2016 Bored of Studies Trial Examinations

Mathematics Extension 1

3rd October 2016

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A reference sheet has been provided.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 7 – 18

60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

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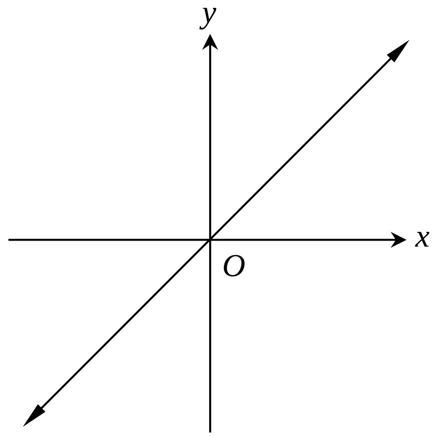
1 Which of the following equations best represents the graph below?



(A) $y = \sin\left(\sin^{-1} x\right)$.

(B) $y = \sin^{-1}(\sin x)$.

(C) $y = e^{\ln x}$.

(D) $y = \ln\left(e^{x}\right)$.

2 Let $P(x)$ and $Q(x)$ be polynomials with real coefficients.

Which of the following statements is ALWAYS true?

(A) If $P(x)$ and $Q(x)$ have the same set of roots, then $P(x) = Q(x)$.

(B) If $P(0) = Q(0)$, then $P(x)$ and $Q(x)$ have the same constant term.

(C) If $\dfrac{P(x)}{Q(x)}$ has a constant remainder, then $P(x)$ and $Q(x)$ have the same degree.

(D) If $P(x)$ is an even function and $P(x) = Q'(x)$, then $Q(x)$ is an odd function.

3 Let α , β and γ be the real roots of the polynomial $P(x)$, satisfying

$$\alpha\beta\gamma = 1$$

$$\alpha + \beta + \gamma = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

Which of the following statements is NOT always true?

(A) The sum of the roots of $P(x)$ has magnitude at least 3.

(B) The sum of the squares of the roots of $P(x)$ is at least 3.

(C) One of the roots of $P(x)$ is $x = 1$.

(D) One of the roots of $P(x)$ is the reciprocal of the other root.

4 Consider the binomial expansion of $(ax+b)^n$, where n is a positive integer and $a, b > 0$.

The coefficient of x^n is the largest in the binomial expansion.

Which of the following statements is true?

(A) $n \geq \dfrac{a}{b}$.

(B) $n \leq \dfrac{a}{b}$.

(C) $n \geq \dfrac{b}{a}$.

(D) $n \leq \dfrac{b}{a}$.

5 Let p be the probability of success in a trial. If the probability of having k successes in n trials is higher than the probability of having the same number of successes in $n-1$ trials, which of the following is true?

(A) $p < \dfrac{k}{n-1}$.

(B) $p > \dfrac{k}{n-1}$.

(C) $p < \dfrac{k}{n}$.

(D) $p > \dfrac{k}{n}$.

6 Let P be a point that divides the interval AB externally in the ratio $m:n$, where the length of AP is larger than the length of BP. What ratio does the point B divide the interval PA internally?

(A) $n:(m-n)$.

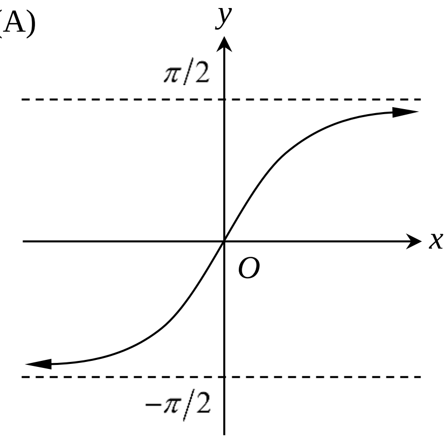
(B) $(m-n):n$.

(C) $m:(m-n)$.

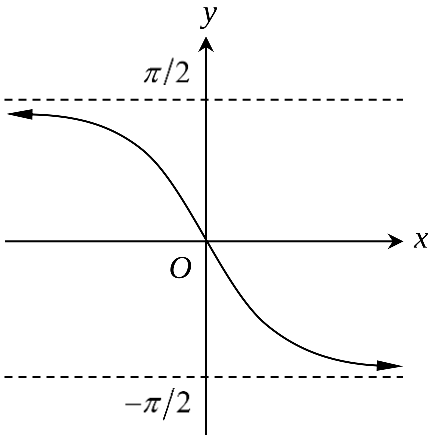
(D) $(m-n):m$.

7 Which of the following best represents the graph of $y = \tan^{-1}\left(\dfrac{x}{1+\sqrt{1+x^2}}\right)$?

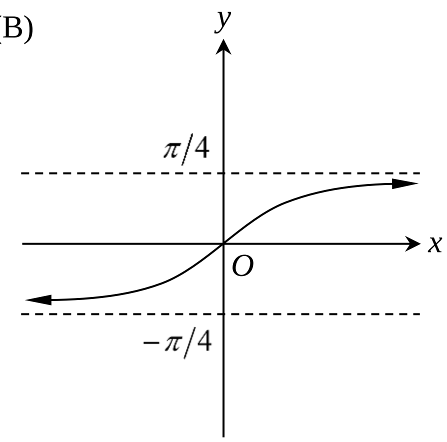
(A)



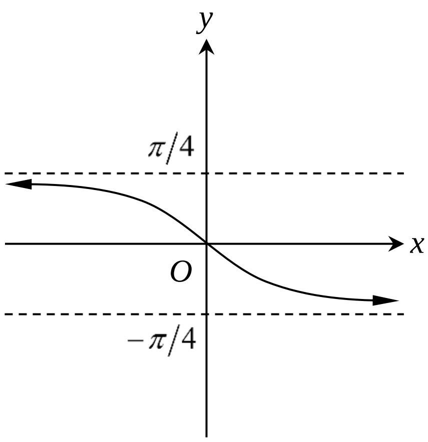
(C)



(B)



(D)



8 Which of the following is the correct value of $\int_0^{2\pi} \sin^4 \theta \, d\theta$?

(A) $\dfrac{3\pi}{16}$.

(B) $\dfrac{3\pi}{8}$.

(C) $\dfrac{3\pi}{4}$.

(D) $\dfrac{3\pi}{2}$.

9 A particle moves in simple harmonic motion with amplitude 0.5 units, period π and centre of motion at $x = 1$.

Which of the following is a possible displacement-time equation of the particle?

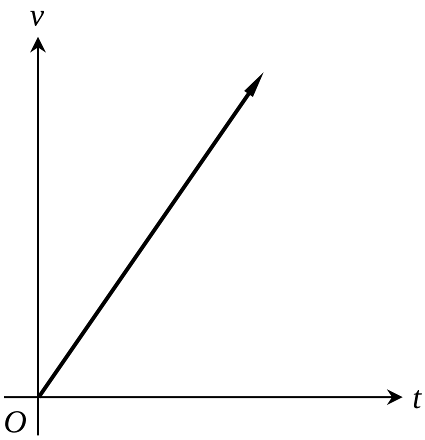
(A) $x = \cos(2t) + 1$.

(B) $x = \dfrac{\sin(t)}{2} + 1$.

(C) $x = \dfrac{\sin(t)\cos(t)}{2} + 1$.

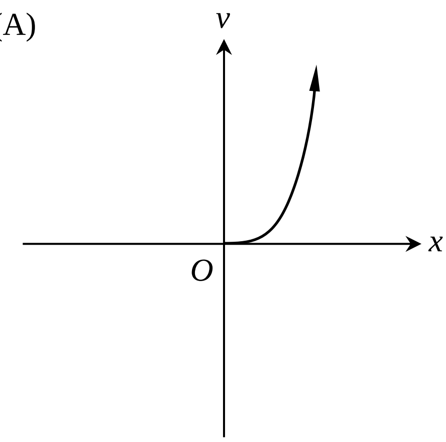
(D) $x = \sin^2(t) + \dfrac{1}{2}$.

10 Consider the following velocity-time graph of a particle.

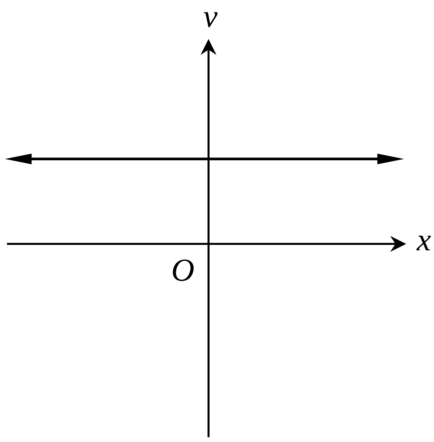


Which of the following is a possible velocity-displacement graph of the same particle?

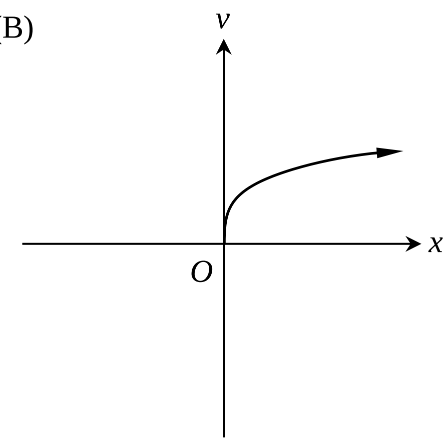
(A)



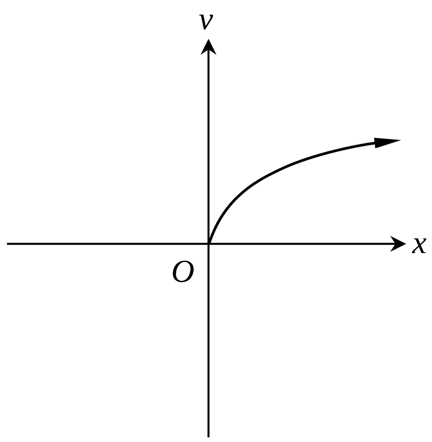
(C)



(B)



(D)



Total marks – 90
Attempt Questions 11 – 16
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) A glucose solution enters the bloodstream at a constant rate *r*. The human body metabolises the glucose and removes it from the bloodstream at a rate proportional to the concentration at the time. The concentration satisfies the differential equation

$$\frac{dC}{dt} = r - kC,$$

where *k* is positive. Let the initial concentration be C_0.

(i) Show that the concentration at time *t* is **2**

$$C = \frac{r}{k} - \left(\frac{r - kC_0}{k}\right)e^{-kt}.$$

(ii) Suppose $C_0 < \frac{r}{k}$. Describe how $C(t)$ changes, as *t* gets large. **1**

(b) Use the substitution $u = \sin x - \cos x$ to find **3**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{\tan x} + \sqrt{\cot x}\, dx.$$

(c) Find the set of solutions to the inequality **3**

$$\frac{\sqrt{x}}{|x-1|} > \frac{1}{\sqrt{x}-1}.$$

Question 11 continues on page 8

(d) (i) Write down the general solution for θ if $\tan\theta = \alpha$. **1**

(ii) Hence, or otherwise, show that if *x* and *y* are positive where **2**

$xy > 1$, then

$$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

(e) Use mathematical induction to prove that **3**

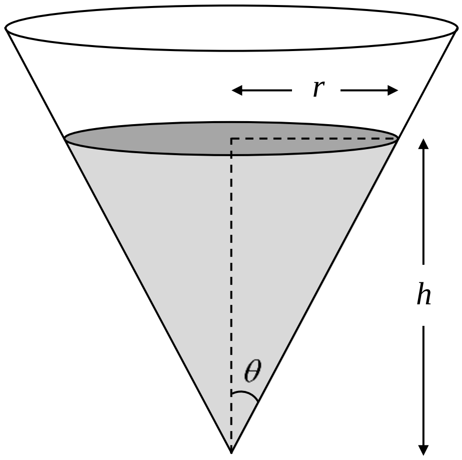
$$\sum_{k=1}^{2^n} \frac{1}{k} < n+1.$$

for positive integer values of *n*.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) An inverted cone with semi-vertical angle θ holds some amount of water. **3**



The volume of the water decreases at a rate proportional to the exposed area. At time t, the radius of the exposed area is r and the depth of the water from the vertex is h.

Show that the depth is decreasing at a constant rate.

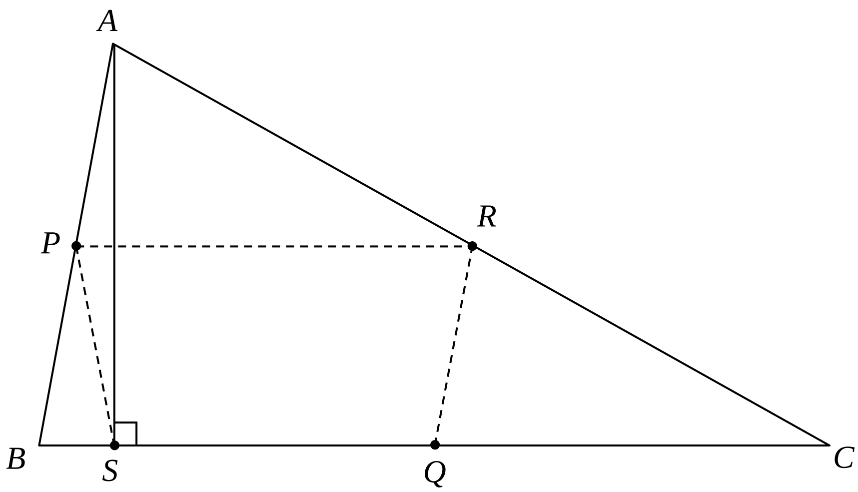
(b) Suppose $\alpha + \beta + \gamma = \dfrac{\pi}{2}$. **3**

Show that

$$\tan\alpha \tan\beta + \tan\beta \tan\gamma + \tan\alpha \tan\gamma = 1.$$

Question 12 continues on page 10

(c) The diagram below shows $\triangle ABC$ where P, Q and R are the midpoints of AB, BC and AC respectively, shown in the diagram below. **3**



Let S be the foot of the perpendicular from A onto BC.

Prove that $PRQS$ is a cyclic quadrilateral.

Question 12 continues on page 11

(d) A particle X with displacement function $x(t)$ is said to be *periodic* with period T if

$$x(t+T) = x(t)$$

The particle X moves in simple harmonic motion about the origin with period $\dfrac{2\pi}{n}$ and amplitude A.

(i) Another particle Y has the displacement equation **3**

$$y(t) = \left(x(t) - b\right)^2,$$

where b is a constant.

Find the acceleration equation of particle Y in terms of the displacement $y(t)$ and the constants n, A and b.

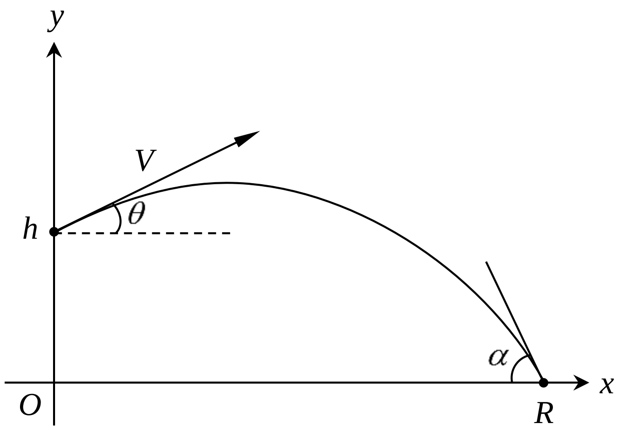
(ii) Deduce that if $b = 0$, then particle Y also moves in simple harmonic motion. **1**

(iii) Show that if $b \neq 0$, then particle Y is periodic, but does not move in simple harmonic motion. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A particle is projected from a cliff of height h with a fixed initial acute angle θ from the horizontal axis, with variable initial speed V. **3**



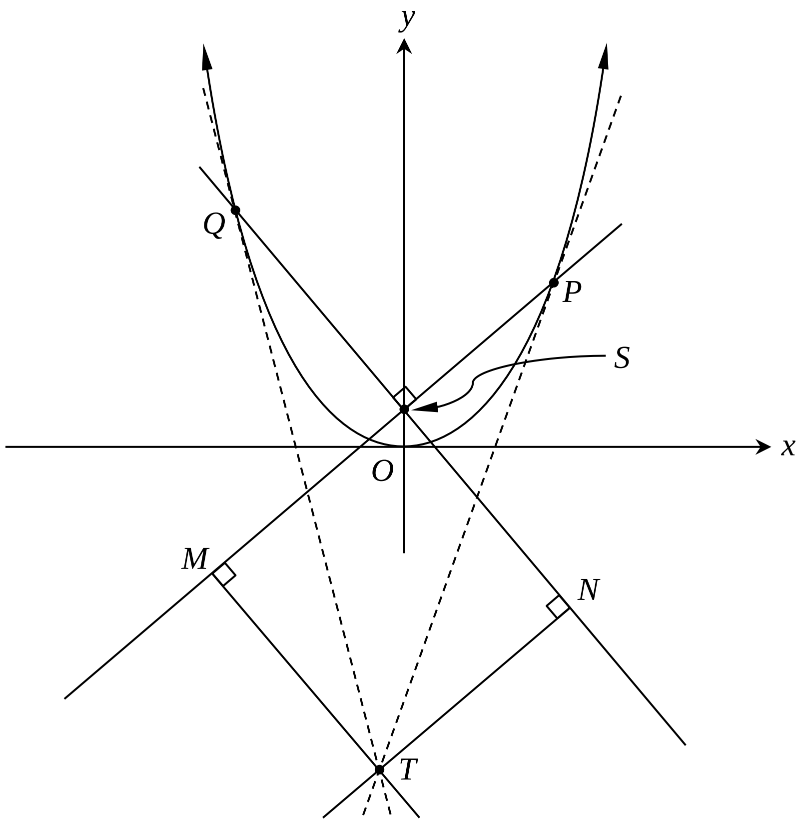
The particle lands on the ground at an acute angle α *at most* 45° from the horizontal axis.

Show that $V^2 \geq 2gh$.

You may state, without proof, any relevant equations of motion.

Question 13 continues on page 13

(b) Let $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ be two points on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the focus S, and the tangents at P and Q intersect at T.



Let the feet of the perpendiculars from T to the lines PS and QS be M and N respectively.

(i) Find the coordinates of T. **2**

(ii) Find the equation of the chords PS and QS. **2**

(iii) Hence, or otherwise, prove that $SNTM$ is a square. **3**

Question 13 continues on page 14

(c) Consider a sequence of n identical dollar symbols and $m-1$ identical dots arranged in a row. The diagram below shows the case for when $n=12$ and $m=5$. **1**

$$\$\$\bullet\$\$\$\bullet\$\$\$\$\bullet\$\$\bullet\$$$

Explain why there are $\binom{n+m-1}{n}$ ways of arranging the dollar symbols and dots.

(d) Travis has n identical coins. He distributes the n coins amongst his m friends.

(i) Use part (c) to explain why the number of ways that Travis may distribute the coins so that all his friends have at least one coin is **2**

$$\binom{n-1}{m-1}.$$

(ii) Hence, or otherwise, simplify the sum **2**

$$\binom{m}{1}\binom{n-1}{0}+\binom{m}{2}\binom{n-1}{1}+\binom{m}{3}\binom{n-1}{2}+\ldots+\binom{m}{m}\binom{n-1}{m-1}.$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function

$$f(x) = \frac{1}{(x+1)(x+2)\ldots(x+n)}.$$

It is possible to express $f(x)$ as the sum

$$f(x) = \sum_{k=1}^{n} \frac{A_k}{x+k},$$

where A_k is some real number.

(i) Show that **1**

$$A_k = \frac{(-1)^{k-1}}{(k-1)!(n-k)!}.$$

(ii) Hence, or otherwise, simplify **2**

$$\binom{n}{1}\frac{1}{n+1} - \binom{n}{2}\frac{2}{n+2} + \binom{n}{3}\frac{3}{n+3} - \binom{n}{4}\frac{4}{n+4} + \ldots + (-1)^{n-1}\binom{n}{n}\frac{n}{n+n}.$$

Question 14 continues on page 16

(b) Let $g(x)$ be a smooth continuous function in the domain $\mathcal{D}:\ a \le x \le b$, where $g(a) = g(b)$.

(i) With the aid of a diagram, briefly explain why there exists $x = x_0$ in $\mathcal{D}$ such that $g'(x_0) = 0$. **1**

(ii) Let $f(x)$ be a smooth continuous function defined in $\mathcal{D}$. **1**

Define

$$g(x) = f(x) - \left[f(a) + (x-a) f'(a) \right] - \frac{f(b) - \left[f(a) + f'(a)(b-a) \right]}{(b-a)^2}(x-a)^2.$$

Use (i) to show that there exists $x = x_1$ in $\mathcal{D}$ such that $g'(x_1) = 0$.

(iii) Hence, show that there exists $x = x_2$ in the interval $a \le x \le x_1$ such that **2**

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2} f''(x_2)$$

Question 14 continues on page 17

(c) Let $f(x)$ be any smooth continuous function defined over the interval $\mathcal{D}:\ a \le x \le b$, with a real root $x = \alpha$ in $\mathcal{D}$ and $f'(\alpha) \ne 0$.

Define $A(x) = x - \dfrac{f(x)}{f'(x)}$.

Let $x_1,\ x_2,\ x_3, \dots$ be a sequence of numbers defined by the recurrence

$$A(x_k) = x_{k+1},$$

where $k = 1, 2, 3, \dots$, with starting point x_0.

(i) Use the result in (b) (iii) to show that there exists $x = \beta_k$ in $\mathcal{D}$ such that **2**

$$|x_{k+1} - \alpha| = \frac{(x_k - \alpha)^2}{2} |A''(\beta_k)|.$$

(ii) Hence show that if $|x_0 - \alpha| < 1$ and $|A''(\beta_k)| < 2$ for all $k = 0, 1, 2, 3, \dots$, then $x_{k+1} \to \alpha$ as $k \to \infty$. **2**

Question 14 continues on page 18

Question 14 (continued)

(d) Let $f(x) = 1 - \ln x$, which has $x = e$ as a solution. Newton's Method is used to approximate e using the starting point $x_0 = 2.71$.

(i) Let $A(x)$ be defined as in part (c). **1**

Show that if $2 < x < 3$, then

$$\frac{1}{3} < \left|A''(x)\right| < \frac{1}{2}.$$

(ii) Use part (c) to find how many applications of Newton's Method are needed to obtain an approximation of e that is guaranteed to be correct to *at least* 2016 decimal places. **3**

End of Exam