MA2002 Trial Run 2022

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Module Information

This is a course in single-variable calculus. We will introduce precise definitions of limit, continuity, the derivative and the Riemann integral. Students will be exposed to computational techniques and applications of differentiation and integration. This course concludes with an introduction to first order differential equations. Major topics: Functions, precise definitions of limit and continuity. Definition of the derivative, velocities and rates of change, Intermediate Value Theorem, differentiation formulas, chain rule, implicit differentiation, higher derivatives, the Mean Value Theorem, curve sketching. Definition of the Riemann integral, the Fundamental Theorem of Calculus. The elementary transcendental functions and their inverses. Techniques of integration: substitution, integration by parts, trigonometric substitutions, partial fractions. Computation of area, volume and arc length using definite integrals. First order differential equations: separable equations, homogeneous equations, integrating factors, linear first order equations, applications.

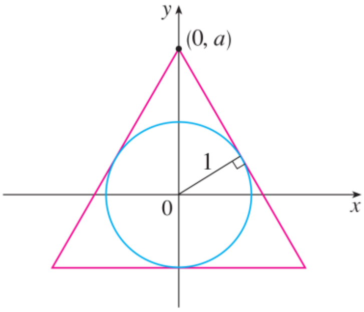
Duration of Test: 2 hours
Total Marks: 100

Problem 1 (6)
A Bernoulli Differential Equation is of the form

$$\frac{dy}{dx} + yP(x) = y^n Q(x).$$

Use the substitution $u = y^{1-n}$ to transform the Bernoulli Equation into a linear differential equation and solve for *y*, expressing it in terms of *x*.

Problem 2 (8)
An isosceles triangle is *circumscribed* about the unit circle so that the equal sides meet at the point $(0, a)$ on the *y*-axis. Using differentiation, find the value of *a* that minimises the lengths of the equal sides.



Problem 3 (12)
Elliptic curves are curves defined by a certain type of cubic equation in two variables. It is of significant importance in various branches of Mathematics like Group Theory and Algebraic Number Theory. In general, an elliptic curve C can be written as such:

$$y^2 = x^3 + ax + b, \text{ where } a, b \in \mathbb{R}$$

An elliptic curve C_1 is constructed by setting $a = 4$ and $b = -3$.

(i) Sketch C_1, stating the coordinates of the axial intercept.

(ii) Use the $\varepsilon - \delta$ definition of a limit to show that C_1 does not have any vertical asymptotes.

Suppose the line $y = ax + b$ is tangential to the curve $y = x^3$.

(iii) Show that $27b^2 = 4a^3$.

Another elliptic curve C_2 is given by $a = -3$ and $b = 2$. This curve has a self-intersecting point, or rather, a *loop*. It is well-known that a necessary and sufficient condition that for this loop to exist is $27b^2 + 4a^3 \neq 0$.

(iv) Verify that C_1 does not contain a loop but C_2 contains it. Also, use another analytical approach to show that C_2 contains a loop.

(v) For $1 \le x \le 2$, C_1 is rotated 2π radians about the *y*-axis. Find the volume of the solid of revolution.

Problem 4 (7)

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Suppose $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Use the Intermediate Value Theorem to prove that there exists some $c \in \mathbb{R}$ such that

$$f(c) = \frac{1}{3} f(x_1) + \frac{1}{12} f(x_2) + \frac{5}{12} f(x_3) + \frac{1}{6} f(x_4).$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function such that f has a continuous derivative and $f'(0) = 0$. Suppose $a_n < 0 < b_n$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Use the Mean Value Theorem to prove that

$$\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(0).$$

Problem 5 (18)

This question deals with a rigorous proof of the following integral:

$$\int_0^{\frac{\pi}{2}}\cos x\,dx=1$$

(i) Using the idea of a Riemann Sum, prove that

$$\int_0^{\frac{\pi}{2}}\cos x\,dx=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{2n}\cos\left(\frac{k\pi}{2n}\right).$$

(ii) Using Euler's Formula, show that the above limit can be expressed as

$$\frac{\pi}{4}\lim_{n\to\infty}\left[\frac{\cot\left(\frac{\pi}{4n}\right)-1}{n}\right].$$

(iii) It is given that

$$\frac{d}{dx}\left(\cot\left(\frac{1}{x}\right)\right)=\frac{1}{x^2}\csc\left(\frac{1}{x}\right)$$

$$\frac{d^2}{dx^2}\left(\cot\left(\frac{1}{x}\right)\right)=\frac{2\csc^2\left(\frac{1}{x}\right)\left(\cot\left(\frac{1}{x}\right)-x\right)}{x^4}$$

$$\frac{d^3}{dx^3}\left(\cot\left(\frac{1}{x}\right)\right)=\frac{2\csc^2\left(\frac{1}{x}\right)\left(3x^2-6x\cot\left(\frac{1}{x}\right)+\csc^2\left(\frac{1}{x}\right)+2\cot^2\left(\frac{1}{x}\right)\right)}{x^6}$$

By considering the successive derivatives of $\cot(1/x)$, explain why L'Hôpital's Rule would not be viable in evaluating the limit in **(ii)**.

(iv) Use an analytical approach to evaluate the limit in **(ii)**, giving full justification.

Problem 6 (8)

Differential equations have vast applications in our everyday lives. This problem studies one application of it on pursuit curves. At noon, ship P sets sail from port O at a constant speed of V ms^{-1} in pursuit of ship S which is travelling due north at constant speed of U ms^{-1}, where $U<V$. At noon, S is D metres due east of O.

(i) Write down the coordinates of S at time t after noon and an expression for dy/dx.

(ii) Explain why, with reference to the distance travelled by P,

$$\frac{d}{dx}(Vt)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

(iii) Show that

$$(D-x)\frac{d^2y}{dx^2}=\frac{U}{V}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Problem 7 (7)

Evaluate the following limits wherever possible. If the limit does not exist, give an explanation.

(i)

$$\lim_{x\to 1}\left(\lfloor x \rfloor + |x|\right)$$

(ii)

$$\lim_{x\to\infty}\left(\frac{1}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+2}}+\frac{1}{\sqrt{x^2+3}}+\ldots+\frac{1}{\sqrt{x^2+x}}\right)$$

Problem 8 (10)

Let $x, y \in \mathbb{R}$ such that $x < y$. A continuous function $f : D \to \mathbb{R}$ is concave up on its domain *D* if

$$f\left(\lambda x+(1-\lambda)y\right)\le \lambda f(x)+(1-\lambda)f(y) \text{ for all } x, y \in D,\ 0<\lambda<1.$$

(i) Show that $x < \lambda x + (1-\lambda)y < y$.

Define $f:(1,\infty)\to\mathbb{R}$, $g:(-\infty,0)\cup(0,\infty)\to\mathbb{R}$ and $h:(0,\infty)\to\mathbb{R}$ such that

$$f(x)=\ln\left(\frac{1}{x}\right) \text{ and } g(x)=h(x)=\ln\left(\frac{\sin x}{x}\right).$$

(ii) Prove, using two different approaches, that *f* is concave up for all *x* in its domain.

(iii) Evaluate

$$\lim_{x\to 0} g(x)$$

and determine if $g'(0)$ exists. If it does, calculate its value. You are to show full working.

Let *S* be the following disjoint union of open intervals.

$$S=\bigcup_{n\in\mathbb{N}}\left(2n\pi,(2n+1)\pi\right)$$

William claims that for the graph of $y = h(x)$, in each open interval, there exists a local maximum.

(iv) Determine if William's claim is valid.

Problem 9 (24)

An ellipse has a semi major axis of length a and a semi minor axis of length b, where $a > b > 0$. The eccentricity, e, is defined to be the deviation of a conic section from a circle. The ellipse has an eccentricity strictly between 0 and 1. For an ellipse, e satisfies the equation $b^2 = a^2\left(1-e^2\right)$.

(i) By parametrising the ellipse with $x = a\cos\theta$ and $y = b\sin\theta$, show that the perimeter of the ellipse can be expressed as

$$E = 4a\int_0^{\frac{\pi}{2}} \sqrt{1-e^2\sin^2\theta}\ d\theta,$$

where E belongs to a class of elliptic integrals, known as the Elliptic Integral of the Second Kind. It is worth noting that a closed form for this antiderivative cannot be found.

(ii) Using integration by parts, establish

$$\int_0^{\frac{\pi}{2}} \sin^{2i}\theta\ d\theta = \frac{\pi}{2}\prod_{j=1}^{i}\frac{2j-1}{2j}$$

and use this recursion to express the integral on the left side only in terms of i.

(iii) Using the binomial theorem and assuming that the order of summation and integration can be swapped, prove that the perimeter of the ellipse can be expressed as the following infinite series, for which a product is nested within it:

$$2\pi a\left[1-\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}}{2i-1}\right]$$