MA2002 Trial Run 2022 Solutions

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Module Information

This is a course in single-variable calculus. We will introduce precise definitions of limit, continuity, the derivative and the Riemann integral. Students will be exposed to computational techniques and applications of differentiation and integration. This course concludes with an introduction to first order differential equations. Major topics: Functions, precise definitions of limit and continuity. Definition of the derivative, velocities and rates of change, Intermediate Value Theorem, differentiation formulas, chain rule, implicit differentiation, higher derivatives, the Mean Value Theorem, curve sketching. Definition of the Riemann integral, the Fundamental Theorem of Calculus. The elementary transcendental functions and their inverses. Techniques of integration: substitution, integration by parts, trigonometric substitutions, partial fractions. Computation of area, volume and arc length using definite integrals. First order differential equations: separable equations, homogeneous equations, integrating factors, linear first order equations, applications.

Duration of Test: 2 hours
Total Marks: 100

Problem 1

A Bernoulli Differential Equation is of the form

$$\frac{dy}{dx}+yP(x)=y^nQ(x).$$

Use the substitution $u=y^{1-n}$ to transform the Bernoulli Equation into a linear differential equation and solve for *y*, expressing it in terms of *x*.

Solution: Setting $u=y^{1-n}$,

$$\frac{du}{dx}=(1-n)y^{-n}\frac{dy}{dx}.$$

Multiplying the Bernoulli Equation by $(1-n)y^{-n}$ and using the new substitution for the derivative,

$$\frac{du}{dx}+(1-n)uP(x)=(1-n)Q(x)$$

The integrating factor is $e^{\int(1-n)P(x)\,dx}$. Multiplying both sides by the integrating factor,

$$\frac{d}{dx}\left(ue^{(1-n)\int P(x)\,dx}\right)=(1-n)e^{(1-n)\int P(x)\,dx}Q(x)$$

$$ue^{(1-n)\int P(x)\,dx}=(1-n)\int e^{(1-n)\int P(x)\,dx}Q(x)\ dx+c$$

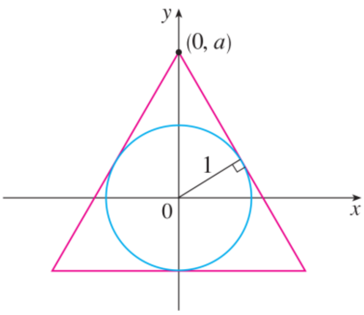
$$u=\frac{(1-n)\int e^{(1-n)\int P(x)\,dx}Q(x)\ dx+c}{e^{(1-n)\int P(x)\,dx}}$$

$$y=\left(\frac{(1-n)\int e^{(1-n)\int P(x)\,dx}Q(x)\ dx+c}{e^{(1-n)\int P(x)\,dx}}\right)^{\frac{1}{1-n}}$$

where *c* is an arbitrary constant of integration. □

Problem 2

An isosceles triangle is *circumscribed* about the unit circle so that the equal sides meet at the point $(0,a)$ on the *y*-axis. Using differentiation, find the value of *a* that minimises the lengths of the equal sides.



Solution: Note that the base of the triangle lies on $y=-1$ since it is tangential to the circle. The equation of the circle is $x^2+y^2=1$, so performing implicit differentiation,

$$\frac{dy}{dx}=-\frac{x}{y}.$$

Suppose the tangent on the right is tangential to the circle at $\left(b,\sqrt{1-b^2}\right)$. Then, its equation is

$$y-\sqrt{1-b^2}=-\frac{b}{\sqrt{1-b^2}}(x-b).$$

As the tangent passes through $(0,a)$, then

$$\begin{aligned}a-\sqrt{1-b^2}&=\frac{b^2}{\sqrt{1-b^2}}\\ a\sqrt{1-b^2}-1+b^2-b^2&=0\\ a\sqrt{1-b^2}&=1\\ 1-b^2&=\frac{1}{a^2}\\ b^2&=1-\frac{1}{a^2}\end{aligned}$$

Also, when the tangent passes through $y=-1$,

$$\begin{aligned}-1-\sqrt{1-b^2}&=-\frac{b}{\sqrt{1-b^2}}(x-b)\\ \sqrt{1-b^2}+1&=bx\\ x&=\frac{\sqrt{1-b^2}+1}{b}\end{aligned}$$

Thus, the length of each equal side is

$$\begin{aligned}d&=\sqrt{(a+1)^2+\left(\frac{\sqrt{1-b^2}+1}{b}\right)^2}\\ &=\sqrt{(a+1)^2+\frac{\left(\frac{1}{a}+1\right)^2}{1-\frac{1}{a^2}}}\\ &=a(a+1)\sqrt{\frac{1}{a^2-1}}\end{aligned}$$

We can consider minimising d^2.

$$\begin{aligned}d^2&=\frac{a^2(a+1)^2}{a^2-1}\\ \left(d^2\right)'&=\frac{2a\left(a^2-a-1\right)}{(a-1)^2}\end{aligned}$$

Since $a\geq 1$, setting $a^2-a-1=0$ and taking the positive root yields

$$a=\frac{1+\sqrt{5}}{2},$$

which is the famous golden ratio! □

Problem 3

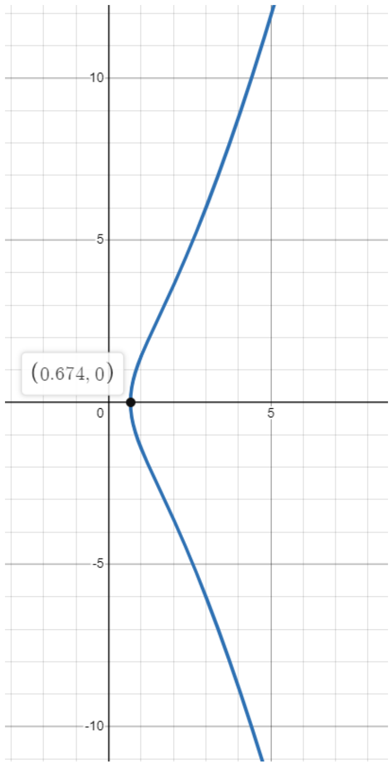
Elliptic curves are curves defined by a certain type of cubic equation in two variables. It is of significant importance in various branches of Mathematics like Group Theory and Algebraic Number Theory. In general, an elliptic curve C can be written as such:

$$y^2 = x^3 + ax + b, \text{ where } a, b \in \mathbb{R}$$

An elliptic curve C_1 is constructed by setting $a = 4$ and $b = -3$.

(i) Sketch C_1, stating the coordinates of the axial intercept.

Solution:



(ii) Use the $\varepsilon - \delta$ definition of a limit to show that C_1 does not have any vertical asymptotes.

Solution: Since the graph is symmetrical about the x-axis, by considering the positive square root of y^2, we have $y = \sqrt{x^3 + 4x - 3}$. It is apparent that as x tends to infinity, then y tends to infinity as well. Hence,

$$\lim_{x \to \infty} \left(x^3 + 4x - 3 \right) = \infty .$$

Let us use the $\varepsilon - \delta$ definition to establish this result. Note that $M > 0$ is arbitrary. We choose $N = \sqrt[3]{M + 3}$ such that for all $x > N$,

$$x^3 + 4x - 3 > N^3 + 4N - 3 = M + 4\sqrt[3]{M+3} > M ,$$

so the proof is complete. Alternatively, you can apply the $\varepsilon - \delta$ definition to both $y = \sqrt{x^3 + 4x - 3}$ and $y = -\sqrt{x^3 + 4x - 3}$ and work from there.

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Suppose the line $y = ax + b$ is tangential to the curve $y = x^3$.

(iii) Show that $27b^2 = 4a^3$.

Solution: Since both the line and the curve have the same gradient, then $a = 3x^2$. For any arbitrary point $P(x, y) = P(k, k^3)$, the equation of the tangent at it is $y - k^3 = 3k^2(x-k)$, which upon expansion yields

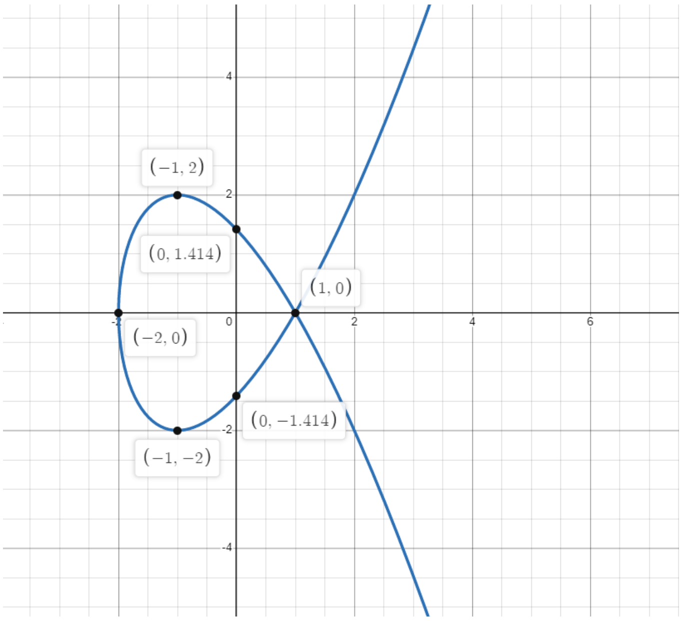
$$y = 3k^2x - 2k^3.$$

We note that $a = 3k^2$ and $b = -2k^3$. With some simple algebra, the result follows. □

Another elliptic curve C_2 is given by $a = -3$ and $b = 2$. This curve has a self-intersecting point, or rather, a *loop*. It is well-known that a necessary and sufficient condition that for this loop to exist is $27b^2 + 4a^3 \neq 0$.

(iv) Verify that C_1 does not contain a loop but C_2 contains it. Also, use another analytical approach to show that C_2 contains a loop.

Solution: The verification is trivial.



Consider the sketch of C_2 as shown above. We use an analytical approach to prove that C_2 contains a loop. Note that C_2 passes through $(-2,0)$ and $(1,0)$. Note that elliptic curves are symmetrical about the *x*-axis. We claim that in the interval where $x \in (-2,1)$, a region is enclosed. By Rolle's Theorem, there exists at least two extrema (due to symmetry so each branch contributes one extremum), and the result follows. □

(v) For $1 \leq x \leq 2$, C_1 is rotated 2π radians about the *y*-axis. Find the volume of the solid of revolution.

Solution: By the method of cylindrical shells, the volume is

$$2\int_1^2 2\pi x\sqrt{x^3 + 4x - 3}\, dx = 49.8 \text{ units}^3.$$ □

Problem 4

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Suppose $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Use the Intermediate Value Theorem to prove that there exists some $c \in \mathbb{R}$ such that

$$f(c) = \frac{1}{3}f(x_1) + \frac{1}{12}f(x_2) + \frac{5}{12}f(x_3) + \frac{1}{6}f(x_4).$$

Solution: Observe that the sum of the coefficients of the $f(x_i)$'s is 1.

Define $m=\min\left(f(x_i)\right)$ and $M=\max\left(f(x_i)\right)$ for $1\le i\le 4$.

$$\begin{aligned} m &= \frac{1}{3}m+\frac{1}{12}m+\frac{5}{12}m+\frac{1}{6}m \\ &\le \frac{1}{3}f(x_1)+\frac{1}{12}f(x_2)+\frac{5}{12}f(x_3)+\frac{1}{6}f(x_4) \\ &\le \frac{1}{3}M+\frac{1}{12}M+\frac{5}{12}M+\frac{1}{6}M \\ &= M \end{aligned}$$

Setting $m=f(x_i)$ and $M=f(x_j)$, by the Intermediate Value Theorem, there exists c between x_i and x_j such that the initial equation holds. □

(ii) Let $f:\mathbb{R}\to\mathbb{R}$ be a real-valued function such that f has a continuous derivative and $f'(0)=0$. Suppose $a_n<0<b_n$ such that $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=0$. Use the Mean Value Theorem to prove that

$$\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(0).$$

Solution: By the Mean Value Theorem, there exists c_n such that

$$\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(c_n).$$

As n tends to infinity, c_n tends to 0 by the Squeeze Theorem. Since the limit of a derivative is the derivative of a limit, then

$$\lim_{n\to\infty}f'(c_n)=f'\left(\lim_{n\to\infty}c_n\right)=f'(0)=0$$ □

Problem 5

This question deals with a rigorous proof of the following integral:

$$\int_0^{\frac{\pi}{2}}\cos x\,dx=1$$

(i) Using the idea of a Riemann Sum, prove that

$$\int_0^{\frac{\pi}{2}}\cos x\,dx=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{2n}\cos\left(\frac{k\pi}{2n}\right).$$

Solution: Partition the interval $[0,\pi/2]$ into n equally spaced ones. Construct n rectangles erected at each ordinate $x=\pi/2n, 2\pi/2n,\ldots,n\pi/2n$. Note that each rectangle is of width $\pi/2n$ and height $\cos(k\pi/2n)$, where $1\le k\le n$. Note that the sum of areas of the rectangles becomes closer to the integral as we decrease the strip width (by increasing the number of rectangles), so we let n tend to infinity. □

(ii) Using Euler's Formula, show that the above limit can be expressed as

$$\frac{\pi}{4}\lim_{n\to\infty}\left[\frac{\cot\left(\frac{\pi}{4n}\right)-1}{n}\right].$$

Solution: By Euler's Formula, by setting $\theta=\pi/2n$,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\pi}{2n}\cos\left(\frac{k\pi}{2n}\right) &= \operatorname{Re}\left\{\frac{\pi}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{ik\theta}\right\}\\
&= \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{n}\operatorname{Re}\left(\frac{e^{i\theta}\left(1-e^{in\theta}\right)}{1-e^{i\theta}}\right)\\
&= \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{n}\operatorname{Re}\left(\frac{e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}}\left(e^{i\frac{n\theta}{2}}e^{-i\frac{n\theta}{2}}-e^{i\frac{n\theta}{2}}e^{i\frac{n\theta}{2}}\right)}{e^{i\frac{\theta}{2}}e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}e^{i\frac{\theta}{2}}}\right)\\
&= \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{n}\operatorname{Re}\left(\frac{e^{i\frac{\theta}{2}}e^{i\frac{n\theta}{2}}\left(e^{-i\frac{n\theta}{2}}-e^{i\frac{n\theta}{2}}\right)}{e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}}\right)\\
&= \frac{\pi}{2}\lim_{n\to\infty}\frac{1}{n}\operatorname{Re}\left(\frac{e^{i\frac{\theta}{2}}e^{i\frac{n\theta}{2}}\left(-2i\sin\frac{n\theta}{2}\right)}{-2i\sin\frac{\theta}{2}}\right)\\
&= \frac{\pi}{2\sin\frac{\theta}{2}}\lim_{n\to\infty}\frac{1}{n}\cos\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\\
&= \frac{\pi}{2}\lim_{n\to\infty}\frac{\cos\frac{n\theta}{2}\cos\frac{\theta}{2}\sin\frac{n\theta}{2}-\sin^2\frac{n\theta}{2}\sin\frac{\theta}{2}}{n\sin\frac{\theta}{2}}\\
&= \frac{\pi}{2}\lim_{n\to\infty}\frac{\cos\frac{n\theta}{2}\cos\frac{\theta}{2}\sin\frac{n\theta}{2}-\sin^2\frac{n\theta}{2}\sin\frac{\theta}{2}}{n\sin\frac{\theta}{2}}\\
&= \frac{\pi}{2}\lim_{n\to\infty}\left(\frac{1}{2n}\sin n\theta\cot\frac{\theta}{2}-\frac{1}{n}\sin^2\frac{n\theta}{2}\right)\\
&= \frac{\pi}{4}\lim_{n\to\infty}\left[\frac{\cot\left(\frac{\pi}{4n}\right)-1}{n}\right]
\end{aligned}$$

The transition from the second last to last step is clear because $\theta=\pi/2n$. □

(iii) It is given that

$$\frac{d}{dx}\left(\cot\left(\frac{1}{x}\right)\right)=\frac{1}{x^2}\csc\left(\frac{1}{x}\right)$$

$$\frac{d^2}{dx^2}\left(\cot\left(\frac{1}{x}\right)\right)=\frac{2\csc^2\left(\frac{1}{x}\right)\left(\cot\left(\frac{1}{x}\right)-x\right)}{x^4}$$

$$\frac{d^3}{dx^3}\left(\cot\left(\frac{1}{x}\right)\right)=\frac{2\csc^2\left(\frac{1}{x}\right)\left(3x^2-6x\cot\left(\frac{1}{x}\right)+\csc^2\left(\frac{1}{x}\right)+2\cot^2\left(\frac{1}{x}\right)\right)}{x^6}$$

By considering the successive derivatives of $\cot(1/x)$, explain why L'Hôpital's Rule would not be viable in evaluating the limit in **(ii)**.

Solution: Note that $\csc\left(\frac{1}{x}\right)$ and $\cot\left(\frac{1}{x}\right)$ tend to infinity as x tends to infinity, so each derivative is still in indeterminate form. Though all three expressions of $\frac{d^n}{dx^n}$ for $n=1,2,3$ contain $\csc\left(\frac{1}{x}\right)$ in the numerator, we cannot simply substitute a large value of x into each derivative to obtain the limit in **(ii)**. □

(iv) Use an analytical approach to evaluate the limit in **(ii)**, giving full justification.

Solution: We prove that

$$\lim_{n\to\infty}\left[\frac{\cot\left(\frac{\pi}{4n}\right)-1}{n}\right]=\frac{4}{\pi}.$$

Using series expansion,

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{\cot\left(\frac{\pi}{4n}\right)-1}{n}\right]&=\lim_{n\to\infty}\left[\frac{\dfrac{\cos\left(\frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}-1}{n}\right]\\&=\lim_{n\to\infty}\left[\frac{1-\frac{1}{2!}\left(\frac{\pi}{4n}\right)^2-\left(\frac{\pi}{4n}+\dots\right)}{n\left(\frac{\pi}{4n}+\dots\right)}\right]\\&=\lim_{n\to\infty}\left[\frac{1-\frac{1}{2!}\left(\frac{\pi}{4n}\right)^2-\frac{\pi}{4n}}{\frac{\pi}{4}}+\dots\right]\\&=\frac{4}{\pi}\end{aligned}$$

and we are done. □

Problem 6

Differential equations have vast applications in our everyday lives. This problem studies one application of it on pursuit curves. At noon, ship *P* sets sail from port *O* at a constant speed of V ms^{-1} in pursuit of ship *S* which is travelling due north at constant speed of U ms^{-1}, where $U < V$. At noon, *S* is *D* metres due east of *O*.

(i) Write down the coordinates of *S* at time *t* after noon and an expression for dy/dx.

Solution: Since *S* moves at a constant speed *U* vertically, the coordinates of *S* at time *t* is (D, Ut). dy/dx denotes the gradient of the tangent at *P*, or the gradient of the line *PS*. Since $P(x, y)$ lies on the curve,

$$\frac{dy}{dx} = \frac{Ut - y}{D - x}$$ □

(ii) Explain why, with reference to the distance travelled by *P*,

$$\frac{d}{dx}(Vt) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} .$$

Solution: Since *V* is the speed of ship *P*, then the total distance travelled by *P* is *Vt*, which is the same as the arc length of the curve from $u = 0$ to $u = x$. Hence,

$$Vt = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, du .$$

By the Fundamental Theorem of Calculus, differentiating both sides yields the result. □

(iii) Show that

$$(D - x)\frac{d^2 y}{dx^2} = \frac{U}{V}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} .$$

Solution: Note that

$$(D - x)\frac{dy}{dx} = Ut - y .$$

Differentiating with respect to *x*,

$$(D - x)\frac{dy}{dx} = Ut - y$$

$$(D - x)\frac{d^2 y}{dx^2} - \frac{dy}{dx} = U\frac{dt}{dx} - \frac{dy}{dx}$$

$$(D - x)\frac{d^2 y}{dx^2} = U\frac{dt}{dx}$$

From **(ii)**, differentiating the left side,

$$V\frac{dt}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and the result follows. □

Problem 7

Evaluate the following limits wherever possible. If the limit does not exist, give an explanation.

(i)

$$\lim_{x\to 1}\left(\lfloor x\rfloor+|x|\right)$$

Solution:

$$\lim_{x\to 1^+}\left(\lfloor x\rfloor+|x|\right)=\lim_{x\to 1^+}(1+1)=2$$
$$\lim_{x\to 1^-}\left(\lfloor x\rfloor+|x|\right)=\lim_{x\to 1^-}(0+1)=1$$

Since the right limit is not equal to the left limit, then the limit does not exist. □

(ii)

$$\lim_{x\to\infty}\left(\frac{1}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+2}}+\frac{1}{\sqrt{x^2+3}}+\ldots+\frac{1}{\sqrt{x^2+x}}\right)$$

Solution: We use the Squeeze Theorem to help us. As

$$\frac{x}{\sqrt{x^2+x}}\le\sum_{k=1}^{x}\frac{1}{\sqrt{x^2+k}}\le\sum_{k=1}^{x}\frac{1}{\sqrt{x^2}}$$
$$\frac{x}{\sqrt{x^2+x}}\le\sum_{k=1}^{x}\frac{1}{\sqrt{x^2+k}}\le 1$$

then as x tends to infinity,

$$\frac{x}{\sqrt{x^2+x}}=\frac{\sqrt{x^2}}{\sqrt{x^2+x}}=\sqrt{\frac{x^2}{x^2+x}}=\sqrt{1-\frac{x}{x^2+x}}\to 1.$$

Thus, the required limit is 1 by Squeeze Theorem. □

Problem 8

Let $x,y\in\mathbb{R}$ such that $x<y$. A continuous function $f:D\to\mathbb{R}$ is strictly concave up on its domain D if

$$f\left(\lambda x+(1-\lambda)y\right)<\lambda f(x)+(1-\lambda)f(y)\text{ for all }x,y\in D,\ 0<\lambda<1.$$

(i) Show that $x<\lambda x+(1-\lambda)y<y$.

Solution:

$$\lambda x+(1-\lambda)y-x=x(\lambda-1)+y(1-\lambda)=(y-x)(1-\lambda)>0$$
$$y-\lambda x-(1-\lambda)y=\lambda(y-x)>0$$

□

Define $f:(1,\infty)\to\mathbb{R}$, $g:(-\infty,0)\cup(0,\infty)\to\mathbb{R}$ and $h:(0,\infty)\to\mathbb{R}$ such that

$$f(x)=\ln\left(\frac{1}{x}\right)\text{ and }g(x)=h(x)=\ln\left(\frac{\sin x}{x}\right).$$

(ii) Prove, using two different approaches, that f is strictly concave up for all x in its domain.

Solution: We find the second derivative, so $f(x) = -\ln x \Rightarrow f'(x) = -1/x \Rightarrow f''(x) = 1/x^2 > 0$

Another way is to use the inequality mentioned. Note that *f* is negative in its domain.

$$\begin{aligned}\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) &= -\lambda \ln x - (1-\lambda)\ln y + \ln(\lambda x + (1-\lambda) y) \\ &> -\lambda \ln x - (1-\lambda)\ln y + \ln y \\ &= \lambda(\ln y - \ln x) \\ &= \lambda \ln\left(\frac{y}{x}\right) > 0\end{aligned}$$

□

(iii) Evaluate

$$\lim_{x\to 0} g(x)$$

and determine if $g'(0)$ exists. If it does, calculate its value. You are to show full working.

Solution:

$$\begin{aligned}\lim_{x\to 0} g(x) &= \lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right) \\ &= \ln\left(\lim_{x\to 0}\frac{\sin x}{x}\right) \\ &= \ln 1 \\ &= 0\end{aligned}$$

Thus, *g* is continuous on all real numbers.

$$g'(x) = \frac{x\cos x - \sin x}{x \sin x} = \cot x - \frac{1}{x}$$

By series expansion, write $\cot x$ as $\cos x / \sin x$, so $g'(0) = 0$. □

Let *S* be the following disjoint union of open intervals.

$$S = \bigcup_{n\in\mathbb{N}} (2n\pi, (2n+1)\pi)$$

William claims that for the graph of $y = h(x)$, in each open interval, there exists a local maximum.

(iv) Determine if William's claim is valid.

Solution: Note that $y = h(x)$ has asymptotes at $x = n\pi$, for positive integers *n*.

$$h'(x) = \cot x - \frac{1}{x}$$

Setting $h'(x) = 0$, $x\cot x = 1$. By the Intermediate Value Theorem, there exists a root in the interval $[4,5]$, for which it is a subset of $[\pi, 2\pi]$. Note that the solutions to $x\cot x = 0$ form an arithmetic progression with common difference π, which indicates that $y = h'(x)$ has asymptotes at every positive integer multiple of π and the shape of each curve in the interval between the asymptotes are the same.

One can verify that each of the extrema is a local maximum by using the first derivative test. □

Problem 9

An ellipse has a semi major axis of length a and a semi minor axis of length b, where $a > b > 0$. The eccentricity, e, is defined to be the deviation of a conic section from a circle. The ellipse has an eccentricity strictly between 0 and 1. For an ellipse, e satisfies the equation $b^2 = a^2\left(1-e^2\right)$.

(i) By parametrising the ellipse with $x = a\cos\theta$ and $y = b\sin\theta$, show that the perimeter of the ellipse can be expressed as

$$E = 4a\int_0^{\frac{\pi}{2}} \sqrt{1-e^2\sin^2\theta}\ d\theta,$$

where E belongs to a class of elliptic integrals, known as the Elliptic Integral of the Second Kind. It is worth noting that a closed form for this antiderivative cannot be found.

Solution:

Using the formula for arc length,

$$\begin{aligned}
\int_0^{2\pi}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta &= \int_0^{2\pi}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2\sin^2\theta+a^2\left(1-e^2\right)\cos^2\theta}\ \mathrm{d}\theta \\
&= 4\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2e^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\cos^2\theta}\ \mathrm{d}\theta \\
&= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\left(\theta+\frac{\pi}{2}\right)}\ \mathrm{d}\theta \\
&= 4a\int_{\frac{\pi}{2}}^{\pi}\sqrt{1-e^2\sin^2 t}\ \mathrm{d}t \\
&= 4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\theta}\ \mathrm{d}\theta\ (*)
\end{aligned}$$

□

(ii) Using integration by parts, establish

$$\int_0^{\frac{\pi}{2}}\sin^{2i}\theta\ d\theta = \frac{\pi}{2}\prod_{j=1}^{i}\frac{2j-1}{2j}$$

and use this recursion to express the integral on the left side only in terms of i.

Solution:

$$\int_0^{\frac{\pi}{2}} \sin^{2i}\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sin^{2i-1}\theta \sin\theta\, d\theta$$

$$= \sin^{2i-1}\theta\left(-\cos\theta\right)\Big|_0^{\frac{\pi}{2}} + (2i-1)\int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right)\sin^{2i-2}\theta\, d\theta$$

$$= (2i-1)\int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right)\sin^{2i-2}\theta\, d\theta$$

$$= (2i-1)\int_0^{\frac{\pi}{2}} \sin^{2i-2}\theta - \sin^{2i}\theta\, d\theta$$

Let $I_i = \int_0^{\frac{\pi}{2}} \sin^i\theta\, d\theta$. Then, $2iI_{2i} = (2i-1)I_{2i-2}$. Using a telescoping product,

$$\frac{I_{2i}}{I_{2i-2}} = \frac{2i-1}{2i}$$

$$\prod_{j=1}^{i} \frac{I_{2j}}{I_{2j-2}} = \prod_{j=1}^{i} \frac{2j-1}{2j}$$

$$\frac{I_{2i}}{I_0} = \prod_{j=1}^{i} \frac{2j-1}{2j}$$

Since $I_0 = \pi/2$, the result follows. □

(iii) Using the binomial theorem and assuming that the order of summation and integration can be swapped, prove that the perimeter of the ellipse can be expressed as the following infinite series, for which a product is nested within it:

$$2\pi a\left[1-\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}}{2i-1}\right]$$

Solution:

By the binomial theorem,

$$4a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\theta}\ \mathrm{d}\theta=4a\int_0^{\frac{\pi}{2}}\sum_{i=0}^{\infty}\binom{1/2}{i}\left(-e^2\sin^2\theta\right)^i\ \mathrm{d}\theta$$

Swapping the order of summation and integration, the integral becomes

$$4a\sum_{i=0}^{\infty}\left(e^{2i}\prod_{j=1}^{i}\frac{3/2-j}{j}\int_0^{\frac{\pi}{2}}\sin^{2i}\theta\ \mathrm{d}\theta\right)=2\pi a\sum_{i=0}^{\infty}\left(e^{2i}\prod_{j=1}^{i}\frac{3/2-j}{j}\prod_{j=1}^{i}\frac{2j-1}{2j}\right)$$

$$=2\pi a\sum_{i=0}^{\infty}\left(e^{2i}\prod_{j=1}^{i}\frac{3-2j}{2j}\prod_{j=1}^{i}\frac{2j-1}{2j}\right)$$

$$=2\pi a\sum_{i=0}^{\infty}\left(e^{2i}(-1)^i\prod_{j=1}^{i}\frac{2j-3}{2j}\prod_{j=1}^{i}\frac{2j-1}{2j}\right)$$

$$=2\pi a\sum_{i=0}^{\infty}\left[\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}(-1)^i}{1-2i}\right]$$

$$=2\pi a\sum_{i=0}^{\infty}\left[\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}(-1)^i}{1-2i}\right]$$

$$=2\pi a\left[1-\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{2j-1}{2j}\right)^2\frac{e^{2i}}{2i-1}\right]$$

□