Contents

Preface

The notes are closely aligned to the NUS Mathematics lecture notes (and some extras along the way) and a reference book titled 'Principles of Mathematical Analysis' by Walter Rudin. For the latter, the chapters in the book that are covered in the notes are Chapter 1 - The Real and Complex Number Systems, Chapter 3 - Numerical Sequences and Series, Chapter 4 - Continuity, Chapter 5 - Differentiation, Chapter 6 - The Riemann-Stieltjes Integral, Chapter 7 - Sequences and Series of Functions and Chapter 8 - Some Special Functions.

1 The Real Numbers, $\mathbb{R}$

1.1 Set Operations

Given two sets A and B, if every element of A is an element of B, then A is a subset of B. That is, $A \subseteq B$.

- The union of A and B is defined by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- The intersection of A and B is defined by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- The complement of A relative to B is defined by

$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

A set with no elements is known as the empty set, or the null set. It is denoted by $\varnothing$.

1.2 Number Systems

$\mathbb{N}$ is the set of natural numbers (also known as positive integers), $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

A number n_1 is rational if $n_1 = p/q$, where $p, q \in \mathbb{Z}$ but $q \neq 0$.

A number n_2 is complex if $n_2 = a + bi$, where $a, b \in \mathbb{R}$. The real part of n_2, or $\Re(n_2)$, is a, and similarly, the imaginary part of n_2, or $\Im(n_2)$, is b.

We have the following result:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

The real numbers can be constructed as **Dedekind cuts of rational numbers**. The interested can see a proof quoted in the early stages of Rudin's book or the appendix.

1.3 The Natural Numbers, $\mathbb{N}$

1.3.1 Principle of Mathematical Induction

If we want to prove a statement is true using mathematical induction, we let $P(n)$ denote the proposition, where $n \in \mathbb{N}$. If $P(1)$ is true and $P(k)$ is true $\implies$ $P(k+1)$ is true, then $P(n)$ is true for all $n \in \mathbb{N}$. Mathematical induction can be used to establish a given result involving series and recurrence relations, derivatives, inequalities, or divisibility. We can also form conjectures and prove them via induction.

1.3.2 Strong Induction

Let $P(n)$ denote the proposition, where $n \in \mathbb{N}$. Suppose $1 \leq i \leq k$. If all the $P(i)$'s are true $\implies$ $P(k+1)$ is true, then $P(n)$ is true for all $n \in \mathbb{N}$.

1.3.3 Well-Ordering Principle

Well-Ordering Principle

Every non-empty subset A of $\mathbb{N}$ has a least element, i.e. there exists $p \in A$ such that $p \leq a$ for all $a \in A$.

1.4 The Rational Numbers, $\mathbb{Q}$

1.5 Countability of the Rationals

THEOREM

$\mathbb{Q}$ is countable

Proof: We use the **Cantor Method** in our proof. The idea is to establish a bijection from $\mathbb{Q}$ to the Cartesian product $\mathbb{Z} \times \mathbb{N}$. That is, $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$. Define $f(p/q) = (p, q)$, where $p \in \mathbb{Z}, q \in \mathbb{N}$. We can represent the output values on a lattice grid, or rather, an array. From here, we can generate $\mathbb{Q}^+$, that is

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \ldots.$$

Note that there is more than one point on the lattice grid denoting $1/2$ (and other rational numbers too). For example, the lattice points $(2, 4), (3, 6), \ldots, (k, 2k)$, where $k \geq 1$, can be plotted by considering $f(1/2)$. This shows that f is injective. f is surjective too. Note that in our proof, we have also taken into consideration Q^- as well as 0, and since $\mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\} = \mathbb{Q}$, we are done. $\square$

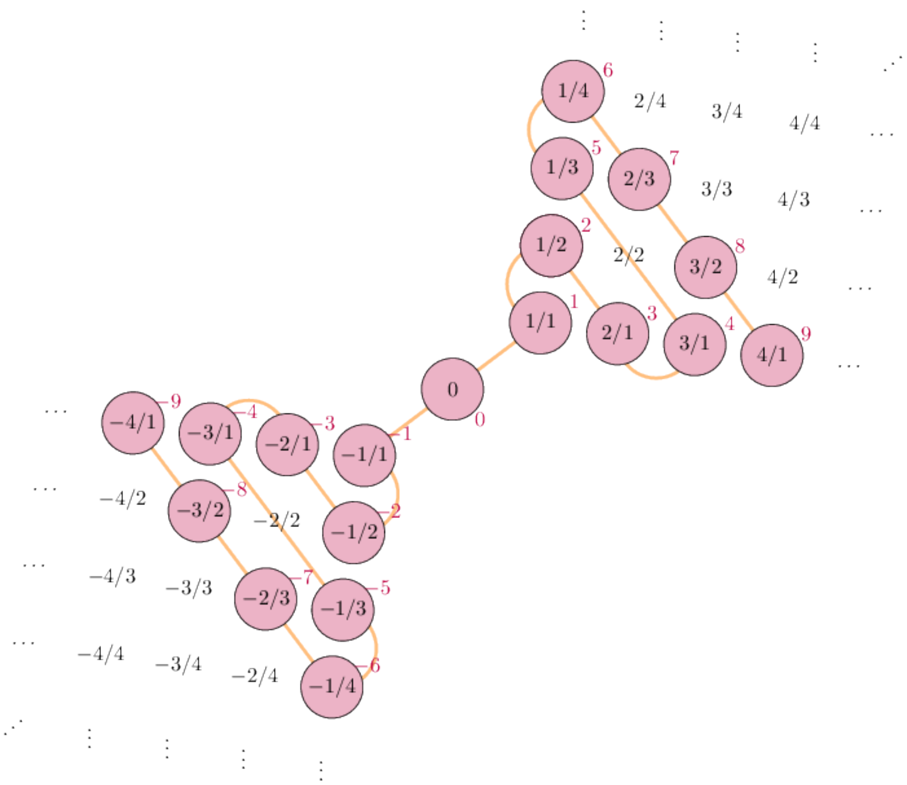

Figure 1: Visual Proof of Cantor's Method

1.5.1 Calkin-Wilf Sequence

The interested reader can search up on the Calkin-Wilf Tree, or the Calkin-Wilf Sequence, which provides an explicit formula to generate the rationals. Fun fact, I didn't come across it until a couple of days ago when my friend, Russell Saerang, shared with me a post on Instagram.

The Calkin-Wilf Sequence can be generated by the formula

$$q_{i+1} = \frac{1}{2\lfloor q_i \rfloor - q_i + 1},$$

where q_i denotes the i^{th} number in the sequence, with $q_1 = 1$ and $\lfloor q_i \rfloor$ is the floor function of q_i. It is also known as the greatest integer function.

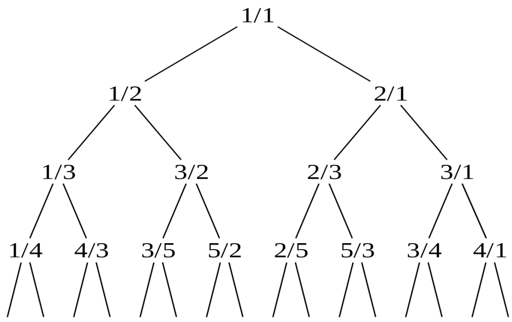

Figure 2: Calkin-Wilf Tree

1.6 The Real Numbers, $\mathbb{R}$

$\mathbb{R}$ is a complete ordered field since it satisfies the nine field axioms. Sidetrack to a bit of Abstract Algebra, the best known fields are those of $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Many other fields, such as p-adic fields are commonly used and studied in Mathematics, particularly in Number Theory. There are certain properties which are deemed trivial and shall not be discussed. For example, the trichotomy property states that if $a, b \in \mathbb{R}$, then either $a < b$, $a > b$ or $a = b$, which is intuitive.

1.6.1 Completeness

Let $S \subseteq \mathbb{R}$ be non-empty. A number u is an upper bound for S if $x \leq u$ for all $x \in S$ and conversely, u is a lower bound for S if $x \geq u$ for all $x \in S$. Similarly, the definitions of bounded above, bounded below, bounded and unbounded are not discussed.

The real numbers satisfy the completeness axiom. As coined by Dedekind, the term *completeness* means that there are no gaps or missing points in the real number line.

1.6.2 Supremum and Infimum

A real number M is the supremum (least upper bound) of S if M is an upper bound of S and $M \leq u$ for every upper bound $u \in S$. That is, $M = \sup(S)$. A real number L is the infimum (greatest lower bound) of S if L is an upper bound of S and $L \geq v$ for every lower bound $v \in S$. That is, $L = \inf(S)$.

The supremum and infimum of a set may or may not be elements of the set.
Example: Consider $S = \{x \in \mathbb{R} : 0 < x < 1\}$, where $\inf(S) = 0$ and $\sup(S) = 1$, but $0, 1 \notin S$.

LEMMA

Let u be an upper bound of $S \subseteq \mathbb{R}$. Then $u = \sup(S)$ if and only if $\forall \varepsilon > 0$, $\exists\, x_\varepsilon \in S$ such that $u - \varepsilon < x_\varepsilon$.

Now, we wish to state the supremum and infimum properties of $\mathbb{R}$. This is known as the completeness axioms. These state that every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum, and similarly, every non-empty subset of $\mathbb{R}$ which is bounded below has an infimum.

1.6.3 Archimedean Property

Archimedean Property

If $x, y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{N}$ such that $nx > y$.

COROLLARY

$\mathbb{N}$ is not bounded above

Proof: For any $\varepsilon > 0$, $\exists\, n \in \mathbb{N}$ such that $1/n < \varepsilon$. This can be justified by setting $x = \varepsilon$ and $y = 1$.

1.6.4 Existence of n^{th} Root and Rational Exponents

Let $a > 0$ and $n \in \mathbb{N}$. There exists a unique positive real number u such that $u^n = a$. The number u is known as the positive n^{th} root of a and thus,

$$u = \sqrt[n]{a} = a^{1/n}.$$

THEOREM

In general, if n is non-square, then $\sqrt{n}$ is irrational.

Proof: Suppose on the contrary that $\sqrt{n}$ is rational, where n is non-square. Then,

$$\sqrt{n} = p/q \implies nq^2 = p^2,$$

where $p, q \in \mathbb{N}, q \neq 0$ and $\gcd(p,q) = 1$. By the Fundamental Theorem of Arithmetic, by considering the prime factorisation of p^2 and q^2, each one of them has an even number of primes. Thus, n must also have an even number of primes. As n is non-square, then there exists at least a prime number with an odd multiplicity, which contradicts the claim made earlier. $\square$

1.6.5 Density

Archimedean Property

The Density Theorem asserts that the rational numbers are *dense* in $\mathbb{R}$, i.e. if $a, b \in \mathbb{R}$ such that $a < b$, then $\exists\, r \in \mathbb{Q}$ such that $a < r < b$. In short, it implies that we are always able to find another rational number that lies between two real numbers.

If $a, b \in \mathbb{R}$ such that $a < b$, then $\exists\, x \in \mathbb{Q}'$ so that $a < x < b$, where $\mathbb{Q}'$ denotes the set of irrationals.

THEOREM

Every interval $I \subseteq \mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

1.7 Important Inequalities

1.7.1 Bernoulli's Inequality

Bernoulli's Inequality (named after Jacob Bernoulli) is an inequality that approximates exponentiations of $1+x$.

Bernoulli's Inequality

One widely-used variant is that for every integer $r \geq 0$ and real number $x \geq -1$, we have

$$(1+x)^r \geq 1 + rx$$

and the inequality is strict if $x \neq 0$ and $r \geq 2$.

Proof: We can prove this via strong induction. Let $P(n)$ be the proposition that $(1+x)^n \geq 1 + nx$ for $n \geq 0$ and $x \in \mathbb{R}, x \geq -1$. We verify that the base cases $P(0)$ and $P(1)$ are true.

For $P(0)$, $(1+x)^0 = 1$ and $1 + 0(x) = 1$, so $P(0)$ is true. For $P(1)$, $(1+x)^1 = 1 + x$ and $1 + 1(1) = 1 + x$, so $P(1)$ is true as well.

Next, we assume that $P(k)$ is true for some $k \in \mathbb{N}$. That is, to assume that $(1+x)^k \geq 1+kx$. We shall prove that $P(k+1)$ is true. That is, $(1+x)^{k+1} \geq 1+(k+1)x$. To establish this inequality, note that

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1+(k+1)x+kx^2 \geq 1+(k+1)x.$$

Since $P(0)$ and $P(1)$ are true and $P(k)$ is true $\implies$ $P(k+1)$ is true, then by Mathematical Induction, $P(n)$ is true for all $n \geq 0$. $\square$

1.7.2 QM-AM-GM-HM Inequality

QM-AM-GM-HM Inequality

For a collection of n non-negative real numbers $a_1,\ a_2,\ldots,\ a_n$, we denote the Quadratic Mean as $Q(n)$, Arithmetic Mean as $A(n)$, Geometric Mean as $G(n)$ and Harmonic Mean as $H(n)$, where

$$Q(n) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2},\ A(n) = \frac{1}{n}\sum_{i=1}^{n} x_i,\ G(n) = \sqrt[n]{\prod_{i=1}^{n} x_i} \text{ and } H(n) = n\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}$$

and the QM-AM-GM-HM Inequality states that $Q(n) \geq A(n) \geq G(n) \geq H(n)$. Equality is attained if and only if $a_1 = a_2 = \ldots = a_n$. The Quadratic Mean $Q(n)$ is also referred to as Root Mean Square, or RMS.

Proof: We first prove that $Q(n) \geq A(n)$. By the Cauchy-Schwarz Inequality,

$$n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2$$

and hence,

$$\frac{n[Q(n)]^2}{n} \geq [nA(n)]^2.$$

With some simple rearrangement, the result follows. $\square$

There are numerous proofs of the AM-GM Inequality like using backward-forward induction (by Cauchy), considering e^x (by Pólya), Method of Lagrange Multipliers etc. This proof hinges on Jensen's Inequality.

Jensen's Inequality

For a concave function $f(x)$,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Consider the logarithmic function $f(x) = \ln x$, where $x \in \mathbb{R}^+$. It can be easily verified that $f(x)$ is concave as $f''(x) = -1/x^2 < 0$. We wish to prove

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right).$$

Using Jensen's Inequality,

$$\frac{1}{n}\sum_{i=1}^{n}\ln(x_i) \leq \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Note that

$$\sum_{i=1}^{n}\ln(x_i) = \ln(x_1)+\ln(x_2)+\ldots+\ln(x_n) = \ln\left(\prod_{i=1}^{n} x_i\right).$$

As such, the inequality becomes
$$\ln\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\ge\frac{1}{n}\ln\left(\prod_{i=1}^{n}x_i\right).$$
With some simple rearrangement, the result follows. $\square$

Lastly, we will prove the GM-HM Inequality using the AM-GM Inequality. Noting that
$$\prod_{i=1}^{n}\frac{1}{x_i}=\left(\prod_{i=1}^{n}x_i\right)^{-1},$$
we have
$$\frac{n/H(n)}{n}\ge\frac{1}{G(n)}$$
and rearranging, we are done. $\square$

1.7.3 Triangle Inequality

Triangle Inequality

For $x,y\in\mathbb{R}$, $|x+y|\le|x|+|y|$ and equality is attained if and only if $xy\ge 0$.

We can derive the following inequalities too:
(1): $|x-y|\le|x|+|y|$ and
(2) Reverse Triangle Inequality: $||x|-|y||\le|x-y|$

(1) can be easily proven by replacing $-y$ with y. We only prove **(2)**.
Proof: We can write x as $x-y+y$ and y as $y-x+x$. Hence,
$$\begin{aligned}|x|&=|x-y+y|\le|x-y|+|y|\\|y|&=|y-x+x|\le|y-x|+|x|=|x-y|+|x|\end{aligned}$$
As such, $|x|-|y|\le|x-y|$ and $|x|-|y|\le-|y-x|$, and taking the absolute value of $|x|-|y|$, the result follows. $\square$

COROLLARY

For $a_i\in\mathbb{R}$, where $1\le i\le n$, by applying the Triangle Inequality repeatedly,
$$\left|\sum_{i=1}^{n}a_i\right|\le\sum_{i=1}^{n}|a_i|.$$

1.8 Appendix: Construction of the Real Numbers using Dedekind Cuts

The following construction is adapted from Rudin's book titled 'Principles of Mathematical Analysis'.

Step 1

The members of $\mathbb{R}$ will be certain subsets of $\mathbb{Q}$, called cuts. A cut is any set $\alpha \subset \mathbb{Q}$ such that
(i) α is non-empty, $\alpha \neq \mathbb{Q}$
(ii) if $p \in \alpha$ and $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$
(iii) if $p \in \alpha$, then $p < r$ for some $r \in \alpha$

Here, $p, q, r, \ldots$ denote the rational numbers, so $p, q, r \in \mathbb{Q}$, whereas $\alpha, \beta, \gamma, \ldots$ denote cuts. Note that **(iii)** says that α has no largest member. Also, **(ii)** implies if $p \in \alpha$ and $q \notin \alpha$, then $p < q$, as well as if $r \notin \alpha$ and $r < s$, then $s \notin \alpha$.

Step 2

Define $\alpha < \beta$ to be such that α is a proper subset of β. We show that $\mathbb{R}$ is an ordered set.

Proof: Suppose $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$. This is because a proper subset of a proper subset is also a proper subset. By the **trichotomy property**, either one of the relations

$$\alpha < \beta, \ \alpha = \beta \text{ or } \alpha > \beta$$

can hold for any pair α, β. To show that at least one of the relations holds true, assume that the first two fail. Then, $\alpha > \beta$, so α is not a subset of β. Hence, there exists $p \in \alpha$ such that $p \notin \beta$. If $q \in \beta$, then $q < p$, and so $q \in \alpha$. Thus, $\beta \subset \alpha$ but since $\beta \neq \alpha$, then $\beta < \alpha$. □

Step 3

$\mathbb{R}$ has the least upper bound property.

Proof: Let A be a non-empty subset of $\mathbb{R}$, and assume that $\beta \in \mathbb{R}$ is an upper bound of A. Define γ to be the union of all $\alpha \in A$. In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. We show that $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.

Since A is non-empty, then there exists some $\alpha_0 \in A$, where α_0 is non-empty. It follows that γ is non-empty too, so with reference to Step 1, it satisfies **(i)**. As $\alpha \subset \beta$ for every $\alpha \in A$, then $\gamma \subset \beta$, so $\gamma \neq \mathbb{Q}$. Pick $p \in \gamma$. Then, $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$, so $q \in \gamma$. Thus, **(ii)** is satisfied. If $r \in \alpha_1$ is chosen such that $r > p$, then $r \in \gamma$, and so γ satisfies **(iii)**. Therefore, $\gamma \in \mathbb{R}$.

It is clear that $\alpha \leq \gamma$ for every $\alpha \in A$. Suppose $\delta < \gamma$. Then, there exists $s \in \gamma$ such that $s \neq \delta$. Since $s \in \gamma$, then $s \in \alpha$ for some $\alpha \in A$, so $\delta < \alpha$, and therefore, δ is not an upper bound of A. We conclude that $\gamma = \sup A$. □

Step 4

If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, we define $\alpha + \beta$ to be the set of all sums $r + s$, where $r \in \alpha$ and $s \in \beta$. First, we define 0^* to be the set of all negative rational numbers, that is $\mathbb{Q}^-$. Note that $\mathbb{Q}^-$ is a cut. We verify that the five axioms for addition hold in $\mathbb{R}$, known as the **Peano Axioms**.

(A1) Closure: $\alpha + \beta$ is a cut
Proof: It is clear that $\alpha + \beta$ is a non-empty subset of $\mathbb{Q}$. Take $r' \notin \alpha$ and $s' \notin \beta$. Then, $r' + s' > r + s$ for all $r \in \alpha$ and $s \in \beta$. Thus, $r' + s' \notin \alpha + \beta$ and so $\alpha + \beta$ satisfies **(i)**. Next, choose $p \in \alpha + \beta$, so $p = r + s$ for some $r \in \alpha$ and $s \in \beta$. If $q < p$, then $q - s < r$, so $q - s \in \alpha$. Writing q as $(q - s) + s$, we see that $q \in \alpha + \beta$, so **(ii)** holds. Lastly, choosing $t \in \alpha$ such that $t > r$, we see that $p < t + s$ and $t + s \in \alpha + \beta$, so **(iii)** is satisfied. □

(A2) Commutativity: $\alpha+\beta=\beta+\alpha$
Proof: $\alpha+\beta$ is the set of all $r+s$ with $r\in\alpha$ and $s\in\beta$. In a similar vein, $\beta+\alpha$ is the set of all $s+r$. Since $r+s=s+r$ for all $r,s\in\mathbb{Q}$, the result follows. □

(A3) Associativity: This is obvious from **(A2)**.

(A4) Existence of identity element: $\alpha+0^*=\alpha$
Proof: If $r\in\alpha$ and $s\in 0^*$, then $r+s<r$, hence $r+s\in\alpha$. So, $\alpha+0^*\subset\alpha$. Next, pick $p,r\in\alpha$ such that $r>p$. Then, $p-r\in 0^*$. Writing p as $r+(p-r)$, we see that $p\in\alpha+0^*$, so $\alpha\subset\alpha+0^*$. The result follows. □

(A5) Existence of inverse: The inverse of β is $-\alpha$ and vice versa
Proof: Fix $\alpha\in\mathbb{R}$. Let β be the set of all p such that

$$\text{there exists } r>0 \text{ such that } -p-r\notin\alpha.$$

That is, there exists some rational number smaller than $-p$ that fails to be in α. It suffices to show that $\beta\in\mathbb{R}$ and $\alpha+\beta=0^*$.

If $s\neq\alpha$ and $p=-s-1$, then $-p-1\notin\alpha$, so $p\in\beta$, and thus, β is non-empty. If $q\in\alpha$, then $-q\notin\beta$, so $\beta\neq\mathbb{Q}$. Hence, β satisfies **(i)**. Next, pick $p\in\beta$ and $r>0$ so that $-p-r\notin\alpha$. If $q<p$, then $-q-r>-p-r$, hence, $-q-r\notin\alpha$. Thus, $q\in\beta$, and **(ii)** holds. Setting $t=p+(r/2)$, we have $t>p$ and $-t-(r/2)=-p-r\notin\alpha$, so $t\in\beta$. As such, β satisfies **(iii)**. This shows that $\beta\in\mathbb{R}$.

Next, if $r\in\alpha$ and $s\in\beta$, then $-s\notin\alpha$, so $r<-s$ and so $r+s<0$. Thus, $\alpha+\beta\subset 0^*$. To prove the opposite inclusion, pick $v\in 0^*$ and put $w=-v/2$. Then, $w>0$, so there exists $n\in\mathbb{Z}$ such that $nw\in\alpha$ but $(n+1)w\notin\alpha$. We remark that *this depends whether $\mathbb{Q}$ has the Archimedean Property*. Put $p=-(n+2)w$. Then, $p\in\beta$ since $-p-w\notin\alpha$ and

$$v=nw+p\in\alpha+\beta,$$

so $0^*\subset\alpha+\beta$. We conclude that $\alpha+\beta=0^*$.

Step 5

If $\alpha,\beta,\gamma\in\mathbb{R}$ and $\beta<\gamma$, then $\alpha+\beta<\alpha+\gamma$.

This is obvious from the definition of $+$ in $\mathbb{R}$ that $\alpha+\beta\subset\alpha+\gamma$. If we had $\alpha+\beta=\alpha+\gamma$, the cancellation law would imply $\beta=\gamma$. It also follows that $\alpha>0^*$ if and only if $-\alpha<0^*$.

Step 6

Multiplication is more cumbersome since the products of negative rationals are positive. We'll first confine ourselves to $\mathbb{R}^+$, namely, the set of all $\alpha\in\mathbb{R}$ with $\alpha>0^*$.

If $\alpha\in\mathbb{R}^+$ and $\beta\in\mathbb{R}^+$, define $\alpha\beta$ to be the set of all p such that $p\leq rs$ for some choice of $r\in\alpha, s\in\beta$, where $r>0$ and $s>0$. Also, define 1^* to be the set of all $q<1$. Note that if $\alpha>0^*$ and $\beta>0^*$, then $\alpha\beta>0^*$. One can also check that the distributive law holds in $\mathbb{R}^+$.

We remark the axioms for multiplication. Note that this applies to a field $\mathbb{F}$ in general.
(M1) Closure: If $x\in\mathbb{R}$ and $y\in\mathbb{R}$, then $xy\in\mathbb{R}$
(M2) Commutativity: $xy=yx$ for all $x,y\in\mathbb{R}$
(M3) Associativity: $(xy)z=x(yz)$ for all $x,y,z\in\mathbb{R}$
(M4) Existence of identity element: $\mathbb{R}$ contains an element $1\neq 0$ such that $1x=x$ for every $x\in\mathbb{R}$
(M5) Existence of inverse: If $x\in\mathbb{R}$ and $x\neq 0$, then there exists an element $1/x\in\mathbb{R}$ such that $x\cdot(1/x)=1$

Step 7

We will now complete the definition of multiplication by setting $\alpha 0^* = 0^* \alpha = 0^*$ and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if} \alpha < 0^*, \beta > 0^*, \\ -[\alpha(-\beta)] & \text{if} \alpha > 0^*, \beta < 0^*. \end{cases}$$

The products on the right were defined in Step 6.

It is simple to prove the axioms for multiplication in $\mathbb{R}$ as we claimed that they hold in $\mathbb{R}^+$ as mentioned. This is merely a consequence of the repeated application of $\gamma = -(-\gamma)$.

In addition, the proof of the distributive law

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$$

involves some casework.
Proof: For instance, suppose $\alpha > 0^*$, $\beta < 0^*$ and $\beta + \gamma > 0^*$. Then, $\gamma = (\beta+\gamma) + (-\beta)$, and since the distributive law holds in $\mathbb{R}^+$, we have

$$\alpha\gamma = \alpha(\beta+\gamma) + \alpha\cdot(-\beta).$$

However, $\alpha \cdot (-\beta) = -(\alpha\beta)$, so

$$\alpha\beta + \alpha\gamma = \alpha(\beta+\gamma).$$

The other cases have similar proofs. □

We conclude that $\mathbb{R}$ is an ordered field with the least upper bound property.

Step 8

We associate with each $r \in \mathbb{Q}$ the set r^* which contains of all $p \in \mathbb{Q}$ such that $p < r$. Each r^* is a cut. That is, $r^* \in \mathbb{R}$. These cuts satisfy the following relations:
(a) $r^* + s^* = (r+s)^*$
(b) $r^* s^* = (rs)^*$
(c) $r^* < s^*$ if and only if $r < s$

We first prove **(a)**.
Proof: Choose $p \in r^* + s^*$, so $p = u + v$, where $u < r$ and $v < s$. Hence, $p < r + s$, so $p \in (r+s)^*$. Conversely, suppose $p \in (r+s)^*$. Then, $p < r+s$. Choose t so that $2t = r + s - p$, and put

$$r' = r - t \text{ and } s' = s - t.$$

Then, $r' \in r^*$, $s' \in s^*$ and $p = r' + s'$. Hence, $p \in r^* + s^*$. □

The proof for **(b)** is similar to **(a)**. We now prove **(c)**.
Proof: If $r < s$, then $r \in s^*$, but $r \notin r^*$, so $r^* < s^*$. If $r^* < s^*$, then there exists $p \in s^*$ such that $p \notin r^*$. Hence, $r \le p < s$, so $r < s$. □

Step 9

In Step 8, the replacement of the rational numbers r by the corresponding rational cuts $r^* \in \mathbb{R}$ preserves sums, products and order. This may be expressed by saying that the ordered field $\mathbb{Q}$ is *isomorphic* (another word for a bijective homomorphism) to the ordered field $\mathbb{Q}^*$ whose elements are the rational cuts. It is this identification of $\mathbb{Q}$ with $\mathbb{Q}^*$ which allows us to regard $\mathbb{Q}$ as a subfield of $\mathbb{R}$.

2 Sequences

A sequence in $\mathbb{R}$ is a real-valued function X with domain $\mathbb{N}$. That is

$$X : \mathbb{N} \to \mathbb{R}.$$

The sequence X is usually denoted by x_n or $X(n)$.

2.1 Limit of a Sequence

2.1.1 ε-K Definition of a Sequence

Let $a \in \mathbb{R}$ and $\varepsilon > 0$. The ε-neighbourhood of a is the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon).$$

ε-K Definition of a Sequence

For a sequence of numbers x_n, L is the limit of the sequence if for every $\varepsilon > 0$, $\exists K \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \ \forall n \geq K.$$

If L exists, then x_n is convergent. x_n diverges otherwise.

Uniqueness of Limit of Sequence

The limit of a sequence x_n is unique. That is, suppose

$$\lim_{n\to\infty} x_n = L \text{ and } \lim_{n\to\infty} x_n = L'.$$

Then, $L = L'$.

Proof: We will use the Triangle Inequality in our proof. Suppose L and L' are the limits of x_n. Then, let $\varepsilon > 0$ be arbitrary, and set $\varepsilon' = \varepsilon/2$. Since $x_n \to L$, $\exists K_1 = K_1(\varepsilon') \in \mathbb{N}$ such that for all $n \geq K_1$,

$$|x_n - L| < \varepsilon' = \frac{\varepsilon}{2}.$$

Similarly, as $x_n \to L'$, $\exists K_2 = K_2(\varepsilon') \in \mathbb{N}$ such that for all $n \geq K_2$,

$$|x_n - L'| < \varepsilon' = \frac{\varepsilon}{2}.$$

We wish to prove that $|L - L'| < \varepsilon$. For all $n \geq K$, by the Triangle Inequality, we can establish that

$$\begin{aligned} |L - L'| &= |L - x_n + x_n - L'| \\ &\leq |x_n - L| + |x_n - L'| \\ &< 2\varepsilon' = \varepsilon \end{aligned}$$

Since ε is arbitrary, we can set $|L - L'| = 0$, resulting in $L = L'$. $\square$

The Triangle Inequality is a helpful tool when finding limits. Note that changing a finite number of terms in a sequence does not affect its convergence or its limit.

Proving that a Sequence converges to L

We state a method to prove that a given sequence x_n converges to L.
Step 1: Express $|x_n - L|$ in terms of n, and find a simple upper bound $L(n)$ for it.
Step 2: Let $\varepsilon > 0$ be arbitrary. Find $K \in \mathbb{N}$ such that for all $n \geq K$, $L(n) < \varepsilon$. Then, $|x_n - L| < \varepsilon$.

Example: Prove that

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

Proof: Let $\varepsilon>0$. By the Archimedean Property, $\exists K=K(\varepsilon)\in\mathbb{N}$ such that $K>1/\varepsilon$. Thus, if $n\geq K$, then $n>1/\varepsilon$, and $1/n<\varepsilon$. Hence, for all $n\geq K$,

$$\left|\frac{1}{n}-0\right|<\varepsilon.$$

□

Example: Prove that

$$\lim_{n\to\infty}\frac{2n^2+1}{n^2+3n}=2.$$

Proof:

$$\begin{aligned}\left|\frac{2n^2+1}{n^2+3n}-2\right|&=\left|\frac{1-6n}{n^2+3n}\right|\\&\leq\frac{1+6n}{n^2+3n}\\&<\frac{1+6n}{n^2}\\&<\frac{n+6n}{n^2}=\frac{7}{n}\end{aligned}$$

Let $\varepsilon>0$ be given. Choose $K\in\mathbb{N}$ such that $K>7/\varepsilon$. Then, for all $n\geq K$, we have

$$\left|\frac{2n^2+1}{n^2+3n}-2\right|<\frac{7}{n}\leq\frac{7}{K}<\varepsilon.$$

□

2.1.2 Limit Theorems

(1) Every convergent sequence is bounded:
That is, if

$$\lim_{n\to\infty}x_n=L,$$

then $|x_n|\leq M$. However, the converse is not true. **Not all bounded sequences are convergent**.
Example: The sequence $x_n=(-1)^n$ is bounded by -1 and 1 and it oscillates about only these two values. However, as $n\to\infty$, the limit does not exist!

Let us prove that every convergent sequence is bounded.
Proof: Suppose

$$\lim_{n\to\infty}x_n=L.$$

Set $\varepsilon=1$ and take $K\in\mathbb{N}$ such that $|x_n-L|<1$ for all $n\geq K$. Then, $L-1<x_n<L+1$. Let $|x_n|=\max(|L-1|,|L+1|)$ for all $n\geq N$. Since $\{|x_1|,|x_2|,\ldots,|x_{N-1}|\}$ is a finite set, then it must contain a maximum. Hence, for $1\leq n\leq N-1$, $|x_n|\leq A$. Consider $M=\max(|L-1|,|L+1|,A)$, setting $|x_n|\leq M$ and we are done. □

(2) Linearity:
Just like how linear operators (i.e. derivatives and integrals) work, we have a similar result for limits. Suppose $\alpha,\beta\in\mathbb{R}$ and

$$\lim_{n\to\infty}x_n=L_1\text{ and }\lim_{n\to\infty}y_n=L_2,$$

then

$$\lim_{n\to\infty}(\alpha x_n\pm\beta y_n)=\alpha L_1\pm\beta L_2.$$

Proof: We shall prove that

$$\lim_{n\to\infty}(x_n+y_n)=L_1+L_2.$$

There exists $K_1,K_2\in\mathbb{N}$ such that $|x_n-L_1|<\varepsilon/2$ for all $n\ge K_1$ and $|y_n-L_2|<\varepsilon/2$ and $n\ge K_2$. Set $K=\max(K_1,K_2)$. By the Triangle Inequality,

$$|x_n-L_1+y_n-L_2|<|x_n-L_1|+|y_n-L_2|<\varepsilon$$

and we are done. □

(3) Product and Quotient:
Considering the sequences x_n and y_n as mentioned in **(2)**, we can establish the following results:

$$\lim_{n\to\infty}x_ny_n=L_1L_2$$

and

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\frac{L_1}{L_2},$$

provided that $y_n,y\neq 0\ \forall n\in\mathbb{N}$.

We only prove the result involving the product of two sequences.
Proof: Since $|x_n|$ is convergent, then it is bounded, so $|x_n|\le M_1$ for all $n\in\mathbb{N}$. We have

$$\begin{aligned}|x_ny_n-L_1L_2|&=|x_ny_n-x_nL_2+x_nL_2-L_1L_2|\\&\le|x_ny_n-x_nL_2|+|x_nL_2-L_1L_2|\\&=|x_n||y_n-L_2|+|L_2||x_n-L_1|\\&\le M_1|y_n-L_2|+|L_2||x_n-L_1|\end{aligned}$$

Set $M=\max(M_1,|L_2|)>0$. So, $M_1|y_n-L_2|+|L_2||x_n-L_1|\le M(|y_n-L_2|+|x_n-L_1|)$. Let $\varepsilon>0$ be given. Then, there exists $K_1,K_2\in\mathbb{N}$ such that

$$\begin{aligned}|x_n-L_1|&<\varepsilon/2M\text{ for all }n\ge K_1\\|y_n-L_2|&<\varepsilon/2M\text{ for all }n\ge K_2\end{aligned}$$

Let $K=\max(K_1,K_2)$. Hence,

$$|x_ny_n-L_1L_2|<M\left(\frac{\varepsilon}{2M}+\frac{\varepsilon}{2M}\right)<\varepsilon.$$

and we are done. □

COROLLARY

If x_n converges and $k\in\mathbb{N}$, then

$$\lim_{n\to\infty}x_n^k=\left(\lim_{n\to\infty}x_n\right)^k.$$

(4): If $|x_n|\to 0$, then $x_n\to 0$.

(5): If $0<b<1$, then

$$\lim_{n\to\infty}b^n=0.$$

Hint: Write $a=1/b-1$ and use Bernoulli's Inequality.

Proof: We have $b=1/(1+a)$. By Bernoulli's Inequality, $(1+a)^n\ge 1+na$. Hence,

$$\frac{1}{(1+a)^n}\le\frac{1}{1+na}<\frac{1}{na}.$$

By the Archimedean Property, $1/(na) < \varepsilon$ and we are done. □

(6): If $c > 0$, then

$$\lim_{n\to\infty} c^{1/n} = 1.$$

If $c = n$, the limit is still the same.
Hint: Consider $0 \le c \le 1$ and $c > 1$. For the first case, write $d_n = c^{1/n} - 1$ and use Bernoulli's Inequality.

(7):

$$\lim_{n\to\infty} x_n = L \implies \lim_{n\to\infty} |x_n| = |L|.$$

(8): Suppose $x_n \ge 0 \ \forall n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} x_n = L \implies \lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}.$$

(9): If $x_n \ge 0 \ \forall n \in \mathbb{N}$ and x_n converges, then

$$\lim_{n\to\infty} x_n \ge 0.$$

COROLLARY

If x_n and y_n are convergent sequences, and $x_n \ge y_n \ \forall n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n \ge \lim_{n\to\infty} y_n.$$

COROLLARY

If $a, b \in \mathbb{R}$ and $a \le x_n \le b \ \forall n \in \mathbb{N}$ and x_n is convergent, then

$$a \le \lim_{n\to\infty} x_n \le b.$$

Example: Suppose we wish to evaluate the following limit:

$$\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}}$$

Solution: For this example, we need to recognise that for $0 \le a < 1$, then $a^n \to 0$ as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}} &= \lim_{n\to\infty} \frac{2^n + 3\left(3^n\right) + 25\left(5^n\right)}{4\left(2^n\right) + 3^n + 5\left(5^n\right)} \\
&= 5 - \lim_{n\to\infty} \frac{19\left(2^n\right) + 2\left(3^n\right)}{4\left(2^n\right) + 3^n + 5\left(5^n\right)} \\
&= 5 - \lim_{n\to\infty} \frac{19\left(\frac{2}{5}\right)^n + 2\left(\frac{3}{5}\right)^n}{4\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n + 5} \\
&= 5
\end{aligned}$$

□

2.1.3 Squeeze Theorem

Squeeze Theorem

Let x_n, y_n and z_n be sequences of numbers such that for all $n \in \mathbb{N}$, $x_n \le y_n \le z_n$. If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L,$$

then

$$\lim_{n\to\infty} y_n = L.$$

Proof: Let $\varepsilon > 0$. Since $x_n \to L$ and $z_n \to L$, then $\exists K \in \mathbb{N}$ such that for all $n \geq K$,

$$|x_n - a| < \varepsilon \text{ and } |z_n - a| < \varepsilon.$$

Working with the modulus,

$$-\varepsilon < x_n - a < \varepsilon \text{ and } -\varepsilon < z_n - a < \varepsilon.$$

Thus,

$$-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon,$$

which implies that $|y_n - a| < \varepsilon$, and the result follows. □

Example: Suppose we wish to evaluate the following limit:

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \right)$$

Solution: Even though one might think that the Riemann Sum comes into play, it actually does not work in this case because

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \right) = \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+\frac{k}{n^2}}} \right)$$

and setting

$$f\left(\frac{k}{n}\right) = \sqrt{1+\frac{k}{n^2}},$$

it is impossible to obtain an explicit expression for $f(x)$. We use the Squeeze Theorem to help us. As

$$\frac{n}{\sqrt{n^2+n}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}},$$

then

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq \frac{n}{\sqrt{n^2}} \\ \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq 1 \\ 1 &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \leq 1 \end{aligned}$$

By the Squeeze Theorem, the required limit is 1. □

2.1.4 L'Hopital's Rule

L'Hopital's Rule

If f and g are differentiable functions such that $g'(x) \neq 0$ on an open interval I containing a,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty \text{ or } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2.1.5 Stolz-Cesàro Theorem

Stolz-Cesàro Theorem

Let x_n and y_n be two sequences of real numbers. Assume that y_n is strictly monotone and divergent and the following limit exists:

$$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L$$

Then,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

Another case of the Stolz-Cesàro Theorem is that if x_n and y_n are two sequences of real numbers, where $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ while y_n is strictly decreasing and

$$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L,$$

then,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

2.2 Monotone Sequences

2.2.1 Monotone Convergence Theorem

Monotone Convergence Theorem

If x_n is a monotone and bounded sequence, then we can find an expression for $\lim_{n\to\infty} x_n$. If x_n is increasing, then $\lim_{n\to\infty} x_n = \sup(x_n)$. Similarly, if x_n is decreasing, then $\lim_{n\to\infty} x_n = \inf(x_n)$.

2.3 Babylonian Method

Methods of computing square roots are numerical analysis algorithms for approximating the principal, or non-negative, square root of a real number, say S.

For the Babylonian Method, we start with an initial value somewhere near $\sqrt{S}$. That is $x_0 \approx \sqrt{S}$. We then use the following recurrence relation to find a better estimate for $\sqrt{S}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right)$$

Note that

$$\lim_{n\to\infty} x_n = \sqrt{S}.$$

Proof: Suppose

$$\lim_{n\to\infty} x_n = L.$$

Then, substituting this into the recurrence relation yields

$$L = \frac{1}{2}\left(L + \frac{S}{L}\right).$$

Rearranging and the result follows. □

2.4 Nested Interval Theorem

Nested Interval Theorem

Let $I_n = [a_n, b_n]$, where $n \in \mathbb{N}$, be a nested sequence of closed bounded sequences. That is, $I_n \supseteq I_{n+1}$. Then, the intersection

$$\bigcap_{n=1}^{\infty} I_n = \{x : x \in I_n \ \forall n \in \mathbb{N}\}$$

is non-empty. In addition, if $b_n - a_n \to 0$ (i.e. length of I_n tends to 0), then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

2.5 Harmonic Series

The Harmonic Numbers, H_n, are defined to be

$$\sum_{k=1}^{n} \frac{1}{k}.$$

The Harmonic Series is defined to be the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Note that the Harmonic Numbers are increasing and $\lim_{n\to\infty} H_n = 0$, however, the Harmonic Series is divergent! Another interesting property is that other than H_1, the Harmonic Numbers are never integers, whose proof hinges on Analytic Number Theory.

2.6 Euler's Number, e

Euler's Number, $e \approx 2.71828$, is defined to be the following limit:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Consider the sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n.$$

THEOREM

x_n is a strictly increasing sequence. That is, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

Proof: It is easier to prove $x_n > x_{n-1}$, so we wish to prove

$$\left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

First, we write $1 + 1/n$ as

$$1 + \frac{1}{n-1} = \frac{n}{n-1} = \frac{1}{1 - 1/n}.$$

Hence,

$$\begin{aligned}\frac{(1+1/n)^n}{(1+1/(n-1))^{n-1}} &= \left(1 + \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{n-1} \\ &= \left(1 + \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{-1} \\ &= \left(1 - \frac{1}{n^2}\right)^n \left(1 - \frac{1}{n}\right)^{-1}\end{aligned}$$

By Bernoulli's Inequality, this is greater than 1, and so $x_n > x_{n-1}$. $\square$

THEOREM

$2 \le e \le 3$

Proof: We use the series expansion of x_n.

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\ldots+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3\\ &= 1+1+\frac{n-1}{(2!)n}+\frac{(n-1)(n-2)}{3!(n^2)}+\ldots\end{aligned}$$

It is clear that $e \ge 2$. To prove that $e \le 3$, we consider the infinite series, but starting from the third term of the expansion of x_n. It suffices to show that

$$\frac{n-1}{2n}+\frac{(n-1)(n-2)}{6n^2}+\frac{(n-1)(n-2)(n-3)}{24n^3}+\ldots \le 1.$$

Observe that the r^{th} term can be written as

$$\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r}=\frac{1}{(r+1)!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\left(1-\frac{r}{n}\right)\le\frac{1}{(r+1)!}.$$

It is clear that

$$\frac{1}{(r+1)!}\le\frac{1}{2^r},$$

since the factorial grows much faster than the geometric series, and so taking the reciprocal, the result follows. To conclude,

$$\sum_{r=1}^{\infty}\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r}\le\sum_{r=1}^{\infty}\frac{1}{2^r}=1,$$

and we are done. □

Though the incredible constant is named after the Swiss Mathematician Leonhard Euler, its discovery is actually accredited to another Swiss Mathematician, Jacob Bernoulli. Just like π, e is also irrational, which can be proven by contradiction.

THEOREM

e is irrational

Proof: Suppose on the contrary that e is rational. Then, $e = p/q$, where $p, q \in \mathbb{Z}$ but $q \neq 0$. Then, as e can be expressed as the following infinite series

$$\sum_{k=0}^{\infty}\frac{1}{k!},$$

then

$$\begin{aligned}e=\frac{p}{q}&=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{m!}+\frac{1}{(m+1)!}+\ldots\\ m!e&=\frac{m!}{0!}+\frac{m!}{1!}+\frac{m!}{2!}+\frac{m!}{3!}+\frac{m!}{4!}+\ldots+\frac{m!}{m!}+\frac{m!}{(m+1)!}+\ldots\end{aligned}$$

By setting $q = m!$, we see that $m!e$ is an integer. Next, we take a look at the right side of the equation. Observe that

$$\frac{m!}{0!}+\frac{m!}{1!}+\frac{m!}{2!}+\frac{m!}{3!}+\frac{m!}{4!}+\ldots+\frac{m!}{m!}$$

is an integer but

$$\frac{m!}{(m+1)!}+\frac{m!}{(m+2)!}+\frac{m!}{(m+3)!}+\ldots=\frac{1}{m+1}+\frac{1}{(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}+\ldots$$

is not an integer, so we obtained a contradiction. □

2.7 Euler-Mascheroni Constant

This is an epic constant. The Euler-Mascheroni Constant, $\gamma \approx 0.5772$, is defined to be the limiting difference between the Harmonic Series and the natural logarithm. That is,

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

From the figure below, the Euler-Mascheroni Constant can be regarded as the sum of areas of the orange rectangles minus the area under the curve $y = 1/x$ for $x \geq 1$.

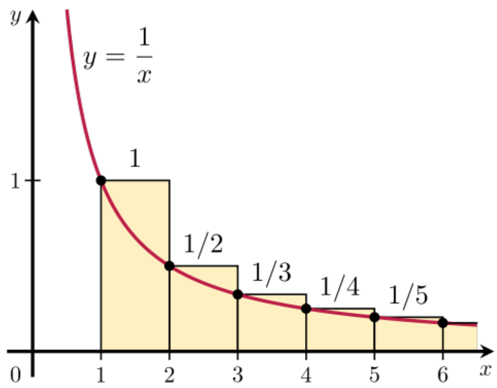

Figure 3: Geometric Interpretation of the Euler-Mascheroni Constant

It is interesting to note that the Euler-Mascheroni converges even though the Harmonic Series diverges and $\ln n$ tends to infinity as n tends to infinity. Let us prove this result using the Monotone Convergence Theorem. We make the following claims. First, we define x_n to be the following sequence:

$$x_n = \sum_{k=1}^{n}\frac{1}{k} - \ln n$$

(1): x_n is a strictly decreasing sequence
Proof: We wish to prove $x_n > x_{n+1}$. Consider

$$\begin{aligned} x_n - x_{n+1} &= \sum_{k=1}^{n}\frac{1}{k} - \ln n - \sum_{k=1}^{n+1}\frac{1}{k} + \ln(n+1) \\ &= \ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1} \end{aligned}$$

Consider the graph of $f(x) = 1/x$, for $n \leq x \leq n+1$. We can regard $\ln((n+1)/n)$ as the area under the curve from $x = n$ to $x = n+1$, whereas $1/(n+1)$ is the area of a rectangle bounded by the ordinates $x = n$, $x = n+1$ and $y = 1/n$. Since f is strictly decreasing and concave up, then the area under the curve is less than the area of the rectangle. Hence, $x_n - x_{n+1} > 0$ and the result follows. □

(2): $0 < x_n \leq 1$ (i.e. x_n is bounded)
Proof: Note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then

$$1 = x_1 > x_2 > x_3 > \ldots$$

and so x_n is bounded above by 1.

Write x_n as

$$\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{1}{x}\,dx.$$

Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up. □

With these two facts, by the Monotone Convergence Theorem, x_n converges! And it converges to γ. Actually, it is still unknown whether γ is rational or irrational. This remains an open problem.

2.8 Subsequences

Let a_n be a sequence. Then, a subsequence of a_n can be formed by deleting elements of a_n. The subsequence is usually written as a_{n_k}, where n_k is an increasing sequence of positive integers.

THEOREM

If x_n converges to a limit, L, then any subsequence x_{n_k} also converges to L. Conversely, if x_n has a subsequence that is divergent, then x_n is also divergent.

COROLLARY

If x_n has two convergent subsequences whose limits are not equal, then x_n is divergent.

2.8.1 Monotone Subsequence Theorem

Squeeze Theorem

Every sequence has a monotone subsequence.

2.8.2 Bolzano-Weierstrass Theorem

Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

2.9 Cluster Point, Limit Superior and Limit Inferior

2.9.1 Cluster Point

Let x_n be a sequence. A point x is a *cluster point*, or accumulation point, of x_n if x_n has a subsequence x_{n_k} which converges to x. That is, $x_{n_k} \to x$.

2.9.2 Limit Superior and Limit Inferior

Let $C(x_n)$ be the set of all cluster points of x_n. Suppose x_n is bounded. Then, by the Bolzano-Weierstrass Theorem, x_n has a convergent subsequence, so $C(x_n)$ is non-empty. We define the limit superior and limit inferior of x_n to be the following:

$$\limsup(x_n) = \sup C(x_n) \text{ and } \liminf(x_n) = \inf C(x_n).$$

THEOREM

Let x_n be a bounded sequence and let $M = \limsup(x_n)$. Then,

(1): For each $\varepsilon > 0$, there are at most finitely many n's such that $x_n \geq M + \varepsilon$. Equivalently, $\exists\, k \in \mathbb{N}$ such that

$$n \geq K \implies x_n < M + \varepsilon.$$

(2): For each $\varepsilon > 0$, there are infinitely many n's such that $x_n > M - \varepsilon$.

We have a similar theorem related to the limit inferior.

THEOREM

Let x_n be a bounded sequence and let $m = \liminf(x_n)$. Then,
(1): For each $\varepsilon > 0$, there are at most finitely many n's such that $x_n \le M + \varepsilon$. Equivalently, $\exists\ k \in \mathbb{N}$ such that

$$n \ge K \implies x_n > M + \varepsilon.$$

(2): For each $\varepsilon > 0$, there are infinitely many n's such that $x_n < M - \varepsilon$.

COROLLARY

If x_n is a bounded sequence, then, x_n converges if and only if $\limsup(x_n) = \liminf(x_n)$.

LEMMA

Suppose x_n and y_n are bounded sequences such that $x_n \le y_n\ \forall n \in \mathbb{N}$. Then,

$$\limsup(x_n) \le \limsup(y_n) \text{ and } \liminf(x_n) \le \liminf(y_n).$$

2.10 Cauchy Sequences

A sequence x_n is a Cauchy Sequence if for every $\varepsilon > 0, \exists\ K \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon\ \forall m, n \ge K.$$

This implies that for large n, the x_n's are very close to each other.

2.10.1 Properties of Cauchy Sequences

(1): A sequence is a Cauchy Sequence if and only if it is convergent.
Example: The sequence $x_n = n$ is not Cauchy since it is not convergent. However, $y_n = 2^{-n}$ and $z_n = 1/n^2$ are Cauchy Sequences. y_n is a geometric sequence and z_n is related to one of the most famous mathematical constants, known as $\zeta(2) = \pi^2/6$, which is the solution to the Basel Problem.

(2): Every Cauchy Sequence is bounded.

2.10.2 Contractive Sequences

A sequence x_n is contractive if there exists $C \in (0, 1)$ such that

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|,$$

where $n \in \mathbb{N}$. Note that every contractive sequence is convergent, and hence Cauchy.

LEMMA

By repeatedly applying the inequality, we obtain the result

$$|x_{n+1} - x_n| \le C^{n-1}|x_2 - x_1|.$$

2.11 Properly Divergent Sequences

A sequence x_n tends to ∞ if for every $M > 0$, then there exists $K \in \mathbb{N}$ such that

$$x_n > M\ \forall n \ge K.$$

We write

$$\lim_{n\to\infty} x_n = \infty.$$

Similarly, a sequence x_n tends to $-\infty$ if for every $M < 0$, then there exists $K \in \mathbb{N}$ such that

$$x_n < M\ \forall n \ge K.$$

We write

$$\lim_{n\to\infty} x_n = -\infty.$$

To conclude, a sequence x_n is properly divergent if either $x_n \to \infty$ or $x_n \to -\infty$.

3 Infinite Series

A sequence, a_n, has sum to n terms, or partial sum,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n.$$

As $n \to \infty$, we are able to find the sum to infinity (if it exists). The sum to infinity is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$$

3.1 Geometric Series

A geometric sequence, u_n, has first term a and common ratio r. The first few terms are

$$u_1 = a,\ u_2 = ar,\ u_3 = ar^2,\ u_4 = ar^3.$$

The general term, u_n, is $u_n = ar^{n-1}$, where $n \in \mathbb{N}$.

3.1.1 Sum to n Terms, S_n, and Sum to Infinity, S_∞

The sum to n terms of a geometric sequence is denoted by S_n. We establish the formula

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

For the sum to infinity, S_∞, we impose a restriction on r for the sum to exist. That is, $|r| < 1$. Hence, S_∞ is

$$S_\infty = \frac{a}{1-r}.$$

REMARK

If $r = -1$, we obtain the famous Grandi's Series, $1 - 1 + 1 - 1 + \ldots$

3.1.2 Arithmetic-Geometric Series

Suppose we have an arithmetic sequence a_n with first term a and common difference d and a geometric sequence b_n with first term b and common ratio r. Then, an arithmetic geometric sequence, c_n, is defined to be the product of the n^{th} term of the arithmetic sequence and the n^{th} term of the geometric sequence. In other words, $c_n = a_n b_n$.

An explicit formula for c_n is

$$c_n = \left[a + d\left(n-1\right)\right] br^{n-1}.$$

The sum to n terms of c_n is

$$S_n = \frac{ab - \left[a + d\left(n-1\right)\right] br^n}{1-r} + \frac{bdr\left(1-r^{n-1}\right)}{\left(1-r\right)^2}.$$

Proof:

$$
\begin{aligned}
(1-r)\,S_n &= (1-r)\sum_{k=1}^{n}(a+d\,(k-1))\,br^{k-1} \\
&= \sum_{k=1}^{n}(a+d\,(k-1))\,br^{k-1} - \sum_{k=1}^{n}(a+d\,(k-1))\,br^{k} \\
&= ab + (a+d)\,br + (a+2d)\,br^2 + (a+3d)\,br^3 + \ldots + (a+d\,(n-1))\,br^{n-1} \\
&- \left[abr + (a+d)\,br^2 + (a+2d)\,br^3 + (a+3d)\,br^4 + \ldots + (a+d\,(n-1))\,br^n\right] \\
&= ab + (a+d)\,br - abr + (a+2d)\,br^2 - (a+d)\,br^2 + (a+3d)\,br^2 - (a+2d)\,br^3 \\
&+ \ldots + (a+d\,(n-1))\,br^{n-1} - (a+d\,(n-2))\,br^{n-1} - (a+d\,(n-1))\,br^n \\
&= ab + bdr + bdr^2 + bdr^3 + \ldots + bdr^{n-1} - (a+d\,(n-1))\,br^n \\
&= ab - (a+d\,(n-1))\,br^n + bd\left[\frac{r\left(1-r^{n-1}\right)}{1-r}\right] \\
&= ab - (a+d\,(n-1))\,br^n + \frac{bdr}{1-r}\left(1-r^{n-1}\right)
\end{aligned}
$$

Dividing both sides by $1-r$ yields the result. □

3.1.3 Gabriel's Staircase

In relation to Probability Theory, Gabriel's Staircase is a useful result in the computation of the expectation of a random variable. It states the following result, for $0<r<1$:

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$$

Proof: Using the formula for S_n of an arithmetic-geometric series, note that $a=d=1$ and $b=r$. Hence, the sum is

$$
\begin{aligned}
S_\infty &= \lim_{n\to\infty}\left[\frac{r-nr^{n+1}}{1-r} + \frac{r^2}{(1-r)^2}\left(1-r^{n-1}\right)\right] \\
&= \lim_{n\to\infty}\frac{\left(r-nr^{n+1}\right)(1-r) + r^2\left(1-r^{n-1}\right)}{(1-r)^2} \\
&= \lim_{n\to\infty}\frac{r-nr^{n+1}+nr^{n+2}-r^{n+1}}{(1-r)^2} \\
&= \frac{r}{(1-r)^2}
\end{aligned}
$$

□

3.1.4 Telescoping Series

A telescoping series is a series whose general term can be written in the form $a_n - a_{n-1}$. Let $b_n = a_n - a_{n-1}$. Then,

$$\sum_{k=1}^{n} b_k = a_n - a_0.$$

This process is known as the method of differences. There are times when the partial fraction decomposition method has to be used on b_n.

LEMMA

Let $k,m\in\mathbb{N}$, where $k\neq m$. Then,

$$\sum_{n=1}^{\infty}\frac{1}{(n+k)(n+k+1)\ldots(n+m-1)(n+m)} = \frac{k!}{(m-k)m!}.$$

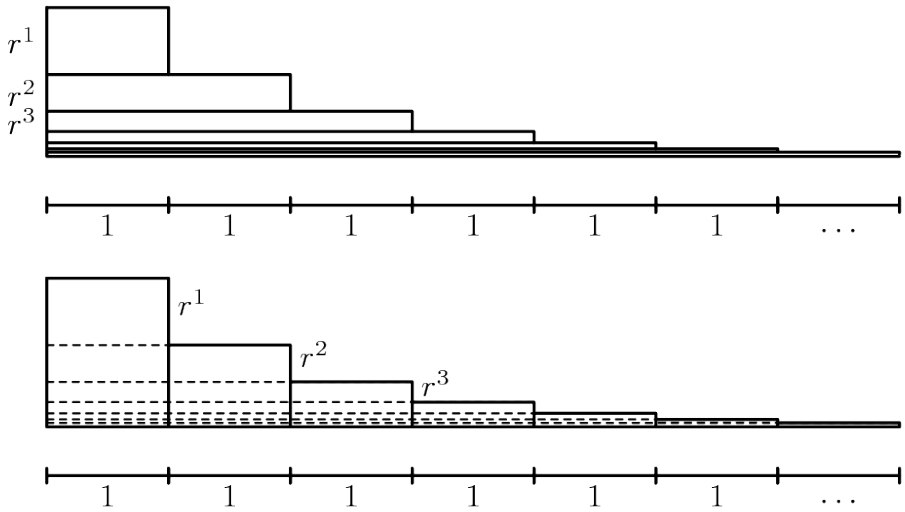

Figure 4: Geometric Proof of Gabriel's Staircase

3.1.5 Lagrange's Trigonometric Identities

Lagrange's Trigonometric Identities, which can be proven by using an appropriate telescoping series, state that

$$\sum_{n=1}^{N} \sin(n\theta) = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$$

$$\sum_{n=1}^{N} \cos(n\theta) = -\frac{1}{2} + \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}$$

where θ is not an integer multiple of 2π.

REMARK

The Dirichlet Kernel can be derived from Lagrange's Second Trigonometric Identity.

3.2 Properties of Convergence and Divergence

Some properties on convergence are regarded as trivial. For instance, if two series are convergent, then their sum is also convergent.

(1): If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
However, the converse is not true as what was mentioned in our discussion about the Harmonic Series.

(2): If $a_n \ge 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums, s_n, is bounded, where $s_{n+1} - s_n = a_n$.

3.2.1 Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent, which is the contrapositive of the previous theorem mentioned!

3.3 Cauchy Criterion for Series

Cauchy Criterion

$\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$, then there exists $K \in \mathbb{N}$ such that

$$|a_{n+1} + a_{n+2} + \ldots + a_m| < \varepsilon \ \forall m > n \ge K.$$

3.4 Series with Non-Negative Terms

3.5 p-series

The p-series is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

If $p > 1$, the p-series converges. If $0 < p \leq 1$, the p-series diverges.

3.6 Tests for Convergence

3.6.1 Direct Comparison Test

Suppose $0 \leq a_n \leq b_n$ for all $n \geq K$ for some $K \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty} b_n \text{ converges } \implies \sum_{n=1}^{\infty} a_n \text{ converges and}$$

$$\sum_{n=1}^{\infty} a_n \text{ diverges } \implies \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

3.6.2 Limit Comparison Test

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series of positive terms. Define

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L.$$

If $L > 0$, then the series are either both convergent or both divergent.
If $L = 0$ and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ will also converge.

3.6.3 Alternating Series Test

An alternating series is a series of the form

$$\sum_{n=1}^{\infty} a_n (-1)^n = a_1 - a_2 + a_3 - a_4 + \ldots,$$

where all a_n are positive or all a_n are negative. If

$$\left| \frac{a_{n+1}}{a_n} \right| \leq 1 \ \forall n \geq 1$$

(i.e. a_n decreases monotonically) and $\lim_{n \to \infty} a_n = 0$, then a_n converges.

3.6.4 Absolute Convergence

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too (i.e. $\sum_{n=1}^{\infty} a_n$ converges absolutely).

3.6.5 D'Alembert's Ratio Test

Let $\sum_{i=1}^{\infty} a_i$ be a series of positive terms. Define

$$L = \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, the series converges, if $L > 1$ the series diverges and if $L = 1$, the test is inconclusive.

3.6.6 Cauchy's Root Test

We wish to determine if the series $\sum_{i=1}^{\infty} a_i$ of positive terms is absolutely convergent. Define

$$L = \limsup_{n \to \infty} \sqrt[n]{a_n}.$$

If $L < 1$, the series is absolutely convergent, if $L > 1$, the series diverges and if $L = 1$, the test is inconclusive.

3.6.7 Cauchy's Condensation Test

For a non-increasing sequence of non-negative real numbers $f(n)$, the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the condensed series $\sum_{n=0}^{\infty} 2^n f(2^n)$ converges. Observe the difference in the lower indices. One of them is 1 and another is 0.

Proof: Suppose the original series converges. We wish to prove that the condensed series converges. Consider twice the original series.

$$\begin{aligned}
2\sum_{n=1}^{\infty} f(n) &= (f(1)+f(1)) + (f(2)+f(2)+f(3)+f(3)) + \ldots \\
&\geq (f(1)+f(2)) + (f(2)+f(4)+f(4)+f(4)) + \ldots \\
&= f(1) + (f(2)+f(2)) + (f(4)+f(4)+f(4)+f(4)) + \ldots \\
&= \sum_{n=0}^{\infty} 2^n f(2^n)
\end{aligned}$$

Dividing both sides by 2, the condensed series converges.

Now, suppose the condensed series converges. We wish to prove the original series converges.

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n f(2^n) &= f(1)+f(2)+f(2)+f(4)+f(4)+f(4)+f(4)+\ldots \\
&\geq f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+\ldots \\
&= \sum_{n=1}^{\infty} f(n)
\end{aligned}$$

This concludes the proof. □

COROLLARY

If both series converge, the sum of the condensed series is no more than twice as large as the sum of the original. We have the inequality

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2\sum_{n=1}^{\infty} f(n).$$

COROLLARY

Consider a variant of the p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

If $p > 1$, the series converges. If $p \leq 1$, the series diverges.

Proof: We use Cauchy's Condensation Test to help us. Note that

$$f(n) = \frac{1}{n(\ln n)^p},$$

so

$$2^n f(2^n) = \frac{2^n}{2^n(\ln(2^n))^p} = \frac{1}{n^p(\ln 2)^p}.$$

We have

$$\frac{1}{(\ln 2)^p}\sum_{n=2}^{\infty} \frac{1}{n^p}$$

so the result follows by the conventional p-series test. □

3.7 Grouping and Rearrangement of Series

3.7.1 Grouping of Series

If the series $\sum_{n=1}^{\infty} a_n$ converges, then any series obtained by grouping the terms of $\sum_{n=1}^{\infty} a_n$ is also convergent and has the same value as $\sum_{n=1}^{\infty} a_n$.

3.7.2 Rearrangement of Series

A series $\sum_{n=1}^{\infty} b_n$ is a rearrangement of the series $\sum_{n=1}^{\infty} a_n$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$.

3.7.3 Sum of Natural Numbers: $1 + 2 + 3 + 4 + \ldots$

The series

$$1 + 2 + 3 + 4 + \ldots$$

is an interesting one. Although it is a divergent series, by certain methods such as rearrangement of the original series or by Ramanujan Summation, we obtain the formula

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}.$$

4 Limits of Functions

4.1 Real-Valued Functions

Let A and B be sets. A function f from A into B is a rule which assigns to each element $x \in A$ a unique element $f(x)$ in B. In this case, we write $f : A \to B$.

A is the domain of f, B is the codomain of f and $f(A)$ is the range of f, where $f(A) = \{f(x) : x \in A\}$. If $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is a real-valued function of a real variable. We shall only consider real-valued functions whose domain is either an interval or a union of intervals.

4.2 ε-δ Definition of a Limit

Let f be a function defined on some open interval I that contains a, except possibly at a itself. Then, $\lim_{x\to a} f(x) = L$ if given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow 0 < |f(x) - L| < \varepsilon.$$

4.3 Limit Theorems

The limit theorems discussed under the section of sequences apply here too. In addition, we have other results:

(1): If $f(x) = g(x)$ for all x in a deleted neighbourhood of $x = a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x),$$

provided that one of these limits exists.

(2): If $f(x) \le g(x)$ for all x in a deleted neighbourhood of $x = a$ and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

4.3.1 Sequential Criterion for Limits

$\lim_{x\to a} f(x) = L \iff$ if x_n is any sequence in the domain of f such that $x_n \ne a$ for all n and $x_n \to a$, then $f(x_n) \to L$.

COROLLARY

$\lim_{x\to a} f(x) \ne L \iff$ there is a sequence x_n in the domain of f such that $x_n \ne a$ for all n and $x_n \to a$, but $f(x_n) \nrightarrow L$.

There is also sequential criteria for left-hand and right-hand limits, as well as infinite limits and limits at infinity, for which they will be discussed under one of the next few sections.

4.3.2 Divergent Criterion

The divergent criterion is to prove that $\lim_{x\to a} f(x)$ does not exist.

Test for Divergence

Method 1: Find a sequence x_n in the domain of f such that $x_n \ne a\ \forall\ n$, $x_n \to a$, but $f(x_n)$ diverges.
Method 2: Find two sequences x_n and y_n in the domain of f such that $x_n \to a$ and $y_n \to a$ for all n and $x_n \to a$, $y_n \to a$ but

$$\lim_{n\to\infty} f(x_n) \ne \lim_{n\to\infty} f(y_n).$$

LEMMA

Let $c \in \mathbb{R}$. Then,
(1): There exists a sequence x_n such that $x_n \in \mathbb{Q}$ for all n, $x_n \neq c$ for all n and $x_n \to c$.
(2): There exists a sequence y_n such that $y_n \in \mathbb{Q}'$ for all n, $y_n \neq c$ for all n and $y_n \to c$.

Example: It is a well-known fact that

$$\lim_{x \to 0} \cos\left(\frac{1}{x^2}\right)$$

does not exist, but how do we prove it?

Solution: We make use of the fact that $\cos(n\pi) = (-1)^n$ for all $n \in \mathbb{N}$ to establish the result. We set $f(x) = \cos(1/x^2)$ and $x_n = 1/\sqrt{n\pi}$. Note that $x_n \neq 0$. Hence, $f(x_n) = \cos(n\pi) = (-1)^n$ but $f(x_n)$ is divergent. By the divergent criterion (Method 1), the limit as $x \to 0$ does not exist. □

4.4 One-Sided Limits

$\lim_{x \to a} f(x) = L$ exists if and only if both $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$ exist and

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

REMARK

$\lim_{x \to a^+} f(x) = L$ means that we are approaching towards $x = a$ from the right, and similarly, $\lim_{x \to a^-} f(x) = L$ means that we are approaching $x = a$ from the left.

COROLLARY

If either one of the one-sided limits of f at $x = a$ does not exist or

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x),$$

then $\lim_{x \to a} f(x)$ does not exist.

4.4.1 Signum Function

The signum function, or $\operatorname{sgn}(x)$, is defined by the following piece-wise function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Note that $\lim_{x \to 0^+} \operatorname{sgn}(x) = 1$, $\lim_{x \to 0^-} \operatorname{sgn}(x) = -1$, but $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist!

4.4.2 Floor and Ceiling Functions

The floor function of a number x, which is denoted by $\lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lfloor x \rfloor = n \text{ if } x \in [n, n+1).$$

Example: $\lfloor \pi \rfloor = 3$ and $\lfloor -4.8 \rfloor = -5$.

The ceiling function of a number x, which is denoted by $\lceil x \rceil$, is defined to be the least integer greater than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lceil x \rceil = n \text{ if } x \in (n, n+1].$$

Example: $\lceil 6.1 \rceil = 7$ and $\lceil -7.8 \rceil = -7$.

Two important inequalities in relation to the floor and ceiling function respectively are that for $n \in \mathbb{Z}$,

$$n \leq \lfloor x \rfloor < n+1 \text{ and } n < \lceil x \rfloor \leq n+1,$$

which can be used to solve equations, inequalities and limits involving them.

4.4.3 Fractional Part of an Integer

For any number x, the fractional part of it is defined by $\{x\}$. We establish the identity

$$\{x\} = x - \lfloor x \rfloor,$$

where $x > 0$.

4.5 Other Sequential Criteria

4.5.1 Sequential Criterion for Right-Hand Limits

$\lim_{x \to a^+} f(x) = L \iff$ if x_n is a sequence in the domain of f such that $x_n > a$ for all n and $x_n \to a$, then $f(x_n) \to L$.

4.5.2 Sequential Criterion for Infinite Limits

$\lim_{x \to a} f(x) = \infty \iff$ if x_n is any sequence in the domain of f such that $x_n \neq a$ for all n and $x_n \to a$, then $f(x_n) \to \infty$.

4.5.3 Sequential Criterion for Limits at Infinity

$\lim_{x \to \infty} f(x) = L \iff$ for any sequence x_n in the domain of f such that $x_n \to \infty$, then $f(x_n) \to L$.

Example: For any $n \in \mathbb{N}$,

$$\lim_{x \to \infty} x^n = \infty$$

and

$$\lim_{x \to -\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}.$$

5 Continuous Functions

A function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

5.1 Types of Discontinuity

These are covered in MA2002 (Calculus) so we shall not emphasise much here.

5.1.1 Removable Discontinuity

A removable discontinuity is a point on the graph that is undefined or does not fit the rest of the graph.

Example: The graph of $f(x) = x^2/x$ is discontinuous at the point $x = 0$, even though the right side can be simplified to $f(x) = x$. However, based on the original domain of the function, if $x = 0$, then the denominator will be 0 as well, which is impossible!

5.1.2 Infinite Discontinuity

Consider the graph of $g(x) = 1/x$, where

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = \infty.$$

The point $x = 0$ is a point of infinite discontinuity.

5.1.3 Jump Discontinuity

In relation to the signum function we discussed earlier, the point $x = 0$ is that of a jump discontinuity.

5.1.4 Oscillating Discontinuity

An oscillating discontinuity exists when the values of the function appear to be approaching two or more values simultaneously.

Consider the graph of $h(x) = \sin(1/x)$, where $x = 0$ is regarded as a point of oscillating discontinuity.

5.2 ε-δ Definition of Continuity

A function f is continuous at $x = a$ if $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

5.2.1 Sequential Criterion for Continuity

f is continuous at $x = a$ if and only if for every sequence x_n in the domain of f such that $x_n \to a$, we have $f(x_n) \to f(a)$.

5.3 Special Functions

5.3.1 Dirichlet Function

Named after Mathematician Peter Gustav Lejeune Dirichlet, the Dirichlet Function, $f(x)$, is defined to be the following:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

It is an example of a function that is nowhere continuous.

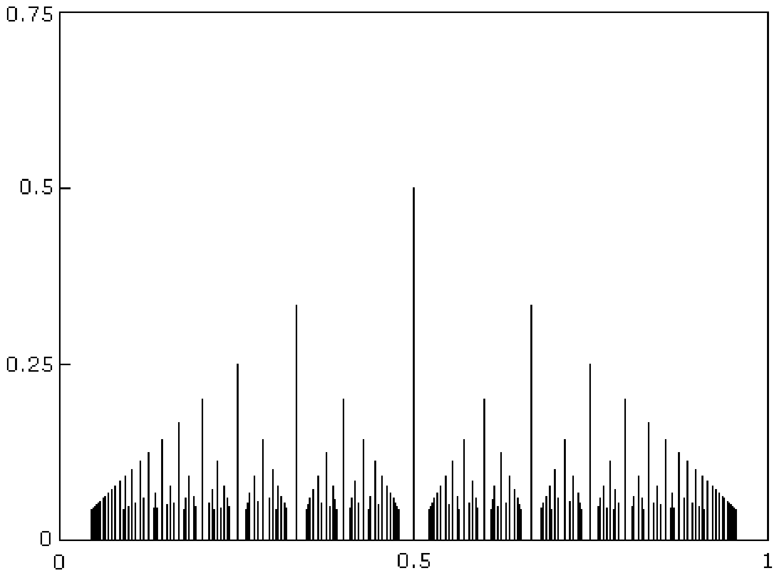

Figure 5: Dirichlet Function

THEOREM

The Dirichlet Function is nowhere continuous.

Proof: Suppose $x \in \mathbb{Q}$, so $f(x) = 1$. We show that f is discontinuous at x. Let $\delta > 0$ be arbitrary and $y \in \mathbb{Q}$ such that $|x - y| < \delta$. Choose $\varepsilon = 1/2$. Without a loss of generality, assume $x < y$. Since there exists $z \in \mathbb{Q}'$ such that $x < z < y$ (due to the density of the irrationals in the reals), then

$$|f(x) - f(z)| = |1 - 0| = 1 > \frac{1}{2} = \varepsilon.$$

In a similar fashion, we now consider the case where $x > y$. There exists $z' \in \mathbb{Q}'$ such that $y < z' < x$, so

$$|f(x) - f(z')| = |1 - 0| = 1 > 1/2 = \varepsilon.$$

Therefore, if $x \in \mathbb{Q}$, f is discontinuous at x.

For the case where $x \in \mathbb{Q}'$, the proof is very similar. $\square$

LEMMA

The Dirichlet Function can be constructed as the double pointwise limit of a sequence of continuous function. That is,

$$f(x) = \lim_{m \to \infty} \lim_{n \to \infty} \cos^{2n}(m! \pi x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

5.3.2 Thomae's Function

Thomae's function is a real-valued function of a real variable. It maps all real numbers to the unit interval $[0,1]$. The function can be defined as such:

$$f(x)=\begin{cases}0 & \text{if } x\notin\mathbb{Q}\\ 1/q & \text{if } x=p/q,\ p,q\in\mathbb{N} \text{ and } \gcd(p,q)=1\end{cases}.$$

It is named after Carl Johannes Thomae, and the function is also known as the popcorn function due to its nature. It is a well-known fact that Thomae's Function is not continuous at all rational points but continuous at all irrational points.

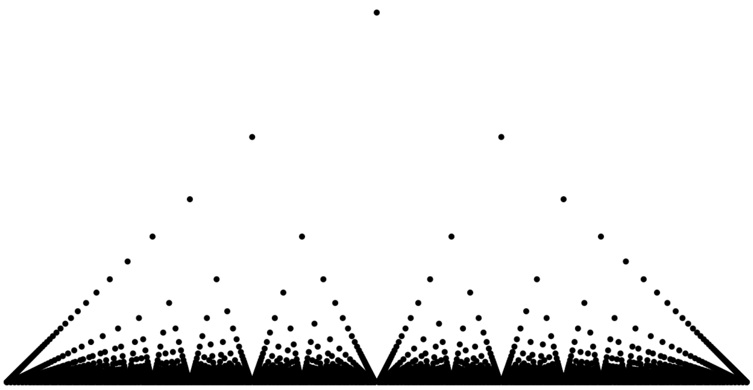

Figure 6: Thomae's Function

5.4 Properties of Continuous Functions

(1): Combinations

Suppose f and g are continuous at $x=a$. Then, $f\pm g, fg$ and αf are also continuous at $x=a$, where $\alpha\in\mathbb{R}$. If $g(a)\neq 0$, then f/g is also continuous at $x=a$.

(2): Composite Functions

Suppose f and g are such that $g\circ f$ is defined. If f is continuous at $x=a$ and g is continuous at $f(a)$, then $g\circ f$ is continuous at $x=a$. Moreover, suppose $f:A\to\mathbb{R}$, $g:B\to\mathbb{R}$ and $f(A)\subseteq B$, so that $g\circ f$ is defined. If f is continuous on A and g is continuous on B, then $g\circ f$ is continuous on A.

(3): Extreme Value Theorem

If f is continuous on $[a,b]$, then f is bounded on $[a,b]$.

5.5 Extreme-Value Theorem

Extreme-Value Theorem

If f is continuous on $[a,b]$, then there exists $x_1,x_2\in[a,b]$ such that

$$f(x_1)\leq f(x)\leq f(x_2)\ \forall x\in[a,b].$$

5.6 Intermediate Value Theorem

Intermediate Value Theorem

If f is continuous on $[a,b]$, and $f(a)<k<f(b)$, then there exists a point $c\in(a,b)$ such that $f(c)=k$.

COROLLARY

If f is continuous on $[a,b]$, then
$$f([a,b]) = [m,M],$$
where $m = \inf f([a,b])$ and $M = \sup f([a,b])$.

5.6.1 Location of Roots

If f is continuous on $[a,b]$, and $f(a) < 0 < f(b)$, then there exists a point $c \in (a,b)$ such that $f(c) = 0$.

5.7 Monotone and Inverse Functions

Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $x_1, x_2 \in A$. Then,
(i): f is increasing on A if $x_1 \le x_2 \implies f(x_1) \le f(x_2)$.
(ii): f is strictly increasing on A if $x_1 < x_2 \implies f(x_1) < f(x_2)$.
(iii): f is decreasing on A if $x_1 \le x_2 \implies f(x_1) \ge f(x_2)$.
(iv): f is strictly decreasing on A if $x_1 < x_2 \implies f(x_1) > f(x_2)$.
(v): f is monotone if it is either increasing or decreasing.
(vi): f is strictly monotone if it either strictly increasing or strictly decreasing.

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function. If $c \in I$ is not an end-point of I, then $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and they are

$$\lim_{x\to c^-} f(x) = \sup\{f(x) : x \in I, x < c\} \text{ and } \lim_{x\to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}.$$

5.8 Continuous Inverse Theorem

Continuous Inverse Theorem

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If f is continuous on I and $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is strictly monotone and continuous on J.

5.9 Uniform Continuity

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. f is uniformly continuous on I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$x, y \in I, |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

COROLLARY

If a function is uniformly continuous on I, then it is continuous on I.

Example: We claim that $f(x) = x^2$ is uniformly continuous on $[0,1]$.

Proof: Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/2$. For $x, y \in [0,1]$, suppose $|x-y| < \delta$. Then,

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x+y||x-y| \\ &< 2 \cdot \delta = \varepsilon \end{aligned}$$

and we are done. □

It is worth noting that $f(x) = x^2$ is uniformly continuous on $[a,b]$ in general, where $a, b \in \mathbb{R}$, but it is not uniformly continuous on $(-\infty, \infty)$!

REMARK

The converse of the corollary is not true! If f is continuous on a closed bounded interval $[a,b]$, then it is uniformly continuous on $[a,b]$.

Sequential Criterion for Uniform Continuity

$f : I \to \mathbb{R}$ is uniformly continuous on I if and only if for any two sequences $x_n, y_n \in I$ such that if $x_n - y_n \to 0$, then $f(x_n) - f(y_n) \to 0$.

5.9.1 Lipschitz Condition

Let I be an interval and $f : I \to \mathbb{R}$ satisfies the Lipschitz Condition on I. Then, there is $K > 0$ such that

$$|f(x) - f(y)| \le K|x - y|, \ \forall x, y \in I,$$

then f is uniformly continuous on I. However, the converse is not true. That is, if f is uniformly continuous on I, it does not imply that it satisfies the Lipschitz Condition.

Example: We verify that $f(x) = x^2$, in the interval $[0,1]$, satisfies the Lipschitz Condition.

Solution: Since $f(x) - f(y) = x^2 - y^2$, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{x^2 - y^2}{x - y}\right| = |x + y| \le 2,$$

and since $2 > 0$, $f(x) = x^2$, in $[0,1]$, is said to satisfy the Lipschitz Condition. In other words, f is Lipschitz continuous. $\square$

5.9.2 Relation to Cauchy Sequences

THEOREM

If $f : I \to \mathbb{R}$ is uniformly continuous on I and x_n is a Cauchy Sequence in I, then $f(x_n)$ is a Cauchy Sequence.

If the function $f : (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b), then $f(a)$ and $f(b)$ can be defined so that the extended function is continuous on $[a,b]$.

6 Differentiable Functions

Recall the ε-δ definition of a limit and sequential criteria for various types of limits, namely two-sided limits, one-sided limits, limit at infinity, finite limits and infinite limits.

6.1 First Principles

A function f is differentiable at a point a if f is defined in some open interval containing a and the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, $f'(a)$ is the derivative of f at $x = a$. Geometrically, $f'(a)$ is the slope of the tangent to the curve $y = f(x)$ at $x = a$.

The above formula is similar to one that students have learnt in H2 Mathematics (9758), which is the derivative of $f(x)$, denoted by $f'(x)$, can be expressed as

$$\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

6.1.1 Differentiable Functions on Open and Closed Intervals

Open Intervals
If f is differentiable at every point in (a, b), then f is differentiable on (a, b).

Closed Intervals
If the function $f : [a, b] \to \mathbb{R}$ is such that f is differentiable on (a, b) and the one sided limits

$$L_1 = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \text{ and } L_2 = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b}$$

exist, then f is differentiable on $[a, b]$. In this case, $f'(a) = L_1$ and $f'(b) = L_2$.

6.2 Continuity and Differentiability

f is continuously differentiable on I if f is differentiable on I and f' is continuous on I. The collection of all functions which are continuously differentiable on I is denoted by $C^1(I)$.

Differentiability implies Continuity

If f is differentiable at a, then it is continuous at a.

Proof:

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} (f(x) - f(a)) + \lim_{x \to a} f(a) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right) + f(a) \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \to a} x - a \right) + f(a) \\
&= f'(a) \cdot 0 + f(a)
\end{aligned}$$

which is just $f(a)$. □

The converse is not true. A continuous function at $x = a$ may not be differentiable at $x = a$. A counterexample can be produced (i.e. $f(x) = |x|$).

Proof: We prove that $f(x)=|x|$ is not differentiable at $x=0$. It is clear that $f(x)$ is continuous $\forall x\in\mathbb{R}$. By the definition of the absolute value function, $f(x)=x$ for $x\geq 0$ and $f(x)=-x$ for $x<0$. Consider

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}1=1$$

and

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}-1=-1,$$

and since the left limit is not equal to the right limit, $|x|$ is not differentiable at $x=0$. □

6.2.1 Weierstrass Function

The Weierstrass Function is an example of a real-valued function that is continuous everywhere but differentiable nowhere. It is an example of a fractal curve named after its discoverer German Mathematician Karl Weierstrass.

In Weierstrass's original paper, the function was defined as the following Fourier Series:

$$f(x)=\sum_{n=0}^{\infty}a^n\cos(b^n\pi x),$$

where $0<a<1$, b is a positive odd integer and $ab>1+3\pi/2$.

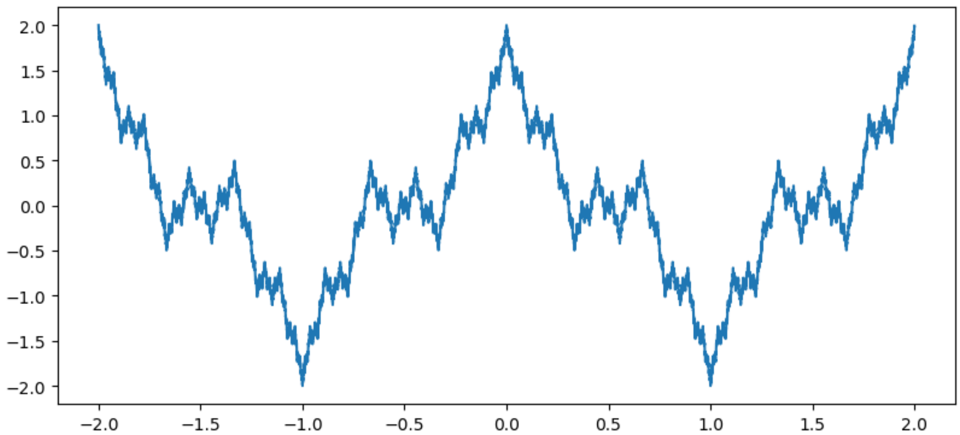

Figure 7: The Weierstrass Function

This link provides an analysis of the Weierstrass Function involving its uniform convergence and it being nowhere differentiable. This involves using the Weierstrass M-test. It is created by Brent Nelson from UC Berkeley Department of Mathematics.

6.3 Derivative Rules and Theorems

6.3.1 Carathéodory's Theorem

Carathéodory's Theorem

Let I be an interval, $f : I \to \mathbb{R}$ and $c \in I$. Then $f'(c)$ exists if and only if there exists a function ϕ on I such that ϕ is continuous at c and

$$f(x) - f(c) = \phi(x)(x - c) \ \forall x \in I.$$

6.3.2 Chain Rule

Let I and J be intervals, and let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be such that $f(J) \subseteq I$. If $a \in J$, f is differentiable at a and g is differentiable at $f(a)$, then $h = g \circ f$ is differentiable at a, and

$$h'(a) = g'(f(a))f'(a).$$

6.3.3 Inverse Function Theorem

Inverse Function Theorem

If f is a continuously differentiable function with non-zero derivative at a; then f is invertible in a neighbourhood of a, the inverse is continuously differentiable, and the derivative of the inverse function at $b = f(a)$ is the reciprocal of the derivative of f at a. As an equation, we have

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

6.4 Mean Value Theorem and Applications

6.4.1 Relative Extremum

Let I be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$.
(a): If $f(x_0) \geq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute maximum of f on I.
(b): If $f(x_0) \leq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute minimum of f on I.
(c): If $\exists\, \delta > 0$ such that $f(x) \leq f(x_0)\ \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f.
(d): If $\exists\, \delta > 0$ such that $f(x) \leq f(x_0)\ \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f.
(e): If $\exists\, \delta > 0$ such that $f(x) \geq f(x_0)\ \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative minimum of f.
If $f(x_0)$ is either a relative minimum or relative maximum of f, then $f(x_0)$ is a relative extremum of f.

REMARK

A relative extremum can only occur at an interior point, but an absolute extremum may occur at one of the end points of the interval. So if a function has an absolute maximum at a point x_0, it may not have a relative maximum at x_0. If f has an absolute maximum at an interior point x_0 of I, then $f(x_0)$ is also a relative maximum of f.

LEMMA

Let $f : (a, b) \to \mathbb{R}$ and $f'(c)$ exists for some $c \in (a, b)$.
(i): If $f'(c) > 0$, then there exists $\delta > 0$ such that

$$f(x) < f(c) \text{ for every } x \in (c - \delta, c) \text{ and } f(x) > f(c) \text{ for every } x \in (c, c + \delta).$$

(ii): If $f'(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) > f(c) \text{ for every } x \in (c - \delta, c) \text{ and } f(x) < f(c) \text{ for every } x \in (c, c + \delta).$$

6.4.2 Fermat's Theorem

Fermat's Theorem

Suppose c is an interior point of an interval I and $f : I \to \mathbb{R}$ is differentiable at c. If f has a relative extremum at c, then $f'(c) = 0$. This is also known as the Interior Extremum Theorem.

Proof: Without a loss of generality, assume that f has a relative maximum at c (the proof if f has a relative minimum is similar). Suppose on the contrary that either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then by the lemma above, there exists $\delta > 0$ such that $f(x) < f(c)$ for every $x \in (c - \delta, c)$ and $f(x) > f(c)$ for every $x \in (c, c + \delta)$. This contradicts the assumption that f has a relative maximum at c. The proofs for other cases are similar. □

REMARK

A function f may have a relative extremum at x_0, but $f'(x_0)$ does not exist.

Example: Consider $f(x) = |x|$. There is a relative (absolute) minimum at $x = 0$, but $f'(0)$ does not exist.

The converse of the Interior Extremum Theorem is false.
Example: Consider $f(x) = x^3$, where $f'(0) = 0$ but $x = 0$ is not a relative extremum point of f. It is merely a point of inflection.

6.4.3 Rolle's Theorem

Rolle's Theorem

If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then $\exists\ c \in (a, b)$ such that $f'(c) = 0$.

Before we proceed with the proof, we take a look at several graphs which satisfy the above conditions.

Firstly, it is clear that as constant function satisfies Rolle's Theorem. For the other three graphs, it is apparent that there exists at least one point $c \in (a, b)$ where the tangent is horizontal, and hence, $f'(c) = 0$.

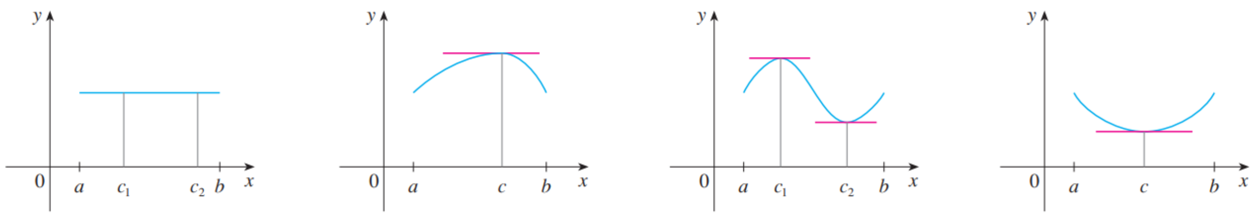

Figure 8: Examples of graphs which satisfy Rolle's Theorem

Proof: The proof where $f(x)$ is a constant will not be discussed since it is trivial. For the more meaningful cases, we have $f(x) > f(a)$ or $f(x) < f(a)$ for some $x \in (a, b)$. Without a loss of generality, we shall prove the former case since the proof for the latter is similar.

By the Extreme Value Theorem, we know that $f(x)$ has a maximum, M in the closed interval $[a, b]$. As $f(a) = f(b)$, the maximum value is attained at $x = c$. That is, $f(c) = M$. So, f has a local maximum at c. Since f is differentiable, the result follows. □

6.4.4 Mean Value Theorem

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We first provide a geometric understanding of the formula.

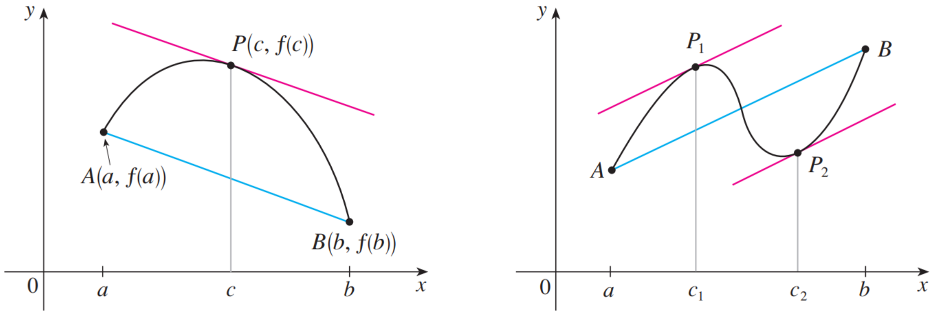

Figure 9: Geometric Interpretation of the Mean Value Theorem

Based on the left diagram, we have a look at the gradient of the secant AB, denoted by m_{AB}. The coordinates of A and B are $(a, f(a))$ and $(b, f(b))$ respectively. Note that

$$m_{AB} = \frac{f(b) - f(a)}{b - a},$$

which this is precisely the gradient of the tangent at P, with coordinates $(c, f(c))$. In short, the theorem states that there is a point P where the tangent is parallel to the secant AB.

Proof: We wish to construct a function $g : [a, b] \to \mathbb{R}$ such that $g(a) = g(b) = 0$, with a point $c \in (a, b)$ such that $g'(c) = 0$. Observe that our proof hinges on Rolle's Theorem. Suppose

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

It is clear that g is continuous on $[a, b]$ and differentiable on (a, b), and $g(a) = g(b) = 0$. By Rolle's Theroem, there exists $c \in (a, b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

Rearranging the equation and we are done. □

COROLLARY

If f is continuous on $[a, b]$, differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

6.4.5 Cubics, Parabolas, Circles

What I would like to discuss now is a very nice question from the H3 Mathematics (9820) 2021 paper involving cubics and the intersection of a circle with a parabola.

(a): Given that the line $y = mx + c$ is a tangent to the curve $y = x^3$, show that $27c^2 = 4m^3$.
(b): It is given that the cubic equation $x^3 = mx + c$ has three distinct roots. By using a sketch, or otherwise, explain why $27c^2 < 4m^3$.
(c): A circle passes through the origin and has four distinct intersections with the parabola $y = x^2$. Describe

the possible positions of the centre of the circle.

Let us discuss the first part.
Solution: Since both the line and the curve have the same gradient, then $m = 3x^2$. For any arbitrary point $P(x, y) = P(a, a^3)$, the equation of the tangent at it is

$$y - a^3 = 3a^2(x - a),$$

which upon expansion yields $y = 3a^2x - 2a^3$. We note that $m = 3a^2$ and $c = -2a^3$. Hence,

$$\left(\frac{m}{3}\right)^3 = \left(-\frac{c}{2}\right)^2,$$

and with some simple algebra, the result follows. □

Great, let us move on to the second part!
Solution: As the equation $f(x) = x^3 - mx - c$ has three distinct roots, suppose the roots are α, β and γ. Hence,

$$\begin{aligned}(x-\alpha)(x-\beta)(x-\gamma) &= x^3 - mx - c \\ x^3 + x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\alpha\gamma+\beta\gamma) + \alpha\beta\gamma &= x^3 - mx - c\end{aligned}$$

Comparing coefficients, we have

$$\begin{aligned}\alpha+\beta+\gamma &= 0 \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -m \\ \alpha\beta\gamma &= -c\end{aligned}$$

Hence, by some painful algebraic expansion,

$$\begin{aligned}4m^3 - 27c^2 &= -4(\alpha\beta+\alpha\gamma+\beta\gamma)^3 - 27\alpha^2\beta^2\gamma^2 \\ &= -4\left(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 + 3\left(\alpha^3\beta^2\gamma + \alpha^2\beta^3\gamma + \alpha^3\beta\gamma^2 + \alpha^2\beta\gamma^3 + \alpha\beta^3\gamma^2 + \alpha\beta^2\gamma^3\right)\right) - 51\alpha^2\beta^2\gamma^2 \\ &= -4\left(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 + 3c\left(\alpha^2\beta + \alpha\beta^2 + \alpha^2\gamma + \alpha\gamma^2 + \beta^2\gamma + \beta\gamma^2\right)\right) - 51c^2 \\ &= -4\left(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3 + 3c\left(\alpha\beta\left(\alpha+\beta\right) + \alpha\gamma\left(\alpha+\gamma\right) + \beta\gamma\left(\beta+\gamma\right)\right)\right) - 51c^2 \\ &= -4\left(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3\right) - 87c^2\end{aligned}$$

There are $3! = 6$ choices for the polarity of α, β and γ. Regardless of this, as the cube of any number *preserves* its polarity, the expression $-4\left(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3\right) - 87c^2$ is always negative, and the result follows. □

However, this is not an elegant solution. Instead, we shall exploit the geometrical property of the Mean Value Theorem to help us! Firstly, we shall investigate

Solution: Examine the following diagram, where the blue line is the tangent at the point $P(a, a^3)$, so it has equation $y = mx + c$, where m and c satisfy the relation claimed in **(a)**. The purple line is a secant. We shall make some inferences with these lines and the curve $y = x^3$.
Let c and c' denote the y-intercepts of the tangent and the secant to the curve respectively. We claim that $c' > c$. Suppose otherwise, then the secant will have one point of intersection with the curve $y = x^3$. By the Mean Value Theorem, there exists at least one point such that the tangent to that point is parallel to the secant passing through these two points.

To wrap up, we conclude that the secant must lie above the tangent and $|c| < |c'|$, and the result follows. □

The last part is interesting as it combines the results from the first two parts. Let's delve into it.
Solution: Recall that a circle of radius r centered at the origin has equation $(x-a)^2 + (y-b)^2 = r^2$. Suppose it intersects the parabola $y = x^2$. Then,

$$\begin{aligned}(x-a)^2 + (x^2-b)^2 &= r^2 \\ x^2 - 2ax + a^2 + x^4 - 2bx^2 + b^2 - r^2 &= 0 \\ x^4 + x^2(1-2b) + x(-2a) + a^2 + b^2 - r^2 &= 0\end{aligned}$$

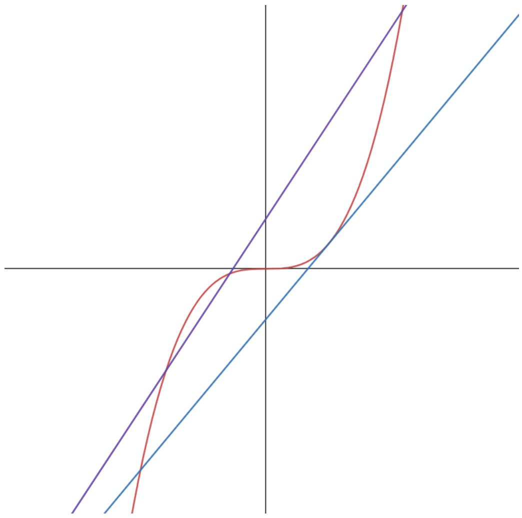

Figure 10: Comparison of the tangent and the secant with the graph of $y = x^3$

Note that $r^2 = a^2 + b^2$ so the equation becomes

$$x(x^3 + x(1 - 2b) - 2a) = 0.$$

As $x = 0$ is a solution, it affirms that the circle passes through the origin. What is of interest is the equation $x^3 = x(2b - 1) + 2a$. Making a comparison with **(b)**, we have $m = 2b - 1$ and $c = 2a$. We need to ensure that this cubic equation has three distinct roots, and $x = 0$ does not satisfy it, otherwise if we put everything together into the quartic equation, we will have a root of multiplicity two. We make use of **(b)**, which claims that $27c^2 < 4m^3$, and so

$$\begin{aligned}
27(2a)^2 &< 4(2b-1)^3 \\
27a^2 &< (2b-1)^3 \\
2b - 1 &> 3a^{2/3} \\
\therefore b &> \frac{1}{2}\left(3a^{2/3} + 1\right)
\end{aligned}$$

Before we conclude, as $x \neq 0$, then $a \neq \frac{1}{2}$.

Hence, the possible positions of the centre of the circle can be described by the following inequality:

$$b > \frac{1}{2}\left(3a^{2/3} + 1\right), \ a \neq \frac{1}{2}$$

and we are done. □

6.4.6 Increasing and Decreasing Functions

Let f be differentiable on (a, b).
(i): If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).
(ii): If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

6.4.7 First Derivative Test

Let f be a continuous function on $[a, b]$ and $c \in (a, b)$. Suppose f is differentiable on (a, b) except possibly at c.

(i): If there is a neighbourhood $(c - \delta, c + \delta) \subseteq I$ of c such that $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$, then $f(c) \geq f(x)\ \forall x \in (c - \delta, c + \delta)$. Hence, f has a relative maximum at c.
(ii): If there is a neighbourhood $(c - \delta, c + \delta) \subseteq I$ of c such that $f'(x) \leq 0$ for $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c + \delta)$, then $f(c) \leq f(x)\ \forall x \in (c - \delta, c + \delta)$. Hence, f has a relative minimum at c.

6.4.8 Differentiability Classes

Consider a function $f(x)$. Its first derivative is denoted by $f'(x)$, second derivative is denoted by $f''(x) = f^{(2)}(x)$, and so on. In general, for $n \in \mathbb{N}$, the n^{th} derivative of f at c is defined as

$$f^{(n)}(c) = (f^{n-1})'(c).$$

Let I be an interval. Then, for $n \in \mathbb{N}$, $C^n(I) = \left\{f : f^{(n)} \text{ exists and is continuous on } I\right\}$. Note that

$$C^\infty(I) = \bigcap_{n=1}^{\infty} C^n(I).$$

If $\in C^\infty(I)$, then f is infinitely differentiable on I.

For $m > n \geq 1$, where $m, n \in \mathbb{Z}$,

$$C^\infty(I) \subset C^m(I) \subset C^n(I) \subset C(I).$$

6.4.9 Second Derivative Test

Let f be defined on an interval I and let its derivative f' exist on I. Suppose c is an interior point of f such that $f'(c) = 0$ and $f''(c)$ exists.
(i): If $f''(c) > 0$, then f has a relative minimum at c.
(ii): If $f''(c) < 0$, then f has a relative maximum at c.
(iii): If $f''(c) = 0$, then the test is inconclusive. Hence, we have to use the first derivative test to prove whether c is a relative minimum, relative maximum or point of inflection.

6.4.10 Cauchy's Mean Value Theorem

Cauchy's Mean Value Theorem

Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$. Then, there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof: We first claim that $g(a) \neq g(b)$. Suppose otherwise, then $g(a) = g(b)$, so by Rolle's Theorem, there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$, contradicting the assumption that $g'(x) \neq 0$ for all $x \in (a, b)$. Next, define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)} \cdot ((g(x) - g(a)) - (f(x) - f(a)),$$

where $x \in [a, b]$. Since h is continuous on $[a, b]$, differentiable on (a, b) and $h(a) = h(b) = 0$, by Rolle's Theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. The result follows. $\square$

6.4.11 L'Hôpital's Rule

L'Hôpital's Rule for Right-Hand Limit (0/0 Case)
Let f and g be differentiable on (a, b) and assume that $g(x) \neq 0$ for all $x \in (a, b)$. Suppose $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$.
(i):

$$\text{If } \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L, \text{ where } L \in \mathbb{R}, \text{ then } \lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

(ii):

$$\text{If } \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \pm\infty, \text{ then } \lim_{x \to a^+} \frac{f(x)}{g(x)} = \pm\infty.$$

L'Hôpital's Rule for Right-Hand Limit (∞/∞ Case)

Let f and g be differentiable on (a,b) and assume that $g(x) \neq 0$ for all $x \in (a,b)$. Suppose $\lim_{x\to a^+} g(x) = \infty$.

(i):

$$\text{If } \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L, \text{ where } L \in \mathbb{R}, \text{ then } \lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

(ii):

$$\text{If } \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \pm\infty, \text{ then } \lim_{x\to a^+} \frac{f(x)}{g(x)} = \pm\infty.$$

6.4.12 Taylor's Theorem

Taylor's Theorem

Let f be a function such that $f \in C^n([a,b])$ and $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$, then for any $x \in [a,b]$, there exists a point c between x and x_0 such that

$$f(x) = \sum_{k=0}^{n+1} \frac{f^k(c)}{k!}(x-x_0)^k.$$

COROLLARY

If $n=0$, then $f(x) = f(x_0) + f'(c)(x-x_0)$, which is an extension of the Mean Value Theorem.

The polynomial $P_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

is the n^{th} Taylor Polynomial for f at x_0.

By Taylor's Theorem, as $f(x) = P_n(x) + R_n(x)$, then

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1}$$

for some point c_n between x and x_0. This formula for R_n is the Lagrange Form of the Remainder.

Let f be infinitely differentiable on $I = (x_0 - r, x_0 + r)$ and $x \in I$. Then, recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

if and only if

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1} = 0,$$

where each c_n is between x and x_0.

7 The Riemann–Stieltjes Integral

7.1 Definition and Existence of the Darboux Integral

Let $I = [a,b]$. A finite set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where

$$a < x_0 < x_1 < x_2 < \ldots < x_n < b$$

is a partition of I. It divides I into the subintervals as

$$I = [x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \ldots \cup [x_{n-1}, x_n] = \bigcup_{i=1}^{n} [x_{i-1}, x_i].$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$. For each $1 \le i \le n$, let

$$M_i = M_i(f,P) = \sup \{f(x) : x \in [x_{i-1}, x_i]\},$$

$$m_i = m_i(f,P) = \inf \{f(x) : x \in [x_{i-1}, x_i]\} \text{ and}$$

$\Delta x_i = x_i - x_{i-1}$. Define the upper sum and lower sum of f with respect to P to be

$$U(f,p) = \sum_{i=1}^{n} M_i \Delta x_i \text{ and } L(f,p) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Note that each partition may not be of uniform length.

By setting $m = \inf \{f(x) : x \in [a,b]\}$ and $M = \sup \{f(x) : x \in [a,b]\}$, then

$$m(b-a) \le L(f,p) \le U(f,p) \le M(b-a).$$

Furthermore,

$$m(b-a) \le \int_a^b f \le M(b-a)$$

and if $f(x) \ge 0$ for all $x \in [a,b]$, then

$$\int_a^b f \ge 0.$$

The upper Darboux Integral of f on $[a,b]$ is defined to be

$$U(f) = \overline{\int_a^b} f(x)\, dx = \inf \{U(f,P) : P \text{ is a partition of } [a,b]\}$$

and the lower Darboux Integral of f on $[a,b]$ is defined to be

$$L(f) = \underline{\int_a^b} f(x)\, dx = \sup \{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

LEMMA

We observe that $L(f) \le U(f)$.

Proof: We prove by contradiction. Suppose $U(f) < L(f)$. Then, there exists a partition P_1 of $[a,b]$ such that

$$U(f) \le U(f,P_1) < L(f).$$

Also, there exists a partition P_2 of $[a,b]$ such that

$$U(f) \le U(f,P_1) < L(f,P_2) \le L(f).$$

However, $L(f, P_2) \le U(f, P_1)$, which is a contradiction. □

Hence, for partitions P and Q of $[a, b]$,

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q),$$

and consequently,

$$L(f) = \underline{\int_a^b} f(x)\ dx \le \overline{\int_a^b} f(x)\ dx \le U(f).$$

REMARK

If P and Q are partitions of $[a, b]$, then Q is a refinement of P if $P \subseteq Q$. Hence,

$$L(f, P) \le L(f, Q) \text{ and } U(f, Q) \le U(f, P).$$

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if

$$\underline{\int_a^b} f(x)\ dx \le \overline{\int_a^b} f(x)\ dx.$$

The Riemann Integral is only defined for bounded functions (i.e. if f is unbounded on $[a, b]$, it is not integrable on $[a, b]$) .
Example:

$$\int_{-1}^{1} \frac{1}{x^2}\ dx$$

is not integrable since $\lim_{x\to 0} 1/x^2 = \infty$, implying that the function is unbounded on $[-1, 1]$.

7.1.1 Dirichlet Function

Consider the Dirichlet Function and denote it by $f(x)$. Since the rational and irrational numbers are both dense subsets of $\mathbb{R}$, then f takes on the value of 0 and 1 on every sub-interval of any partition. Thus for any partition P, $U(f, P) = 1$ and $L(f, P) = 0$. By noting that the upper and lower Darboux Integrals are unequal, we conclude that f is not Riemann integrable on $[0, 1]$.

A fun fact is that the Dirichlet Function is actually Lebesgue integrable.

7.2 Riemann Integrability Criterion and Consequences

7.2.1 Riemann Integrability Criterion

Riemann Integrability Criterion

For a bounded function $f : [a, b] \to \mathbb{R}$, then f is integrable on $[a, b]$ if and only if for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Proof: We first prove that if $U(f, P) - L(f, P) < \varepsilon$, then f is integrable on $[a, b]$. Note that $\varepsilon > 0$ is arbitrary. Recall that

$$L(f, P) \le L(f) \le U(f) \le U(f, P).$$

Hence,

$$U(f) - L(f) \le U(f, P) - L(f, P) < \varepsilon,$$

and we are done.

Now, suppose f is integrable on $[a,b]$. We wish to prove that $U(f,P)-L(f,P)<\varepsilon$. Note that there exists a partition P_1 on $[a,b]$ such that $U(f,P_1)<U(f)$ so

$$U(f,P_1)<U(f)+\frac{\varepsilon}{2}.$$

In a similar fashion, there exists a partition P_2 such that

$$L(f,P_2)>L(f)-\frac{\varepsilon}{2}.$$

Let $P=P_1\cup P_2$ be the common refinement of the previous two partitions. Since

$$0\le U(f,P)-L(f,P),$$

then

$$0\le U(f,P)-L(f,P)<U(f)-L(f)+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon.$$

□

COROLLARY

If $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$, then f is integrable on $[a,b]$.

COROLLARY

If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

COROLLARY

Let $f,g:[a,b]\to\mathbb{R}$ be bounded functions, P be a partition of $[a,b]$ and $c\in\mathbb{R}$. Then,
(i):

$$L\left(cf,P\right)=\begin{cases}cL(f,P) & \text{if } c>0\\ cU(f,P) & \text{if } c<0\end{cases}$$

(ii):

$$U\left(cf,P\right)=\begin{cases}cU(f,P) & \text{if } c>0\\ cL(f,P) & \text{if } c<0\end{cases}$$

(iii):

$$L(f,P)+L(g,P)\le L(f+g,P)\le U(f+g,P)\le U(f,P)+U(g,P)$$

COROLLARY

Let S be a non-empty bounded subset of $\mathbb{R}$ and $K>0$ such that

$$|s-t|\le K \text{ whenever } s,t\in S.$$

Then, $\sup S-\inf S\le K$.

7.2.2 Properties of Intgerals

Let $f,g:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $c\in\mathbb{R}$. Then,
(1): Just like linear transformations, the function $cf+g$ is integrable on $[a,b]$ and

$$\int_a^b (cf+g)=c\int_a^b f+\int_a^b g.$$

(2): If $f(x)\le g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f\le\int_a^b g.$$

(3): $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f\right|\le\int_a^b |f|.$$

(4): fg is integrable on $[a, b]$.

If f is integrable on $[a, b]$, then for any $c \in (a, b)$, f is integrable on $[a, c]$ and $[c, b]$. The converse is true and we have the following result:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

7.2.3 Length of Curve

Let f be a continuous function on $[a, b]$. If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a, b]$, define

$$L(f, P) = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$

where the supremum is taken over all possible partitions $a = x_0 < x_1 < x_2, \ldots < x_n = b$. This definition as the supremum of the all possible partition sums is also valid if f is merely continuous, not differentiable.

If f is continuously differentiable on $[a, b]$, then the arc length is

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

7.3 Fundamental Theorems of Calculus

7.3.1 First Fundamental Theorem of Calculus

Let f be integrable on $[a, b]$ and for $x \in [a, b]$, let

$$F(x) = \int_a^x f.$$

If f is continuous at a point $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

COROLLARY

If f is continuous on $[a, b]$, then the indefinite integral F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$. F is known as the anti-derivative of f.

REMARK

Not all functions have an antiderivative. That is, for example, there do not exist functions $F(x)$ and $G(x)$ such that

$$F(x) = \int e^{-x^2}\, dx \text{ and } G(x) = \int \frac{1}{\ln x},$$

where $f(x) = e^{-x^2}$ is known as the integral of the error function (or simply Gaussian Integral) and $g(x) = 1/\ln x$ is the logarithmic integral.

COROLLARY

If f is continuous on $[a, b]$ and g is differentiable on $[c, d]$, where $g([c, d]) \subseteq [a, b]$, then for $x \in [c, d]$,

$$G(x) = \int_a^{g(x)} f$$

is differentiable on $[c, d]$ and

$$G'(x) = f(g(x))g'(x)$$

for all $x \in [c, d]$.

7.3.2 Second Fundamental Theorem of Calculus

Let g be a differentiable function on $[a,b]$ and assume that g' is continuous on $[a,b]$. Then,

$$\int_a^b g' = g(b) - g(a).$$

7.3.3 Cauchy's Fundamental Theorem of Calculus

Let g be a differentiable function on $[a,b]$ and assume that g' is integrable on $[a,b]$. Then,

$$\int_a^b g' = g(b) - g(a).$$

It is possible for the derivative of a function to not be integrable. Consider the following function $f(x)$:

$$f(x) = \begin{cases} x^2 \sin\left(1/x^2\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

For $x \neq 0$,

$$f'(x) = -\frac{2}{x}\cos\left(\frac{1}{x^2}\right) + 2x\sin\left(\frac{1}{x^2}\right)$$

but $f'(x)$ is not integrable on $[-1,1]$ as this is a region of oscillating discontinuity!

7.4 Integration by Parts

Suppose the functions $u, v : [a,b] \to \mathbb{R}$ are differentiable on $[a,b]$ and their derivatives u' and v' are integrable on $[a,b]$. Then,

$$\int_a^b uv' = u(b)v(b) - u(a)v(a) - \int_a^b vu'.$$

Integration by parts can help to prove certain recurrence relations involving integrals.
Example: Let

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \; dx.$$

The recurrence relation, also known as the *reduction formula*, states that

$$I_n = \frac{n-2}{n} I_{n-2}.$$

7.5 Integration by Substitution

Suppose the function $\phi : [a,b] \to \mathbb{R}$ is such that ϕ' exists and is integrable on $[a,b]$. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\phi([a,b])$, then

$$\int_a^b f(\phi(t)\phi'(t) \; dt = \int_{\phi(a)}^{\phi(b)} f(x) \; dx.$$

7.6 Taylor's Theorem with Integral Form of Remainder

Let f be a function such that $f, f', \ldots, f^{n+1}$ exist on $[a,x]$ and $f^{(n+1)}$ is integrable on $[a,x]$. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n.$$

7.7 Riemann Sum

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and let $\Delta x = x_i - x_{i-1}$ for $1 \le i \le n$. Then, the norm of P, denoted by $\|P\|$, is defined by

$$\|P\| = \max\{\Delta x_i : 1 \le i \le n\}.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$, $\|P\| < \delta$ implies that

$$U(f, P) < \overline{\int_a^b} f + \varepsilon \text{ and } L(f, P) > \underline{\int_a^b} f - \varepsilon.$$

We are now ready to define the Riemann Sum of f with respect to P.

Now, we let ξ_i be a point in the i^{th} sub-interval $[x_{i-1}, x_i]$ for $1 \le i \le n$. The sum

$$S(f, P)(\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is the Riemann Sum of f with respect to P and $\xi = (\xi_1, \ldots, \xi_n)$.

If there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ and any choice of $\xi = (\xi_1, \ldots, \xi_n)$,

$$\|P\| < \delta \implies |S(f, P)(\xi) - A| < \varepsilon,$$

then

$$\lim_{\|P\| \to 0} S(f, P)(\xi) = A.$$

Note that

$$L(f, P) \le S(f, P)(\xi) \le U(f, P).$$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then,

$$\lim_{\|P\| \to 0} U(f, P) = \overline{\int_a^b} f \text{ and } \lim_{\|P\| \to 0} L(f, P) = \underline{\int_a^b} f.$$

Hence, f is integrable on $[a, b]$ and $\int_a^b f = A$ if and only if

$$\lim_{\|P\| \to 0} S(f, P)(\xi) = A.$$

COROLLARY

Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. For each $n \in \mathbb{N}$, let $P_n = \left\{x_0^{(n)}, x_1^{(n)}, \ldots, x_{m_n}^{(n)}\right\}$ be a partition of $[a, b]$ and let $\xi^{(n)} = \left(\xi_1^{(n)}, \ldots, \xi_{m_n}^{(n)}\right)$ be such that $\xi_i^{(n)} \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right]$ for all $1 \le i \le m_n$. Define the sequence y_n as follows:

$$y_n = S(f, p)(\xi^{(n)})$$

If $\lim_{n \to \infty} \|P_n\| = 0$, then

$$\lim_{n \to \infty} y_n = \int_a^b f.$$

7.8 Improper Integrals

An improper integral is one such that either the integrand, f, is unbounded on (a, b) or the interval of integration is unbounded.

7.8.1 Bounded Intervals

Suppose f is defined on $[a, b)$ and f is integrable on $[a, c]$ for every $c \in (a, b)$. If the limit

$$L = \lim_{c \to b^-} \int_a^c f(x)\ dx$$

exists, then the improper integral $\int_a^b f(x)\ dx$ converges and

$$\int_a^b f(x)\ dx = L.$$

If the limit does not exist, then the improper integral diverges.

Similarly, if f is defined on $(a, b]$ and f is integrable on $[c, b]$ for every $c \in (a, b)$, then

$$\int_a^b f(x)\ dx = \lim_{c \to a^+} f(x)\ dx,$$

provided that the limit exists.

7.8.2 Unbounded Intervals

Suppose f is defined on $[a, \infty)$ and f is integrable on $[a, c]$ for every $c > a$. If the limit

$$L = \lim_{c \to \infty} \int_a^c f(x)\ dx$$

exists, then the improper integral $\int_a^\infty f(x)\ dx$ converges and define

$$\int_a^\infty f(x)\ dx = L.$$

If the limit does not exist, then the improper integral diverges.

If f is defined on $(-\infty, b]$ and f is integrable on $[c, b]$ for every $c < b$, then we define

$$\int_{-\infty}^b f(x)\ dx = \lim_{c \to -\infty} \int_c^b f(x)\ dx,$$

proided that the limit exists.

8 Sequences and Series of Functions

An example of a sequence of functions $f_n(x)$, where $n \in \mathbb{N}$ is

$$f_n(x) = \frac{x + x^n}{2 + x^n}$$

for $x \in [0, 1]$. Then, consider the integral

$$\int_0^{\frac{1}{2}} f_n(x).$$

As $n \to \infty$, what can be deduced?

This section deals with questions like these. To start off, we need to introduce the ideas of pointwise convergence and uniform convergence.

8.1 Pointwise and Uniform Convergence

8.1.1 Pointwise Convergence

Let E be a non-empty subset of $\mathbb{R}$. Suppose for each $n \in \mathbb{N}$, we have a function $f_n : E \to \mathbb{R}$. Then, f_n is a sequence of functions on E. For each $x \in E$, the sequence $f_n(x)$ of real numbers converges. Define the function $f : E \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in E$. Then, f_n converges to f pointwise on E, and so $f_n \to f$ pointwise on E.

Hence, $f_n \to f$ pointwise on E if and only if for every $x \in E$ and for every $\varepsilon > 0$, there exists $K = K(\varepsilon, x) \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon.$$

8.1.2 Relationship between Pointwise Convergence, Differentiability and Integrability

(1): If $f_n \to f$ pointwise on I and each f_n is continuous on I, then f is **not necessarily continuous** on I.
Example: Consider $f_n(x) = x^n$ for $x \in [0, 1]$. Note that each f_n is continuous on $[0, 1]$. However, f is not continuous at $x = 1$ since for $x \in [0, 1)$, then

$$\lim_{n\to\infty} f_n(x) = 0$$

but for $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = 1.$$

(2): If $f_n \to f$ pointwise on $[a, b]$ and each f_n is integrable on $[a, b]$, then **(a)** f is not necessarily integrable on $[a, b]$ and **(b)** the pointwise convergence

$$\int_a^b g_n \to \int_a^b g$$

is not necessarily true.

(3): If $f_n \to f$ pointwise on $[a, b]$ and each f_n and f are differentiable on $[a, b]$, then $f_n' \to f'$ not necessarily pointwise on $[a, b]$.
Example: Consider $f_n(x) = \sin(nx)/\sqrt{n}, x \in \mathbb{R}$. $f(x) = 0$ for all $x \in \mathbb{R}$, and thus $f_n \to f$ pointwise on $\mathbb{R}$. As $f'(x) = \sqrt{n}\cos(nx)$, for each $n \in \mathbb{N}$, $f' = 0$, but $f_n' \to f'$ pointwise on $\mathbb{R}$. Then, $f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$, but $f'(0) = 0$.

8.1.3 Uniform Convergence

A sequence of functions f_n converges uniformly to f on E if for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. In this case, $f_n \to f$ uniformly on E. We say that the sequence f_n of functions converges uniformly on E if there exists a function f such that f_n converges to f uniformly on E.

Let $E \subseteq \mathbb{R}$ and let $\phi : E \to \mathbb{R}$ be a bounded function. The uniform norm of ϕ on E is defined as

$$\|\phi\|_E = \sup\{|\phi(x)| : x \in E\}.$$

Then, $|\phi(x)| \leq \|\phi\|_E$ for all $x \in E$.

LEMMA

A sequence of functions f_n converges to f uniformly on E if and only if $\|f_n - f\|_E \to 0$.

8.1.4 Cauchy Criterion for Uniform Convergence

A sequence of functions f_n converges uniformly on E if and only if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_E < \varepsilon$$

for all $m, n \geq K$.

LEMMA

A sequence f_n does not converge uniformly to f on E if and only if for some $\varepsilon_0 > 0$, there is a subsequence f_{n_k} of f_n and a sequence x_k in E such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

for all $k \in \mathbb{N}$.

(1): If f_n converges uniformly on E, then f_n converges pointwise on E. The converse is not true.
(2): If f_n converges uniformly on E and $F \subseteq E$, then f_n converges uniformly on F.

8.1.5 Properties preserved by Uniform Convergence

If f_n converges uniformly to f on an interval I and each f_n is continuous at $x_0 \in I$, then f is continuous at x_0.

COROLLARY

If f_n converges uniformly to f on I and each f_n is continuous on I, then f is continuous on I. Hence,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x),$$

implying that we can interchange the order of the two limit operations.

8.1.6 Relationship between Uniform Convergence, Differentiability and Integrability

Suppose $f_n \to f$ uniformly on $[a, b]$ and each f_n is integrable on $[a, b]$. Then,
(i): f is integrable on $[a, b]$ and

(ii): for each $x_0 \in [a,b]$, the sequence of functions

$$F_n(x) = \int_{x_0}^{x} f_n(t)\ dt$$

converges uniformly to the function

$$F(x) = \int_{x_0}^{x} f(t)\ dt$$

on $[a,b]$. Hence,

$$\lim_{n\to\infty} \int_{x_0}^{x} f_n(t)\ dt = \int_{x_0}^{x} \lim_{n\to\infty} f_n(t)\ dt$$

and in particular,

$$\lim_{n\to\infty} \int_{a}^{b} f_n(t)\ dt = \int_{a}^{b} f(t)\ dt.$$

Now, suppose f_n is a sequence of differentiable functions on $[a,b]$ such that
(i): $f_n(x_0)$ converges for some $x_0 \in [a,b]$,
(ii): f_n' converges uniformly on $[a,b]$.

Then, f_n converges uniformly on $[a,b]$ to a differentiable function f and for $a \le x \le b$,

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

8.2 Infinite Series of Functions

If f_n is a sequence of functions on E, then $S = \sum_{n=1}^{\infty} f_n$ is an infinite series of functions. For each $n \in \mathbb{N}$ and $x \in E$, the n^{th} partial sum of S is the function

$$S_n(x) = \sum_{i=1}^{n} f_i(x).$$

(1): S converges pointwise to a function S on E if the sequence S_n of functions converges pointwise to S on E
(2): S converges uniformly to a function S on E if the sequence S_n of functions converges uniformly to S on E
(3): S converges absolutely on E if the series $\sum_{n=1}^{\infty} |f_n|$ converges pointwise on E

We make the following claims:
(1): S converges pointwise on E if and only if $\sum_{n=1}^{\infty} f_n(x)$ converges $\forall x \in E$
(2): S converges uniformly on E S converges pointwise on E. However, the converse is false.

8.2.1 Cauchy Criterion for Uniform Convergence

Let f_n be a sequence of functions on E. Then, $\sum_{n=1}^{\infty} f_n$ converges uniformly on E if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$n > m \ge K \Rightarrow \left\| \sum_{i=m+1}^{n} f_i \right\|_E < E.$$

COROLLARY

If $\sum_{n=1}^{\infty} f_n$ converges uniformly on E, then $f_n \to 0$ uniformly on E.

8.2.2 Weierstrass M-Test

Weierstrass M-Test

Let f_n be a sequence of functions on E and M_n be a sequence of positive real numbers such that $\|f_n\|_E \le M_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely on E.

Example: We can prove that the series expansion of the exponential function can be uniformly convergent on any bounded subset $S \subset \mathbb{C}$.

Solution: Let $z \in \mathbb{C}$. Note that the series expansion of the complex exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Any bounded subset is a subset of some disc D_R of radius R centred on the origin on the complex plane. The Weierstrass M-test requires us to find an upper bound, M_n, on the terms of the series, with M_n independent of the position in the disc. Observe that

$$\left|\frac{z^n}{n!}\right| \le \frac{|z|^n}{n!} \le \frac{R^n}{n!}$$

so by setting $M = R^n/n!$, we are done. □

8.2.3 Relationship between Infinite Series of Functions, Differentiability and Integrability

If $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on an interval I and each f_n is continuous on each $x_0 \in I$, then f is continuous at x_0.

COROLLARY

If $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on I and each f_n is continuous on I, then f is continuous on I.

We now state some properties related to differentiability and integrability.
If $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a, b]$ and each f_n is integrable on $[a, b]$, then
(1): f is integrable on $[a, b]$ and
(2): for every $x \in [a, b]$,

$$\sum_{n=1}^{\infty} \int_a^x f_n(t)\ dt = \int_a^x f(t)\ dt = \int_a^x \sum_{n=1}^{\infty} f_n(t)\ dt$$

where the convergence is uniform on $[a, b]$. The interchange of the sum and integral is due to the Fubini-Tonelli Theorem.

Now, suppose f_n is a sequence of differentiable functions on $[a, b]$ such that
(1): $\sum_{n=1}^{\infty} f_n(x_0)$ converges for some $x_0 \in [a, b]$ and
(2): $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$.

Then, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ to a differentiable function f and for $a \le x \le b$,

$$\sum_{n=1}^{\infty} f_n'(x) = f'(x).$$

9 Power Series

A seires of functions of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

where $x_0, a_1, a_2, \ldots$ are constants, is a power series in $x - x_0$. So,

$$\sum_{n=0}^{\infty} (x-x_0)^n = \sum_{n=0}^{\infty} f_n(x)$$

where for each n, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = a_n(x-x_0)^n$.

If $x_0 = 0$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

We pose a question: on what set does a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converge?

Recall from H2 Mathematics (9758) the following expansions and the range of values of x for which the expansion is valid. These can be found in the List of Formulae (MF26).
(1): Binomial Series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots, \ |x| < 1$$

(2): Exponential Function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots, \ x \in \mathbb{R}$$

(3): Sine Function

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots, \ x \in \mathbb{R}$$

(4): Cosine Function

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots, \ x \in \mathbb{R}$$

(5): Natural Logarithm (Mercator Series)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots, \ \ -1 < x \le 1$$

Before we give an introduction to the radius of convergence of a power series, we state a theorem:

Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series.
(1): If it converges at $x = x_1$, then it is absolutely convergent for all values of x for which $|x-x_0| < |x_1 - x_0|$.
(2): If it diverges for $x = x_2$, then it diverges for all values of x such that $|x - x_0| > |x_2 - x_0|$.

9.1 Radius of Convergence

Given a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, let

$$S = \left\{ |x-x_0| : x \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges.} \right\}$$

The radius of convergence of the series, R, is defined as follows:
(i): $R = 0$ if $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges only for $x = x_0$.
Example: The series

$$\sum_{n=0}^{\infty} n!x^n$$

converges only at $x = 0$, implying that $R = 0$.

(ii): $R = \infty$ if $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges $\forall x \in \mathbb{R}$.
Example: The exponential function e^x converges at every point of $\mathbb{R}$, and so $R = \infty$.

(iii): $R = \sup S$ if $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for some x and diverges for others.
Example: Consider the geometric series $1 + x + x^2 + x^3 + \ldots$, which converges $\forall x \in \mathbb{R}$ and diverges for all oher x's. Hence, $R = 1$.

9.1.1 Absolute Convergence

$\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for all $x \in (x_0 - R, x_0 + R)$ and diverges for all x with $|x - x_0| > R$.

9.1.2 Ratio Test

Suppose a_n is non-zero for all n. Let

$$\rho = \lim_{n=0} \left| \frac{a_{n+1}}{a_n} \right|.$$

(i): If the limit ρ exists, then the radius of convergence, R, of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \begin{cases} 1/\rho & \text{if } \rho > 0 \\ \infty & \text{if } \rho = 0 \end{cases}.$$

(ii): If $\rho = \infty$, then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = 0$.

9.1.3 Ratio Test or Root Test?

The ratio test is frequently easier to apply than the root test since it is usually easier to evaluate ratios than n^{th} roots.

However, the root test is a *stronger* test for convergence. This means that whenever the ratio test shows convergence, the root test does too and whenever the root test is inconclusive, the ratio test is too (merely the contrapositive statement).

For any sequence x_n of positive numbers,

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \liminf_{n\to\infty} \sqrt[n]{x_n} \text{ and } \limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

9.1.4 Cauchy-Hadamard Formula

Cauchy-Hadamard Formula

Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series and let $\rho = \limsup |a_n|^{\frac{1}{n}}$. The radius of convergence, R, is

$$R = \begin{cases} 0 & \text{if } \rho = \infty \\ 1/\rho & \text{if } 0 < \rho < \infty \\ \infty & \text{if } \rho = 0 \end{cases}.$$

9.2 Properties of Power Series

Suppose

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

has a radius of convergence $R>0$. Then, f is infinitely differentiable on (x_0-R, x_0+R), i.e.

$$f'(x)=\sum_{n=1}^{\infty} n a_n(x-x_0)^{n-1},$$

where $x \in (x_0-R, x_0+R)$ too. In general, for every $k \in \mathbb{N}$, we have the following result:

$$f^{(k)}(x)=\sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1)a_n(x-x_0)^{n-k},$$

where $x \in (x_0-R, x_0+R)$ and the radius of convergence of each of these derived series is also R.

LEMMA

If $\lim_{n\to\infty} a_n = a$, where $a>0$ and b_n is a bounded sequence, then

$$\limsup a_n b_n = a \limsup b_n.$$

Although a power series and its derived series have the same values of R, they may converge on different sets. *Example:* Consider the power series

$$f(x)=\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

By the ratio test, $R=1$, so the series converges in $(-1,1)$. The series also converges at $x=\pm 1$. In fact, when $x=1$, we obtain the famous p-series for which $p=2$, and it is also known as the Basel Problem. When $x=-1$, we obtain a variant of the Basel Problem which can be evaluated as well. Hence, the series converges in $[-1,1]$.

Differentiating both sides of the power series gives

$$f'(x)=\sum_{n=1}^{\infty} \frac{x^{n-1}}{n},$$

where $x \in (-1,1)$. $f'(x)$ converges at $x=-1$ but diverges at $x=1$, which is the Harmonic Series. Hence, $f'(x)$ converges on $[-1,1)$.

COROLLARY

If

$$f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

for all $x \in (x_0-r, x_0+r)$ for some $r>0$, then

$$a_k=\frac{f^{(k)}(x_0)}{k!}.$$

for all $k \in \mathbb{N}\cup\{0\}$.

COROLLARY

This is related to the uniqueness of power series. If

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n=\sum_{n=0}^{\infty} b_n(x-x_0)^n$$

for all $x \in (x_0-r, x_0+r)$ for some $r>0$, then $a_n=b_n$ for all $n \in \mathbb{N}\cup\{0\}$.

COROLLARY

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ have a non-zero radius of convergence R. Then, for any a and b for which $x_0 - R < a < b < x_0 + R$,

$$\int_a^b f(x)\ dx = \int_a^b \sum_{n=0}^{\infty} a_n(x-x_0)^n\ dx = \sum_{n=0}^{\infty} \int_a^b a_n(x-x_0)^n\ dx.$$

In other words, a power series can be integrated term-by-term over any closed interval $[a, b]$ contained in the interval of convergence.

9.2.1 Abel's Summation Formula

Let b_n and c_n be sequences of real numbers, and for each pair of integers $n \ge m \ge 1$, set

$$B_{n,m} = \sum_{k=m}^{n} b_k.$$

Then,

$$\sum_{k=m}^{n} b_k c_k = B_{n,m}c_n - \sum_{k=m}^{n-1} B_{k,m}(c_{k+1} - c_k).$$

for all $n > m \ge 1$, $n, m \in \mathbb{N}$.

9.2.2 Abel's Theorem

Abel's Theorem

Suppose $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has a finite non-zero radius of convergence R.
(i): If the series converges at $x = x_0 + R$, then it converges uniformly on $[x_0, x_0 + R]$.
(ii): If the series converges at $x = x_0 - R$, then it converges uniformly on $[x_0 - R, x_0]$.

COROLLARY

Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

where $x \in (x_0 - R, x_0 + R)$ and R is the radius of convergence of the series, where $0 \le R \le \infty$.
(i): If the series converges at $x = x_0 + R$, then

$$\lim_{x \to (x_0+R)^-} f(x) = \sum_{n=0}^{\infty} a_n R^n.$$

(ii): If the series converges at $x = x_0 - R$, then

$$\lim_{x \to (x_0-R)^+} f(x) = \sum_{n=0}^{\infty} (-1)^n a_n R^n.$$

9.3 Taylor Series

A function f is infinitely differentiable on (a, b) if $f^{(n)}(x)$ exists $\forall x \in (a, b)$ and for all $n \in \mathbb{N}$. This class of functions is denoted by C^∞.

Let f be infinitely differentiable on $(x_0 - r, x_0 + r)$ for some $r > 0$. The power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is known as the Taylor Series of f about x_0.

REMARK

The Taylor Series of f about x_0 does not always converge to f near x_0.

If f has a power series representation of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all $x \in (x_0 - r, x_0 + r)$ for some $r > 0$, then the power series of f must be the Taylor Series of f about x_0.

9.3.1 Maclaurin Series

Considering the Taylor Series, set $x_0 = 0$. We then obtain the Maclaurin Series of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

9.4 Analytic Functions

A function f is analytic on (a, b) if f is infinitely differentiable on (a, b) and for any $x_0 \in (a, b)$, the Taylor Series of f about x_0 converges to f in a neighbourhood of x_0.

Example: The functions e^x, $\sin x$ and $\cos x$ are analytic on $\mathbb{R}$ and the infinite geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is analytic on $(-1, 1)$.

9.5 Arithmetic Operations with Power Series

9.5.1 Cauchy Product

The Cauchy Product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where for each $n \in \mathbb{N}$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0.$$

Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \ |x - x_0| < R_1 \text{ and } g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n, \ |x - x_0| < R_2.$$

For $\alpha, \beta \in \mathbb{R}$, we have

(1):

$$\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x - x_0)^n \text{ for } |x - x_0| < \min(R_1, R_2)$$

(ii):

$$f(x) g(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \ |x - x_0| < \min(R_1, R_2) \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

9.5.2 Merten's Theorem

Merten's Theorem

If $\sum_{n=0}^{\infty} a_n$ converges absolutely and $\sum_{n=0}^{\infty} b_n$ converges, then the Cauchy Product of these two series, $\sum_{n=0}^{\infty} c_n$, converges. Thus,

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

9.5.3 Conditional Convergence

Recall that for Merten's Theorem, we just need at least one of the series to converge absolutely.

A series is conditionally convergent if it converges but does not converge absolutely. If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge conditionally, then their Cauchy Product may not converge.

Example: Set

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}},$$

where $n \geq 0$. It is clear that both series are conditionally convergent (but not absolutely convergent) by the Alternating Series Test. The Cauchy Product of these two series is

$$c_n = \sum_{k=0}^{\infty} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}, \quad \forall n \in \mathbb{N}.$$

Note that $n \geq k$ so $n+1 \geq k+1$ and $n+1 \geq n-k+1$ so we are able to obtain a lower bound for $|c_n|$. Hence,

$$|c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1,$$

implying that $\sum_{n=0}^{\infty} c_n$ diverges.

9.5.4 Riemann Rearrangement Theorem

Riemann Rearrangement Theorem

Suppose a_n is a sequence of real numbers, and that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Let M be a real number. Then there exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

There also exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \infty.$$

The sum can also be rearranged to diverge to $-\infty$ or to fail to approach any limit, finite or infinite.

9.6 Some Special Functions

9.6.1 Exponential Function

The function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for all $x \in \mathbb{R}$, is the exponential function.

$E : \mathbb{R} \to \mathbb{R}$ has the following properties:
(1): $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$
(2): $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$
(3): $E(x) > 0$ for all $x \in \mathbb{R}$
(4): E is strictly increasing (i.e. $E'(x) > 0$ for all $x \in \mathbb{R}$)
(5): $\lim_{x\to\infty} E(x) = \infty$ and $\lim_{x\to-\infty} E(x) = 0$

For **(1)**, any function $f(x)$ that has this property is invariant under successive levels of differentiation. Actually, one can verify that the exponential function is indeed the only function that is invariant under the differential operator by treating the differential equation $f'(x) = f(x)$ as a separable one.

The functional equation

$$f(x+y) = f(x)f(y)$$

holds true only for the exponential function.

Euler's Number, e (≈ 2.71828459045) is defined as the following limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

By considering the Maclaurin Series of e^x, setting $x = 1$ gives the expansion

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

In relation to sequences, for $x \in \mathbb{R}$, e^x is defined as

$$e^x = \lim_{n\to\infty} e^{r_n},$$

where r_n is an increasing rational sequence which converges to x.

For $x \in \mathbb{R}$, e^x is continuous on $\mathbb{R}$.

9.6.2 Logarithmic Function

Since the exponential function E is strictly increasing on $\mathbb{R}$ and $E(\mathbb{R}) = (0,\infty)$, then it implies that E is injective and thus has an inverse function $L : (0,\infty) \to \mathbb{R}$, which is also strictly increasing.

We have the following composition of functions

$$L(E(x)) = x \ \forall x \in \mathbb{R}$$

and

$$E(L(y)) = y \ \forall y > 0.$$

By the Fundamental Theorem of Calculus, we define $L(y)$ to be the following integral:

$$L(y) = \int_1^y \frac{1}{t}\,dt$$

The function $L : (0,\infty) \to \mathbb{R}$ is the natural logarithm, $\ln(x)$.

The natural logarithm $\ln : (0,\infty) \to \mathbb{R}$ has the following properties:
(1):

$$\frac{d}{dy}\ln y = \frac{1}{y}$$

for all $y > 0$
(2):

$$\ln y = \int_1^y \frac{1}{t}\,dt$$

for all $y > 0$
(3): $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y, > 0$
(4): $\ln(1) = 0$ and $\ln(e) = 1$
(5): For $x > 0$ and $\alpha \in \mathbb{R}$, $x^\alpha = e^{\alpha \ln x}$

The functional equation

$$f(xy) = f(x) + f(y)$$

holds true only for the logarithmic function.

COROLLARY

Let $\alpha \in \mathbb{R}$. Then, the function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$f(x) = x^{\alpha}$$

for all $x > 0$ is differentiable on $(0, \infty)$ and

$$f'(x) = \alpha x^{\alpha - 1}$$

for all $x > 0$ as well.

9.6.3 Trigonometric Functions

The function

$$C(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$, is the cosine function.

The function

$$S(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all $x \in \mathbb{R}$, is the sine function.

These two trigonometric functions have the following relationship, that for all $x \in \mathbb{R}$,

$$C'(x) = -S(x) \text{ and } S'(x) = C(x).$$

Differentiating both sides of each equation will yield

$$C''(x) = -C(x) \text{ and } S''(x) = -S(x),$$

which are second order linear homogeneous differential equations which constant coefficients.

Thus, we make the claim that if $g : \mathbb{R} \to \mathbb{R}$ has the property that $g''(x) = -g(x)$ for all $x \in \mathbb{R}$, then

$$g(x) = \alpha C(x) + \beta S(x)$$

for all $x \in \mathbb{R}$ too, where $\alpha = g(0)$ and $\beta = g'(0)$.

The two functions satisfy the identity $(C(x))^2 + (S(x))^2 = 1$ for all $x \in \mathbb{R}$, which is also known as the Pythagorean Identity.

The cosine function is *even*. That is, $C(-x) = C(x)$ (i.e. the graph is symmetrical about the y-axis). It satisfies the following addition formula:

$$C(x+y) = C(x)C(y) - S(x)S(y) \text{ for all } x, y \in \mathbb{R}.$$

The sine function is *odd*. That is, $S(-x) = -S(x)$ (i.e. the graph is symmetrical about the origin). It satisfies the following addition formula:

$$S(x+y) = S(x)C(y) + C(x)S(y) \text{ for all } x, y \in \mathbb{R}.$$

The four other trigonometric functions, as well as all the inverse trigonometric functions, will not be discussed. Moreover, respective small angle approximations will not be discussed too.

9.6.4 Gamma Function

Gamma Function

The Gamma Function is one commonly used extension of the factorial function to complex numbers. Denoted by $\Gamma(z)$, the Gamma Function is defined by the following convergent improper integral:

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt,\ \Re(z) > 0.$$

THEOREM

For $z \geq 0$, we have the following relationship:

$$\Gamma(z+1) = z\Gamma(z),$$

which has some semblance to the functional equation $f(x+1) = xf(x)$. It can be established using integration by parts, and by repeatedly applying the recurrence relation, we obtain

$$\Gamma(n) = (n-1)!,$$

where $n \in \mathbb{N}$.

Proof: Using integration by parts,

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z\,dt = -t^z e^{-t}\big|_0^\infty + z\int_0^\infty e^{-t}t^{z-1}\,dt = z\Gamma(z)$$

and we are done.

Next, to prove the closed form for $\Gamma(n)$, as $\Gamma(1) = 1$, so

$$\begin{aligned}\prod_{i=1}^{n-1}\frac{\Gamma(i+1)}{\Gamma(i)} &= \prod_{i=1}^{n-1} i\\ \frac{\Gamma(n)}{\Gamma(1)} &= (n-1)!\end{aligned}$$

and the result follows by the telescoping product. □

THEOREM

$\ln\Gamma(z)$ is convex on $(0,\infty)$

REMARK

The Bohr-Mullerup Theorem characterises the Gamma Function, claiming that the Gamma Function is the only function satisfying $f(1) = 1$, $f(x+1) = xf(x)$ and f is logarithmically convex.

REMARK

There are two types of Euler Integral. The Gamma Function is also known as the Euler Integral of the First Kind and the Beta Function (discussed in the next section) is also known as the Euler Integral of the Second Kind.

Euler's Reflection Formula and Legendre's Duplication Formula are examples of functional equations closely related to the Gamma Function.

Euler's Reflection Formula

For $z \notin \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z).$$

Legendre's Duplication Formula

The Legendre Duplication Formula states that

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

9.6.5 Beta Function

Beta Function

For $x, y \in \mathbb{C}$, where $\Re(x) > 0$ and $\Re(y) > 0$, the Beta Function $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Note that by using the substitution $x = y$, we obtain the relationship $B(x,y) = B(y,x)$, implying that the Beta Function is symmetric.

THEOREM

The Beta Function is closely related to the Gamma Function and the binomial coefficients by the following equation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

The proof hinges on writing $\Gamma(x)\Gamma(y)$ as a double integral and using the technique of change of variables.

See Raffles Institution 2017 H3 Mathematics Preliminary Examination Question 2 for an appetiser on the Beta Function.