2023 **BORED OF STUDIES TRIAL EXAMINATION**

3rd October

Mathematics Extension 1

General instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5–10)

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the following table of values for a function $f(x)$.

x	$f(x)$	$f'(x)$
0	-1	1
1	0	2
2	2	3

Suppose that $f(x)$ has a unique inverse function $g(x)$ for all real x.
What is the value of $g'(0)$?

(A) -1

(B) 1

(C) $\frac{1}{3}$

(D) $\frac{1}{2}$

2 When the polynomial $P(x) = acx^3 + bcx + d$ is divided by $D(x)$, it gives cx with a remainder of d. What is $D(x)$?

(A) $-ax^2 - b$

(B) $-ax^2 + b$

(C) $ax^2 - b$

(D) $ax^2 + b$

3 What is the range of $y = \sin^{-1}\left(1 - \sqrt{1 + 2x^2}\right)$

(A) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

(B) $\left[-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right]$

(C) $\left[-\frac{\pi}{2}, 0\right]$

(D) $\left[0, \frac{\pi}{2}\right]$

4 Suppose that the vectors $a\underset{\sim}{i} + b\underset{\sim}{j}$ and $c\underset{\sim}{i} + d\underset{\sim}{j}$ are perpendicular. Which of the following *could* be true?

(A) $a < 0, b < 0, c < 0$ and $d < 0$

(B) $a < 0, b < 0, c > 0$ and $d > 0$

(C) $a > 0, b < 0, c > 0$ and $d < 0$

(D) $a > 0, b > 0, c < 0$ and $d > 0$

5 Let $(3x+1)^8 = c_0 + c_1x + c_2x^2 + \cdots + c_8x^8$ for some constants $c_0, c_1, c_2, \ldots, c_7$ and c_8. Which of the following has the largest value?

(A) c_1

(B) c_3

(C) c_5

(D) c_7

6 Suppose that $\sqrt{2}\sin x + \cos\left(x + \dfrac{\pi}{4}\right) + R\cos(x+\alpha) = 0$ where $R > 0$ and $0 < \alpha < 2\pi$.

What is the value of α?

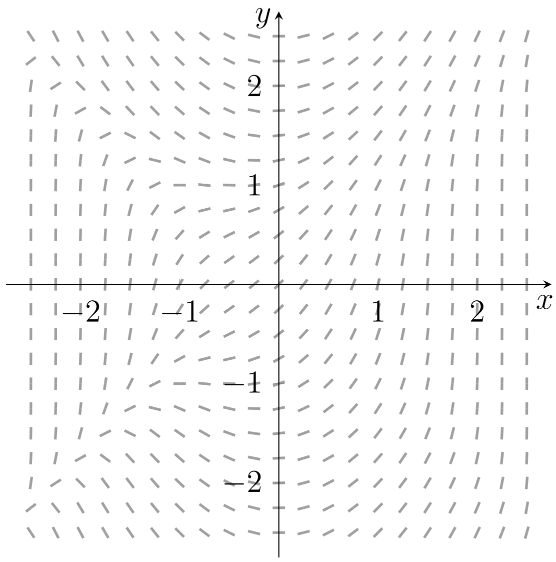
(A) $\dfrac{\pi}{4}$

(B) $\dfrac{3\pi}{4}$

(C) $\dfrac{5\pi}{4}$

(D) $\dfrac{7\pi}{4}$

7 Which of the following differential equations best represents the direction field below?



(A) $\dfrac{dy}{dx} = x + e^{x^2+y^2}$

(B) $\dfrac{dy}{dx} = x + e^{-x^2+y^2}$

(C) $\dfrac{dy}{dx} = x + e^{x^2-y^2}$

(D) $\dfrac{dy}{dx} = x + e^{-x^2-y^2}$

8 A group of 6 people have reserved seating tickets for a concert. The six seats are split into 4 adjacent seats in one row and the remaining 2 adjacent seats in another row.

There is a couple in the group who must sit next to each other. How many possible seating arrangements are there for the group?

(A) 48 (B) 120 (C) 144 (D) 192

9 Let $f(x) = 8\cos^6 x - 12\cos^4 x + 6\cos^2 x - 1$. Which expression is equal to $\displaystyle\int f(x)\,dx$?

(A) $\dfrac{1}{24}\sin 6x + \dfrac{3}{8}\sin 2x + c$

(B) $\dfrac{1}{12}\sin 6x + \dfrac{3}{4}\sin 2x + c$

(C) $\dfrac{(\sin 2x)^4}{4} + c$

(D) $\dfrac{(\sin 2x)^4}{8} + c$

10 Suppose that a polynomial $P(x)$ is such that there exists some $a > 0$ where

$$\int_0^a P(x)\,dx = 0.$$

Which of the following polynomials satisfies this condition?

(A) $-x(x^2+1)$ (B) $-x(x-1)(x-4)$

(C) $(x+1)(x-1)(x-2)$ (D) $(x+1)(x+2)(x+3)$

Section II

60 marks
Attempt Questions 11—14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet

(a) Find the projection of the vector $\underset{\sim}{i} - 2\underset{\sim}{j}$ onto the line $2x + 3y + 6 = 0$. **2**

(b) Show that $\cos 20° \cos 40° \cos 80° = \dfrac{1}{8}$. **1**

(c) The polynomial $P(x) = x^3 - 3x + 2$ has a double root.

(i) Find all the roots of $P(x)$. **2**

(ii) Hence, or otherwise, solve $\dfrac{x^2 - x + 2}{4x} \leq \dfrac{1}{x+1}$. **3**

(d) A circular metallic disc initially has a radius of 1 cm. As it gets heated, the rate at which its radius R expands at time t is given by

$$\frac{dR}{dt} = 10 - 2R.$$

(i) Find the specific solution to the differential equation of R in terms of t. **1**

(ii) When is the growth rate of disc's **area** the fastest? **2**

(e) A boat is sailing at constant velocity in a straight line across a lake.
At time t hours, its displacement in kilometres is given by the position vector $\underset{\sim}{r}$.
Let the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be directed at due east and due north respectively.
The boat is initially at $-5\underset{\sim}{i} + 10\underset{\sim}{j}$ and 3 hours later it is at $4\underset{\sim}{i} - 2\underset{\sim}{j}$.

(i) Find the displacement equation $\underset{\sim}{r}$ of the boat in terms of t. **2**

(ii) Find the velocity of the boat in terms of magnitude and direction in true bearings (to the nearest degree). **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet

(a) A multiple choice test of 25 questions is designed such that there are four possible options and only one correct answer for each question. Each correct answer gains 3 marks but each incorrect answer loses 1 mark.

Suppose a certain student chooses to make a non-serious attempt and answers all 25 questions at random.

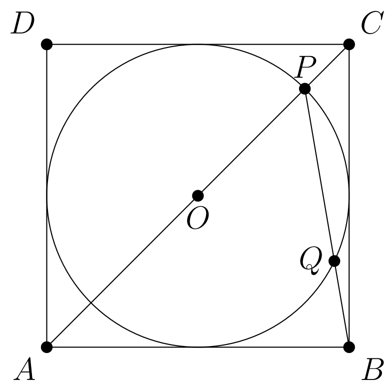
(i) Find the probability the student will get exactly 15 marks in total. **1**

(ii) Now suppose that for each question, the student can choose to make a serious attempt (not answering at random) or a non-serious attempt (answering at random), both of which are equally likely. **2**

If the student makes a serious attempt at the question, the probability that they get the question correct is $\frac{5}{12}$.

By considering a normal approximation, find the probability that the student will score at least 27 marks.

(b) A circle with centre O is inscribed in a square $ABCD$ with a side length of 1 unit as shown in the diagram below. Let P be the point on the circle where the diagonal AC intersects the circle and is closest to C. Let Q be the point where PB intersects the circle. Suppose that the vectors representing $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $\underset{\sim}{i}$ and $\underset{\sim}{j}$ respectively.



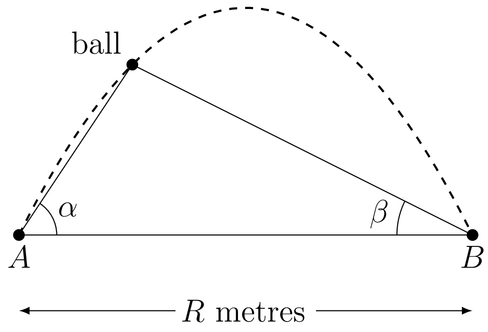
(i) Show that the vector presenting $\overrightarrow{BP}$ is $\left(\dfrac{\sqrt{2}-2}{4}\right)\underset{\sim}{i} + \left(\dfrac{\sqrt{2}+2}{4}\right)\underset{\sim}{j}$. **2**

(ii) Hence, show that $\cos\angle POQ = \dfrac{1}{3}$. **2**

Question 12 continues on page 7

(c) Student A launches a ball into the air from the ground at an angle of θ to the horizontal and an initial speed of u. Student B is on the ground R metres away from student A and will catch the ball.

Whilst the ball is in flight, student A and student B are both stationary and view the ball at an angle of elevation of α and β respectively at time t.



The displacement vector of the ball relative to the origin at student A is given by

$$\underset{\sim}{r} = ut\cos\theta\,\underset{\sim}{i} + \left(ut\sin\theta - \frac{gt^2}{2}\right)\underset{\sim}{j} \qquad \textbf{(Do NOT prove this)}$$

where g is the acceleration due to gravity.

(i) Show that $\tan\alpha$ decreases linearly over time. **1**

(ii) Show that $\tan\alpha + \tan\beta$ is independent of time. **2**

(d) It can be shown that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **(Do NOT prove this)**

(i) Use this result to show that **2**

$$\sin 7\theta + \sin\theta = 8\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

(ii) Hence, find the roots of $64x^6 - 112x^4 + 56x^2 - 7$. **1**

(iii) Deduce that **2**

$$\operatorname{cosec}^2\frac{\pi}{7} + \operatorname{cosec}^2\frac{2\pi}{7} + \operatorname{cosec}^2\frac{3\pi}{7} = 8.$$

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet

(a) By considering an appropriate t-formula, or otherwise, show that **2**

$$\tan^{-1} x = \begin{cases} -\dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{for } x < 0 \\[2ex] \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) & \text{for } x \geq 0. \end{cases}$$

(b) Let (x, y) be a point on the circle $x^2 + y^2 = r^2$ for some constant $r > 0$.
Let $f(z)$ be a probability density function on all real z, with a global maximum turning point at $(0, 1)$.

Suppose that $f(z)$ has the property that for some values x and y on the circle

$$g(r) = f(x)f(y)$$

where $g(r)$ is some function of r, independent of x and y.

(i) By considering the parametric equations of the circle, show that for $x \neq 0$ and $y \neq 0$ **2**

$$\frac{f'(x)}{xf(x)} = \frac{f'(y)}{yf(y)}.$$

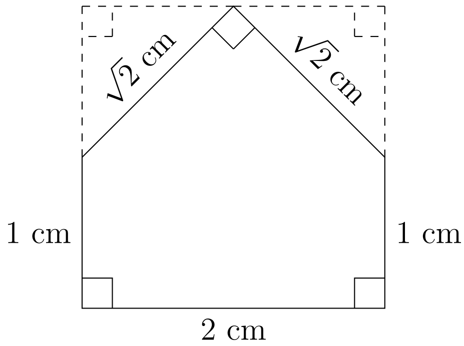
(ii) Explain why $\dfrac{f'(x)}{xf(x)} = -k$ for some constant $k > 0$. **2**

(iii) Hence, use the substitution $u = x\sqrt{k}$ to find a specific solution to the differential equation in part (ii). **3**

Question 13 continues on page 9

Question 13 (continued)

(c) Consider a pentagon with side lengths as shown in the diagram below. Note that the top vertex bisects the side length of a 2 cm by 2 cm square.



(i) Explain why the longest distance between any two points within the pentagon must be between two non-adjacent vertices. **1**

(ii) Hence, show that the longest distance between any two points within the pentagon is $\sqrt{5}$ cm. **1**

(iii) Six points are randomly placed within a 3 cm by 4 cm rectangle. Use the pigeonhole principle and the result of part (ii) to show that there exists at least one pair of points that are within $\sqrt{5}$ cm of each other. **1**

(d) Prove by mathematical induction that for positive integers n **3**

$$\frac{\sin x}{\cos x+\cos 3x}+\frac{\sin x}{\cos x+\cos 5x}+\cdots+\frac{\sin x}{\cos x+\cos(2n+1)x}=\frac{\tan(n+1)x-\tan x}{2}.$$

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet

(a) (i) By considering the graph of $y = x^2 - 3x + 3$, or otherwise, sketch the graph of $y = \dfrac{1}{x^2 - 3x + 3}$, showing any turning points and asymptotes. **2**

(ii) On the same set of axes, sketch the graph of $y = \tan^{-1}\left(\dfrac{1}{x^2 - 3x + 3}\right)$. **1**

(iii) Explain why the graphs of $y = \dfrac{1}{x^2 - 3x + 3}$ and $y = \tan^{-1}\left(\dfrac{1}{x^2 - 3x + 3}\right)$ do not have a point of intersection. **1**

(iv) Differentiate $(x + c)\tan^{-1}(x + c) - \ln\left(\sqrt{1 + (x + c)^2}\right)$ with respect to x for some constant c. **1**

(v) Hence, use the expansion of $\tan(A + B)$ to find the area between the curves **4**

$$y = \frac{1}{x^2 - 3x + 3} \quad \text{and} \quad y = \tan^{-1}\left(\frac{1}{x^2 - 3x + 3}\right)$$

for $1 \leq x \leq 2$.

(b) A soccer player performs a training drill where she needs to score exactly r goals. Her training session is complete after she scores her r^{th} goal. **2**

She is given a maximum of n attempts and manages to score r goals and complete her training session. By considering two different methods to count the number of ways she can do this, show that

$$\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \cdots + \binom{n-1}{r-1} = \binom{n}{r}.$$

(c) (i) Show that for integer $n \geq 4$ **1**

$$n^4 = 24\binom{n}{4} + 36\binom{n}{3} + 14\binom{n}{2} + \binom{n}{1}.$$

(ii) Hence, using the results in parts (b) and (c)(i), show *without* induction that **3**

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1).$$

End of paper

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