EXERCISE 8.1 INDEX LAWS WITH INTEGERS AS INDICES

2 D

$$\begin{aligned}\frac{(-2xy)^2\times 2\left(x^2y^{-1}\right)^3}{8(xy)^{-3}}&=\frac{4x^2y^2\times 2x^6y^{-3}}{8x^{-3}y^{-3}}\\&=\frac{8x^8y^{-1}}{8x^{-3}y^{-3}}\\&=x^{11}y^2\end{aligned}$$

4 (a) $\frac{1}{4}=\frac{1}{2^2}$

$=2^{-2}$

(b) $\frac{1}{16}=\frac{1}{2^4}$

$=2^{-4}$

(c) $\frac{1}{32}=\frac{1}{2^5}$

$=2^{-5}$

(d) $0.125=\frac{1}{8}$

$=\frac{1}{2^3}$

$=2^{-3}$

(e) $\frac{1}{64}=\frac{1}{2^6}$

$=2^{-6}$

(f) $\frac{1}{128}=\frac{1}{2^7}$

$=2^{-7}$

(g) $0.25=\frac{1}{4}$

$=\frac{1}{2^2}$

$=2^{-2}$

(h) $8^{-3}=\left(2^3\right)^{-3}$

$=2^{-9}$

6 (a)

$$\begin{aligned}\frac{3^{2n}\times 25^{2n-1}}{15^{n-1}}&=\frac{3^{2n}\times\left(5^2\right)^{2n-1}}{(3\times 5)^{n-1}}\\&=\frac{3^{2n}\times 5^{4n-2}}{3^{n-1}\times 5^{n-1}}\\&=3^{n+1}\times 5^{3n-1}\end{aligned}$$

(b)

$$\begin{aligned}\left(x^{-1}+y^{-1}\right)\left(x^{-1}-y^{-1}\right)&=\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right)\\&=\frac{1}{x^2}-\frac{1}{y^2}\end{aligned}$$

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(c) $\dfrac{2^n \times 4^{n+1}}{8^{n-2}} = \dfrac{2^n \times \left(2^2\right)^{n+1}}{\left(2^3\right)^{n-2}}$

$$= \frac{2^n \times 2^{2n+2}}{2^{3n-6}}$$
$$= \frac{2^{3n+2}}{2^{3n-6}}$$
$$= 2^8$$

(d) $\left(x^{-2} + x^{-1}\right)^2 = \left(x^{-2}\right)^2 + 2 \times x^{-2} \times x^{-1} + \left(x^{-1}\right)^2$

$$= x^{-4} + 2x^{-3} + x^{-2}$$
$$= \frac{1}{x^4} + \frac{2}{x^3} + \frac{1}{x^2}$$

(e) $\dfrac{x - 5 + 6x^{-1}}{1 - 2x^{-1}} \times \dfrac{x}{x} = \dfrac{x^2 - 5x + 6}{x - 2}$

$$= \frac{(x-2)(x-3)}{x-2}$$
$$= x - 3$$

(f) $\dfrac{x - 4x^{-1}}{1 + 2x^{-1}} \times \dfrac{x}{x} = \dfrac{x^2 - 4}{x + 2}$

$$= \frac{(x-2)(x+2)}{x+2}$$
$$= x - 2$$

(g) $4^{-2} \times 6^3 \times 8^4 \times 12^{-2} = \left(2^2\right)^{-2} \times (2 \times 3)^3 \times \left(2^3\right)^4 \times \left(3 \times 2^2\right)^{-2}$

$$= 2^{-4} \times 2^3 \times 3^3 \times 2^{12} \times 3^{-2} \times 2^{-4}$$
$$= 3 \times 2^7$$

(h) $\dfrac{15^{n+1} \times 25 \times 5^{3n-4}}{9^{n-1} \times 25^{n-2}} = \dfrac{(3 \times 5)^{n+1} \times 5^2 \times 5^{3n-4}}{\left(3^2\right)^{n-1} \times \left(5^2\right)^{n-2}}$

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$$= \frac{3^{n+1} \times 5^{n+1} \times 5^{3n-2}}{3^{2n-2} \times 5^{2n-4}}$$

$$= \frac{3^{n+1} \times 5^{4n-1}}{3^{2n-2} \times 5^{2n-4}}$$

$$= \frac{5^{2n+3}}{3^{n-3}}$$

8 (a) $\left(a^{-1}+b\right)\left(a^{-1}-b\right) = a^{-2} - b^2$

$$= \frac{1}{a^2} - b^2$$

(b) $\left(x^{-1}+y\right)\left(x+y^{-1}\right) = 1 + x^{-1}y^{-1} + xy + 1$

$$= 2 + xy + \frac{1}{xy}$$

(c) $\left(x^{-2}+y^{-2}\right)\left(x^{-2}-y^{-2}\right) = x^{-4} - y^{-4}$

$$= \frac{1}{x^4} - \frac{1}{y^4}$$

(d) $\left(a^2-2b^{-1}\right)\left(a^{-2}-b\right) = 1 - a^2b - 2a^{-2}b^{-1} + 2$

$$= 3 - a^2b - \frac{2}{a^2b}$$

(e) $\dfrac{a^{-1}+b^{-1}}{a+b} \times \dfrac{ab}{ab} = \dfrac{b+a}{ab(a+b)}$

$$= \frac{1}{ab}$$

(f) $\dfrac{y^{-1}+y}{1+y^2} \times \dfrac{y}{y} = \dfrac{1+y^2}{y\left(1+y^2\right)}$

$$= \frac{1}{y}$$

EXERCISE 8.2 INDEX LAWS WITH FRACTIONAL INDICES

2 C

$$\begin{aligned}\sqrt[5]{8}\times\sqrt[5]{4} &= \left(2^3\right)^{\frac{1}{5}}\times\left(2^2\right)^{\frac{1}{5}}\\ &= 2^{\frac{3}{5}}\times 2^{\frac{2}{5}}\\ &= 2^{\frac{3}{5}+\frac{2}{5}}\\ &= 2^{\frac{5}{5}}\\ &= 2\end{aligned}$$

4 (a) $$\begin{aligned}x^{\frac{2}{3}}\times x^{\frac{3}{2}} &= x^{\frac{2}{3}+\frac{3}{2}}\\ &= x^{\frac{13}{6}}\end{aligned}$$

(b) $$\begin{aligned}\left(a^{-1}b\right)^2\times\left(\frac{1}{b^{-2}}\right)^{\frac{1}{2}} &= a^{-2}b^2\times\left(b^2\right)^{\frac{1}{2}}\\ &= \frac{b^2}{a^2}\times b\\ &= \frac{b^3}{a^2}\end{aligned}$$

(c) $$\begin{aligned}\left(x^{\frac{1}{2}}\right)^2-\left(x^{-2}\right)^{\frac{1}{2}} &= x^1-x^{-1}\\ &= x-\frac{1}{x}\end{aligned}$$

(d) $$\begin{aligned}\left(x^{\frac{1}{3}}\right)^2\times\left(x^{-1}y^3\right)^{-1}\times x^{-\frac{5}{3}}y^2 &= x^{\frac{2}{3}}\times xy^{-3}\times x^{-\frac{5}{3}}y^2\\ &= x^0y^{-1}\\ &= \frac{1}{y}\end{aligned}$$

(e) $$\begin{aligned}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right) &= \left(x^{\frac{1}{2}}\right)^2-\left(y^{\frac{1}{2}}\right)^2\\ &= x-y\end{aligned}$$

(f) $\sqrt[6]{x^2y^3}\times\dfrac{x^{\frac{1}{3}}}{y^{-\frac{1}{2}}}=\left(x^2y^3\right)^{\frac{1}{6}}\times x^{\frac{1}{3}}y^{\frac{1}{2}}$

$$\begin{aligned}&=x^{\frac{1}{3}}y^{\frac{1}{2}}\times x^{\frac{1}{3}}y^{\frac{1}{2}}\\&=x^{\frac{2}{3}}y\end{aligned}$$

(g) $\dfrac{54^{\frac{1}{4}}}{6^{\frac{3}{4}}\times12^{-\frac{1}{2}}}=\dfrac{\left(3^3\times2\right)^{\frac{1}{4}}}{\left(2\times3\right)^{\frac{3}{4}}\times\left(2^2\times3\right)^{-\frac{1}{2}}}$

$$\begin{aligned}&=\frac{3^{\frac{3}{4}}\times2^{\frac{1}{4}}}{2^{\frac{3}{4}}\times3^{\frac{3}{4}}\times2^{-1}\times3^{-\frac{1}{2}}}\\&=\frac{3^{\frac{3}{4}}\times2^{\frac{1}{4}}}{2^{-\frac{1}{4}}\times3^{\frac{1}{4}}}\\&=3^{\frac{1}{2}}\times2^{\frac{1}{2}}\\&=6^{\frac{1}{2}}\end{aligned}$$

(h) $\dfrac{\left(x^{m+1}\right)^n\times x^{m+n}}{\left(x^m\right)^{n+1}\times x^{2n}}=\dfrac{x^{mn+n}\times x^{m+n}}{x^{mn+m}\times x^{2n}}$

$$\begin{aligned}&=\frac{x^{mn+m+2n}}{x^{mn+m+2n}}\\&=1\end{aligned}$$

EXERCISE 8.3 SOLVING EQUATIONS WITH EXPONENTS

2 C

$$\begin{aligned}9^x&=\frac{1}{3}\\\left(3^2\right)^x&=3^{-1}\\3^{2x}&=3^{-1}\\2x&=-1\\x&=-\frac{1}{2}\end{aligned}$$

EXERCISE 8.4 LOGARITHMS

2 A

$$8^3 = 512$$
$$\log_8 512 = 3$$

4 (a) $16 = 2^4, 8 = 2^3$

$$\log_2 16 + \log_2 8 = 4 + 3 = 7$$

(b) $\log_{10} 2 + \log_{10} 5 = \log_{10}(2 \times 5)$

$$= \log_{10} 10$$
$$= 1$$

(c) $16 = 2^4, 8 = 2^3$

$$(\log_2 16)(\log_2 4) = 4 \times 2 = 8$$

(d) $\log_3 54 - \log_3 18 = \log_3\left(\frac{54}{18}\right)$

$$= \log_3 3$$
$$= 1$$

(e) $\frac{\log_a 8}{\log_a 2} = \frac{\log_a 2^3}{\log_a 2}$

$$= \frac{3\log_a 2}{\log_a 2}$$
$$= 3$$

(f) $\log_a 5 + \log_a \frac{1}{5} = \log_a\left(5 \times \frac{1}{5}\right)$

$$= \log_a 1$$
$$= 0$$

(g) $\log_2 18 - 2\log_2 3 = \log_2 18 - \log_2 3^2$

$$= \log_2 18 - \log_2 9$$
$$= \log_2 \frac{18}{9}$$
$$= \log_2 2$$
$$= 1$$

(h) $81 = 3^4, 125 = 5^3$

$$\log_3 81 \times \log_5 125 = 4 \times 3 = 12$$

6 (a) $\frac{1}{2}\log_{10} 16 + 2\log_{10} 5 = \log_{10} 16^{\frac{1}{2}} + \log_{10} 5^2$

$$= \log_{10} 4 + \log_{10} 25$$
$$= \log_{10}(4 \times 25)$$
$$= \log_{10} 100$$
$$= 2$$

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(b) $\log_2\left(2^x\right) = x\log_2 2$

$= x\times 1$

$= x$

(c) $10^{\log_{10} 3} = 3$

(d) $\dfrac{\log_{10} 25}{\log_{10} 5} = \dfrac{\log_{10} 5^2}{\log_{10} 5}$

$= \dfrac{2\log_{10} 5}{\log_{10} 5}$

$= 2$

(e) $\log_{10} 125 + \log_{10} 25 + \log_{10} 5 = \log_{10} 5^3 + \log_{10} 5^2 + \log_{10} 5$

$= 3\log_{10} 5 + 2\log_{10} 5 + \log_{10} 5$

$= 6\log_{10} 5$

(f) $\dfrac{\log\left(x^3\right)}{\log x} = \dfrac{3\log x}{\log x} = 3$

(g) $\log_{10}\dfrac{1+\sqrt{5}}{2} + \log_{10}\dfrac{3+\sqrt{5}}{2} = \log_{10}\left(\dfrac{1+\sqrt{5}}{2}\times\dfrac{3+\sqrt{5}}{2}\right)$

$= \log_{10}\left(\dfrac{3+\sqrt{5}+3\sqrt{5}+5}{4}\right)$

$= \log_{10}\left(\dfrac{8+4\sqrt{5}}{4}\right)$

$= \log_{10}\left(2+\sqrt{5}\right)$

(h) $\dfrac{\log x}{\log\sqrt{x}} = \dfrac{\log x}{\log x^{\frac{1}{2}}}$

$= \dfrac{\log x}{\frac{1}{2}\log x}$

$= 2$

8 **(a)** $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2}$

$= 2.322$

(b) $\log_3 12 = \dfrac{\log_{10} 12}{\log_{10} 3}$

$= 2.262$

(c) $\log_5 20 = \dfrac{\log_{10} 20}{\log_{10} 5}$

$= 1.861$

(d) $\log_4 3 = \dfrac{\log_{10} 3}{\log_{10} 4}$

$= 0.792$

(e) $\log_3 16 = \dfrac{\log_{10} 16}{\log_{10} 3}$

$= 2.524$

(f) $\log_6 4 = \dfrac{\log_{10} 4}{\log_{10} 6}$

$= 0.774$

(g) $\log_5 3 = \dfrac{\log_{10} 3}{\log_{10} 5}$

$= 0.683$

(h) $\log_2 10 = \dfrac{\log_{10} 10}{\log_{10} 2}$

$= 3.22$

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EXERCISE 8.5 SOLVING EQUATIONS WITH LOGARITHMS

2 **(a)** $\log_{10} x = \log_{10} 4 + \log_{10} 2$

$$\log_{10} x = \log_{10} 8$$
$$x = 8$$

(b) $\log_{10} x = \log_{10} 4 - \log_{10} 2$

$$\log_{10} x = \log_{10} 2$$
$$x = 2$$

(c) $\log_{10} x = \dfrac{\log_{10} 4}{\log_{10} 2}$

$$\log_{10} x = \frac{2\log_{10} 2}{\log_{10} 2}$$
$$\log_{10} x = 2$$
$$x = 10^2$$
$$x = 100$$

(d) $\log_{10} x = \dfrac{1}{2}\log_{10}\left(\dfrac{1}{4}\right)$

$$\log_{10} x = \frac{1}{2}\log_{10}\left(2^{-2}\right)$$
$$\log_{10} x = \left(\frac{1}{2}\times 2\right)\log_{10}\left(2^{-1}\right)$$
$$\log_{10} x = \log_{10}\left(\frac{1}{2}\right)$$
$$x = \frac{1}{2}$$

(e) $2\log_{10} x + 3 = \log_{10}\left(x^5\right)$

$$2\log_{10} x + 3 = 5\log_{10} x$$
$$3\log_{10} x = 3$$
$$\log_{10} x = 1$$
$$x = 10$$

(f) $\log_{10} x^2 = 2$

$$2\log_{10} x = 2$$
$$\log_{10} x = 2$$
$$x = 10$$

4 **(a)** $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$

$$\log_{10} \frac{10x}{3} = 2$$
$$\frac{10x}{3} = 10^2 = 100$$
$$10x = 300$$
$$x = 30$$

(b) $2\log_{10} x + 3 = 5\log_{10} x$

$$3\log_{10} x = 3$$
$$\log_{10} x = 1$$
$$x = 10$$

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(c) $\log_{10} 2 + 5\log_{10} x - \log_{10} 5 - \log_{10}\left(x^3\right) = \log_{10} 40$

$$\begin{aligned}
5\log_{10} x - 3\log_{10} x &= \log_{10} 40 - \log_{10} 2 + \log_{10} 5 \\
2\log_{10} x &= \log_{10} \frac{40\times 5}{2} \\
2\log_{10} x &= \log_{10} 100 \\
2\log_{10} x &= 2 \\
\log_{10} x &= 1 \\
x &= 10
\end{aligned}$$

(d) $\log_{10} x = 4\log_{10} 2 - 2\log_{10} x$

$$\begin{aligned}
3\log_{10} x &= \log_{10}\left(2^4\right) \\
\log_{10}\left(x^3\right) &= \log_{10}(16) \\
x^3 &= 16 \\
x &= \sqrt[3]{16}
\end{aligned}$$

(e) $\log_{10} x - \log_{10}(x-1) = 1$

$$\begin{aligned}
\log_{10} \frac{x}{x-1} &= 1 \\
\frac{x}{x-1} &= 10 \\
x &= 10x - 10 \\
9x &= 10 \\
x &= \frac{10}{9}
\end{aligned}$$

Checking, this solution is valid, since when $x = \frac{10}{9}$, $x > 0$ and $x - 1 > 0$.

(f) $\log_{10} x = 2\log_{10} 3 + \log_{10} 5 - \log_{10} 2 - 1$

$$\log_{10} x = \log_{10} \frac{3^2 \times 5}{2} - \log_{10} 10$$
$$\log_{10} x = \log_{10} \frac{45}{2\times 10}$$
$$\log_{10} x = \log_{10} \frac{9}{4}$$
$$x = \frac{9}{4}$$

6 (a) $2^x = 7$

$$\log_{10} 2^x = \log_{10} 7$$
$$x\log_{10} 2 = \log_{10} 7$$
$$x = \frac{\log_{10} 7}{\log_{10} 2}$$
$$x = 2.807$$

(b) $3^x = 18$

$$\log_{10} 3^x = \log_{10} 18$$
$$x\log_{10} 3 = \log_{10} 18$$
$$x = \frac{\log_{10} 18}{\log_{10} 3}$$
$$x = 2.631$$

(c) $5^x = 2$

$$\log_{10} 5^x = \log_{10} 2$$
$$x\log_{10} 5 = \log_{10} 2$$
$$x = \frac{\log_{10} 2}{\log_{10} 5}$$
$$x = 0.431$$

(d) $0.4^x = 2$

$$\log_{10} 0.4^x = \log_{10} 2$$
$$x\log_{10} 0.4 = \log_{10} 2$$
$$x = \frac{\log_{10} 2}{\log_{10} 0.4}$$
$$x = -0.756$$

(e) $6^x = 21$

$$\log_{10} 6^x = \log_{10} 21$$
$$x\log_{10} 6 = \log_{10} 21$$
$$x = \frac{\log_{10} 21}{\log_{10} 6}$$
$$x = 1.699$$

(f) $3^{-x} = 0.1$

$$\log_{10} 3^{-x} = \log_{10} 0.1$$
$$x\log_{10} 3^{-1} = \log_{10} 0.1$$
$$x = \frac{\log_{10} 0.1}{\log_{10} 3^{-1}}$$
$$x = 2.096$$

(g) $5^x = 16$

$$\log_{10} 5^x = \log_{10} 16$$
$$x\log_{10} 5 = \log_{10} 16$$
$$x = \frac{\log_{10} 16}{\log_{10} 5}$$
$$x = 1.723$$

(h) $4^x = 5$

$$\log_{10} 4^x = \log_{10} 5$$
$$x\log_{10} 4 = \log_{10} 5$$
$$x = \frac{\log_{10} 5}{\log_{10} 4}$$
$$x = 1.161$$

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8 B

$$y = a10^{bx}$$
$$\frac{y}{a} = 10^{bx}$$
$$bx = \log_{10} \frac{y}{a}$$
$$x = \frac{1}{b}\log_{10} \frac{y}{a}$$

10 $\log y = \log a + n\log x$

$$\log y = \log a + \log\left(x^n\right)$$
$$\log y = \log\left(ax^n\right)$$
$$y = ax^n$$

12 $x = a^2\sqrt{b^3c}$

$$\log x = \log\left(a^2\sqrt{b^3c}\right)$$
$$\log x = \log a^2 + \log\left(b^3c\right)^{\frac{1}{2}}$$
$$\log x = 2\log a + \frac{1}{2}\log\left(b^3c\right)$$
$$\log x = 2\log a + \frac{1}{2}\log b^3 + \frac{1}{2}\log c$$
$$\log x = 2\log a + \frac{3\log b}{2} + \frac{\log c}{2}$$

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14 $y = ae^{4t}$

$$\frac{y}{a} = e^{4t}$$
$$4t = \log_e \frac{y}{a}$$
$$t = \frac{1}{4}\log_e \frac{y}{a}$$

16 $\log_a 2 = \log_b 16$

Use the change of base rule to express $\log_b 16$ in terms of a.

$$\log_a 2 = \frac{\log_a 16}{\log_a b}$$
$$\log_a 2 = \frac{\log_a 2^4}{\log_a b}$$
$$\log_a 2 = \frac{4\log_a 2}{\log_a b}$$
$$\log_a b = \frac{4\log_a 2}{\log_a 2}$$
$$\log_a b = 4$$
$$b = a^4$$

18 (a) $10\ 000 = 5000 \times \left(1 + \frac{0.06}{12}\right)^n$

$$\frac{10\ 000}{5000} = 1.005^n$$
$$2 = 1.005^n$$
$$\log_{10} 2 = \log_{10}\left(1.005^n\right)$$
$$\log_{10} 2 = n\log_{10} 1.005$$
$$n = \frac{\log_{10} 2}{\log_{10} 1.005}$$
$$n = 138.97...$$

It will take approximately 139 months to double in value.

139 months is 11 years and 7 months.

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(b) $20\ 000 = 5000 \times \left(1 + \frac{0.06}{12}\right)^n$

$$\frac{20\ 000}{5000} = 1.005^n$$
$$4 = 1.005^n$$
$$\log_{10} 4 = \log_{10}\left(1.005^n\right)$$
$$\log_{10} 4 = n\log_{10} 1.005$$
$$n = \frac{\log_{10} 4}{\log_{10} 1.005}$$
$$n \approx 277.95...$$

It will take approximately 278 months to grow to $\$20000$.

Alternatively, this is doubling twice, which will take twice as long as doubling once, so $n = 2 \times 138.97... \approx 278$.

(c) $30\ 000 = 5000 \times \left(1 + \frac{0.06}{12}\right)^n$

$$\frac{30\ 000}{5000} = 1.005^n$$
$$6 = 1.005^n$$
$$\log_{10} 6 = \log_{10}\left(1.005^n\right)$$
$$\log_{10} 6 = n\log_{10} 1.005$$
$$n = \frac{\log_{10} 6}{\log_{10} 1.005}$$
$$n \approx 359.24...$$

It will take approximately 359.2 months to grow to $\$30000$.

20 (a) $y_n = 5 \times (1 + 0.05)^n$

$$= 5 \times 1.05^n$$

(b) $z_n = 4 \times (1 + 0.07)^n$

$$= 4 \times 1.07^n$$

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(c) $z_n > y_n$

$$
\begin{aligned}
4\times1.07^n &> 5\times1.05^n \\
\frac{1.07^n}{1.05^n} &> \frac{5}{4} \\
\left(\frac{1.07}{1.05}\right)^n &> \frac{5}{4} \\
\log_{10}\left(\frac{1.07}{1.05}\right)^n &> \log_{10}\left(\frac{5}{4}\right) \\
n\log_{10}\left(\frac{1.07}{1.05}\right) &> \log_{10}\left(\frac{5}{4}\right) \\
n &> \frac{\log_{10}\left(\frac{5}{4}\right)}{\log_{10}\left(\frac{1.07}{1.05}\right)} \approx 11.826...
\end{aligned}
$$

If the merger goes ahead, it will take nearly 12 years for a shareholder to be better off.

EXERCISE 8.6 EXPONENTIAL FUNCTIONS

2 B

Let $u = x^2$ so $y = e^{x^2} = e^u$.

$\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = 2x$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx} \\
&= e^u \times 2x \\
&= 2xe^{x^2}
\end{aligned}
$$

Alternatively,

$\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ where $f(x) = x^2$.

$$\frac{d}{dx}e^{f(x)} = 2xe^{x^2}$$

4 (a) $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$

$\dfrac{d}{dx}e^{2x+3} = 2e^{2x+3}$

(b) $\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ where $f(x) = x^2 - 2x$.

$\dfrac{d}{dx}e^{f(x)} = (2x-2)e^{x^2-2x} = 2(x-1)e^{x^2-2x}$

(c) $\dfrac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ where $f(x) = -x^3$.

$\dfrac{d}{dx}3e^{f(x)} = 3\dfrac{d}{dx}e^{f(x)} = 3\times -3x^2e^{-x^3} = -9x^2e^{-x^3}$

(d) $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$

$\dfrac{d}{dx}2e^{3x-1} = 2\dfrac{d}{dx}e^{3x-1} = 2\times 3e^{3x-1} = 6e^{3x-1}$

(e) $\dfrac{d}{dx}e^{ax+b} = ae^{ax+b}$

$\dfrac{d}{dx}\left(e^{3x-1} + e^{4x+2}\right) = 3e^{3x-1} + 4e^{4x+2}$

(f) Let $y = \sqrt{x}e^{-x}$

Let $u = \sqrt{x}$ and $v = e^{-x}$

$\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$ and $\dfrac{dv}{dx} = -e^{-x}$

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$$\begin{aligned}\frac{dy}{dx}&=v\frac{du}{dx}+u\frac{dv}{dx}\\&=e^{-x}\times\frac{1}{2\sqrt{x}}+\sqrt{x}\times -e^{-x}\\&=\left(\frac{1}{2\sqrt{x}}-\sqrt{x}\right)e^{-x}\\&=\left(\frac{1}{2\sqrt{x}}-\sqrt{x}\times\frac{2\sqrt{x}}{2\sqrt{x}}\right)e^{-x}\\&=\frac{(1-2x)e^{-x}}{2\sqrt{x}}\end{aligned}$$

(g) $\frac{d}{dx}e^{f(x)}=f'(x)e^{f(x)}$ where $f(x)=2x^2$.

$$\frac{d}{dx}3e^{f(x)}=3\frac{d}{dx}e^{f(x)}=3\times 4xe^{2x^2}=12xe^{2x^2}$$

(h) $\frac{d}{dx}e^{ax+b}=ae^{ax+b}$

$$\frac{d}{dx}3e^{2x-1}=3\frac{d}{dx}e^{2x-1}=3\times 2e^{2x-1}=6e^{2x-1}$$

(i) Let $y=xe^{x^2}$

Let $u=x$ and $v=e^{x^2}$

First find $\frac{dv}{dx}$.

$\frac{d}{dx}e^{f(x)}=f'(x)e^{f(x)}$ where $f(x)=x^2$.

$$\frac{dv}{dx}=\frac{d}{dx}e^{f(x)}=2xe^{x^2}$$

$$\frac{du}{dx}=1$$

$$\begin{aligned}\frac{dy}{dx}&=v\frac{du}{dx}+u\frac{dv}{dx}\\&=e^{x^2}\times 1+x\times 2xe^{x^2}\\&=\left(1+2x^2\right)e^{x^2}\end{aligned}$$

6 Let $u=1+t$ and $v=e^{5t}$

$\frac{du}{dt}=1$ and $\frac{dv}{dx}=5e^{5t}$

$$\begin{aligned}\frac{dx}{dt}&=v\frac{du}{dt}+u\frac{dv}{dt}\\&=e^{5t}\times 1+\left(1+t\right)\times 5e^{5t}\\&=\left(1+5+5t\right)e^{5t}\\&=\left(6+5t\right)e^{5t}\end{aligned}$$

Let $u=6+5t$ and $v=e^{5t}$

$\frac{du}{dt}=5$ and $\frac{dv}{dx}=5e^{5t}$

$$\begin{aligned}\frac{dx'}{dt}&=v\frac{du}{dt}+u\frac{dv}{dt}\\&=e^{5t}\times 5+\left(6+5t\right)\times 5e^{5t}\\&=\left(5+30+25t\right)e^{5t}\end{aligned}$$

$$\frac{d^2x}{dt^2}=\left(35+25t\right)e^{5t}$$

$$x=\left(1+t\right)e^{5t}$$

$$\frac{dx}{dt}=\left(6+5t\right)e^{5t}$$

$$\frac{d^2x}{dt^2}=\left(35+25t\right)e^{5t}$$

Note: In this case it is better not to factorise $\frac{d^2x}{dt^2}$ since we will be expanding brackets next.

$$\begin{aligned}\frac{d^2x}{dt^2}-10\frac{dx}{dt}+25x&=\left(35+25t\right)e^{5t}-10\left(6+5t\right)e^{5t}+25\left(1+t\right)e^{5t}\\&=\left(35+25t-60-50t+25+25t\right)e^{5t}\\&=0\times e^{5t}\\&=0\end{aligned}$$

$$\therefore \frac{d^2x}{dt^2}-10\frac{dx}{dt}+25x=0$$

8 $y=e^{-x}$ crosses the y-axis when $x=0$.

$y = e^{-0} = 1$

$\frac{dy}{dx} = -e^{-x}$

When $x = 0$, $\frac{dy}{dx} = -e^{-(0)} = -1$

The gradient of the line is -1 and the y-intercept is 1.

Equation of tangent: $y = -x + 1$

10 $y = 2 + e^{-x}$

When $x = 0, y = 2 + e^{0} = 2 + 1 = 3$.

The y-intercept of both tangent and normal is 3.

$\frac{dy}{dx} = -e^{-x}$

$x = 0, \frac{dy}{dx} = -e^{0} = -1$

The tangent has gradient -1 and y-intercept 3.

$\therefore y_T = -x + 3$

To calculate the gradient of the normal use $m_1 \times m_2 = -1$

$m_N = -\frac{1}{-1} = 1$

The normal has gradient 1 and y-intercept 3.

$\therefore y_N = x + 3$

12 $y = 500\left(1 - e^{-0.2t}\right) = 500 - 500e^{-0.2t}$

The derivative of $e^{-0.2t}$ is $-0.2e^{-0.2t}$. $y = 500 - 500e^{-0.2t} = 500 - 500e^{u}$ where $u = -0.2$

$\frac{dy}{dx} = 0 - 500 \times -0.2e^{-0.2t} = 100e^{-0.2t}$

This is the instantaneous rate of change of y.

EXERCISE 8.7 SOME APPLICATIONS OF EXPONENTIAL FUNCTIONS

2 Let $u = x,\ v = e^{-0.5x}$

$$\frac{du}{dx} = 1,\ \frac{dv}{dx} = -0.5e^{-0.5x}$$

$$y = xe^{-0.5x}$$

$$\frac{dy}{dx} = \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= e^{-0.5x} \times 1 + x \times -0.5e^{-0.5x}$$

$$= e^{-0.5x}\left(1 - \frac{x}{2}\right)$$

$$= \frac{(2-x)e^{-0.5x}}{2}$$

The turning point occurs when $\frac{dy}{dx} = 0$

$$\frac{(2-x)e^{-0.05x}}{2} = 0$$

$$2 - x = 0$$

$$x = 2$$

$$y = 2 \times e^{-0.5\times 2}$$

$$= 2e^{-1}$$

$$= \frac{2}{e}$$

x	1	2	3
$\frac{dy}{dx}$	$\frac{1}{2e^{0.5}}$	0	$-\frac{1}{2e^{0.5}}$

$\therefore \left(2, \frac{2}{e}\right)$ is a maximum turning point.

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(a) Note that $e^{-0.5x}$ is always positive, so $y > 0 \Rightarrow xe^{-0.5x} > 0 \Rightarrow x > 0$

(b) $\frac{dy}{dx} > 0 \Rightarrow \frac{(2-x)e^{-0.5x}}{2} > 0 \Rightarrow 2 - x > 0 \Rightarrow x < 2$

4 **(a)** $\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$ where $f(x) = -x^2$.

$$f(x) = e^{-x^2}$$
$$f'(x) = -2xe^{-x^2}$$

(b) **(i)** $f'(x) = 0$

$$-2xe^{-x^2} = 0$$
$$x = 0$$

(ii) $f'(x) > 0$

$$-2xe^{-x^2} > 0$$
$$-2x > 0$$
$$x < 0$$

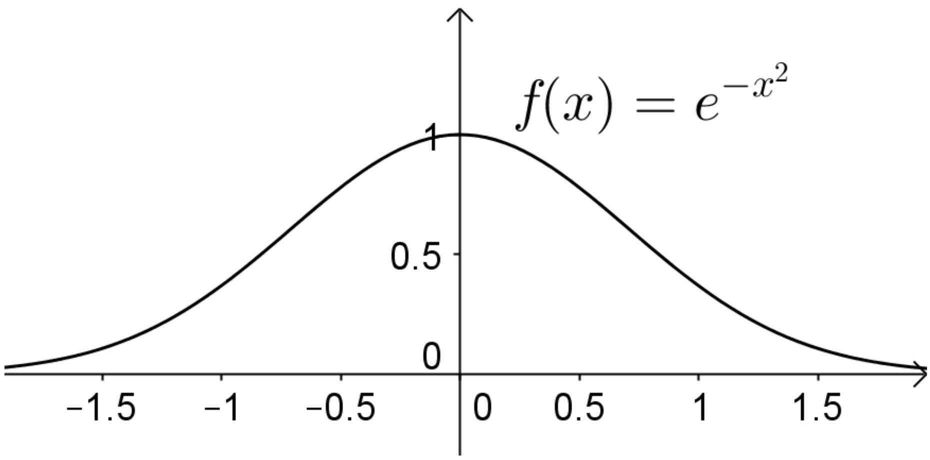
(iii) $f'(x) < 0$

$$-2xe^{-x^2} < 0$$
$$-2x < 0$$
$$x > 0$$

(c) The maximum value of $f(x) = e^{-x^2}$ will occur when x^2 is a minimum, i.e. when $x = 0$.

When $x = 0$, $f(x) = e^0 = 1$.

As $x \to \pm\infty$, $f(x) \to 0$.



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6 $y = e^t + 4e^{-t}$

$$y' = e^t - 4e^{-t}$$
$$y'' = e^t + 4e^{-t}$$

Stationary points occur when $y' = 0$.

$$e^t - 4e^{-t} = 0$$
$$\frac{e^{2t} - 4}{e^t} = 0$$
$$e^{2t} = 4$$

Since $e^t > 0$, then $e^t = 2$.

When $e^t = 2$,

$$y = 2 + \frac{4}{2}$$
$$= 4$$

$y'' > 0$ for all values of x.

$\therefore$ at the stationary point, the gradient changes from negative to positive, so 4 is a minimum value.

(a) correct **(b)** incorrect

(c) correct **(d)** correct

8 **(a)** $A = 2xy$

$$= 2xe^{-x^2}$$

Let $u = 2x,\ v = e^{-x^2}$

$$\frac{du}{dx} = 2,\ \frac{dv}{dx} = -2xe^{-0.5x}$$

$$A = 2xe^{-x^2}$$
$$\frac{dA}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= e^{-x^2} \times 2 + 2x \times -2xe^{-x^2}$$
$$= e^{-x^2}\left(2 - 4x^2\right)$$
$$= 2\left(1 - 2x^2\right)e^{-x^2}$$

Let $u = 2\left(1-2x^2\right),\ v = e^{-x^2}$

$$\frac{du}{dx} = -8x,\ \frac{dv}{dx} = -2xe^{-x^2}$$

$$\begin{aligned}\frac{d^2A}{dx^2} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= e^{-x^2}\times -8x + 2\left(1-2x^2\right)\times -2xe^{-x^2}\\ &= -8xe^{-x^2} - 4x\left(1-2x^2\right)e^{-x^2}\\ &= 4xe^{-x^2}\left(-2-1+2x^2\right)\\ &= 4xe^{-x^2}\left(2x^2-3\right)\end{aligned}$$

$$\begin{aligned}\frac{dA}{dx} &= 0\\ 2e^{-x^2}\left(1-2x^2\right) &= 0\\ 2x^2 &= 1\\ x &= \pm\frac{1}{\sqrt{2}}\end{aligned}$$

When $x = \frac{1}{\sqrt{2}}$

$$\begin{aligned}\frac{d^2A}{dx^2} &= 4\times\frac{1}{\sqrt{2}}\times e^{-\frac{1}{2}}\left(2\times\frac{1}{2}-3\right)\\ &= -\frac{8}{\sqrt{2e}}\\ &< 0\end{aligned}$$

$\therefore\ x = \frac{1}{\sqrt{2}}$ gives the maximum area.

From the diagram, $x = -\frac{1}{\sqrt{2}}$ will just be a reflection in the y-axis and give the same area as when $x = \frac{1}{\sqrt{2}}$.

(b) $A = 2xe^{-x^2}$

$$= 2 \times \frac{1}{\sqrt{2}} \times e^{-\frac{1}{2}}$$

$$= \frac{\sqrt{2}}{\sqrt{e}}$$

$$\approx 0.86 \text{ units}^2$$

EXERCISE 8.8 NATURAL LOGARITHMS

2 **(a)** $\log_e e^{3x+5} = 2$

$$\begin{aligned} 3x + 5 &= 2 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

(b) Take logarithms to base e of both sides.

$$\begin{aligned} e^{\frac{x}{4}} &= 3 \\ \frac{x}{4} &= \log_e 3 \\ x &= 4\log_e 3 \end{aligned}$$

(c) Take logarithms to base e of both sides.

$$\begin{aligned} 5e^{4x} &= 8 \\ e^{4x} &= \frac{8}{5} \\ 4x &= \log_e \frac{8}{5} \\ x &= \frac{1}{4}\log_e \frac{8}{5} \end{aligned}$$

4 **(a)** $\log_e (x+2) = 3$

$$\begin{aligned} x + 2 &= e^3 \\ x &= e^3 - 2 \end{aligned}$$

Check the validity of answer:

$$x + 2 = \left(e^3 - 2\right) + 2 = e^3 > 0$$

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$\therefore x = e^3 - 2$.

(b) $\log_e(2x-2) = 4$

$$2x - 2 = e^4$$
$$x = \frac{e^4 + 2}{2}$$

Check the validity of answer:

$$2x - 2 = 2 \times \frac{e^4 + 2}{2} - 2 = e^4 > 0$$

$\therefore x = \dfrac{e^4 + 2}{2}$.

(c) $\log_e(x+2) - \log_e(x-2) = 1$

$$\log_e \frac{x+2}{x-2} = 1$$
$$\frac{x+2}{x-2} = e^1$$
$$x + 2 = e(x-2)$$
$$x + 2 = ex - 2e$$
$$x - ex = -2e - 2$$
$$ex - x = 2e + 2$$
$$x(e-1) = 2(e+1)$$
$$x = \frac{2(e+1)}{e-1}$$

Check the validity of answer:

$x - 2 > 0$ and $x + 2 > 0$

So $x > 2$.

$\dfrac{2(e+1)}{e-1} \approx 4.33$ so $x - 2 > 0$ and $x + 2 > 0$.

$\therefore x = \dfrac{2(e+1)}{e-1}$.

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6 **(a)** $\log_e x + \log_e (x+5) = \log_e (x+2) + \log_e 6$

$$\begin{aligned}
\log_e x(x+5) &= \log_e 6(x+2) \\
x(x+5) &= 6(x+2) \\
x^2+5x &= 6x+12 \\
x^2-x-12 &= 0 \\
(x+3)(x-4) &= 0 \\
x &= -3,\ x=4
\end{aligned}$$

Check the validity of answers:

$\log_e x$ is undefined when $x=-3$.

All logarithms are defined when $x=4$.

$\therefore\ x=4$ is the only valid answer.

(b) $\log_e x - \log_e (x+5) = \log_e (x-4) - \log_e (x+2)$

$$\begin{aligned}
\log_e \frac{x}{x+5} &= \log_e \frac{x-4}{x+2} \\
\frac{x}{x+5} &= \frac{x-4}{x+2} \\
x(x+2) &= (x-4)(x+5) \\
x^2+2x &= x^2+x-20 \\
x &= -20
\end{aligned}$$

Check the validity of answers:

$\log_e x$ is undefined when $x=-20$.

There is no valid solution to this equation.

EXERCISE 8.9 GRAPHS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

2 Intersections of $f(x)$ and $g(x)$:

$$\begin{aligned}
e^x &= 3e^x \\
2e^x &= 0
\end{aligned}$$

Since $e^x \neq 0$, there is no point of intersection for $f(x)$ and $g(x)$.

Intersections of $f(x)$ and $h(x)$:

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$$
\begin{aligned}
e^x &= e^{3x} \\
e^{3x} - e^x &= 0 \\
e^x\left(e^{2x} - 1\right) &= 0 \\
e^{2x} - 1 &= 0 \\
e^{2x} &= 1 \\
2x &= 0 \\
x &= 0
\end{aligned}
$$

$f(0) = h(0) = e^0 = 1$

$f(x)$ and $h(x)$ intersect at $(0,\ 1)$.

Intersections of $g(x)$ and $h(x)$:

$$
\begin{aligned}
3e^x &= e^{3x} \\
3e^x - e^{3x} &= 0 \\
e^x\left(3 - e^{2x}\right) &= 0 \\
e^{2x} &= 3 \\
2x &= \log_e 3 \\
x &= \frac{1}{2}\log_e 3
\end{aligned}
$$

$$g\left(\frac{1}{2}\log_e 3\right) = h\left(\frac{1}{2}\log_e 3\right) = 3e^{\frac{1}{2}\log_e 3} = 3e^{\log_e \sqrt{3}} = 3\sqrt{3}$$

$g(x)$ and $h(x)$ intersect at $\left(\frac{1}{2}\log_e 3,\ 3\sqrt{3}\right)$.

$f(x)$ and $h(x)$ both approach zero from above as $x \to -\infty$.

When $x < 0$, $h(x)$ is below $f(x)$, when $x > 0$, $h(x)$ is above $f(x)$.

The values of $g(x)$ are three times the values of $f(x)$ for a given value of x. $g(x)$ is always above $f(x)$.

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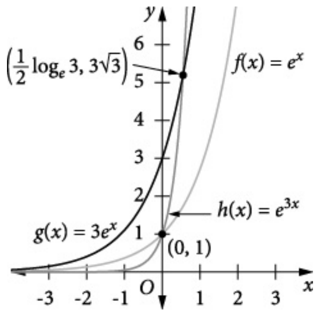


$g(x)$ is the graph of $f(x)$ stretched vertically by a factor of 3.

$h(x)$ is the graph of $f(x)$ compressed horizontally towards the x-axis by $\frac{1}{3}$.

4 Intersections of $f(x)$ and $g(x)$:

$$\begin{aligned}\log_e x &= \log_e 3x \\ x &= 3x \\ 2x &= 0\end{aligned}$$

Since $x \neq 0$, there is no point of intersection for $f(x)$ and $g(x)$.

Intersections of $f(x)$ and $h(x)$:

$$\begin{aligned}\log_e x &= \log_e (x+3) \\ x &= x+3\end{aligned}$$

This is not possible, there is no point of intersection for $f(x)$ and $h(x)$.

Intersections of $g(x)$ and $h(x)$:

$$\begin{aligned}\log_e (3x) &= \log_e (x+3) \\ 3x &= x+3 \\ 2x &= 3 \\ x &= 1.5\end{aligned}$$

$g(1.5) = h(1.5) = \log_e (3\times1.5) = \log_e 4.5$

$g(x)$ and $h(x)$ intersect at $(1.5, \log_e 4.5)$.

$f(x) = \log_e x$ cuts the x-axis at $(1, 0)$.

As $x \to 0$ from the right, $f(x) \to -\infty$.

$$\begin{aligned}g(x) &= \log_e (3x) \\ &= \log_e 3 + \log_e x \\ &= f(x) + \log_e 3\end{aligned}$$

So $g(x)$ is $f(x)$ moved up by $\log_e 3$ units.

$$\begin{aligned} g(x) &= 0 \\ \log_e(3x) &= 0 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

So $g(x)$ cuts the x-axis at $\left(\frac{1}{3}, 0\right)$.

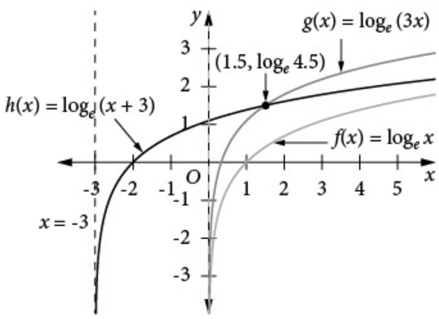
As $x \to 0$ from the right, $g(x) \to -\infty$.

$h(x) = \log_e(x+3)$ is $f(x)$ moved 3 units to the left.

$$\begin{aligned} h(x) &= 0 \\ \log_e(x+3) &= 0 \\ x+3 &= 1 \\ x &= -2 \end{aligned}$$

So $h(x)$ cuts the x-axis at $(-2, 0)$.

As $x \to -3$ from the right, $g(x) \to -\infty$.



6 $$\begin{aligned} f(x) &= \log_e(2x) \\ &= \log_e 2 + \log_e x \end{aligned}$$

$f(x)$ is the graph of $y = \log_e x$ moved up by $\log_e 2$.

$$\begin{aligned} f(x) &= 0 \\ \log_e(2x) &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

so $f(x)$ cuts the x-axis at $\left(\frac{1}{2}, 0\right)$.

$$\begin{aligned} h(x) &= \log_e\left(\frac{x}{2}\right) \\ &= \log_e x - \log_e 2 \end{aligned}$$

$h(x)$ is the graph of $y = \log_e x$ moved down by $\log_e 2$.

$$\begin{aligned} h(x) &= 0 \\ \log_e\left(\frac{x}{2}\right) &= 0 \\ \frac{x}{2} &= 1 \\ x &= 2 \end{aligned}$$

so $h(x)$ cuts the x-axis at $(2, 0)$.

$f(x)$ and $h(x)$ does not intersect.

$$\begin{aligned} g(x) &= \log_e x^2 \\ &= 2\log_e|x| \end{aligned}$$

$g(x)$ has the graph of $y = 2\log_e x$ and its reflection along the y-axis.

$$\begin{aligned} g(x) &= 0 \\ \log_e x^2 &= 0 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

$g(x)$ cuts the x-axis at $(-1, 0)$ and $(1, 0)$.

Intersections of $g(x)$ and $f(x)$:

$$\begin{aligned} \log_e x^2 &= \log_e(2x) \\ x^2 &= 2x \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x &= 0,\ x = 2 \end{aligned}$$

$x = 0$ is not a valid answer in the given domain, so $x = 2$ is the only valid answer.

$$\begin{aligned} f(2) &= g(2) \\ &= \log_e 4 \end{aligned}$$

So $f(x)$ and $g(x)$ intersect at $(2, \log_e 4)$.

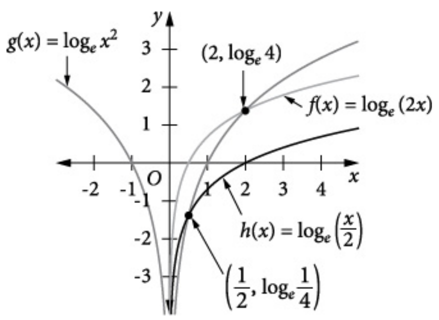
Intersections of $g(x)$ and $h(x)$:

$$\begin{aligned}\log_e x^2 &= \log_e\left(\frac{x}{2}\right)\\ x^2 &= \frac{x}{2}\\ 2x^2 - x &= 0\\ x(2x-1) &= 0\\ x = 0,\ x &= \frac{1}{2}\end{aligned}$$

$x = 0$ is not a valid answer in the given domain, so $x = \frac{1}{2}$ is the only valid answer.

$$\begin{aligned}h\left(\frac{1}{2}\right) &= g\left(\frac{1}{2}\right)\\ &= \log_e \frac{1}{4}\end{aligned}$$

So $h(x)$ and $g(x)$ intersect at $\left(\frac{1}{2}, \log_e \frac{1}{4}\right)$.



8 $f(x) = e^x$ and $g(x) = e^{-x}$ are reflections of each other in the y-axis.

Both $f(x)$ and $g(x)$ intersect the y-axis at $(0,1)$.

Both graphs have the x-axis as an asymptote.

$$\begin{aligned}h(x) &= f(x) + g(x)\\ &= e^x + e^{-x}\end{aligned}$$

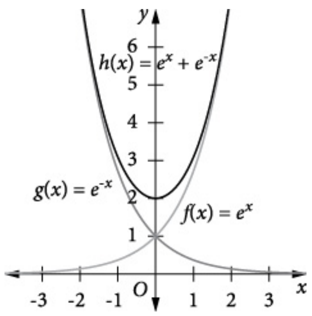
$h(x)$ is symmetrical about the y-axis.

Its minimum value occurs on the y-axis at $(0, 2)$.

$h(x)$ is contained entirely between the branches of $f(x)$ and $g(x)$ above the point (0, 2).

$h(x)$ approaches $f(x)$ as x increases, $x > 0$.

$h(x)$ approaches $g(x)$ as x decreases, $x < 0$.



EXERCISE 8.10 LOGARITHMS IN THE REAL WORLD

2 **(a)** $10\ \log_{10}\left(\frac{P_2}{P_1}\right)=20$

$$\log_{10}\left(\frac{P_2}{P_1}\right)=2$$

$$\frac{P_2}{P_1}=10^2=100$$

100 times as loud

(b) The first sound is $75-35=40$ dB louder than the second.

$$10\ \log_{10}\left(\frac{P_2}{P_1}\right)=40$$

$$\log_{10}\left(\frac{P_2}{P_1}\right)=4$$

$$\frac{P_2}{P_1}=10^4=10\ 000$$

$10\ 000$ times as loud

(c) The first sound is $79-72=7$ dB louder than the second.

$$10\ \log_{10}\left(\frac{P_2}{P_1}\right)=7$$

$$\log_{10}\left(\frac{P_2}{P_1}\right)=0.7$$

$$\frac{P_2}{P_1}=10^{0.7}=5.011...$$

About 5 times as loud

4 The difference on the Richter scale is $8.7-6.5=2.2$

$$\log_{10}\left(\frac{P_2}{P_1}\right)=2.2$$

$$\frac{P_2}{P_1}=10^{2.2}$$

$$=158.489...$$

The second earthquake is about 160 times stronger than the first.

Note: The measurements are given to two significant figures accuracy, and therefore the answer should be written to the same level of accuracy.

6 The pH is $-\log_{10}\left(2.3\times10^{-5}\right)=-\log_{10}2.3-(-5)$

$$=-0.3617...+5$$
$$\approx 4.6$$

Since the pH level is less than 7, it is acidic.

8 A sound of zero intensity on the decibel scale would be $10\ \log_{10}0$ which is undefined. This is considered a flaw in the decibel scale (as well as the Richter scale).

CHAPTER REVIEW 8

2 **(a)** $\log_3 18+2\log_3 9-\log_3 54=\log_3 18+\log_3 9^2-\log_3 54$

$$=\log_3\frac{18\times81}{54}$$
$$=\log_3 27$$
$$=3$$

(b) $\log_a\left(xy^2\right)+\log_a\left(yz^2\right)-\log_a\left(xz^2\right)=\log_a\frac{xy^2\times yz^2}{xz^2}$

$$=\log_a\frac{xy^3z^2}{xz^2}$$
$$=\log_a y^3$$
$$=3\log_a y$$

(c) $\log_{10}\frac{6+4\sqrt{6}}{5}+\log_{10}\frac{2\sqrt{6}-3}{2}=\log_{10}\left(\frac{6+4\sqrt{6}}{5}\times\frac{2\sqrt{6}-3}{2}\right)$

$$=\log_{10}\left(\frac{12\sqrt{6}-18+48-12\sqrt{6}}{10}\right)$$
$$=\log_{10}\left(\frac{30}{10}\right)$$
$$=\log_{10}3$$

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(d)

$$\begin{aligned}2\log(x+1)-\log(x-1)-2\log(y+1)+\log(y-1) &= 2\log(5+1)-\log(5-1)-2\log(2+1)+\log(2-1)\\ &= 2\log 6-\log 4-2\log 3+\log 1\\ &= \log 6^2-\log 4-\log 3^2\\ &= \log\frac{36}{4\times 9}\\ &= \log 1\\ &= 0\end{aligned}$$

4 **(a)** Let $y=\left(x^2+2x\right)e^x$

Let $u=x^2+2x$ and $v=e^x$

$\frac{du}{dx}=2x+2$ and $\frac{dv}{dx}=e^x$

$$\begin{aligned}\frac{dy}{dx}&=v\frac{du}{dx}+u\frac{dv}{dx}\\ &=e^x\times(2x+2)+\left(x^2+2x\right)\times e^x\\ &=\left(2x+2+x^2+2x\right)e^x\\ &=\left(x^2+4x+2\right)e^x\end{aligned}$$

(b) Let $y=\left(x^2+3x\right)e^{-3x}$

Let $u=x^2+3x$ and $v=e^{-3x}$

$\frac{du}{dx}=2x+3$ and $\frac{dv}{dx}=-3e^{-3x}$

$$\begin{aligned}\frac{dy}{dx}&=v\frac{du}{dx}+u\frac{dv}{dx}\\ &=e^{-3x}\times(2x+3)+\left(x^2+3x\right)\times -3e^{-3x}\\ &=e^{-3x}\left(2x+3-3x^2-9x\right)\\ &=e^{-3x}\left(3-7x-3x^2\right)\end{aligned}$$

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(c) Let $y = e^{\sqrt{x}}$

$$\frac{d}{dx}e^{\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}e^{\sqrt{x}} = \frac{1}{2\sqrt{x}}e^{\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

6 (a) $\left(3^x - 1\right)\left(2^x - \frac{1}{8}\right) = 0$

$$3^x = 1,\ 2^x = \frac{1}{8} = 2^{-3}$$
$$3^x = 1 \Rightarrow x = 0$$
$$2^x = 2^{-3} \Rightarrow x = -3$$
$$x = -3, 0$$

(b) Use prime factors and index laws.

$$2^{3x+1} = \frac{1}{32}$$
$$2^{3x+1} = \frac{1}{2^5}$$
$$2^{3x+1} = 2^{-5}$$
$$3x + 1 = -5$$
$$3x = -6$$
$$x = -2$$

(c) $e^{4x+1} = \frac{1}{e^3}$

$$e^{4x+1} = e^{-3}$$
$$4x + 1 = -3$$
$$4x = -4$$
$$x = -1$$

8 (a) $\log_e\left(x+2\right) = \log_e\left(2x\right)$

$$x + 2 = 2x$$
$$2 = x$$

Check: Both $\log_e\left(x+2\right)$ and $\log_e\left(2x\right)$ are defined if $x = 2$.

The solution $x = 2$ is valid.

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(b) $\log_e(2x+3)=\log_e(x^2)$

$$\begin{aligned}2x+3&=x^2\\0&=x^2-2x-3\\0&=(x-3)(x+1)\\x&=3,-1\end{aligned}$$

Check: Both $\log_e(2x+3)$ and $\log_e(x^2)$ are defined if $x=3$ and $x=-1$.

So, both solutions are valid.

(c) $\log_e(x^2)=\log_e\left(\frac{x}{3}\right)$

$$\begin{aligned}x^2&=\frac{x}{3}\\3x^2&=x\\3x^2-x&=0\\x(3x-1)&=0\\x&=0,\ \frac{1}{3}\end{aligned}$$

Check: Both logarithms are undefined for $x=0$ but defined for $x=\frac{1}{3}$.

So, only $x=\frac{1}{3}$ is valid.

10 $A=P(1+i)^n$

Here, $P=4000$ and $i=3\%=0.03$ so $A=4000(1.03)^n$.

(a) $8000=4000(1.03)^n$

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$$
\begin{aligned}
2 &= 1.03^n \\
\log_e 2 &= \log_e (1.03)^n \\
\log_e 2 &= n\log_e (1.03) \\
n &= \frac{\log 2}{\log_e (1.03)} \\
&= 23.44977
\end{aligned}
$$

This equates to 23 years and 5.4 months.

So, it takes about 23 years 5 months to double.

(b) $10\,000 = 4000(1.03)^n$

$$
\begin{aligned}
\frac{10\,000}{4000} &= 1.03^n \\
\frac{5}{2} &= 1.03^n \\
\log_e\left(\frac{5}{2}\right) &= \log_e (1.03)^n \\
\log_e\left(\frac{5}{2}\right) &= n\log_e (1.03) \\
n &= \frac{\log_e\left(\frac{5}{2}\right)}{\log_e (1.03)} \\
&= 30.9989
\end{aligned}
$$

So, it takes about 31 years to reach $\$10\,000$.

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(c) $80\,000 = 4000(1.03)^n$

$$\begin{aligned}
\frac{80\,000}{4000} &= 1.03^n \\
20 &= 1.03^n \\
\log_e(20) &= \log_e(1.03)^n \\
\log_e(20) &= n\log_e(1.03) \\
n &= \frac{\log_e 20}{\log_e(1.03)} \\
&= 101.34822
\end{aligned}$$

This equates to 101 years 4.2 months.

So, it takes about 101 years 4 months to reach $\$80\,000$

12 $\theta = \theta_0 e^{-kt}$

$$\begin{aligned}
\frac{d\theta}{dt} &= -k \times \theta_0 e^{-kt} \\
&= -k \times \theta \\
&= -k\theta
\end{aligned}$$

14 (a) $370\,000\times1.005^n=2998\times\left(\dfrac{1.005^n-1}{0.005}\right)$

$$\frac{370\,000}{2998}\times0.005\times1.005^n=1.005^n-1$$
$$\frac{1850}{2998}\times1.005^n-1.005^n=-1$$
$$1.005^n\left(1-\frac{1850}{2998}\right)=1$$
$$1.005^n\times\frac{1148}{2998}=1$$
$$1.005^n=\frac{2998}{1148}$$

Take logarithms of both sides.

$$n\log1.005=\log\left(\frac{2998}{1148}\right)$$
$$n\approx192.46$$

(b) $48\,500\times1.006^n-780\left(\dfrac{1.006^n-1}{0.006}\right)=0$

$$0.006\times48\,500\times1.006^n-780\left(1.006^n-1\right)=0$$
$$291\times1.006^n-780\times1.006^n+780=0$$
$$780=780\times1.006^n-291\times1.006^n$$
$$780=\left(780-291\right)\times1.006^n$$
$$780=489\times1.006^n$$
$$1.006^n=\frac{780}{489}$$

Take logarithms of both sides.

$$n\log1.006=\log\left(\frac{780}{489}\right)$$
$$n=78.055...\approx78$$

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