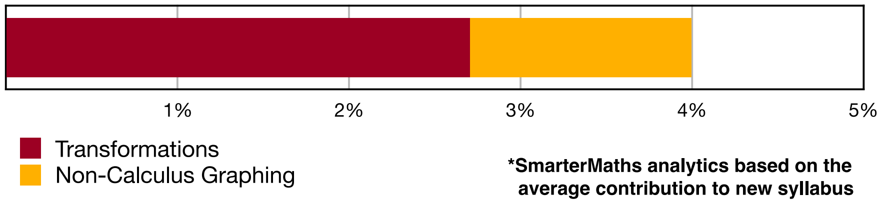
ADVANCED MATHEMATICS
Functions (Adv), F2 Graphing (Adv)

Transformations (Y12)
Non-Calculus Graphing (Y12)

Teacher: Cathyanne Horvat
Exam Equivalent Time: 61.5 minutes *(based on allocation of 1.5 minutes per mark)*

F2 Graphing



***SmarterMaths analytics based on the average contribution to new syllabus Advanced Maths exams since 2020.**

HISTORICAL CONTRIBUTION

- *F2 Graphing Techniques* has contributed an average of 4.0% per Adv exam since the new syllabus was introduced in 2020.
- We have split the topic into 2 categories for analysis purposes: *1-Transformations (2.7%) and 2-Non-Calculus Graphing (1.3%).*
- This analysis looks at *Transformations.*

HSC ANALYSIS - What to expect and common pitfalls

- *Transformations* represents new syllabus content that explicitly looks at translations and dilations of several function types, including the introduction of the aforementioned terminology.
- The *2022 Adv* exam required students to calculate translations and dilations in three separate steps, producing a mean mark of 51%. This question is on the back of *2021 Q21* which combined vertical and horizontal dilations and similarly caused problems with a 48% state mean mark. Revision attention here goes without saying.
- The NESA sample HSC exam, released in March 2020, has been instructive in developing this challenging database area. Pay careful attention to *F2 EQ-Bank* questions.
- There have been some examples in past HSC exams that looked at similar content. Please review of *F2 2013 HSC 15c* which proved very challenging for a majority of students.
- We note that Trig transformations, which we regard as an extremely important transformation sub-topic, are covered separately under *T3 Trig Graphs*.
- This topic area provides scope for examiners to ask both low and high difficulty questions, with a variety of underlying functions.

Questions

1. Functions, 2ADV F2 SM-Bank 1

i. Draw the graph $y = \ln x$. ***(1 mark)***

ii. Explain how the above graph can be transformed to produce the graph

$$y = 3\ln(x+2)$$

and sketch the graph, clearly identifying all intercepts. ***(3 marks)***

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2. Functions, 2ADV F2 EQ-Bank 16

$y = -\dfrac{(x+2)^4}{3}$ has been produced by three successive transformations: a translation, a dilation and then a reflection.

i. Describe each transformation and state the equation of the graph after each transformation. ***(2 marks)***

ii. Sketch the graph. ***(1 mark)***

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3. Functions, 2ADV F2 2021 HSC 19

Without using calculus, sketch the graph of $y = 2 + \dfrac{1}{x+4}$, showing the asymptotes and the x and y intercepts. ***(3 marks)***

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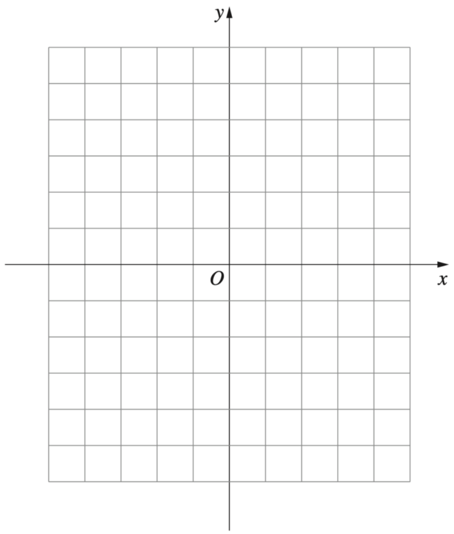
4. Functions, 2ADV F2 2023 HSC 19

a. Sketch the graphs of the functions $f(x) = x - 1$ and $g(x) = (1 - x)(3 + x)$ showing the x-intercepts. ***(2 marks)***



b. Hence, or otherwise, solve the inequality $x - 1 < (1 - x)(3 + x)$. ***(2 marks)***

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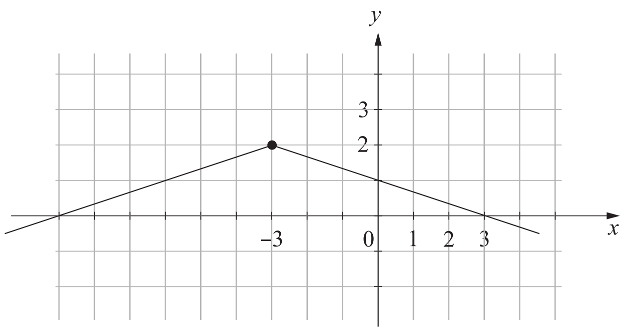
5. Functions, 2ADV F2 EQ-Bank 1

The function $f(x) = |x|$ is transformed and the equation of the new function is $y = kf(x + b) + c$.

The graph of the new function is shown below.



What are the values of k, b and c. ***(2 marks)***

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6. Functions, 2ADV F2 EQ-Bank 14

List a set of transformations that, when applied in order, would transform $y = x^2$ to the graph with equation $y = 1 - 6x - x^2$. ***(3 marks)***

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7. Functions, 2ADV F2 SM-Bank 16

Let $f(x) = x^2 - 4$

Let the graph of $g(x)$ be a transformation of the graph of $f(x)$ where the transformations have been applied in the following order:

• dilation by a factor of $\dfrac{1}{2}$ from the vertical axis (parallel to the horizontal axis)

• translation by two units to the right (in the direction of the positive horizontal axis

Find $g(x)$ and the coordinates of the horizontal axis intercepts of the graph of $g(x)$. ***(3 marks)***

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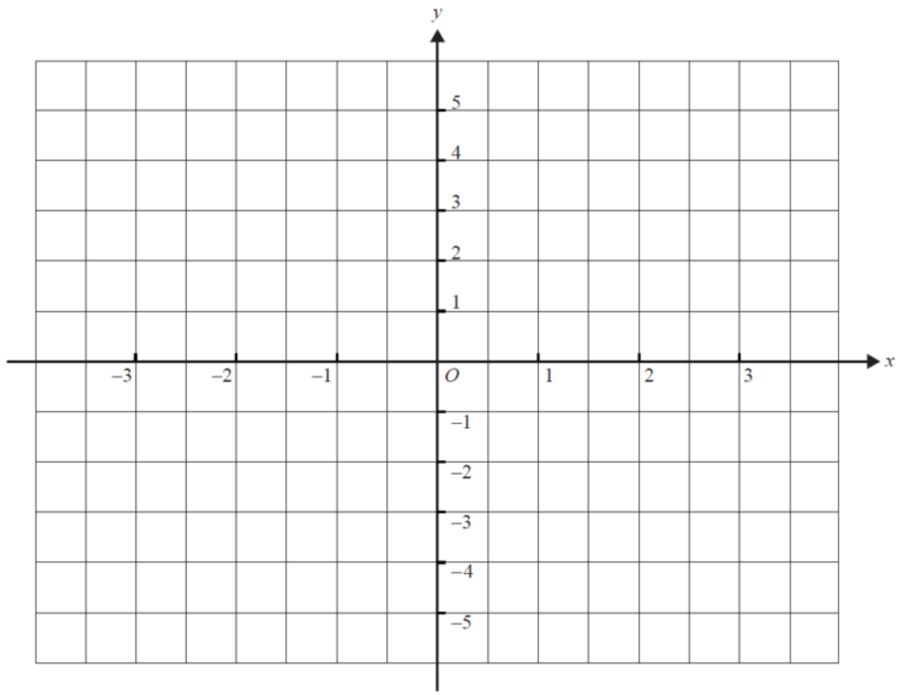
8. Functions, 2ADV F2 SM-Bank 12

Sketch the graph of $f(x) = \dfrac{2x + 1}{x - 1}$. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation. ***(4 marks)***



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9. Functions, 2ADV F2 EQ-Bank 13

The curve $y = kx^2 + c$ is subject to the following transformations

- Translated 2 units in the positive x-direction
- Dilated in the positive y-direction by a factor of 4
- Reflected in the y-axis

The final equation of the curve is $y = 8x^2 + 32x - 8$.

i. Find the equation of the graph after the dilation. ***(1 mark)***

ii. Find the values of k and c. ***(2 marks)***

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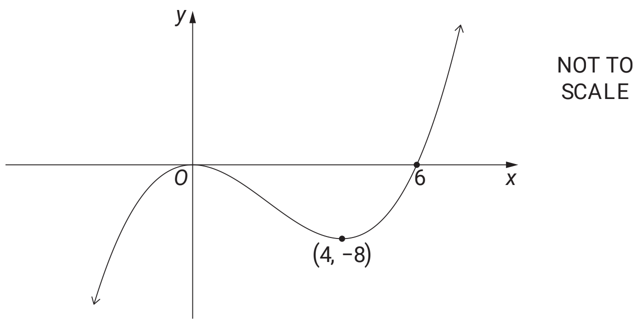
10. Functions, 2ADV F2 2021 HSC 21

Consider the graph of $y = f(x)$ as shown.



Sketch the graph of $y = 4f(2x)$ showing the x-intercepts and the coordinates of the turning points. ***(2 marks)***

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11. Functions, 2ADV F2 2022 HSC 19

The graph of the function $f(x) = x^2$ is translated m units to the right, dilated vertically by a scale factor of k and then translated 5 units down. The equation of the transformed function is $g(x) = 3x^2 - 12x + 7$.

Find the values of m and k. ***(3 marks)***

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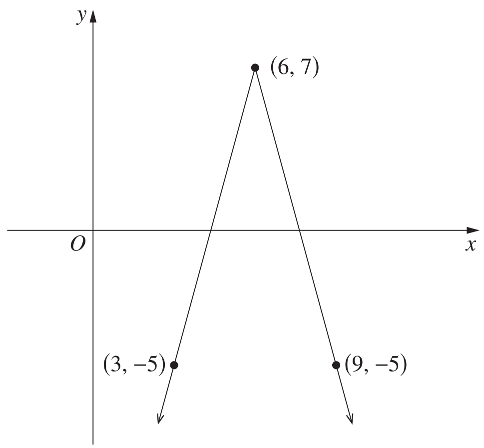
12. Functions, 2ADV F2 2023 HSC 27

The graph of $y = f(x)$, where $f(x) = a|x - b| + c$, passes through the points $(3, -5), (6, 7)$ and $(9, -5)$ as shown in the diagram.



a. Find the values of a, b and c. ***(3 marks)***

b. The line $y = mx$ cuts the graph of $y = f(x)$ in two distinct places.

Find all possible values of m. ***(2 marks)***

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13. Functions, 2ADV F2 2013 HSC 15c

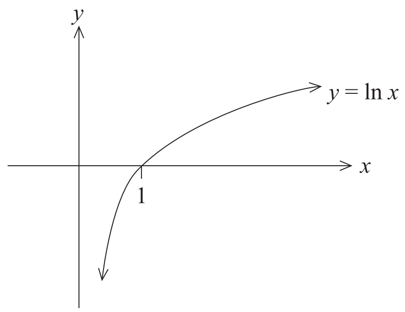
i. Sketch the graph $y = |\,2x - 3\,|$. ***(1 mark)***

ii. Using the graph from part (i), or otherwise, find all values of m for which the equation $|\,2x - 3\,| = mx + 1$ has exactly one solution. ***(2 marks)***

Worked Solutions

1. Functions, 2ADV F2 SM-Bank 1

i.

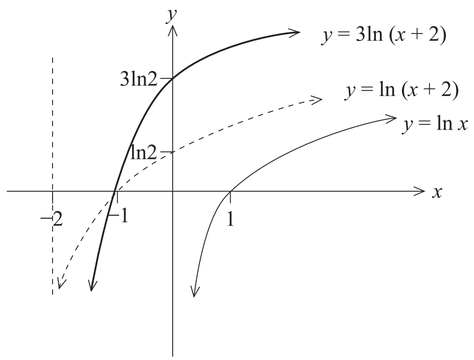


ii. Transforming $y = \ln x \Rightarrow y = \ln(x+2)$

$y = \ln x \quad \Rightarrow$ shift 2 units to left.

Transforming $y = \ln(x+2)$ to $y = 3\ln(x+2)$

$\Rightarrow$ increase each y value by a factor of 3



2. Functions, 2ADV F2 EQ-Bank 16

i. Transformation 1:

Translate $y = x^4$ 2 units to the left.

$y = x^4 \ \Rightarrow \ y = (x+2)^4$

Transformation 2:

Dilate $y = (x+2)^4$ by a factor of $\dfrac{1}{3}$ from the x-axis

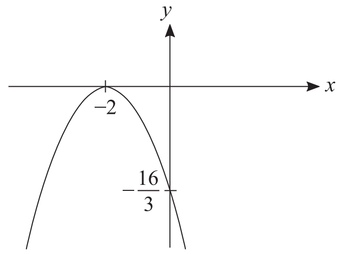
$$y = (x+2)^4 \ \Rightarrow \ y = \frac{(x+2)^4}{3}$$

Transformation 3:

Reflect $y = \dfrac{(x+2)^4}{3}$ in the x-axis.

$$y = \frac{(x+2)^4}{3} \ \Rightarrow \ y = -\frac{(x+2)^4}{3}$$

ii.



3. Functions, 2ADV F2 2021 HSC 19

Asymptotes: $x = -4$

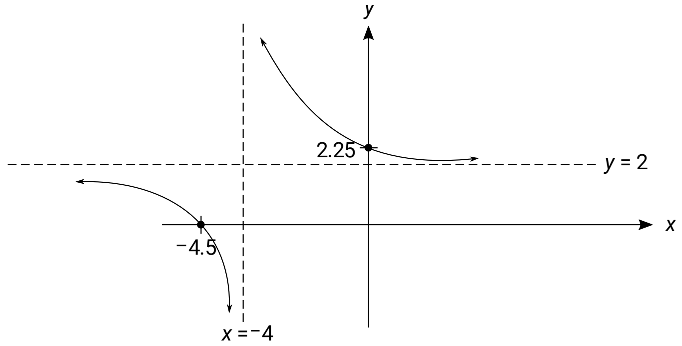
As $x \to \infty, y \to 2$

y-intercept occurs when $x = 0$:

$y = 2.25$

x-intercept occurs when $y = 0$:

$$2 + \frac{1}{x+4} = 0 \;\Rightarrow\; x = -4.5$$



4. Functions, 2ADV F2 2023 HSC 19

a. $g(x)$ cuts x-axis at 1 and -3.

$g(x)_{\text{max}} = g(-1) = 4$

Find intersection of graphs:

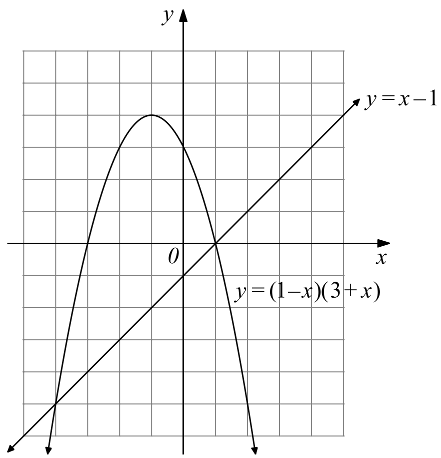
$$(1-x)(3+x) = x - 1$$

$$3 + x - 3x - x^2x - 1$$

$$x^2 + 3x - 4 = 0$$

$$(x+4)(x-1) = 0$$

Intersections at: $(1, 0), (-4, -5)$



b. From the graph:

$x - 1 < (1-x)(3+x)$ when $-4 < x < 1$

Test $x = 0$:

$0 - 1 < (1-0)(3+0) \quad \Rightarrow \quad -1 < 3$ (correct)

5. Functions, 2ADV F2 EQ-Bank 1

$y = |x|$

Translate 3 units left $\Rightarrow\ y = |x+3|$

Reflect in the x-axis $\Rightarrow\ y = -\,|x+3|$

Dilate by $\dfrac{1}{3}$ from the x-axis

$\Rightarrow$ Multiply by $\dfrac{1}{3}$ $\Rightarrow\ y = -\dfrac{1}{3}|x+3|$

Translate 2 units up $\Rightarrow\ y = -\dfrac{1}{3}|x+3| + 2$

$\therefore k = -\dfrac{1}{3}, b = 3, c = 2$

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6. Functions, 2ADV F2 EQ-Bank 14

$y = x^2$

Transformation 1:
Translate 3 units in negative x-direction

$y = (x+3)^2$

$y = x^2 + 6x + 9$

Transformation 2:
Translate 10 units in negative y-direction

$y = x^2 + 6x - 1$

Transformation 3:
Reflect in the x-axis

$y = -\left(x^2 + 6x - 1\right)$

$\quad = 1 - 6x - x^2$

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7. Functions, 2ADV F2 SM-Bank 16

1st transformation

Dilation by a factor of $\dfrac{1}{2}$ from the y-axis:

$x^2 - 4 \ \Rightarrow\ \left(\dfrac{x}{\frac{1}{2}}\right)^2 - 4 = 4x^2 - 4$

2nd transformation
Translation by 2 units to the right:

$4x^2 - 4 \ \Rightarrow\ g(x) = 4(x-2)^2 - 4$

x-axis intercept of $g(x)$:

$$\begin{aligned} 4(x-2)^2 - 4 &= 0 \\ (x-2)^2 &= 1 \\ x - 2 &= \pm 1 \end{aligned}$$

$x - 2 = 1 \ \Rightarrow\ x = 3$
$x - 2 = -1 \ \Rightarrow\ x = 1$

$\therefore$ Horizontal axis intercepts occur at $(1,0)$ and $(3,0)$.

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8. Functions, 2ADV F2 SM-Bank 12

$$\begin{aligned}\frac{2x+1}{x-1} &= \frac{2x-2+3}{x-1}\\ &= \frac{2(x-1)+3}{x-1}\\ &= 2+\frac{3}{x-1}\end{aligned}$$

COMMENT: Manipulation of the equation makes graphing much easier.

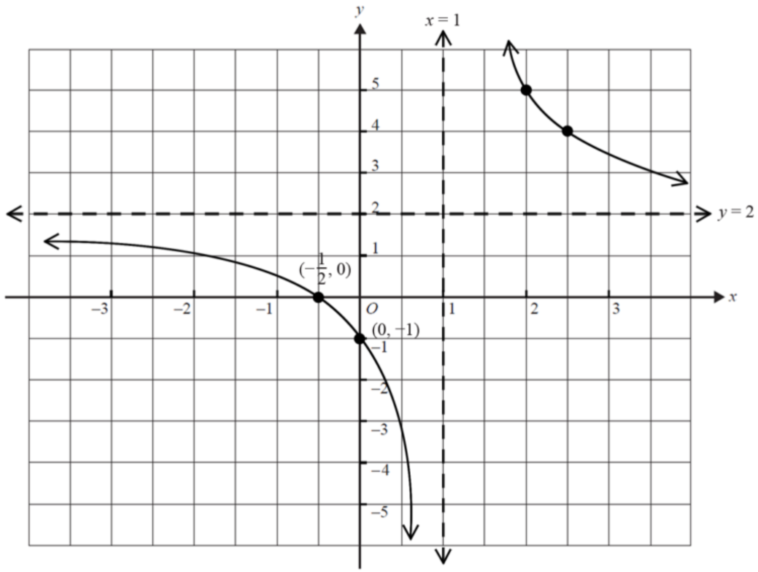
Asymptotes: $x=1,\ \ y=2$

As $x \to \infty,\ \ y \to 2(+)$

As $x \to -\infty,\ \ y \to 2(-)$

As $x \to -1(-),\ \ y \to -\infty$

As $x \to -1(+),\ \ y \to \infty$



9. Functions, 2ADV F2 EQ-Bank 13

i. $y = kx^2 + c$

Translate 2 units in positive x-direction.

$y = kx^2 + c \ \Rightarrow\ y = k(x-2)^2 + c$

Dilate in the positive y-direction by a factor of 4.

$y = k(x-2)^2 + c \ \Rightarrow\ y = 4k(x-2)^2 + 4c$

ii. $y = 4k\left(x^2 - 4x + 4\right) + 4c$

$\quad = 4kx^2 - 16kx + 16k + 4c$

Reflect in the y-axis.

$\Rightarrow$ Swap: $x \to -x$

$y = 4k(-x)^2 - 16k(-x) + 16k + 4c$

$\quad = 4kx^2 + 16kx + 16k + 4c$

COMMENT: Using "swap" terminology for reflections in the y-axis is simpler and more intelligible for students in our view.

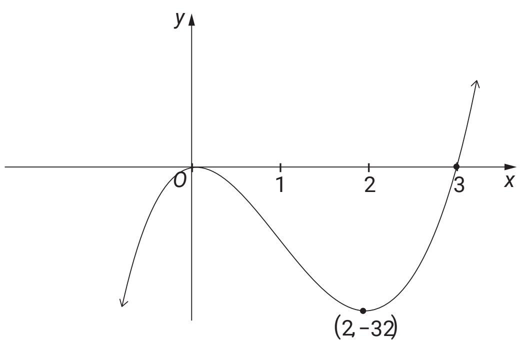
Equating co-efficients:

$4k = 8$

$\therefore k = 2$

$$\begin{aligned}16k + 4c &= -8\\ 4c &= -40\\ \therefore c &= -10\end{aligned}$$

10. Functions, 2ADV F2 2021 HSC 21



♦ *Mean mark 48%.*

11. Functions, 2ADV F2 2022 HSC 19

Horizontal translation m units to the right:

$x^2 \ \rightarrow \ (x-m)^2$

♦ Mean mark 51%.

Dilated vertically by scale factor k:

$(x-m)^2 \ \rightarrow \ k(x-m)^2$

Vertical translation 5 units down:

$k(x-m)^2 \ \rightarrow \ k(x-m)^2 - 5$

$$\begin{aligned} y &= k(x-m)^2 - 5 \\ &= k\left(x^2 - 2mx + m^2\right) - 5 \\ &= kx^2 - 2kmx + \left(km^2 - 5\right) \end{aligned}$$

$\therefore k = 3$

$-2km = -12$

$\therefore m = 2$

12. Functions, 2ADV F2 2023 HSC 27

a. Consider the transformation of $y = -|x|$

Translate 6 units to the right

$y = -|x| \ \rightarrow \ y = -|x-6|$

$\therefore b = 6$

Translate 7 units vertically up

$y = -|x-6| \ \rightarrow \ y = -|x-6| + 7$

$\therefore c = 7$

$f(x) = a|x-6| + 7$ passes through $(3, -5)$:

$-5 = a|3-6| + 7$

$-5 = 3a + 7$

$3a = -12$

$\therefore a = -4$

b. $y = mx$ passes through (0, 0)

One solution when $y = mx$ passes through (0, 0) and (6, 7)

$$m = \frac{7-0}{6-0} = \frac{7}{6}$$

As graph gets flatter and turns negative $\Rightarrow$ 2 solutions

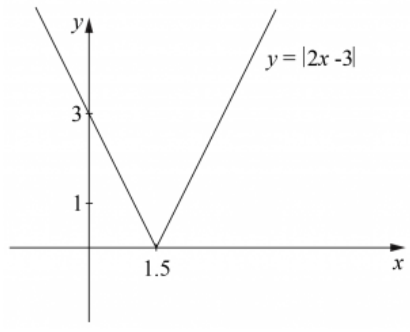
2 solutions continue until $y = mx$ is parallel to the line joining (6, 7) to $(9, -5)$, where:

$$m = \frac{7-(-5)}{6-9} = -\frac{12}{3} = -4$$

$\therefore$ 2 solutions when $-4 < m < 7/6$

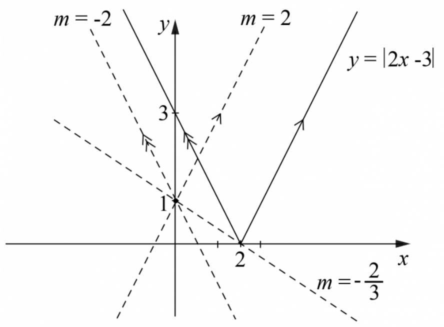
13. Functions, 2ADV F2 2013 HSC 15c

i.



♦ *Mean mark 49%*
MARKER'S COMMENT: Many students drew diagrams that were "too small", didn't use rulers or didn't use a consistent scale on the axes!

ii.



♦♦ *Mean mark 25%.*
COMMENT: Students need a clear graphical understanding of what they are finding to solve this very challenging, Band 6 question.

Line of intersection $y = mx + 1$ passes through $(0, 1)$
If it also passes through $(1.5, 0) \Rightarrow 1$ solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{1 - 0}{0 - \frac{3}{2}}$$
$$= -\frac{2}{3}$$

Gradients of $y = |\,2x - 3\,|$ are 2 or -2

Considering a line through $(0, 1)$:
If $m \geq 2$, only intersects once.

Similarly,
If $m < -2$, only intersects once.

$\therefore$ Only one solution when $m = -\dfrac{2}{3},\ \ m \geq 2$ or $m < -2$