MA4211 Functional Analysis

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Reference books:

(1) Kreyszig, E. (1989). '*Introductory Functional Analysis with Applications*'. Wiley.

Contents

1. Metric Spaces

1.1. *Metric Spaces*

Functional Analysis is essentially the study of infinite-dimensional Linear Algebra.

Example 1.1 (Euclidean metric/distance). Recall the familiar metric in Euclidean space $\mathbb{R}$

$$d(x,y) = |x-y|.$$

We call this the Euclidean metric or Euclidean distance. Naturally, we can extend this to the Euclidean 2-space $\mathbb{R}^2$. Consider $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Then,

$$d(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

We give the definition of a metric space (Definition 1.1).

Definition 1.1 (metric space). Let X be a set. A metric space is an ordered pair (X,d) equipped with a distance function $d : X \times X \to \mathbb{R}$ such that the following properties are satisifed:

(i) Non-negativity: $d(x,y) \geq 0$

(ii) Positive-definiteness: $x = y$ if and only if $d(x,y) = 0$

(iii) Symmetry: for all $x,y \in SX$, we have $d(x,y) = d(y,x)$

(iv) Triangle inequality: for all x,y,z, we have $d(x,z) \leq d(x,y) + d(y,z)$

Definition 1.2 ($\mathbb{R}^\infty$). Define $\mathbb{R}^\infty$ to be the space of all infinite sequences of real numbers, i.e. $(x_1, x_2, \ldots)$ where $x_1, x_2, \ldots \in \mathbb{R}$.

Example 1.2 ($\mathbb{R}^\infty$). We have the infinite sequences $(0,0,\ldots,)$ and $(1,2,3,\ldots,100,\ldots)$ in $\mathbb{R}^\infty$.

Example 1.3. For $X = \mathbb{R}$, we can define

$$d(x,y) = \min\{|x-y|, 1\} \quad \text{such that it is a metric.}$$

Example 1.4. For $X = \mathbb{R}^\infty$, let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, where each element is in $\mathbb{R}$. Then, one can check that

$$d(\mathbf{x},\mathbf{y}) = \sup d(x_i, y_i) \quad \text{is a metric.}$$

Example 1.5 (ℓ^∞). We give an introduction to the sequence space ℓ^∞. This example gives one an impression of how surprisingly general the concept of a metric space is. We can define

$$X = \{\text{bounded sequences of complex numbers}\}.$$

So, every element of X is a complex sequence ξ_j such that for all $j = 1, 2, \ldots$, we have

$$|\xi_j| \leq c_x \quad \text{where} \quad c_x \text{ is a real number which may depend on } x.$$

Then, the following is a metric:

$$d(\mathbf{x},\mathbf{y}) = \sup_{j\in\mathbb{N}} |\xi_j - \eta_j| \quad \text{where} \quad y = (\eta_1, \eta_2, \ldots) \in X$$

Definition 1.3 (function space). Let

$$\mathcal{C}[a,b] \quad \text{denote} \quad \text{the set of continuous functions on } [a,b].$$

Example 1.6 (function space). Let $X = \mathcal{C}[a,b]$, for which we recall that this refers to the set of continuous functions on $[a,b]$. Let $f,g \in \mathcal{C}[a,b]$. Then,

$$d(f,g) = \max_{x\in[a,b]} |f(x)-g(x)| \quad \text{and} \quad d(f,g) = \sqrt{\int_0^L |f(x)-g(x)|^2\, dx} \quad \text{are metrics.}$$

Example 1.7 (Hamming distance). Consider the two English words 'word' and 'wind' of the same length for which the second and third letters differ. Since two letters differ, we say that their Hamming distance is 2. We write

$$d(\text{wind}, \text{word}) = 2.$$

In this case, d is a metric. The reader can read Kreyszig p. 9 Question 10 to prove that the Hamming distance is indeed a metric.

Definition 1.4 (ℓ^p-space). Let $p \geq 1$ be a fixed real number. Each element in the space ℓ^p is a sequence $(x_1,\ldots)$ such that $|x_1|^p + \ldots$ converges. So,

$$\ell^p = \left\{ \mathbf{x} \in \mathbb{R}^\infty : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

Definition 1.5 (p-norm). Every element in ℓ^p-space is equipped with a norm, known as the p-norm. We define it as follows (will not be strict with the use of either x or $\mathbf{x}$):

$$\text{if } x \in \ell^p \quad \text{then} \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

Theorem 1.1 (Young's inequality). Suppose $\alpha, \beta > 0$. Then,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Example 1.8. Let $t = 1/p$. Then,

$$\begin{aligned} \ln\left(t\alpha^p + (1-t)\beta^q\right) &\geq t\ln(\alpha^p) + (1-t)\ln(\beta^q) \quad \text{since } \ln \text{ is concave down} \\ &= \frac{1}{p}\ln(\alpha^p) + \frac{1}{q}\ln(\beta^q) \\ &= \ln\alpha + \ln\beta \\ &= \ln\alpha\beta \end{aligned}$$

Taking exponentials on both sides yields the desired result.

Theorem 1.2 (Hölder's inequality). We have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q .$$

Theorem 1.3 (Minkowski's inequality). We have

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

Please refer to my MA4262 notes for proofs of Theorems 1.2 and 1.3.

1.2. *Some Topology*

Definition 1.6 (convergence of sequence). Let x_n be a sequence in $\mathbb{R}$. We say that

$$x_n \to x \quad \text{if} \quad \text{there exists } x \in \mathbb{R} \text{ such that } \lim_{n\to\infty} d(x_n, x) = 0.$$

One should recall from MA2108 that this is equivalent to saying that

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } n \geq N \text{ we have } d(x_n, x) < \varepsilon.$$

Definition 1.7 (continuous function). Let (X, d_X) and (Y, d_Y) be metric spaces. We say that f is continuous at $x_0 \in X$ if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that } d_X(x, x_0) < \delta \text{ implies } d_Y(f(x), f(x_0)) < \varepsilon.$$

Theorem 1.4. Let (X, d_X) and (Y, d_Y) be metric spaces and $T : X \to Y$ be a map. Then,

$$T \text{ is continuous} \quad \text{if and only if} \quad x_n \to x \text{ implies } T(x_n) \to T(x).$$

Definition 1.8 (isometry). Let (X, d_X) and (Y, d_Y) be metric spaces and $T : X \to Y$. We say that

$$T \text{ is an isometry} \quad \text{if} \quad d_X(x_1, x_2) = d_Y(T(x_1), T(x_2)).$$

Definition 1.9 (isometric spaces). If there exists a bijective isometry T between two metric spaces X and Y, then we say that X and Y are isometric.

Definition 1.10 (open and closed intervals). Let

$$(a, b) \text{ and } [a, b] \quad \text{denote} \quad \text{the open interval and closed interval in } \mathbb{R} \text{ respectively.}$$

Definition 1.11 (open and closed balls). Define

$$B(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\} \quad \text{denote} \quad \text{the open ball in } \mathbb{R}^n$$
$$\overline{B}(x,r) = \{y \in \mathbb{R}^n : d(x,y) \leq r\} \quad \text{denote} \quad \text{the closed ball in } \mathbb{R}^n$$

Here, each ball is centred at x and is of radius r.

Definition 1.12 (open set). A subset $A \subseteq X$ is open if

$$\text{for all } x \in A \text{ there exists } r > 0 \quad \text{such that} \quad B(x,r) \subseteq A.$$

Definition 1.13 (closed set). A subset of a metric space $S \subseteq X$ is closed if

$$\text{its complement } S^c \quad \text{is open.}$$

Definition 1.14 (topology). Given a set X, a topology $\mathcal{T}$ on X is a collection of subsets of X satisfying the following properties:

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under arbitrary unions

(iii) $\mathcal{T}$ is closed under finite intersection

Definition 1.15 (limit point). In a topological space, a point x is a limit point of a sequence x_n if for every neighbourhood of x, there exists $N \in \mathbb{N}$ such that for all $n \geq n$, x_n belongs to that neighbourhood.

Definition 1.16 (continuous function). If $f : X \to Y$ is a function between two topological spaces X and Y, then f is continuous if the pre-image of every open set in Y is open in X.

Definition 1.17 (closure). Let X be a topological space. For $A \subseteq X$, the closure of A, denoted by $\operatorname{cl}(A)$ or $\overline{A}$, is defined as follows:

$$\overline{A} = A \cup \{\text{limit points of } A\}$$

Definition 1.18 (dense set). Let X be a topological space. If $D \subseteq X$ such that

$$\overline{D} = X \quad \text{then} \quad D \text{ is dense in } X.$$

Definition 1.19 (separable space). A topological space is separable if it has a countable dense subset.

Theorem 1.5. Let $1 \leq p < \infty$. Then, ℓ^p is separable.

Proof. Define $X \subseteq \ell^p$ to be the collection of sequences of the form

$$(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \quad \text{where } x_i \in \mathbb{Q}.$$

As X is a countable union of countable sets, X is countable. Let $y \in \ell^p$ and $\varepsilon > 0$ be arbitrary. That is,

$$\sum_{i=1}^{\infty} |y_i|^p < \infty.$$

Also, there exists $n \in \mathbb{N}$ such that

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon^p}{2}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$. We can choose $x \in X$ such that

$$\sum_{i=1}^{n} |y_i - x_i|^p < \frac{\varepsilon^p}{2}.$$

Then,

$$\|y - x\|_p^p = \sum_{i=1}^{n} |y_i - x_i|^p + \sum_{i=n+1}^{\infty} |y_i|^p < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p.$$

Taking the p^{th} root, we obtain $d(x, y) = \|x - y\|_p < \varepsilon$. $\square$

Theorem 1.6. ℓ^∞ is not separable.

Proof. Let $y \in \mathbb{R}^\infty$ be a sequence of 0s and 1s. Define $z \in \mathbb{R}$ to be as follows:

$$z = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \ldots \quad \text{so} \quad \text{we infer that } 0 \leq z \leq 1.$$

Because there are uncountably many real numbers in $[0, 1]$, it follows there are uncountably many distinct sequences $y \in \{0, 1\}^{\mathbb{N}}$. Denote this uncountable family by $\mathcal{Y} \subset \ell^\infty$. Note that for any two distinct sequences

$$y = (y_1, y_2, y_3, \ldots) \quad y' = (y'_1, y'_2, y'_3, \ldots) \quad \in \mathcal{Y},$$

there is at least one index i such that $y_i \neq y'_i$. Because each coordinate is either 0 or 1, at index i, we have$|y_i - y'_i| = 1$. Hence,

$$\|y - y'\|_\infty = \sup_{n \in \mathbb{N}} |y_n - y'_n| \geq 1.$$

In fact, it is exactly 1 if the two sequences differ in at least one place (and cannot exceed 1 because each coordinate difference is 0 or 1). Around each $y \in \mathcal{Y}$, consider the open ball $B(y, 1/3)$ of radius $1/3$. Since any two distinct y, y' are at distance $\|y - y'\|_\infty = 1$, their balls $B(y, 1/3)$ and $B(y', 1/3)$ cannot overlap. In other words, these balls are pairwise disjoint.

Suppose on the contrary that $D \subseteq \ell^\infty$ is a countable dense set. Then, for each $y \in \mathcal{Y}$, $B(y, 1/3)$ must contain at least one point of D. However, there are uncountably many such disjoint balls $B(y, 1/3)$ since $\mathcal{Y}$ is uncountable. A single countable set D cannot meet each of these uncountably many disjoint balls in a distinct point. This leads to a contradiction. $\square$

Definition 1.20 (Cauchy sequence)**.** A sequence x_n in a metric space X is Cauchy if

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } m, n \geq N \text{ we have } d(x_m, x_n) < \varepsilon.$$

Proposition 1.1. Every convergent sequence is Cauchy.

Remark 1.1. The converse of Proposition 1.1 is not true, i.e. not every Cauchy sequence is convergent.

Definition 1.21 (complete metric space)**.** A metric space is called complete if every Cauchy sequence in the space converges.