Cubic Formula

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Consider the general expression of a cubic polynomial, which is

$$ax^3 + bx^2 + cx + d, \text{ where } a \neq 0. \tag{1}$$

We denote it by $f(x)$. Say we wish to find the roots of the cubic equation $f(x) = 0$. Hence,

$$f'(x) = 3ax^2 + 2bx + c \tag{2}$$
$$f''(x) = 6ax + 2b \tag{3}$$

Setting $f''(x) = 0$, we have $x = -b/3a$. The purpose of finding this is to get rid of the term in x^2 in our original cubic equation so that it will be in the form of a *depressed cubic.* Replacing x with $x - b/3a$ in (1), we have

$$\begin{aligned} f\left(x - \frac{b}{3a}\right) &= a\left(x - \frac{b}{3a}\right)^3 + b\left(x - \frac{b}{3a}\right)^2 + c\left(x - \frac{b}{3a}\right) + d && (4)\\ &= a\left(x^3 - \frac{3bx^2}{3a} + \frac{3b^2x}{9a^2} - \frac{b^3}{27a^3}\right) + b\left(x^2 - \frac{2bx}{3a} + \frac{b^2}{9a^2}\right) + cx - \frac{bc}{3a} + d && (5)\\ &= ax^3 - bx^2 + \frac{b^2x}{3a} - \frac{b^3}{27a^2} + bx^2 - \frac{2b^2x}{3a} + \frac{b^3}{9a^2} + cx - \frac{bc}{3a} + d && (6)\\ &= ax^3 + \left(-\frac{b^2}{3a} + c\right)x - \frac{2b^3}{27a^2} - \frac{bc}{3a} + d && (7)\end{aligned}$$

Now, we have to solve $f(x - b/3a) = 0$, and to make things simpler, consider equating (7) to 0 and divide both sides by a. This is justified because $a \neq 0$. As such,

$$x^3 + \left(-\frac{b^2}{3a^2} + \frac{c}{a}\right)x - \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = 0. \tag{8}$$

Here, we can set p and q to be

$$p = -\frac{b^2}{3a^2} + \frac{c}{a} \text{ and } q = -\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} \tag{9}$$

so that our cubic equation is reduced to

$$x^3 + px + q = 0. \tag{10}$$

We say that (10) is a *depressed cubic.* Note that every depressed cubic equation has no term in x^2.

Now, the trick is to set $x = u + v$. The binomial expansion of $(u + v)^3$ gives us

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \tag{11}$$

so it is easy to see that

$$(u + v)^3 - 3uv(u + v) - u^3 - v^3 = 0. \tag{12}$$

This has semblance to (10), and we see that $p = -3uv$ and $q = -u^3 - v^3$. As

$$-qv^3 = u^3v^3 + v^6, \tag{13}$$

we have

$$-qv^3 = \left(-\frac{p}{3}\right)^3 + v^6. \tag{14}$$

By the substitution $t = v^3$, this can be easily reduced to a quadratic equation in t.

$$v^6 + qv^3 - \frac{p^3}{27} = 0 \tag{15}$$

$$t^2 + qt - \frac{p^3}{27} = 0 \tag{16}$$

By the quadratic formula,

$$t = \frac{-q \pm \sqrt{r}}{2}, \text{ where } r = q^2 + \frac{4p^3}{27}. \tag{17}$$

As such, due to symmetry,

$$u = \sqrt[3]{\frac{-q + \sqrt{r}}{2}} \text{ and } v = \sqrt[3]{\frac{-q - \sqrt{r}}{2}}. \tag{18}$$