2016 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

1. D	3. A	5. C	7. B	9. D
2. B	4. B	6. A	8. C	10. B

Working/Justification

Question 1

Note that the graph has an unrestricted domain and range.

$y = \sin(\sin^{-1} x)$ has a restricted domain $-1 \leq x \leq 1$

$y = \sin^{-1}(\sin x)$ has a restricted range $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$

$y = e^{\ln x}$ has a restricted domain $x > 0$

$y = \ln(e^x)$ has an unrestricted domain and range

So the answer is (D)

Question 2

Option (A) is not always true because it may be possible that $P(x) = kQ(x)$ for constant $k \neq 0$ yet $P(x)$ and $Q(x)$ have the same set of roots

Option (C) is not always true because if $P(x) = A(x)Q(x) + c$ for some constant c and some polynomial $A(x)$ then $A(x)Q(x)$ should have the same degree as $P(x)$. This does not imply that $Q(x)$ has the same degree as $P(x)$.

Option is (D) is not always true because $Q(x)$ is the primitive of an even function $P(x)$ plus a constant, which is not necessarily an odd function.

Option (B) is always true because $P(0)$ generates the constant term of the polynomial $P(x)$ and similarly $Q(0)$ generates the constant term of the polynomial $Q(x)$. If $P(0) = Q(0)$ then the constant terms of the polynomials $P(x)$ and $Q(x)$ must be equal.

Hence the answer is (B)

Question 3

Since $\alpha\beta\gamma = 1$ then $\alpha + \beta + \gamma = \alpha\beta + \beta\gamma + \alpha\gamma$. If we let $\alpha + \beta + \gamma = s$ then the cubic polynomial $P(x)$ has the form $P(x) = x^3 - sx^2 + sx - 1$, which by substitution has a root at $x = 1$.

Since one of the roots is 1 then the other two roots must be reciprocals of each other in order for $\alpha\beta\gamma = 1$ to hold. Thus, the roots are α, $\frac{1}{\alpha}$ and 1.

When considering the sum of the squares of the roots, we note that $\alpha^2 + \frac{1}{\alpha^2}$ is at least 2 (this can be shown by noting that the lowest value of $\left(\alpha - \dfrac{1}{\alpha}\right)^2$ is zero). Thus, the sum of the squares of the roots must be at least 3.

The sum of the roots may have a magnitude less than 3. For example, if the roots are $-2, -\frac{1}{2}$ and 1 then the conditions are satisfied but sum of the roots has magnitude $\frac{3}{2}$. Note that this can be shown more formally by considering the discriminant of the quadratic factor after $P(x)$ is divided by $(x - 1)$.

Hence the statement that is not always true is (A)

Question 4

The coefficient of x^n is a^n. If it is the largest coefficient then it must be greater than or equal to the coefficient of x^{n-1} which is $na^{n-1}b$. Solving the inequality

$$a^n \geq na^{n-1}b$$

$$n \leq \frac{a}{b} \quad \text{hence the answer is } (B)$$

Question 5

The probability of having k successes in n trials is $\binom{n}{k}p^k(1-p)^{1-k}$. The probability of having k successes in $n-1$ trials is $\binom{n-1}{k}p^k(1-p)^{n-k-1}$. Solving the inequality

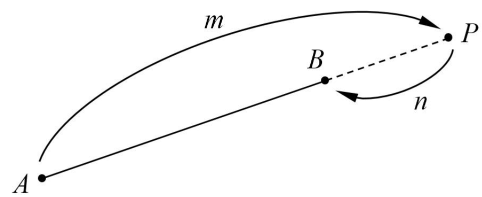
$$\binom{n}{k}p^k(1-p)^{n-k} > \binom{n-1}{k}p^k(1-p)^{n-k-1}$$

$$\frac{n!}{k!(n-k)!}(1-p) > \frac{(n-1)!}{k!(n-k-1)!}$$

$$p < \frac{k}{n} \quad \text{hence the answer is } (C)$$

Question 6

If P divides the interval AB externally in the ratio $m : n$ then $\dfrac{PA}{PB} = \dfrac{m}{n}$.



The ratio at which B divides the interval PA is therefore $\dfrac{PB}{AB} = \dfrac{n}{m-n}$.

Hence the answer is (A),

Question 7

First note that as $x \to \infty$, $\dfrac{x}{1+\sqrt{1+x^2}} \to 1$. This means that $y \to \dfrac{\pi}{4}$. Also, note that when $x > 0$ then $y > 0$ and when $x < 0$ then $y < 0$. Hence the answer is (B).

Question 8

$$\int_0^{2\pi} \sin^4 x\, dx = \int_0^{2\pi} \sin^2 x(1-\cos^2 x)\, dx \qquad \text{but } \sin 2x = 2\sin x \cos x$$

$$= \int_0^{2\pi} \left(\sin^2 x - \frac{1}{4}\sin^2 2x\right) dx \qquad \text{but } \cos 2\theta = 1 - 2\sin^2\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2}(1-\cos 2x) - \frac{1}{8}(1-\cos 4x)\right) dx$$

$$= \left|\frac{1}{2}(x - \frac{1}{2}\sin 2x) - \frac{1}{8}(x - \frac{1}{4}\sin 4x)\right|_0^{2\pi}$$

$$= \frac{3\pi}{4} \qquad \text{hence the answer is } (C)$$

Question 9

$x = \cos(2t) + 1$ has period π and centre $x = 1$, but amplitude 1.

$x = \frac{1}{2}\sin(t) + 1$ has amplitude $\frac{1}{2}$ and centre $x = 1$, but period 2π.

$x = \frac{1}{2}\sin(t)\cos(t) + 1$ can be rewritten as $x = \frac{1}{4}\sin(2t) + 1$ which has period π and centre $x = 1$, but amplitude $\frac{1}{4}$.

$x = \sin^2(t) + \frac{1}{2}$ can be rewritten as $x = -\frac{1}{2}\cos(2t) + 1$ which has period π, centre $x = 1$ and amplitude $\frac{1}{2}$.

Hence the answer is (D).

Question 10

From the sketch note the velocity is linear over time, so the acceleration must be constant over time. This means that $\frac{d}{dx}\left(\frac{v^2}{2}\right)$ must be a constant.

For this to be possible, $\frac{v^2}{2}$ must be a linear function with respect to displacement. This can only occur if the velocity is a square root function with respect to displacement. The square root function has a vertical gradient at the origin.

Hence the answer is (B).

Section II

Question 11

(a)

(i) Start with the concentration equation

$$C = \frac{r}{k} - \left(\frac{r - kC_0}{k}\right) e^{-kt}$$

$$\frac{dC}{dt} = (r - kC_0)e^{-kt} \quad \text{but } C = \frac{r}{k} - \left(\frac{r - kC_0}{k}\right) e^{-kt} \Rightarrow (r - kC_0)e^{-kt} = k\left(\frac{r}{k} - C\right)$$

$$= k\left(\frac{r}{k} - C\right)$$

$$= r - kC$$

(ii) If $C_0 < \frac{r}{k}$ and $k > 0$ then $r - kC_0 > 0$. This means that $\frac{dC}{dt} > 0$ when $t = 0$. Since C behaves according to an exponential function then $\frac{dC}{dt} > 0$ for all t. This means that as t gets large, the concentration C increases and approaches $\frac{r}{k}$ from below.

(b) When $u = \sin x - \cos x$ then $du = (\cos x + \sin x)\, dx$

When $x = \frac{\pi}{3}$ then $u = \frac{\sqrt{3} - 1}{2}$

When $x = \frac{\pi}{6}$ then $u = \frac{1 - \sqrt{3}}{2}$

Also note that

$$u^2 = \sin^2 x - 2\sin x \cos x + \cos^2 x$$

$$\sin x \cos x = \frac{1 - u^2}{2}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right)\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)\, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\right)\, dx$$

$$= \sqrt{2}\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{du}{\sqrt{1-u^2}}$$

$$= \sqrt{2}[\sin^{-1} u]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

$$= \sqrt{2}\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)\right) \qquad \text{but } \sin^{-1}(-x) = -\sin^{-1} x$$

$$= 2\sqrt{2}\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

(c) First rearrange the inequality as follows, noting it is only valid for the domain $0 \leq x < 1$ and $x \geq 1$

$$\sqrt{x} > \frac{|x-1|}{\sqrt{x}-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$\sqrt{x} > \frac{|x-1|(\sqrt{x}+1)}{x-1}$$

Case 1: $x > 1$

$|x-1| = x-1$ so the inequality reduces to

$$\sqrt{x} > \sqrt{x} + 1$$

This is a false statement hence there are no solutions in the domain $x > 1$

Case 2: $0 \leq x < 1$

$|x-1| = 1-x$ so the inequality reduces to

$$\sqrt{x} > -\sqrt{x} - 1$$

$$\sqrt{x} > -\frac{1}{2}$$

This statement holds for all x in the domain $0 \leq x < 1$ so the final solution is $0 \leq x < 1$

(d)

(i) The general solution is $\theta = n\pi + \tan^{-1}\alpha$ for integer n

(ii) Let $a = \tan^{-1} x$ and $b = \tan^{-1} y$ then

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$a + b = n\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \qquad \text{for integer } n$$

$$\tan^{-1} x + \tan^{-1} y = n\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

However, note that since $xy > 1$, $x > 0$ and $y > 0$ then $\dfrac{x+y}{1-xy} < 0$ so $-\dfrac{\pi}{2} < \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) < 0$.
Also, since $x > 0$ and $y > 0$ then $0 < \tan^{-1} x < \dfrac{\pi}{2}$ and $0 < \tan^{-1} y < \dfrac{\pi}{2}$. So

$$0 < \tan^{-1} x + \tan^{-1} y < \pi$$
$$0 < \tan^{-1} x + \tan^{-1} y - \tan^{-1}\left(\frac{x+y}{1-xy}\right) < \frac{3\pi}{2}$$
$$0 < n\pi < \frac{3\pi}{2}$$

The only integer value of n which satisfies this inequality is $n = 1$ hence

$$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

For $x > 0$, $y > 0$ and $xy > 1$.

(e) When $n = 1$

$$LHS = 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

$$RHS = 2$$

Since $LHS < RHS$ the statement is true for $n = 1$

Assume the statement is true for $n = m$

$$\sum_{k=1}^{2^m} \frac{1}{k} < m + 1$$

Required to prove the statement is true for $n = m + 1$

$$\sum_{k=1}^{2^{m+1}} \frac{1}{k} < m + 2$$

$$LHS = \sum_{k=1}^{2^{m+1}} \frac{1}{k}$$

$$= \sum_{k=1}^{2^m} \frac{1}{k} + \frac{1}{2^m + 1} + \frac{1}{2^m + 2} + ... + \frac{1}{2^m + 2^m}$$

$$< m + 1 + \frac{1}{2^m + 1} + \frac{1}{2^m + 2} + ... + \frac{1}{2^m + 2^m} \quad \text{by assumption}$$

$$< m + 1 + \underbrace{\frac{1}{2^m} + \frac{1}{2^m} + ... + \frac{1}{2^m}}_{2^m \text{ terms}}$$

$$= m + 1 + \frac{2^m}{2^m}$$

$$= m + 2$$

$$= RHS$$

Since the statement is true for $n = 1$ then by induction the statement is true for all positive integer values of n.

Question 12

(a) Let k be the constant of proportionality.

$$V = \frac{1}{3}\pi r^2 h \quad \text{but } \tan\theta = \frac{r}{h}$$

$$= \frac{1}{3}\pi h^3 \tan^2\theta$$

$$\frac{dV}{dh} = \pi h^2 \tan^2\theta$$

$$\text{but } \frac{dV}{dt} = -k\pi r^2$$

$$= -k\pi h^2 \tan^2\theta$$

$$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$$

$$= \frac{1}{\pi h^2 \tan^2\theta} \times -k\pi h^2 \tan^2\theta$$

$$= -k$$

Hence the depth is decreasing at a constant rate.

(b) Since $\alpha + \beta + \gamma = \frac{\pi}{2}$ then $\gamma = \frac{\pi}{2} - (\alpha + \beta)$

$$LHS = \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\alpha\tan\gamma$$

$$= \tan\alpha\tan\beta + \tan\gamma(\tan\alpha + \tan\beta)$$

$$= \tan\alpha\tan\beta + \tan\left(\frac{\pi}{2} - (\alpha + \beta)\right)(\tan\alpha + \tan\beta)$$

$$= \tan\alpha\tan\beta + \frac{\tan\alpha + \tan\beta}{\tan(\alpha + \beta)}$$

$$= \tan\alpha\tan\beta + (\tan\alpha + \tan\beta) \times \frac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$$

$$= \tan\alpha\tan\beta + 1 - \tan\alpha\tan\beta$$

$$= 1$$

$$= RHS$$

$\therefore \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\alpha\tan\gamma = 1$

(c)

Since $AS \perp BC$ then $\angle BSA$ is a right angle

This means that the points A, B and S are concyclic where AB is the diameter of the circle enclosing the points.

But P is the midpoint of AB so P must be the centre of this circle.

$\Rightarrow PS = PB$ (equal radii)

$\therefore \Delta PBS$ is isosceles (two sides equal)

$\Rightarrow \angle PBS = \angle PSB$ (equal angles opposite equal sides)

but P, Q and R are the midpoints of AB, BC and AC so

$PR \parallel BC$ (ratio of intercepts of parallel lines)

Similarly $QR \parallel PB$

$\therefore PRQB$ is a parallelogram (opposite sides parallel)

$\angle PRQ = \angle PBQ$ (opposite angles of parallelogram)

$\Rightarrow \angle PSB = \angle PRQ$

$\therefore PRQS$ is a cyclic quadrilateral (external angle equals opposite interior angle)

(d)

(i) Using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 2(x-b)\frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(2(x-b))\frac{dx}{dt} + 2(x-b)\frac{d^2x}{dt^2} \quad \text{by the product rule}$$

$$= 2\left(\frac{dx}{dt}\right)^2 + 2(x-b)\frac{d^2x}{dt^2}$$

But X moves in simple harmonic motion about the origin with period $\frac{2\pi}{n}$ and amplitude A so

$\left(\frac{dx}{dt}\right)^2 = n^2(A^2 - x^2)$ and $\frac{d^2x}{dt^2} = -n^2x$.

Substitute this into the expression

$$\frac{d^2y}{dt^2} = 2n^2(A^2 - x^2) - 2n^2x(x-b)$$

$$= -2n^2(2x^2 - bx - A^2)$$

But $y = (x-b)^2$ so $x = b \pm \sqrt{y}$ hence $x^2 = b^2 + y \pm 2b\sqrt{y}$, so

$$\frac{d^2y}{dt^2} = -2n^2(b^2 + 2y + 3b\sqrt{y} - A^2) \quad \text{OR}$$

$$\frac{d^2y}{dt^2} = -2n^2(b^2 + 2y - 3b\sqrt{y} - A^2)$$

(ii) If $b = 0$, the acceleration equation reduces to

$$\frac{d^2y}{dt^2} = -2n^2(2y - A^2)$$

$$= -4n^2\left(y - \frac{A^2}{2}\right)$$

Since the acceleration equation has the general form $\frac{d^2y}{dt^2} = -k(y - B)$ for constants B and $k > 0$, then the particle Y moves in simple harmonic motion.

(iii) If $b \neq 0$, there exists different powers of y in the acceleration equation so it cannot be expressed in the general form $\dfrac{d^2y}{dt^2} = -k(y - B)$ for constants $k > 0$ and B, hence does not move in simple harmonic motion.

However, particle X is periodic with period $\frac{2\pi}{n}$ so $x\left(t + \frac{2\pi}{n}\right) = x(t)$. Note that

$$\begin{aligned} y\left(t + \frac{2\pi}{n}\right) &= \left(x\left(t + \frac{2\pi}{n}\right) - b\right)^2 \\ &= (x(t) - b)^2 \\ &= y(t) \end{aligned}$$

Hence, although Y is periodic, it does not move in simple harmonic motion.

Question 13

(a) The equations of motion are

$$\ddot{y} = -g \qquad\qquad \ddot{x} = 0$$

$$\dot{y} = -gt + V\sin\theta \qquad\qquad \dot{x} = V\cos\theta$$

$$y = -\frac{1}{2}gt^2 + Vt\sin\theta + h \qquad\qquad x = Vt\cos\theta$$

When $y = 0$ then $gt^2 - 2Vt\sin\theta - 2h = 0$, hence

$$t = \frac{2V\sin\theta \pm \sqrt{4v^2\sin^2\theta + 8gh}}{2g}$$

$$= \frac{V\sin\theta + \sqrt{V^2\sin^2\theta + 2gh}}{g} \qquad \text{since } t > 0$$

$$gt - V\sin\theta = \sqrt{V^2\sin^2\theta + 2gh}$$

But since $0 < \alpha \leq \frac{\pi}{4}$ when the particle hits the ground then

$$0 < \tan\alpha \leq 1 \qquad \text{where } \tan\alpha = \left|\frac{\dot{y}}{\dot{x}}\right|$$

$$0 < \left|\frac{-gt + V\sin\theta}{V\cos\theta}\right| \leq 1$$

$$0 < \frac{\sqrt{V^2\sin^2\theta + 2gh}}{V\cos\theta} \leq 1$$

$$0 < V^2\sin^2\theta + 2gh \leq V^2\cos^2\theta$$

$$V^2(\cos^2\theta - \sin^2\theta) \geq 2gh$$

$$V^2\cos 2\theta \geq 2gh \quad \text{but } \cos 2\theta \leq 1 \quad \text{so } V^2 \geq V^2\cos 2\theta$$

$$\therefore V^2 \geq 2gh$$

(b)

(i) From the chain rule

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

$$= \frac{2ap}{2a}$$

$$= p$$

The equation of the tangent at P is

$$y - ap^2 = p(x - 2ap)$$

$$y = px - ap^2$$

Similarly the equation of the tangent at Q is $y = qx - aq^2$.

Equating the y values to solve for T

$$px - ap^2 = qx - aq^2$$

$$x(p - q) = a(p - q)(p + q)$$

$$x = a(p + q) \quad \text{since } p \neq q$$

Subtitute this back into the equation of the tangent

$$y = ap(p + q) - ap^2$$

$$= apq$$

Hence the coordinates of T are $(a(p + q), apq)$

(ii) The gradient of PS is

$$m_{PS} = \frac{ap^2 - a}{2ap - 0}$$

$$= \frac{p^2 - 1}{2p}$$

The equation of the chord PS is

$$y = \left(\frac{p^2 - 1}{2p}\right) x + a$$

$$(p^2 - 1)x - 2py + 2ap = 0$$

Similarly, the equation of the chord QS is $(q^2 - 1)x - 2qy + 2aq = 0$

(iii) First note that $\angle QSP = 90°$ so $\angle MSN = 90°$ by vertically opposite angles.

Also, $\angle TMS = \angle TNS = 90°$ by definition.

Hence $SNTM$ is a rectangle. To prove it is a square, two adjacent sides need to be proved as equal. By perpendicular distance

$$\begin{aligned}
TM &= \frac{|(p^2 - 1)(a(p + q)) - 2p(apq) + 2ap|}{\sqrt{(p^2 - 1)^2 + 4p^2}} \\
&= \frac{a|p^3 + p^2 q - p - q - 2p^2 q + 2p|}{\sqrt{p^4 - 2p^2 + 1 + 4p^2}} \\
&= \frac{a|p^3 - p^2 q + p - q|}{\sqrt{p^4 + 2p^2 + 1}} \\
&= \frac{a|p^2(p - q) + (p - q)|}{\sqrt{(p^2 + 1)^2}} \\
&= \frac{a|(p - q)(p^2 + 1)|}{p^2 + 1} \\
&= a|p - q|
\end{aligned}$$

Note that from the first line, if p and q are interchanged, the expression becomes the length of TN. Hence, $TN = a|q - p|$. But since $|p - q| = |q - p|$ then $TM = TN$. Therefore, $SNTM$ is a square.

(c)

Of the $n + m - 1$ positions, choose n to be dollar symbols. The remaining $m - 1$ must be dots. There are $\binom{n+m-1}{n}$ ways of arranging the dollar symbols and dots.

(d)

(i) The number of ways of distributing the coins is analogous to using the dots as dividers between friends where each $ refers to one coin.

In order to have all friends to have at least one coin, the dividers cannot sit on the start and end points of the sequence or be adjacent to each other.

For n coins, one can think of the sequence as having $n+1$ gaps between each coin for which $m-1$ dots must be chosen to place within those available gaps. This ensures the the dots cannot be adjacent to each other.

Since the two gaps at the start and end points of the sequence are not available then there $n-1$ gaps to distribute the $m-1$ dots which has a total of $\binom{n-1}{m-1}$ combinations.

Another way to think of it is to suppose that each friend receives exactly one coin at first (there is only one way of doing so). This leaves $n-m$ coins to distribute amongst the friends using the same divider approach but without restriction. The number of ways of arranging is therefore $\binom{(n-m)+m-1}{n-m}$ which simplifies to $\binom{n-1}{n-m}$ and is equivalent to $\binom{n-1}{m-1}$

(ii) There are $\binom{m}{k}$ ways of choosing k friends.

The number of ways of distributing coins to these k friends (where $k<m$) so that at least one received a coin is $\binom{n-1}{k-1}$. Hence

$$\sum_{k=1}^{m}\binom{m}{k}\binom{n-1}{k-1}$$

is the total number of ways of distributing the coins to the friends, where it is possible that some of them may not have received any coins at all.

This is equivalent to the scenario in (c) of simply distributing the coins to m friends (which includes all cases) allowing for some to get zero coins hence

$$\binom{m}{1}\binom{n-1}{0}+\binom{m}{2}\binom{n-1}{1}+\binom{m}{3}\binom{n-1}{2}+\ldots+\binom{m}{m}\binom{n-1}{m-1}=\binom{n+m-1}{n}$$

Question 14

(a)

(i) Consider a particular term when $k = m$

$$\frac{1}{(x+1)...(x+(m-1))(x+m)(x+(m+1))...(x+n)} = \frac{A_1}{x+1} + ... + \frac{A_m}{x+m} + ... + \frac{A_n}{x+n}$$

Multiply both sides by $x + m$

$$\frac{1}{(x+1)...(x+(m-1))(x+(m+1))...(x+n)} = \frac{A_1(x+m)}{x+1} + ... + A_m + ... + \frac{A_n(x+m)}{x+n}$$

$$\text{let } x = -m \quad A_m = \frac{1}{(-m+1)...(-1) \times \underbrace{(1)...(-m+n)}_{(n-m)!}}$$

$$= \frac{1}{(-1)^{m-1}\underbrace{(m-1)...(1)}_{(m-1)!} \times (n-m)!}$$

$$= \frac{(-1)^{m-1}}{(m-1)!(n-m)!}$$

Hence for all k, $A_k = \dfrac{(-1)^{k-1}}{(k-1)!(n-k)!}$

(ii) Let $x = n$

$$\frac{1}{(n+1)(n+2)...(n+n)} = \sum_{k=1}^{n} \frac{1}{n+k} \times \frac{(-1)^{k-1}}{(k-1)!(n-k)!}$$

$$\frac{1}{(n+1)(n+2)...(n+n)} \times \frac{n!}{n!} = \sum_{k=1}^{n} \frac{1}{n+k} \times \frac{(-1)^{k-1}}{(k-1)!(n-k)!} \times \frac{k}{k}$$

$$\frac{n!}{(2n)!} = \sum_{k=1}^{n} \frac{k}{n+k} \times \frac{(-1)^{k-1}}{k!(n-k)!}$$

$$\frac{n!}{(2n)!} \times n! = \sum_{k=1}^{n} \frac{k}{n+k} \times \frac{(-1)^{k-1}}{k!(n-k)!} \times n!$$

$$\frac{1}{\binom{2n}{n}} = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}k}{n+k}$$

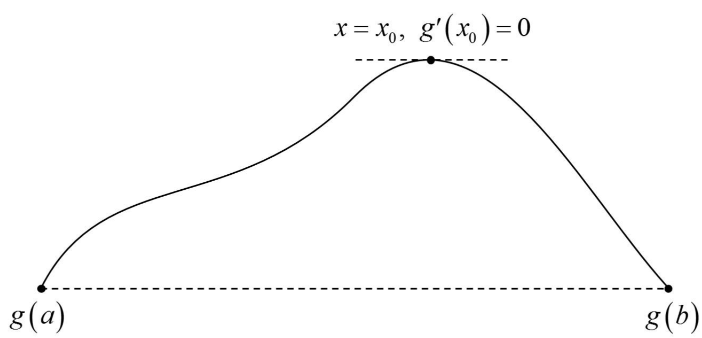
$$\therefore \binom{n}{1}\frac{1}{n+1} - \binom{n}{2}\frac{2}{n+2} + \binom{n}{3}\frac{3}{n+3} - ... + (-1)^{n-1}\binom{n}{n}\frac{n}{n+n} = \frac{1}{\binom{2n}{n}}$$

(b)

(i) If $g(a) = g(b)$ and $g(x)$ is a smooth continuous function then $g'(a)$ and $g'(b)$ cannot be both positive or both negative. If they were the same sign and $g'(x) \neq 0$ anywhere in the domain, then this suggests the function is monotonic increasing or decreasing which makes it impossible to obtain $g(a) = g(b)$.

There are two main cases where condition $g(a) = g(b)$ is possible.

Case 1: $g'(a)$ and $g'(b)$ are opposite sign (i.e. one is positive and the other is negative) in which case there must exist some $x = x_0$ in the domain such that $g'(x_0) = 0$ for the change in sign of the derivative to be possible.



Case 2: At least one of $g'(a)$ and $g'(b)$ is zero in which case $x_0 = a$ or $x_0 = b$ which trivially satisfies the required condition.

Hence there exists an $x = x_0$ in $\mathcal{D}$ such that $g'(x_0) = 0$

(ii) Consider $g(a)$ and $g(b)$

$$g(a) = f(a) - [f(a) + (a-a)f'(a)] - \frac{f(b) - [f(a) + f'(a)(b-a)]}{(b-a)^2}(a-a)^2$$

$$= 0$$

$$g(b) = f(b) - [f(a) + (b-a)f'(a)] - \frac{f(b) - [f(a) + f'(a)(b-a)]}{(b-a)^2}(b-a)^2$$

$$= f(b) - [f(a) + (b-a)f'(a)] - f(b) + [f(a) + f'(a)(b-a)]$$

$$= 0$$

Since $g(a) = g(b)$ then from (i), there exists an $x = x_1$ in $\mathcal{D}$ such that $g'(x_1) = 0$

(iii) Consider $g'(a)$

$$g'(x) = f'(x) - f'(a) - \frac{f(b) - [f(a) + f'(a)(b-a)]}{(b-a)^2} \times 2(x-a)$$

$$g'(a) = f'(a) - f'(a) - \frac{f(b) - [f(a) + f'(a)(b-a)]}{(b-a)^2} \times 2(a-a)$$

$$= 0$$

Recall that $g'(x_1) = 0$ from part (ii). Since $g'(a) = g'(x_1)$ then from (i), there exists an $x = x_2$, where $a \leq x_2 \leq x_1$, such that $g''(x_2) = 0$

$$g''(x) = f''(x) - 2 \times \frac{f(b) - [f(a) + f'(a)(b-a)]}{(b-a)^2}$$

$$g''(x_2) = f''(x_2) - 2 \times \frac{f(b) - [f(a) + f'(a)(b-a)]}{(b-a)^2} \qquad \text{but } g''(x_2) = 0$$

$$\frac{(b-a)^2}{2} f''(x_2) = f(b) - [f(a) + f'(a)(b-a)]$$

$$\therefore f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(x_2)$$

Note: The relation still holds if $a \geq b$ with the only difference being that $x_1 \leq x_2 \leq a$ instead.

(c)

(i) First note that

$$A(x) = x - \frac{f(x)}{f'(x)}$$

$$A'(x) = 1 - \frac{[f'(x)]^2 - f(x) \times f''(x)}{[f'(x)]^2}$$

$$= \frac{f(x)f''(x)}{[f'(x)]^2}$$

From part (b)(iii), let $f(x) = A(x), a = \alpha$ and $b = x_k$. There exists a β_k in $\mathcal{D}$ such that

$$A(x_k) = A(\alpha) + (x_k - \alpha)A'(\alpha) + \frac{(x_k - \alpha)^2}{2}A''(\beta_k) \quad \text{but } A'(\alpha) = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = 0$$

$$x_{k+1} = A(\alpha) + \frac{(x_k - \alpha)^2}{2}A''(\beta_k) \quad \text{but } A(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)} = \alpha$$

$$x_{k+1} = \alpha + \frac{(x_k - \alpha)^2}{2}A''(\beta_k)$$

$$\therefore |x_{k+1} - \alpha| = \frac{(x_k - \alpha)^2}{2}|A''(\beta_k)| \quad \text{noting that } \frac{(x_k - \alpha)^2}{2} \geq 0$$

(ii) From the previous part

$$|x_{k+1} - \alpha| = (x_k - \alpha)^2 \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$= \left((x_{k-1} - \alpha)^2 \left|\frac{A''(\beta_{k-1})}{2}\right|\right)^2 \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$= \left((x_{k-2} - \alpha)^2 \left|\frac{A''(\beta_{k-2})}{2}\right|\right)^4 \times \left|\frac{A''(\beta_{k-1})}{2}\right|^2 \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$= \left((x_{k-3} - \alpha)^2 \left|\frac{A''(\beta_{k-3})}{2}\right|\right)^8 \times \left|\frac{A''(\beta_{k-2})}{2}\right|^4 \times \left|\frac{A''(\beta_{k-1})}{2}\right|^2 \times \left|\frac{A''(\beta_k)}{2}\right|$$

......

$$= \left((x_0-\alpha)^2 \left| \frac{A''(\beta_0)}{2} \right| \right)^{2^k} \times \left| \frac{A''(\beta_1)}{2} \right|^{2^{k-1}} \times ... \times \left| \frac{A''(\beta_k)}{2} \right|$$

$$= (x_0-\alpha)^{2^{k+1}} \times \left| \frac{A''(\beta_0)}{2} \right|^{2^k} \times \left| \frac{A''(\beta_1)}{2} \right|^{2^{k-1}} \times ... \times \left| \frac{A''(\beta_k)}{2} \right|$$

Aside: First note that since $0 \le |x_0-\alpha| < 1$ then $(x_0-\alpha)^{2^{k+1}} \to 0$ as $k \to \infty$

Also, since $|A''(\beta_k)| < 2$ for all k then the other products will approach zero when $k \to \infty$.

This means that $|x_{k+1}-\alpha| \to 0$ which is equivalent to $x_{k+1} \to \alpha$ when $k \to \infty$

Aside: To see this a bit more formally, observe that

$$0 \le \left| \frac{A''(\beta_0)}{2} \right|^{2^k} \times \left| \frac{A''(\beta_1)}{2} \right|^{2^{k-1}} \times ... \times \left| \frac{A''(\beta_k)}{2} \right| \le \left[\max \left\{ \left| \frac{A''(\beta_0)}{2} \right|^{2^k}, \left| \frac{A''(\beta_1)}{2} \right|^{2^{k-1}}, ..., \left| \frac{A''(\beta_k)}{2} \right| \right\} \right]^{k+1}$$

But since $|A''(\beta_k)| < 2$ for all k then $\max \left\{ \left| \frac{A''(\beta_0)}{2} \right|^{2^k}, \left| \frac{A''(\beta_1)}{2} \right|^{2^{k-1}}, ..., \left| \frac{A''(\beta_k)}{2} \right| \right\} < 1$ so the right hand side of the inequality approaches zero when $k \to \infty$.

$$\therefore \left| \frac{A''(\beta_0)}{2} \right|^{2^k} \times \left| \frac{A''(\beta_1)}{2} \right|^{2^{k-1}} \times ... \times \left| \frac{A''(\beta_k)}{2} \right| \to 0 \text{ as } k \to \infty$$

(d)

(i) From the definition of $A(x)$

$$A'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$= \frac{(1-\ln x) \times \frac{1}{x^2}}{\left(-\frac{1}{x}\right)^2}$$

$$= 1 - \ln x$$

$$A''(x) = -\frac{1}{x}$$

$$|A''(x)| = \frac{1}{x}$$

Since $2 < x < 3$ then

$\frac{1}{3} < \frac{1}{x} < \frac{1}{2}$

or equivalently

$\frac{1}{3} < |A''(x)| < \frac{1}{2}$

(ii) Since $|A''(x)| < 2$ and $|x_0 - e| < 1$ then from (c)(ii), each successive application of Newton's method will give a result that will ultimately converge to the actual root.

For ease, rewrite the first approximation x_0 as

$$x_0 = 2 \times 10^0 + 7 \times 10^{-1} + 1 \times 10^{-2}$$

The true root $x = e$ can be similarly written as

$$e = 2 \times 10^0 + 7 \times 10^{-1} + 1 \times 10^{-2} + 8 \times 10^{-3} + 2 \times 10^{-4} + 8 \times 10^{-5} + ...$$

Therefore

$$|x_0 - e| = 8 \times 10^{-3} + 2 \times 10^{-4} + 8 \times 10^{-5} + ...$$

The initial approximation of 2.71 is correct to the true root to **2 decimal places** because the error term has significant figures at the third decimal place.

Since $\frac{1}{6} < \left|\frac{A''(x)}{2}\right| < \frac{1}{4}$ then it is not certain (without doing the actual numerical computations) whether the factor $\left|\frac{A''(\beta_k)}{2}\right|$ alone will affect the number of decimal places.

For example, take 0.008 multipled by 0.2 which gives 0.0016 and the answer remains significant to 3 decimal places compared to 0.008. Only multiplying by a number less than or equal to 0.1 guarantees a shift in the number of decimal places.

This means that $\left(\frac{A''(x)}{2}\right)^2 < 0.1$ **will** change the number of decimal places upon multiplication.

Using the result in (c), note that after $k+1$ applications of Newton's method

$$|x_{k+1}-e| = (x_0-e)^{2^{k+1}} \times \left|\frac{A''(\beta_0)}{2}\right|^{2^k} \times ... \times \left|\frac{A''(\beta_{k-1})}{2}\right|^2 \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$= (8\times 10^{-3} + 2\times 10^{-4} + 8\times 10^{-5} + ...)^{2^{k+1}} \times \left[\left(\frac{A''(\beta_0)}{2}\right)^2\right]^{2^{k-1}} \times ... \times \left[\left(\frac{A''(\beta_{k-1})}{2}\right)^2\right]^1 \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$< (10\times 10^{-3} + 10\times 10^{-4} + 10\times 10^{-5} + ...)^{2^{k+1}} \times (10^{-1})^{2^{k-1}} \times ... \times (10^{-1})^1 \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$= (10^{-2} + 10^{-3} + 10^{-4} + ...)^{2^{k+1}} \times (10^{-1})^{1+2+4+...+2^{k-1}} \times \left|\frac{A''(\beta_k)}{2}\right|$$

$$= (10^{-2} + 10^{-3} + 10^{-4} + ...)^{2^{k+1}} \times (10^{-1})^{2^k-1} \times \left|\frac{A''(\beta_k)}{2}\right|$$

If the upper bound is expanded in full, the largest term is where the decimal place begins to have significant figures. Recall that $\left|\frac{A''(\beta_k)}{2}\right|$ has no known impact on the decimal places. The largest term is therefore some integer (between 0 and 9) multiplied by $10^{-(2\times 2^{k+1}+2^k-1)}$.

Thus, in order to guarantee the approximation is correct to 2016 decimal places, the error term must have significant figures at the 2017th or greater decimal place. Considering the powers of 10 in the bound for the error of the $(k+1)$th application of Newton's method above, this means that

$$2\times 2^{k+1} + 2^k - 1 \geq 2017$$

$$2^k \geq \frac{2018}{5}$$

$$k+1 > \frac{\ln\left(\frac{2018}{5}\right)}{\ln 2} + 1$$

The right hand side of the inequality is approximately 9.66. Thus, it requires at least 10 applications of Newton's method to guarantee the approximation to be correct to 2016 decimal places.