2018 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

Answers

1	D	**6**	A
2	D	**7**	C
3	B	**8**	A
4	C	**9**	B
5	D	**10**	A

Brief explanations

1 The inequality is equivalent to $-4 < \dfrac{x^2+4}{x} < 4$. Simplifying the left inequality gives $\dfrac{x^2+4x+4}{x} > 0$ or equivalently $x(x+2)^2 > 0$ which has a solution of $x > 0$. Using a similar approach with the right inequality gives $x(x-2)^2 < 0$ which has a solution of $x < 0$. It is not possible for left and right inequalities to hold at the same time, hence there are no real solutions so the answer is (D).

2 Use the identities $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta = \sec^2\theta - 1$ and $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ which simplify the expression to $2\sec^2\theta + 2\operatorname{cosec}^2\theta - 1$. Using $\sin 2\theta = 2\sin\theta\cos\theta$ and further simplification leads to $\dfrac{8}{\sin^2 2\theta} - 1$. The minimum value of this occurs when $\sin 2\theta$ is maximum. Hence, the answer is 7 or (D).

3 The general term of the binomial expansion is $\dbinom{n}{k} x^{n-k}\left(\dfrac{1}{x^2}\right)^k$, or equivalently, $\dbinom{n}{k} x^{n-3k}$. A non-zero constant term occurs when $n = 3k$. Since k is an integer, this suggests that n must be divisible by 3. The only choice which satisfies this is 2016, hence the answer is (B).

4 Notice that the equation of the tangent of $y = \tan^{-1}(ax)$ at the origin is $y = ax$. Consider the graph of $y = \tan^{-1}(ax)$. If $a > 0$, then for three points of intersection the gradient of $y = bx$ must be less than the gradient of the tangent at the origin so $a > b$. If $a < 0$, then for three points of intersection the gradient of $y = bx$ must be lower in *magnitude* compared to the gradient of the tangent at the origin. However, since the a and b must both be negative then $a < b < 0$. Hence, the answer is (C).

5 Substitute $(0, ka)$ into the equation of the normal leads to $kap = 2ap + ap^3$, or equivalently, $ap(p^2 + 2 - k) = 0$. Since $k > 2$ there must be three real solutions for p (namely $p = 0, \pm\sqrt{k-2}$), which correspond to 3 different possible equations of the normals. Hence, the answer is (D).

6 Since the integrand is non-negative in the domain and we are choosing a value for just a, then the value of a which gives the largest integrand will result in the largest value of the definite integral, which is equivalent to the smallest denominator of the integrand. Hence, the answer is (A).

7 The general solution to $\sin x = \dfrac{1}{2}$ is $x = n\pi + (-1)^n \dfrac{\pi}{6}$ for integer n. Since the largest domain amongst the choices is $-4\pi \leq x \leq 4\pi$, then consider the solutions when $n = -4, -3, -2, -1, 0, 1, 2, 3$. These are

$$x = -\frac{23\pi}{6}, -\frac{19\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

Testing each of the choices, it can be seen that the answer is (C).

8 Let the parametric coordinates of the points on the parabola $y^2 = 4ax$ which intersect the chord be $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$. Since $OP \perp OQ$ then $\dfrac{2ap}{ap^2} \times \dfrac{2aq}{aq^2} = -1$ which simplifies to $pq = -4$. Substitute the equation of the chord $x = -by - c$ into the parabola $y^2 - 4ax = 0$ which gives $y^2 + 4aby + 4ac = 0$. Since the y-coordinates of P and Q must be roots of this equation, then using the product of the roots $2ap \times 2aq = 4ac$, which means $4a + c = 0$. Hence, the answer is (A).

9 Manipulate the angle as follows

$$\begin{aligned}
\sin(x) + \sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{2\pi}{3}\right) &= \sin\left(x + \frac{\pi}{3} - \frac{\pi}{3}\right) + \sin\left(x + \frac{\pi}{3}\right) + \sin\left(x + \frac{\pi}{3} + \frac{\pi}{3}\right) \\
&= 2\sin\left(x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\left(x + \frac{\pi}{3}\right) \\
&= 2\sin\left(x + \frac{\pi}{3}\right)
\end{aligned}$$

This has a maximum value of 2 so the answer is (B).

10 The probability of choosing at most one black marble is equal to choosing at most one white marble. Since these are complementary outcomes then the probability of either must be $\frac{1}{2}$, so the answer is (A).

Question 11

(a) Let $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta$. When $x = 1$, $\theta = 0$ and when $x = 0$, $\theta = \dfrac{\pi}{4}$

$$\begin{aligned}
\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx &= -2\int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times \sin 2\theta\, dx \\
&= 2\int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \times 2\sin\theta\cos\theta\, dx \\
&= 4\int_0^{\frac{\pi}{4}} \tan\theta \sin\theta\cos\theta\, dx \qquad \text{noting that } \tan\theta \ge 0 \text{ for } 0 \le x \le \frac{\pi}{4} \\
&= 4\int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta \\
&= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\, d\theta \\
&= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2} - 1
\end{aligned}$$

(b) Using the fact that $a = \dfrac{1}{2}\dfrac{dv^2}{dx}$ and $v = \dfrac{dv}{dt}$

$$\begin{aligned}
K &= \frac{1}{2}mv^2 \\
\frac{dK}{dt} &= \frac{1}{2}m\frac{dv^2}{dt} \\
&= \frac{1}{2}m\frac{dv^2}{dx}\frac{dx}{dt} \\
&= mav
\end{aligned}$$

(c)

$$\begin{aligned}
P(x) &= ax^3 + bx^2 + cx + d \\
P'(x) &= 3ax^2 + 2bx + c \\
P''(x) &= 6ax + 2b
\end{aligned}$$

The point of inflexion occurs when $P''(x) = 0$ which gives $x = -\dfrac{b}{3a}$. Since $P''(x)$ is a linear function, then the x-values on either side of $x = -\dfrac{b}{3a}$ give a different sign for $P''(x)$, which confirms the point of inflexion. Using the relationship between the sum of roots and coefficients $x_1 + x_2 + x_3 = -\dfrac{b}{a}$, the point of inflexion therefore occurs when

$$x = \frac{x_1 + x_2 + x_3}{3}$$

(d) Since $P(x)$ has a double root at $x = \alpha$ then $P(x) = (x-\alpha)^2 g(x)$ for some polynomial $g(x)$. From Newton's method

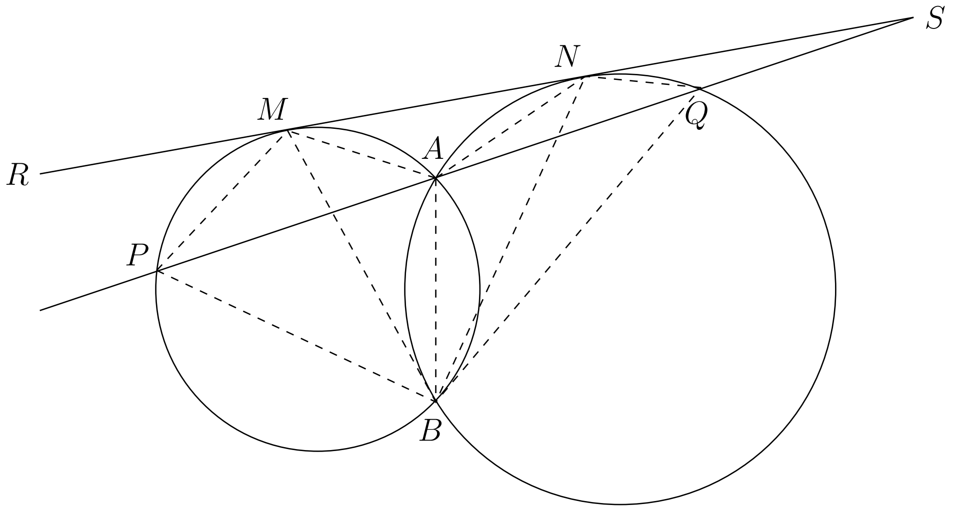
$$\begin{aligned}
x_n &= x_{n-1} - \frac{P(x_{n-1})}{P'(x_{n-1})} \\
x_n - \alpha &= x_{n-1} - \alpha - \frac{(x_{n-1}-\alpha)^2 g(x_{n-1})}{2(x_{n-1}-\alpha)g(x_{n-1}) + (x_{n-1}-\alpha)^2 g'(x_{n-1})} \\
\frac{x_n - \alpha}{x_{n-1} - \alpha} &= 1 - \frac{g(x_{n-1})}{2g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1})} \\
&= \frac{2g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1}) - g(x_{n-1})}{2g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1})} \\
&= \frac{g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1})}{2g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1})}
\end{aligned}$$

However, in the given domain $P(x)$ is concave up which means that graphically $P(x) \geq 0$ (at least in this domain), in order to have the double root at $x = \alpha$. Since $(x-\alpha)^2 \geq 0$, this implies that $g(x) \geq 0$ in the domain, so it can be deduced that

$$g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1}) \leq 2g(x_{n-1}) + (x_{n-1}-\alpha)g'(x_{n-1})$$

Hence, $\left|\dfrac{x_n - \alpha}{x_{n-1} - \alpha}\right| < 1$

(e) Construct the lines BM, BN, BP, BQ, AM, AN and AB. Define the points R and S as shown below.



Let $\angle RMP = \alpha$, $\angle QNS = \beta$, $\angle NMA = x$ and $\angle MNA = y$.

$$\begin{aligned}
\angle RMP &= \angle MBP \quad \text{(angle in alternate segment)} \\
\angle MBP &= \angle MAP \quad \text{(angle in same segment)} \\
\therefore \angle MAP &= \alpha \\
\text{Similarly } \angle NAQ &= \beta \\
\angle NMA &= \angle ABM \quad \text{(angle in alternate segment)} \\
\therefore \angle ABM &= x \\
\text{Similarly } \angle ABN &= y
\end{aligned}$$

$$
\begin{aligned}
\angle MAN &= \pi - (x+y) \quad \text{(angle sum of triangle)} \\
\angle MAP + \angle MAN + \angle NAQ &= \pi \quad \text{(angles on a straight line)} \\
\Rightarrow \alpha + \beta &= x + y \\
\angle PBQ &= \angle MBP + \angle ABM + \angle ABN + \angle NBQ \\
&= \alpha + \beta + x + y \\
&= 2(x+y) \\
\angle MBN &= \angle ABM + \angle ABN \\
&= x + y \\
\therefore \angle PBQ &= 2 \times \angle MBN
\end{aligned}
$$

(f) Let $\alpha = \sqrt{\dfrac{x(x+y+z)}{yz}}$, $\beta = \sqrt{\dfrac{y(x+y+z)}{xz}}$ and $\gamma = \sqrt{\dfrac{z(x+y+z)}{xy}}$

$$
\begin{aligned}
&\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma \\
&= \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) + \pi + \tan^{-1}\gamma \quad \text{since } \alpha\beta = \sqrt{\frac{xy(x+y+z)^2}{xyz^2}} = \frac{x+y+z}{z} > 1 \\
&= \tan^{-1}\left(\frac{\frac{\alpha+\beta}{1-\alpha\beta} + \gamma}{1-\gamma\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)}\right) + \pi \quad \text{since } \gamma\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) < 0 < 1 \text{ given } \alpha\beta > 1 \text{ and } \gamma > 0 \\
&= \tan^{-1}\left(\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\alpha\gamma-\beta\gamma}\right) + \pi
\end{aligned}
$$

But

$$
\begin{aligned}
\alpha+\beta+\gamma &= \sqrt{\frac{x(x+y+z)}{yz}} + \sqrt{\frac{y(x+y+z)}{xz}} + \sqrt{\frac{z(x+y+z)}{xy}} \\
&= \sqrt{x+y+z}\left(\sqrt{\frac{x}{yz}} + \sqrt{\frac{y}{xz}} + \sqrt{\frac{z}{xy}}\right) \\
&= (x+y+z)\sqrt{\frac{x+y+z}{xyz}} \\
&= \sqrt{\frac{xyz(x+y+z)^3}{x^2y^2z^2}} \\
&= \alpha\beta\gamma
\end{aligned}
$$

which implies that $\tan^{-1}\left(\dfrac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-\alpha\beta-\alpha\gamma-\beta\gamma}\right) = 0$ hence

$$
\tan^{-1}\sqrt{\frac{x(x+y+z)}{yz}} + \tan^{-1}\sqrt{\frac{y(x+y+z)}{xz}} + \tan^{-1}\sqrt{\frac{z(x+y+z)}{xy}} = \pi.
$$

Question 12

(a) (i) For $k > 0$

$$\begin{aligned} f(x) &= x + \frac{k^2}{x} \\ f'(x) &= 1 - \frac{k^2}{x^2} \\ f''(x) &= \frac{2k^2}{x^3} \end{aligned}$$

Since $f''(k) > 0$ and $f''(-k) < 0$ then $(k, 2k)$ and $(-k, -2k)$ are minimum and maximum turning points respectively.

(ii) Since $x = 0$ is an asymptote that the curve approaches, then the minimum turning point is the only turning point in the domain $x > 0$. Similarly, the maximum turning point is the only turning point in the domain $x < 0$.

Hence the range of $f(x)$ is $y \geq 2k$ and $y \leq -2k$.

(b) (i) The equations of the tangents to P and Q are

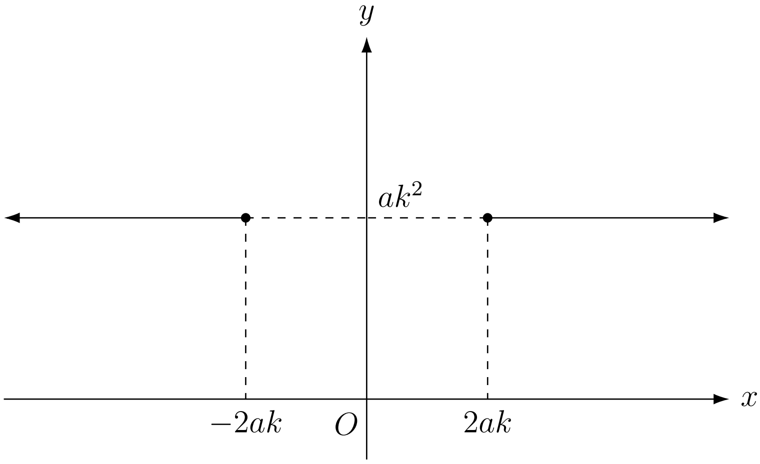
$$\begin{aligned} y &= px - ap^2 \\ y &= qx - aq^2 \\ x(p-q) - a(p^2 - q^2) &= 0 \\ x &= a(p+q) \qquad \text{noting that } p \neq q \\ y &= ap(p+q) - ap^2 \\ &= apq \end{aligned}$$

Hence the coordinates of R are $(a(p+q), apq)$.

(ii) Since pq is a positive constant then let $pq = k^2$ for some constant k. This means that

$$\begin{aligned} x &= a\left(p + \frac{k^2}{p}\right) \\ y &= ak^2 \end{aligned}$$

As p varies, there is a restriction on x as shown in part (a). In this case, $x \geq 2ak$ and $x \leq -2ak$ is the domain of the locus $y = ak^2$.



(c) There are $2 \times E$ and $3 \times S$, with the remaining letters (P, R and O) only appearing once. The following cases need to be considered when selecting four letters:

Case 1: All four letters are distinct

There are five distinct letters to choose from to make a four-letter code, which means there are $\binom{5}{4}$ possible combinations. Since the ordering of the code matters, each combination has 4! ways of rearrangement. Hence, there are a total of $\binom{5}{4} \times 4!$, or equivalently, 120 possible codes.

Case 2: One letter is repeated twice and the other two are distinct

The only letters that could possibly repeat twice are E and S, which means there are $\binom{2}{1}$ combinations of repeated letters possible. For a given choice of repeated letter, there are now 4 remaining letters to choose two from (to ensure the other letter is repeated twice only), which has $\binom{4}{2}$ possible combinations. However, since the ordering of the code matters, each combination has $\frac{4!}{2!}$ ways of arrangement. Hence there are total of $\binom{2}{1} \times \binom{4}{2} \times \frac{4!}{2!}$, or equivalently, 144 possible codes.

Case 3: One letter is repeated three times and the other one is distinct

This is only possible if S is repeated three times. This leaves 4 remaining letters to choose one from, which has $\binom{4}{1}$ combinations. However, since the ordering of the code matters, each combination has $\frac{4!}{3!}$ ways of arrangement. Hence there are a total of $\binom{4}{1} \times \frac{4!}{3!}$, or equivalently, 16 possible codes.

Case 4: One letter is repeated twice and the other one is repeated twice

This is only possible with the one combination which is two E and two S. Again, since the ordering of the code matters, there are $\frac{4!}{2!2!}$, or equivalently, 6 possible codes.

Combining all the different cases together there are 286 possible four-letter codes.

(d) *Method 1*: By writing 7 as 10 − 3 and using the binomial expansion

$$\begin{aligned}
7^{100} - 3^{100} &= (10-3)^{100} - 3^{100} \\
&= \binom{100}{0}10^{100} + \binom{100}{1}10^{99}3^1 + ... + \binom{100}{99}10^1 3^{99} + \binom{100}{100}3^{100} - 3^{100} \\
&= \binom{100}{0}10^{100} + \binom{100}{1}10^{99}3^1 + ... + \binom{100}{99}10^1 3^{99} \\
&= 10^3\left[\binom{100}{0}10^{97} + \binom{100}{1}10^{96}3^1 + ... + \binom{100}{97}3^{97}\right] + \binom{100}{98}10^2 3^{98} + \binom{100}{99}10^1 3^{99}
\end{aligned}$$

But $\binom{100}{99} = 100$ and $\binom{100}{98} = 4950$ so

$$\begin{aligned}
\binom{100}{98}10^2 3^{98} + \binom{100}{99}10^1 3^{99} &= 10^3 \times 495 \times 3^{98} + 10^3 \times 3^{99} \\
&= 10^3(495 \times 3^{98} + 3^{99})
\end{aligned}$$

$$\therefore 7^{100} - 3^{100} = 10^3\left[\binom{100}{0}10^{97} + \binom{100}{1}10^{96}3^1 + ... + \binom{100}{3}3^{97} + 495 \times 3^{98} + 3^{99}\right]$$

Hence $7^{100} - 3^{100}$ is divisible by 1000.

Method 2: By writing 7 as $5+2$ and 3 as $5-2$, then using the binomial expansion

$$\begin{aligned}
7^{100} - 3^{100} &= (5+2)^{100} - (5-2)^{100} \\
&= \sum_{k=0}^{100}\binom{100}{k}5^{100-k}2^k - \sum_{k=0}^{100}\binom{100}{k}5^{100-k}(-2)^k \\
&= \sum_{k=0}^{100}\binom{100}{k}5^{100-k}\left(2^k - (-2)^k\right)
\end{aligned}$$

The even terms cancel out, leaving behind odd terms which double up so

$$\begin{aligned}
7^{100} - 3^{100} &= 2\binom{100}{1}5^{99}2^1 + 2\binom{100}{3}5^{97}2^3 + 2\binom{100}{5}5^{95}2^5 + ... + 2\binom{100}{99}5^1 2^{99} \\
&= \binom{100}{1}5^{99}2^2 + \binom{100}{3}5^{97}2^4 + \binom{100}{5}5^{95}2^6 + ... + \binom{100}{99}5^1 2^{100} \\
&= 10^3\left[\binom{100}{3}5^{94}2^1 + \binom{100}{5}5^{92}2^3 + ... + \binom{100}{97}2^{95}\right] + \binom{100}{1}5^{99}2^2 + \binom{100}{99}5^1 2^{100}
\end{aligned}$$

Note that $\binom{100}{1} = \binom{100}{99} = 100$ so

$$\begin{aligned}
\binom{100}{1}5^{99}2^2 + \binom{100}{99}5^1 2^{100} &= 100(5^{98}2^1 \times 10 + 2^{99} \times 10) \\
&= 1000(5^{98}2^1 + 2^{99})
\end{aligned}$$

Hence

$$7^{100} - 3^{100} = 10^3\left[5^{98}2^1 + \binom{100}{3}5^{94}2^1 + \binom{100}{3}5^{94}2^1 + ... + \binom{100}{97}2^{95} + 2^{99}\right]$$

Therefore $7^{100} - 3^{100}$ is divisible by 1000.

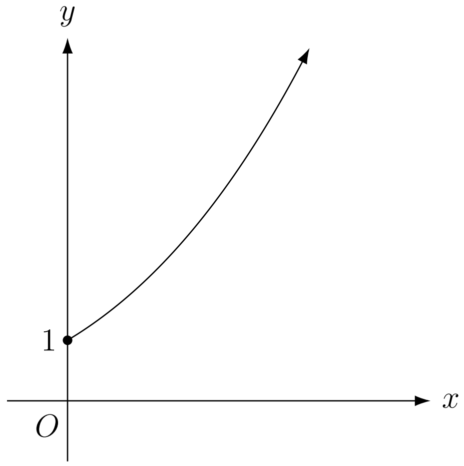
(e) (i) The differential equation $\dfrac{dx}{dt} = x + 1$ is a modified exponential growth and so has general solution $x = -1 + Ae^t$ for some constant A.

When $t = 0$, $x = 0$ so $A = 1$ hence $x = e^t - 1$.

(ii) The differential equation $\dfrac{dy}{dt} = 3y$ is an exponential growth equation with general solution $y = Be^{3t}$ for some constant B.

When $t = 0$, $y = 1$ so $B = 1$ hence $y = (e^t)^3$. Since $x = e^t - 1$ then $y = (x + 1)^3$.

(iii) Since $t \geq 0$, then $e^t - 1 \geq 0$ hence the path $y = (x + 1)^3$ has a domain of $x \geq 0$ and consequently a range of $y \geq 1$.



Question 13

(a) Using the given expression

$$\begin{aligned} T(t) &= R + (T(0) - R)e^{-kt} \\ \frac{dT}{dt} &= -k(T(0) - R)e^{-kt} \qquad \text{but } T - R = (T(0) - R)e^{-kt} \\ &= -k(T - R) \end{aligned}$$

Hence $T(t) = R + (T(0) - R)e^{-kt}$ is a solution to the differential equation.

(b) (i) Since the temperature of the coffee satisfies Newton's law of cooling then from part (a)

$$\begin{aligned} C_1(t) &= R + (C_1(0) - R)e^{-kt} \\ &= R + (\alpha C_0(0) + (1 - \alpha)M(0) - R)e^{-kt} \end{aligned}$$

(ii) Using the result from part (i)

$$\begin{aligned} C_1(t_0) - C_2(t_0) &= R + (\alpha C_0(0) + (1 - \alpha)M(0) - R)e^{-kt_0} - \alpha C_0(t_0) - (1 - \alpha)M(t_0) \\ &= R(1 - e^{-kt_0}) + \alpha(C_0(0)e^{-kt_0} - C_0(t_0)) + (1 - \alpha)(M(0)e^{-kt_0} - M(t_0)) \end{aligned}$$

But since $C_0(t)$ satisfies Newton's law of cooling then from part (a)

$$\begin{aligned} C_0(t) &= R + (C_0(0) - R)e^{-kt} \\ C_0(t_0) - C_0(0)e^{-kt_0} &= R(1 - e^{-kt_0}) \end{aligned}$$

Thus

$$\begin{aligned} C_1(t_0) - C_2(t_0) &= R(1 - e^{-kt_0}) - \alpha(R(1 - e^{-kt_0})) + (1 - \alpha)(M(0)e^{-kt_0} - M(t_0)) \\ &= (1 - \alpha)\left[R(1 - e^{-kt_0}) - M(t_0) + M(0)e^{-kt_0}\right] \end{aligned}$$

(iii) Since $M(t)$ also satisfies Newton's law of cooling then from part (a)

$$\begin{aligned} M(t) &= R + (M(0) - R)e^{-k_M t} \\ M(t_0) - M(0)e^{-kt_0} &= R(1 - e^{-k_M t_0}) \end{aligned}$$

Hence from part (ii)

$$\begin{aligned} C_1(t_0) - C_2(t_0) &= (1 - \alpha)\left[R(1 - e^{-kt_0}) - R(1 - e^{-k_M t_0})\right] \\ &= R(1 - \alpha)(e^{-k_M t_0} - e^{-kt_0}) \end{aligned}$$

If $k_M < k$ then $-k_M t_0 > -kt_0$ hence $e^{-k_M t_0} > e^{-kt_0}$. Since $0 < \alpha < 1$ and $R > 0$ then $C_1(t_0) > C_2(t_0)$.

(iv) If $k_M < k$ then the rate of change of the temperature of the milk warming is slower than the rate of change of the temperature of the coffee cooling. Under this condition, $C_1(t_0) > C_2(t_0)$, which means that the adding the milk initially and waiting until $t = t_0$ leads to a warmer coffee compared to waiting first and adding the milk later at $t = t_0$.

(c) When $n = 4$, the left hand side is $\ln 6!$ which is approximately 6.58 which is greater than 6 on the right hand side. Hence the statement is true for $n = 4$.

Assume the statement is true for some $n = k$

$$\ln[(k+2)!] > k+2$$

Required to prove the case for $n = k+1$

$$\ln[(k+3)!] > k+3$$

$$\begin{aligned}
LHS &= \ln[(k+3)!] \\
&= \ln[(k+3)(k+2)!] \\
&= \ln(k+3) + \ln[(k+2)!] \\
&> \ln(k+3) + k + 2 \quad \text{by assumption} \\
&> 1 + k + 2 \quad \text{since } k \geq 4 \text{ then} \ln(k+3) \geq \ln(4+3) > 1 \\
&= k+3 \\
&= RHS
\end{aligned}$$

Hence, the statement is true by induction.

(d) Since $g(x)$ has a period of T and $h(x)$ has a period of $2T$ then

$$\begin{aligned}
g(x+T) &= g(x) \\
h(x+2T) &= h(x)
\end{aligned}$$

Define $p(x) = g(x) + h(x)$ then

$$\begin{aligned}
p(x+2T) &= g(x+2T) + h(x+2T) \quad \text{but } g(x+2T) = g((x+T)+T) = g(x+T) \\
&= g(x+T) + h(x) \\
&= g(x) + h(x)
\end{aligned}$$

Hence $g(x) + h(x)$ has a period of $2T$.

Remark

Since it is only given that $h(x) = h(x+2T)$ then it cannot be deduced that $h(x) = h(x+T)$ for all real x. That is, it is possible there are some values of x where $h(x) \neq h(x+T)$. This means that we cannot claim that $g(x) + h(x) = g(x+T) + h(x+T)$ for all real x.

(e) (i) Using the definition of y_1

$$\begin{aligned}
y_1 &= x_A + x_B \\
\ddot{y_1} &= \ddot{x_A} + \ddot{x_B} \\
&= \frac{1}{5}\left(-5x_A - 3x_B\right) + \frac{1}{5}\left(-5x_B - 3x_A\right) \\
&= -\frac{1}{5}(8x_A + 8x_B) \\
&= -\frac{8}{5}y_1
\end{aligned}$$

Since this is in the form $\ddot{y_1} = -n^2 y_1$ for positive constant n then y_1 satisfies the acceleration equation for simple harmonic motion.

Using the definition of y_2

$$\begin{aligned}
y_2 &= x_A - x_B \\
\ddot{y_2} &= \ddot{x_A} - \ddot{x_B} \\
&= \frac{1}{5}\left(-5x_A - 3x_B\right) - \frac{1}{5}\left(-5x_B - 3x_A\right) \\
&= \frac{1}{5}(-2x_A + 2x_B) \\
&= -\frac{2}{5}y_2
\end{aligned}$$

Using a similar argument, y_2 satisfies the acceleration equation for simple harmonic motion.

(ii) By solving simultaneously for x_A and using the general displacement equations for simple harmonic motion

$$\begin{aligned}
x_A &= \frac{1}{2}(y_1 + y_2) \\
&= \frac{1}{2}\left(A_1 \cos\left(n_1 t + \alpha_1\right) + A_2 \cos\left(n_2 t + \alpha_2\right)\right)
\end{aligned}$$

for some constants $A_1, A_2, n_1, n_2, \alpha_1$ and α_2.

From part (i), it can deduced that $n_1 = \sqrt{\frac{8}{5}}$ and $n_2 = \sqrt{\frac{2}{5}}$ which means that the periods of y_1 and y_2 are $\pi\sqrt{\frac{5}{2}}$ and $2\pi\sqrt{\frac{5}{2}}$ respectively.

Since the period of y_2 is twice the period of y_1 then using the result in (d), it must be that the period of $y_1 + y_2$ is $2\pi\sqrt{\frac{5}{2}}$. Since multiplying the displacement equations by $\frac{1}{2}$ has has no effect on the period, then $\frac{1}{2}(y_1+y_2)$, or equivalently x_A, must have a period of $2\pi\sqrt{\frac{5}{2}}$, which simplifies to $\pi\sqrt{10}$.

Question 14

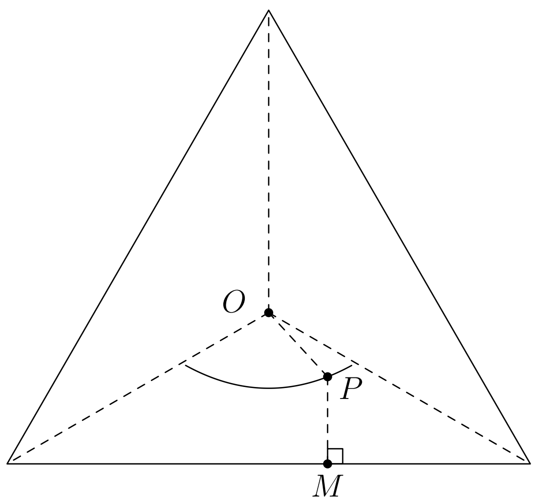
(a) (i) First find the x-coordinate of the points of intersection by solving

$$\begin{aligned} x^2\sqrt{3} + \frac{1}{4\sqrt{3}} &= \frac{x}{\sqrt{3}} + \frac{1}{2\sqrt{3}} \\ 12x^2 - 4x - 1 &= 0 \\ (6x+1)(2x-1) &= 0 \\ x &= -\frac{1}{6} \quad \text{since } x < 0 \end{aligned}$$

Hence, the area A is given by

$$\begin{aligned} A &= \int_{-\frac{1}{6}}^{0} \left(\frac{x}{\sqrt{3}} + \frac{1}{2\sqrt{3}}\right) dx - \int_{-\frac{1}{6}}^{0} \left(x^2\sqrt{3} + \frac{1}{4\sqrt{3}}\right) dx \\ &= \frac{1}{4\sqrt{3}} \int_{-\frac{1}{6}}^{0} \left(1 + 4x - 12x^2\right) dx \\ &= \frac{1}{4\sqrt{3}} \left[x + 2x^2 - 4x^3\right]_{-\frac{1}{6}}^{0} \\ &= \frac{5}{216\sqrt{3}} \end{aligned}$$

(ii) First consider the case where the closest side to the dart is the base of the triangle. The boundary of the area $\mathcal{R}$ is the case when $OP = PM$. The locus of this boundary is in fact a parabola where O is the focus and the side of the triangle is a directrix.



The case where $OP < OM$ is the area needed. In this case, it is twice the area found in part (i) (see remark for full explanation).

By symmetry the total area (define this as A_R) is six times the area found in part (i).

Let the area of the equilateral triangle board be A_T, then the ratio of the areas is

$$\begin{aligned}\frac{A_R}{A_T} &= \frac{6 \times \dfrac{5}{256\sqrt{3}}}{\dfrac{1}{2}\sin\dfrac{\pi}{3}} \\ &= \frac{5}{27}\end{aligned}$$

This value can be interpreted as the probability that the dart will land closer to the centre than the edge.

Remark:

The focal length of the parabola drawn in the diagram is half the vertical distance from O to the base. By trigonometry of appropriate right-angled triangles this is $\dfrac{1}{2} \times \dfrac{1}{2}\tan\dfrac{\pi}{6}$, or equivalently, $\dfrac{1}{4\sqrt{3}}$.

From part (i), the focal length of $x^2\sqrt{3} + \dfrac{1}{4\sqrt{3}}$ is $\dfrac{1}{4\sqrt{3}}$. This can be found by finding the focal length of the equivalently shaped parabola $y = \sqrt{3}x^2$.

If we treat the triangle board as a number plane with centre O, then the equation of the left dotted line shown in the diagram is $y = x\tan\dfrac{\pi}{6}$, or equivalently, $y = \dfrac{x}{\sqrt{3}}$. The equation of the parabola is $y = x^2\sqrt{3} - \dfrac{1}{4\sqrt{3}}$. Notice, this has similarities to the parabola and line described in part (i), only shifted by $\dfrac{1}{2\sqrt{3}}$, so the area found can therefore be used accordingly.

(b) (i) The Cartesian equations of the trajectories are

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

$$y = x\tan(\theta+\alpha) - \frac{gx^2}{2V^2}\sec^2(\theta+\alpha)$$

Solving simultaneously for x, noting the required solution is non-zero

$$\begin{aligned}
\frac{gx^2}{2V^2}\left(\sec^2(\theta+\alpha) - \sec^2\theta\right) &= x\left(\tan(\theta+\alpha) - \tan\theta\right) \\
x &= \frac{2V^2(\tan(\theta+\alpha) - \tan\theta)}{g\left(\sec^2(\theta+\alpha) - \sec^2\theta\right)} \\
&= \frac{2V^2(\tan(\theta+\alpha) - \tan\theta)}{g\left(\tan^2(\theta+\alpha) - \tan^2\theta\right)} \\
&= \frac{2V^2}{g\left(\tan(\theta+\alpha) + \tan\theta\right)}
\end{aligned}$$

Substitute to obtain the y-coordinate

$$\begin{aligned}
y =& \frac{2V^2\tan\theta}{g\left(\tan(\theta+\alpha) + \tan\theta\right)} - \frac{4V^4}{g^2\left(\tan(\theta+\alpha) + \tan\theta\right)^2} \times \frac{g}{2V^2}\sec^2\theta \\
=& \frac{2V^2\tan\theta}{g\left(\tan(\theta+\alpha) + \tan\theta\right)} - \frac{2V^2(1+\tan^2\theta)}{g\left(\tan(\theta+\alpha) + \tan\theta\right)^2}
\end{aligned}$$

(ii) When $\alpha \to 0$ then

$$\begin{aligned}
x &\to \frac{2V^2}{g\left(\tan\theta + \tan\theta\right)} \\
&= \frac{V^2\cot\theta}{g} \\
y &\to \frac{2V^2\tan\theta}{g\left(\tan\theta + \tan\theta\right)} - \frac{2V^2(1+\tan^2\theta)}{g\left(\tan\theta + \tan\theta\right)^2} \\
&= \frac{V^2}{g} - \frac{V^2(1+\tan^2\theta)}{2g\tan^2\theta} \\
&= \frac{V^2}{2g}\left(2 - \frac{1+\tan^2\theta}{\tan^2\theta}\right) \\
&= \frac{V^2}{2g}\left(\frac{\tan^2\theta - 1}{\tan^2\theta}\right) \\
&= \frac{V^2(1-\cot^2\theta)}{2g}
\end{aligned}$$

(iii) From part (ii), the equation of the locus of P is

$$
\begin{aligned}
y &= \frac{V^2(1-\cot^2\theta)}{2g} \qquad \text{but } x = \frac{V^2\cot\theta}{g} \\
&= \frac{V^2\left(1-\dfrac{g^2x^2}{V^4}\right)}{2g} \\
&= \frac{V^2}{2g} - \frac{gx^2}{2V^2}
\end{aligned}
$$

(c) (i) If Jack has k lollies in the remaining pocket, then it must be that $n-k$ lollies were eaten in that pocket and n lollies were eaten in the other pocket. This means that there are $2n-k$ 'trials' where the outcomes are either eaten or not eaten. The probability of $n-k$ lollies being eaten is therefore

$$\binom{2n-k}{n-k}\left(\frac{1}{2}\right)^{n-k}\left(\frac{1}{2}\right)^{n}$$

or equivalently, $\displaystyle\binom{2n-k}{n}\frac{1}{2^{2n-k}}$.

(ii) The lowest possible value for k is zero, where Jack draws the last lolly in one pocket, draws the last lolly in the other pocket, then in the next draw he discovers one of them is empty.

The highest possible value for k is n, where Jack draws all the lollies from the same pocket and discovers it to be empty.

Summing the probabilities across all possible values of k covers all possible outcomes, so it must equal 1.

$$
\begin{aligned}
\binom{2n}{n}\frac{1}{2^{2n}} + \binom{2n-1}{n}\frac{1}{2^{2n-1}} + \binom{2n-2}{n}\frac{1}{2^{2n-2}} + \dots + \binom{n}{n}\frac{1}{2^{n}} &= 1 \\
\sum_{k=0}^{n}\binom{n+k}{n}\frac{1}{2^{n+k}} &= 1 \\
\sum_{k=0}^{n}2^{-k}\binom{n+k}{k} &= 2^{n}
\end{aligned}
$$