Racah Institute of Physics
The Hebrew University

Final exam (home exam) in Computational Physics of Complex Systems
Course number: 77732, presented by Prof. Ofer Biham
Semester B, 2023; Dates: July 18-20, 2023

Solve 5 problems of your choice out of the following 6 problems.

1. Show how to design a Wolff algorithm for the simulation of an XY model on a two-dimensional square lattice. Show that the algorithm satisfies detailed balance and that it is ergodic. No need to write codes or show simulation results, only to explain how the algorithm works.

2. Calculate the fractal dimension of a variant of the Cantor set, which is constructed starting from an interval of a unit length, in which at each stage one removes the middle 1/5 of each segment. Is the resulting dimension larger or smaller than the dimension of the canonical Cantor set in which the middle 1/3 is removed?

3. You are given a random number generator that produces numbers which are uniformly distributed in the range $[0, 1)$. Show how to use this random number generator in order to obtain random numbers distributed according to the exponential distribution

$$P(x) = Ke^{-\alpha x},$$

within the range $x > 0$, where K is a normalization constant that is determined by the values of $\alpha > 0$.

4. Consider a model of network growth, in which at each time step one node is added to the network. The new node forms a link to one random node among the existing nodes in the network, which is selected randomly. The initial seed of the network consists of a single node. Write down the master equation for the time evolution of the degree distribution $P_t(k)$ of the network

at time t. The resulting network exhibits a tree structure. Assuming that the degree distribution $P_t(k)$ converges towards a steady-state form $P(k)$, try to solve the master equation under steady state conditions and find a closed-form exression for $P(k)$. Calculate the mean degree $\langle K \rangle$.

5. Consider a finite one dimensional lattice of length $N+1$. The sites of the lattice are indexed by $i = 0, 1, 2, \ldots, N$ (from left to right). At time $t = 0$ a random walker (RW) is located at $i = 0$. At each time step the RW hops one step to the right or to the left with equal probabilities (except for the boundary site at $i = 0$ from which it can only hop to the right). Calculate the mean first passage time from the initial node $i = 0$ to the target node $i = N$, namely the mean number of time steps that are required for an RW starting from $i = 0$ to reach the site $i = N$ for the first time. Also, calculate the mean first return time, namely the mean number of times steps that are required for an RW to return to the initial node $i = 0$ for the first time. Calculate the probability $P_{\rm FR}$ that the RW will return to $i = 0$ before it will reach reach $i = N$ for the first time. The probability $P_{\rm FP}$ that the RW will reach $i = N$ for the first time before returning to $i = 0$ is given by $P_{\rm FP} = 1 - P_{\rm FR}$.

6. In class we considered the Nagel-Schreckenberg model for highway traffic [for details of the model see Nagel and Schreckenberg, J. Phys. I France 2, 2221 (1992)]. Write down the rate equations for the densities c_0, c_1 and c_2 of cars whose instantaneous speeds are 0, 1 and 2, respectively. Try to solve these equations for steady-state conditions. In case you find the general equations too complicated, it is sufficient to solve them for the deterministic case, in which the probability p of randomly slowing down satisfies $p = 0$. Use the results to obtain a closed-form expression for the flux as a function of the density of cars, c. Plot the flux vs. c (the fundamental diagram) and discuss its properties. Are the mean field results expected to under-estimate or over-estimate the flux compared to simulation results of the model?

Good Luck!!