Idempotent for Fun!

(a).	An idempotent matrix, $\mathbf{A}$, is such that $\mathbf{A}^2 = \mathbf{A}$. Prove that the eigenvalues of an idempotent matrix are 0 or 1.
	SOLUTION Consider $\mathbf{Ax} = \lambda\mathbf{x}$. Then, $\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{A}^2 = \mathbf{A}$, $\lambda^2\mathbf{x} = \lambda\mathbf{x}$. $\left(\lambda^2 - \lambda\right)\mathbf{x} = \mathbf{0}$ $\lambda(\lambda - 1)\mathbf{x} = \mathbf{0}$ Since the eigenvector $\mathbf{x}$ is non-zero, then $\lambda(\lambda - 1) = 0$. Solving, we have $\lambda = 0$ or $\lambda = 1$.
(b).	If $\mathbf{A}$ is an $n \times n$ matrix and λ is an eigenvalue of $\mathbf{A}$, then the union of $\mathbf{0}$ and the set of all the eigenvectors $\mathbf{e}_k$ corresponding to eigenvalues λ_k, is a subspace of $\mathbb{R}^n$. The subspace is known as an eigenspace. Prove that the eigenspace is a subspace of $\mathbb{R}^n$ under usual matrix addition and scalar multiplication.
	SOLUTION $\mathbf{e}_k = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \lambda_k \mathbf{x}\right\}$ Let $\mathbf{Ax} = \lambda_k \mathbf{x}$, $\mathbf{Av} = \lambda_k \mathbf{v}$. Then, $\mathbf{A}(\mathbf{x} + \mathbf{v}) = \mathbf{Ax} + \mathbf{Av}$ $= \lambda_k \mathbf{x} + \lambda_k \mathbf{v}$ $= \lambda_k (\mathbf{x} + \mathbf{v})$ $\Rightarrow$ Eigenspace is closed under usual matrix addition. Let $\alpha \in \mathbb{R}$. Then, $\mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{Ax}$ $= \alpha\lambda_k \mathbf{x}$ $= \lambda_k (\alpha\mathbf{x})$ $\Rightarrow$ Eigenspace is closed under scalar multiplication.
(c).	Using results from $(\mathbf{a})$ and $(\mathbf{b})$, show that any idempotent matrix is diagonalisable.

SOLUTION

$\mathbf{e}_k = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \lambda_k \mathbf{x}\}$

Note that $\lambda = 0$ and 1.

Consider eigenspaces $\mathbf{e}_0 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{0}\}$ and $\mathbf{e}_1 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{x}\}$.

Let $\operatorname{rank}(\mathbf{A}) = r$. Then $\ker(\mathbf{A}) = n - r$ by Rank-Nullity Theorem.

Suppose the vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r$ form a basis for the range space of $\mathbf{A}$.

Then $\mathbf{Ax}_i = \mathbf{y}_i \ \forall i = [1, r]$.

$\Rightarrow \mathbf{A}^2\mathbf{x}_i = \mathbf{Ay}_i$

Since $\mathbf{A}$ is idempotent, $\mathbf{Ax}_i = \mathbf{Ay}_i$.

Then, $\mathbf{x}_i = \mathbf{y}_i \Rightarrow \mathbf{y}_i \in \mathbf{e}_1$.

$\therefore \dim(\mathbf{e}_1) = r$

$\dim(\mathbb{R}^n) = n$

$\dim(\mathbf{e}_0) + \dim(\mathbf{e}_1) = n - r + r$ since $\mathbf{e}_0 \cap \mathbf{e}_1 = \mathbf{0}$

$= n$

$\Rightarrow \dim(\mathbb{R}^n) = \dim(\mathbf{e}_0) + \dim(\mathbf{e}_1)$

$\therefore \mathbb{R}^n = \mathbf{e}_0 \oplus \mathbf{e}_1$