H2 Further Mathematics

Proofs in Linear Algebra

1. Let $U_1, U_2, ..., U_n$ be subspaces of $\mathbb{R}^n$. Prove that $\bigcap_{i=1}^{n} U_i$ is a subspace of $\mathbb{R}^n$.

SOLUTION

$U_1, U_2, ..., U_n$ are non-empty.

Thus, $\mathbf{0} \in U_1$, $\mathbf{0} \in U_2, ...,$ $\mathbf{0} \in U_n \Rightarrow \mathbf{0} \in \bigcap_{i=1}^{n} U_i$.

Let $\mathbf{u}_1$, $\mathbf{u}_2, ...,$ $\mathbf{u}_n \in \bigcap_{i=1}^{n} U_i$.

$\because U_1, U_2, ..., U_n$ are closed under usual addition, $\sum_{i=1}^{n} \mathbf{u}_i \in U_1$.

Similarly, $\sum_{i=1}^{n} \mathbf{u}_i \in U_2, ..., \sum_{i=1}^{n} \mathbf{u}_i \in U_n \Rightarrow \sum_{i=1}^{n} \mathbf{u}_i \in \bigcap_{i=1}^{n} U_i$.

Let $\alpha \in \mathbb{R}$.

$\because U_1, U_2, ..., U_n$ are closed under scalar multiplication, $\alpha \mathbf{u} \in U_1$.

Similarly, $\alpha \mathbf{u} \in U_2, ..., \alpha \mathbf{u} \in U_n \Rightarrow \alpha \mathbf{u} \in \bigcap_{i=1}^{n} U_i$.

$\therefore \bigcap_{i=1}^{n} U_i$ is a subspace of $\mathbb{R}^n$.

2. A Markov matrix is used to represent steps in a Markov chain. If all the entries of a $n \times n$ matrix are non-negative and the sum of each column vector equals 1, then the matrix is called a Markov matrix. Show that one of the eigenvalues of a 2×2 Markov matrix is 1. If the entries of the Markov matrix are positive, show that the other eigenvalue must be less than 1.

SOLUTION

Let the Markov matrix be A and its eigenvalues be denoted by λ.

$$A = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a-\lambda & b \\ 1-a & 1-b-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow (a-\lambda)(1-b-\lambda) - b(1-a) = 0$$

$$\lambda^2 + \lambda(b-a-1) + (a-b) = 0$$

Solving the above quadratic equation gives us $\lambda = 1$ (shown) or $a-b$.

The other eigenvalue is $a-b$. Since the new condition is $0 < a \le b \le 1$, then $a-b < 1$.

4. If $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, $\ker(T)=0$ and $\{\mathbf{x}_1,\ \mathbf{x}_2,...,\mathbf{x}_k\}$ is a linearly independent subset of $\mathbb{R}^n$, show that $\{T(\mathbf{x}_1),\ T(\mathbf{x}_2),...,T(\mathbf{x}_k)\}$ is a linearly independent subset of $\mathbb{R}^n$.

SOLUTION

Suppose $\sum_{i=1}^{k} \alpha_i T(\mathbf{x}_i) = \mathbf{0}.$

T is a linear transformation $\Rightarrow \sum_{i=1}^{k} T(\alpha_i \mathbf{x}_i) = \mathbf{0}.$

$\Rightarrow \sum_{i=1}^{k} \alpha_i \mathbf{x}_i \in \ker(T) \Rightarrow \sum_{i=1}^{k} \alpha_i \mathbf{x}_i = \mathbf{0}$

$\{\mathbf{x}_1,\ \mathbf{x}_2,...,\mathbf{x}_k\}$ are linearly independent $\Rightarrow a_i\text{'s} = 0\ \forall i = 1,\ 2,...,k$

5. Consider a $n \times n$ matrix $\mathbf{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. It is given that $\det(\mathbf{A}) = a$. With the aid of eigenvalues, prove that $\det(k\mathbf{A}) = ak^n$ for all $k \in \mathbb{R}$. Next, prove the same statement using mathematical induction for all $k \in \mathbb{Z}^+$.

SOLUTION

$$\det(\mathbf{A}) = a \Rightarrow \prod_{i=1}^{n} \lambda_i = a$$

$$\det(k\mathbf{A}) = \underbrace{(k \times k \times \ldots \times k)}_{n \text{ times}} \det(\mathbf{A})$$

$$= k^n \det(\mathbf{A})$$

$$= ak^n$$

Let P_k denote the proposition that $\det(k\,\mathbf{A}) = ak^n \ \forall k \in \mathbb{Z}^+$.

For $k = 1$, $LHS = \det(\mathbf{A}) = a$.

$RHS = ak^0 = a$.

$\Rightarrow P_1$ is true.

Assume P_m is true for some $k \in \mathbb{Z}^+$, i.e. $\det(m\,\mathbf{A}) = am^n$.

To show P_{m+1} is true, we have to prove that $\det\left[(m+1)\mathbf{A}\right] = a(m+1)^n$.

$LHS = \det\left[(m+1)\mathbf{A}\right]$

$= \underbrace{\left[(m+1)\times(m+1)\times\ldots\times(m+1)\right]}_{n \text{ times}}\det(\mathbf{A})$

$= (m+1)^n \det(\mathbf{A})$

$= (m+1)^n \times a$

$= RHS$

Since P_1 is true and P_m is true $\Rightarrow P_{m+1}$ is true, by mathematical induction, P_k is true $\forall k \in \mathbb{Z}^+$.

6. Consider a $n \times n$ matrix $\mathbf{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Prove that the eigenvectors of $\mathbf{A}$ are linearly independent.

SOLUTION

Suppose $\sum_{i=1}^{n} \alpha_i \mathbf{e}_i = \mathbf{0}$.

Then, $\mathbf{A} \sum_{i=1}^{n} \alpha_i \mathbf{e}_i = \mathbf{0}$.

$$\Rightarrow \sum_{i=1}^{n} \alpha_i \left(\mathbf{A} \mathbf{e}_i \right) = \mathbf{0}$$

By definition of eigenvalues and eigenvectors, we have $\mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i \ \ \forall i \in [1, n]$.

$$\Rightarrow \sum_{i=1}^{n} \alpha_i \lambda_i \mathbf{e}_i = \mathbf{0}$$

Consider multiplying both sides of $\sum_{i=1}^{n} \alpha_i \mathbf{e}_i = \mathbf{0}$ by λ_1.

Then, $\sum_{i=1}^{n} \alpha_i \lambda_1 \mathbf{e}_i = \mathbf{0}$.

Hence, $\sum_{i=1}^{n} \alpha_i \lambda_1 \mathbf{e}_i - \sum_{i=1}^{n} \alpha_i \lambda_i \mathbf{e}_i = \mathbf{0}$.

$$\sum_{i=1}^{n} \alpha_i \left(\lambda_1 - \lambda_i \right) \mathbf{e}_i = \mathbf{0}.$$

Since all the eigenvalues are distinct, $\lambda_1 - \lambda_i \neq 0 \ \forall i \in [1, n]$.

Since all the $\mathbf{e}_i$'s are nonzero vectors, we must have $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

7. Suppose $n \in \mathbb{Z}^{+}$. Let $T: \mathbb{R}^{n} \to \mathbb{R}$ be a non-zero linear transformation. Prove the following statements.

(i). $\ker(T) = n-1$

(ii). Let $B = \{\mathbf{v}_1,\ \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$ be a basis for $\ker(T)$ and **w** be the *n*-dimensional vector that is not in $\ker(T)$. Then,

$B' = \{\mathbf{v}_1,\ \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{w}\}$ is a basis for $\mathbb{R}^{n}$.

(iii). Each vector **u** in $\mathbb{R}^{n}$ can be expressed as $\mathbf{u} = \mathbf{v} + \dfrac{\mathbf{w}T(\mathbf{u})}{T(\mathbf{w})}$ for some vector **v** in $\ker(T)$.

SOLUTION

(i). Let **A** be the matrix representation of *T*. Then **A** is a $1 \times n$ matrix.

$\Rightarrow \operatorname{rank}(T) = 1$

Using the Rank-Nullity Theorem, we have $\ker(T) = n-1$.

(ii). Suppose $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_{n-1} \mathbf{v}_{n-1} + \alpha_n \mathbf{w} = \mathbf{0}$ for some a_i's $\in \mathbb{R}$.

If $a_n \neq 0$, then $\mathbf{w} = -\dfrac{1}{\alpha_n}\left(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_{n-1} \mathbf{v}_{n-1}\right)$.

$\Rightarrow \mathbf{w}$ can be written as a linear combination of the other vectors in *B*, which is a contradiction.

Hence, $\alpha_n = 0$.

Since *B* is a basis, we must have $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$.

We conclude the proof by stating that the α_i's $= 0\ \forall i \in [1,n]$.

Hence, B' is a basis for $\mathbb{R}^{n}$.

(iii). Note that $\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_{n-1} \mathbf{v}_{n-1} + \alpha_n \mathbf{w}$.

Since $\mathbf{v}$ is in $\ker(T)$, $\mathbf{u} = \mathbf{v} + \alpha_n \mathbf{w}$.

$T(\mathbf{u}) = T(\mathbf{v} + \alpha_n \mathbf{w})$

$= T(\mathbf{v}) + \alpha_n T(\mathbf{w})$

$= \mathbf{0} + \alpha_n T(\mathbf{w})$

$= \alpha_n T(\mathbf{w})$

Then, $\dfrac{T(\mathbf{u})}{T(\mathbf{w})} = \alpha_n$.

$\therefore \mathbf{u} = \mathbf{v} + \dfrac{\mathbf{w} T(\mathbf{u})}{T(\mathbf{w})}$