Topic 1 - Probability and Venn Diagram

$P(A \cup B) = P(A) + P(B)$ ← mutually exclusive

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Non-mutually exclusive)

$P(A) = P(A \cap B) + P(A \cap \overline{B})$

Use a Lattice Diagram (table of all possible outcomes) for **Dice questions**

When independent,

$P(A \cap B) = P(A) \times P(B)$

$P(A|B) = P(A)$

When dependant,

$P(A \cap B) = P(A) \times P(B|A)$

$P(B|A) = \frac{P(A \cap B)}{P(A)}$

Topic 2 - Permutations and Combinations

REFERENCE SHEET FOR FORMULAS

Placing groups

Treat a special group as one element, then find the ways of placing all the elements multiplied by the number of internal arrangements of the special group

Elements to be separate

Place the separating objects first, then use ${}^{n}P_{r}$, where n is the gaps between each number (and on the ends) and r is the number of elements to be separated

Identical objects

For set/s of identical objects, use normal methods, then **divide by** $n! \times p! \times q!$... for as many sets

Bangles - Divide answer by 2

Circle - place 1 person/group down

Combination - teams/groups of the same size

Divide result by factorial (!) number of groups/teams of the same size

CASE BY CASE method ← use it when stuck

Alphabetical order - eg: if 1,2,2,3,5 are arranged and odd numbers are ascending, do $\frac{5!}{2! \times 3!}$ as 1,3,5 has $3!$ permutations, one of which will be ascending

Pigeonhole Principle

By the Pigeonhole principle, there must be $n + 1$ pigeons to ensure until multiple objects are chosen/event is repeated.

By the PHP, The average number of pigeons per hole is $\frac{n}{k} = a$.Therefore at least a pigeons must be in 1 hole

By the PHP, there must be $nk + 1$ pigeons for the min num of pigeons so one hole (k) has $n + 1$ pigeons

Topic 3 - Discrete probability distributions

$E(X)$ = sum of (each possible value × its probability)

$Var(X) = E(X^2) - [E(X)]^2$ or $= E(X^2) - \mu^2$

$\sigma = \sqrt{Var(x)}$

Topic 4 - Algebraic techniques

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^m \times x^n = x^{m+n}$, $x^m \div x^n = x^{m-n}$, $x^{\frac{n}{m}} = \sqrt[m]{x^n}$

Topic 5 - Binomial Expansion + Pascal's Triangle

$(a + b)^n = C_0 a^n + {}^{n}C_1 a^{n-1} b + {}^{n}C_2 a^{n-2} b^2 + ... + {}^{n}C_n b^n$

$(a + b)^n = \sum_{k=0}^{n} {}^{n}C_k a^{n-k} b^k$

$T_{k+1} = {}^{n}C_k a^{n-k} b^k$

$\frac{T_{k+1}}{T_k} \geq 1$ when T_{k+1} is the **greatest coefficient**

Topic 6 - Linear, Quadratic and Cubic Functions

$y - y_1 = m(x - x_1)$ **-** point - gradient form

Perpendicular, $m_2 = -\frac{1}{m_1}$

Real, Rational root, $\Delta = 0$ or a perfect square

Axis of symmetry: $\frac{-b}{2a}$

Always **positive,** positive definite

Parallelogram if **ONE** of the following is true:

- Both pairs of opposite angles are equal
- Both pairs of opposite sides are equal
- Both pairs of opposite sides are parallel
- Diagonals bisect each other
- One pair of opposite sides are equal/parallel

Rectangle if **ONE** of the following is true:

- All angles are 90^o
- Diagonals are equal AND bisect each other

Rhombus if **ONE** of the following is true:

- All sides are equal
- Diagonals bisect each other at 90^o

Square if **ONE** of the following is true:

- All sides are equal and one angle is 90^o
- All angles are 90^o and two adjacent sides are equal
- Diagonals are equal and bisect each other at 90^o

Topic 7 - Introduction to Functions

Vertical test fails, Horizontal test fails, **many-to-many**
Only vertical test fails, **one-to-many**
Only horizontal test fails, **many-to-one**
Passes both, **one-to-one**

$f(x) = f(-x)$, even
$f(-x) = -f(x)$, odd
$x \in [0, 2)$ is an example of new notation

Topic 8 - Further Functions and Relations

$Q\alpha\frac{1}{x}$, $Q = \frac{k}{x}$ (If Q varies inversely with x)

$|a| = a$**,** if $a \geq 0$, $= \ -a$, if $a < 0$

$|a - b| = a - b$, if $a > b$, $= 0$, if $a = b$,
$= b - a$ if $a < b$

$\sqrt{a^2} = |a|$, thus following the rules above

To graph, $|x| + |y| = 1$, find the sign of each variable in each quadrant, creating separate equations for each quadrant

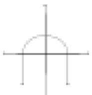
Topic 9 - Graphical Relations

See booklet for reciprocal, square root and squared functions, along with addition and multiplication of functions

Topic 10 - Inequalities

If inequality is $\leq or \geq$, boundary line is **SOLID**
If inequality is $< or >$, boundary line is **DOTTED**

Semi circles boundary line extend infinitely



Solve inequalities by changing inequality by change inequality to an equals sign, then solving for x (restrictions also count) **OR** Graph it

For Absolute value inequalities, graph it **OR** create 2 (or more) equations from the inequality and solve for x

Topic 11 - Inverse Functions

Derivative of the Inverse function:

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Only an inverse **function**, if the original line is **one-to-one**. If not, then an **inverse relation**

Inverse function is found by reflecting graph across $y = x$

(a,b) on $f(x)$ becomes (b,a) on $f^{-1}(x)$

If an Inverse if a function,

$f^{-1}[f(x)] = x$ AND $f[f^{-1}(x)] = x$

Topic 12 - Applied Trigonometry

REMEMBER Angle of **elevation** (from ground) and angle of **depression** (from imaginary horizontal line down to the line)

$sin\,(A) = cos(90^o - A)$, and vice-versa
$tan\,(A) = cot(90^o - A)$, and vice-versa
$sec\,(A) = cosec(90^o - A)$, and vice-versa

Topic 13 - Trigonometric Equations and Radians

'All Stations To Central'

Period = interval which the function repeats at
Amplitude = half the distance from min to max value

In $y = a\,sin(bx)$ and $y = a\,cos(bx)$**,** amplitude = $|a|$, period = $2\pi\ (or\ 360^o) \div b$

Circle parts
Sector = 'slice' of the circle
Segment = straight line (called a secant) which splits the circle into a major segment and minor segment

Degrees to radians, multiply by $\frac{\pi}{180}$

Radians to degrees, multiply by $\frac{180}{\pi}$

Topic 13 - Trigonometric Equations and Radians

Arc length $= \frac{\theta}{2\pi} \times 2\pi r = r\theta$, where θ is in **radians**

$= \frac{\theta}{360} \times 2\pi r$ where θ is in **degrees**

Area of Sector $= \frac{\theta r^2}{2}$, where θ is in **radians**

$= \frac{\theta}{360} \times \pi r^2$, where θ is in **degrees**

Area of a Segment $= \frac{\theta r^2}{2} - \frac{1}{2} r^2 sin(\theta)$, where θ is in **radians** (Ie: Area of sector - triangle made by line with centre of the circle)

Topic 14 - Trigonometric Functions and Identities

$1 + cot^2\theta = cosec^2\theta$

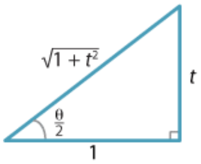
$tan^2\theta + 1 = sec^2\theta$

Topic 15 - Further Trigonometric Identities

$$\begin{aligned} cos\,2A &= cos^2A - sin^2A \\ &= 2cos^2A - 1 \\ &= 1 - 2sin^2A \end{aligned}$$

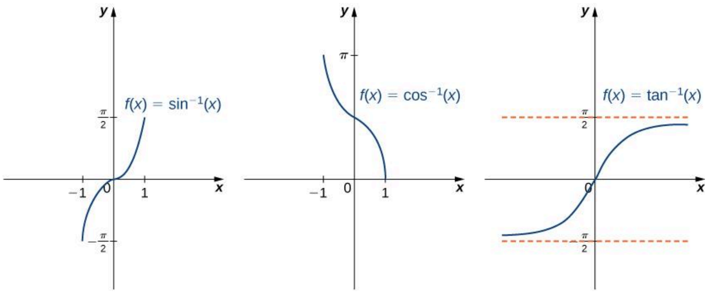
$tan\,2A = \frac{2tanA}{1-tan^2A}$

Prove t-angle formula with the following:



(use double angle formula)

Topic 16 - Inverse Trigonometric Functions



See page 10 to prove the following:

$sin^{-1}(-x) = -sin^{-1}(x)$

$tan^{-1}(-x) = -tan^{-1}(x)$

$cos^{-1}(-x) = \pi - cos^{-1}(x)$

$sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

See page 16 of Topic 16 to see how to find the domain and range of equations like $y = tan^{-1}(1 - x^2)$

Topic 17 - Parametric Forms

Parametric equations have a **third variable**, called the **parameter**

Make a cartesian equation by rearranging one equation in terms of the third variable, and substitute it into the second equation (**Remember restrictions**)

Topic 18 - Calculus

$f(x)$ isn't differentiable at points of **discontinuity**

Secant - line passing through two points on a curve

Differentiate from first principles:

$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Normal = line perpendicular to the **tangent**

$m_{normal} = -\frac{1}{m_{tangent}}$

If $x = displacement$,
when $x > 0$, particle is right of the origin
when $x < 0$, particle is left of the origin
(same thing for velocity, but particle is described as moving)

- When acceleration and velocity are **same direction**, particle is speeding up
- When acceleration and velocity are **different directions**, particle is slowing down

Topic 19 - Exponentials and Logs

e is euler's number, which is where $f'(0) = 1$ in the equation $y = a^x$ (a is euler's number)

$f(x) = e^x$, $f'(x) = e^x$

Richter scale formula: $M = log_{10}(\frac{A}{A_0})$

(A = amplitude of the wave, A_0 is the reference value that corresponds to a zero-level earthquake)

Topic 19 - Exponentials and Logs

Decibel formula:

$L = 10log_{10}\frac{P_2}{P_1}$, where P_2 **and** P_1 are two sounds

PH formula:

$pH = -log_{10}[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in moles per litre

Logarithms:

$y = a^x \Leftrightarrow log_a y = x$

$log_a x + log_a y = log_a xy$

$log_a x - log_a y = log_a \frac{x}{y}$

$log_a x^p = p\ log_a x$

$log_a x = \frac{log_b x}{log_b a}$

$y = log_a x$ **and** $y = a^x$ **are inverse functions**

Topic 20 - Applications of Calculus

- If a line is straight, the variable is increasing or decreasing at a constant rate.
- If it is curved, it is increasing/decreasing at an increasing/decreasing rate

If $N = Ae^{kt}$ then A **is the value of** N when $t = 0$, and k is the **growth constant/rate**

Exponential Decay

$\frac{dN}{dt} = -kN$, where **k** is a positive constant

In $N = P + Ae^{kt}$, $\frac{dN}{dt} = k(N - P)$

Newton's law of cooling

$\frac{dT}{dt} = -k(T - M)$ from $T = M + Ae^{kt}$

Where T is the temperature of the object and M is the temperature of the environment

Chain rule (related rate of change)

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

This chain can technically be infinite

Topic 20 - Applications of Calculus

Volume of:

Sphere = $\frac{4}{3}\pi r^3$ Cone = $\frac{1}{3}\pi r^2 h$

Cylinder = $\pi r^2 h$ Pyramid = $\frac{1}{3}Ah$

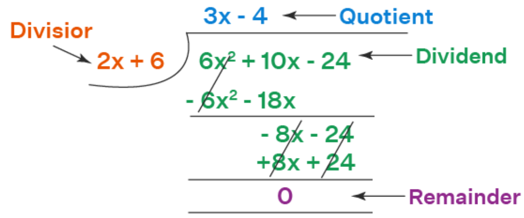
Surface Area of:

Sphere = $4\pi r^2$ Cone = $\pi r^2 + \pi rl$

Cylinder = $2\pi r^2 + 2\pi rh$

Topic 21 - Polynomials

Long division example



$\frac{6x^2+10x-24}{2x+6} = 3x - 4 + \frac{0\ (remainder)}{2x+6}$

(multiply by the divisor to rearrange the equation)

$P(x) = A(x) \times Q(x) + R(x)$

$P(x)$ is the dividend $Q(x)$ is the quotient

$A(x)$ is the divisor $R(x)$ is the remainder

Remainder theorem

If a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$ (**only when divisor is linear**)

Factor theorem

If a polynomial $P(x)$ is divided by $(x - a)$, and if $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$

Multiple root theorem

If a polynomial $P(x)$ has a root of multiplicity m, then its derivative $P'(x)$ has a root of multiplicity $(m - 1)$ and $P''(x)$ has a root of multiplicity $(m - 2)$, etc

Quadratic Polynomials (α *and* β are roots)

$ax^2 + bx + c = 0$

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$

$\alpha + \beta = -\frac{b}{a}$

$\alpha\beta = \frac{c}{a}$

Cubic Polynomial

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$,

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

Quartic Polynomial - Same pattern (see booklet)

YR 12

Topic 1 - Induction

$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Summation

Test $n = 1$

LHS:

...

RHS:

... = LHS

$LHS = RHS$ so true for $n = 1$

Assume true for $n = k$

Ie. (...)

Show true for $n = k + 1$

Ie. show (…)

… (‘by assumption’ when used)

= RHS

Hence, if the result is true for $n = k$ then it is true for $n = k + 1$

Since the result is true for $n = 1$ and for $n = k + 1$, assuming true for $n = k$, then by the process of mathematical induction, the result is true for all positive integer values of n

Divisibility

Test $n = 1$

…, which is divisible by __. Therefore true for $n = 1$

Assume true for $n = k$

Ie. … = __M, where M is an integer (**rearrange**)

Show true for $n = k + 1$

Ie. show … is divisible by __

… (**remember ‘by assumption’** when needed)

= __(...), which is divisible by _ because it is a multiple of _ and … is an integer.

(**GO TO THE TOP OF THE PAGE)**

Divisibility CONTINUED

Hence, if the result is divisible for $n = k$, it is divisible for $n = k + 1$

Since the result is true when $n = 1$ and $n = k + 1$, assuming true for $n = k$, then by the process of mathematical induction, the result is true for all positive integer values of n

Topic 2 - Graphing Techniques

Consider functions of the form:

$y = k\,f[a(x + b)] + c$

- **Vertical dilation** with **scale factor** k (k stretches or compresses the function vertically
- c **translates** the graph **vertically**
- **Horizontal dilation** with scale factor of $\frac{1}{a}$ ($\frac{1}{a}$ stretches or compresses the function horizontally)
- b **translates** the graph **left** or **right**

ORDER OF TRANSFORMATIONS

1. Horizontal dilation with a factor $\frac{1}{a}$
2. Horizontal translation b units left/right
3. Vertical dilation with scale factor k
4. Vertical translation c units up/down

Topic 3A - Trigonometric Functions and Graphs

Use booklet to see how to do questions where a variable changes in a trigonometric pattern (wave tides, height on ferris wheels, etc)

Topic 3B - Trigonometric Equations

Auxiliary Angle

$a\,sin(x) + b\,cos(x) = r\,sin(x + \alpha)$

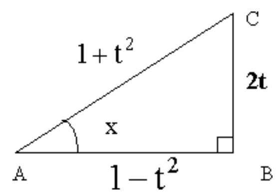
$a\,sin(x) - b\,cos(x) = r\,sin(x - \alpha)$

$a\,cos(x) - b\,sin(x) = r\,cos(x + \alpha)$

$a\,cos(x) + b\,sin(x) = r\,cos(x - \alpha)$

Where $r = \sqrt{a^2 + b^2}$ and $tan(\alpha) = \frac{b}{a}$

T formulae



When using t-results to solve equations, remember to test for when $tan\frac{x}{2}$ is undefined ($x = 180, 540,...$)

Topic 6 - Differentiation of Trig, Exponential and logarithmic function → Rules for Differentiation

$\lim_{x\to 0} \frac{sin(x)}{x} = 1$ $\lim_{x\to 0} \frac{cos(x)-1}{x} = 0$

$\lim_{x\to 0} \frac{tan(x)}{x} = 1$

Reference sheet!

Topic 7 - Introduction to Vectors

Include the squiggle below vectors!

$\binom{u}{v}$ = Vector column notation (NO FRACTION LINE)

(x, y) = position vector (tail is always at origin)

$\left|\vec{AB}\right|$ = magnitude/length of a vector

$\vec{v} = x\vec{i} + y\vec{j}$ → Component form (squiggle)

Parallel vectors can be in the same or opposite direction!

$\vec{AB} = \vec{OB} - \vec{OA}$

$\left|\vec{OA}\right| = \sqrt{a^2 + b^2}$, where $\vec{OA} = \binom{a}{b}$

$\hat{u} = \frac{u}{|u|}$ (all have squiggles)

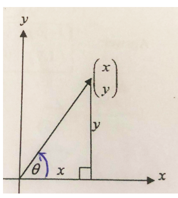
$a = |a|cos(\theta)\ \vec{i} + |a|sin(\theta)\ \vec{j}$
(resolving into component form)

Unit vector: If $a = x\vec{i} + y\vec{j}$, then
$\hat{a} = \frac{1}{\sqrt{x^2+y^2}}(x\vec{i} + y\vec{j})$ → Rationalise

Direction of a vector:

$tan\theta = \frac{y}{x}$,

θ is measured counterclockwise from the positive x axis → **It isn't always acute**



Topic 8 - Further Operations with Vectors

$a \bullet b = |a||b|cos\theta$
$a \bullet b = x_1x_2 + y_1y_2$ → for component form vectors

$cos\theta = \frac{a \bullet b}{|a||b|}$

Parallel:
$a \bullet b = |a||b|$ → when in the same direction
$a \bullet b = -|a||b|$ → when in opposite directions

Equal:
$a \bullet a = |a|^2$ → will be 1 for unit vectors

Perpendicular/Orthogonal:
$a \bullet b = 0$

Note:
$(a + b) \bullet (a - b) = |a|^2 - |b|^2$

$\left|proj_b a\right| = \frac{a \bullet b}{|b|}$

$proj_b a = \frac{a \bullet b}{|b|^2}(b) = \left|proj_b a\right| \times \frac{b}{|b|}$

Projection of vector a **perpendicular** to b
$a - proj_b a$

See Booklet for proving quadrilaterals

Topic 9 - The first + second derivatives → Applications of the derivative

$y' > 0$ for all x, Increasing function
$y' < 0$ for all x, Decreasing function

Test for a **change** in **gradient sign/concavity** for stationary points or points of inflection

Test Endpoints for optimisation

Points of inflection include:

- an **oblique** point of inflection ($y' \neq 0$ at the point),
- a **horizontal** point of inflection ($y'\ and\ y'' = 0$ at the point),
- a **vertical** point of inflection (y' and y'' are undefined at the point)

(concavity must always change)