Q - 1 &= \frac{3}{4}x_Q
\end{aligned}$$

Substitute this into $\left|\overrightarrow{YQ}\right|$

$$\begin{aligned}
\left|\overrightarrow{YQ}\right| &= \frac{|(x_t\underset{\sim}{i} + (y_t - 1)\underset{\sim}{j}) \cdot (x_Q\underset{\sim}{i} + \frac{3}{4}x_Q\underset{\sim}{j})|}{\sqrt{x_Q^2 + \frac{9}{16}x_Q^2}} \\
&= \frac{|x_Q||x_t + \frac{3}{4}(y_t - 1)|}{|x_Q|\sqrt{\frac{25}{16}}} \\
&= \frac{|4x_t + 3y_t - 3|}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
d^2 &= \left|\overrightarrow{YP}\right|^2 - \left|\overrightarrow{YQ}\right|^2 \\
&= x_t^2 + (y_t - 1)^2 - \frac{(4x_t + 3y_t - 3)^2}{25} \\
&= \frac{1}{25}[25x_t^2 + 25y_t^2 - 50y_t + 25 \\
&\qquad - (16x_t^2 + 9y_t^2 + 9 + 2(4x_t)(3y_t) + 2(3y_t)(-3) + 2(4x_t)(-3))] \\
&= \frac{1}{25}[9x_t^2 + 16y_t^2 + 16 - 24x_ty_t - 32y_t + 24x_t] \\
&= \frac{1}{25}[9x_t^2 + 16y_t^2 + 16 + 2(3x_t)(-4y_t) + 2(4)(-4y_t) + 2(3x_t)(4)] \\
&= \frac{(3x_t - 4y_t + 4)^2}{25}
\end{aligned}$$

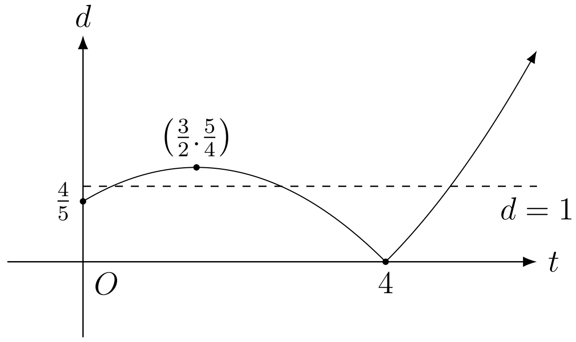
(ii) Substitute $x_t = t$ and $y_t = \dfrac{t^2}{4}$ into d^2

$$\begin{aligned}
d^2 &= \frac{(3t - t^2 + 4)^2}{25} \\
d^2 &= \frac{(t^2 - 3t - 4)^2}{25} \quad \text{but } d \geq 0 \\
d &= \frac{|(t+1)(t-4)|}{5}
\end{aligned}$$

Consider the parabola $y = \dfrac{(x+1)(x-4)}{5}$. The stationary points occur at the midpoint of the x-intercepts which is $x = \dfrac{3}{2} \Rightarrow y = -\dfrac{5}{4}$.

The graph of $y = \left|\dfrac{(x+1)(x-4)}{5}\right|$ will reflect the negative y-values about the x-axis.

Now apply this to the equation for d as a function of t, noting that $t \geq 0$ and $d \geq 0$, with the turning point $\left(\frac{3}{2}.\frac{5}{4}\right)$.



(iii) From part (ii), the local maximum is at $\left(\frac{3}{2}.\frac{5}{4}\right)$ and the d-intercept is at $d = \frac{5}{4}$. Hence, the particle is exactly 1 unit away from the line (represented by the horizontal line $d = 1$ above) exactly three times.

Question 13

(a) (i) Given the velocity vector

$$\dot{\underset{\sim}{r}} = \frac{u}{\sqrt{2}}\underset{\sim}{i} + \left(\frac{u}{\sqrt{2}} - gt\right)\underset{\sim}{j}$$

$$\underset{\sim}{r} = \left(\frac{ut}{\sqrt{2}} + C_x\right)\underset{\sim}{i} + \left(\frac{ut}{\sqrt{2}} - \frac{gt^2}{2} + C_y\right)\underset{\sim}{j}$$

When $t = 0$, $\underset{\sim}{r} = 0\underset{\sim}{i} + 0\underset{\sim}{j} \Rightarrow C_x = 0, C_y = 0$, so

$$\begin{aligned} x &= \frac{ut}{\sqrt{2}} \\ t &= \frac{x\sqrt{2}}{u} \quad \text{substitute into } y \\ y &= \frac{u}{\sqrt{2}} \times \frac{x\sqrt{2}}{u} - \frac{g}{2} \times \frac{2x^2}{u^2} \\ &= x - \frac{gx^2}{u^2} \end{aligned}$$

(ii) On the x-y plane, the particle lands on the point $(10\cos\theta, 10\sin\theta)$. Substitute this into the Cartesian equation to get

$$\begin{aligned} 10\sin\theta &= 10\cos\theta - \frac{100g\cos^2\theta}{u^2} \qquad \text{noting } 0 < \theta < \frac{\pi}{2} \\ \tan\theta &= 1 - \frac{10g\cos\theta}{u^2} \end{aligned}$$

(iii) On the x-y plane, the particle lands on the point $(20\cos\theta, -20\sin\theta)$. Substitute this into the Cartesian equation to get

$$\begin{aligned} -20\sin\theta &= 20\cos\theta - \frac{400g\cos^2\theta}{u^2} \qquad \text{noting } 0 < \theta < \frac{\pi}{2} \\ \tan\theta &= \frac{20g\cos\theta}{u^2} - 1 \end{aligned}$$

(iv) From the result in part (ii)

$$\begin{aligned} \tan\theta &= 1 - \frac{10g\cos\theta}{u^2} \\ \frac{10g\cos\theta}{u^2} &= 1 - \tan\theta \end{aligned}$$

Substitute this into the result from part (iii)

$$\begin{aligned} \tan\theta &= \frac{20g\cos\theta}{u^2} - 1 \\ &= 2(1 - \tan\theta) - 1 \\ \tan\theta &= \frac{1}{3} \end{aligned}$$

(b) First note that

$$\begin{aligned}
\sin x + \cos x &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)\\
&= \sqrt{2}\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)\\
&= \sqrt{2}\cos\left(\frac{\pi}{4} - x\right)\\
(\sin x + \cos x)^{2n} &= \left[\sqrt{2}\cos\left(\frac{\pi}{4} - x\right)\right]^{2n}\\
&= 2^n\left[\cos^2\left(\frac{\pi}{4} - x\right)\right]^n\\
\frac{1}{(\sin x + \cos x)^{2n}} &= \frac{1}{2^n}\left[\sec^2\left(\frac{\pi}{4} - x\right)\right]^n\\
&= \frac{1}{2^n}\left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]^n
\end{aligned}$$

Using the given substitution

$$\begin{aligned}
t &= \tan\left(\frac{\pi}{4} - x\right)\\
dt &= -\sec^2\left(\frac{\pi}{4} - x\right)\,dx\\
&= -\left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]\,dx
\end{aligned}$$

When $x = 0 \Rightarrow t = 1$ and when $x = \frac{\pi}{2} \Rightarrow -1$ so

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x + \cos x)^{2n}} &= \frac{1}{2^n}\int_0^{\frac{\pi}{2}} \left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]^n\,dx\\
&= \frac{1}{2^n}\int_0^{\frac{\pi}{2}} \left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]^{n-1}\left[1 + \tan^2\left(\frac{\pi}{4} - x\right)\right]\,dx\\
&= -\frac{1}{2^n}\int_1^{-1} (1 + t^2)^{n-1}\,dt\\
&= \frac{1}{2^n}\int_{-1}^{1} (1 + t^2)^{n-1}\,dt \quad \text{but } (1 - t^2)^{n-1} \text{ is an even function}\\
&= \frac{2}{2^n}\int_0^1 \left[\binom{n-1}{0} + \binom{n-1}{1}t^2 + \binom{n-1}{2}t^4 + \cdots + \binom{n-1}{n-1}t^{2n-2}\right]\,dt\\
&= \frac{1}{2^{n-1}}\left[\frac{\binom{n-1}{0}t}{1} + \frac{\binom{n-1}{1}t^3}{3} + \frac{\binom{n-1}{2}t^5}{5} + \cdots + \frac{\binom{n-1}{n-1}t^{2n-1}}{2n-1}\right]_0^1\\
&= \frac{\binom{n-1}{0}}{1 \times 2^{n-1}} + \frac{\binom{n-1}{1}}{3 \times 2^{n-1}} + \frac{\binom{n-1}{2}}{5 \times 2^{n-1}} + \cdots + \frac{\binom{n-1}{n-1}}{(2n-1) \times 2^{n-1}}
\end{aligned}$$

(c) (i) Using the given results

$$\begin{aligned}
\cos\theta P(\theta) &= 0\\
8\cos^4\theta + 4\cos^3\theta - 6\cos^2\theta - 2\cos\theta &= 0\\
\cos 4\theta + 8\cos^2\theta - 1 + \cos 3\theta + 3\cos\theta - 6\cos^2\theta - 2\cos\theta &= 0\\
\cos 4\theta + \cos 3\theta + 2\cos^2\theta + \cos\theta - 1 &= 0\\
(\cos\theta + \cos 4\theta) + (\cos 2\theta + \cos 3\theta) &= 0\\
\left[\cos\left(\frac{5\theta}{2} - \frac{3\theta}{2}\right) + \cos\left(\frac{5\theta}{2} + \frac{3\theta}{2}\right)\right] + \left[\cos\left(\frac{5\theta}{2} - \frac{\theta}{2}\right) + \cos\left(\frac{5\theta}{2} + \frac{\theta}{2}\right)\right] &= 0\\
2\cos\frac{5\theta}{2}\cos\frac{3\theta}{2} + 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2} &= 0\\
\cos\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right) &= 0\\
\cos\frac{5\theta}{2}\left[\cos\left(\theta + \frac{\theta}{2}\right) + \cos\left(\theta - \frac{\theta}{2}\right)\right] &= 0\\
\cos\frac{5\theta}{2}\cos\theta\cos\frac{\theta}{2} &= 0
\end{aligned}$$

If $\cos\frac{\theta}{2} = 0$ then for $0 \le \theta \le 2\pi$, or equivalently, $0 \le \frac{\theta}{2} \le \pi$

$$\frac{\theta}{2} = \frac{\pi}{2} \quad \Rightarrow \theta = \pi$$

If $\cos\theta = 0$ then for $0 \le \theta \le 2\pi$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

If $\cos\frac{5\theta}{2} = 0$ then for $0 \le \theta \le 2\pi$, or equivalently, $0 \le \frac{5\theta}{2} \le 5\pi$

$$\frac{5\theta}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2} \quad \Rightarrow \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$$

Hence, the full solution set is

$$\theta = \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{9\pi}{5}$$

(ii) Let the other roots be α and β. By the sum and product of roots

$$\begin{aligned}
\alpha + \beta - 1 &= -\frac{1}{2} \quad \Rightarrow \alpha + \beta = \frac{1}{2}\\
\alpha\beta(-1) &= \frac{1}{4} \quad \Rightarrow \alpha\beta = -\frac{1}{4}
\end{aligned}$$

This implies that α and β are the roots of $4x^2 - 2x - 1 = 0$ which solves to

$$\begin{aligned}
x &= \frac{2 \pm \sqrt{4 - 4(4)(-1)}}{8}\\
&= \frac{1 \pm \sqrt{5}}{4}
\end{aligned}$$

Hence, the roots of $P(x)$ are -1, $\frac{1-\sqrt{5}}{4}$ and $\frac{1+\sqrt{5}}{4}$.

(iii) The solutions of $P(\cos\theta) = 0$ are derived from $\cos\dfrac{\theta}{2}\cos\dfrac{5\theta}{2} = 0$, which are

$$\begin{aligned}
\theta &= \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5} \\
\cos\theta &= \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}
\end{aligned}$$

However, note that $\cos\pi = -1$ and

$$\begin{aligned}
\cos\frac{9\pi}{5} &= \cos\left(2\pi - \frac{\pi}{5}\right) \\
&= \cos\frac{\pi}{5} \\
\cos\frac{7\pi}{5} &= \cos\left(2\pi - \frac{3\pi}{5}\right) \\
&= \cos\frac{3\pi}{5}
\end{aligned}$$

Hence, there are three distinct roots of $P(x)$, which are $\cos\dfrac{\pi}{5}, \cos\dfrac{3\pi}{5}$ and -1.

Since $\dfrac{\pi}{2} < \dfrac{3\pi}{5} < \pi$ then $\cos\dfrac{3\pi}{5} < 0$, so from part (ii)

$$\begin{aligned}
\cos\frac{3\pi}{5} &= \frac{1-\sqrt{5}}{2} \\
\cos\left(\pi - \frac{2\pi}{5}\right) &= \frac{1-\sqrt{5}}{2} \\
-\cos\frac{2\pi}{5} &= \frac{1-\sqrt{5}}{4} \\
\cos\frac{2\pi}{5} &= \frac{\sqrt{5}-1}{4}
\end{aligned}$$

Question 14

(a) (i) There are ${}^{16}C_{12}$ possible combinations of successful tenants.

There are ${}^{20}P_{12}$ possible arrangements of 12 successful tenants amongst the 20 vacant homes.

Hence, there are ${}^{16}C_{12} \times {}^{20}P_{12}$ ways the interested tenants be arranged amongst the houses, if they were successful.

(ii) If there are two particular tenants that are successful then there are ${}^{14}C_{10}$ combinations of the other successful tenants.

There are ${}^{18}P_{10}$ possible arrangements of the other 10 successful tenants amongst the 18 remaining houses.

There are 19 possible pairs of houses where the two tenants can be neighbours.

There are 2! ways to arrange the two particular tenants between each of the neighbouring houses.

The number of arrangements where two particular tenants are successful and are also neighbours is ${}^{14}C_{10} \times {}^{18}P_{10} \times 19 \times 2!$.

Hence, the probability this occurs is $\dfrac{{}^{14}C_{10} \times {}^{18}P_{10} \times 19 \times 2!}{{}^{16}C_{12} \times {}^{20}P_{12}}$, or equivalently $\dfrac{11}{200}$.

(b) (i) After each minute there will be 200 litres of water flowing into the tank. Since the water coming in will have $e^{-0.2t}$ grams of radioactive chemical per litre, then there will be $200e^{-0.2t}$ grams of radioactive chemical coming into the tank.

If m is the mass of radioactive chemicals inside the tank after t minutes, then there are $\frac{m}{V}$ grams of it per litre in the tank, where V is the volume of the tank after time t minutes.

After each minute there will be 100 litres of water flowing out of the tank. This carries $\frac{100m}{V}$ grams of radioactive chemical out of the tank. Therefore, the rate of change of m is

$$\frac{dm}{dt} = 200e^{-0.2t} - \frac{100m}{V}$$

To find V, note that after time t minutes, there are $200t$ litres of water flowing in and $100t$ litres water flowing out. Since the tank is initially holding 100 litres of water then the volume of water in the tank after t minutes is $V = 100 + 200t - 100t$, or equivalently $V = 100(1+t)$. Hence,

$$\frac{dm}{dt} = 200e^{-0.2t} - \frac{m}{1+t}$$

(ii) Noting that m is a function of t and using the given result

$$\begin{aligned}
\frac{d}{dt}\left[m(1+t)\right] &= \frac{dm}{dt}(1+t)+m \\
&= 200e^{-0.2t}(1+t)-m+m \\
&= 200e^{-0.2t}(1+t) \\
m(1+t) &= 200\int e^{-0.2t}\,dt + 200\int te^{-0.2t}\,dt \\
&= -1000e^{-0.2t} + 200\left(-\frac{1}{(0.2)^2}e^{-0.2t}(1+0.2t)\right)+C \\
&= -1000e^{-0.2t} - 5000e^{-0.2t}(1+0.2t)+C
\end{aligned}$$

When $t=0, m=0 \Rightarrow C=6000$

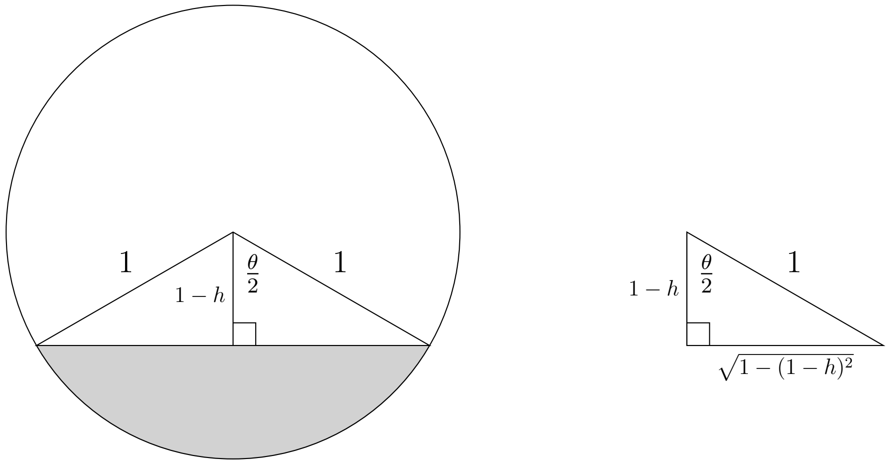
$$\begin{aligned}
m &= \frac{-6000e^{-0.2t}-1000te^{-0.2t}+6000}{1+t} \\
&= \frac{6000-1000e^{-0.2t}(t+6)}{1+t}
\end{aligned}$$

(iii) Let M be the mass of radioactive chemical which flowed out of the tank.

From the derivation in part (i), it was shown that

$$\begin{aligned}
\frac{dM}{dt} &= \frac{m}{1+t} \\
&= 200e^{-0.2t} - \frac{dm}{dt} \\
M &= \int_0^4 \left(200e^{-0.2t}-\frac{dm}{dt}\right)\,dt \\
&= -1000[e^{-0.2t}]_0^4 - \left[\frac{6000-1000e^{-0.2t}(t+6)}{1+t}\right]_0^4 \\
&= 1000(1-e^{-0.8}) - \left(\frac{6000-10000e^{-0.8}}{5}\right) \\
&\approx 249 \text{ grams}
\end{aligned}$$

(c) Let θ be the angle subtended by the chord at the centre.



From the right-angled triangle

$$\begin{aligned}\sin\frac{\theta}{2} &= \sqrt{1-(1-h)^2}\\ &= \sqrt{2h-h^2}\\ \cos\frac{\theta}{2} &= 1-h\\ \frac{\theta}{2} &= \cos^{-1}(1-h)\end{aligned}$$

Consider the area of the segment A

$$\begin{aligned}A &= \frac{1}{2}\times 1^2\times(\theta-\sin\theta)\\ &= \frac{\theta}{2}-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= \cos^{-1}(1-h)-(1-h)\sqrt{2h-h^2}\end{aligned}$$

Let V be the volume of the oil at time t.

$$\begin{aligned}V &= 10A\\ &= 10\cos^{-1}(1-h)-10(1-h)\sqrt{2h-h^2}\\ \frac{dV}{dh} &= \frac{-10(-1)}{\sqrt{1-(1-h)^2}}-10(-1)\sqrt{2h-h^2}-\frac{10(1-h)(2-2h)\times\frac{1}{2}}{\sqrt{2h-h^2}}\\ &= \frac{10[1+(2h-h^2)-(1-h)^2]}{\sqrt{2h-h^2}}\\ &= \frac{10(1+2h-h^2-1+2h-h^2)}{\sqrt{2h-h^2}}\\ &= \frac{20(2h-h^2)}{\sqrt{2h-h^2}} \quad \text{note that } 0<h<1\\ &= 20\sqrt{2h-h^2}\end{aligned}$$

It is given that S is the area of the tank's cylindrical surface that is in contact with the oil after t weeks. This is equal to the rectangular area at the bottom of the cylindrical tank plus twice the area of the segment at each end of the tank.

The area of the rectangle is the arc length of the segment multiplied by 10 metres and the area of the segment is A, as defined earlier above.

$$\begin{aligned} S &= 1 \times \theta \times 10 + 2A \quad \text{but } \frac{\theta}{2} = \cos^{-1}(1-h) \text{ and } V = 10A \\ &= 20\cos^{-1}(1-h) + \frac{V}{5} \\ \frac{dS}{dh} &= \frac{-20(-1)}{\sqrt{1-(1-h)^2}} + \frac{1}{5}\cdot\frac{dV}{dh} \\ &= \frac{20}{\sqrt{2h-h^2}} + 4\sqrt{2h-h^2} \end{aligned}$$

Using the chain rule and given $\dfrac{dV}{dt} = -5$

$$\begin{aligned} \frac{dS}{dh} &= \frac{dS}{dh} \times \frac{dh}{dV} \times \frac{dV}{dt} \\ &= \left(\frac{20}{\sqrt{2h-h^2}} + 4\sqrt{2h-h^2}\right) \times \frac{1}{20\sqrt{2h-h^2}} \times (-5) \\ &= -\frac{5}{2h-h^2} - 1 \\ &= \frac{5}{h(h-2)} - 1 \end{aligned}$$

(d) A new streak is defined everytime the current outcome differs from the previous outcome. For example, in the sequence $\{HTTTTHHHTH\}$ there are 5 streaks but there are 4 occasions where the outcome changes, which can be visualised as $\{H|TTTT|HHH|T|H\}$. This means that for $k+1$ streaks to occur, there must be k instances where the outcome changes in the sequence.

For any pair of coin tosses, let X_j be a Bernoulli random variable for $j = 1, 2, 3, \ldots, n-1$ which takes the value 0 if the outcomes between two consecutive tosses is the same and 1 if the outcomes between two consecutive tosses are different. Note that these are equally likely so $X_j \sim \text{Bin}\left(1, \frac{1}{2}\right)$.

Repeating this across $n-1$ pairs (which is equivalent to n coin tosses) means the sum of these Bernoulli random variables $Y = X_1 + X_2 + X_3 + \cdots + X_{n-1}$, represents number of occasions that two consecutive tosses have a different outcome, so $Y \sim \text{Bin}\left(n-1, \frac{1}{2}\right)$. However, S is the random variable represent the number of streaks, so $S = Y + 1$. Hence

$$\begin{aligned} E(S) &= E(Y+1) \\ &= E(Y) + 1 \\ &= (n-1) \times \frac{1}{2} + 1 \\ &= \frac{n+1}{2} \end{aligned}$$