2021 BORED OF STUDIES TRIAL EXAMINATION

Mathematics Extension 1

Solutions

Section I

Answers

1	B	**6**	B
2	B	**7**	D
3	A	**8**	C
4	B	**9**	D
5	A	**10**	C

Brief explanations

1 If $P(x)$ is a cubic polynomial, the graph of $y = \dfrac{1}{P(x)}$ has vertical asymptotes at the roots of $P(x)$.

Since there are only two vertical asymptotes then two of the roots are equal (i.e. a double root). Hence, the answer is (B).

2 Evaluating the integral directly gives

$$\begin{aligned}\int \frac{dx}{\sqrt{16-9x^2}} &= \frac{1}{3}\int \frac{3}{\sqrt{16-9x^2}}\,dx \\ &= \frac{1}{3}\sin^{-1}\frac{3x}{4} + c\end{aligned}$$

Noting that $\sin^{-1}\dfrac{3x}{4} = \dfrac{\pi}{2} - \cos^{-1}\dfrac{3x}{4}$, it is not possible for the answer to be (A) or (C).

Let $u = \sin^{-1}\dfrac{3x}{4}$, noting that $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This implies that

$$\begin{aligned}\sin u &= \frac{3x}{4} \\ \cos u &= \frac{\sqrt{16-9x^2}}{4} \qquad \text{noting that } u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ so } \cos u \geq 0 \\ \tan u &= \frac{3x}{\sqrt{16-9x^2}} \\ u &= \tan^{-1}\left(\frac{3x}{\sqrt{16-9x^2}}\right)\end{aligned}$$

Hence, the answer is (B).

3 Since $X \sim \text{Bin}\left(100, \frac{1}{10}\right)$ then $E(X) = 10$ and $\text{Var}(X) = 9$.

Approximate X with a normal random variable $Z = \dfrac{X - 10}{3}$ where $Z \sim N(0, 1)$.

For option (A), $P(4 \le X \le 16) \approx P(-2 \le Z \le 2)$

For option (B), $P(8 \le X \le 28) \approx P(-\frac{2}{3} \le Z \le 6)$

For option (C), $P(0 \le X \le 95) \approx P(-\frac{10}{3} \le Z \le \frac{85}{3})$

For option (D), $P(3 \le X \le 98) \approx P(-\frac{7}{3} \le Z \le \frac{88}{3})$

As per the reference sheet, it is known that 95% of scores have z-scores between -2 and 2. The probability value is closest to 0.95 in (A).

Remark: For completeness, the options (C) and (D) far exceed the interval $-2 \le Z \le 2$ and so can be ruled out.

For (B), note that as per the reference sheet 68% of scores have z-scores between -1 and 1. By symmetry of the normal curve, 34% of scores have z-scores between -1 and 0. This implies that 16% of scores have z-scores less than -1. Hence, $P(-\frac{2}{3} \le Z \le 6)$ cannot be close to 0.95 as there is at least 16% that is not covered by this interval.

4 Test the point $(1, 1)$, where the direction field suggests a relatively stationary point. This only works for options (B) and (D). Test another point $(2, 1.5)$, where the direction field suggests a positive gradient. This only choice which satisfies this is (B).

5 Rearranging the differential equation in the standard exponential growth/decay structure

$$\begin{aligned}\frac{dv}{dt} &= a + bv \\ &= b\left(v + \frac{a}{b}\right)\end{aligned}$$

The general solution to this differential equation is $v = -\dfrac{a}{b} + Ae^{bt}$, for some constant A. For an exponential decay, it is necessary that $b < 0$. For a negative limiting value, $\dfrac{a}{b} > 0$ is necessary.

Since $b < 0$, then it is necessary that $a < 0$. Hence, the answer is (A).

6 There are five cases to consider.

When there are *no ties* then the number of possible outcomes is 4!.

When *two students tie only*, then there are $\binom{4}{2}$ possible choices in that tie. There are 3! ways to arrange the positioning of the pair of ties versus the remaining non-ties. The number of possible outcomes is therefore $3! \times \binom{4}{2}$.

When *two pairs of students tie*, then there are $\binom{4}{2}$ possible choices in one of the ties. Note that the choice of one of the ties automatically determines the choice of the other tie, so the arrangements between the ties are already counted.

When *three students tie*, then there are $\binom{4}{3}$ possible choices in that tie. There are 2! ways to arrange the positioning of the triplet of ties versus the single non-tie. The number of possible outcomes is therefore $2! \times \binom{4}{3}$.

When *four students tie*, then there are only $\binom{4}{4}$ possible choices in that tie.

The probability that there will be two pairs of students tied is therefore

$$\frac{\binom{4}{2}}{4! + 3!\binom{4}{2} + \binom{4}{2} + 2!\binom{4}{3} + 1} \text{ or equivalently } \frac{2}{25}. \text{ Hence, the answer is (B).}$$

7 Testing $u = x + y$

$$\begin{aligned} u &= x + y \\ \frac{du}{dx} &= 1 + \frac{dy}{dx} \\ &= \frac{x^2 + y^2 + xy}{xy} \\ &= \frac{x^2 + (u - x)^2 + x(u - x)}{x(u - x)} \\ &= \frac{x^2 - xu + u^2}{x(u - x)} \end{aligned}$$

This is not separable in u and x. A similar argument can be made for $u = x - y$.

Testing $u = xy$

$$\begin{aligned} u &= xy \\ \frac{du}{dx} &= y + x\frac{dy}{dx} \\ &= \frac{u}{x} + \frac{x\left(x^2 + \frac{u^2}{x^2}\right)}{u} \\ &= \frac{2u^2 + x^4}{ux} \end{aligned}$$

This has a similar structure to $\dfrac{dy}{dx}$ and is not separable in u and x.

Testing $u = \frac{y}{x}$

$$\begin{aligned}
u &= \frac{y}{x} \\
\frac{du}{dx} &= \frac{x\frac{dy}{dx} - y}{x^2} \\
&= \frac{\frac{x^2+y^2}{y} - y}{x^2} \\
&= \frac{x^2 + x^2u^2 - x^2u^2}{x^3u} \\
&= \frac{1}{xu}
\end{aligned}$$

This is a separable differential equation between u and x. Hence, the answer is (D).

8 There are 10 ways to select n adjacent numbers from the circle in the given arrangement.

There is always one choice of n adjacent numbers that starts with 1, and another set of n adjacent numbers that starts with a 2, and so on. This means that the sum of all the first numbers must be $1 + 2 + \cdots + 10$ or equivalently, 55.

This can be similarly applied to the 2nd, 3rd,..., nth card position in the chosen set. Hence, the sum of all the numbers on the cards across all the possibilities is $55n$.

Now the problem can thought of as distributing $55n$ across 10 possible sums of n numbers. The lowest possible sum in the set of n cards is $5.5n$ (with rounding up to integers). From the pigeonhole principle, it is guaranteed that there is one set of n cards, which sums to $5.5n$ (with rounding up to integers) or above.

When $n = 3$ or $n = 4$ it is guaranteed that one set of cards will sum to at least 17 or 22 respectively. This is insufficient to guarantee that there is at least one set of cards that will sum to at least 28.

When $n = 5$ or $n = 6$ it is guaranteed that one set of cards will sum to at least 28 or 33 respectively. These cases are sufficient to guarantee that there is at least one set of cards that will sum to at least 28. The lowest is when $n = 5$, hence the answer is (C).

9 Given that

$$\begin{aligned}
a\sin x + b\cos x &= R\sin(x + \alpha) \\
&= R\sin x\cos\alpha + R\cos x\sin\alpha
\end{aligned}$$

Equating coefficients gives $a = R\cos\alpha$ and $b = R\sin\alpha$. If $ab < 0$ then $R^2\sin\alpha\cos\alpha < 0$. This suggests that α must be such that $\sin\alpha$ and $\cos\alpha$ have opposite signs. This occurs in the 2nd quadrant (where $\sin\alpha > 0$ and $\cos\alpha < 0$) and the 4th quadrant (where $\sin\alpha < 0$ and $\cos\alpha > 0$). Hence, the answer is (D).

10 Let X be the number of correct answers, where $X \sim \text{Bin}\left(5, \frac{1}{4}\right)$. The probability that the student gets exactly 3 questions correct, *given* she knows she got at least one correct is

$$\begin{aligned} P(X = 3 | X \in \{1, 2, 3, 4, 5\}) &= \frac{P(X = 3 \cap X \in \{1, 2, 3, 4, 5\})}{P(X \in \{1, 2, 3, 4, 5\})} \\ &= \frac{P(X = 3)}{1 - P(X = 0)} \\ &= \frac{\binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2}{1 - \binom{5}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^5} \\ &= \frac{90}{781} \end{aligned}$$

Hence, the answer is (C).

Question 11

(a) Using the fact that $3 + 7 = 10$ and $\binom{n}{0} = \binom{n}{n} = 1$

$$\begin{aligned}
10^n - 3^n - 7^n &= (3+7)^n - 3^n - 7^n \\
&= \binom{n}{0}3^n + \binom{n}{1}3^{n-1}7^1 + \binom{n}{2}3^{n-2}7^2 + \cdots + \binom{n}{n}7^n - 3^n - 7^n \\
&= \binom{n}{1}3^{n-1}7^1 + \binom{n}{2}3^{n-2}7^2 + \cdots + \binom{n}{n-1}3^1 7^{n-1} \\
&= 21\left[\binom{n}{1}3^{n-2} + \binom{n}{2}3^{n-3}7^1 + \cdots + \binom{n}{n-1}7^{n-2}\right] \\
&= 21M \quad \text{where } M = \binom{n}{1}3^{n-2} + \binom{n}{2}3^{n-3}7^1 + \cdots + \binom{n}{n-1}7^{n-2}
\end{aligned}$$

Since $\binom{n}{k}$ is an integer for any value of k, then M is also an integer. Hence, $10^n - 3^n - 7^n$ is divisible by 21.

(b) Rearranging the inequality

$$\begin{aligned}
\frac{\sqrt{2-x}}{x} &< 1 \\
\frac{\sqrt{2-x} - x}{x} &< 0 \\
x(\sqrt{2-x} - x) &< 0 \\
x(x - \sqrt{2-x}) &> 0
\end{aligned}$$

There are two cases to consider.

Case 1 - $x < 0$ and $x < \sqrt{2-x}$

Since $\sqrt{2-x} > 0$ then $\sqrt{2-x} > x$ is always true if $x < 0$. Hence, the solution in this case is $x < 0$.

Case 2 - $x > 0$ and $x > \sqrt{2-x}$

Since both sides of the inequality are positive, squaring both sides preserves the sign

$$\begin{aligned}
x &> \sqrt{2-x} \\
x^2 &> 2 - x \\
(x+2)(x-1) &> 0 \\
x < -2 \quad &\text{or} \quad x > 1
\end{aligned}$$

However, $x > 0$ in this case, so only take $x > 1$. Also, $\sqrt{2-x}$ has an upper bound restriction that $x \leq 2$, so the full solution in this case is $1 < x \leq 2$.

Hence, the final solutions are $x < 0$ and $1 < x \leq 2$.

(c) (i) Differentiating

$$
\begin{aligned}
f(x) &= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \\
f'(x) &= -\frac{1}{\sqrt{1-\frac{1}{1+x^2}}} \times -\frac{1}{2} \times 2x \times (1+x^2)^{-\frac{3}{2}} \\
&= \frac{x}{(1+x^2)\sqrt{(1+x^2)\left(1-\frac{1}{1+x^2}\right)}} \\
&= \frac{x}{\sqrt{x^2}(1+x^2)} \\
&= \frac{x}{|x|(1+x^2)}
\end{aligned}
$$

When $x = 0$, then $f'(x)$ is undefined. For the remainder of the domain

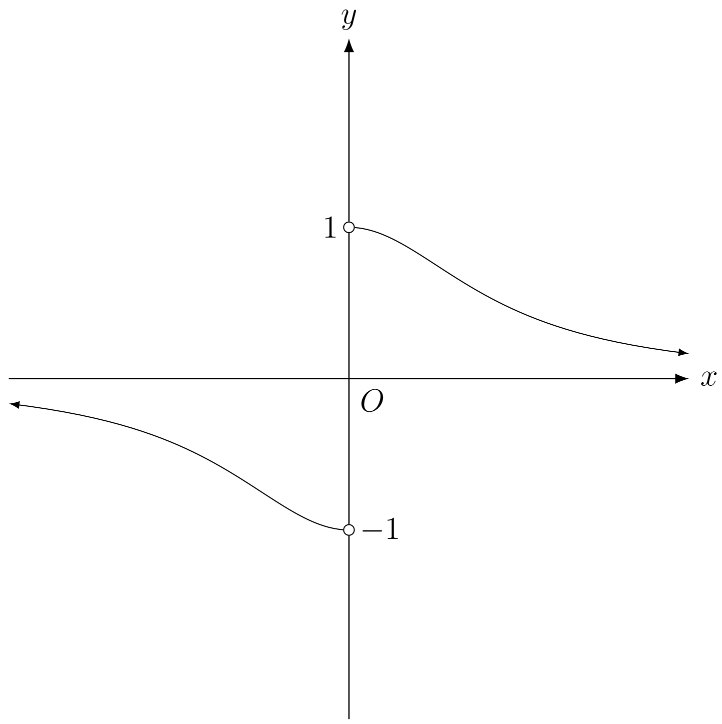
$$
f'(x) = \begin{cases} -\dfrac{1}{1+x^2}, & \text{for } x < 0 \\[2ex] \dfrac{1}{1+x^2}, & \text{for } x > 0 \end{cases}
$$

To sketch this, first consider the graph of $y = x^2 + 1$ which is a parabola with minimum turning point at $(0, 1)$. When $x \to \infty$ then $y \to \infty$ and when $x \to -\infty$ then $y \to \infty$.

For the graph of $y = \dfrac{1}{1+x^2}$ for $x > 0$, note that when $x \to \infty$ then $y \to 0^+$.

For the graph of $y = -\dfrac{1}{1+x^2}$ for $x < 0$ note that when $x \to -\infty$ then $y \to 0^-$.

Hence, the graph of $y = f'(x)$ is as follows.

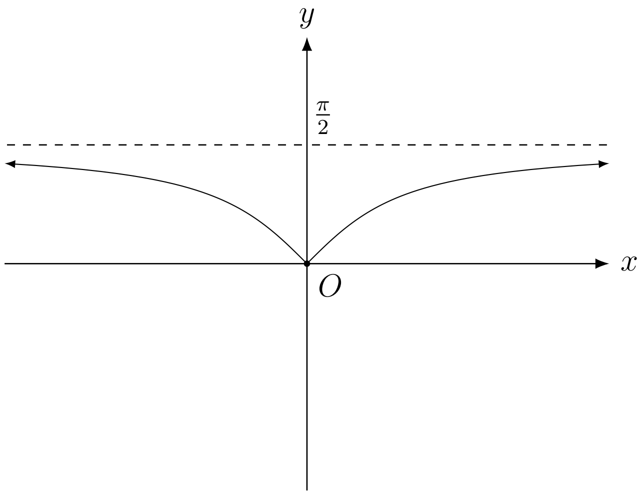


(ii) Since $f'(x) = \dfrac{1}{1+x^2}$ when $x > 0$ then $f(x) = \tan^{-1} x + c$. However, $f(1) = \frac{\pi}{4}$ hence $c = 0$.

Similarly, it can deduced that $f(x) = -\tan^{-1} x$ when $x < 0$. This means that

$$f(x) = \begin{cases} -\tan^{-1} x\,, & \text{for } x < 0 \\ 0\,, & \text{for } x = 0 \\ \tan^{-1} x\,, & \text{for } x > 0 \end{cases}$$

This gives the following graph



(iii) The volume is given by

$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{4}} x^2\,dy \quad \text{but } y = \tan^{-1} x \quad \text{for } x > 0 \\ &= \pi \int_0^{\frac{\pi}{4}} \tan^2 y\,dy \\ &= \pi \int_0^{\frac{\pi}{4}} (\sec^2 y - 1)\,dy \\ &= \pi \left[\tan y - y\right]_0^{\frac{\pi}{4}} \\ &= \pi \left(1 - \frac{\pi}{4}\right) \quad \text{cubic units} \end{aligned}$$

(d) (i) The projection of the unit vector $\underset{\sim}{\hat{\ell}}$ onto the direction of $\underset{\sim}{\hat{m}}$ is given by $(\underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}}$.

The projection of the unit vector $\underset{\sim}{\hat{r}}$ onto the direction of $\underset{\sim}{\hat{m}}$ is given by $(\underset{\sim}{\hat{r}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}}$.

Since $\underset{\sim}{\hat{\ell}}$ and $\underset{\sim}{\hat{r}}$ are equally inclined to the horizontal and have equal length, then the length of the projection of $\underset{\sim}{\hat{r}}$ is the same as the length of the projection of $\underset{\sim}{\hat{\ell}}$ so $(\underset{\sim}{\hat{r}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}} = (\underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}}$. Also, the vector $\underset{\sim}{\hat{\ell}} + \underset{\sim}{\hat{r}}$ must be parallel to the mirror.

The length of $\underset{\sim}{\hat{\ell}} + \underset{\sim}{\hat{r}}$ is in fact the sum of the projections of $\underset{\sim}{\hat{\ell}}$ and $\underset{\sim}{\hat{r}}$ in the direction of $\underset{\sim}{\hat{m}}$. This means that

$$\begin{aligned} \underset{\sim}{\hat{\ell}} + \underset{\sim}{\hat{r}} &= (\underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}} + (\underset{\sim}{\hat{r}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}} \\ &= 2(\underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}} \\ \underset{\sim}{\hat{r}} &= 2(\underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{\hat{m}})\underset{\sim}{\hat{m}} - \underset{\sim}{\hat{\ell}} \end{aligned}$$

(ii) The length of $\underset{\sim}{s}$ is given by

$$\begin{aligned} |\underset{\sim}{s}| &= |\underset{\sim}{\hat{\ell}} \cdot \underset{\sim}{\hat{m}}| + |\underset{\sim}{\hat{r}} \cdot \underset{\sim}{\hat{m}}| \\ &= 2\cos\theta \end{aligned}$$

With the second mirror reflecting the beam back to the source, the angle between $\underset{\sim}{s}$ and $-\underset{\sim}{\hat{r}}$ is θ by alternate angles. This means that

$$\begin{aligned} \alpha + \theta + \alpha &= \pi \\ \alpha &= \frac{\pi}{2} - \frac{\theta}{2} \end{aligned}$$

Hence

$$\begin{aligned} \underset{\sim}{s} \cdot \underset{\sim}{\hat{n}} &= |\underset{\sim}{s}| \cos\alpha \\ &= 2\cos\theta\cos\alpha \\ &= 2\cos\theta\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\ &= 2\cos\theta\sin\frac{\theta}{2} \\ &= \sin\left(\theta + \frac{\theta}{2}\right) - \sin\left(\theta - \frac{\theta}{2}\right) \\ &= \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} \end{aligned}$$

Question 12

(a) Multiplying the LHS by $\dfrac{2\sin\left(\dfrac{\pi}{2n}\right)}{2\sin\left(\dfrac{\pi}{2n}\right)}$ gives

$$\begin{aligned}
\text{LHS} &= \sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right) \\
&= \frac{2\sin\left(\frac{\pi}{2n}\right)\left[\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right)\right]}{2\sin\left(\frac{\pi}{2n}\right)}
\end{aligned}$$

For the k^{th} term in the numerator, apply the products to sums transformation.

$$2\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{k\pi}{n}\right) = \cos\left(\left(k-\frac{1}{2}\right)\frac{\pi}{n}\right) - \cos\left(\left(k+\frac{1}{2}\right)\frac{\pi}{n}\right)$$

This means that

$$\begin{aligned}
2\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{\pi}{n}\right) &= \cos\left(\frac{\pi}{2n}\right) - \cos\left(\frac{3\pi}{2n}\right) \\
2\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{2\pi}{n}\right) &= \cos\left(\frac{3\pi}{2n}\right) - \cos\left(\frac{5\pi}{2n}\right) \\
2\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{3\pi}{n}\right) &= \cos\left(\frac{5\pi}{2n}\right) - \cos\left(\frac{7\pi}{2n}\right) \\
&\vdots \\
2\sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{n\pi}{n}\right) &= \cos\left(\frac{(2n-1)\pi}{2n}\right) - \cos\left(\frac{(2n+1)\pi}{2n}\right)
\end{aligned}$$

Summing these gives the numerator of the LHS, with many terms cancelling out. Hence

$$\begin{aligned}
\text{LHS} &= \frac{\cos\left(\frac{\pi}{2n}\right) - \cos\left(\frac{(2n+1)\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)} \\
&= \frac{\cos\left(\frac{\pi}{2n}\right) - \cos\left(\pi + \frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)} \\
&= \frac{2\cos\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)} \\
&= \cot\left(\frac{\pi}{2n}\right) \\
&= \text{RHS}
\end{aligned}$$

(b) Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Using the formulas for the dot product and the area of triangle

$$\begin{aligned}
\underset{\sim}{a} \cdot \underset{\sim}{b} &= |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta \\
A &= \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta \\
(2A)^2 + (\underset{\sim}{a} \cdot \underset{\sim}{b})^2 &= |\underset{\sim}{a}|^2|\underset{\sim}{b}|^2 \\
4A^2 &= (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 \\
&= a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 - a_1^2b_1^2 - 2a_1a_2b_1b_2 - a_2^2b_2^2 \\
&= a_1^2b_2^2 + a_2^2b_1^2 - 2a_1b_2a_2b_1 \\
&= (a_1b_2 - a_2b_1)^2 \\
A &= \frac{1}{2}|a_1b_2 - a_2b_1|
\end{aligned}$$

(c) (i) Note that $\underset{\sim}{r}(t) = f(t)\underset{\sim}{i} + tf(t)\underset{\sim}{j}$. Using the result in part (b) for the area of the triangle with $\underset{\sim}{a} = \underset{\sim}{r}(t)$ and $\underset{\sim}{b} = \underset{\sim}{r}(t+h)$

$$\begin{aligned}
A &= \frac{1}{2}\left|f(t)(t+h)f(t+h) - tf(t)f(t+h)\right| \\
&= \frac{1}{2}\left|hf(t)f(t+h)\right| \\
&= \frac{h}{2}f(t)f(t+h) \quad \text{(noting that } h > 0 \text{ and } f(t) \geq 0 \text{ for } t \geq 0\text{)}
\end{aligned}$$

(ii) Using the definition of the derivative as a limit

$$\begin{aligned}
S'(t) &= \lim_{h\to 0} \frac{S(t+h) - S(t)}{h} \\
&= \lim_{h\to 0} \frac{A}{h} \\
&= \lim_{h\to 0} \frac{1}{2}f(t)f(t+h) \\
&= \frac{1}{2}[f(t)]^2 \\
&= \frac{2a^2t^2}{(1+t^2)^2}
\end{aligned}$$

(iii) Let $t = \tan\theta \Rightarrow dt = \sec^2\theta\, d\theta$.

When $t = 0$ then $\theta = 0$ and when $t = T$ then $\theta = \tan^{-1} T$

$$\begin{aligned}
S(T) &= S(T) - S(0) \\
&= \int_0^T S'(t)\, dt \quad \text{(by the Fundamental Theorem of Calculus)} \\
&= \int_0^T \frac{2a^2t^2}{(1+t^2)^2}\, dt \\
&= a^2 \int_0^{\tan^{-1} T} \frac{2\tan^2\theta}{(1+\tan^2\theta)^2} \times \sec^2\theta\, d\theta \\
&= a^2 \int_0^{\tan^{-1} T} \frac{2\tan^2\theta}{\sec^4\theta} \times \sec^2\theta\, d\theta \\
&= a^2 \int_0^{\tan^{-1} T} 2\sin^2\theta\, d\theta \\
&= a^2 \int_0^{\tan^{-1} T} (1 - \cos 2\theta)\, d\theta \\
&= a^2 \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\tan^{-1} T} \\
&= a^2 \left[\theta - \frac{\tan\theta}{1+\tan^2\theta}\right]_0^{\tan^{-1} T} \qquad \text{(using } t\text{-formula)} \\
&= a^2 \tan^{-1} T - \frac{a^2T}{1+T^2}
\end{aligned}$$

(iv) For the Cartesian equation of $\underset{\sim}{r}(t)$, note that

$$\begin{aligned}
y &= tx \\
t &= \frac{y}{x} \qquad \text{substitute into } x \\
x &= \frac{2at}{1+t^2} \\
x &= \frac{2a \times \frac{y}{x}}{1 + \frac{y^2}{x^2}} \\
x^2 + y^2 &= 2ay \\
x^2 + (y-a)^2 &= a^2
\end{aligned}$$

The Cartesian Equation is a circle of radius a. Also, note that from part (iii)

$$\lim_{T\to\infty} S(T) = \frac{\pi a^2}{2}.$$

This is half the area of the circle in the Cartesian equation. In fact, the particle moves from the origin and sweeps out the right half of the circle, and the above limit is the area of the swept region.

(d) (i) Let the other two roots be α and β. From the sum and product of roots

$$\begin{aligned} \alpha + \beta + 1 &= 3 \\ \alpha + \beta &= 2 \\ \alpha\beta(1) &= -1 \\ \alpha\beta &= -1 \end{aligned}$$

This means that α and β are the roots of $x^2 - 2x - 1 = 0$.

Hence, the other two roots are

$$\begin{aligned} x^2 - 2x - 1 &= 0 \\ x^2 - 2x + 1 &= 2 \\ (x-1)^2 &= 2 \\ x &= 1 \pm \sqrt{2} \end{aligned}$$

(ii) Since the required result comes from the quadratic factor $(x^2 - 2x - 1)$, apply the substitution $x = \tan\theta$ to solve the quadratic equation

$$\begin{aligned} \tan^2\theta - 2\tan\theta - 1 &= 0 \\ \tan^2\theta - 1 &= 2\tan\theta \\ -1 &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ \tan 2\theta &= -1 \\ \theta &= \frac{3\pi}{8}, -\frac{\pi}{8} \end{aligned}$$

Since $\frac{3\pi}{8}$ lies in the first quadrant, then $\tan\frac{3\pi}{8} = 1 + \sqrt{2}$.

Since $-\frac{\pi}{8}$ lies in the fourth quadrant, then $\tan\left(-\frac{\pi}{8}\right) = 1 - \sqrt{2}$.

Using the fact that $\tan\theta$ is an odd function, it follows that $\tan\frac{\pi}{8} = \sqrt{2} - 1$.

Question 13

(a) Differentiating both sides of the equation of the curve with respect to t

$$
\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\
\frac{1}{a^2}\frac{dx^2}{dt} + \frac{1}{b^2}\frac{dy^2}{dt} &= \frac{d}{dt}(1) \\
\frac{1}{a^2}\frac{dx^2}{dx} \times \frac{dx}{dt} + \frac{1}{b^2}\frac{dy^2}{dy} \times \frac{dy}{dt} &= 0 \\
\frac{2x}{a^2}\frac{dx}{dt} + \frac{2y}{b^2}\frac{dy}{dt} &= 0 \\
2y\frac{dy}{dt} &= -\frac{2b^2x}{a^2}\frac{dx}{dt} \quad (*) \\
\frac{dy}{dt} \times \frac{dt}{dx} &= -\frac{b^2x}{a^2y} \quad \text{if } y \neq 0, \frac{dx}{dt} \neq 0 \\
\frac{dy}{dx} &= -\frac{b^2x}{a^2y}
\end{aligned}
$$

In the fourth quadrant, $x > 0$ and $y < 0$, so $\dfrac{dy}{dx} > 0$.

From its starting point, the y-value of P changes from zero to negative. Since the y-value is decreasing over time, then the x-value must also be decreasing over time.

This means that $\dfrac{dx}{dt} < 0$ in the fourth quadrant.

Let L be the length of OP, so $L^2 = x^2 + y^2$. Similarly applying the chain rule

$$
\begin{aligned}
\frac{dL^2}{dt} &= \frac{dx^2}{dt} + \frac{dy^2}{dt} \\
&= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\
&= 2x\frac{dx}{dt} - \frac{2b^2x}{a^2}\frac{dx}{dt} \quad \text{from } (*) \\
&= 2x\frac{dx}{dt}\left(1 - \frac{b^2}{a^2}\right)
\end{aligned}
$$

Since $a < b$ then $1 - \dfrac{b^2}{a^2} < 0$. Given $\dfrac{dx}{dt} < 0$ and $x > 0$ in fourth quadrant then, $\dfrac{dL^2}{dt} > 0$.

However, when it reaches the point $(0, -b)$, then $\dfrac{dL^2}{dt} = 0$.

The point is then moves from the y-intercept $(0, -b)$ into the third quadrant, where $x < 0$.

Since x changes from zero to negative then $\dfrac{dx}{dt} < 0$ in the third quadrant. Hence, $\dfrac{dL^2}{dt} < 0$.

This means that from the first derivative test, L^2 increases until it hits a maximum at $(0, -b)$ before decreasing. Since $L > 0$, this assertion also holds for L.

(b) (i) Since the random variables are independent of each other then the probability of getting a combination of particular values is the product of the individual probabilities.

$$L = P(X_1 = x_1)P(X_2 = x_2)P(X_3 = x_3)\cdots P(X_n = x_n)$$

Since each random variable has a Bernoulli distribution, then for some integer k

$$P(X_k = x_k) = p^{x_k}(1-p)^{1-x_k}.$$

Note that x_k can only take the values of 0 (with probability $1-p$) and 1 (with probability p), as $X_k \sim \text{Bin}\,(1, p)$. This means that

$$\begin{aligned} L &= p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x_2} \times p^{x_3}(1-p)^{1-x_3} \times \cdots \times p^{x_n}(1-p)^{1-x_n} \\ &= p^{x_1+x_2+x_3+\cdots+x_n}(1-p)^{n-(x_1+x_2+x_3+\cdots+x_n)} \\ &= p^{n\bar{x}}(1-p)^{n(1-\bar{x})} \quad \text{where } \bar{x} = \frac{x_1+x_2+\cdots+x_n}{n} \end{aligned}$$

(ii) Firstly, note that $p = 0$ and $p = 1$ trivially give the minimum possible value of L which is zero. To get the maximum value of L, it must be where $0 < p < 1$.

$$\begin{aligned} L &= p^{n\bar{x}}(1-p)^{n(1-\bar{x})} \\ \frac{dL}{dp} &= n\bar{x}p^{n\bar{x}-1}(1-p)^{n(1-\bar{x})} + n(1-\bar{x})p^{n\bar{x}-1}(1-p)^{n(1-\bar{x})-1}(-1) \\ &= nL\left(\frac{\bar{x}}{p} - \frac{1-\bar{x}}{1-p}\right) \\ &= \frac{nL(\bar{x}-p)}{p(1-p)} \end{aligned}$$

Stationary points occur when $\dfrac{dL}{dp} = 0$ which gives $p = \bar{x}$ for $L > 0$.

If $p < \bar{x}$ then $\dfrac{dL}{dp} > 0$ and if $p > \bar{x}$ then $\dfrac{dL}{dp} < 0$, noting $n > 0$ and $0 < p < 1$.

This means that a local maximum occurs at $p = \bar{x}$. Since it is the only stationary point for $0 < p < 1$, then it is also the global maximum.

Alternative Method: Take the natural logarithm of L

$$\begin{aligned} \ln L &= n\bar{x}\ln p + n(1-\bar{x})\ln(1-p) \\ \frac{d\ln L}{dp} &= \frac{n\bar{x}}{p} - \frac{n(1-\bar{x})}{1-p} \\ &= \frac{n(\bar{x}-p)}{p(1-p)} \end{aligned}$$

Stationary points occur when $\dfrac{d\ln L}{dp} = 0$ which gives $p = \bar{x}$.

If $p < \bar{x}$ then $\dfrac{d \ln L}{dp} > 0$ and if $p > \bar{x}$ then $\dfrac{d \ln L}{dp} < 0$, noting $n > 0$ and $0 < p < 1$.

This means that a local maximum occurs at $p = \bar{x}$. Since it is the only stationary point for $0 < p < 1$, then it is also the global maximum.

Remark: Note that the maximum of $\ln L$ is also the maximum of L because the ordering of L is unchanged by the logarithmic function (as it is increasing). Alternatively, it can be seen that

$$\begin{aligned} \frac{d \ln L}{dp} &= \frac{d \ln L}{dL} \times \frac{dL}{dp} \\ &= \frac{1}{L} \frac{dL}{dp} \end{aligned}$$

Since $L > 0$ then wherever $\dfrac{d \ln L}{dp} = 0$ also is where $\dfrac{dL}{dp} = 0$.

Also, the sign of $\dfrac{d \ln L}{dp}$ is the same as the sign of $\dfrac{dL}{dp}$.

(iii) Notice that the mean of the outcomes $\bar{x}$ is in fact the *sample proportion* $\hat{p}$. The result in part (ii) means that the sample proportion is the *most likely* estimate of the unknown parameter p that can give the particular combination of outcomes/sample $X_1 = x_1, X_2 = x_2, X_3 = x_3, \ldots, X_n = x_n$.

Remark: The above is known as *maximum likelihood* estimate. It is a commonly used method in Statistics to estimate unknown population parameters (e.g. the exact probability of success) when only having data from a sufficiently large sample drawn from that population. If the observed sample is assumed to be the most likely sample possible, then the parameters of a population can be reasonably estimated using that sample only.

(c) Since α is a non-zero root of $P(x)$ then

$$\binom{n}{0}\alpha^4 - \binom{n}{1}\alpha^3 + \binom{n}{2}\alpha^2 - \binom{n}{3}\alpha + \binom{n}{4} = 0$$
$$\binom{n}{0}\alpha^3 - \binom{n}{1}\alpha^2 + \binom{n}{2}\alpha - \binom{n}{3} + \binom{n}{4}\alpha^{-1} = 0$$

This means that $\alpha^3 = \binom{n}{1}\alpha^2 - \binom{n}{2}\alpha + \binom{n}{3} - \binom{n}{4}\dfrac{1}{\alpha}$. Similarly

$$\begin{aligned} \beta^3 &= \binom{n}{1}\beta^2 - \binom{n}{2}\beta + \binom{n}{3} - \binom{n}{4}\frac{1}{\beta} \\ \gamma^3 &= \binom{n}{1}\gamma^2 - \binom{n}{2}\gamma + \binom{n}{3} - \binom{n}{4}\frac{1}{\gamma} \\ \delta^3 &= \binom{n}{1}\delta^2 - \binom{n}{2}\delta + \binom{n}{3} - \binom{n}{4}\frac{1}{\delta} \end{aligned}$$

Since $\binom{n}{0} = 1$ then

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= \binom{n}{1} \\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= \binom{n}{2} \\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= \binom{n}{3} \\
\alpha\beta\gamma\delta &= \binom{n}{4}
\end{aligned}$$

Hence

$$\begin{aligned}
\alpha^3+\beta^3+\gamma^3+\delta^3 &= \binom{n}{1}(\alpha^2+\beta^2+\gamma^2+\delta^2) - \binom{n}{2}(\alpha+\beta+\gamma+\delta) \\
&\quad + 4\binom{n}{3} - \binom{n}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) \\
&= \binom{n}{1}[(\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)] \\
&\quad - \binom{n}{1}\binom{n}{2} + 4\binom{n}{3} - \binom{n}{4}\left(\frac{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}{\alpha\beta\gamma\delta}\right) \\
&= \binom{n}{1}^3 - 2\binom{n}{1}\binom{n}{2} - \binom{n}{1}\binom{n}{2} + 4\binom{n}{3} - \binom{n}{4}\frac{\binom{n}{3}}{\binom{n}{4}} \\
&= \binom{n}{1}^3 - 3\binom{n}{1}\binom{n}{2} + 3\binom{n}{3} \\
&= n^3 - 3n \times \frac{n!}{2!(n-2)!} + 3 \times \frac{n!}{3!(n-3)!} \\
&= n^3 - \frac{3n^2(n-1)}{2} + \frac{n(n-1)(n-2)}{2} \\
&= n^3 + \frac{-3n^3+3n^2+n^3-3n^2+2n}{2} \\
&= n^3 + \frac{-2n^3+2n}{2} \\
&= n
\end{aligned}$$

(d) (i) There are two cases to consider.

Case 1 - The chameleon is not selected

Out of the n lizards, there are $\binom{n}{4}$ possible ways of selecting four of them for the two pairs. Let the chosen colours be A, B, C and D. There are three possible pairings namely:

- (A, B) and (C, D)
- (A, C) and (B, D)
- (A, D) and (B, C)

Note that the choice of one pair automatically determines the choice of the other pair. Altogether, there are $3\binom{n}{4}$ possible pairings.

Case 2 - The chameleon is selected

Out of the remaining n lizards, there are $\binom{n}{3}$ possible ways of selecting three of them for the two pairs. Let the chosen colours be P, Q and R. The chameleon can change colour to either P, Q or R overnight. There are three possible pairings where no pair contains the same colour:

- If the chameleon changes to colour P, the pairing must be (P, Q) and (P, R)
- If the chameleon changes to colour Q, the pairing must be (Q, P) and (Q, R)
- If the chameleon changes to colour R, the pairing must be (R, P) and (R, Q)

Note again that the choice of one pair automatically determines the choice of the other pair. Altogether, there are $3\binom{n}{3}$ possible pairings.

Combing the two cases gives a total of $3\binom{n}{4} + 3\binom{n}{3}$ ways, or equivalently, $3\binom{n+1}{4}$ ways using Pascal's triangle relation.

(ii) Consider n lizards, each with different colours. There are $\binom{n}{2}$ possible pairings. There are $\binom{\binom{n}{2}}{2}$ ways to select any two of those pairings.

This includes cases where the two pairs could have a common lizard.

If the two pairs have a common lizard then the counting of colours is equivalent to choosing four lizards with a chameleon that can change colour as described above in case 2. If they do not have a common lizard the counting of colours is equivalent to case 1 described above.

The equivalence of these scenarios means that

$$3\binom{n+1}{4} = \binom{\binom{n}{2}}{2}.$$

Question 14

(a) Since the particle is decelerating then

$$\begin{aligned}
\frac{dv}{dt} &= -a \\
\frac{d}{dx}\left(\frac{v^2}{2}\right) &= -a \\
\frac{v^2}{2} &= -ax + c \quad \text{when } x = 0, v = u \Rightarrow c = \frac{u^2}{2} \\
v^2 &= u^2 - 2ax \\
v &= \sqrt{u^2 - 2ax} \quad \text{since particle is moving in positive direction } v \geq 0 \\
&= \sqrt{u^2 - 2a\ell} \quad \text{when } x = \ell
\end{aligned}$$

(b) (i) Once the particle leaves the pipe, having travelled a distance of ℓ, it enters the motion described by $\underset{\sim}{r}$. The particle hits the ground at $y = 0$ so

$$\begin{aligned}
h - \frac{gt^2}{2} &= 0 \\
t^2 &= \frac{2h}{g} \\
t &= \sqrt{\frac{2h}{g}} \quad \text{noting that } t \geq 0
\end{aligned}$$

The horizontal distance travelled from the edge of the cliff is

$$\begin{aligned}
R &= \ell + v_0\sqrt{\frac{2h}{g}} \\
&= \ell + \sqrt{\frac{2h(u^2 - 2a\ell)}{g}}
\end{aligned}$$

Noting that when the object exits the pipe, its speed is the initial speed of the projectile motion so $v_0 = \sqrt{u^2 - 2a\ell}$ from part (a).

(ii) Differentiating R with respect to ℓ

$$R = \ell + \sqrt{\frac{2h(u^2 - 2a\ell)}{g}}$$

$$\frac{dR}{d\ell} = 1 - a\sqrt{\frac{2h}{g}} \times \frac{1}{\sqrt{u^2 - 2a\ell}}$$

Stationary points occur when $\dfrac{dR}{d\ell} = 0$

$$\begin{aligned}
a\sqrt{\frac{2h}{g}} \times \frac{1}{\sqrt{u^2 - 2a\ell}} &= 1 \\
\sqrt{u^2 - 2a\ell} &= a\sqrt{\frac{2h}{g}} \\
2a\ell &= u^2 - \frac{2a^2h}{g} \\
\ell &= \frac{gu^2 - 2a^2h}{2ag}
\end{aligned}$$

However, a condition is that $a \geq u\sqrt{\dfrac{g}{2h}} \Rightarrow 2a^2h \geq gu^2$. This implies $\ell \leq 0$.

This means that the only valid value for the length of the pipe $\ell = 0$ (i.e. no pipe at all). To see this, note that

$$\begin{aligned}
\frac{dR}{d\ell} &= 1 - a\sqrt{\frac{2h}{g}} \times \frac{1}{\sqrt{u^2 - 2a\ell}} \qquad \text{but } a \geq u\sqrt{\frac{g}{2h}} \\
&\leq 1 - \frac{u}{\sqrt{u^2 - 2a\ell}}
\end{aligned}$$

Since $\ell \geq 0$ and $a > 0$ then

$$\begin{aligned}
\sqrt{u^2 - 2a\ell} &\leq \sqrt{u^2} \\
&= u \quad \text{noting that } u > 0 \\
\frac{u}{\sqrt{u^2 - 2a\ell}} &\geq 1
\end{aligned}$$

This implies that $\dfrac{dR}{d\ell} \leq 0$, so R is a non-increasing function over ℓ.

Hence, the maximum value of R is found at the endpoint where $\ell = 0$.

(c) Using the given result, note that for the kth term

$$\begin{aligned}\tan^{-1}\left(\frac{2}{k^2}\right) &= \tan^{-1}\left(\frac{k+1-k+1}{1+k^2-1}\right)\\ &= \tan^{-1}\left(\frac{(k+1)-(k-1)}{1+(k+1)(k-1)}\right)\\ &= \tan^{-1}(k+1)-\tan^{-1}(k-1)\end{aligned}$$

Alternative Method:

For the k^{th} term, let $\alpha-\beta=2$ and $1+\alpha\beta=k^2$. Solving simultaneously gives

$$\begin{aligned}1+(2+\beta)\beta &= k^2\\ 1+2\beta+\beta^2 &= k^2\\ (1+\beta)^2 &= k^2\\ \beta &= -1\pm k\\ \alpha &= 1\pm k\end{aligned}$$

For the positive solution, this leads to

$$\tan^{-1}\left(\frac{2}{k^2}\right) = \tan^{-1}(k+1)-\tan^{-1}(k-1)$$

For the negative solution, this leads to

$$\begin{aligned}\tan^{-1}\left(\frac{2}{k^2}\right) &= \tan^{-1}(-k+1)-\tan^{-1}(-k-1) \quad \text{but } \tan^{-1}x \text{ is an odd function}\\ &= \tan^{-1}(k+1)-\tan^{-1}(k-1)\end{aligned}$$

Both solutions lead to the same result.

This means that

$$\begin{aligned}\tan^{-1}\left(\frac{2}{1^2}\right) &= \tan^{-1}2-\tan^{-1}0\\ \tan^{-1}\left(\frac{2}{2^2}\right) &= \tan^{-1}3-\tan^{-1}1\\ \tan^{-1}\left(\frac{2}{3^2}\right) &= \tan^{-1}4-\tan^{-1}2\\ &\vdots\\ \tan^{-1}\left(\frac{2}{(n-1)^2}\right) &= \tan^{-1}n-\tan^{-1}(n-2)\\ \tan^{-1}\left(\frac{2}{n^2}\right) &= \tan^{-1}(n+1)-\tan^{-1}(n-1)\end{aligned}$$

Taking the sum of the above and cancelling out many terms gives

$$\begin{aligned}S &= -\tan^{-1}0-\tan^{-1}1+\tan^{-1}n+\tan^{-1}(n+1)\\ &= \tan^{-1}n+\tan^{-1}(n+1)-\frac{\pi}{4}\end{aligned}$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} \tan S &= \tan\left(\lim_{n\to\infty}\left[\tan^{-1} n + \tan^{-1}(n+1) - \frac{\pi}{4}\right]\right) \\
&= \tan\left(\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= \tan\left(\frac{3\pi}{4}\right) \\
&= -1
\end{aligned}$$

Remark: Moving the limit inside the tangent function is allowed since:
1. $\tan\theta$ is a continuous function over the restricted domain $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.
2. For sufficiently large n, the sum is always inside this domain.
3. For any continuous function $f(x)$ and convergent sequence a_n, $\lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n)$.

(d) (i) Note that

$$\begin{aligned}
T_{n,k} &= \binom{2k}{k}\binom{2(n-k)}{n-k} \\
&= \binom{2(n-k)}{n-k}\binom{2k}{k} \\
&= \binom{2(n-k)}{n-k}\binom{2(n-(n-k))}{(n-(n-k))} \\
&= T_{n,n-k}
\end{aligned}$$

Hence

$$\begin{aligned}
b_n &= T_{n,1} + 2T_{n,2} + 3T_{n,3} + \cdots + nT_{n,n} \\
&= T_{n,n-1} + 2T_{n,n-2} + 3T_{n,n-3} + \cdots + nT_{n,0} \\
&= nT_{n,0} + (n-1)T_{n,1} + (n-2)T_{n,2} + \cdots + T_{n,n-1}
\end{aligned}$$

(ii) Using the result from part (i)

$$\begin{aligned}
b_n &= T_{n,1} + 2T_{n,2} + 3T_{n,3} + \cdots + nT_{n,n} \\
b_n &= nT_{n,0} + (n-1)T_{n,1} + (n-2)T_{n,2} + \cdots + T_{n,n-1} \\
2b_n &= nT_{n,0} + (n-1+1)T_{n,1} + (n-2+2)T_{n,2} + \cdots + (1+n-1)T_{n,n-1} + nT_{n,n} \\
2b_n &= n(T_{n,0} + T_{n,1} + T_{n,2} + \cdots + T_{n,n}) \\
b_n &= \frac{n}{2}a_n
\end{aligned}$$

(iii) Consider the $(k+1)^{\text{th}}$ term of $4b_{n-1}+2a_{n-1}$ for $k=0,1,2,...,n-1$

$$\begin{aligned}4kT_{n-1,k}+2T_{n-1,k} &= (4k+2)T_{n-1,k}\\ &= 2(2k+1)\binom{2k}{k}\binom{2(n-1-k)}{n-1-k}\end{aligned}$$

Note that in this case $T_{n-1,k}$ is the $(k+1)^{\text{th}}$ term of a_{n-1}, so n must be replaced by $n-1$ compared to the $T_{n,k}$ provided (which was the $(k+1)^{\text{th}}$ term of a_n).

Manipulating $\binom{2k}{k}$

$$\begin{aligned}\binom{2k}{k} &= \frac{(2k)!}{k!k!}\times\frac{(2k+2)(2k+1)}{(2k+2)(2k+1)}\\ &= \frac{1}{2(2k+1)}\times\frac{(2k+2)!}{k!(k+1)!}\times\frac{k+1}{k+1}\\ &= \frac{k+1}{2(2k+1)}\times\frac{(2k+2)!}{(k+1)!(k+1)!}\\ &= \frac{k+1}{2(2k+1)}\binom{2(k+1)}{k+1}\end{aligned}$$

Hence

$$\begin{aligned}4kT_{n-1,k}+2T_{n-1,k} &= (k+1)\binom{2(k+1)}{k+1}\binom{2(n-1-k)}{n-1-k}\\ &= (k+1)\binom{2(k+1)}{k+1}\binom{2(n-(k+1))}{n-(k+1)}\\ &= (k+1)T_{n,k+1}\end{aligned}$$

where $T_{n,k+1}$ is the $(k+2)^{\text{th}}$ term of a_n.

This means that

$$\begin{aligned}4b_{n-1}+2a_{n-1} &= 4(T_{n-1,1}+2T_{n-1,2}+3T_{n-1,3}+\cdots+(n-1)T_{n-1,n-1})\\ &\quad +2(T_{n-1,0}+T_{n-1,1}+T_{n-1,2}+\cdots+T_{n-1,n-1})\\ &= [4(0)+2]T_{n-1,0}+[4(1)+2]T_{n-1,1}+\cdots+[4(n-1)+2]T_{n-1,n-1}\\ &= T_{n,1}+2T_{n,2}+3T_{n,3}+\cdots+nT_{n,n}\\ &= b_n\end{aligned}$$

Hence, $b_n=4b_{n-1}+2a_{n-1}$.

(iv) When $n = 1$

$$\begin{aligned} a_1 &= T_{1,0} + T_{1,1} \\ &= \binom{2\times 0}{0}\binom{2}{1} + \binom{2}{1}\binom{2\times 0}{0} \\ &= 1\times 2 + 2\times 1 \\ &= 4 \end{aligned}$$

The statement is true for $n = 1$.

Assume the statement is true for $n = k$

$$a_k = 4^k$$

Required to prove the statement is true for $n = k + 1$

$$a_{k+1} = 4^{k+1}$$

$$\begin{aligned} \text{LHS} &= a_{k+1} \\ &= \frac{2b_{k+1}}{k+1} \quad \text{from part (ii)} \\ &= \frac{2}{k+1}\left(4b_k + 2a_k\right) \quad \text{from part (iii)} \\ &= \frac{2}{k+1}\left(2ka_k + 2a_k\right) \quad \text{from part (ii)} \\ &= \frac{4a_k(k+1)}{k+1} \quad \text{but } a_k = 4^k \text{ by assumption} \\ &= 4^{k+1} \\ &= \text{RHS} \end{aligned}$$

Since the statement is true for $n = 1$, then by induction it holds true for all positive integers n.