2017 Bored of Studies Trial Examinations

Mathematics Extension 1

Solutions

Section I

1. B	3. B	5. D	7. C	9. B
2. A	4. D	6. B	8. D	10. C

Working/Justification

Question 1

For a simple expansion of $(1-x)^{4n+2}$, the magnitudes of the coefficients increase from 1 to a maximum in the middle terms before decreasing back to 1 again (see Pascal's triangle). However, the comparisons of the choices are of the coefficients themselves, not the magnitude. This means consideration needs to be given to the signs of the coefficients, which alternate in this case.

Since there are $4n+3$ terms, the middle term (which is $a_{2n+1}x^{2n+1}$) has a negative coefficient. The term preceding it (which is $a_{2n}x^{2n}$) may have a smaller *magnitude* in the coefficient but since it is positive rather than negative then $a_{2n} \geq a_{2n+1}$. Hence, the answer is (B).

Question 2

A function and its inverse should intersect on some point on the line $y = x$. Let (α, α) be that point of intersection between $f(x)$ and $f^{-1}(x)$. The gradient of the tangent to $f(x)$ at the point of intersection is $f'(\alpha)$. The gradient of the inverse function at that point is $\dfrac{1}{f'(\alpha)}$.

If the angle between two lines is $\dfrac{\pi}{6}$ then

$$\tan\frac{\pi}{6} = \left| \frac{f'(\alpha) - \dfrac{1}{f'(\alpha)}}{1 + f'(\alpha) \times \dfrac{1}{f'(\alpha)}} \right|$$

$$\frac{2}{\sqrt{3}} = \left| f'(\alpha) - \frac{1}{f'(\alpha)} \right|$$

By directly substituting each of the four choices for $f'(\alpha)$, it can be shown that the answer is (A).

Question 3

The solution to the differential equation is $N = A + Be^{-kt}$.

When $t = t_0$, $N = 2A$ which implies that $Be^{-kt_0} = A$.

When $t = mt_0$, $N = (n+1)A$ which implies that

$$\begin{aligned} Be^{-kmt_0} &= nA \\ B\left(e^{-kt_0}\right)^m &= nA \\ B\left(\frac{A}{B}\right)^m &= nA \\ \left(\frac{A}{B}\right)^{m-1} &= n \end{aligned}$$

Hence the answer is (B)

Question 4

Expanding the expression

$$\begin{aligned} \sqrt{2}\sin x + \cos\left(x + \frac{\pi}{4}\right) &= \sqrt{2}\sin x + \frac{1}{\sqrt{2}}(\cos x - \sin x) \\ &= \left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)\sin x + \frac{1}{\sqrt{2}}\cos x \\ &= \frac{1}{\sqrt{2}}(\sin x + \cos x) \\ &= \sin\left(x + \frac{\pi}{4}\right) \end{aligned}$$

The maximum of value is therefore 1 so the answer is (D).

Question 5

When R divides PQ externally in the ratio $m : n$, the resulting x-coordinate is $x_0 = \dfrac{mx_2 - nx_1}{m-n}$. Thus $\alpha = \dfrac{m}{m-n}$ and $\beta = -\dfrac{n}{m-n}$. This means that $\alpha + \beta = 1$ so the answer is (D).

Question 6

Let p be the probability of winning in one attempt. In the game, there are 3 trials to consider and the probability of winning at least twice is $3p^2(1-p)+p^3$. The probability of winning in one attempt is $3p(1-p)^2$. The condition is therefore

$$3p^2(1-p)+p^3 \geq 3p(1-p)^2$$

Testing each of the choices, only one does not satisfy this condition which is $\frac{1}{3}$, so the answer is (B).

Question 7

Start with the fact that

$$\int_a^b \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} b - \sin^{-1} a$$

Let $u = \sin^{-1} a$ and $v = \sin^{-1} b$. Equivalently $a = \sin u$ and $b = \sin v$.

Also, since $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ then $\cos u = \sqrt{1-\sin^2 u}$ and similarly $\cos v = \sqrt{1-\sin^2 v}$.

Consider

$$\sin(v-u) = \sin v \cos u - \sin u \cos v$$

$$= \sin v\sqrt{1-\sin^2 u} - \sin u\sqrt{1-\sin^2 v}$$

$$v - u = \sin^{-1}\left(b\sqrt{1-a^2} - a\sqrt{1-b^2}\right)$$

$$\sin^{-1} b - \sin^{-1} a = \sin^{-1}\left(b\sqrt{1-a^2} - a\sqrt{1-b^2}\right)$$

Hence the answer is (C).

Question 8

Since the polynomial is monic and cubic then it must take the form

$$P(x) = (x+a)(x^2+4x+1)+x+1$$

Since $P(x)$ divided by $(x-1)$ gives 2 then by the remainder theorem $P(1)=2$ so

$$P(1) = (1+a)(1+4+1)+1+1$$

$$2 = 6(a+1)+2$$

$$a = -1$$

Hence

$$P(x) = (x-1)(x^2+4x+1)+x+1$$

$$= x^3+4x^2+x-x^2-4x-1+x+1$$

$$= x^3+3x^2-2x$$

Hence the answer is (D).

Question 9

The surface area and volume of a sphere is $4\pi r^2$ and $\frac{4}{3}\pi r^3$ respectively.

For a hemisphere, the surface area is $3\pi r^2$ (half the surface area of the sphere plus the circular base). The volume of the hemisphere is $\frac{2}{3}\pi r^3$.

Let S and V denote the surface area and volume of the hemisphere respectively.

$$S = 3\pi r^2$$

$$\frac{dS}{dr} = 6\pi r$$

$$V = \frac{2}{3}\pi r^3$$

$$\frac{dV}{dr} = 2\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dS}\frac{dS}{dt}$$

$$= 2\pi r^2 \times \frac{1}{6\pi r} \times k$$

$$= \frac{k}{3}r$$

Hence the answer is (B).

Question 10

Let $F(x)$ be the primtive function of e^{-x^2} so that $F'(x) = e^{-x^2}$.

$$\begin{aligned}
\lim_{h\to 0}\left(\frac{1}{h}\int_{-h}^{h} e^{-x^2}\,dx\right) &= \lim_{h\to 0}\frac{F(h)-F(-h)}{h} \\
&= 2\lim_{2h\to 0}\frac{F(h)-F(-h)}{2h} \\
&= 2F'(0) \\
&= 2
\end{aligned}$$

Note that $\lim\limits_{2h\to 0}\dfrac{F(h)-F(-h)}{2h}$ has the form $\lim\limits_{x\to c}\dfrac{F(x)-F(c)}{x-c}$ which is in fact the first principles definition of the derivative and is equal to $F'(c)$. Hence the answer is (C).

Section II

Question 11

(a)

(i) Substitute $\sin\theta = \dfrac{2t}{1+t^2}$ and $\cos\theta = \dfrac{1-t^2}{1+t^2}$ into $\sin 2\theta - \sin\theta + \cos\theta + 1 = 0$

$$2 \times \frac{2t}{1+t^2} \times \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1 = 0$$

$$4t(1-t^2) - 2t(1+t^2) + (1-t^2)(1+t^2) + (1+t^2)^2 = 0$$

$$4t - 4t^3 - 2t - 2t^3 + 1 - t^4 + 1 + 2t^2 + t^4 = 0$$

$$3t^3 - t^2 - t - 1 = 0$$

(ii) Using the relationship between roots and coefficients for the polynomial in part (i):

$$\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) + \tan\left(\frac{\gamma}{2}\right) = \frac{1}{3} \qquad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)\tan\left(\frac{\gamma}{2}\right) = \frac{1}{3}$$

Consider the following

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \frac{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)}{1 - \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}$$

$$= \frac{\frac{1}{3} - \tan\left(\frac{\gamma}{2}\right)}{1 - \frac{1}{3\tan\left(\frac{\gamma}{2}\right)}}$$

$$= -\tan\left(\frac{\gamma}{2}\right)$$

$$= \tan\left(-\frac{\gamma}{2}\right)$$

The general solution is $\dfrac{\alpha}{2} + \dfrac{\beta}{2} = k\pi - \dfrac{\gamma}{2}$ for some integer k, or equivalently $\alpha + \beta + \gamma = 2k\pi$

(b)

(i) The domain of $\sin^{-1} x$ is $-1 \leq x \leq 1$. Hence, the domain of $\sin^{-1}\sqrt{x}$ is $-1 \leq \sqrt{x} \leq 1$, which is equivalent to $0 \leq x \leq 1$.

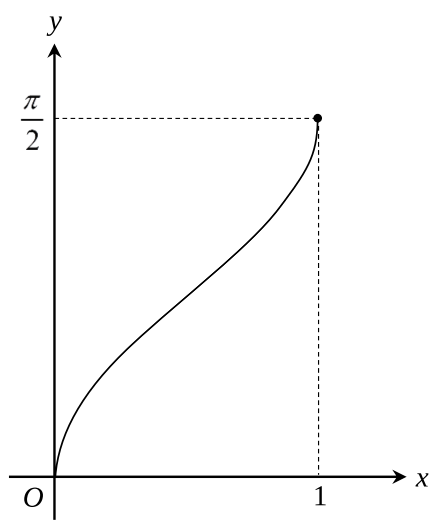
Given this domain, the range of $f(x)$ must be $0 \leq y \leq \dfrac{\pi}{2}$

(ii)

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \times \frac{1}{\sqrt{1-(\sqrt{x})^2}} \\ &= \frac{1}{2\sqrt{x(1-x)}} \end{aligned}$$

Since $\sqrt{x(1-x)} > 0$ then $f'(x) > 0$, so $f(x)$ is increasing.

(iii)

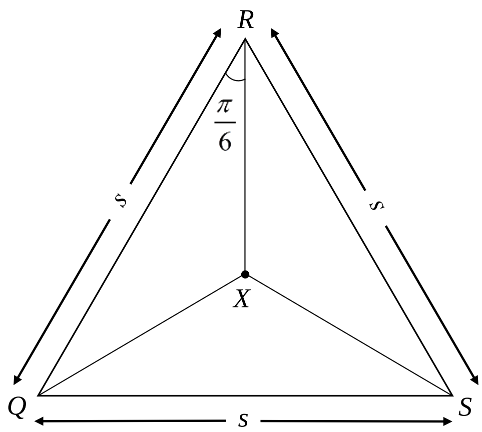


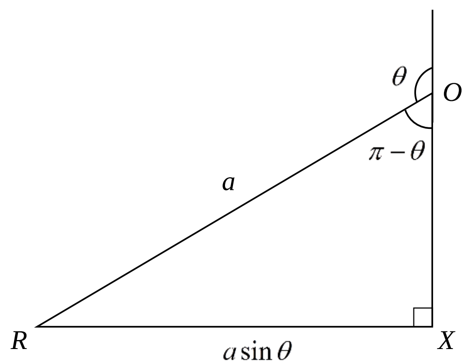
(iv) For $y = \sin^{-1}\sqrt{x}$ then $x = \sin^2 y$. Let A be the area of the region. From the graph

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} x\,dy \\ &= \int_0^{\frac{\pi}{4}} \sin^2 y\,dy \\ &= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2y}{2}\,dy \\ &= \left[\frac{y}{2} - \frac{\sin 2y}{4}\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{8} - \frac{1}{4} \end{aligned}$$

(c)

(i) Let PO extend to meet the centre of the base equilateral $\triangle QRS$ at point X.





By symmetry of the point X, $\angle QXR = \dfrac{2\pi}{3}$ and $\angle QRX = \dfrac{\pi}{6}$. By the sine rule

$$\frac{s}{\sin\dfrac{2\pi}{3}} = \frac{XR}{\sin\dfrac{\pi}{6}}$$

$$s = \sqrt{3}XR$$

Looking at $\triangle RXO$, note that $OR = a$ and $\angle XOR = \pi - \theta$. This means that

$$\sin\angle ROX = \frac{XR}{OR}$$

$$XR = a\sin(\pi - \theta)$$

$$s = a\sqrt{3}\sin\theta$$

(ii) From cosine rule in $\triangle POR$

$$PR^2 = PO^2 + RO^2 - 2 \times PO \times RO \times \cos \angle POR$$

$$s^2 = 2a^2 - 2a^2 \cos\theta$$

$$3a^2 \sin^2\theta = 2a^2 - 2a^2 \cos\theta$$

$$3(1 - \cos^2\theta) = 2 - 2\cos\theta$$

$$3\cos^2\theta - 2\cos\theta - 1 = 0$$

$$(3\cos\theta + 1)(\cos\theta - 1) = 0 \quad \text{since } \theta \text{ is obtuse then } \cos\theta \neq 1$$

$$\cos\theta = -\frac{1}{3}$$

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Question 12

(a)

(i) $u = -x$ so $du = -dx$. When $x = -a$, $u = a$ and when $x = a$, $u = -a$.

As u and x are variables of definite integration, they can be treated identically. Replace u with x to obtain an equivalent transformed integral in terms of x

$$\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = -\int_{-a}^{a} \frac{f(-u)}{1+e^{-u}}\,du$$

$$= -\int_{-a}^{a} \frac{f(-x)}{1+e^{-x}}\,dx$$

$$= -\int_{a}^{-a} \frac{e^x f(x)}{e^x+1}\,dx$$

$$= \int_{-a}^{a} \frac{e^x f(x) + f(x) - f(x)}{e^x+1}\,dx$$

$$= \int_{-a}^{a} f(x)\,dx - \int_{-a}^{a} \frac{f(x)}{e^x+1}\,dx$$

$$2\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = \int_{-a}^{a} f(x)\,dx \quad \text{but } f(x) \text{ is an even function so } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} \frac{f(x)}{1+e^x}\,dx = \int_{0}^{a} f(x)\,dx$$

(ii) From part (i), since $f(x) = \left(\dfrac{1-e^x}{1+e^x}\right)^2$ is an even function then

$$\int_{-1}^{1} \frac{(1-e^x)^2}{(1+e^x)^3}\,dx = \int_0^1 \left(\frac{1-e^x}{1+e^x}\right)^2\,dx$$

Let $u = 1+e^x$ so $du = e^x\,dx \Rightarrow dx = \dfrac{du}{u-1}$. When $x=0$, $u=2$ and when $x=1$, $u=1+e$.

$$\begin{aligned}
\int_{-1}^{1} \frac{(1-e^x)^2}{(1+e^x)^3}\,dx &= \int_2^{1+e} \frac{(2-u)^2}{u^2(u-1)}\,du \\
&= \int_2^{1+e} \frac{u^2-4u+4}{u^2(u-1)}\,du \\
&= \int_2^{1+e} \frac{u^2-4(u-1)}{u^2(u-1)}\,du \\
&= \int_2^{1+e} \frac{1}{u-1} - \frac{4}{u^2}\,du \\
&= \left[\ln(u-1) + \frac{4}{u}\right]_2^{1+e} \\
&= \frac{4}{1+e} - 1
\end{aligned}$$

Remark: Whilst not required for the question, it can be shown that $f(x) = \left(\dfrac{1-e^x}{1+e^x}\right)^2$ is an even function.

$$\begin{aligned}
f(-x) &= \left(\frac{1-e^{-x}}{1+e^{-x}}\right)^2 \\
&= \left(\frac{e^x-1}{e^x+1}\right)^2 \\
&= \left(\frac{1-e^x}{1+e^x}\right)^2 \\
&= f(x)
\end{aligned}$$

(b)

(i) Recognise that the acceleration equations take the form $a = -n^2 x$ and therefore each follow simple harmonic motion about the origin. Since the particles start at their maximum displacements of $x = a$ and $y = b$ then the displacement equations are

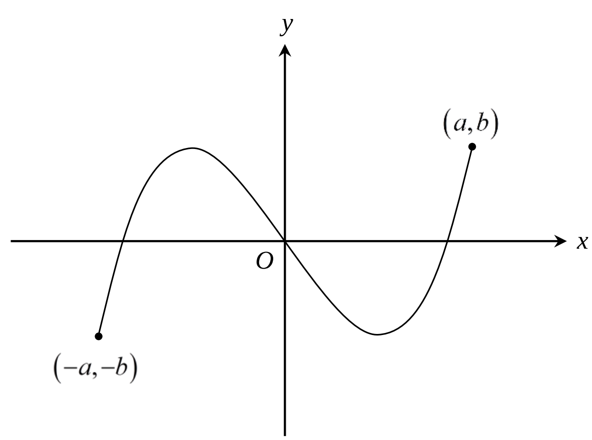
$$x = a\cos t$$

$$y = b\cos 3t$$

(ii) To find the Cartesian equation of the particle's displacement expand $\cos 3t$

$$\begin{aligned} y &= b\cos 3t \\ &= b(\cos 2t\cos t - \sin 2t\sin t) \\ &= b(\cos t(2\cos^2 t - 1) - 2\cos t(1-\cos^2 t)) \\ &= b\cos t(4\cos^2 t - 3) \\ &= \frac{bx}{a}\left(\frac{4x^2}{a^2} - 3\right) \end{aligned}$$

This is a cubic curve with the domain $-a \le x \le a$ and $-b \le y \le b$



(iii) The particle is initially at the point (a, b) and then traces the path of the curve $y = \dfrac{bx}{a}\left(\dfrac{4x^2}{a^2} - 3\right)$ until it reaches the point $(-a, -b)$ before reversing back along that curve to (a, b). The motion then repeats periodically.

(c)

(i) By Newton's method and using the definition that $E_n = x_n - \sqrt{a}$ and similarly $E_{n-1} = x_{n-1} - \sqrt{a}$

$$x_n = x_{n-1} - \frac{x_{n-1}^2 - a}{2x_{n-1}}$$

$$x_n - \sqrt{a} = x_{n-1} - \frac{x_{n-1}^2 - a}{2x_{n-1}} - \sqrt{a}$$

$$E_n = E_{n-1} - \frac{(E_{n-1} + \sqrt{a})^2 - a}{2(E_{n-1} + \sqrt{a})}$$

$$= \frac{2E_{n-1}^2 + 2\sqrt{a}E_{n-1} - E_{n-1}^2 - 2\sqrt{a}E_{n-1} - a + a}{2(E_{n-1} + \sqrt{a})}$$

$$= \frac{E_{n-1}^2}{2(E_{n-1} + \sqrt{a})}$$

(ii) After one application of Newton's method, observe that from (i)

$$E_1 = \frac{E_0^2}{2(E_0 + \sqrt{a})}$$

$$= \frac{(x_0 - \sqrt{a})^2}{2x_0}$$

Since $x_0 > 0$ then $E_1 \geq 0$ which implies that $x_1 \geq \sqrt{a}$. Assuming that x_0 does not happen to equal the actual root exactly (in which case Newton's method is redundant) then the approximation is always greater than the exact value of the root.

(iii) Since $\sqrt{a} > 0$ then $E_{n-1} + \sqrt{a} > E_{n-1}$. So

$$E_n = \frac{E_{n-1}^2}{2(E_{n-1} + \sqrt{a})}$$

$$< \frac{E_{n-1}^2}{2E_{n-1}}$$

$$< \frac{1}{2}E_{n-1} \quad \text{now applying this repeatedly}$$

$$< \frac{1}{2^2}E_{n-2}$$

......

$$< \frac{1}{2^n}E_0$$

Hence $E_0 > 2^n E_n$

Question 13

(a)

(i) Obtain the Cartesian equation of the path by substituting $t = \dfrac{x}{V\cos\alpha}$ into y

$$y = -\frac{gx^2}{2V^2\cos^2\alpha} + x\tan\alpha + h \quad \text{when } y = 0, x = R$$

$$0 = \frac{gR^2}{2V^2\cos^2\alpha} - R\tan\alpha - h$$

$$0 = gR^2 - 2V^2\sin\alpha\cos\alpha R - 2hV^2\cos^2\alpha$$

$$R = \frac{2V^2\sin\alpha\cos\alpha \pm \sqrt{4V^4\sin^2\alpha\cos^2\alpha + 8ghV^2\cos^2\alpha}}{2g} \qquad \text{take the positive root since } R > 0$$

$$= \frac{V^2\sin\alpha\cos\alpha + V\cos\alpha\sqrt{V^2\sin^2\alpha + 2gh}}{g}$$

$$= \frac{V\cos\alpha}{g}\left(V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}\right)$$

(ii) The angle of projection α is to be varied such that R is a maximum. For ease of notation denote the range as $R(\alpha)$ so that it represents a function of α. From the Cartesian equation:

$$\frac{gR^2(\alpha)}{2V^2\cos^2\alpha} - R(\alpha)\tan\alpha - h = 0$$

$$\frac{gR^2(\alpha)}{2V^2}\sec^2\alpha - R(\alpha)\tan\alpha - h = 0$$

Differentiate both sides with respect to α

$$\frac{gR(\alpha)}{V^2}\sec^2\alpha R'(\alpha) + \frac{gR^2(\alpha)}{V^2}\sec^2\alpha\tan\alpha - R'(\alpha)\tan\alpha - R(\alpha)\sec^2\alpha = 0$$

When $\alpha = \alpha_{\max}$ then $R'(\alpha) = 0$

$$\frac{gR^2(\alpha_{\max})}{V^2}\sec^2\alpha_{\max}\tan\alpha_{\max} - R(\alpha_{\max})\sec^2\alpha_{\max} = 0$$

$$R(\alpha_{\max}) = \frac{V^2}{g\tan\alpha_{\max}} \qquad \text{noting that } R(\alpha_{max}) \neq 0$$

Substitute this back into the Cartesian equation when $\alpha = \alpha_{\max}$

$$-\frac{V^2}{2g\sin^2\alpha_{\max}} + \frac{V^2}{g} + h = 0$$

$$\sin^2\alpha_{\max} = \frac{V^2}{2(V^2+gh)}$$

$$\tan^2\alpha_{max} = \frac{\sin^2\alpha_{\max}}{1-\sin^2\alpha_{\max}}$$

$$= \frac{V^2}{V^2+2gh}$$

Remark: Whilst not required for the question, it can be shown that this condition produces the absolute maximum range given α is acute. When setting $R'(\alpha) = 0$ to find the stationary points there are actually two conditions obtained for $R(\alpha)$ which are 0 and $\dfrac{V^2}{g\tan\alpha}$.

Since $R(\alpha) = 0$ is the absolute minimum, which occurs at $\alpha = \frac{\pi}{2}$ then it can be shown that stationary point, which occurs at $R(\alpha) = \dfrac{V^2}{g\tan\alpha}$ is an absolute maximum.

To see this, note that $R'(\alpha)$ can be made as the subject from the differentiated Cartesian equation to get

$$R'(\alpha) = \frac{R(\alpha)\sec^2\alpha(V^2 - gR(\alpha)\tan\alpha)}{V^2\tan\alpha - gR(\alpha)\sec^2\alpha}$$

When $\alpha = 0$, $R'(0) = -\dfrac{R(0)V^2}{gR} < 0$. This suggests that when $\alpha = \alpha_{\max}$ for $0 < \alpha_{\max} < \frac{\pi}{2}$ it cannot be at a local minimum.

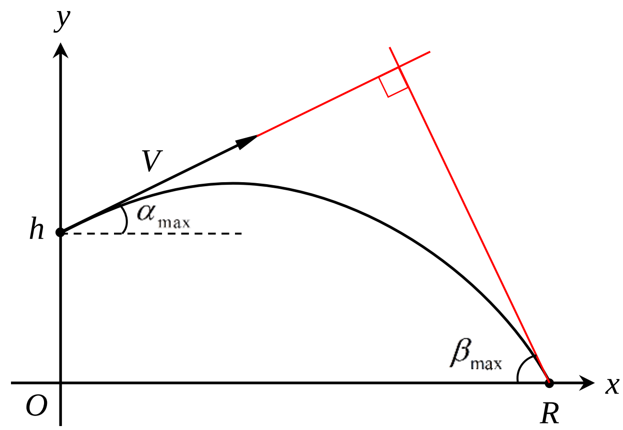
However, since the absolute minimum occurs at $\alpha = \frac{\pi}{2}$ and there are no other stationary points for $0 < \alpha < \frac{\pi}{2}$ then $R(\alpha) = \dfrac{V^2}{g\tan\alpha}$ is an absolute maximum.

(iii) The time of flight is $\dfrac{R}{V\cos\alpha}$ which from (i) is $\dfrac{V\sin\alpha+\sqrt{V^2\sin^2\alpha+2gh}}{g}$. This means that

$$\begin{aligned}
\tan\beta_{\max} &= -\frac{\dot{y}}{\dot{x}}\\
&= \frac{gt - V\sin\alpha_{\max}}{V\cos\alpha_{\max}}\\
&= \frac{\sqrt{V^2\sin^2\alpha_{\max}+2gh}}{V\cos\alpha_{\max}}\\
&= \sqrt{\tan^2\alpha_{\max}+\frac{2gh}{V^2}\sec^2\alpha_{\max}}\\
&= \sqrt{\tan^2\alpha_{\max}\left(1+\frac{2gh}{V^2}\right)+\frac{2gh}{V^2}}\\
&= \sqrt{\tan^2\alpha_{\max}\left(\frac{V^2+2gh}{V^2}\right)+\frac{2gh}{V^2}}\\
&= \sqrt{1+\frac{2gh}{V^2}}\\
&= \sqrt{\frac{V^2+2gh}{V^2}}\\
&= \frac{1}{\tan\alpha_{\max}} \quad \text{noting that } \alpha_{\max} \text{ is acute so } \tan\alpha_{\max}>0
\end{aligned}$$

In order for this to be true $\alpha_{\max}$ and $\beta_{\max}$ must be complementary angles, that is $\alpha_{\max}+\beta_{\max}=\dfrac{\pi}{2}$

(iv) Note that the resultant vectors describing the point of projection and the landing lie on the tangents of P and Q respectively.



The angle to the positive horizontal of P is α_{max} so the gradient of the tangent at P is given by $m_P = \tan\alpha_{\text{max}}$. The angle to the positive horizontal of Q is $\pi - \beta_{\text{max}}$ so the gradient of the tangent at Q is given by $m_Q = \tan(\pi - \beta_{\text{max}})$. Consider the product of the gradients of the two tangents:

$$\begin{aligned} m_P \times m_Q &= \tan\alpha_{\text{max}} \tan(\pi - \beta_{\text{max}}) \\ &= -\tan\alpha_{\text{max}} \tan\beta_{\text{max}} \\ &= -\tan\alpha_{\text{max}} \tan\left(\frac{\pi}{2} - \alpha_{\text{max}}\right) \\ &= -\tan\alpha_{\text{max}} \times \frac{1}{\tan\alpha_{\text{max}}} \\ &= -1 \end{aligned}$$

Since the tangents at P and Q are perpendicular to each other then PQ must be a focal chord (recall the parametric condition $pq = -1$).

(b) When $n = 1$

$$\begin{aligned} LHS &= \left(\frac{a+b}{2}\right)^1 \\ &= \frac{a^1 + b^1}{2} \\ &= RHS \end{aligned}$$

Hence the statement is true for $n = 1$.

Assume the statement is true for $n = k$

$$\left(\frac{a+b}{2}\right)^k \leq \frac{a^k + b^k}{2}$$

Required to prove the statement is true for $n = k + 1$

$$\left(\frac{a+b}{2}\right)^{k+1} \leq \frac{a^{k+1} + b^{k+1}}{2}$$

$$\begin{aligned}
LHS &= \left(\frac{a+b}{2}\right)^{k+1} \\
&= \left(\frac{a+b}{2}\right)^k \left(\frac{a+b}{2}\right) \\
&\leq \left(\frac{a^k + b^k}{2}\right) \left(\frac{a+b}{2}\right) \\
&= \frac{a^{k+1} + b^{k+1} + a^k b + ab^k}{4} \\
&= \frac{2a^{k+1} + 2b^{k+1} + a^k b + ab^k - a^{k+1} - b^{k+1}}{4} \\
&= \frac{2a^{k+1} + 2b^{k+1} + a^k(b-a) + (a-b)b^k}{4} \\
&= \frac{2a^{k+1} + 2b^{k+1} + (a^k - b^k)(b-a)}{4} \qquad \text{but } (a^k - b^k)(b-a) \leq 0 \\
&\leq \frac{2a^{k+1} + 2b^{k+1}}{4} \\
&= \frac{a^{k+1} + b^{k+1}}{2} \\
&= RHS
\end{aligned}$$

Since the statement is true for $n = 1$, it is true for all positive integers n by induction.

(c)

(i) Using $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{1}{(x-1)^2} + \frac{1}{x^3}$$

$$\frac{v^2}{2} = -\frac{1}{x-1} - \frac{1}{2x^2} + C \quad \text{when } v = 0, x = \frac{1}{2} \Rightarrow C = 0$$

$$v^2 = -\frac{2}{x-1} - \frac{1}{x^2}$$

(ii) Using the fact that generally $v^2 \geq 0$

$$-\frac{2}{x-1} - \frac{1}{x^2} \geq 0$$

$$\frac{2x^2 + x - 1}{x^2(x-1)} \leq 0$$

$$x^2(x-1)(2x-1)(x+1) \leq 0$$

$$(x-1)(2x-1)(x+1) \leq 0$$

$$x \leq -1 \text{ or } \frac{1}{2} \leq x \leq 1$$

However, there are also a number of restrictions that need to be considered.

Firstly, x cannot be 0 or 1 as these values cause the acceleration and velocity to be undefined.

Also, since the particle is initially at $x = \dfrac{1}{2}$ and moves continuously along the number line, it cannot cross over to the domain $x \leq -1$ without crossing x-values on the number line which make $v^2 < 0$.

Thus, the particle can only be allowed to have a displacement such that $\dfrac{1}{2} \leq x < 1$.

Question 14

(a)

(i) Since PQ is a focal chord, then the intersection of the tangents at P and Q must lie on the directrix. Hence the points T and D lie on the directrix. Also,

$$PS^2 = PM \times PT \quad \text{(tangent-secant ratio)}$$

$$PD^2 = PM \times PT \quad \text{(since } PS = PD \text{ by definition of the parabola)}$$

$$TD^2 = PT^2 - PD^2 \quad \text{(Pythagoras' theorem)}$$

$$= PT^2 - PM \times PT$$

$$= PT \times (PT - PM)$$

$$= TM \times TP$$

(ii) By part (i), we can use the converse of the tangent-secant ratio to deduce that DT is tangential to the circle through PMD. Since PD is perpendicular to DT, then PD must be a diameter since the radius is perpendicular to the tangent at the point of contact.

(iii) By the reflection property of the parabola, and vertically opposite angles, $\angle SPM = \angle DPM$.

PM is common and $PS = PD$ by definition of the parabola.

$\therefore \triangle SPM \equiv \triangle DPM \quad (SAS)$

$\angle PMD = \frac{\pi}{2}$ (angle in a semi-circle with diameter PD)

$\therefore \angle PMD = \angle PMS = \frac{\pi}{2}$ (corresponding angles of congruent triangles).

Hence D, M and S are collinear.

Furthermore, $MD = MS$ (corresponding angles of congruent triangles).

Hence, M is the midpoint of SD.

(iv) Note that D has the coordinates $(2ap, -a)$. But M is the midpoint of SD, so M has the coordinates $(ap, 0)$. Similarly, N has the coordinates $(aq, 0)$.

$$\begin{aligned} OM \times ON &= |ap| \times |aq| \\ &= a^2\,|pq| \qquad \text{but since } PQ \text{ is a focal chord } pq = -1 \\ &= a^2 \times |-1| \\ &= a^2 \end{aligned}$$

(b)

(i) Out of all n people, there are $\binom{n}{k}$ ways to choose a subset of k to form the committee. The treasurer can be chosen in k ways. Similarly, the chairman can also be chosen in k ways since a single person may be both chairman and treasurer. So the number of possible committees is $k^2\binom{n}{k}$

(ii) We will count the number of all possible committees in two different ways. The first way is on a case-by-case basis. By summing up the result in (i) from $k = 1$ to $k = n$, we go through all possible committee sizes ranging from a one-person committee all the way to the other extreme where everybody is in the committee. This yields the left hand sum $\displaystyle\sum_{k=1}^{n} k^2\binom{n}{k}$.

We can actually achieve the same result by counting all possible committees formed in one go. First note that there are two cases to consider.

Case #1: The treasurer and chairman are the same person.

There are n ways to choose the treasurer/chairman. After that, the remaining $n-1$ people are either in the committee or not in the committee. Hence, we have $\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n \text{ times}}$.

Hence, this case has $n2^{n-1}$ possibilities.

Case #2: The treasurer and chairman are different people.

There are n ways to choose the chairman and $n-1$ ways to choose the treasurer. We now have $n-2$ people left. Each of these people again have two options: to be in the committee or not to be in it. Hence, we have $n(n-1)2^{n-2}$ ways.

Combine these mutually exclusive cases to get $n2^{n-1} + n(n-1)2^{n-2}$. Since this also counts the total number of committees of all sizes, we can conclude this this must be equal to $\displaystyle\sum_{k=1}^{n} k^2\binom{n}{k}$.

(iii)

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} \times \frac{k!(n-k)!}{n!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} \times \frac{k!(n-k)!}{n!}$$

$$= \frac{k}{n}$$

(iv) From (iii)

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Also, from Pascal's relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

So

$$\sum_{k=1}^{n} k^3\binom{n}{k} = n\sum_{k=1}^{n} k^2\binom{n-1}{k-1}$$

$$= n\sum_{k=1}^{n-1} k^2\binom{n-1}{k-1} + n \times n^2\binom{n-1}{n-1}$$

$$= n\sum_{k=1}^{n-1} k^2\left(\binom{n}{k} - \binom{n-1}{k}\right) + n \times n^2\binom{n-1}{n-1} \quad \text{but } \binom{n-1}{n-1} = \binom{n}{n}$$

$$= n\sum_{k=1}^{n-1} k^2\binom{n}{k} - n\sum_{k=1}^{n-1} k^2\binom{n-1}{k} + n \times n^2\binom{n}{n}$$

$$= n\sum_{k=1}^{n} k^2\binom{n}{k} - n\sum_{k=1}^{n-1} k^2\binom{n-1}{k}$$

But from part (ii),

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = n2^{n-1} + n(n-1)2^{n-2}$$

Similarly

$$\sum_{k=1}^{n-1} k^2 \binom{n-1}{k} = (n-1)2^{n-2} + (n-1)(n-2)2^{n-3}$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} k^3 \binom{n}{k} &= n\left(n2^{n-1} + n(n-1)2^{n-2}\right) - n\left((n-1)2^{n-2} + (n-1)(n-2)2^{n-3}\right) \\
&= n\left(n(n+1)2^{n-2}\right) - n\left(n(n-1)2^{n-3}\right) \\
&= n^2 2^{n-3}\left(2(n+1) - (n-1)\right) \\
&= n^2(n+3)2^{n-3}
\end{aligned}$$