ADV: Statistics (Adv), S1 Probability & Probability Distributions (Adv)

Discrete Probability Distributions (Y11)

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Exam Equivalent Time: 66 minutes *(based on HSC allocation of 1.5 minutes approx. per mark)*

HISTORICAL CONTRIBUTION

- *S1 Probability* is a Year 11 topic that contains a mixture of old course and new syllabus content.
- The new *Advanced* course in this topic area has a significant overlap with the *Standard 2* syllabus and also introduces some brand new content.
- *S1 Probability* has been split into four sub-categories for the purposes of this analysis: *1-Multi-Stage Events, 2-Relative Frequency, 3-Conditional Probability and Venn Diagrams and 4-Discrete Probability Distributions.*
- This analysis looks at *Discrete Probability Distributions.*

HSC ANALYSIS - What to expect and common pitfalls

- *Discrete Probability Distributions* is a new sub-topic within *Advanced* probability which attracted two questions in the 2021 exam (it was not examined in 2020).
- *2021 Q34* was an extremely challenging question requiring students to calculate $E(X)$ using algebraic probabilities that produced an 18% mean mark. Please review closely.
- *NESA's HSC sample exam* and *Topic Guidance* provides an indication of concepts that will likely be examined and some exemplar problems which are the basis for multiple database questions.
- Our research has also looked at the examination of this topic area by other States. This history has been analysed and high quality revision examples that test numerous important concepts are included in the database.

Questions

1. Probability, 2ADV S1 2007 MET2 19 MC

The discrete random variable X has probability distribution as given in the table. The mean of X is 5.

x	0	2	4	6	8
$\Pr(X=x)$	a	0.2	0.2	0.3	b

The values of a and b are

A. $a=0.05$ and $b=0.25$

B. $a=0.1$ and $b=0.29$

C. $a=0.2$ and $b=0.9$

D. $a=0.3$ and $b=0$

2. Probability, 2ADV S1 2010 MET2 15 MC

The discrete random variable X has the following probability distribution.

X	0	1	2
$\Pr(X=x)$	a	b	0.4

If the mean of X is 1 then

A. $a=0.3$ and $b=0.1$

B. $a=0.2$ and $b=0.2$

C. $a=0.4$ and $b=0.2$

D. $a=0.1$ and $b=0.3$

3. Probability, 2ADV S1 2015 MET2 14 MC

Consider the following discrete probability distribution for the random variable X.

x	1	2	3	4	5
$\Pr(X=x)$	p	$2p$	$3p$	$4p$	$5p$

The mean of this distribution is

A. 2

B. 3

C. $\frac{7}{2}$

D. $\frac{11}{3}$

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4. Probability, 2ADV S1 2016 MET2 7 MC

The number of pets, X, owned by each student in a large school is a random variable with the following discrete probability distribution.

x	0	1	2	3
$\Pr(X=x)$	0.5	0.25	0.2	0.05

If two students are selected at random, the probability that they own the same number of pets is

A. 0.3

B. 0.305

C. 0.355

D. 0.405

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5. Probability, 2ADV S1 2021 HSC 2 MC

The probability distribution table for a discrete random variable X is shown.

x	$P(X = x)$
1	0.6
2	0.3
3	0.1

What is the expected value of X?

A. 0.6

B. 1.0

C. 1.5

D. 2.0

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6. Probability, 2ADV S1 2019 MET2-N 5 MC

Consider the probability distribution for the discrete random variable X shown in the table below.

x	-1	0	1	2	3
$\Pr(X=x)$	b	b	b	$\frac{3}{5}-b$	$\frac{3b}{5}$

The value of $E(X)$ is

A. $\frac{76}{65}$

B. 1

C. 0

D. $\frac{2}{13}$

E. $\frac{86}{65}$

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7. Probability, 2ADV S1 2009 MET2 10 MC

The discrete random variable X has a probability distribution as shown.

x	0	1	2	3
$\Pr(X = x)$	0.4	0.2	0.3	0.1

The median of X is

A. 0

B. 1

C. 1.1

D. 2

8. Probability, 2ADV S1 2017 MET2 14 MC

The random variable X has the following probability distribution, where $0 < p < \dfrac{1}{3}$.

x	−1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

The variance of X is

A. $2p(1 - 3p)$

B. $p(5 - 9p)$

C. $(1 - 3p)^2$

D. $6p - 16p^2$

9. Probability, 2ADV S1 2016 MET2 19 MC

Consider the discrete probability distribution with random variable X shown in the table below.

x	−1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $\mathrm{E}(X)$ are respectively

A. -0.8 and 1

B. -0.8 and 1.6

C. 0 and 2.4

D. 0 and 1

10. Probability, 2ADV S1 EQ-Bank 42

The discrete random variable X has the probability distribution shown in the table below.

$X = x$	4	5	6	7	8
$P(x)$	0.3	a	0.1	0.15	0.2

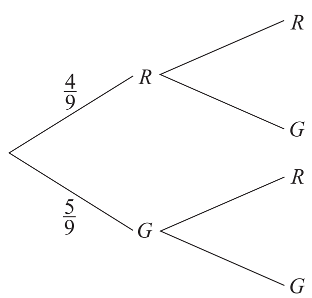
Find the value of a, and hence calculate the the expected value and variance of X. ***(3 marks)***

11. Probability, 2ADV S1 EQ-Bank 40

One bag contains red and green balls.

Kalyn randomly chooses one ball from the bag. Without replacement, he then chooses a second ball from the bag.

Complete the tree diagram below and then draw a probability distribution table for the number of red balls that could be drawn out of the bag. ***(3 marks)***



12. Probability, 2ADV S1 SM-Bank 41

Evaluate p and q in the discrete probability distribution table below, given that $E(X) = 3$. *(3 marks)*

x	1	2	3	4
$P(X = x)$	p	q	0.2	0.4

13. Probability, 2ADV S1 2008 MET1 7

Jane drives to work each morning and passes through three intersections with traffic lights. The number X of traffic lights that are red when Jane is driving to work is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X = x)$	0.1	0.2	0.3	0.4

i. What is the mode of X? *(1 mark)*

ii. Jane drives to work on two consecutive days. What is the probability that the number of traffic lights that are red is the same on both days? *(2 marks)*

14. Probability, 2ADV S1 2009 MET1 7

The random variable X has this probability distribution.

X	0	1	2	3	4
$\Pr(X = x)$	0.1	0.2	0.4	0.2	0.1

Find

i. $P(X > 1 \mid X \leq 3)$ *(2 marks)*

ii. $P(X)$, the variance of X. *(3 marks)*

15. Probability, 2ADV S1 2010 MET1 8

The discrete random variable X has the probability distribution

x	-1	0	1	2
$\Pr(X = x)$	p^2	p^2	$\frac{p}{4}$	$\frac{4p+1}{8}$

Find the value of p. *(3 marks)*

16. Probability, 2ADV S1 2012 MET1 4

On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

i. Find the mean of X. *(2 marks)*

ii. What is the probability that Daniel receives only one telephone call on each of three consecutive days? *(1 mark)*

iii. Daniel receives telephone calls on both Monday and Tuesday.
What is the probability that Daniel receives a total of four calls over these two days? *(3 marks)*

17. Probability, 2ADV S1 2013 MET1 7

The probability distribution of a discrete random variable, X, is given by the table below.

x	0	1	2	3	4
$\Pr(X = x)$	0.2	$0.6p^2$	0.1	$1 - p$	0.1

a. Show that $p = \dfrac{2}{3}$ or $p = 1$. *(3 marks)*

b. Let $p = \dfrac{2}{3}$.

i. Calculate $E(X)$. Answer in exact form. *(2 marks)*

ii. Find $P(X \geq E(X))$. *(1 mark)*

18. Probability, 2ADV S1 2021 HSC 34

A discrete random variable has probability distribution as shown in the table where n is a finite positive integer.

x	r	r^2	r^3	$\dots$	r^k	$\dots$	r^n
$P(X=x)$	r^n	r^{n-1}	r^{n-2}	$\dots$	r^{n-k+1}	$\dots$	r

Show that $E(X) = n(2r-1)$. ***(3 marks)***

Worked Solutions

1. Probability, 2ADV S1 2007 MET2 19 MC

$$\text{Sum of probabilities} = 1$$
$$a + 0.2 + 0.2 + 0.3 + b = 1$$

$$\text{Since } E(X) = 5,$$

$$5 = (0 \times a) + (2 \times 0.2) + (4 \times 0.2) + (6 \times 0.3) + 8b$$
$$8b = 2$$
$$\therefore b = 0.25$$

$$\therefore a = 0.05, \quad b = 0.25$$
$$\Rightarrow \quad A$$

2. Probability, 2ADV S1 2010 MET2 15 MC

$$E(X) = 1,$$

$$1 \times b + 2 \times 0.4 = 1$$
$$b = 0.2$$

$$\text{Sum of probabilities} = 1,$$

$$a + 0.2 + 0.4 = 1$$
$$a = 0.4$$
$$\Rightarrow \quad C$$

3. Probability, 2ADV S1 2015 MET2 14 MC

Find p:

$p+2p+3p+4p+5p=1$

$$\therefore p=\frac{1}{15}$$

$$\begin{aligned}E(X)&=1\times p+2(2p)+3(3p)+4(4p)+5(5p)\\&=55p\\&=55\times\left(\frac{1}{15}\right)\\&=\frac{11}{3}\end{aligned}$$

$\Rightarrow\ D$

4. Probability, 2ADV S1 2016 MET2 7 MC

$$\begin{aligned}&P(0,0)+P(1,1)+P(2,2)+P(3,3)\\&\quad=0.5^2+0.25^2+0.2^2+0.05^2\\&\quad=0.355\end{aligned}$$

$\Rightarrow\ C$

5. Probability, 2ADV S1 2021 HSC 2 MC

$$\begin{aligned}E(X)&=1\times0.6+2\times0.3+3\times0.1\\&=1.5\end{aligned}$$

$\Rightarrow\ C$

6. Probability, 2ADV S1 2019 MET2-N 5 MC

$$1=b+b+b+\frac{3}{5}-b+\frac{3b}{5}$$

$$\frac{2}{5}=\frac{13b}{5}$$

$$b=\frac{2}{13}$$

$$\begin{aligned}E(X)&=-\frac{2}{13}+0+\frac{2}{13}+2\left(\frac{3}{5}-\frac{2}{13}\right)+3\left(\frac{6}{65}\right)\\&=\frac{76}{65}\end{aligned}$$

7. Probability, 2ADV S1 2009 MET2 10 MC

$P(X\le 0)=0.4$

$P(X\le 1)=0.6$

$\therefore\ \text{Median}=1$

$\Rightarrow\ B$

8. Probability, 2ADV S1 2017 MET2 14 MC

$$\begin{aligned}\text{Var}(X)&=E\left(X^2\right)-[E(X)]^2\\&=\left[(-1)^2p+0^2\times2p+1^2(1-3p)\right]-[-p+0+1-3p]^2\\&=6p-16p^2\end{aligned}$$

$\Rightarrow D$

9. Probability, 2ADV S1 2016 MET2 19 MC

Smallest $E(X)$ occurs when $a=0.8$,

$$\therefore \text{ Smallest } E(X) = 0.8\times -1+0.2\times 4$$
$$= 0$$

♦♦♦ Mean mark 15%.

Consider the value of b,

Sum of probabilities $=1$

$\therefore 0\le 4b\le 0.8 \to 0\le b\le 0.2$

Largest $E(X)$ occurs when $a=0$, and $b=0.2$,

$\therefore$ Largest $E(X)$

$$= 0.2\times 0+0.2\times 0.2+(2\times 0.2)\times(2\times 0.2)+0.2\times 4$$
$$= 0.04+0.16+0.8$$
$$= 1$$

$\Rightarrow D$

10. Probability, 2ADV S1 EQ-Bank 42

$0.3+a+0.1+0.15+0.2=1$

$\Rightarrow a=0.25$

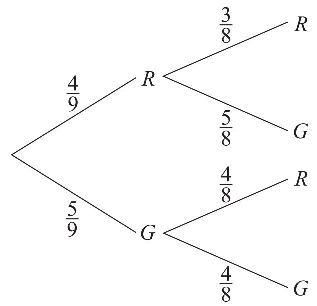
$$E(X)=\sum xP(x)$$

$X=x$	4	5	6	7	8
$P(x)$	0.3	0.25	0.1	0.15	0.2
$x\times P(x)$	1.2	1.25	0.6	1.05	1.6

$$E(X) = 1.2+1.25+0.6+1.05+1.6$$
$$= 5.7$$

$$\text{Var}(X) = E(X^2)-[E(X)]^2$$
$$= (4^2\times 0.3)+(5^2\times 0.25)+(6^2\times 0.1)+(7^2\times 0.15)+(8^2\times 0.2)-5.7^2$$
$$= 2.31$$

11. Probability, 2ADV S1 EQ-Bank 40



$$x=2:\ P(RR)=\frac{4}{9}\times\frac{3}{8}=\frac{1}{6}$$
$$x=1:\ P(R \text{ and } G)=\frac{4}{9}\times\frac{5}{8}+\frac{5}{9}\times\frac{4}{8}=\frac{5}{9}$$
$$x=0:\ P(GG)=\frac{5}{9}\times\frac{4}{8}=\frac{5}{18}$$

x	0	1	2
$P(X=x)$	$\frac{5}{18}$	$\frac{5}{9}$	$\frac{1}{6}$

12. Probability, 2ADV S1 SM-Bank 41

$$p+q+0.2+0.4=1$$
$$p+q=0.4\ \ldots(1)$$

Given $E(X)=3$,

$$p+2q+0.6+1.6=3$$
$$p+2q=0.8\ \ldots(2)$$

Subtract: (2) - (1)

$$q=0.4$$
$$\therefore p=0$$

13. Probability, 2ADV S1 2008 MET1 7

i. 3

ii. $P(0,0)+P(1,1)+P(2,2)+P(3,3)$
$$\begin{aligned}&=0.1^2+0.2^2+0.3^2+0.4^2\\&=0.3\end{aligned}$$

14. Probability, 2ADV S1 2009 MET1 7

i. $P(X>1 \mid X\leq 3)$
$$\begin{aligned}&=\frac{P(X=2)+P(X=3)}{1-P(X=4)}\\&=\frac{0.4+0.2}{1-0.1}\\&=\frac{0.6}{0.9}\\&=\frac{2}{3}\end{aligned}$$

ii. $E(X)=0.1(0)+1(0.2)+2(0.4)+3(0.2)+4(0.1)$
$$\begin{aligned}&=0+0.2+0.8+0.6+0.4\\&=2\end{aligned}$$

$$\begin{aligned}E\left(X^2\right)&=0^2(0.1)+1^2(0.2)+2^2(0.4)+3^2(0.2)+4^2(0.1)\\&=0+0.2+1.6+1.8+1.6\\&=5.2\end{aligned}$$

$$\begin{aligned}\therefore\ \mathrm{Var}(X)&=E\left(X^2\right)-[E(X)]^2\\&=5.2-(2)^2\\&=1.2\end{aligned}$$

15. Probability, 2ADV S1 2010 MET1 8

$$\begin{aligned}\text{Sum of probabilities}&=1\\p^2+p^2+\frac{p}{4}+\frac{4p+1}{8}&=1\\16p^2+2p+4p+1&=8\\16p^2+6p-7&=0\\(2p-1)(8p+7)&=0\end{aligned}$$

$$\therefore p=\frac{1}{2},\quad (p>0)$$

16. Probability, 2ADV S1 2012 MET1 4

i. $E(X)=0\times 0.2+1\times 0.2+2\times 0.5+3\times 0.1$
$$\begin{aligned}&=0+.2+1+0.3\\&=1.5\end{aligned}$$

ii. $P(1,1,1)=0.2\times 0.2\times 0.2$
$$=0.008$$

iii. Conditional Probability:

♦ Mean mark 36%.

$P(x=4 \mid x\geq 1\text{ both days})$
$$\begin{aligned}&=\frac{P(1,3)+P(2,2)+P(3,1)}{P(x\geq 1\text{ both days})}\\&=\frac{0.2\times 0.1+0.5\times 0.5+0.1\times 0.2}{0.8\times 0.8}\\&=\frac{0.02+0.25+0.02}{0.64}\\&=\frac{0.29}{0.64}\\&=\frac{29}{64}\end{aligned}$$

17. Probability, 2ADV S1 2013 MET1 7

a. Since probabilities must sum to 1:

$$\begin{aligned} 0.2+0.6p^2+0.1+1-p+0.1 &= 1 \\ 0.6p^2-p+0.4 &= 0 \\ 6p^2-10p+4 &= 0 \\ 3p^2-5p+2 &= 0 \\ (p-1)(3p-2) &= 0 \end{aligned}$$

$$\therefore p=1 \text{ or } p=\frac{2}{3}$$

b.i. $$\begin{aligned} E(X) &= \sum x\cdot P(X=x) \\ &= 1\times\left(\frac{3}{5}\times\frac{2^2}{3^2}\right)+2\left(\frac{1}{10}\right)+3\left(1-\frac{2}{3}\right)+4\left(\frac{1}{10}\right) \\ &= \frac{4}{15}+\frac{1}{5}+1+\frac{2}{5} \\ &= \frac{28}{15} \end{aligned}$$

♦♦ *Part (b)(ii) mean mark 32%.*

ii. $$\begin{aligned} P\left(X\geq\frac{28}{15}\right) &= P(X=2)+P(X=3)+P(X=4) \\ &= \frac{1}{10}+\frac{1}{3}+\frac{1}{10} \\ &= \frac{8}{15} \end{aligned}$$

18. Probability, 2ADV S1 2021 HSC 34

♦♦♦ *Mean mark 18%.*

$$\begin{aligned} E(X) &= \sum x\cdot P(X=x) \\ &= r\cdot r^n+r^2\cdot r^{n+1}+\ldots+r^n\cdot r \\ &= r^{n+1}+r^{n+1}+\ldots+r^{n+1} \\ &= nr^{n+1}\ldots(1) \end{aligned}$$

Sum of probabilities $=1$

$$\underbrace{r+r^2+r^3+\ldots+r^{n-1}+r^n}_{\text{GP where } a=r,\ r=r}=1$$

$$\begin{aligned} \frac{r(1-r^n)}{1-r} &= 1 \\ r-r^{n+1} &= 1-r \\ r^{n+1} &= 2r-1\ldots(2) \end{aligned}$$

Substitute (2) into (1):

$$E(X)=n(2r-1)$$