2014 Bored of Studies Trial Examinations

Mathematics Extension 1

Written by Carrotsticks & Trebla.

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11 – 14.

Total Marks – 70

Section I Pages 1 – 4

10 marks
- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II Pages 5 – 13

60 marks
- Attempt Questions 11 – 13
- Allow about 1 hour 45 minutes for this section.

Total marks – 10
Attempt Questions 1 – 10
All questions are of equal value

Shade your answers in the appropriate box in the Multiple Choice answer sheet provided.

---

1 The curves $y = x^2$ and $y = (x-2k)^2$, where $k > 0$, intersect at 45°.

How many possible values of k are there?

(A) One.

(B) Two.

(C) Four.

(D) None.

2 Consider the equation $a\sin\theta + b\cos\theta + c = 0$, where $0 \le \theta \le 2\pi$.

Let $\Delta = a^2 + b^2 - c^2$.

Which of the following statements is false?

(A) When $\Delta > 0$, there are always exactly two solutions.

(B) When $\Delta < 0$, there are always no solutions.

(C) When $\Delta = 0$, there is always exactly one solution.

(D) For any value of Δ, there are at most two solutions.

3 Let $P(x) = x^3 + px + q$, where p and q are constants, have roots α, β and γ.

Find the value of

$$\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\beta} + \frac{\beta}{\alpha}.$$

(A) -6.

(B) -3.

(C) 0.

(D) 3.

4 Two particles move in simple harmonic motion. The maximum velocity of Particle X is twice the maximum velocity of Particle Y.

Consider the following statements.

(I) If they have the same amplitude, Particle X has half the period of Particle Y.

(II) If they have the same amplitude, Particle X has twice the period of Particle Y.

(III) If they have the same period, Particle X has half the amplitude of Particle Y.

(IV) If they have the same period, Particle X has twice the amplitude of Particle Y.

Which of the following is correct?

(A) (I) and (III) are true.

(B) (I) and (IV) are true.

(C) (II) and (III) are true.

(D) (II) and (IV) are true.

5 Consider the following expressions.

(I) $(-1)^k \frac{\pi}{12} - \frac{k\pi}{2}$

(II) $(-1)^k \frac{\pi}{12} + \frac{k\pi}{2}$

(III) $(-1)^k \frac{\pi}{6} - k\pi$

Which of the following are NOT general solutions to $\sin 2x = \frac{1}{2}$, for some integer k?

(A) (I) and (II)

(B) (I) and (III)

(C) (II) and (III)

(D) (III)

6 A real polynomial $P(x)$ of degree n is divided by another polynomial $A(x)$ which has degree k, where $0 \le k \le n$. Let $R(x)$ be the remainder term from the division. Which of the following statements is always true?

(A) The degree of $R(x)$ is greater than k

(B) The degree of $R(x)$ is greater than or equal to k

(C) The degree of $R(x)$ is less than k

(D) The degree of $R(x)$ is less than or equal to k

7 What is the smallest number of people required in a room such the probability that at least two of people have the same birthday is at least 50%? Assume there are 365 days in a year.

(A) 23

(B) 24

(C) 25

(D) 26

8 Which of the following values of k allows the inequality $|x-a|+|x-b| \le k$ to have the solution $a \le x \le b$?

(A) $a-b$

(B) $b-a$

(C) $a+b$

(D) $|a|+|b|$

9 Consider a function $f(x)$ for some unrestricted domain. Let $f^{-1}(x)$ be an inverse function of $f(x)$. Which of the following statements is NOT necessarily true?

(A) $f(f^{-1}(x)) = x$

(B) $f^{-1}(f(x)) = x$

(C) If $f'(x)$ is zero at the point (a,b) then the derivative of $f^{-1}(x)$ is undefined at the point (b,a) for some fixed points (a,b) and (b,a)

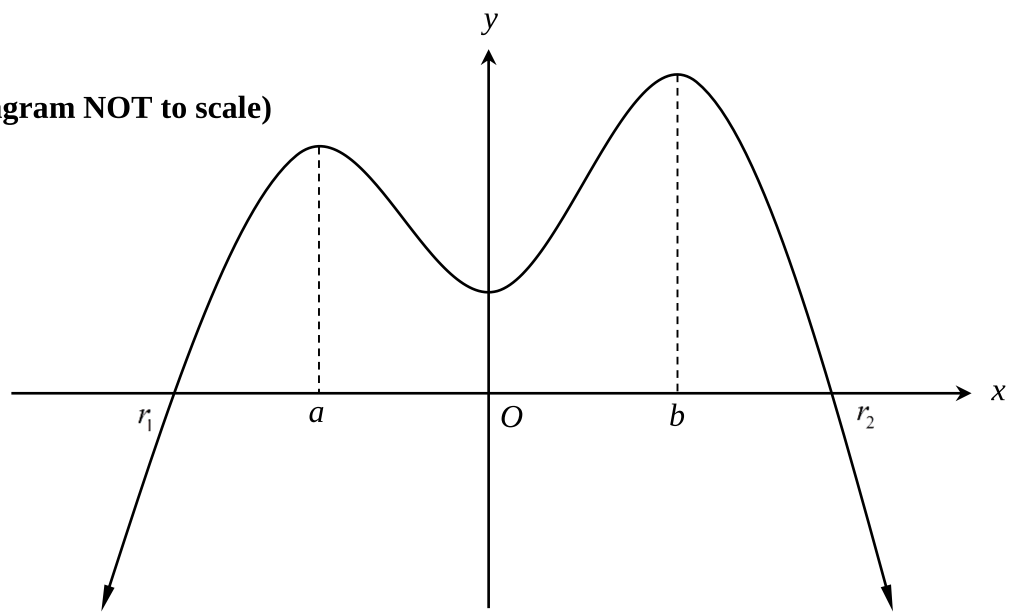
(D) If the derivative of $f^{-1}(x)$ is zero at the point (a,b) then $f'(x)$ is undefined at the point (b,a) for some fixed points (a,b) and (b,a).

10 The diagram below shows a polynomial with three stationary points and two real roots.

Newton's Method is used, with $x = x_0$ as the initial value.

Which of the following statements about Newton's Method is always false?

(Diagram NOT to scale)



(A) If $x_0 < a$, then further iterations can approach r_1.

(B) If $a < x_0 < b$, then further iterations cannot approach either r_1 or r_2.

(C) If $0 < x_0 < b$, then further iterations can approach r_1.

(D) If $x_0 > b$, then further iterations can approach r_1.

Total marks – 60
Attempt Questions 11 – 14
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the inequality $\dfrac{e^x}{1-e^x} \geq e^x - 1$ **3**

(b) Show that $\lim_{x \to 0}\left(\dfrac{\sin ax}{\sin bx}\right) = \dfrac{a}{b}$. **1**

(c) Suppose that $P(x_0, y_0)$ divides the interval $A(x_1, y_1)$ and $B(x_2, y_2)$. **3**

Show that if P divides the interval AB externally, then

$$(x_0 y_1 - x_1 y_0)(x_0 y_2 - x_2 y_0) > 0$$

(d) Sketch the function $f(x) = \cos^{-1}\left(\dfrac{1}{x^2}\right)$, labelling intercepts and asymptotes. **3**

(e)

(i) Use the substitution $x = a\sin\theta$, where $a > 0$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, to evaluate $\displaystyle\int \sqrt{a^2 - x^2}\, dx$. **4**

(ii) Hence, or otherwise, evaluate $\displaystyle\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx$. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The region bounded by the graph $y=\sqrt{\sin^4 x+\cos^4 x}$, the x axis and the lines $x=0$ and $x=\frac{\pi}{2}$ is rotated about the x axis to form a solid. **3**

Find the volume of the solid formed.

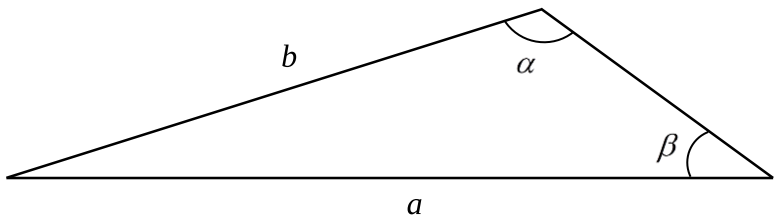
(b)

(i) Prove that **2**

$$\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta},$$

and hence write down a similar expression for $\tan\left(\frac{\alpha-\beta}{2}\right)$.

(ii) The diagram shows ΔABC with angles α and β where $\alpha>\beta$, and corresponding sides a and b respectively. **1**



Prove that

$$\frac{a+b}{a-b}=\frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)}.$$

Question 12 continues on the next page

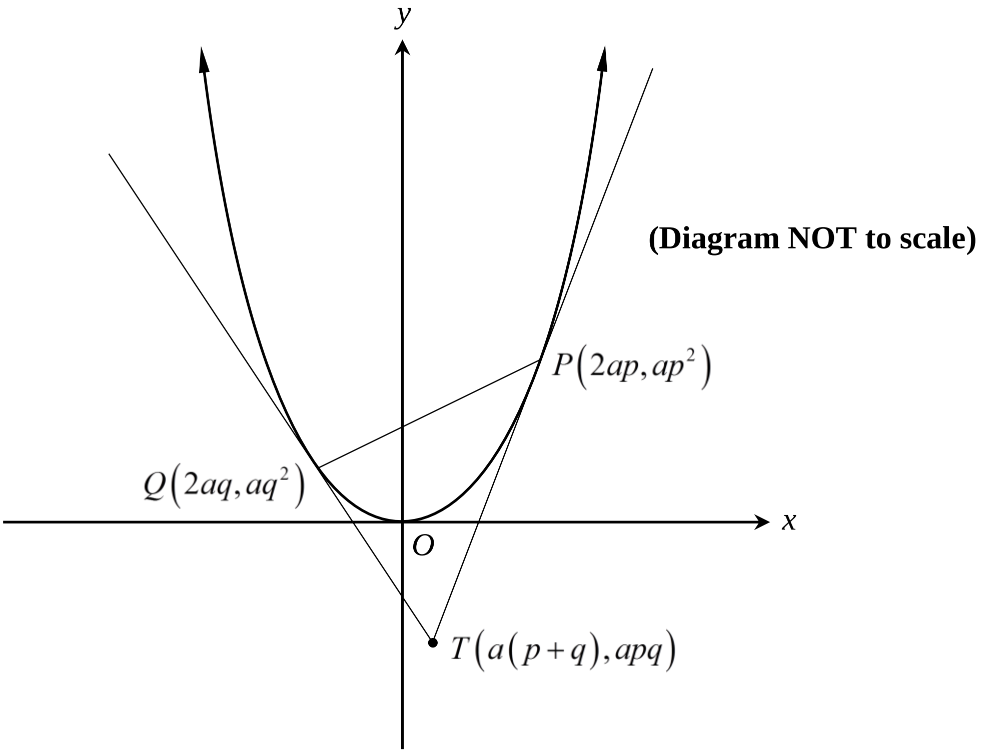
(c) The acceleration of a particle is given by $\ddot{x} = -2k(1-kx)$, where x is the displacement of the particle from the origin, in metres, and k is a positive constant.

The particle is initially at the origin, and has initial velocity $\dot{x} = \sqrt{2}$.

(i) Find the displacement-time equation of the particle. **3**

(ii) Hence, find the limiting value of x as $t \to \infty$. **1**

(d) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$, where $a > 0$ such that PQ is a focal chord. Tangents drawn at P and Q intersect at $T(a(p+q), apq)$ **(Do NOT prove this)**.



(i) Show that the area of ΔPQT is $\frac{a^2}{2}\left|p+\frac{1}{p}\right|^3$. **3**

(ii) The x value of P moves a rate of 1 unit per second. **2**

For what values of p is the area of the triangle increasing?

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a)

(i) Show that $(k+1)(k+2)(k+3)...(k+n)$ is divisible by $n!$, where n and k are positive integers. **1**

(ii) Let m be an arbitrary positive integer. **3**

Use mathematical induction on n to prove that $(mn)!$ is divisible by $(m!)^n$ for all positive integers m and n.

(b) An n-sided die is rolled m times, where $m \geq n$, and the number facing upwards is recorded.

(i) Show that the probability of acquiring a particular number exactly k times is given by **1**

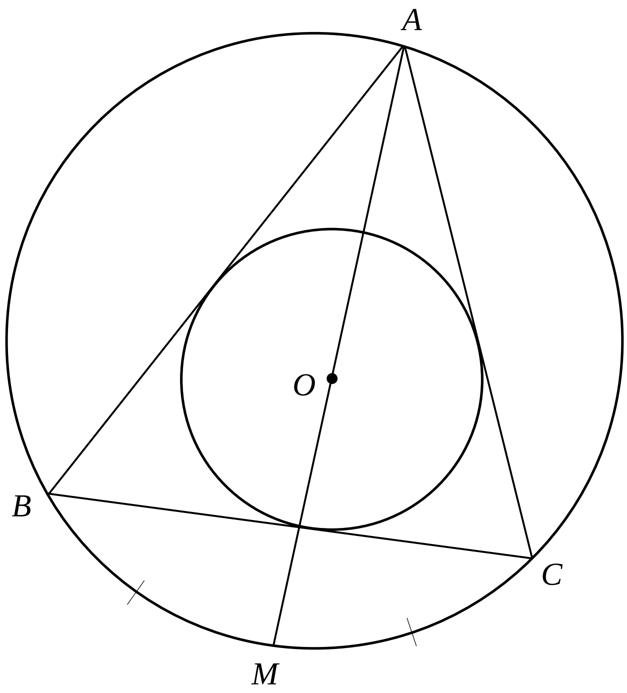
$$P(k) = \binom{m}{k}\frac{(n-1)^{m-k}}{n^m}$$

(ii) Deduce that if $m+1$ is not divisible by n, then the chosen number is most likely to appear approximately $\frac{m+1}{n}$ times. **2**

(iii) Explain why if $m+1$ is divisible by n, then the chosen number is most likely to appear exactly $\frac{m+1}{n}$ or $\frac{m-n+1}{n}$ times. **1**

Question 13 continues on the next page

(c) In ΔABC, a circle is inscribed such that it is tangential to all three sides. Let the centre of this circle be O. Another circle is drawn such that the vertices of ΔABC lie on the circumference of the circle. From vertex A, a line is drawn to O. From O, a line is drawn to meet the midpoint M of arc BC. **4**



Prove that ΔOCM is isosceles.

(d) Prove that for some positive integer $0 \leq k \leq n$. **3**

$$\sum_{k=1}^{n}\binom{k-1}{0}+\sum_{k=2}^{n}\binom{k-1}{1}+\sum_{k=3}^{n}\binom{k-1}{2}+\ldots+\sum_{k=n-1}^{n}\binom{k-1}{n-2}+\sum_{k=n}^{n}\binom{k-1}{n-1}=\sum_{k=1}^{n}\binom{n}{k}$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) An n digit password is made using digits 0, 1, 2, … , 9, with repetition allowed. **3**
The password is entered by pressing a series of buttons in the correct order.

An observer notices that Alice uses $1 \leq k \leq 10$ distinct numbers for her password.

How many possible combinations of numbers are there, in terms of k and n?

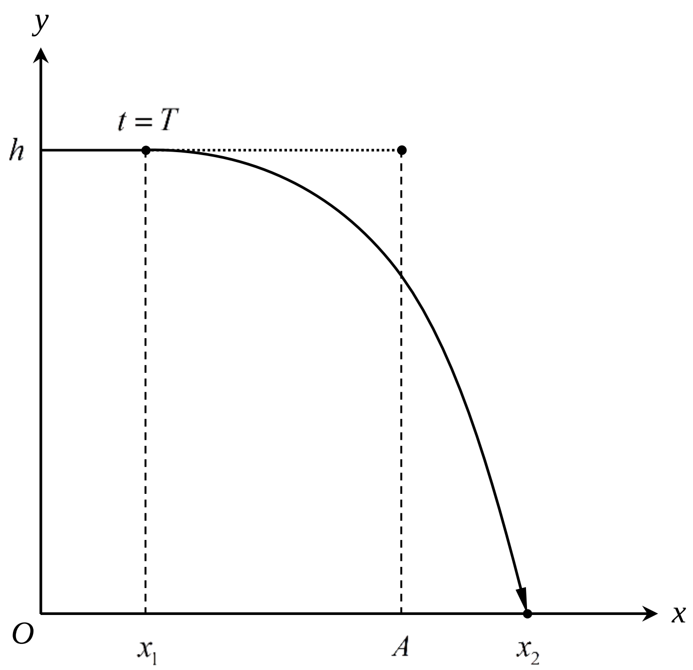
Justify your answer.

Question 14 continues on the next page

(b) A particle moves in simple harmonic motion h metres in the air along a horizontal rod. The particle's displacement is given by $x = A\sin(nt)$, where A, $n > 0$ and $0 \le nt \le \dfrac{\pi}{2}$.

At some moment $t = T$, the particle is released from the horizontal rod and undergoes projectile motion. The particle lands x_2 metres away from the origin.

Let t_0 denote the time between when the particle is released, to when the particle hits the ground.



(i) Prove that when $\tan(nT) = \dfrac{1}{nt_0}$, the horizontal range x_2 is maximised with respect to T. **4**

(ii) Hence, or otherwise, show that when x_2 is maximised, then $x_1 x_2 = A^2$. **2**

Question 14 continues on the next page

(c) Suppose that the growth at time t of a population of P (in millions) can be modeled by the differential equation

$$\frac{dP}{dt} = -aP\left(1 - \frac{P}{k}\right),$$

where a is a positive integer.

(i) Suppose that the initial population is $\frac{k}{2}$ million. **1**

Show that $P = \frac{k}{1+e^{-at}}$ satisfies the above differential equation.

(ii) Now consider the rate of change of another population N (in millions) **1**

which can be modeled by the differential equation

$$\frac{dN}{dt} = b(k - N),$$

where b is a positive integer.

The initial population of N is also $\frac{k}{2}$ million and the limiting population of N is the same as the limiting population of P.

Show that $N = k - \frac{k}{2}e^{-bt}$ satisfies the differential equation above.

(iii) Prove that if $b < a < 2b$, then there exists a time $t > 0$ where the two populations P and N will be equal to each other. **4**

End of Exam

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$