Prove that $\int_0^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x = (\ln 2)^2 - 2\ln 2 + 2 - \zeta(2)$, where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

SOLUTION:

Integrand is an Even Function

Firstly, we note that $\mathrm{f}(x) = \ln(1+x)\ln(1-x)$ is an **even function**. Replacing x with $-x$ gives

$$\mathrm{f}(-x) = \ln(1-x)\ln(1+x)$$

so it is clear that $\mathrm{f}(x) = \mathrm{f}(-x)$.

$$\Rightarrow \int_0^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x = \frac{1}{2}\int_{-1}^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x$$

Performing a Substitution

Consider the substitution $1+x=2t$. Then, $\mathrm{d}x = 2\ \mathrm{d}t$.

$$\begin{aligned}\frac{1}{2}\int_{-1}^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x &= \int_0^1 \ln 2t \ln(2-2t)\ \mathrm{d}t \\ &= \int_0^1 \{\ln 2 + \ln t\}\{\ln 2 + \ln(1-t)\}\ \mathrm{d}t \\ &= \int_0^1 (\ln 2)^2 + \ln 2 \ln t + \ln 2 \ln(1-t) + \ln t \ln(1-t)\ \mathrm{d}t\end{aligned}$$

The integral $\int_0^1 (\ln 2)^2 + \ln 2 \ln t + \ln 2 \ln(1-t)\ \mathrm{d}t$ is trivial. One can evaluate it using integration by parts.

$$\therefore \int_0^1 (\ln 2)^2 + \ln 2 \ln t + \ln 2 \ln(1-t)\ \mathrm{d}t = (\ln 2)^2 - 2\ln 2$$

The Challenging Part of the Integral

Note that using the **Maclaurin Series**,

$$\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n}$$

so replacing t with $-t$, we have

$$\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}.$$

$$\begin{aligned}\int_0^1 \ln t \ln(1-t)\ \mathrm{d}t &= -\int_0^1 \ln t \sum_{n=1}^{\infty} \frac{t^n}{n}\ \mathrm{d}t \\ &= -\sum_{n=1}^{\infty} \frac{1}{n}\int_0^1 t^n \ln t\ \mathrm{d}t\end{aligned}$$

Let $I_n = \int_0^1 t^n \ln t \ \mathrm{d}t$. Then, using integration by parts,

$$I_n = -\frac{1}{\left(n+1\right)^2}.$$

$$-\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 t^n \ln t \ \mathrm{d}t = \sum_{n=1}^{\infty} \frac{1}{n\left(n+1\right)^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} - \frac{1}{\left(n+1\right)^2}$$

$$= 2 - \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Introduction to the Basel Problem

Note that $\zeta\left(2\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ and surprisingly, this sum **converges**! Its value is apparently $\frac{\pi^2}{6}$. This infinite series, known as the **Basel Problem**, was solved by the great Swiss Mathematician Leonhard Euler.

Convergence of the Basel Problem

Consider the inequality

$$n^2 > n\left(n-1\right).$$

Taking reciprocals and summing *n* from 1 to ∞, it is clear that the resulting series converges. Alternatively, one can use the **Limit Comparison Test** to prove it.

Solution to the Basel Problem

Consider $\mathrm{f}\left(x\right) = \frac{\sin x}{x}$. From the graph of $\mathrm{f}\left(x\right)$, the roots of the equation $\mathrm{f}\left(x\right) = 0$ are $\pm k\pi$, where $k \in \mathbb{Z} \setminus \left\{0\right\}$. Using the **Weierstrass Factorisation Theorem**,

$$\mathrm{f}\left(x\right) = \prod_{k \in \mathbb{N}} \left(1 - \frac{x^2}{k^2 \pi^2}\right).$$

Also, note that using the Maclaurin Series of $\sin x$, we have

$$\sin x = \sum_{r=0}^{\infty} \frac{\left(-1\right)^r x^{2r+1}}{\left(2r+1\right)!}$$

so

$$\mathrm{f}\left(x\right) = \sum_{r=0}^{\infty} \frac{\left(-1\right)^r x^{2r}}{\left(2r+1\right)!}$$

When $r = 1$, the coefficient of x^2 is clearly $-\frac{1}{6}$.

Also, the coefficient of x^2 using the Weierstrass Factorisation Theorem is

$$-\sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2}.$$

Equating the two, we establish that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

$$\therefore \int_0^1 \ln(1+x)\ln(1-x)\ \mathrm{d}x = (\ln 2)^2 - 2\ln 2 + 2 - \zeta(2)$$