An ellipse has a semi major axis of length a and a semi minor axis of length b, where $a > b > 0$. Prove that the perimeter is given by

$$C = 2\pi a\left[1-\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2 \frac{e^{2i}}{2i-1}\right],$$

where e denotes the eccentricity of the ellipse.

SOLUTION:

We first parametrise the ellipse. That is, $x = a\cos\theta$ and $y = b\sin\theta$.

Firstly, we note that the eccentricity, e, is given by

$$e^2 = 1-\frac{b^2}{a^2}.$$

Hence,

$$b^2 = a^2\left(1-e^2\right).$$

Thus, the perimeter is given by

$$\begin{aligned}
\int_0^{2\pi}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta &= \int_0^{2\pi}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+a^2\left(1-e^2\right)\cos^2\theta}\ \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2e^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\left(\theta+\frac{\pi}{2}\right)}\ \mathrm{d}\theta \\
&= 4a\int_{\frac{\pi}{2}}^{\pi}\sqrt{1-e^2\sin^2 t}\ \mathrm{d}t \\
&= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\theta}\ \mathrm{d}\theta \quad (*)
\end{aligned}$$

By the Reduction Formula (involves integration by parts),

$$\int_0^{\frac{\pi}{2}} \sin^{2i}\theta \,\mathrm{d}\theta = \frac{\pi}{2}\prod_{j=1}^{i}\frac{2j-1}{2j}.$$

Using the Binomial Theorem on $(*)$,

$$4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\theta}\,\mathrm{d}\theta = 4a\int_0^{\frac{\pi}{2}}\sum_{i=0}^{\infty}\binom{1/2}{i}\left(-e^2\sin^2\theta\right)^i\,\mathrm{d}\theta$$

$$= 4a\sum_{i=0}^{\infty}\left(e^{2i}\prod_{j=1}^{i}\frac{3/2-j}{j}\int_0^{\frac{\pi}{2}}\sin^{2i}\theta\,\mathrm{d}\theta\right) \quad \because \text{Fubini's Theorem}$$

$$= 2\pi a\sum_{i=0}^{\infty}\left(e^{2i}\prod_{j=1}^{i}\frac{3/2-j}{j}\prod_{j=1}^{i}\frac{2j-1}{2j}\right)$$

$$= 2\pi a\sum_{i=0}^{\infty}\left(e^{2i}\prod_{j=1}^{i}\frac{3-2j}{2j}\prod_{j=1}^{i}\frac{2j-1}{2j}\right)$$

$$= 2\pi a\sum_{i=0}^{\infty}\left(e^{2i}(-1)^i\prod_{j=1}^{i}\frac{2j-3}{2j}\prod_{j=1}^{i}\frac{2j-1}{2j}\right)$$

$$= 2\pi a\sum_{i=0}^{\infty}\left[\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}(-1)^i}{1-2i}\right]$$

$$= 2\pi a\sum_{i=0}^{\infty}\left[\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}(-1)^i}{1-2i}\right]$$

$$= 2\pi a\left[1-\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}}{2i-1}\right]$$

REMARK:

$$\prod_{j=1}^{i}\frac{2j-1}{2j} = \frac{(2i)!}{2^{2i}(i!)^2}$$

COMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

The complete elliptic integral of the second kind E is defined as

$$E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\ \mathrm{d}\theta=\int_0^1\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\ \mathrm{d}t\ .$$

The latter is known as the **Legendre Normal Form**.
As a power series, the above can be expressed as

$$E(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2\frac{k^{2n}}{1-2n}\ .$$