This question deals with a rigorous proof of the following integral:

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1$$

(a) Consider the graph of $y = \cos x$ on the closed interval $\left[0, \frac{\pi}{2}\right]$. By subdividing the region into n equally-sized rectangles, prove that

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{2n} \cos\left(\frac{k\pi}{2n}\right).$$

(b) **(i)** Using Euler's Formula, express

$$e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}$$

in terms of a single trigonometric function.

(ii) By considering the real part of the aforementioned limit, show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{2n} \cos\left(\frac{k\pi}{2n}\right) = \frac{\pi}{4} \lim_{n\to\infty} \left[\frac{\cot\left(\frac{\pi}{4n}\right) - 1}{n}\right].$$

This part hinges on **(bii)**. You will evaluate the same limit using three different approaches.

(c) **(i)** A theorem on limits is that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

provided that $\lim_{x\to a} g(x) \neq 0$.

Hence, show that

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1.$$

(ii) L'Hôpital's Rule states that if $f(x)$ and $g(x)$ are differentiable functions such that

$g'(x) \neq 0$ on an open interval I containing a,

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists,

then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Sketch the graph of $y = \cot\left(\dfrac{\pi}{4x}\right)$ for $x \geq 0.5$. Hence, use L'Hôpital's Rule to show that

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1 .$$

(iii) The Mittag-Leffler's Theorem states that

$$\cot x = \frac{1}{x} + 2x \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2 \pi^2} .$$

Hence, show that

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = 1 .$$