Racah Inst. of Physics, The Hebrew University
Computational Physics of Complex Systems: 77732
given by Prof. Ofer Biham

Problem set number 2

This is the second and last problem set for the course. There are 6 problems in this set. It is sufficient to submit solutions to 4 of these problems to get full credit for this set.

This problem set can be submitted as a joint project of two students. The solution should include a detailed explanation of the work that was done, the methods, a description of the results (referring to the graphs) and their interpretation. The codes themselves should not be included in the submission.

Problem 1

In class we considered the Nagel-Schreckenberg model for highway traffic. Use mean-field theory (rate equations) to calculate the fundamental diagram (flux vs. density) for the case $v_{max} = 2$. For details of the model see: Nagel and Schreckenberg, J. Phys. I France 2, 2221 (1992); see also Phys. Rev. E51, 2939 (1995).

Simulate the cellular automaton model with $v_{max} = 2$ and compare the results to the mean field results. Try to explain why the mean-field results underestimate the flux.

Consider a variant of the model, in which whenever the speed of a car is $v = 0$ it has a probability q to pull over and stop on the side of the road and leave the system. What will be the asymptotic solution of the rate equations in this case? What would be the asymptotic solution of an actual simulation of this variant of the model? How do you explain the difference?

Problem 2

You are given a random number generator that produces numbers in the range $[0, 1)$. Show how to obtain random numbers distributed according to the power-law distribution

$$f(x) = Kx^{-(\alpha+1)} \tag{1}$$

within the range $x > x_{\min}$, where K is a normalization constant that is determined by α and $x_{\min}$. For α in the range $1 < \alpha < 2$ the first moment of $f(x)$ is finite but the second moment diverges. For $\alpha > 2$ both the first and second moments are finite. To verify the applicability of this approach you can compare the histogram that is obtained from the numerical calculation to the theoretical distribution. It may also be useful to examine the convergence of the mean value $\langle x \rangle$ as the number of sample points is increased.

Problem 3

Consider a random walker (RW) on a finite one dimensional lattice of L sites, indexed by i from 1 to L. The lattice has a closed boundary on the left (to the left of site 1) such that when the RW is in site 1 it can move only to the right. It has an open boundary on the right hand size of site L from which particles can exit the system. The time is discrete and the RW makes one move each time step with equal probabilities to the right or to the left. Calculate the *average* number of steps that is required for an RW starting from site 1 to exit the system through the other side. This is an example of a large class of problems called first passage processes. This is done by using recursion equations of the form

$$\langle t_i \rangle = 1 + \frac{1}{2} \left(\langle t_{i-1} \rangle + \langle t_{i+1} \rangle \right), \tag{2}$$

where $\langle t_i \rangle$ is the mean number of steps it would take an RW starting from site i to leave the system via the boundary on the right hand side.

Problem 4

Simulate the dynamics of the deterministic sandpile model introduced by Bak, Tang and Wiesenfeld [Phys. Rev. Lett. 59, 381 (1987)] on a square lattice in two dimensions with open boundaries. Develop ways to make the simulation as efficient as possible and explain them.

Plot the avalanche size distribution, $P(s)$, on a log-log scale where the avalanche size s is the number of times during the avalanche in which unstable sites distributed 4 grains to adjacent sites. Use linear regression to extract the exponent τ, where $P(S) \sim s^{-\tau}$. A reasonable choice of the lattice size is for example $L = 128$.

Problem 5

Consider a random walk (RW) on an undirected graph. The graph includes N nodes of degrees k_i, $i = 1, \ldots, N$. The time is discrete and at each step the RW moves from its current node to one of its neighbors in the graph. Using considerations analogous to the detailed balance condition, calculate the frequency at which the RW visits a node of a given degree k_i. Is it possible to extend this result to the case of a directed graph?

Problem 6

Consider the network growth model of Barabasi and Albert. In this model one starts with a small seed network of a few nodes, where some of the pairs are connected. The growth algorithm goes as follows. At each time step one adds a new node to the network. The new node sends a fixed number (such as $m = 2$) of links to existing nodes in the network. For each existing node, the probability that it will receive a link is proportional to its degree (namely the number of links it already has). This is called the preferential attachment principle. Write a code that simulates the growth of such networks and plot the degree distribution. Such networks exhibit a power-law degree distribution. Try to evaluate the exponent.

Good Luck!!