2019 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

Answers

1	D	**6**	B
2	B	**7**	C
3	C	**8**	B
4	A	**9**	D
5	A	**10**	B

Brief explanations

1 $AB = m$ and $BC = n$, so B must divide AC internally with ratio $m : n$. Also, A divides BC externally in the ratio $m : (m + n)$ or C divides AB externally in the ratio $(m + n) : n$. The only option which matches these is (D).

2 Differentiating each of the options, it can be seen that the only answer which cannot satisfy $\dfrac{dN}{dt} = k(N - b)$ is option (B).

3 From t-formula results $\sin 2x = \dfrac{2t}{1 + t^2}$ where $t = \tan x$. The equation then simplifies to $\tan 2x = 1$, which has general solution $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$. Hence, the answer is (C).

4 Since $|x - 1| \geq 0$ then the inequality implies $\dfrac{1}{x - 1} > 0$. This means that the only valid domain is $x > 1$, which implies $|x - 1| = x - 1$ and the inequality reduces to $(x - 1)^2 < 1$. Solving this leads to $0 < x < 2$ but since $x > 1$ then the final solution is $1 < x < 2$, which is (A).

5 By the remainder theorem, $P(x) = (x + 1)^3 Q(x) + R(x)$ where $Q(x)$ and $R(x)$ are polynomials. The remainder $R(x)$ must be a quadratic, otherwise it can be further divided by the cubic polynomial $(x + 1)^3$. Hence, $R(x) = ax^2 + bx + c$ for some coefficients a, b and c. Since $P(-1) = 2$ then $a - b + c = 2$. The coefficients of all the options satisfy this equality except for (A).

6 Applying long division or inspection, the other factor of x^4+4 is x^2+2x+2 so

$$\begin{aligned}\int_{-1}^{1}\frac{(x+1)^2+1}{x^4+4}\,dx &= \int_{-1}^{1}\frac{x^2+2x+2}{(x^2+2x+2)(x^2-2x+2)}\,dx\\ &= \int_{-1}^{1}\frac{1}{x^2-2x+2}\,dx\\ &= \int_{-1}^{1}\frac{1}{(x-1)^2+1}\,dx\\ &= \left[\tan^{-1}(x-1)\right]_{-1}^{1}\\ &= -\tan^{-1}(-2)\\ &= \tan^{-1}2\end{aligned}$$

Hence, the answer is (B).

7 Let $x=\sec\theta$. When $x=-1$ then $\theta=\pi$. When $x=-2$ then $\theta=\frac{2\pi}{3}$ or $\theta=\frac{4\pi}{3}$. This means the answer cannot be (A) or (D).

Note that $\frac{(x^2-1)^{\frac{3}{2}}}{x}\le 0$ in the domain $-2\le x\le -1$ so the value of the definite integral must be negative. Since $\tan^4\theta$ is non-negative, then the only way to get a negative definite integral is for the lower limit to be greater than the upper limit. Hence, the answer is (C).

8 The general term of the binomial expansion of $(1-x)^n$ is $\binom{n}{k}(-1)^k x^k$. Note that

$$\begin{aligned}1+2+3+...+(n-1) &= \frac{n(n-1)}{2}\\ &= \frac{n(n-1)(n-2)!}{(n-2)!2!}\\ &= \frac{n!}{2!(n-2)!}\\ &= \binom{n}{2}\quad\text{or}\quad\binom{n}{n-2}\end{aligned}$$

This means $k=2$ or $k=n-2$ are possible candidate choices. The coefficient of x^{n-2} is $\binom{n}{n-2}(-1)^{n-2}$, but since n is odd then the coefficient simplifies to $-\binom{n}{n-2}$. On the other hand, the coefficient of x^2 is $\binom{n}{2}(-1)^2$, which is $\binom{n}{2}$. Hence, the answer is (B).

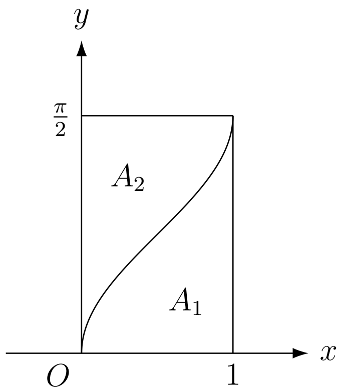
9 Since there are no restrictions on the seating arrangement of the students, there are $(2n)!$ ways of arranging them anywhere in the two rows. Each student has two possible exams they could be allocated with. Therefore, the total number of arrangements is $2\times(2n)!$, so the answer is (D).

10 Since $y = \sin^{-1}\sqrt{x}$ is increasing in the domain $0 < x < 1$ (noting its actual domain is $0 \leq x \leq 1$), the range must be $0 \leq y \leq \frac{\pi}{2}$.

Let A_1 be the area bounded by the curve $y = \sin^{-1}\sqrt{x}$, the x-axis and $x = 1$. Let A_2 be the area bounded by the curve $y = \sin^{-1}\sqrt{x}$, the y-axis and $y = \frac{\pi}{2}$.

$$\begin{aligned} A_1 + A_2 &= \int_0^1 y\,dx + \int_0^{\frac{\pi}{2}} x\,dy \\ &= \int_0^1 \sin^{-1}\sqrt{x}\,dx + \int_0^{\frac{\pi}{2}} \sin^2 y\,dy \end{aligned}$$

However, the sum of the areas A_1 and A_2 form a rectangle of side lengths 1 and $\frac{\pi}{2}$.



By inspection and deduction of the available options, the most likely answer is (B). This can be formally shown as follows

$$\begin{aligned} \int_0^1 \sin^{-1}\sqrt{x}\,dx + \int_0^{\frac{\pi}{2}} \sin^2 y\,dy &= \frac{\pi}{2} \times 1 \\ \Rightarrow \int_0^1 \sin^{-1}\sqrt{x}\,dx &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin^2 y\,dy \\ &= \frac{\pi}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2y - 1)\,dy \\ &= \frac{\pi}{2} + \frac{1}{2}\left[\frac{1}{2}\sin 2y - y\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4} \end{aligned}$$

Hence, the answer is (B).

Question 11

(a) Rearranging the given result $\dfrac{\sin\alpha+\sin\beta+\sin\gamma}{\sin(\alpha+\beta+\gamma)} = \dfrac{\cos\alpha+\cos\beta+\cos\gamma}{\cos(\alpha+\beta+\gamma)}$ gives

$$(\cos\alpha+\cos\beta+\cos\gamma)\sin(\alpha+\beta+\gamma) - (\sin\alpha+\sin\beta+\sin\gamma)\cos(\alpha+\beta+\gamma) = 0$$

Note that

$$\begin{aligned}\sin(\alpha+\beta+\gamma)\cos\alpha - \cos(\alpha+\beta+\gamma)\sin\alpha &= \sin(\alpha+\beta+\gamma-\alpha)\\ &= \sin(\beta+\gamma)\end{aligned}$$

Similarly

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha+\beta+\gamma)\cos\gamma - \cos(\alpha+\beta+\gamma)\sin\gamma\\ \sin(\alpha+\gamma) &= \sin(\alpha+\beta+\gamma)\cos\beta - \cos(\alpha+\beta+\gamma)\sin\beta\end{aligned}$$

Adding all the equations gives

$$\begin{aligned}&\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\alpha+\gamma)\\ &= (\cos\alpha+\cos\beta+\cos\gamma)\sin(\alpha+\beta+\gamma) - (\sin\alpha+\sin\beta+\sin\gamma)\cos(\alpha+\beta+\gamma)\\ &= 0\end{aligned}$$

(b) (i) Using the distance formula noting that $a>0$ and $1+p^2>0$

$$\begin{aligned}PT^2 &= (2ap-a(p+q))^2 + (ap^2-apq)^2\\ &= a^2(2p-p-q)^2 + a^2p^2(p-q)^2\\ &= a^2(p-q)^2(1+p^2)\\ PT &= a|p-q|\sqrt{1+p^2}\end{aligned}$$

(ii) The point T is the point of intersection of the tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$. The coordinates of T are $(a(p+q), apq)$.

Similarly, the point P' is the point of intersection of the tangents at $R(2ar, ar^2)$ and $P(2ap, ap^2)$. The coordinates of P' are $(a(p+r), apr)$.

Using the distance formula noting that $a>0$ and $1+p^2>0$

$$\begin{aligned}P'T^2 &= (a(p+r)-a(p+q))^2 + (apr-apq)^2\\ &= a^2(r-q)^2 + a^2p^2(r-q)^2\\ &= a^2(r-q)^2(1+p^2)\\ P'T &= a|q-r|\sqrt{1+p^2}\end{aligned}$$

(iii) Using the results in (i) and (iii), and interchanging the parameter p with q.

$$QT = a|p-q|\sqrt{1+q^2}$$

$$Q'T = a|p-r|\sqrt{1+q^2}$$

$$\begin{aligned}\frac{P'T}{PT} + \frac{Q'T}{QT} &= \frac{a|q-r|\sqrt{1+p^2}}{a|p-q|\sqrt{1+p^2}} + \frac{a|p-r|\sqrt{1+q^2}}{a|p-q|\sqrt{1+q^2}} \\ &= \frac{|q-r|+|p-r|}{|p-q|}\end{aligned}$$

Since R lies between P and Q on the parabola, then either $p < r < q$ or $q < r < p$.

If $p < r < q$ then $|q-r| = q-r$, $|p-r| = -(p-r)$ and $|p-q| = -(p-q)$ so

$$\begin{aligned}\frac{|q-r|+|p-r|}{|p-q|} &= \frac{q-r-p+r}{q-p} \\ &= 1\end{aligned}$$

If $q < r < p$ then $|q-r| = -(q-r)$, $|p-r| = p-r$ and $|p-q| = p-q$ so

$$\begin{aligned}\frac{|q-r|+|p-r|}{|p-q|} &= \frac{-q+r+p-r}{p-q} \\ &= 1\end{aligned}$$

Hence, in all cases

$$\frac{P'T}{PT} + \frac{Q'T}{QT} = 1$$

(c) From the sum of roots

$$\alpha+\beta+\gamma = -2$$

Using this result

$$\begin{aligned}\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) &= \alpha^2(-2-\alpha)+\beta^2(-2-\beta)+\gamma^2(-2-\gamma) \\ &= -(\alpha^3+\beta^3+\gamma^3)-2(\alpha^2+\beta^2+\gamma^2)\end{aligned}$$

Since α, β and γ are the roots of $P(x)$ then

$$\begin{aligned}\alpha^3+2\alpha^2-3\alpha-1 &= 0 \\ \beta^3+2\beta^2-3\beta-1 &= 0 \\ \gamma^3+2\gamma^2-3\gamma-1 &= 0 \\ -(\alpha^3+\beta^3+\gamma^3)-2(\alpha^2+\beta^2+\gamma^2) &= -3(\alpha+\beta+\gamma)-3 \\ &= 3 \\ \therefore \alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) &= 3\end{aligned}$$

(d) For the $(k+1)$th application of Newton's method

$$
\begin{aligned}
x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} \\
\frac{f(x_k)}{f'(x_k)} &= x_k - x_{k+1} \\
\sum_{k=0}^{n} \frac{f(x_k)}{f'(x_k)} &= \sum_{k=0}^{n} (x_k - x_{k+1}) \\
&= (x_0 - x_1) + (x_1 - x_2) + (x_2 - x_3) + \dots + (x_{n-1} - x_n) + (x_n - x_{n+1}) \\
&= x_0 - x_{n+1} \quad \text{but } x_{n+1} = x_0 \\
&= 0
\end{aligned}
$$

(e) Let X_m be the number of heads Matthew tosses and let X_j be the number of heads Jonathan tosses.

(i) Since Matthew and Jonathan toss the coins independently then the probability that both toss exactly k heads is the product of the probabilities that each of them toss k heads.

$$
\begin{aligned}
P(X_m = k, X_j = k) &= P(X_m = k) \times P(X_j = k) \\
&= \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \times \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\
&= \binom{n}{k}^2 \frac{1}{2^{2n}}
\end{aligned}
$$

(ii) Computing the probability that Matthew tosses k heads and Jonathan tosses $n-k$ heads

$$
\begin{aligned}
P(X_m = k, X_j = n-k) &= P(X_m = k) \times P(X_j = n-k) \\
&= \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \times \binom{n}{n-k} \left(\frac{1}{2}\right)^{n-k} \left(\frac{1}{2}\right)^{k} \\
&= \binom{n}{k}^2 \frac{1}{2^{2n}} \quad \text{since } \binom{n}{k} = \binom{n}{n-k}
\end{aligned}
$$

Hence, the probability that Matthew tosses k heads and Jonathan tosses $n-k$ heads is equal to the probability that Matthew tosses k heads and Jonathan tosses k heads computed in part (i).

Alternative solution:

The probability that Jonathan tosses $n-k$ heads is equal to the probability that Jonathan tosses $n-k$ tails by symmetry of the coin toss. Tossing $n-k$ tails is equivalent to tossing k heads. Hence, the probability that Matthew tosses k heads and Jonathan tosses $n-k$ heads is equal to the probability that both of them each toss k heads, as computed in part (i).

(iii) From part (ii), it was shown that

$$P(X_m = k, X_j = k) = P(X_m = k, X_j = n - k)$$

For Matthew and Jonathan to toss the same number of heads then

$$\begin{aligned}
&P(X_m = X_j) \\
&= P(X_m = 0, X_j = 0) + P(X_m = 1, X_j = 1) + ... + P(X_m = n, X_j = n) \\
&= P(X_m = 0, X_j = n) + P(X_m = 1, X_j = n - 1) + ... + P(X_m = n, X_j = 0) \\
&= P(X_m + X_j = n)
\end{aligned}$$

If Matthew tosses k heads and Jonathan tosses $n - k$ heads, then out of the entire $2n$ tosses there are n heads. This is true for all possible integer values of $0 \leq k \leq n$. The probability that in $2n$ tosses there are n heads is given by

$$\begin{aligned}
P(X_m + X_j = n) &= \binom{2n}{n} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^n \\
\therefore P(X_m = X_j) &= \binom{2n}{n} \frac{1}{2^{2n}}
\end{aligned}$$

(iv) Using the result in part (i)

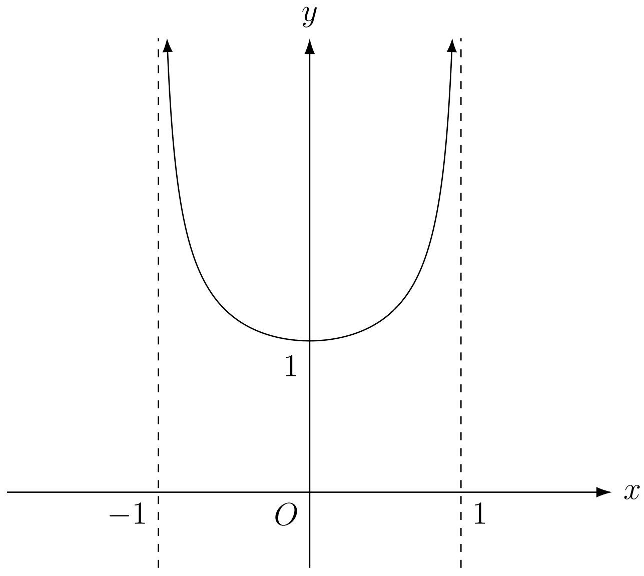
$$\begin{aligned}
P(X_m = X_j) &= P(X_m = 0, X_j = 0) + P(X_m = 1, X_j = 1) + ... + P(X_m = n, X_j = n) \\
&= \binom{n}{0}^2 \frac{1}{2^{2n}} + \binom{n}{1}^2 \frac{1}{2^{2n}} + ... + \binom{n}{n}^2 \frac{1}{2^{2n}} \\
&= \sum_{k=0}^{n} \binom{n}{k}^2 \frac{1}{2^{2n}}
\end{aligned}$$

Equating this to the result in (iii)

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k}^2 \frac{1}{2^{2n}} &= \binom{2n}{n} \frac{1}{2^{2n}} \\
\therefore \sum_{k=0}^{n} \binom{n}{k}^2 &= \binom{2n}{n}
\end{aligned}$$

Question 12

(a) The domain of the curve is $-1 < x < 1$ with asymptotes at $x = -1$ and $x = 1$. The function is also even so there is symmetry about the y-axis. Using this and the given minimum point of $(0, 1)$, the following sketch can be obtained.



(b) (i) The domain is the solution to $-1 \leq 2x^2 - 1 \leq 1$

$$0 \leq 2x^2 \leq 2$$
$$0 \leq x^2 \leq 1$$
$$\therefore \text{Domain is } -1 \leq x \leq 1$$

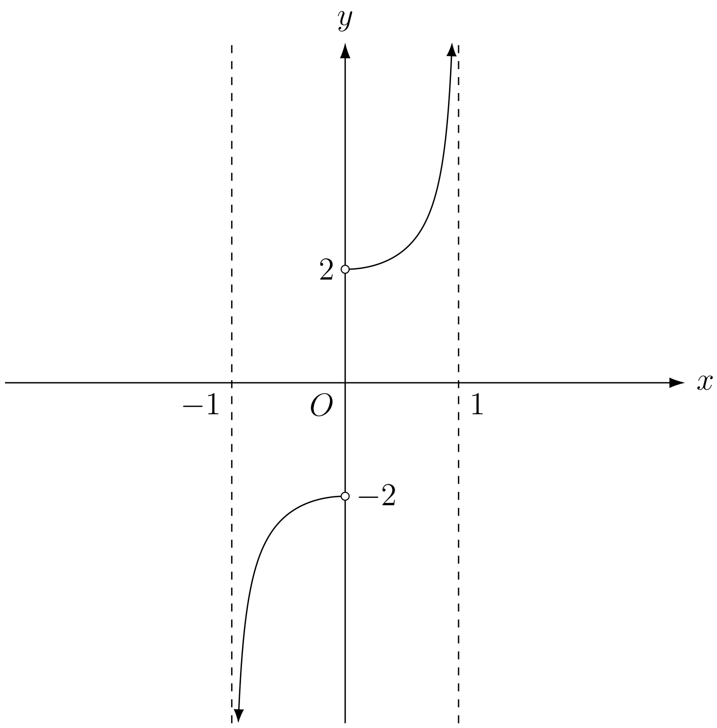
(ii) Finding the derivative of $f(x) = \sin^{-1}(2x^2 - 1)$

$$\begin{aligned} f'(x) &= \frac{4x}{\sqrt{1-(2x^2-1)^2}} \\ &= \frac{4x}{\sqrt{1-4x^4+4x^2-1}} \\ &= \frac{4x}{\sqrt{4x^2(1-x^2)}} \\ &= \frac{2x}{|x|\sqrt{1-x^2}} \end{aligned}$$

Note that $f'(x)$ is not defined at $x = 0$ and so

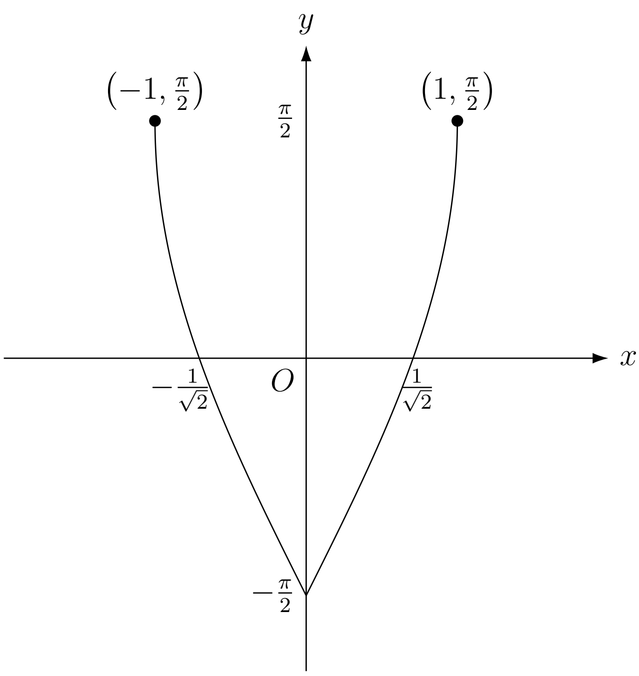
$$f'(x) = \begin{cases} \dfrac{2}{\sqrt{1-x^2}}, & \text{for } x > 0 \\ -\dfrac{2}{\sqrt{1-x^2}}, & \text{for } x < 0 \end{cases}$$

Using the result in (a), the following sketch of $y = f'(x)$ can be obtained



(iii) From $f'(x)$ obtained in (ii), $f(x)$ is increasing in the domain $0 < x < 1$ and decreasing in the domain $-1 < x < 0$. Also, observe that $f(x)$ is an even function. Hence, the minimum value of $f(x)$ occurs at $f(0) = -\frac{\pi}{2}$. The maximum value of $f(x)$ occurs at $f(1)$ or $f(-1)$ which is $\frac{\pi}{2}$. Hence, the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

(iv) Using all the information above to sketch $y = f(x)$.



(c) (i) Let v be the velocity of the particle at time t. When the particle moves from its initial displacement at the origin

$$\begin{aligned}
\frac{d}{dx}\left(\frac{v^2}{2}\right) &= -x \\
\frac{v^2}{2} &= -\frac{x^2}{2} + c \quad \text{when } x = 0, v = v_0 \Rightarrow c = \frac{v_0^2}{2} \\
v^2 &= v_0^2 - x^2
\end{aligned}$$

If $|v_0| < 2$ or equivalently $v_0^2 < 4$ then

$$\begin{aligned}
4 - x^2 &> v^2 \quad \text{but } v^2 \geq 0 \\
\Rightarrow 4 - x^2 &> 0 \\
\Rightarrow -2 &< x < 2
\end{aligned}$$

Since the displacement equation satisfies $x < 2$ then the acceleration equation will always be $\dfrac{d^2x}{dt^2} = -x$. Hence, provided $|v_0| < 2$ the particle will exhibit simple harmonic motion.

(ii) From the equation $v^2 = v_0^2 - x^2$, the extreme points occur when $v = 0$

$$\begin{aligned}
v_0^2 - x^2 &= 0 \\
x^2 &= v_0^2 \\
x &= \pm v_0
\end{aligned}$$

The particle exhibits simple harmonic motion and from the acceleration equation, it can be seen that the centre is the origin. Hence, the amplitude of the particle's motion is $|v_0|$.

(iii) If $v_0 = 2$ then the velocity equation becomes $v^2 = 4 - x^2$ for $x < 2$. However, since the particle will reach the displacement $x = 2$ after moving initially from the origin, the other piece of the acceleration equation must be considered.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{v^2}{2}\right) &= 4 - x \\
\frac{v^2}{2} &= 4x - \frac{x^2}{2} + c \quad \text{when } x \to 2^-, v \to 0^+ \Rightarrow c = -6 \\
v^2 &= -12 + 8x - x^2 \\
&= -(x-6)(x-2)
\end{aligned}$$

The extreme points of the displacement occur when $v = 0$, which are $x = 2$ and $x = 6$. Thus, the maximum displacement of the particle is $x = 6$.

(iv) The particle is initially at the origin before moving in simple harmonic motion (described by $\ddot{x} = -x$) up to $x = 2$. Once it reaches $x = 2$, it transitions to a different simple harmonic motion (described by $\ddot{x} = 4 - x$). From that point, it then continually oscillates between $x = 2$ and $x = 6$.

Question 13

(a) (i) For $y = a^x$

$$\begin{aligned}\frac{dy}{dx} &= a^x \ln a \\ m &= a^p \ln a\end{aligned}$$

For $y = \log_a x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x \ln a} \\ m &= \frac{1}{p \ln a}\end{aligned}$$

Multiplying the two expressions for m

$$\begin{aligned}m^2 &= \frac{a^p \ln a}{p \ln a} \\ &= \frac{a^p}{p} \\ &= 1\end{aligned}$$

since (p, p) lies on $y = a^x$ so $p = a^p$

Alternative solution:

For a function $y = f(x)$ at (p, p)

$$\begin{aligned}\frac{dy}{dx} &= f'(x) \\ m &= f'(p)\end{aligned}$$

For its inverse $y = f^{-1}(x)$ or equivalently $x = f(y)$ at (p, p)

$$\begin{aligned}\frac{dx}{dy} &= f'(y) \\ \frac{dy}{dx} &= \frac{1}{f'(y)} \\ m &= \frac{1}{f'(p)}\end{aligned}$$

Multiplying the two expressions for m gives the result $m^2 = 1$

(ii) From part (i), $m^2 = 1$ gives two cases for m

When $m = 1$, then $\dfrac{1}{p\ln a} = 1$ and $a^p \ln a = 1 \Rightarrow a^p = p$

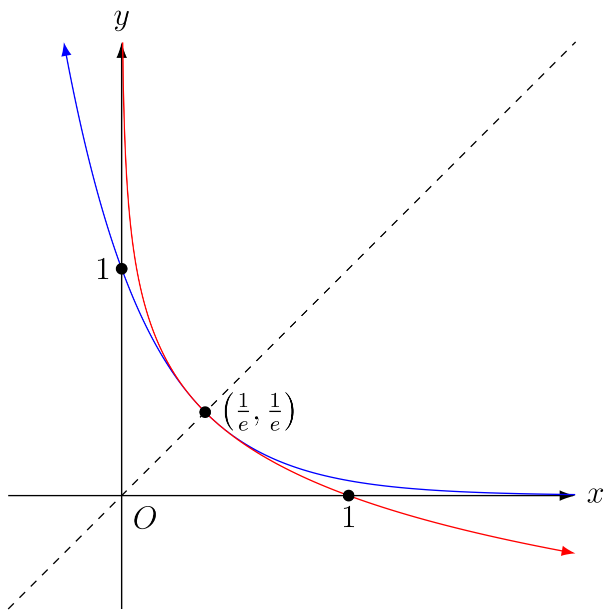
$$\begin{aligned} \frac{1}{p\ln a} &= 1 \\ \ln a^p &= 1 \quad \text{but } p = a^p \\ \ln p &= 1 \\ \Rightarrow p &= e \quad \text{substitute into } p = a^p \\ a^e &= e \\ \Rightarrow a &= e^{\frac{1}{e}} \end{aligned}$$

But $0 < a < 1$ so this cannot be the value for a, so consider the other case. When $m = -1$, then $\dfrac{1}{p\ln a} = -1$ and $a^p \ln a = -1 \Rightarrow a^p = p$

$$\begin{aligned} \frac{1}{p\ln a} &= -1 \\ \ln a^p &= -1 \quad \text{but } a^p = p \\ \ln p &= -1 \\ \Rightarrow p &= \frac{1}{e} \quad \text{substitute into } p = a^p \\ a^{\frac{1}{e}} &= \frac{1}{e} \\ \Rightarrow a &= e^{-e} \end{aligned}$$

Here a satisfies the condition $0 < a < 1$.

(iii) Since $0 < a < 1$, the exponential curve is decreasing as x increases. Hence, sketching the graphs of $y = e^{-ex}$ (blue) and $y = \log_{e^{-e}} x$ (red)



(b) (i) Using the given displacement equations

$$\begin{aligned} t &= \frac{x}{V\cos\theta} \quad \text{substitute into } y \\ y &= V\sin\theta \times \frac{x}{V\cos\theta} - \frac{g}{2} \times \frac{x^2}{V^2\cos^2\theta} \\ &= -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta \end{aligned}$$

The projectile just clears a wall on the point (d_1, d_2). This point satisfies the Cartesian equation of the projectile motion so

$$\begin{aligned} d_2 &= -\frac{gd_1^2}{2V^2\cos^2\theta} + d_1\tan\theta \\ -\frac{g}{2V^2\cos^2\theta} &= \frac{d_2 - d_1\tan\theta}{d_1^2} \end{aligned}$$

The projectile just clears another wall on the point (d_2, d_1). This point satisfies the Cartesian equation of the projectile motion so

$$\begin{aligned} d_1 &= -\frac{gd_2^2}{2V^2\cos^2\theta} + d_2\tan\theta \\ -\frac{g}{2V^2\cos^2\theta} &= \frac{d_1 - d_2\tan\theta}{d_2^2} \end{aligned}$$

Equating the two expressions

$$\begin{aligned} \frac{d_2 - d_1\tan\theta}{d_1^2} &= \frac{d_1 - d_2\tan\theta}{d_2^2} \\ d_2^3 - d_1d_2^2\tan\theta &= d_1^3 - d_1^2d_2\tan\theta \\ \tan\theta(d_1^2d_2 - d_1d_2^2) &= d_1^3 - d_2^3 \\ d_1d_2\tan\theta(d_1 - d_2) &= (d_1 - d_2)(d_1^2 + d_1d_2 + d_2^2) \\ \therefore \tan\theta &= \frac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2} \quad \text{where } d_1 \neq d_2 \end{aligned}$$

(ii) Suppose that the particle clears the walls when $\tan\theta \leq 3$. From part (i), this implies that

$$\begin{aligned} \frac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2} &\leq 3 \\ d_1^2 + d_1d_2 + d_2^2 &\leq 3d_1d_2 \\ d_1^2 - 2d_1d_2 + d_2^2 &\leq 0 \\ (d_1 - d_2)^2 &\leq 0 \end{aligned}$$

Since $d_1 \neq d_2$, there are no real values of d_1 and d_2 which could satisfy this inequality. Hence, the initial assumption must be false. In other words, the particle cannot clear the walls when the angle of projection θ is low enough such that $\tan\theta \leq 3$.

(c) (i) Since n is a positive integer

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \binom{n}{0}+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\ldots+\binom{n}{n}\frac{1}{n^n}\\ &\geq \binom{n}{0}+\binom{n}{1}\frac{1}{n}\\ &= 1+n\times\frac{1}{n}\\ &= 2\end{aligned}$$

(ii) When $n=1$, LHS $=1!$ and RHS $=1^{2\times1-1}$. The statement is true for $n=1$ because both sides equal 1.

Assume the statement is true for some $n=k$

$$(2k-1)! \leq k^{2k-1}$$

Required to prove the statement is true for $n=k+1$

$$(2k+1)! \leq (k+1)^{2k+1}$$

$$\begin{aligned}\text{LHS} &= (2k+1)!\\ &= (2k+1)(2k)(2k-1)!\\ &\leq (2k+1)(2k)k^{2k-1} \quad \text{by assumption}\\ &= 2(2k+1)k^{2k}\\ &= 2(2k+1)\left(\frac{k}{k+1}\right)^{2k}(k+1)^{2k}\\ &\leq 2(2k+1)\left(\frac{1}{2}\right)^2(k+1)^{2k} \quad \text{using the result in (i)*}\\ &= \frac{1}{2}(2k+1)(k+1)^{2k}\\ &\leq \frac{1}{2}(2k+2)(k+1)^{2k}\\ &= (k+1)^{2k+1}\\ &= \text{RHS}\end{aligned}$$

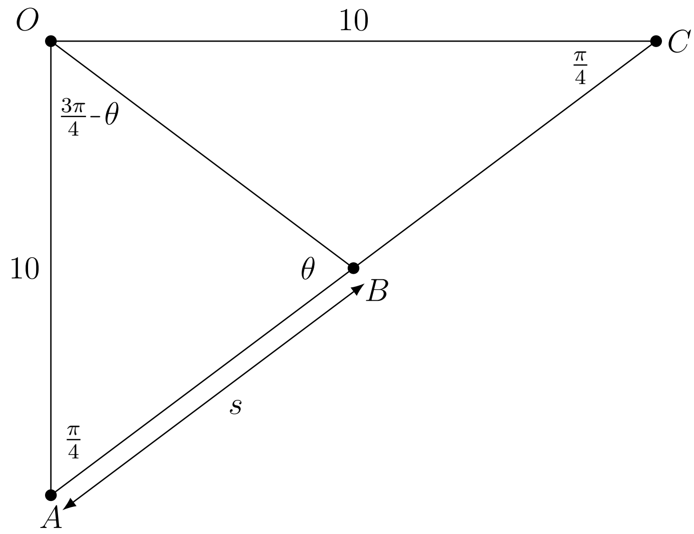
Hence, the statement is true by mathematical induction

*Rearranging the result in (i):

$$\begin{aligned}&\left(1+\frac{1}{n}\right)^n \geq 2\\ \Rightarrow &\left(\frac{n+1}{n}\right)^n \geq 2\\ \Rightarrow &\left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}\end{aligned}$$

Question 14

(a) (i) Viewing from above



Since the angle of elevation to the top of the monument for both visitor centres A and C is $\frac{\pi}{4}$, the associated right-angled triangles are isosceles so $OC = 10$ and $OA = 10$.

This means that $\triangle AOC$ is right-angled isosceles, so $\angle OAB = \frac{\pi}{4}$. Since $\angle ABO = \theta$ then by angle sum of triangle $\angle AOB = \frac{3\pi}{4} - \theta$.

Using the sine rule on $\triangle AOB$

$$\begin{aligned}\frac{AB}{\sin\angle AOB} &= \frac{OA}{\sin\angle ABO}\\ \frac{s}{\sin\left(\frac{3\pi}{4}-\theta\right)} &= \frac{10}{\sin\theta}\\ \therefore s &= \frac{10\sin\left(\frac{3\pi}{4}-\theta\right)}{\sin\theta}\end{aligned}$$

(ii) Using the result in (i)

$$\begin{aligned}s &= \frac{10\sin\left(\frac{3\pi}{4}-\theta\right)}{\sin\theta}\\ \frac{ds}{d\theta} &= \frac{10(-\cos\left(\frac{3\pi}{4}-\theta\right)\sin\theta - \sin\left(\frac{3\pi}{4}-\theta\right)\cos\theta)}{\sin^2\theta}\\ &= -\frac{10\sin\left(\frac{3\pi}{4}-\theta+\theta\right)}{\sin^2\theta}\\ &= -\frac{10\sin\frac{3\pi}{4}}{\sin^2\theta}\\ &= -\frac{10}{\sqrt{2}\sin^2\theta}\end{aligned}$$

Since the visitor cycles from A to C at 5 metres per second then $\dfrac{ds}{dt} = 5$, so

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{d\theta}{ds} \times \frac{ds}{dt} \\
&= -\frac{\sqrt{2}\sin^2\theta}{10} \times 5 \\
&= -\frac{\sin^2\theta}{\sqrt{2}}
\end{aligned}$$

(iii) From the visitor's perspective

$$\begin{aligned}
\tan\beta &= \frac{10}{OB} \\
OB &= 10\cot\beta
\end{aligned}$$

Using the sine rule on $\triangle AOB$

$$\begin{aligned}
\frac{OB}{\sin\angle OAB} &= \frac{10}{\sin\theta} \\
\frac{10\cot\beta}{\sin\frac{\pi}{4}} &= \frac{10}{\sin\theta} \\
\tan\beta &= \sqrt{2}\sin\theta \\
\beta &= \tan^{-1}(\sqrt{2}\sin\theta) \\
\frac{d\beta}{d\theta} &= \frac{\sqrt{2}\cos\theta}{1+2\sin^2\theta}
\end{aligned}$$

Using the result in (ii)

$$\begin{aligned}
\frac{d\beta}{dt} &= \frac{d\beta}{d\theta} \times \frac{d\theta}{dt} \\
&= \frac{\sqrt{2}\cos\theta}{1+2\sin^2\theta} \times -\frac{\sin^2\theta}{\sqrt{2}} \\
&= -\frac{\cos\theta\sin^2\theta}{1+2\sin^2\theta} \times \frac{\text{cosec}^2\theta}{\text{cosec}^2\theta} \\
&= -\frac{\cos\theta}{2+\text{cosec}^2\theta}
\end{aligned}$$

(iv) When the visitor moves from A to B, $\angle AOB$ increases from 0 to $\frac{\pi}{2}$. This means that $\angle ABO$ or θ decreases from $\frac{3\pi}{4}$ to $\frac{\pi}{4}$ over time.

When $\theta = \dfrac{3\pi}{4}$ then $\dfrac{d\beta}{dt} = \dfrac{1}{4\sqrt{2}}$ and when $\theta = \dfrac{\pi}{4}$ then $\dfrac{d\beta}{dt} = -\dfrac{1}{4\sqrt{2}}$.

This means there is a maximum point where $\frac{d\beta}{dt} = 0$. This occurs when $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$. Substitute this into the relationship $\beta = \tan^{-1}(\sqrt{2}\sin\theta)$ and the maximum angle of elevation the visitor can have is $\tan^{-1}\sqrt{2}$.

(b) (i)

$$\angle AMC = \angle ABC \quad \text{(angles in the same segment in } \mathscr{C}_1\text{)}$$
$$\angle KMN = \angle ABC \quad \text{(angles in the same segment in } \mathscr{C}_2\text{)}$$

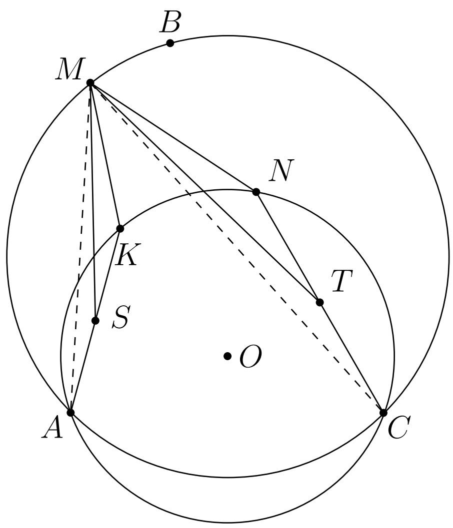
But

$$\angle AMC = \angle AMK + \angle KMC$$
$$\angle KMN = \angle KMC + \angle CMN$$
$$\Rightarrow \angle AMK = \angle CMN$$

Also

$$\angle MAK = \angle MCN \quad \text{(angles in the same segment in } \mathscr{C}_3\text{)}$$
$$\therefore \triangle MKA \,|||\, \triangle MNC \quad \text{(equiangular)}$$

(ii) The points S and T are shown in the diagram below.



Since $\triangle MKA$ and $\triangle MNC$ are similar

$$\frac{NC}{KA} = \frac{MN}{MK} \quad \text{(corresponding sides of similar triangles in proportion)}$$

But $NT = \frac{1}{2}NC$ and $KS = \frac{1}{2}KA \Rightarrow \dfrac{NC}{KA} = \dfrac{NT}{KS}$ so

$$\frac{MN}{MK} = \frac{NT}{KS}$$

Also $\angle MNT = \angle MKS$ (corresponding angles of similar triangles)

$$\therefore \triangle MNT \,|||\, \triangle MKS$$

as corresponding sides are in proportion and included angles are equal.
This means that

$$\angle MSB = \angle MTB \quad \text{(corresponding angles of similar triangles)}$$
$$\therefore MSTB \text{ is a cyclic quadrilateral (angles in same segment)}$$

(iii) Since S bisects chord KA and similarly T bisects chord NC

$$OS \perp KA \quad \text{and} \quad OT \perp NC$$

This is because the line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Since $\angle BSO = \frac{\pi}{2}$ and $\angle BTO = \frac{\pi}{2}$ then $SBTO$ is a cyclic quadrilateral as opposite angles $\angle BSO$ and $\angle BTO$ are supplementary.

(iv) Since S, T and B are three non-collinear points, they uniquely define a circle. In part (ii), it was shown that M lies on this circle and in part (iii), it was shown that O also lies on this circle. This means that S, T, B, M and O are concyclic, so

$$\begin{aligned} & \angle OSB = \angle OMB \quad \text{(angles in the same segment)} \\ \text{but} \quad & \angle OSB = \frac{\pi}{2} \quad (OS \text{ bisects chord } KA \text{ as explained in part (iii)}) \\ & \therefore OM \perp BM \end{aligned}$$