ADVANCED MATHEMATICS
Calculus (Adv), C1 Introduction to Differentiation (Adv)

Rates of Change (Y11)
Tangents (Y11)
Standard Differentiation (Y11)

Teacher: Cathyanne Horvat
Exam Equivalent Time: 58.5 minutes *(based on allocation of 1.5 minutes per mark)*

C1 Introduction to Differentiation
C2 Differential Calculus

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C2 Differential Calculus

***SmarterMaths analytics based on the average contribution to new syllabus Advanced Maths exams since 2020.**

HISTORICAL CONTRIBUTION

- *C1 Introduction to Differentiation* is Y11 content that has contributed an average of 1.3% per exam since the new syllabus was introduced in 2020.
- This topic has been split into three sub-topics for analysis purposes: *1-Standard Differentiation, 2-Tangents, and 3-Rates of Change.*
- This analysis looks at the sub-topic *Standard Differentiation*.

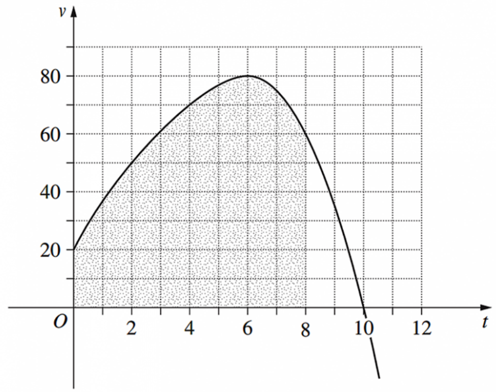
HSC ANALYSIS - What to expect and common pitfalls

- *Standard Differentiation* is the critical gateway content for all the calculus to follow in the Advanced course. It has been examined just once in a dedicated question in new syllabus exams (2022).
- This topic area looks at the product, quotient and chain rules, but importantly excludes log, exponential and trig underlyings which are covered in later sub-topics.
- *Standard Differentiation* has been asked 6 times in the last decade and if asked, provides low hanging fruit for students to score full marks.
- First principles and limit equations are covered in the database and should be reviewed.
- **TIP:** Create a clear cut process for writing formulae and identifying inputs before any calculations are made using the quotient rule. This will minimise errors and practice will develop speed (see worked solutions).

Questions

1. Calculus, 2ADV C1 2008 HSC 6b

The graph shows the velocity of a particle, v metres per second, as a function of time, t seconds.



i. What is the initial velocity of the particle? *(1 mark)*

ii. When is the velocity of the particle equal to zero? *(1 mark)*

iii. When is the acceleration of the particle equal to zero? *(1 mark)*

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2. Calculus, 2ADV C1 2009 HSC 1d

Find the gradient of the tangent to the curve $y = x^4 - 3x$ at the point $(1, -2)$. *(2 marks)*

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3. Calculus, 2ADV C1 EQ-Bank 2

When differentiating $f(x) = 3 - 2x - x^2$ from first principles, a student began the solution as shown below.

Complete the solution. *(2 marks)*

$$f\prime(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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4. Calculus, 2ADV C1 2018 HSC 12d

The displacement of a particle moving along the x-axis is given by

$$x = \frac{t^3}{3} - 2t^2 + 3t,$$

where x is the displacement from the origin in metres and t is the time in seconds, for $t \geq 0$.

i. What is the initial velocity of the particle? *(1 mark)*

ii. At which times is the particle stationary? *(2 marks)*

iii. Find the position of the particle when the acceleration is zero. *(2 marks)*

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5. Calculus, 2ADV C1 2014 HSC 11c

Differentiate $\dfrac{x^3}{x+1}$. *(2 marks)*

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6. Calculus, 2ADV C1 2011 HSC 2c

Find the equation of the tangent to the curve $y = (2x+1)^4$ at the point where $x = -1$. *(3 marks)*

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7. Calculus, 2ADV C1 SM-Bank 2

i. Find the equations of the tangents to the curve $y = x^2 - 3x$ at the points where the curve cuts the x-axis. *(2 marks)*

ii. Where do the tangents intersect? *(2 marks)*

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8. Calculus, 2ADV C1 SM-Bank 3

The displacement x metres from the origin at time t seconds of a particle travelling in a straight line is given by

$$x = 2t^3 - t^2 - 3t + 11 \quad \text{when} \quad t \geq 0$$

i. Calculate the velocity when $t = 2$. *(1 mark)*

ii. When is the particle stationary? *(2 marks)*

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9. Calculus, 2ADV C1 SM-Bank 14

Evaluate $f\prime(4)$, where $f(x) = x\sqrt{2x+1}$. *(3 marks)*

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10. Calculus, 2ADV C1 SM-Bank 11

A particle is moving along the x-axis. Its velocity v at time t is given by

$$v = \sqrt{20t - 2t^2} \text{ metres per second}$$

Find the acceleration of the particle when $t = 4$.

Express your answer as an exact value in its simplest form. *(3 marks)*

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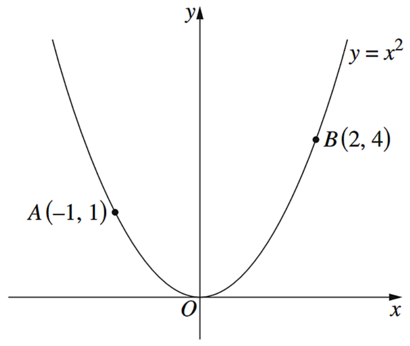
11. Calculus, 2ADV C1 2010 HSC 7b

The parabola shown in the diagram is the graph $y = x^2$. The points $A(-1, 1)$ and $B(2, 4)$ are on the parabola.



i. Find the equation of the tangent to the parabola at A. *(2 marks)*

ii. Let M be the midpoint of AB.
There is a point C on the parabola such that the tangent at C is parallel to AB.
Show that the line MC is vertical. *(2 marks)*

iii. The tangent at A meets the line MC at T.
Show that the line BT is a tangent to the parabola. *(2 marks)*

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12. Calculus, 2ADV C1 2019 HSC 14d

The equation of the tangent to the curve $y = x^3 + ax^2 + bx + 4$ at the point where $x = 2$ is $y = x - 4$.

Find the values of a and b. *(3 marks)*

Worked Solutions

1. Calculus, 2ADV C1 2008 HSC 6b

i. Find v when $t = 0$

$v = 20 \ \ \text{m/s}$

ii. Particle comes to rest at $t = 10$ seconds (from graph)

iii. Acceleration is zero when $t = 6$ seconds (from graph)

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2. Calculus, 2ADV C1 2009 HSC 1d

$$y = x^4 - 3x$$

$$\frac{dy}{dx} = 4x^3 - 3$$

At $x = 1$

$$\frac{dy}{dx} = 4 - 3 = 1$$

$\therefore$ Gradient of tangent at $(1, -2) = 1$.

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3. Calculus, 2ADV C1 EQ-Bank 2

$$\begin{aligned} f\prime(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{3 - 2(x+h) - (x+h)^2 - (3 - 2x - x^2)}{h} \\ &= \lim_{h \to 0} \frac{3 - 2x - 2h - x^2 - 2hx - h^2 - 3 + 2x + x^2}{h} \\ &= \lim_{h \to 0} \frac{-2h - 2hx}{h} \\ &= \lim_{h \to 0} \frac{h(-2x - 2)}{h} \end{aligned}$$

$\therefore \ f\prime(x) = -2x - 2$

Worked Solutions

4. Calculus, 2ADV C1 2018 HSC 12d

i. $x = \dfrac{t^3}{3} - 2t^2 + 3t$

$$v = \frac{dx}{dt} = t^2 - 4t + 3$$

Find v when $t = 0$:

$$\begin{aligned} v &= 0 - 0 + 3 \\ &= 3 \ \text{ms}^{-1} \end{aligned}$$

ii. Particle is stationary when $v = 0$

$$t^2 - 4t + 3 = 0$$

$$(t-3)(t-1) = 0$$

$$t = 1 \ \text{or} \ 3 \ \text{seconds}$$

iii. $a = \dfrac{dv}{dt} = 2t - 4$

Find t when $a = 0$

$$2t - 4 = 0$$

$$t = 2$$

$$\begin{aligned} x(2) &= \frac{2^3}{3} - 2(2^2) + 3(2) \\ &= \frac{8}{3} - 8 + 6 \\ &= \frac{2}{3} \end{aligned}$$

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5. Calculus, 2ADV C1 2014 HSC 11c

$$y = \frac{x^3}{x+1}$$

Using $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$

$$u = x^3 \qquad v = (x+1)$$
$$u' = 3x^2 \qquad v' = 1$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3x^2(x+1) - x^3(1)}{(x+1)^2} \\
&= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} \\
&= \frac{2x^3 + 3x^2}{(x+1)^2} \\
&= \frac{x^2(2x+3)}{(x+1)^2}
\end{aligned}$$

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6. Calculus, 2ADV C1 2011 HSC 2c

$$y = (2x+1)^4$$

Using the Function of a Function Rule (or Chain Rule)

$$\begin{aligned}
\frac{dy}{dx} &= 4 \times (2x+1)^3 \times \frac{d}{dx}(2x+1) \\
&= 8(2x+1)^3
\end{aligned}$$

At $x = -1,\ y = 1$

$$\begin{aligned}
\frac{dy}{dx} &= 8(2(-1)+1)^3 \\
&= 8(-1)^3 \\
&= -8
\end{aligned}$$

Tangent has $m = -8$ through $(-1, 1)$

$$\begin{aligned}
\text{Using}\quad y - y_1 &= m(x - x_1) \\
y - 1 &= -8(x+1) \\
y - 1 &= -8x - 8 \\
8x + y + 7 &= 0
\end{aligned}$$

$\therefore$ Equation of tangent is $8x + y + 7 = 0$

MARKER'S COMMENT: The best setting out clearly showed the derivative function, the gradient, the point and then finally, calculations for the equation of the tangent, as per the Worked Solution.

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7. Calculus, 2ADV C1 SM-Bank 2

i.
$$\begin{aligned}
y &= x^2 - 3x \\
&= x(x-3)
\end{aligned}$$

Cuts x-axis at $x = 0$ or $x = 3$

$$\frac{dy}{dx} = 2x - 3$$

At $x = 0 \Rightarrow \dfrac{dy}{dx} = -3$

T_1 has $m = -3$, through $(0, 0)$

$$\begin{aligned}
y - 0 &= -3(x - 0) \\
y &= -3x
\end{aligned}$$

At $x = 3 \Rightarrow \dfrac{dy}{dx} = 3$

T_2 has $m = 3$, through $(3, 0)$

$$\begin{aligned}
y - 0 &= 3(x - 3) \\
y &= 3x - 9
\end{aligned}$$

ii. Intersection occurs when:

$$\begin{aligned}
3x - 9 &= -3x \\
6x &= 9 \\
x &= \frac{3}{2}
\end{aligned}$$

$$y = -3 \times \frac{3}{2} = -\frac{9}{2}$$

$\therefore$ Intersection at $\left(\dfrac{3}{2}, -\dfrac{9}{2}\right)$

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8. Calculus, 2ADV C1 SM-Bank 3

i. $x = 2t^3 – t^2 – 3t + 11$

$$v = \frac{dx}{dt} = 6t^2 – 2t – 3$$

$\text{When } t = 2,$

$$v = 6 \times 2^2 – 2 \cdot 2 – 3$$
$$= 17 \text{ ms}^{-1}$$

ii. $\text{Particle is stationary when } \ v = 0$

$$6t^2 – 2t – 3 = 0$$

$$\therefore t = \frac{2 \pm \sqrt{(-2)^2 – 4 \cdot 6 \cdot (-3)}}{12}$$
$$= \frac{2 \pm \sqrt{76}}{12}$$
$$= \frac{1 \pm \sqrt{19}}{6}$$
$$= \frac{1 + \sqrt{19}}{6} \quad (t \geq 0)$$

9. Calculus, 2ADV C1 SM-Bank 14

$$f(x) = x\sqrt{2x + 1}$$

$$f\prime(x) = 1\sqrt{2x + 1} + x \times \frac{1}{2} \times 2(2x + 1)^{-\frac{1}{2}}$$
$$= \sqrt{2x + 1} + x(2x + 1)^{-\frac{1}{2}}$$
$$f\prime(4) = \sqrt{9} + 4(9)^{-\frac{1}{2}}$$
$$= 3 + \frac{4}{3}$$
$$= \frac{13}{3}$$

10. Calculus, 2ADV C1 SM-Bank 11

$$v = \sqrt{20t – 2t^2}$$

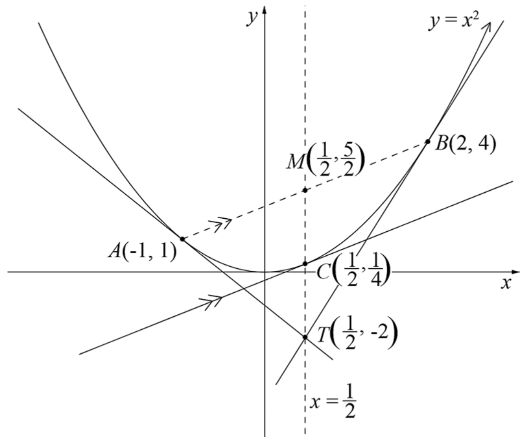
$$\alpha = \frac{dv}{dt}$$
$$= \frac{1}{2} \cdot \left(20t – 2t^2\right)^{-\frac{1}{2}} \cdot (20 – 4t)$$

$\text{When } \ t = 4,$

$$\alpha = \frac{1}{2}\left(20 \cdot 4 – 2 \cdot 4^2\right)^{-\frac{1}{2}}(20 – 16)$$
$$= \frac{2}{\sqrt{48}}$$
$$= \frac{2}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6} \ \text{ ms}^{-2}$$

11. Calculus, 2ADV C1 2010 HSC 7b

i.



$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\text{At } A(-1,1) \Rightarrow \frac{dy}{dx} = -2$$

Tangent has $m = -2$, through $(-1,1)$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2(x+1)$$

$$y - 1 = -2x - 2$$

$$2x + y + 1 = 0$$

$$\therefore \text{ Tangent at } A \text{ is } 2x + y + 1 = 0$$

ii. $A(-1,1) \quad B(2,4)$

♦ *Mean mark 37%.*
IMPORTANT: The critical understanding required for this question is that the gradient of AB needs to be equated to the gradient function (i.e. $\frac{dy}{dx}$).

$$M = \left(\frac{-1+2}{2}, \frac{1+4}{2}\right)$$

$$= \left(\frac{1}{2}, \frac{5}{2}\right)$$

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4-1}{2+1} = 1$$

$$= 1$$

When $\dfrac{dy}{dx}$

$2x = 1$

$x = \dfrac{1}{2}$

$\therefore\ C\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$

$\Rightarrow M$ and C both have x-value $= \dfrac{1}{2}$

$\therefore\ MC$ is vertical ... as required

iii. T is point on tangent when $x = \dfrac{1}{2}$

Tangent $\quad 2x + y + 1 = 0$

At $x = \dfrac{1}{2}$

$2 \times \left(\dfrac{1}{2}\right) + y + 1 = 0$

$\Rightarrow y = -2$

$\therefore\ T\left(\dfrac{1}{2}, -2\right)$

♦♦ *Mean mark 29%.*

Given $B(2, 4)$

$m_{BT} = \dfrac{4+2}{2-\frac{1}{2}}$

$= 4$

At $B(2, 4)$, find gradient of tangent:

$\dfrac{dy}{dx} = 2x = 2 \times 2 = 4$

$\therefore m_{\text{tangent}} = 4 = m_{BT}$

$\therefore BT$ is a tangent

12. Calculus, 2ADV C1 2019 HSC 14d

$y = x^3 + ax^2 + bx + 4$

$\dfrac{dy}{dx} = 3x^2 + 2ax + b$

When $x = 2,\ \dfrac{dy}{dx} = 1$

$12 + 4a + b = 1$

$4a + b = -11 \ldots (1)$

♦ *Mean mark 46%.*

The point $(2, -2)$ lies on y:

$8 + 4a + 2b + 4 = -2$

$4a + 2b = -14 \ldots (2)$

Subtract $(2)-(1)$

$b = -3$

Substitute into (1)

$4a - 3 = -11$

$4a = -8$

$a = -2$