Note that $\displaystyle\sum_{k=0}^{\infty}\frac{1}{16^k\left(8k+r\right)}=2^{r/2}\int_0^{\frac{1}{\sqrt{2}}}\frac{x^{r-1}}{1-x^8}\ \mathrm{d}x.$

So $\displaystyle\sum_{k=0}^{\infty}\left[\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)\right]$

$$=\sum_{k=0}^{\infty}\frac{4}{16^k\left(8k+1\right)}-\sum_{k=0}^{\infty}\frac{2}{16^k\left(8k+4\right)}-\sum_{k=0}^{\infty}\frac{1}{16^k\left(8k+5\right)}-\sum_{k=0}^{\infty}\frac{1}{16^k\left(8k+6\right)}$$

$$=\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{1-x^8}\ \mathrm{d}x-4\int_0^{\frac{1}{\sqrt{2}}}\frac{x^3}{1-x^8}\ \mathrm{d}x-4\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}}\frac{x^4}{1-x^8}\ \mathrm{d}x-8\int_0^{\frac{1}{\sqrt{2}}}\frac{x^5}{1-x^8}\ \mathrm{d}x$$

$$=\int_0^{\frac{1}{\sqrt{2}}}\frac{4\sqrt{2}-8x^3-4\sqrt{2}x^4-8x^5}{1-x^8}\ \mathrm{d}x$$

Letting $y=x\sqrt{2},$ we have $\dfrac{y}{\sqrt{2}}=x$ so the above integral becomes

$$16\int_0^1\frac{4-2y^3-y^4-y^5}{16-y^8}\ \ \mathrm{d}y$$

$$=-16\int_0^1\frac{y^5+y^4+2y^3-4}{16-y^8}\ \ \mathrm{d}y$$

$$=-16\int_0^1\frac{\left(y-1\right)\left(y^2+2\right)\left(y^2+2y+2\right)}{16-y^8}\ \ \mathrm{d}y$$

Note that the denominator is a difference of squares. Hence, $16-y^8\equiv\left(4+y^4\right)\left(4-y^4\right)$

Integral becomes

$$=-16\int_0^1\frac{\left(y-1\right)\left(y^2+2\right)\left(y^2+2y+2\right)}{\left(4+y^4\right)\left(2+y^2\right)\left(2-y^2\right)}\ \ \mathrm{d}y$$

$$=-16\int_0^1\frac{\left(y-1\right)\left(y^2+2y+2\right)}{\left(4+y^4\right)\left(2-y^2\right)}\ \ \mathrm{d}y$$

$$=-16\int_0^1\frac{\left(y-1\right)\left(y^2+2y+2\right)}{\left(y^2+2y+2\right)\left(y^2-2y+2\right)\left(2-y^2\right)}\ \ \mathrm{d}y$$

$$=-16\int_0^1\frac{y-1}{\left(y^2-2y+2\right)\left(2-y^2\right)}\ \ \mathrm{d}y$$

$$\frac{y-1}{\left(y^2-2y+2\right)\left(2-y^2\right)}=\frac{Ay+B}{y^2-2y+2}+\frac{Cy+D}{2-y^2}$$

$$\Rightarrow y-1=(Ay+B)\left(2-y^2\right)+(Cy+D)\left(y^2-2y+2\right)$$

$$\Rightarrow y-1=y^3(C-A)+y^2(D-B-2C)+y(2A+2C-2D)+2B+2D$$

Solving gives $A=C=\dfrac{1}{4},\ B=-\dfrac{1}{2}$ and $D=0$

Integral becomes

$$=-4\int_0^1 \frac{y-2}{y^2-2y+2}+\frac{y}{2-y^2}\ \ \mathrm{d}y$$

Let $I=\displaystyle\int_0^1 \frac{y-2}{y^2-2y+2}\ \mathrm{d}y$ and $J=\displaystyle\int_0^1 \frac{y}{2-y^2}\ \mathrm{d}y$.

$$I=\frac{1}{2}\int_0^1 \frac{2y-2}{y^2-2y+2}\ \mathrm{d}y-\int_0^1 \frac{1}{y^2-2y+2}\ \mathrm{d}y$$

$$=\left[\frac{1}{2}\ln\left|y^2-2y+2\right|\right]_0^1-\int_0^1 \frac{1}{(y-1)^2+1}\ \mathrm{d}y$$

$$=\left[\frac{1}{2}\ln\left|y^2-2y+2\right|\right]_0^1-\left[\tan^{-1}(y-1)\right]_0^1$$

$$=-\frac{1}{2}\ln 2-\frac{\pi}{4}$$

$$J=-\frac{1}{2}\int_0^1 \frac{2y}{2-y^2}\ \mathrm{d}y$$

$$=-\frac{1}{2}\left[\ln\left|2-y^2\right|\right]_0^1$$

$$=\frac{1}{2}\ln 2$$

$$\therefore \text{Answer}=-4\left(-\frac{1}{2}\ln 2-\frac{\pi}{4}+\frac{1}{2}\ln 2\right)=\pi$$