2017 Bored of Studies Trial Examinations

Mathematics Extension 1

9th October 2017

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 6 – 13

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box on the Multiple Choice answer sheet provided.

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1 For some positive integer *n* and real *x*, suppose that

$$(1-x)^{4n+2} = \sum_{k=0}^{4n+2} a_k x^k .$$

Which of the following is greater than all the others?

(A) a_0.

(B) a_{2n}.

(C) a_{2n+1}.

(D) a_{4n+2}.

2 Consider a function $f(x)$ and its inverse function $f^{-1}(x)$. The angle between the tangents of the two curves at their point of intersection is $\frac{\pi}{6}$. Which of the following is a possible value for the gradient of $f(x)$ at that point of intersection?

(A) $\frac{1}{\sqrt{3}}$.

(B) $\frac{\sqrt{3}}{2}$.

(C) $\frac{2}{\sqrt{3}}$.

(D) $3\sqrt{3}$.

3 Suppose that a population of size N undergoes exponential decay at time t according to the differential equation

$$\frac{dN}{dt} = -k\left(N - A\right),$$

where A and k are positive constants. The following properties are observed:

- The initial population is $A + B$ for some positive constant B.
- When $t = t_0$, the population is $2A$.
- When $t = mt_0$, the population is $\left(n+1\right)A$ for some positive constants m and n.

Which of the following is correct?

(A) $\left(\frac{A}{B}\right)^{m-1} = n-1.$

(B) $\left(\frac{A}{B}\right)^{m-1} = n.$

(C) $\left(\frac{A}{B}\right)^{m} = n-1.$

(D) $\left(\frac{A}{B}\right)^{m} = n.$

4 Which of the following is the maximum value of

$$\sqrt{2}\sin x + \cos\left(x + \frac{\pi}{4}\right)?$$

(A) $1+\sqrt{2}.$

(B) $\sqrt{2}.$

(C) $\sqrt{3}.$

(D) $1.$

5 Let P and Q be points on a number plane with coordinates (x_1, y_1) and (x_2, y_2) respectively. Let $R(x_0, y_0)$ be a point that divides the interval PQ externally in the ratio $m:n$. The x-coordinate of R can be expressed in the form $x_0 = \alpha x_1 + \beta x_2$.

What is the value of $\alpha + \beta$?

(A) -1.

(B) $\dfrac{m-n}{m+n}$.

(C) $\dfrac{m+n}{m-n}$.

(D) 1.

6 A game is constructed where the probability of winning at least twice in three attempts is greater than or equal to the probability of winning in one attempt. Assume winning and losing are the only outcomes of the game, and the probability of winning for each attempt is the same.

Which of the following can NOT be a possible probability of winning the game in one attempt?

(A) 0.

(B) $\dfrac{1}{3}$.

(C) $\dfrac{2}{3}$.

(D) 1.

7 Which of the following is equal to $\int_a^b \frac{dx}{\sqrt{1-x^2}}$, where $-1<a<1$ and $-1<b<1$?

(A) $\sin^{-1}\left(a\sqrt{1-b^2}-b\sqrt{1-a^2}\right)$.

(B) $\sin^{-1}\left(a\sqrt{1-b^2}+b\sqrt{1-a^2}\right)$.

(C) $\sin^{-1}\left(b\sqrt{1-a^2}-a\sqrt{1-b^2}\right)$.

(D) $\sin^{-1}\left(b\sqrt{1-a^2}+a\sqrt{1-b^2}\right)$.

8 When a monic cubic polynomial $P(x)$ is divided by $(x-1)$, the remainder is 2. Also, when $P(x)$ is divided by (x^2+4x+1), the remainder is $(x+1)$.

Which of the following is a possible expression for $P(x)$?

(A) x^3-3x^2+2x+2.

(B) x^3+3x^2+2x-4.

(C) x^3-3x^2-2x+6.

(D) x^3+3x^2-2x.

9 Consider a solid hemisphere with radius r. Suppose that the surface area of the hemisphere is growing at a constant rate of k units per second.

What is the rate of change of the volume of the hemisphere?

(A) $\frac{k}{2}r$.

(B) $\frac{k}{3}r$.

(C) $2kr$.

(D) $3kr$.

10 What is the value of $\lim_{h\to 0}\left(\frac{1}{h}\int_{-h}^{h} e^{-x^2}\,dx\right)$?

(A) 0.

(B) 1.

(C) 2.

(D) 4.

Total marks – 60
Attempt Questions 11 – 14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $t = \tan\left(\frac{\theta}{2}\right)$ and consider the trigonometric equation

$$\sin 2\theta - \sin\theta + \cos\theta + 1 = 0 .$$

(i) Rewrite the above equation as a polynomial in terms of t. **1**

(ii) If $\tan\left(\frac{\alpha}{2}\right)$, $\tan\left(\frac{\beta}{2}\right)$ and $\tan\left(\frac{\gamma}{2}\right)$ are roots of the polynomial in (i), **2**

show that $\alpha + \beta + \gamma = 2k\pi$, for some integer k,

(b) Consider the function $f(x) = \sin^{-1}\sqrt{x}$.

(i) State the domain and range of $f(x)$. **2**

(ii) Show that $f(x)$ is increasing. **1**

(iii) Sketch the graph of $y = f(x)$. **2**

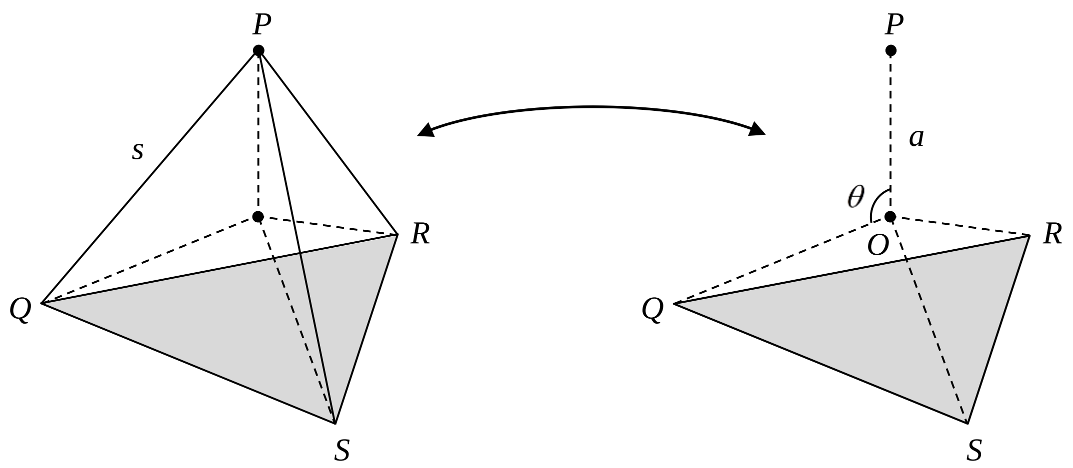
(iv) Hence, find the area of the region bounded by the curve $y = f(x)$, **3**

the y-axis and the line $y = \frac{\pi}{4}$.

Question 11 continues on page 7

(c) Consider a triangular pyramid with vertices P, Q, R and S. Each of the faces of the pyramid are equilateral triangles with side length s.

Let O be the centre of the pyramid such that it is equidistant from each of the vertices with distance a. Let θ be the obtuse angle between any two vertices subtended at O.



(i) Show that $s = a\sqrt{3}\sin\theta$. **2**

(ii) Hence show that $\theta = \cos^{-1}\left(-\frac{1}{3}\right)$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose that $f\left(x\right)$ is an even function and a is some positive constant.

(i) Use the substitution $u=-x$ to show that **2**

$$\int_{-a}^{a}\frac{f\left(x\right)}{1+e^{x}}\,dx=\int_{0}^{a}f\left(x\right)dx.$$

(ii) Use (i) and the substitution $u=1+e^{x}$ to evaluate **3**

$$\int_{-1}^{1}\frac{\left(1-e^{x}\right)^{2}}{\left(1+e^{x}\right)^{3}}\,dx.$$

(b) A particle moves on a number plane with the following horizontal and vertical acceleration equations

$$\ddot{x}=-x$$
$$\ddot{y}=-9y$$

The particle is initially at the point $\left(a,b\right)$ on the number plane where a and b are the maximum horizontal and vertical displacements respectively.

(i) Find the horizontal and vertical displacement equations of the particle at time t. **1**

(ii) By finding the Cartesian equation of the particle's displacement, or otherwise, sketch the path of the particle's motion on the number plane. **3**

(iii) Hence, describe the motion of the particle over time. **1**

Question 12 continues on page 9

(c) Consider the approximation of the positive root of $x^2 = a$ using Newton's method. Suppose that x_0 is the initial approximation of the root, where $x_0 > 0$. Let x_n be the approximation to the root after n applications of Newton's method.

Let

$$E_n = x_n - \sqrt{a}\,,$$

for $n = 0, 1, 2, \ldots$

(i) Show that $E_n = \dfrac{E_{n-1}^2}{2\left(E_{n-1} + \sqrt{a}\right)}$ for $n = 1, 2, 3, \ldots$ **2**

(ii) Suppose that Newton's method was applied once using x_0 to get a better approximation to the root. **1**

Is this approximation less than or greater than the exact value of the root? Justify your answer.

(iii) Deduce that $E_0 > 2^n E_n$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) An object is fired from a cliff of height h metres with initial velocity V at an acute angle of α from the horizontal. You may assume the following equations of motion.

$$x = Vt\cos\alpha$$

$$y = -\frac{gt^2}{2} + Vt\sin\alpha + h$$

(i) Show that the horizontal range of the motion is given by **2**

$$R = \frac{V\cos\alpha}{g}\left(V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}\right).$$

(ii) Let α_{max} be the angle of projection that maximises the range R. **3**

Show that $\tan^2\alpha_{\text{max}} = \dfrac{V^2}{V^2 + 2gh}$.

(iii) Let β_{max} be the acute angle at which the object hits the ground **2**

when it is initially projected at angle α_{max}.

Show that $\alpha_{\text{max}} + \beta_{\text{max}} = \dfrac{\pi}{2}$.

(iv) Let P be the point from which the particle is projected, and let **1**
Q be the point the particle can land with maximum range.

Explain briefly why PQ must be a focal chord of the parabolic trajectory.

Question 13 continues on page 11

(b) Suppose that a and b are positive real numbers. **3**

Use mathematical induction to prove that

$$\left(\frac{a+b}{2}\right)^n \le \frac{a^n+b^n}{2},$$

for positive integer n.

(c) A particle is initially at rest on the number line at $x=\frac{1}{2}$. The particle moves continuously along the number line line according to the acceleration equation:

$$\ddot{x}=\frac{1}{(x-1)^2}+\frac{1}{x^3}.$$

(i) Let v be the velocity of the particle at time t. Show that **1**

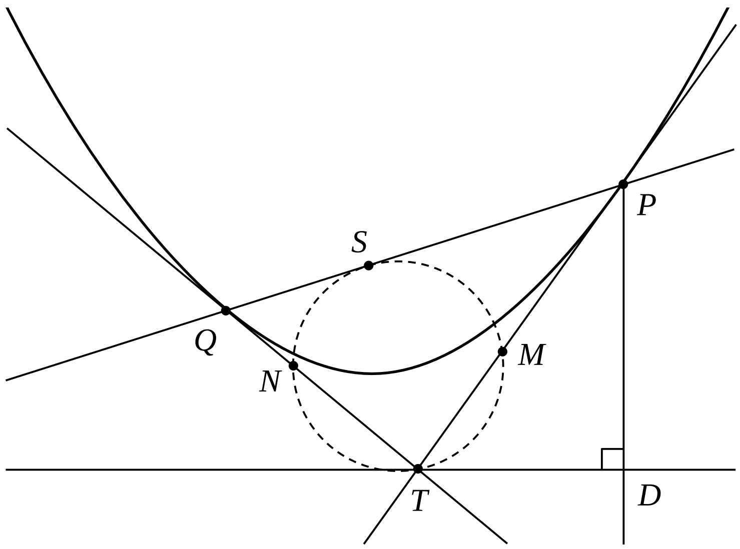
$$v^2=-\frac{2}{x-1}-\frac{1}{x^2}.$$

(ii) Hence, or otherwise, find the range of possible values that the particle's displacement can be during its motion along the number line. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Let PQ be a focal chord on the parabola $x^2 = 4ay$ with vertex O. The tangents from P and Q intersect at T. A circle is drawn through T such that it is tangential to PQ at S. The circle passes through the tangent drawn from P and Q at M and N respectively.



(i) Show that $TD^2 = TM \times TP$. **3**

(ii) Explain why PD is the diameter of the circle passing through D, M and P. **1**

(iii) Hence, deduce that M is the midpoint of SD. **2**

(iv) Hence, or otherwise, show that $OM \times ON = a^2$. **2**

Question 14 continues on page 13

(d) A committee of k is to be formed from a total of n people. From the committee of k, a chairman and treasurer must be chosen. A single person may be both chairman and treasurer.

(i) Explain why the number of possible committees of k people is **1**

$$k^2\binom{n}{k}.$$

(ii) By considering committees of all sizes, explain why **2**

$$\sum_{k=1}^{n} k^2\binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}.$$

(iii) Show that **1**

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

(iv) Hence, or otherwise, deduce that **3**

$$\sum_{k=1}^{n} k^3\binom{n}{k} = n^2(n+3)2^{n-3}.$$

End of Exam