

MA3211 MA3211S MA4247 MA5217 Complex Analysis Notes

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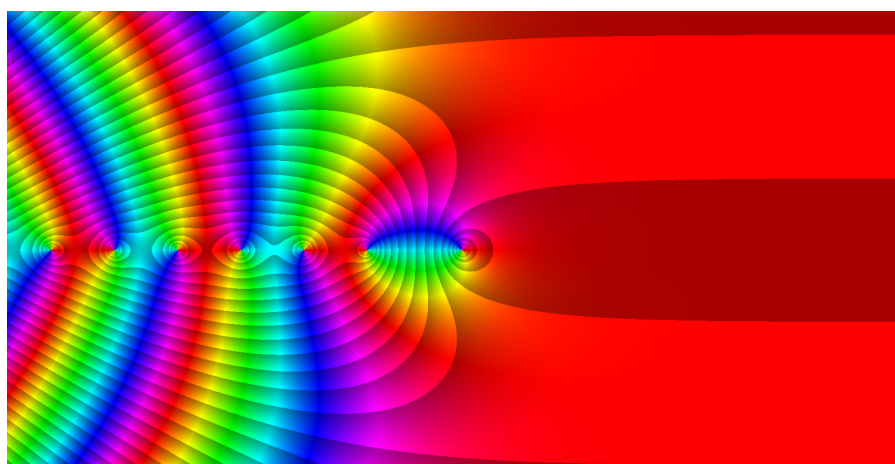
Reference books:

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- (2) Needham, T. (2023). *Visual Complex Analysis: 25th Anniversary Edition*. Oxford.
- (3) Conway, J. B. (1973). *Functions of One Complex Variable I*. Springer.

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1. Complex Numbers

1.1. Some Background Knowledge

First, we define

$\mathbb{R}[x]$ to be the set of polynomials with real coefficients.

The polynomial $x^2 + 1 \in \mathbb{R}[x]$ of degree 2 over \mathbb{R} has no solution in \mathbb{R} since for all $\alpha \in \mathbb{R}$, we have $\alpha^2 + 1 > 0$, so $x^2 + 1$ is irreducible over $\mathbb{R}[x]$. For those who have prior knowledge on Abstract Algebra, since $\mathbb{R}[x]$ is a principal ideal domain (PID)[†], then

$$(x^2 + 1)\mathbb{R}[x] \subseteq \mathbb{R}[x] \text{ is a maximal ideal.}$$

As such, we are now in position to define the complex numbers \mathbb{C} .

Definition 1.1 (complex numbers). Define

$$\mathbb{C} = \mathbb{R}[x] / (x^2 + 1)\mathbb{R}[x]$$

to be the quotient ring of $\mathbb{R}[x]$ modulo the maximal ideal $(x^2 + 1)\mathbb{R}[x]$. This is a field, known as the field of complex numbers.

Proposition 1.1. The image of

$$x \in \mathbb{R}[x] \text{ in } \mathbb{C} \text{ is denoted by } i \in \mathbb{C},$$

called the imaginary unit. i has the property that $i^2 = -1$.

Proposition 1.2 (field extension). The composite of the canonical ring homomorphisms

$$\mathbb{R} \hookrightarrow \mathbb{R}[x] \twoheadrightarrow \mathbb{C} \text{ where } x \mapsto i$$

is an inclusion of fields $\mathbb{R} \hookrightarrow \mathbb{C}$ so \mathbb{C} is a field extension of \mathbb{R} .

Proposition 1.3. As an \mathbb{R} -vector space, \mathbb{C} has dimension 2 with standard ordered \mathbb{R} -basis $\{1, i\}$.

[†]Recall from MA3201 that if F is a field, then $F[x]$ is a Euclidean domain. In fact, recall the chain of inclusions $\text{ED} \subseteq \text{PID} \subseteq \text{UFD}$, where ED and UFD denote Euclidean domain and unique factorisation domain respectively. I recall in one of Sadhukhan's MA2101S that one student asked whether F is a field implies $F[x]$ is also a field. Clearly, this is wrong and Sadhukhan mentioned that $F[x]$ is a UFD. It was only when I crashed one of Bao Haunchen's MA4203 lectures (first lecture actually) where I learnt that the stronger statement $F[x]$ is an ED holds.

Definition 1.2. The \mathbb{R} -linear projection maps

$$\operatorname{Re} : \mathbb{C} \rightarrow \mathbb{R} \text{ where } z \mapsto x \quad \text{and} \quad \operatorname{Im} : \mathbb{C} \rightarrow \mathbb{R} \text{ where } z \mapsto y$$

are called the real part and imaginary part of $z \in \mathbb{C}$. So,

$$\text{for all } z \in \mathbb{C} \quad \text{one has } z = \operatorname{Re} z + i \operatorname{Im} z \text{ in } \mathbb{C}.$$

Proposition 1.4 (field operations). The field operations of \mathbb{C} , expressed in terms of the real/imaginary parts, are:

(i) **Addition/Subtraction:**

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$

(ii) **Multiplication:**

$$(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$$

(iii) **Division:**

$$\frac{(a + ib)}{(c + id)} = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

(iv) **Multiplicative inverse:**

$$(c + id)^{-1} = \frac{c - id}{c^2 + d^2}$$

Definition 1.3 (complex conjugation). The \mathbb{R} -linear map

$$\overline{(\cdot)} : \mathbb{C} \rightarrow \mathbb{C} \quad \text{where} \quad z = x + iy \mapsto \bar{z} = x - iy$$

is called complex conjugation.

Proposition 1.5. We say that complex conjugation is an automorphism of \mathbb{C} as a field over \mathbb{R} . The automorphism group $\operatorname{Aut}(\mathbb{C}/\mathbb{R})$ is of order 2. That is to say,

$$\overline{\bar{z}} = z.$$

Proposition 1.6. The following properties hold for all $z, w \in \mathbb{C}$:

(i) $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$

(ii) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$ and $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$

Definition 1.4 (absolute value). The absolute value of a complex number is the map

$$|\cdot|_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0} \text{ where } z \mapsto |z|_{\mathbb{C}} \quad \text{given by} \quad |z|_{\mathbb{C}} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{z\bar{z}}.$$

As such, if $z = x + iy$ (where $x, y \in \mathbb{R}$), we have

$$|z|_{\mathbb{C}}^2 = x^2 + y^2 = z\bar{z}.$$

Proposition 1.7. For any $a \in \mathbb{R} \subseteq \mathbb{C}$, we have $|a|_{\mathbb{C}} = |a|_{\mathbb{R}}$.

Lemma 1.1. For any $z, w \in \mathbb{C}$, we have

- (i) **Positive-definiteness:** $|z|_{\mathbb{C}} = 0$ in $\mathbb{R}_{\geq 0}$ if and only if $z = 0$ in \mathbb{C}
- (ii) $|\bar{z}|_{\mathbb{C}} = |z|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$
- (iii) **Multiplicativity:** $|zw|_{\mathbb{C}} = |z|_{\mathbb{C}} |w|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$
- (iv) $|\operatorname{Re} z|_{\mathbb{R}}, |\operatorname{Im} z|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$

Proof. (i) and (ii) are trivial. To prove (iii), we have

$$|zw|_{\mathbb{C}}^2 = zw\overline{zw} = z\bar{z} \cdot w\bar{w} = |z|_{\mathbb{C}}^2 |w|_{\mathbb{C}}^2.$$

Taking square roots on both sides, (iii) follows.

For (iv), let $z = x + iy$, where $x, y \in \mathbb{R}$. Then, $x^2, y^2 \leq x^2 + y^2$, so $|x|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$ and $|y|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$. □

Lemma 1.2 (triangle inequality). For any $z, w \in \mathbb{C}$, we have

$$|z + w|_{\mathbb{C}} \leq |z|_{\mathbb{C}} + |w|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\geq 0}.$$

Proof. We have

$$\begin{aligned} |z + w|_{\mathbb{C}}^2 &= (z + w)(\overline{z + w}) = z\bar{z} + w\bar{w} + (z\bar{w} + \bar{z}w) \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2\operatorname{Re}(z\bar{w}) \\ &\leq |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2|z\bar{w}|_{\mathbb{C}} \quad \text{by (iv) of Lemma 1.1} \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2|z|_{\mathbb{C}}|w|_{\mathbb{C}} \\ &= (|z|_{\mathbb{C}} + |w|_{\mathbb{C}})^2 \end{aligned}$$

Taking square roots on both sides, the result follows. □

By (i) and (iii) of Lemma 1.1 on the positive-definiteness and multiplicativity, as well as Lemma 1.2 on the triangle inequality, we infer that

$|\cdot|_{\mathbb{C}}$ is an absolute value of \mathbb{C} in the abstract sense.

Corollary 1.1. We say that

\mathbb{C} equipped with the absolute value function $|\cdot|_{\mathbb{C}}$ as a normed \mathbb{R} -vector space

is isomorphic to \mathbb{R}^2 with the standard Euclidean norm $\|\cdot\|_2$, so \mathbb{C} is said to be *Cauchy complete*.

Corollary 1.2 (generalised triangle inequality). For any $z_1, z_2, \dots, z_n \in \mathbb{C}$, we have

$$|z_1 + \dots + z_n|_{\mathbb{C}} \leq |z_1|_{\mathbb{C}} + \dots + |z_n|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\geq 0}.$$

Proof. Consider the triangle inequality (Lemma 1.2) and use induction. □

Theorem 1.1 (Cauchy-Schwarz inequality for \mathbb{R}^2). For any $z, w \in \mathbb{C}$, we have

$$|\langle z, w \rangle_{\mathbb{R}^2}|_{\mathbb{R}} \leq |z|_{\mathbb{C}} |w|_{\mathbb{C}} \quad \text{with equality if and only if } z \text{ and } w \text{ are } \mathbb{R}\text{-linearly dependent.}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product of the two inputs. That is to say,

$$z = x + iy \text{ and } w = u + iv \quad \text{implies} \quad \langle z, w \rangle_{\mathbb{R}^2} = xu + yv.$$

Proof. The trick is as follows:

$$\begin{aligned} \langle z, w \rangle_{\mathbb{R}^2}^2 + \langle iz, w \rangle_{\mathbb{R}^2}^2 &= (xu + yv)^2 + (-yu + xv)^2 \\ &= x^2u^2 + y^2v^2 + 2xuyv + y^2u^2 + x^2v^2 - 2yuxv \\ &= (x^2 + y^2)(u^2 + v^2) \\ &= |z|_{\mathbb{C}}^2 |w|_{\mathbb{C}}^2 \end{aligned}$$

which implies $|\langle z, w \rangle_{\mathbb{R}^2}|_{\mathbb{R}} \leq |z|_{\mathbb{C}} |w|_{\mathbb{C}}$. Equality holds if and only if $\langle iz, w \rangle_{\mathbb{R}^2} = 0$, or equivalently, $-yu + xv = 0$, i.e. z and w are \mathbb{R} -linearly dependent. Well, to be more explicit, we recall that

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{as vectors in } \mathbb{R}^2.$$

If z and w are linearly dependent, there exists $k \in \mathbb{R}$ such that $(x, y) = k(u, v)$, so $x = ku$ and $y = kv$. As such, $-yu + xv = 0$. □

We can generalise Theorem 1.1 to the Cauchy-Schwarz inequality for \mathbb{C}^n (Theorem 1.2).

Theorem 1.2 (Cauchy-Schwarz inequality for \mathbb{C}^n). For any $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$, we have

$$|z_1 w_1 + \dots + z_n w_n|_{\mathbb{C}}^2 \leq (|z_1|_{\mathbb{C}}^2 + \dots + |z_n|_{\mathbb{C}}^2) (|w_1|_{\mathbb{C}}^2 + \dots + |w_n|_{\mathbb{C}}^2)$$

and

$$\text{equality holds if and only if } \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \text{ and } \begin{bmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{bmatrix} \text{ are } \mathbb{C}\text{-linearly dependent over } \mathbb{C}^n.$$

Equivalently, this means that there exist $\lambda, \mu \in \mathbb{C}$ which are both non-zero such that

$$\text{for all } 1 \leq j \leq n \quad \text{we have} \quad \lambda z_j = \mu \overline{w_j} \quad \text{in } \mathbb{C}.$$

Proof. We have

$$\begin{aligned} 0 &\leq \sum_{i < j} |z_i \overline{w_j} - z_j \overline{w_i}|_{\mathbb{C}}^2 \\ &= \sum_{i < j} (z_i \overline{w_j} - z_j \overline{w_i}) (\overline{z_i \overline{w_j} - z_j \overline{w_i}}) \\ &= \sum_{i < j} |z_i|^2 |w_j|^2 + |z_j|^2 |w_i|^2 - 2 \operatorname{Re} (z_i \overline{z_j} \overline{w_i} w_j) \end{aligned}$$

We now add the following term to both sides of the inequality:

$$\left| \sum_{i=1}^n z_i w_i \right|^2 = \sum_{i=1}^n |z_i|^2 |w_i|^2 + \sum_{i < j} (z_i w_i \overline{z_j w_j} + \overline{z_i w_i} z_j w_j)$$

for which it follows that

$$\begin{aligned} \left| \sum_{i=1}^n z_i w_i \right|^2 &\leq \sum_{i=1}^n |z_i|^2 |w_i|^2 + \sum_{i < j} (|z_i|^2 |w_j|^2 + |z_j|^2 |w_i|^2) \\ &= \left(\sum_{i=1}^n |z_i|^2 \right) \left(\sum_{i=1}^n |w_i|^2 \right) \end{aligned}$$

Equality holds if and only if

$$\sum_{i < j} |z_i \overline{w_j} - z_j \overline{w_i}|_{\mathbb{C}}^2 = 0.$$

This holds if and only if for all $i < j$, one has $z_i \overline{w_j} = z_j \overline{w_i}$. □

Theorem 1.3 (de Moivre's theorem). For $n \in \mathbb{Z}$,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

Proof. By Euler's formula, $e^{i\theta} = \cos \theta + i \sin \theta$. In de Moivre's theorem, the left side of the equation is $e^{in\theta}$ by raising both sides to the power of n . The result follows by using Euler's formula on $e^{in\theta}$. □

Example 1.1 (MA5217 AY24/25 Sem 1 Homework 1). Find all solutions of the equation $e^{e^z} = 1$.

Solution. Note that $1 = e^{2k\pi i}$ for all $k \in \mathbb{Z}$. Since the exponential function is injective, we have $e^z = 2k\pi i$. Hence, $z = \ln |2k\pi| + i\pi/2$. □

1.2. Complex-Valued Functions

Let X be any set. Then, we have the following:

$\operatorname{Maps}(X, \mathbb{R}) = \{\text{all } \mathbb{R}\text{-valued functions on } X\}$ is an \mathbb{R} -vector space

$\operatorname{Maps}(X, \mathbb{C}) = \{\text{all } \mathbb{C}\text{-valued functions on } X\}$ is a \mathbb{C} -vector space

Proposition 1.8. The \mathbb{R} -basis $\{1, i\}$ of \mathbb{C} gives an \mathbb{R} -linear decomposition:

$$\text{Maps}(X, \mathbb{C}) \cong \text{Maps}(X, \mathbb{R}) \oplus i \cdot \text{Maps}(X, \mathbb{R}) \quad \text{where} \quad f \mapsto \text{Re } f + i \cdot \text{Im } f.$$

This is such that for any $x \in X$,

$$\text{Re}(f)(x) = \text{Re}(f) \in \mathbb{R}, \quad \text{Im}(f)(x) = \text{Im}(f) \in \mathbb{R}.$$

Proposition 1.9. The \mathbb{R} -automorphism $(\bar{\cdot})$ of \mathbb{C} also gives an \mathbb{R} -linear automorphism:

$$(\bar{\cdot}) : \text{Maps}(X, \mathbb{C}) \rightarrow \text{Maps}(X, \mathbb{C}) \quad \text{where} \quad f \mapsto \bar{f}.$$

This is such that for any $x \in X$,

$$\bar{f}(x) = \overline{f(x)} \quad \text{in } \mathbb{C}.$$

Proposition 1.10. One has the following decomposition:

$$\text{Re } f = \frac{f + \bar{f}}{2}, \quad \text{Im } f = \frac{f - \bar{f}}{2i}.$$

2. Holomorphic and Analytic Functions

2.1. Holomorphic Functions

Definition 2.1. Let

$\Omega \subseteq \mathbb{C}$ be an open and connected set in \mathbb{C}

$H(\Omega)$ be the set of holomorphic functions in Ω

Definition 2.2 (holomorphic function). Let $\Omega \subseteq \mathbb{C}$ be an open set. A function $f : \Omega \rightarrow \mathbb{C}$ is holomorphic at a or \mathbb{C} -differentiable at a (Proposition 2.3) if and only if

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists in } \mathbb{C}.$$

In this case, the limit, which is uniquely determined by f and a , is called the holomorphic derivative of f at a , denoted by

$$\frac{df}{dz}(a) = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ in } \mathbb{C}.$$

As such,

$f : \Omega \rightarrow \mathbb{C}$ is holomorphic on G if and only if for all $a \in G$, f is holomorphic at a .

Proposition 2.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f, g : \Omega \rightarrow \mathbb{C}$ be functions holomorphic at a . Then, the following hold:

(i) **\mathbb{C} -linearity:** for all $c, d \in \mathbb{C}$,

the function $cf + dg : \Omega \rightarrow \mathbb{C}$ is also holomorphic at a

equipped with

its holomorphic derivative $(cf + dg)'(a) = c \cdot f'(a) + d \cdot g'(a)$ in \mathbb{C}

(ii) **Product rule:** the function $f \cdot g : \Omega \rightarrow \mathbb{C}$ is also holomorphic at a equipped with its holomorphic derivative

$$(fg)'(a) = f'(a)g(a) + g'(a)f(a) \text{ in } \mathbb{C}$$

Remark 2.1. Recall Definition 2.1, which mentioned that $H(\Omega)$ denotes the set of all functions $f : \Omega \rightarrow \mathbb{C}$ which are holomorphic on Ω . We say that

$H(\Omega)$ is a \mathbb{C} -algebra under pointwise \pm, \times of functions.

Note that for any point $a \in \Omega$, we have the evaluation at a map, i.e.

$$\text{ev}_a : H(\Omega) \rightarrow \mathbb{C} \quad \text{where} \quad f \mapsto f(a),$$

which is a \mathbb{C} -algebra homomorphism.

Also, Proposition 2.1 says that the derivative at a map

$$H(\Omega) \rightarrow \mathbb{C} \quad \text{where} \quad f \mapsto f'(a)$$

is a \mathbb{C} -linear derivative of $H(\Omega)$ to the $H(\Omega)$ -module \mathbb{C} via ev_a .

Example 2.1 (identity map). For any open $\Omega \subset \mathbb{C}$, the identity map id is holomorphic with derivative

$$\text{id}'(a) = \frac{dz}{dz}(a) = 1 \quad \text{for all } a \in G.$$

Hence, $z \in H(\Omega)$. In fact, for any polynomial $f \in \mathbb{C}[z]^\dagger$, the function $z \mapsto f(z)$ is also $H(G)$.

Example 2.2. For any open $G = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, the reciprocal function z^{-1} is holomorphic with derivative

$$\frac{dz^{-1}}{dz}(a) = -\frac{1}{a^2} \quad \text{for all } a \in G.$$

Hence, $z^{-1} \in H(\Omega)$. Moreover, for any Laurent polynomial $f \in \mathbb{C}[z, z^{-1}]$ (we will only discuss this when formally defining Laurent series/polynomials in Theorem 5.1), the function $z \mapsto f(z)$ is also in $H(\Omega)$.

Proposition 2.2 (chain rule). Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open sets. Let

$$f : \Omega_1 \rightarrow \mathbb{C} \text{ and } g : \Omega_2 \rightarrow \mathbb{C} \quad \text{such that} \quad f(\Omega_1) \subseteq \Omega_2$$

so $g \circ f : \Omega_1 \rightarrow \mathbb{C}$ is defined. If f is holomorphic at a and g is holomorphic at $f(a)$, then $g \circ f$ is holomorphic at a , equipped with its holomorphic derivative

$$(g \circ f)'(a) = g'(f(a)) f'(a).$$

Proof. Let $b = f(a) \in \Omega_2$. Define the functions $\xi : \Omega_1 \rightarrow \mathbb{C}$ and $\eta : \Omega_2 \rightarrow \mathbb{C}$ by setting

$$\xi(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} - f'(a) & \text{if } z \in \Omega_1 \setminus \{a\} \\ \text{any value} & \text{if } z = a \end{cases} \quad \text{and} \quad \eta(w) = \begin{cases} \frac{g(w) - g(b)}{w - b} - g'(b) & \text{if } w \in \Omega_2 \setminus \{b\} \\ \text{any value} & \text{if } w = b. \end{cases}$$

[†]Here, one should perhaps recall from MA3201 that $\mathbb{C}[z]$ denotes the set of all polynomials in z with complex coefficients. That is, $\mathbb{C}[z] \ni f(z) = a_0 + a_1 z + \dots + a_n z^n$ where $a_0, a_1, \dots, a_n \in \mathbb{C}$.

Then, for all $z \in \Omega_1$ and $w \in \Omega_2$, we have the following in \mathbb{C} :

$$\begin{aligned} f(z) - f(a) &= [f'(a) + \xi(z)](z - a) \\ g(w) - g(b) &= [g'(b) + \eta(w)](w - b) \end{aligned}$$

Thus, for all $z \in \Omega_1$, we have

$$\begin{aligned} g(f(z)) - g(f(a)) &= (g'(f(a)) + \eta(f(z)))(f(z) - f(a)) \\ &= (g'(f(a)) + \eta(f(z)))(f'(a) + \xi(z))(z - a) \end{aligned}$$

so for all $z \in \Omega_1 \setminus \{a\}$, we have

$$\frac{g(f(z)) - g(f(a))}{z - a} = (g'(f(a)) + \eta(f(z)))(f'(a) + \xi(z)).$$

Since

$$\begin{aligned} f &\text{ is holomorphic at } a \in \Omega_1 \quad \text{and} \\ g &\text{ is holomorphic at } b \in \Omega_2 \end{aligned}$$

then

$$\lim_{z \rightarrow a} \xi(z) = 0 \quad \text{and} \quad \lim_{w \rightarrow b} \eta(w) = 0.$$

Also,

$$f \text{ is continuous at } a \quad \text{implies} \quad \lim_{z \rightarrow a} f(z) = f(a) = b.$$

Hence,

$$\lim_{z \rightarrow a} \frac{g(f(z)) - g(f(a))}{z - a} \text{ exists in } \mathbb{C} \quad \text{and} \quad \text{equals } g'(f(a))f'(a).$$

□

Next, recall Definition 2.3 on \mathbb{R} -differentiability from MA3210.

Definition 2.3 (\mathbb{R} -differentiability). We say that f is \mathbb{R} -differentiable at a if and only if there exists an \mathbb{R} -linear map $(Df)(a) : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\text{for all } \varepsilon \in \mathbb{R}_{>0}, \text{ there exists } \delta \in \mathbb{R}_{>0}$$

such that

$$\text{for all } z \in G \text{ with } 0 \leq \|z - a\| < \delta \quad \text{we have} \quad \|f(z) - f(a) - (Df)(a)(z - a)\| \leq \varepsilon \cdot \|z - a\|.$$

When this holds, the \mathbb{R} -linear map $(Df)(a)$ is uniquely determined by f and a and we call this the derivative of f at a .

Proposition 2.3 (\mathbb{C} -differentiability). If f is holomorphic at a (\mathbb{C} -differentiable at a), then f is \mathbb{R} -differentiable at a and

$$(Df)(a) \in \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{is the image of} \quad f'(a) \in \text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

under the following canonical inclusion:

$$\text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \hookrightarrow \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{where} \quad z \mapsto \text{multiplication by } z.$$

Corollary 2.1. Suppose f is holomorphic on Ω and for all $a \in G$, we have $f'(a) = 0$ in \mathbb{C} . Then, f is locally constant on Ω .

Proof. Let $a \in \Omega$ be an arbitrary point. Choose $r \in \mathbb{R}_{>0}$ be sufficiently small such that $B(a, r) \subseteq G$, where

$$B(a, r) \quad \text{is the open ball in } \mathbb{C} \text{ centred at } a \text{ of radius } r.$$

By the mean-value inequality, for any $z \in B(a, r)$, there exists $\xi \in [a, z] \subseteq B(a, r)$ such that

$$\|f(z) - f(a)\| \leq \|f'(\xi)\| \|z - a\|$$

Since $f'(\xi) = 0$, then f is constant of value $f(a)$ on $B(a, r)$. □

Now, identify \mathbb{C} with the standard \mathbb{R} -basis $\{1, i\}$. Then, consider the following comparison:

$$\mathbb{R}^2 \xleftarrow{1, i} \mathbb{C} \xrightarrow{z \mapsto \text{multiplication by } z} \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \xleftarrow{1, i} \mathcal{M}_{2 \times 2}(\mathbb{R})$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + bi \mapsto (x + yi) \mapsto (a + bi)(x + yi) = (ax - by) + i(bx + ay) \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We infer that via 1 and i , the matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R})$$

corresponds to the \mathbb{R} -linear map $\mathbb{C} \rightarrow \mathbb{C}$ given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad x + yi \mapsto (px + qy) + i(rx + sy).$$

This \mathbb{R} -linear map is \mathbb{C} -linear if and only if $p = s$ and $q = -r$ in \mathbb{R} . As such, we can set $a = p$ and $q = -b$.

Now, again via 1 and i , write

$$f : \Omega \rightarrow \mathbb{C} \quad \text{as} \quad x + iy \mapsto f(x + iy) = u(x, y) + iv(x, y).$$

Suppose f is \mathbb{R} -differentiable at a . Then,

$$(Df)(a) \in \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{corresponds to} \quad \begin{bmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a) \end{bmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R}).$$

Hence, $(Df)(a)$ lies in the image of $\mathbb{C} \hookrightarrow \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ if and only if

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \text{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

This is precisely the Cauchy-Riemann equations, for which we will formally introduce in Theorem 2.1!

2.2. The Cauchy-Riemann Equations

Theorem 2.1 (Cauchy-Riemann equations). Let $\Omega \subseteq \mathbb{C}$ be an open set. Let $f : \Omega \rightarrow \mathbb{C}$ be a function written as

$$x + iy \mapsto f(x + iy) = u(x, y) + iv(x, y).$$

Suppose f is \mathbb{R} -differentiable at a . Then,

f is holomorphic at a if and only if $\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a)$ and $\frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a)$ are satisfied.

Theorem 2.2 (polar form of CR equations). If u and v are expressed in terms of polar coordinates (r, θ) , then

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

Proof. Using the substitution $z = re^{i\theta}$, we have $x = r \cos \theta$ and $y = r \sin \theta$. Since $f(z) = u(x, y) + iv(x, y)$, we will now perform change of variables from (x, y) to (r, θ) . By the chain rule for partial derivatives, to compute $\partial u / \partial r$,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta.$$

By the CR equations (Theorem 2.1),

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta.$$

To compute $\partial v / \partial \theta$,

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x} (-r \sin \theta) + \frac{\partial v}{\partial y} (r \cos \theta).$$

It is thus clear that the first equation of the theorem holds true. The proof of the second theorem is left as an exercise. \square

Theorem 2.3. Let $f(z) = u(x, y) + iv(x, y)$. Suppose the first-order partial derivatives of u and v (u_x, u_y, v_x and v_y) exist in a neighbourhood of z . If they are continuous at z and the CR equations hold, then f is differentiable at z .

Example 2.3. Suppose

$$f(z) = \begin{cases} (\bar{z})^2 / z & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but the derivative of f fails to exist at $z = 0$.

Solution. We let $z = x + iy$, where $x, y \in \mathbb{R}$. Then, for $z \neq 0$,

$$f(z) = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \left(\frac{-3x^2y + y^3}{x^2 + y^2} \right)$$

which is of the form $f(z) = u(x, y) + iv(x, y)$. The reader can check that at $(0, 0)$, u_x, u_y, v_x, v_y are all zero, so the CR equations are satisfied. Next, we consider the following limit:

$$L = \lim_{h \rightarrow 0} \frac{(\bar{h})^2 / h - 0}{h} = \lim_{h \rightarrow 0} \left(\frac{\bar{h}}{h} \right)^2 = \lim_{(x, y) \rightarrow (0, 0)} \left(\frac{x - iy}{x + iy} \right)^2.$$

Say we approach along the real axis. Then, $L = 1$. However, if we approach along the line $y = x$,

$$L = \lim_{(x, x) \rightarrow (0, 0)} \left[\frac{x(1 - i)}{x(1 + i)} \right]^2 = -1$$

so we conclude that $f'(0)$ does not exist. □

Example 2.4. Let

$$f(z) = f(x, y) = \begin{cases} \frac{xy(x + iy)}{x^2 + y^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at $z = 0$ but f is not differentiable at $z = 0$.

Solution. We let the reader verify that the CR equations are satisfied at $z = 0$. As for differentiability, let $h = a + ib$, where $a, b \in \mathbb{R}$. Then consider

$$\frac{f(h) - f(0)}{h} = \frac{ab(a + ib)}{(a^2 + b^2)(a + ib)} = \frac{ab}{a^2 + b^2}.$$

We need to prove that as $(a, b) \rightarrow (0, 0)$, the limit L does not exist. Suppose we approach along the x -axis, then $L = 0$. However, if we approach along the line $y = x$, we have

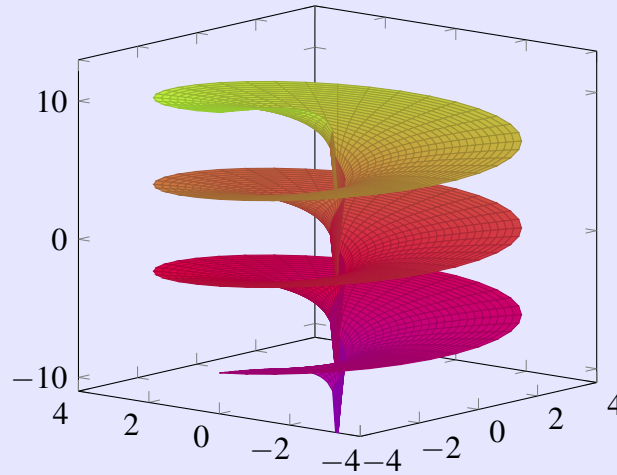
$$L = \lim_{a \rightarrow 0} \frac{a^2}{a^2 + a^2} = \frac{1}{2}.$$

As such, the limit L does not exist so we can conclude that $f'(0)$ does not exist. □

Definition 2.4 (principal logarithm). Define

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z.$$

Note that $\operatorname{Log} z$ is a single-valued function defined on $\mathbb{C} \setminus \{0\}$.



2.3. Analytic Functions and Entire Functions

Definition 2.5 (analytic function). A function f is analytic at z if it is differentiable in a neighbourhood of z . A function f is analytic on a set S if f is analytic at every $z \in S$.

Example 2.5. Show that there are no analytic functions $f = u + iv$ such that $u(x, y) = x^2 + y^2$.

Solution. Suppose on the contrary that there exists some analytic function f . Then, $u_x = 2x$ and $u_y = 2y$, so by the CR equations, $v_y = 2x$ and $v_x = -2y$. $v_y = 2x$ implies that $v(x, y) = 2xy + g(x)$. Taking the partial with respect to x and substituting it into $v_x = -2y$, we have $2y + g'(x) = -2y$. As such, $g'(x) = -4y$, so $g(x) = -4xy + c$, where c is an arbitrary constant. Putting everything together,

$$f(x, y) = x^2 + y^2 + i(-2xy + c).$$

However, this does not satisfy $u_x = v_y$ in the CR equations. So, such an f does not exist. \square

Example 2.6. Suppose f is analytic and real-valued in a domain D . Prove that f is constant in D .

Solution. Suppose $f(z) = u + iv$. We have $\operatorname{Im}(f) = 0$ so by the CR equations, $u_x = 0$ and $u_y = -v_x = 0$. This implies that $f'(z) = u_x + iv_x = 0$ so f is constant in D . \square

Example 2.7. Suppose f and \bar{f} are analytic in a domain D . Show that f is constant in D .

Solution. Observe that $\operatorname{Re}(f) = (f + \bar{f})/2$ which is real-valued and analytic if both f and \bar{f} are analytic. By Example 2.6, $\operatorname{Re}(f)$ is constant, so f is constant. \square

Definition 2.6 (entire function). A function f which is analytic on the whole of \mathbb{C} is entire.

Example 2.8. Let

$$f(z) = x^3 - 3xy^2 + x^2 - y^2 + x + 1 + i(3x^2y - y^3 + 2xy + y).$$

- (a) Show that $f(z)$ is entire.
 (b) Express $f(z)$ as a function of z .

Solution.

- (a) This is a very simple exercise using the CR equations.
 (b) Recall the binomial theorem and see that

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3) + x^2 - y^2 + x + 1 + i(2xy + y) \\ &= x^3 - 3xy^2 + i(3x^2y - y^3) + x^2 - y^2 + 2ixy + x + iy + 1 \\ &= (x + iy)^3 + (x + iy)^2 + x + iy + 1 \\ &= z^3 + z^2 + z + 1 \end{aligned}$$

$$\text{So, } f(z) = z^3 + z^2 + z + 1. \quad \square$$

Example 2.9. Find an entire function f such that $\operatorname{Re}(f) = x^2 - 3x - y^2$ or explain why there is no such function.

Solution. Write $f = u + iv$, where u and v are real-valued functions. Given that $u = x^2 - 3x - y^2$, we apply the CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 3 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y.$$

Solving the first equation yields $v = 2xy + g(y)$, where $g(y)$ is a function in terms of y . Then, $2x + g'(y) = 2x - 3$, which implies that $g(y) = -3y + c$ for some constant c .

Now, we have $v = 2xy - 3y + c$. We conclude that the following function satisfies the hypotheses:

$$\begin{aligned} f(z) &= x^2 - 3x - y^2 + i(2xy - 3y + c) \\ &= x^2 - y^2 + 2ixy - 3x - 3iy + ic \\ &= z^2 - 3z + ci \end{aligned}$$

$$\text{So, } f(z) = z^2 - 3z + ci. \quad \square$$

Example 2.10 (Dinh's 70 problems). Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function such that

$$f(0) = f'(0) = 0 \quad \text{and} \quad \operatorname{Re}(f') = x^2 - y^2 + 6xy.$$

Find f .

Solution. Let $z = x + iy$, so $z^2 = x^2 - y^2 + 2xyi$. As such,

$$x^2 - y^2 + 6xy = \operatorname{Re}(z^2 - 3iz^2)$$

Since $f'(0) = 0$, then $f'(z) = z^2 - 3iz^2$. It follows that $f(z) = z^3/3 - iz^3$ as $f(0) = 0$. \square

2.4. Harmonic Functions

Definition 2.7 (harmonic function). A real-valued function $h(x, y)$ is said to be harmonic if it is twice continuously differentiable and satisfies Laplace's equation. That is,

$$h_{xx} + h_{yy} = 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Example 2.11. Show that u^2 cannot be harmonic for any non-constant harmonic function u .

Solution. Let u be a non-constant harmonic function. Then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Also, we have

$$\frac{\partial^2 (u^2)}{\partial x^2} = 2u \frac{\partial^2 u}{\partial x^2} + 2 \left(\frac{\partial u}{\partial x} \right)^2 \quad \text{and} \quad \frac{\partial^2 (u^2)}{\partial y^2} = 2u \frac{\partial^2 u}{\partial y^2} + 2 \left(\frac{\partial u}{\partial y} \right)^2.$$

However,

$$\frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2} = 2u \frac{\partial^2 u}{\partial x^2} + 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2u \frac{\partial^2 u}{\partial y^2} + 2 \left(\frac{\partial u}{\partial y} \right)^2 = 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial u}{\partial y} \right)^2 \neq 0,$$

which concludes the proof. \square

Definition 2.8 (harmonic conjugate). Let u be a harmonic function. If v is a harmonic function satisfying the Cauchy-Riemann equations, then v is a harmonic conjugate of u .

Example 2.12 (MA5217 AY24/25 Sem 1 Homework 1). Show that the function

$$u(x, y) = e^{x-y} \cos(x+y) + e^{x+y} \cos(x-y)$$

is harmonic in \mathbb{C} and find a harmonic conjugate of u .

Solution. By definition, we need to show that u satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Let $s = x + y$ and $t = x - y$, so

$$u\left(\frac{s+t}{2}, \frac{s-t}{2}\right) = e^t \cos s + e^s \cos t$$

Hence,

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = e^s \cos t - e^t \cos s + e^t \cos s - e^s \cos t = 0$$

so u is harmonic. Finding a harmonic conjugate is trivial. \square

Example 2.13 (Dinh's 70 problems). Find all harmonic functions $u(x, y)$ in \mathbb{C} such that

$$(x^2 - y^2) u(x, y) \quad \text{is harmonic in } \mathbb{C}.$$

Solution. Let $f(x, y) = (x^2 - y^2)u(x, y)$. Then,

$$f_{xx} = (x^2 - y^2)u_{xx} + 4xu_x + 2u \quad \text{and} \quad f_{yy} = (x^2 - y^2)u_{yy} - 4yu_y - 2u.$$

As such,

$$f_{xx} + f_{yy} = 4(xu_x - yu_y),$$

where we used the fact that u is harmonic (i.e. $u_{xx} + u_{yy} = 0$). For f to be harmonic, $xu_x = yu_y$. One can use techniques taught to solve partial differential equations to deduce that $u(x, y) = g(xy)$, where $g : \mathbb{R} \rightarrow \mathbb{R}$. Therefore, $g''(xy) = 0$, so $g(t) = at + b$, where $a, b \in \mathbb{R}$. Hence, $u(x, y) = axy + b$. \square

3. Line Integrals

3.1. Fundamental Results

Theorem 3.1 (fundamental theorem of line integrals). Suppose C is a smooth curve given by $z(t) : a \leq t \leq b$ and $F'(z) = f(z)$. Then,

$$\int_C f(z) dz = F(z(b)) - F(z(a)).$$

Lemma 3.1 (triangle inequality). Suppose f is a continuous complex-valued function of t . Then,

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt.$$

Lemma 3.2 (ML inequality/estimation lemma). If C is a smooth curve of length L , f is continuous on C and $|f| \leq M$ on C , then

$$\left| \int_C f(z) dz \right| \leq ML \quad \text{where} \quad M = \sup_{z \in C} |f(z)| \quad \text{alternatively.}$$

Example 3.1. Let γ be the contour given by $\gamma(t) = 3e^{it}$, where $0 \leq t \leq \pi$. Prove that

$$\left| \int_{\gamma} \frac{\overline{z} e^{iz}}{z^2 - 11z + 30} dz \right| \leq 5.$$

Solution. Obviously, $L = 3\pi$ since $\gamma(t) = 3e^{it}$, where $0 \leq t \leq \pi$ is the equation of the upper half of a circle of radius 3 centred at the origin, so its arc length is 3π . Now, we need to justify that $M \leq 5/3\pi$. Let $z = x + iy$.

We have

$$\left| \frac{\overline{z} e^{iz}}{z^2 - 11z + 30} \right| = \left| \frac{\bar{z} \cdot \overline{e^{iz}}}{(z-5)(z-6)} \right| = \frac{|\bar{z}| e^{-y}}{|z-5||z-6|} = \frac{|z| e^{-y}}{|z-5||z-6|}.$$

Since $|z| \leq 3$ and applying the triangle inequality, we see that

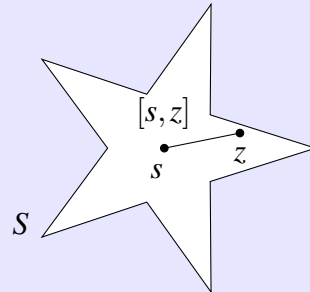
$$\frac{|z| e^{-y}}{|z-5||z-6|} \leq \frac{3 \cdot 1}{||z|-5||z|-6|} \leq \frac{3}{|3-5||3-6|} = \frac{1}{2}$$

so $M = 1/2$. It is clear that $1/2 < 5/3\pi$ so we conclude that $M \leq 5/3\pi$. □

4. The Cauchy-Goursat Theorem

4.1. Some Results in Topology

Definition 4.1 (star-shaped set). A set S is star-shaped if it has a point s , known as the star centre, so that for each $z \in S$, the segment $[s, z]$ lies in S .



Remark 4.1. Note that a star domain is not necessarily convex.

Example 4.1. A cross-shaped figure is a star domain but is not convex.

Theorem 4.1. Let S be an open star-shaped region and f continuous on S . Let T be a closed triangular region and ∂T be the boundary of the triangle traversed in the anticlockwise direction. Suppose

$$\int_{\partial T} f(z) dz = 0$$

for every T in S , then f has an anti-derivative, F , in S .

Definition 4.2 (open ball). Define $B_r(w)$ or $B(w, r)$ to be

the open ball of radius r and centre w .

Throughout this set of notes, we will be using these two representations interchangeably.

Definition 4.3 (boundary point). A point $w \in \mathbb{C}$ is a boundary point of S if

for every $r \in \mathbb{R}^+$ we have $B_r(w) \cap S \neq \emptyset$.

Definition 4.4 (closure). Denote the set of boundary points by ∂S . Given a set S , the closure of S , denoted by \bar{S} , is defined by

$$\bar{S} = S \cup \partial S.$$

Theorem 4.2. A set G is closed if and only if $G = \bar{G}$.

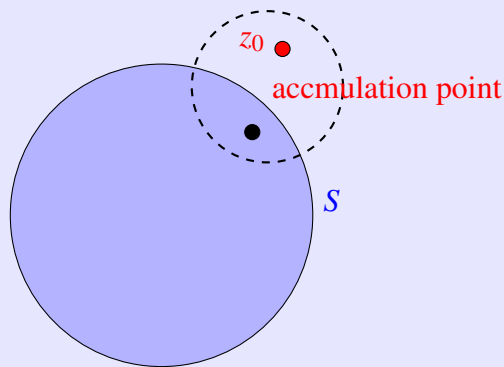
Proof. For the forward direction, suppose G is closed. We wish to prove that $G = G \cup \partial G$, or equivalently, $\partial G \subseteq G$. Suppose on the contrary that $\partial G \not\subseteq G$. Then, there exists $w \in \partial G \setminus G$. For every $\varepsilon > 0$, we have

$$B_\varepsilon(w) \cap G \neq \emptyset \text{ and } B_\varepsilon(w) \cap G' \neq \emptyset \quad \text{which implies} \quad B_\varepsilon(w) \cap G \neq \emptyset.$$

However, $w \notin G$, so $w \in G'$. As G is closed, then G' is open, so there exists $\varepsilon' > 0$ such that $B(w, \varepsilon') \subseteq G'$. Hence, $B(w, \varepsilon') \cap G = \emptyset$ and this is a contradiction, so $\partial G \subseteq G$.

We then prove the reverse direction. Suppose $G = G \cup \partial G$. We wish to prove that G' is open. Let $x \in G'$. As $\partial G \subseteq G$, then $G' \cap \partial G = \emptyset$. There exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap G = \emptyset$ or $B(x, \varepsilon) \cap G = \emptyset$. As $x \in G'$, then $B(x, \varepsilon) \cap G' \neq \emptyset$. Therefore, $B(x, \varepsilon) \cap G = \emptyset$ or $B(x, \varepsilon) \subseteq G'$, which is the definition of G' being open. \square

Definition 4.5 (accumulation point). A point z_0 is an accumulation point of a set S if each neighbourhood of z_0 contains at least one point of S distinct from z_0 .



Remark 4.2. The accumulation point of a set S does not have to be an element of that set.

Example 4.2. Prove that a set S is closed if and only if S contains all its accumulation points.

Solution. For the forward direction, we proceed with contradiction. Let y be an accumulation of S which is not in S . Then, $y \in S'$. As S' is an open set, there exists $\delta > 0$ such that $B_\delta(y) \subseteq S'$. As such, $B_\delta(y) \cap S = \emptyset$, contradicting the assumption that y is an accumulation point for S .

For the reverse direction, suppose S contains all its accumulation points. We need to show that S is closed. It suffices to show that S' is open. Let $x \in S'$. Then, x is not an accumulation of S since S already contains all its accumulation points. So, there exists $\delta > 0$ such that

$$B_\delta(x) \setminus (\{x\} \cap S) = B_\delta(x) \cap S = \emptyset.$$

We conclude that $B_\delta(x) \subseteq S'$, so S' is open. \square

4.2. Cauchy-Goursat Theorem

Theorem 4.3 (Cauchy-Goursat theorem/Cauchy integral theorem). Suppose f is analytic on a star-shaped region S . Then, for every simple closed path C in S traversed in the anticlockwise direction,

$$\int_C f(z) dz = 0.$$

Example 4.3. Let $f(z) = \text{Log}(z + 2)$ and the contour γ be the circle $|z| = 1$ oriented in the anticlockwise direction. Use the Cauchy-Goursat theorem to prove that

$$\int_{\gamma} f(z) dz = 0.$$

Solution. Recall that $\text{Log} z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$. Thus, $f(z) = \text{Log}(z + 2)$ is analytic on $\mathbb{C} \setminus (-\infty, -2]$. However, $(-\infty, -2]$ lies outside the circle $|z| = 1$. Thus, $f(z)$ is analytic inside and on the circle $|z| = 1$, which is a simple closed contour. The result follows by the Cauchy-Goursat theorem. \square

Theorem 4.4. Let f be continuous on a star-shaped region S , and analytic on $S \setminus \{z_0\}$, i.e. the set S but excluding the point z_0 . Then, f has an anti-derivative on S , and consequently,

$$\int_C f(z) dz = 0 \quad \text{for every simple closed curve } C \in S \text{ traversed anticlockwise.}$$

4.3. Cauchy's Integral Formula

Theorem 4.5 (Cauchy's integral formula). Let f be analytic everywhere within and on a simple closed contour C traversed in the anticlockwise direction. If a is interior to C , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz.$$

Proof. Define

$$g(z) = \frac{f(z) - f(a)}{z - a} \quad \text{which is analytic everywhere except at } z = a.$$

Since the derivative of f exists at a , then by the first principles of differentiation,

$$\lim_{z \rightarrow a} g(z) = f'(a).$$

Using the Cauchy-Goursat formula applied to a star-shaped region excluding a (since g is analytic everywhere except a), then

$$\int_C g(z) dz = 0 \quad \text{which implies} \quad \int_C \frac{f(z) - f(a)}{z - a} dz = 0.$$

So,

$$\int_C \frac{f(z)}{z - a} dz = f(a) \int_C \frac{1}{z - a} dz = f(a) \cdot 2\pi i,$$

where the last equality follows since we are taking the contour integral on a loop around a . \square

Example 4.4. Let $z_0 \in \mathbb{C}$ and γ be a simple closed contour enclosing z_0 with positive orientation. Without using Cauchy's integral formula, and using only the fact that

$$\int_{\gamma} \frac{1}{z - z_0} dz = 2\pi i,$$

show that

$$\text{if } p(z) = z_0 + z_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \text{ is a polynomial then } \int_{\gamma} \frac{p(z)}{z - z_0} dz = p(z_0) \cdot 2\pi i.$$

Solution. By the division algorithm for polynomials, there exist polynomials $f(z)$ and r such that $p(z) = (z - z_0)f(z) + r$. So, $p(z_0) = r$.

Hence, $p(z) = (z - z_0)f(z) + p(z_0)$ and we have

$$\int_{\gamma} \frac{p(z)}{z - z_0} dz = \int_{\gamma} f(z) + \frac{p(z_0)}{z - z_0} dz = \int_{\gamma} f(z) dz + p(z_0) \int_{\gamma} \frac{1}{z - z_0} dz = p(z_0) \cdot 2\pi i.$$

Note that the integral

$$\int_{\gamma} f(z) dz = 0$$

by the Cauchy-Goursat theorem. □

Example 4.5. Let C be the circle $|z| = 2$ oriented in the anticlockwise direction. Evaluate

$$\int_C \frac{1}{|z - i|^2} dz.$$

Solution. We use the identity $|z|^2 = z\bar{z}$, so $|z - i|^2 = (z - i)(\bar{z} + i)$. Since $|z| = 2$, then $\bar{z} = 4/z$, so

$$(z - i)(\bar{z} + i) = z\bar{z} + i(z - \bar{z}) + 1 = 5 + i\left(z - \frac{4i}{z}\right) = \frac{iz^2 + 5z + 4}{z} = \frac{(iz + 1)(z - 4i)}{z}.$$

Hence, the contour integral is equivalent to

$$\int_C \frac{z}{(iz + 1)(z - 4i)} dz = \int_C \frac{f(z)}{z - i} dz \quad \text{where } f(z) = -\frac{iz}{z - 4i}.$$

By Cauchy's integral formula, the integral is equivalent to $2\pi i f(i) = -2\pi/3$. □

Corollary 4.1 (Cauchy's differentiation formula). If f is analytic at a point a , then $f^{(n)}(a)$ exists for $n = 1, 2, \dots$ and are also analytic z_0 , and

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - a)^{n+1}} dz,$$

where C is a simple closed curve traversed in the anticlockwise direction that encloses a .

Proof. Apply induction on Cauchy's integral formula. □

Theorem 4.6 (Liouville's theorem). If f is a bounded and entire function, then f is a constant.

Proof. Since f is entire, we can represent it using a Taylor series about $z = 0$, so

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By Cauchy's integral formula,

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz,$$

where C is a circle of radius r centred at the origin. Since f is bounded, then $|f(z)| \leq M$ for some constant M and for all $z \in \mathbb{C}$. We have

$$|a_n| = \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_C \left| \frac{f(z)}{z^{n+1}} \right| |dz| \leq \frac{1}{2\pi} \int_C \frac{M}{|z|^{n+1}} |dz| \leq \frac{M}{2\pi r^{n+1}} \int_C |dz| = \frac{M}{2\pi r^{n+1}} \cdot 2\pi r = \frac{M}{r^n}$$

Now, as $|z| = r$ on the circle C , by setting $r > 0$ to be arbitrary, as r tends to infinity, $a_n = 0$ for all $n \geq 1$. This is because f is entire. Hence, $f(z) = a_0 = M/r$ which is a constant. \square

Example 4.6. Find all entire functions $f(z)$ with $f(0) = 2$ and $|f(z) - e^z| \geq 1$ for all $z \in \mathbb{C}$.

Solution. We note that

$$\frac{1}{|f(z) - e^z|} \leq 1 \quad \text{where} \quad f(z) - e^z \neq 0.$$

So, $1/(f(z) - e^z)$ is bounded and entire. By Liouville's theorem,

$$\frac{1}{f(z) - e^z} = c,$$

where c is a constant. Since $f(0) = 2$, then $c = 1$. As such, $f(z) = e^z + 1$. \square

Example 4.7. Let g be an entire function such that $|g'(z)| < |g'(z) + i|$ for all complex numbers z . Show that there exist $\alpha, \beta \in \mathbb{C}$ such that $g(z) = \alpha z + \beta$ for all $z \in \mathbb{C}$.

Solution. Since g is entire, then g' is also entire. Let

$$h(z) = \frac{g'(z)}{g'(z) + i}.$$

Then h is the quotient of two entire functions such that the denominator is not equal to zero at each $z \in \mathbb{C}$, hence h is entire. It is clear that for all $z \in \mathbb{C}$, $|h(z)| < 1$, so h is bounded on \mathbb{C} . By Liouville's theorem, $h(z) = c$, where c is a constant, so $g'(z) = c g'(z) + ci$. We have

$$g'(z) = \frac{ic}{1-c} = \alpha.$$

Hence, $g(z) = \alpha z + \beta$. \square

Example 4.8. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function such that

$$\lim_{z \rightarrow \infty} f(z) = \infty.$$

Show that f has at least one zero in \mathbb{C} .

Solution. Suppose on the contrary f has no zeros in \mathbb{C} . Consider

$$g(z) = \frac{1}{f(z)}.$$

Note that g is entire. Using the given limit, there exists $R > 0$ such that for all $|z| > R$, $|f(z)| > 1$. This implies that $|g(z)| < 1$ but since g is continuous, it obtains a maximum M on the compact set $\overline{D(0, R)}$. Hence, for all $z \in \mathbb{C}$, $|g(z)| \leq \max\{1, M\}$, so by Liouville's theorem, g is a constant, implying that f is a constant, which is a contradiction. \square

Example 4.9. Find all entire functions $f(z)$ such that

$$|f(z)| \leq \frac{1}{1+x^2+2y^2} \quad \text{for all } z = x+iy \in \mathbb{C}.$$

Solution. Since $x^2, y^2 \geq 0$, then $|f(z)| \leq 1$. By Liouville's theorem, f is a constant, say c . Then,

$$c \leq \frac{1}{1+|z|^2+y^2}.$$

It is clear that

$$\lim_{z \rightarrow \infty} f(z) = 0$$

so $c = 0$. Hence, the only function satisfying the hypothesis is $f(z) = 0$. \square

Example 4.10 (MA5217 AY24/25 Sem 1 Homework 1). Find all entire functions f satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z \in \mathbb{C}$.

Solution. By an inductive argument, for all $n \in \mathbb{Z}$, we have

$$f(z+n) = f(z) \quad \text{and} \quad f(z+ni) = f(z).$$

Hence, it suffices to consider the behaviour of f on the unit square $[0, 1] \times [0, 1]$. Since the unit square is a compact set, it is bounded by the Heine-Borel theorem. Hence, $f(x+iy)$ is bounded for all $x, y \in \mathbb{R}$. Since f is a bounded function, it is constant (follows by Liouville's theorem where we assumed that f is entire). So, $f(z) = c$ for some $c \in \mathbb{R}$. \square

Example 4.11 (Dinh's 70 problems). Let $f = u+iv$ be an entire function. Show that if $u^2(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, then f must be a constant.

Solution. We have $f^2 = u^2 - v^2 + 2uvi$. Consider

$$g = e^{-f^2} = e^{v^2-u^2} e^{-2uvi} \quad \text{which is entire} \quad \text{and} \quad |g| \leq \frac{1}{e}.$$

By Liouville's theorem, g is a constant. So, $e^{-f^2} = k$ for some constant k . Thus, f is a constant. \square

Theorem 4.7 (fundamental theorem of algebra). Every non-constant polynomial with complex coefficients has a zero in \mathbb{C} .

4.4. Applications of Cauchy's Integral Formula

Theorem 4.8 (Morera's theorem). Let f be a continuous function on D . Let T be a closed triangle in D and ∂T be the boundary of T traversed in the anticlockwise direction. Then,

$$\int_{\partial T} f(z) dz = 0.$$

Theorem 4.9. Let f be an entire function. Define $g(z) = f'(a)$ if $z = a$ and

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

if $z \neq a$. Then, g is also entire.

Theorem 4.10 (extended Liouville's theorem). If f is entire and if for some $k \in \mathbb{N}$, there exists constants $A, B > 0$ such that

$$|f(z)| \leq A + B|z|^k,$$

then f is a polynomial of degree at most k .

Example 4.12 (Dinh's 70 problems). Let u be a real-valued harmonic function in the complex plane such that

$$u(z) \leq a|\ln |z|| + b$$

for all z , where a and b are positive constants. Prove that u is constant.

Solution. By Liouville's theorem, since u is harmonic, it suffices to show that u is bounded. Let $f(z) = a|\ln |z|| + b$. Then, by Cauchy's integral formula,

$$|u'(k)| = \left| \frac{1}{2\pi i} \int_{\gamma: |z|=R} \frac{f(z)}{(z-k)^2} dz \right| \leq R \cdot \frac{a|\ln R| + b}{|R - |k||^2},$$

where we have considered γ to be the circle of radius R centred at the origin and naturally, the path is taken to be positively-oriented. To establish the upper bound for $|u'(k)|$, the triangle inequality and reverse triangle inequality are used. Now, note that

$$\lim_{R \rightarrow \infty} R \cdot \frac{a|\ln R| + b}{|R - |k||^2} = 0$$

which implies that $|u'(k)| = 0$, or rather, $u'(k) = 0$. So, $u(k)$ is a constant for all $k \in \mathbb{R}$. □

Theorem 4.11 (Gauss' mean value theorem). If f is analytic in D and $\alpha \in D$, then

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta.$$

Proof. By Cauchy's integral formula, for $a \in D$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

Let C be a circle of radius r centred at a . Then, our parameterisation is $z = a + re^{i\theta}$, so $dz/d\theta = ire^{i\theta}$. Hence,

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{a + re^{i\theta} - a} \cdot ire^{i\theta} d\theta$$

and the result follows with some simple cancellation. \square

Theorem 4.12 (maximum modulus theorem for open balls). Suppose $f(z)$ is analytic throughout a neighbourhood $|z - z_0| < R$ of a point z_0 . If $|f(z)| \leq |f(z_0)|$ for each z in the neighbourhood, then $f(z)$ attains a constant value $f(z_0)$ throughout the neighbourhood.

Theorem 4.13 (maximum modulus principle). If f is analytic in D and

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in D \text{ then } f(z) \text{ is a constant.}$$

Example 4.13 (Dinh's 70 problems). Let $f(z) = a_0 + a_1z + \dots + a_nz^n$ be a complex polynomial of degree $n > 0$. Prove that

$$\frac{1}{2\pi i} \int_{|z|=R} z^{n-1} |f(z)|^2 dz = a_0 \bar{a}_n R^{2n}.$$

Solution. Note that $|f(z)|^2 = f(z) \cdot \overline{f(z)}$. Setting $z = Re^{i\theta}$, the integral becomes

$$\frac{1}{2\pi} \int_0^{2\pi} R^n e^{in\theta} (a_0 + a_1 Re^{i\theta} + \dots + a_n R^n e^{in\theta}) (\bar{a}_0 + \bar{a}_1 R e^{-i\theta} + \dots + \bar{a}_n R^n e^{-in\theta}) d\theta.$$

Since

$$\int_0^{2\pi} e^{ik\theta} d\theta = 0 \text{ for all } k \neq 0,$$

upon multiplying the polynomials $a_0 + a_1 Re^{i\theta} + \dots + a_n R^n e^{in\theta}$ and $\bar{a}_0 + \bar{a}_1 R e^{-i\theta} + \dots + \bar{a}_n R^n e^{-in\theta}$, we wish to extract the coefficient of $e^{-in\theta}$. So, the integral becomes

$$\frac{1}{2\pi i} \int_0^{2\pi} R^n e^{in\theta} a_0 \bar{a}_n R^n e^{-in\theta} d\theta$$

and the result follows. \square

Example 4.14 (Dinh's 70 problems). Suppose $u(z)$ is harmonic on $D(0, r)$, where $r > 1$. Prove that

$$\int_0^{2\pi} u(e^{it}) \cos^2\left(\frac{t}{2}\right) dt = \pi u(0) + \frac{\pi}{2} u'(0) \quad \text{and} \quad \int_0^{2\pi} u(e^{it}) \sin^2\left(\frac{t}{2}\right) dt = \pi u(0) - \frac{\pi}{2} u'(0),$$

where $u'(0) = u_x(0)$.

Solution. Let I_1 and I_2 denote the two integrals respectively. We have

$$I_1 + I_2 = \int_0^{2\pi} u(e^{it}) dt \text{ and } I_1 - I_2 = \int_0^{2\pi} u(e^{it}) \cos t dt.$$

We parametrise each integral using $z = e^{it}$ so $dz/dt = ie^{it}$. Also, recall that $\cos t = (z + z^{-1})/2$. So,

$$I_1 + I_2 = \frac{1}{i} \int_{|z|=1} \frac{u(z)}{z} dz = \pi u(0),$$

where we used Cauchy's integral formula. Also,

$$I_1 - I_2 = \frac{1}{2i} \int_{|z|=1} u(z) + \frac{u(z)}{z^2} dz = \frac{1}{2i} \int_{|z|=1} \frac{u(z)}{z^2} dz = \pi u'(0),$$

where we used Cauchy's integral formula and the fact that $u(z)$ is analytic on $D(0, r)$ (since $u(z)$ is harmonic on $D(0, r)$). \square

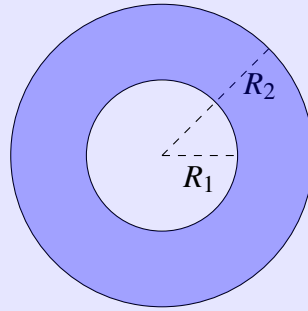
5. Series

5.1. Laurent Series

Definition 5.1 (annulus). Define

$$\text{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$$

to be the shaded region as follows:



Theorem 5.1 (Laurent expansion). If f is analytic in the annulus

$$\text{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\},$$

then it has a Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz.$$

Here, C is a circle of radius R centred at the origin with $R_1 < R < R_2$.

Example 5.1.

(a) Consider the function

$$f(z) = \frac{5z - 3}{(z + 1)(z - 3)}.$$

Find the Laurent series of $f(z)$ for the annular domain $1 < |z| < 3$.

(b) Find the value of the contour integral

$$\int_C \frac{5z - 3}{z^5(z + 1)(z - 3)} dz,$$

where C denotes the circle $|z| = 2$ oriented in the anticlockwise direction.

(c) Find the Laurent series of the function

$$\frac{10z^6 - 6z^4}{(z^2 + 1)(z^2 - 3)}$$

in the annular domain $1 < |z| < \sqrt{3}$.

Solution.

(a) We see that

$$\begin{aligned}\frac{5z-3}{(z+1)(z-3)} &= \frac{2}{z+1} + \frac{3}{z-3} \\ &= \frac{2}{z} \cdot \frac{1}{1+1/z} - \frac{1}{1-z/3} \\ &= \frac{2}{z} \sum_{n=0}^{\infty} (-1)^n (-z)^n - \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n\end{aligned}$$

We note that the first summation is valid for $|1/z| < 1$ while the second summation is valid for $|z/3| < 1$.

(b) We see that the contour integral is equivalent to

$$\int_C \frac{f(z)}{z^5} dz = 2\pi i \left(-\frac{1}{3^4}\right) = -\frac{2\pi i}{81}.$$

(c) Let us make a comparison. Perhaps we can consider $f(z^2)$. Note that

$$f(z^2) = \frac{5z^2-3}{(z^2+1)(z-3)}.$$

Hence, it is clear that the function in (c) is $2z^4 f(z^2)$. Recall that the Laurent series of f in the annulus $1 < |z| < 3$ is

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n$$

so the required answer is

$$4z^4 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} - 2z^4 \sum_{n=0}^{\infty} \left(\frac{z^2}{3}\right)^n = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n-2}} - 2 \sum_{n=0}^{\infty} \frac{z^{2n+4}}{3^n}$$

in the annular domain $1 < |z| < \sqrt{3}$. □

Example 5.2. Suppose $f(z)$ is entire and $|f(z)| > 1$ when $|z| > 1$. Prove that $f(z)$ is a polynomial.

Solution. Since f is entire, then in the closed unit disk, it has a finite number of zeros. Say the zeros are z_1, \dots, z_m . So, we can write

$$f(z) = (z - z_1) \dots (z - z_m) g(z) = p(z) g(z),$$

where g is entire with no zeros and $p(z)$ is a polynomial of degree m . It suffices to show that g is a constant. Let $h(z) = 1/g(z)$ so we shall write h as the following Laurent series:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{h(z)}{z^{n+1}} dz$$

Here, we let γ be $|z| = R$, i.e. the circle of radius R centred at the origin. Letting $z = Re^{i\theta}$, the contour integral becomes

$$\int_0^{2\pi} \frac{iRe^{i\theta} h(Re^{i\theta})}{R^{n+1} e^{i(n+1)\theta}} d\theta.$$

Let $h(Re^{i\theta}) \leq kR^m$ so it is clear that for all $n > m$,

$$\lim_{R \rightarrow \infty} |a_n| \leq \lim_{R \rightarrow \infty} \frac{kR^m}{R^n} = 0.$$

As such, $h(z)$ is a constant, and $g(z)$ is a constant. □

6. Residue Theory

6.1. Introduction

We adopt an alternative representation for the annulus $\text{Ann}(z_0, R_1, R_2)$, so if $f(z)$ is analytic in this annulus,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{-n+1}} ds$$

and C is any positively oriented simple closed contour around z_0 lying inside $\text{Ann}(z_0, R_1, R_2)$.

Definition 6.1 (principal part of Laurent series). The sum

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{is the principal part of } f(z) \text{ at } z_0.$$

Theorem 6.1 (removable singularity). If $b_n = 0$ for all $n \in \mathbb{N}$, then z_0 is a point of removable singularity of $f(z)$. Thus, the Laurent series of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where } 0 < |z - z_0| < R.$$

Example 6.1. The singular point $z = 0$ of $\sin z/z$ is a removable singularity. We have

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

where $0 < |z| < \infty$. This asserts that our claim is true.

Example 6.2 (Dinh's 70 problems). Let $f(z)$ be holomorphic in $\mathbb{C} \setminus \{0\}$ and suppose that

$$\int_{|z|=1} z^n f(z) dz = 0 \quad \text{for any } n \in \mathbb{Z}_{\geq 0}.$$

Show that f has a removable singularity at $z = 0$.

Solution. f has a Laurent series representation around $z = 0$. Write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$$

so the integral becomes

$$\begin{aligned} \int_{|z|=1} z^n \sum_{k=-\infty}^{\infty} a_k z^k dz &= 0 \\ \sum_{k=-\infty}^{\infty} \int_{|z|=1} a_k z^{n+k} dz &= 0 \end{aligned}$$

since that the series converges uniformly on compact sets away from the singularity. Note that

$$\int_{|z|=1} z^k dz = 0 \quad \text{for all } k \neq -1.$$

As such, $n + k = -1$. Since $n \geq 0$, it forces the inequality $k \leq -1$, which implies that $a_k = 0$ for all $k \leq -1$, i.e.

$$\sum_{k=-1}^{\infty} \int_{|z|=1} a_k z^{n+k} dz = 0.$$

It is clear that $a_{-1} = 0$. With all coefficients of negative powers being zero, it shows that $f(z)$ has a removable singularity at $z = 0$. \square

Definition 6.2 (essential singularity). If $b_n \neq 0$ for infinitely many n , then z_0 is a point of essential singularity of $f(z)$. In this case, some of the b_n 's may be zero.

Example 6.3. The point $z = 0$ of $\exp(1/z)$ is an essential singularity as

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \dots$$

where $0 < |z| < \infty$.

Definition 6.3 (pole). If there exists $m \in \mathbb{N}$ such that $b_m \neq 0$ but $b_n = 0$ for all $n > m$ so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^m \frac{b_n}{(z - z_0)^n},$$

then z_0 is a pole of order m of $f(z)$. If $m = 1$, z_0 is a simple pole of $f(z)$; if $m = 2$, z_0 is a double pole of $f(z)$.

Example 6.4. Consider the point $z = 1$ of

$$f(z) = \frac{1}{(z-1)^2} + z.$$

We can rewrite it as

$$f(z) = \frac{1}{(z-1)^2} + 1 + (z-1)$$

Hence, $z = 1$ is a double pole.

Example 6.5 (MA5217 AY24/25 Sem 1 Homework 1). Find all the singularities in \mathbb{C} of the following function $f(z)$ and their types where

$$f(z) = \frac{z^2 + 3z + 2}{z(z^4 - 1)} e^{1/z^2}.$$

Solution. Consider the term $z^4 - 1$ in the denominator of $f(z)$. Then, $z^4 - 1 = (z^2 + 1)(z^2 - 1) = (z^2 + 1)(z + 1)(z - 1)$. Also, the numerator can be factorised as $(z + 2)(z + 1)$. Also, consider

$$\frac{e^{1/z^2}}{z} = \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n \frac{1}{n!} \cdot \frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1} n!}.$$

So, $f(z)$ has simple poles at $z = 1, z = i, z = -i$, a removable singularity at $z = -1$, and an essential singularity at $z = 0$. \square

Theorem 6.2 (residue theorem). Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points z_1, z_2, \dots, z_n interior to C . Let $\text{Res}(f, a_k)$ denote the residue of f at a_k , for all $1 \leq k \leq n$. Then,

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, a_k).$$

Theorem 6.3. If f is analytic everywhere on the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C , then

$$\int_C f(z) dz = 2\pi i \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Example 6.6 (Dinh's 70 problems). Evaluate the integral

$$\int_{C^+(0,2)} e^{e^{1/z}} dz.$$

Solution. Let $w = 1/z$ so $dw/dz = -w^2$. The integral becomes

$$\int_{C^+(0,1/2)} e^{e^w} \cdot \frac{dw}{w^2}.$$

Let $f(w) = e^{e^w}$. By the residue theorem,

$$\int_{C^+(0,1/2)} \frac{f(w)}{w^2} dw = 2\pi i \text{Res}(f(w), 0) = 2\pi i e$$

and we are done. \square

6.2. Residue Computation Methods

There are three methods for computing residues.

Theorem 6.4 (method 1). Suppose for z near z_0 , $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{z - z_0},$$

where $\phi(z)$ is analytic at z_0 and f has a simple pole or a removable singularity at z_0 . Then,

$$\text{Res}_{z=z_0} f(z) = \phi(z_0).$$

Proof. Since $\phi(z)$ is analytic at z_0 , then by Taylor's theorem, for z near z_0 ,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \dots$$

so the Laurent series of $f(z)$ at z_0 is

$$f(z) = \frac{\phi(z)}{z - z_0} = \frac{\phi(z_0) + \phi'(z_0)(z - z_0) + \dots}{z - z_0} = \frac{\phi(z_0)}{z - z_0} + \phi'(z_0) + \dots$$

and the result follows. \square

Theorem 6.5 (method 2). Suppose for z near z_0 , $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $m \geq 1$. Then,

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

Proof. It is inferred that f has a pole of order less than or equal to m or a removable point of singularity at z_0 . Observe that when $m = 1$, it is just method 1 (recall Theorem 6.4). Using Taylor's theorem again, the series expansion of $\phi(z)$ is the same as before. That is,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \dots$$

so

$$\begin{aligned} f(z) &= \frac{\phi(z)}{(z - z_0)^m} \\ &= \frac{1}{(z - z_0)^m} \left[\phi(z_0) + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} (z - z_0)^{m-1} + \dots \right] \\ &= \frac{\phi(z_0)}{(z - z_0)^m} + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \cdot \frac{1}{z - z_0} + \dots \end{aligned}$$

The result follows. \square

Theorem 6.6 (method 3). If $p(z)$ and $q(z)$ are analytic at z_0 and $q(z)$ has a simple zero at z_0 (i.e. $q(z_0) = 0$ but $q'(z_0) \neq 0$), then

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Theorem 6.7 (method 4). If all the above methods fail, use the Laurent series of $f(z)$ and read b_1 .

Example 6.7. For the following function $f(z)$, find all of its singularities in \mathbb{C} , their types and residues at these points:

$$f(z) = \frac{z^2 + 1}{z^6 + 1}.$$

Solution. The singularities of $f(z)$ are the zeros of the denominator $z^6 + 1$, that is the 6 points

$$z_k = \exp\left(\frac{i\pi}{6} + \frac{k\pi i}{3}\right),$$

where $0 \leq k \leq 5$. These points are simple zeros of $z^6 + 1 = 0$. The points $z_1 = i$ and $z_4 = -i$ are the roots of the equation $z^2 + 1 = 0$ (refer to the numerator). Thus, z_1, z_4 are removable and z_0, z_2, z_3, z_5 are simple poles of f .

So, the residues of f at z_1, z_4 are 0, whereas the residue of f at z_k for $k = 0, 2, 3, 5$ is equal to $(z_k^2 + 1)/6z_k^5$. \square

Example 6.8 (classic result). Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx = \frac{\pi}{e}.$$

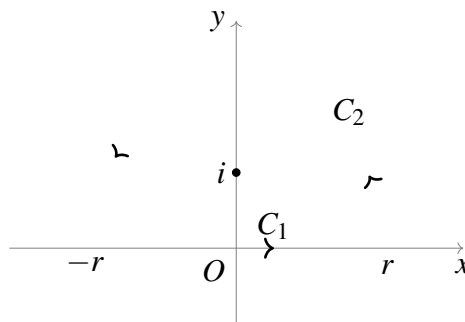
Solution. We consider

$$f(z) = \frac{e^{iz}}{z^2 + 1}$$

so the integral $\text{Re}(f(z))$ over the real numbers is the required answer. Let C be the path $C_1 + C_2$, where C_1 and C_2 are parametrised as follows:

$$C_1(t) = t, \text{ where } t \in [-r, r]$$

$$C_2(t) = re^{it}, \text{ where } t \in [0, \pi]$$



By Cauchy's residue theorem,

$$\sum \text{Res}(f(z)) = \frac{1}{2\pi i} \int_C f(z) dz.$$

Only one of the two poles of $f(z)$, $z = i$, is inside C as we are considering the upper half of the circle centred at the origin. We have

$$\int_C f(z) dz = \int_{C_1} \frac{e^{iz}}{z^2 + 1} dz + \int_{C_2} \frac{e^{iz}}{z^2 + 1} dz.$$

For the integral over C_1 , applying the parametrisation,

$$\int_{C_1} \frac{e^{iz}}{z^2 + 1} dz = \int_{-r}^r \frac{e^{it}}{t^2 + 1} dt = \int_{-r}^r \frac{\cos t}{t^2 + 1} dt + i \int_{-r}^r \frac{\sin t}{t^2 + 1} dt.$$

Since $\sin t$ is an odd function, then the integral of $\sin t / (t^2 + 1)$ is zero. Hence,

$$\int_{C_1} \frac{e^{iz}}{z^2 + 1} dz = \int_{-r}^r \frac{\cos t}{t^2 + 1} dt.$$

As for the integral over C_2 , applying the parametrisation,

$$\int_{C_2} \frac{e^{iz}}{z^2 + 1} dz = \int_0^\pi \frac{\exp(ire^{it})}{r^2 e^{i2t} + 1} \cdot ire^{it} dt.$$

By applying Euler's Formula,

$$\begin{aligned} \int_0^\pi \frac{\exp(ire^{it})}{r^2 e^{i2t} + 1} \cdot ire^{it} dt &= ir \int_0^\pi \frac{e^{i(t+r\cos t)} e^{-r\sin t}}{r^2 e^{i2t} + 1} dt \\ \left| \int_0^\pi \frac{\exp(ire^{it})}{r^2 e^{i2t} + 1} \cdot ire^{it} dt \right| &= r \int_0^\pi \frac{e^{-r\sin t}}{|r^2 e^{i2t} + 1|} dt \\ &\leq \frac{r}{r^2 - 1} \int_0^\pi e^{-r\sin t} dt \end{aligned}$$

Let the radius r of the semicircle tend to infinity so it is then clear that

$$\int_{C_2} \frac{e^{iz}}{z^2 + 1} dz = 0.$$

Therefore, by Cauchy's residue theorem (rearrange the equation at the start of our solution),

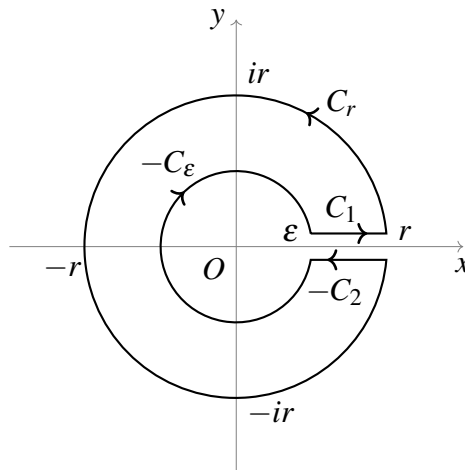
$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + 1} dt = 2\pi i \cdot \frac{e^{i^2}}{2i} = \frac{\pi}{e}.$$

□

Example 6.9 (branch cut). Prove that

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 5x + 6} dx = \pi(\sqrt{3} - \sqrt{2}).$$

Solution. Note that 0 is a branch point of \sqrt{z} . So, \sqrt{z} has a branch cut along the positive real axis, i.e. $[0, \infty)$. Hence, \sqrt{z} is analytic on $\mathbb{C} \setminus [0, \infty)$. We adopt the following keyhole contour.



One should think of the above contour as having ε so small that C_1 and C_2 are essentially on the x -, or rather, real axis. Let the region the contour encloses be D . Then, we shall consider the integral over the boundary (this is denoted by ∂D). That is,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2 + 5z + 6} dz.$$

By Cauchy's residue theorem,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2 + 5z + 6} dz = 2\pi i \left[\frac{\sqrt{z}}{2z+5} \Big|_{z=-3} + \frac{\sqrt{z}}{2z+5} \Big|_{z=-2} \right] = 2\pi (\sqrt{3} - \sqrt{2}).$$

We now evaluate the contour integral by considering the different *pieces*.

$$\begin{aligned} \int_{\partial D} \frac{\sqrt{z}}{z^2 + 5z + 6} dz &= \int_{C_r} - \int_{C_\varepsilon} + \int_{C_1} - \int_{C_2} \\ &= \int_{C_r} - \int_{C_\varepsilon} + 2 \int_\varepsilon^r \frac{\sqrt{x}}{x^2 + 5x + 6} dx \end{aligned}$$

By the estimation lemma,

$$\left| \int_{C_r} f(z) dz \right| \leq 2\pi r \cdot \frac{\sqrt{r}}{r^2 - 5r - 6}$$

which tends to 0 as r tends to infinity. In a similar fashion, one can prove that

$$\left| \int_{C_\varepsilon} f(z) dz \right| \leq 2\pi \varepsilon \cdot \frac{\sqrt{\varepsilon}}{6 - 5\varepsilon - \varepsilon^2}$$

which tends to 0 too as ε tends to 0. As such,

$$2\pi (\sqrt{3} - \sqrt{2}) = 2 \int_0^\infty \frac{\sqrt{x}}{x^2 + 5x + 6} dx$$

and the result follows. □

Example 6.10 (pizza contour). Prove that for $n \geq 2$,

$$\int_0^\infty \frac{1}{x^n + 1} dx = \frac{\pi}{n \sin\left(\frac{\pi}{n}\right)}.$$

Solution. Let

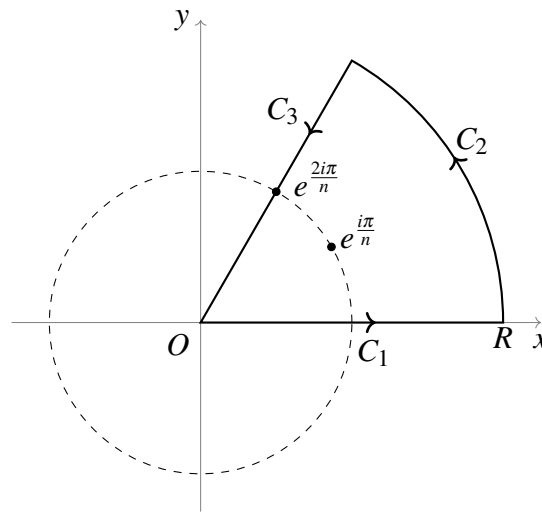
$$f(z) = \frac{1}{z^n + 1}$$

and the required integral to be I . Consider the following parametrisation (informally known as the *pizza contour*):

$$C_1(t) := t, \text{ where } 0 \leq t \leq R$$

$$C_2 : |z| = R \text{ (note that the angle subtended by the arc does not matter)}$$

$$C_3(t) := (R - t) \exp\left(\frac{2\pi i}{n}\right), \text{ where } 0 \leq t \leq R$$



By defining C to be the contour, it is clear that

$$\int_C \frac{1}{z^n + 1} dz = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

Only the pole $z = e^{i\pi/n}$ is in C so by the residue theorem,

$$\int_C \frac{1}{z^n + 1} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/n}} f(z) = -\frac{2\pi i}{n} \exp\left(\frac{i\pi}{n}\right).$$

We focus on C_1 .

$$\int_{C_1} \frac{1}{z^n + 1} dz = \int_0^R \frac{1}{t^n + 1} dt.$$

Letting R tend to infinity, and since t is a dummy variable, it is easy to see that

$$\int_{C_1} \frac{1}{z^n + 1} dz = \int_0^\infty \frac{1}{x^n + 1} dx = I.$$

For C_2 , by the triangle inequality, $|z^n + 1| \geq ||z^n| - |1|| = |R^n - 1|$. Hence,

$$\left| \int_{C_2} \frac{1}{z^n + 1} dz \right| \leq \int_{C_2} \frac{1}{R^n - 1} dz = \frac{c\pi R}{R^n - 1}.$$

Letting R tend to infinity, we see that the integral over C_2 is zero. Earlier, we mentioned that the angle subtended by the arc does not matter and we affirm this statement here.

The integral over C_3 is more complicated. Using the substitution

$$z = (R - t) \exp\left(\frac{2\pi i}{n}\right),$$

we see that

$$\int_{C_3} \frac{1}{z^n + 1} dz = -\exp\left(\frac{2\pi i}{n}\right) \int_0^R \frac{1}{(R - t)^n + 1} dt.$$

This calls for a substitution, say $u = R - t$. Hence, the integral over C_3 becomes

$$-\exp\left(\frac{2\pi i}{n}\right) \int_0^R \frac{1}{u^n + 1} du \xrightarrow{R \rightarrow \infty} -\exp\left(\frac{2\pi i}{n}\right) \int_0^\infty \frac{1}{x^n + 1} dx = -I \exp\left(\frac{2\pi i}{n}\right).$$

To conclude,

$$\begin{aligned}
 -\frac{2\pi i}{n} \exp\left(\frac{i\pi}{n}\right) &= I \left[1 - \exp\left(\frac{2\pi i}{n}\right) \right] \\
 I &= -\frac{2\pi i}{n} \cdot \frac{\exp\left(\frac{i\pi}{n}\right)}{1 - \exp\left(\frac{2\pi i}{n}\right)} \\
 &= -\frac{2\pi i}{n} \cdot \frac{\exp\left(\frac{i\pi}{n}\right)}{\exp\left(\frac{i\pi}{n}\right) \exp\left(-\frac{i\pi}{n}\right) - \exp\left(\frac{i\pi}{n}\right) \exp\left(\frac{i\pi}{n}\right)} \\
 &= \frac{\pi}{n \sin\left(\frac{\pi}{n}\right)}
 \end{aligned}$$

so we have finally derived this beautiful result. □

Example 6.11. Prove that

$$\int_0^{2\pi} \frac{1}{5 + 3 \sin \theta} d\theta = \frac{\pi}{2}.$$

Solution. Set $z = e^{i\theta}$ so $\sin \theta = (z - z^{-1})/2i$. The integral becomes

$$\int_{|z|=1} \frac{1}{5 + 3 \left(\frac{z - z^{-1}}{2i} \right)} \cdot \left(-\frac{i}{z} \right) dz = 2 \int_{|z|=1} \frac{1}{3z^2 + 10iz - 3} dz.$$

Let

$$f(z) = \frac{1}{3z^2 + 10iz - 3}.$$

It has two simple poles $z_1 = -i/3$ and $z_2 = -3i$. The first one is interior to the circle $|z| = 1$ so we shall consider this. By the residue theorem, the answer is

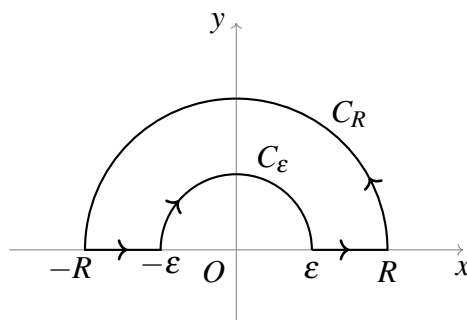
$$2 \cdot 2\pi i \cdot \frac{1}{3(z_1 + 3i)} = \frac{\pi}{2}.$$

□

Example 6.12. Prove that

$$\int_0^\infty \frac{(\log x)^2}{x^2 + 1} dx = \frac{\pi^3}{8}.$$

Solution. We consider the following contour.



Define

$$f(z) = \frac{(\log z)^2}{z^2 + 1}$$

and in our contour, say C , we let $0 < \varepsilon < 1 < R$. $\log z$ denotes the branch of the logarithm function defined on $\{z \in \mathbb{C} : -\pi/2 < \arg z < 3\pi/2\}$. Hence, it is clear that

$$\int_C = \int_{C_R} + \int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_\varepsilon^R.$$

By the residue theorem,

$$\int_C \frac{(\log z)^2}{z^2 + 1} dz = \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}.$$

Now, let us focus on C_R . We use the estimation lemma to help us.

$$\left| \int_{C_R} \right| \leq \pi R \cdot \frac{(\log R + i\theta)^2}{R^2 - 1}$$

which tends to 0 as R tends to infinity. In a similar fashion, one can show that

$$\lim_{\varepsilon \rightarrow 0} \int_{C_\varepsilon} = 0.$$

As such,

$$\begin{aligned} \int_C \frac{(\log z)^2}{z^2 + 1} dz &= \int_{-R}^{-\varepsilon} \frac{(\log z)^2}{z^2 + 1} dz + \int_\varepsilon^R \frac{(\log z)^2}{z^2 + 1} dz \\ &= \int_\varepsilon^R \frac{(\log(-z))^2}{z^2 + 1} dz + \int_\varepsilon^R \frac{(\log z)^2}{z^2 + 1} dz \\ &= \int_\varepsilon^R \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2 + 1} dz \end{aligned}$$

Now we set R to tend to infinity and ε to tend to 0. Also, we computed the value of the integral over C earlier so putting everything together,

$$\begin{aligned} -\frac{\pi^3}{4} &= \int_0^\infty \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2 + 1} dz \\ &= -\pi^2 \int_0^\infty \frac{1}{z^2 + 1} dz + 2i\pi \int_0^\infty \frac{\log z}{z^2 + 1} dz + 2 \int_0^\infty \frac{(\log z)^2}{z^2 + 1} dz \\ &= -\frac{\pi^3}{2} + 2i\pi \int_0^\infty \frac{\log z}{z^2 + 1} dz + 2 \int_0^\infty \frac{(\log z)^2}{z^2 + 1} dz \end{aligned}$$

Lastly, we will show that

$$\int_0^\infty \frac{\log x}{x^2 + 1} dx = 0.$$

Using the substitution $u = 1/x$,

$$\int_0^\infty \frac{\log x}{x^2 + 1} dx = \int_0^\infty \frac{-\log u}{(1/u)^2 + 1} \cdot \left(-\frac{1}{u}\right)^2 du = -\int_0^\infty \frac{\log u}{u^2 + 1} du$$

and the result follows. □

We have the following beautiful corollary:

Corollary 6.1. Let

$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx.$$

Then for all $n \geq 1$, I_{2n} satisfies the recurrence relation

$$I_{2n} = \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}} - \frac{1}{2} \sum_{k=1}^n \binom{2n}{2k} (-1)^k \pi^{2k} I_{2n-2k}.$$

It is not surprising that we only discuss the integrals I_{2n} instead of I_{2n+1} because

$$\int_0^\infty \frac{(\log x)^{2n+1}}{x^2 + 1} dx = 0$$

for all $n \geq 0$ by performing the substitution $u = 1/x$.

The above formula is also equivalent to the following by using the Dirichlet beta function:

Definition 6.4 (Dirichlet beta function). Define the Dirichlet beta function to be

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

Corollary 6.2. Let

$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx.$$

Then for all $n \geq 1$, $I_{2n} = 2(2n)! \beta(2n+1)$.

Proof.

$$\begin{aligned} \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx &= \int_0^1 \frac{(\log u)^{2n}}{u^2 + 1} du \quad \text{using } u = \frac{1}{x} \\ \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx &= 2 \int_0^1 \frac{(\log x)^{2n}}{x^2 + 1} dx \\ &= 2 \int_0^1 (-1)^k \sum_{k=0}^{\infty} (\log x)^{2n} x^{2k} dx \quad \text{using integration by parts} \\ &= 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \end{aligned}$$

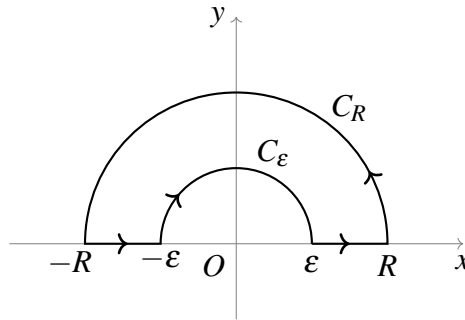
and the result follows. □

Example 6.13 (Dinh's 70 problems). Show that

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)^2} dx = \frac{\pi(1-\alpha)}{2 \cos\left(\frac{\pi\alpha}{2}\right)}$$

for $-1 < \alpha < 3$, $\alpha \neq 1$. What happens if $\alpha = 1$?

Solution. We consider the following contour.



Here, $0 < \epsilon < R$ and C_R and C_ϵ denote the upper-half of the semicircle of radius R and ϵ respectively.

So,

$$\int_C f(z) dz = \int_{C_R} + \int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_{\epsilon}^R.$$

By the residue theorem, it is clear that

$$\int_C f(z) dz = \frac{i\pi e^{ia\pi/2}}{2}.$$

It is clear that

$$\lim_{R \rightarrow \infty} \int C_R = 0 \text{ and } \lim_{\epsilon \rightarrow 0} \int C_\epsilon = 0.$$

So,

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^R = \int_C f(z) dz$$

and the result follows. \square

Example 6.14 (MA5217 AY24/25 Sem 1 Homework 1). Compute the following integrals using the residue formula:

$$\int_{-\infty}^{\infty} \frac{x-4}{(x^2-4x+5)(x^2+4)} dx \quad \text{and} \quad \int_0^{\infty} \frac{x^2}{(x^2+4)^2} dx$$

Solution. We deal with the first integral. Note that $z = 2+i, z = 2-i, z = 2i, z = -2i$ are simple poles of the integral. Let $C = C_1 + C_2$ be the upper half of the semicircle of radius R centred at the origin on the complex plane, where C_1 is the diameter and C_2 is the arc.

So,

$$C_1 = \{z = x + iy \in \mathbb{C} : -R \leq x \leq R\}$$

$$C_2 = \left\{ z = x + iy \in \mathbb{C} : z = Re^{i\theta}, 0 \leq \theta \leq \pi \right\}$$

Let $f(z)$ denote the integrand. We are only interested in the poles interior and on the boundary of C . By the residue theorem,

$$\int_C f(z) dz = 2\pi i \sum \text{Res}(f(z), z = z_k) = 2\pi i \left(\frac{2i}{13} \right) = -\frac{4\pi}{13}$$

Hence,

$$\int_{C_1} f(z) dz = \int_{-R}^R \frac{x-4}{(x^2-4x+5)(x^2+4)} dx.$$

Letting $R \rightarrow \infty$, we see that we obtain the original integral. Also,

$$\begin{aligned} \left| \int_{C_2} f(z) dz \right| &= \left| \int_0^\pi \frac{Re^{i\theta} \cdot iRe^{i\theta}}{(R^2e^{2i\theta} - 4Re^{i\theta} + 5)(R^2e^{2i\theta} + 4)} d\theta \right| \\ &= \left| \int_0^\pi \frac{R^2}{(R^2e^{2i\theta} - 4Re^{i\theta} + 5)(R^2e^{2i\theta} + 4)} d\theta \right| \end{aligned}$$

which is equal to 0 by the triangle inequality. Hence, the answer is $-4\pi/13$.

For the second integral, we note that the function is even. Letting g denote the integrand, we have

$$\int_0^\infty g(z) dz = \frac{1}{2} \int_{-\infty}^\infty g(z) dz.$$

We consider the same contour as the previous part, acknowledging that $z = \pm 2i$ are double poles of g . So, it follows that the sum of residues is $-\pi/8$, and by some tedious computation, the integral evaluates to $\pi/8$.

To compute the residue of the double pole $z = 2i$, we use the formula

$$\lim_{z \rightarrow 2i} \frac{d}{dz} ((z-2i)^2 g(z))$$

which is quite easy. □

Example 6.15 (Dinh's 70 problems). Evaluate

$$\int_{-\infty}^\infty \frac{x \sin x}{(1+x^2)^2} dx.$$

Solution. Let $f(z) = \frac{ze^{iz}}{(1+z^2)^2}$. Define C_1 to be the upper half of the semicircle of radius R centred at the origin and C_2 to be the real axis bounded by $\pm R$. So, C_1 can be parametrised using $z = Re^{it}$ for $t \in [0, \pi]$, whereas C_2 can be parametrised using $z = t$ for $t \in [-R, R]$. Let $C = C_1 \cup C_2$. By the residue theorem,

$$\int_C f(z) dz = 2\pi i \operatorname{Res}(f(z), i).$$

Note that

$$\operatorname{Res}(f(z), i) = \lim_{z \rightarrow i} \frac{d}{dz} \left(\frac{ze^{iz}}{(z+i)^2} \right) = \frac{1}{4e}.$$

Hence,

$$\int_C f(z) dz = \frac{i\pi}{2e}.$$

Now,

$$\lim_{R \rightarrow \infty} \left| \int_{C_1} f(z) dz \right| = \lim_{R \rightarrow \infty} \left| R^2 \int_0^\pi \frac{1}{(1+R^2e^{2i\theta})^2} d\theta \right| = 0.$$

Lastly, we work with C_2 . So, we have

$$\lim_{R \rightarrow \infty} \int_{C_2} f(z) dz = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{t \sin t}{(1+t^2)^2} dt = \int_{-\infty}^{\infty} \frac{x \sin x}{(1+x^2)^2} dx.$$

It follows that the answer is $\pi/2e$. □

Example 6.16 (Dinh's 70 problems). Show that for any $0 < a < 1$,

$$\int_0^{\infty} \frac{x^a}{x(1+x)} dx = \frac{\pi}{\sin(a\pi)}.$$

Solution. Let $t = x/(1+x)$, so

$$x = \frac{t}{1-t} \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{(1-t)^2}.$$

The integral becomes

$$\begin{aligned} \int_0^1 t^{a-1} (1-t)^{-a} dt &= B(a, 1-a) \quad \text{by definition of beta function} \\ &= \frac{\Gamma(a) \Gamma(1-a)}{\Gamma(1)} \quad \text{by relationship with gamma function} \\ &= \Gamma(a) \Gamma(1-a) \end{aligned}$$

and the result follows by Euler's reflection formula. □

7. Further Properties of Holomorphic Functions

7.1. Properties of Holomorphic and Harmonic Functions

Definition 7.1 (extended complex plane). Define

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\} \quad \text{to be the extended complex plane.}$$

Theorem 7.1 (Cauchy's estimate). Let $f \in H(\Omega)$ and let $\overline{D}(z_0, r) \subseteq \Omega$. Then, for all $n = 0, 1, 2, \dots$,

$$|a_n| \leq r^{-n} \sup_{|z-z_0|=r} |f(z)|.$$

Example 7.1 (Dinh's 70 problems). Suppose $f(z)$ is an odd function and holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfies

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^2} \text{ for all } z \neq 0.$$

Prove that

$$f(z) = \frac{a_{-1}}{z} + a_1 z \quad \text{for all } z \in \mathbb{C} \setminus \{0\} \text{ where } a_{-1}, a_1 \in \mathbb{C}.$$

Solution. Since f is holomorphic in $\mathbb{C} \setminus \{0\}$, its Laurent series representation about $z = 0$ is

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

f is odd implies $f(-z) = -f(z)$, so

$$f(z) = \dots + \frac{a_{-3}}{z^3} + \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \dots$$

Note that for $|z| \leq 1$, we have $|z^2 f(z)| \leq |z|^4 + 1 \leq 2$ and

$$z^2 f(z) = \dots + \frac{a_{-3}}{z} + a_{-1} z + a_1 z^3 + a_3 z^5 + \dots$$

so it forces $a_{-3}, a_{-5}, \dots, a_5, a_7, \dots = 0$. The result follows. \square

Here is an alternative solution.

Solution. Again, write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

By Cauchy's estimate (Theorem 7.1), if $|f(z)| \leq M$, we have

$$|f^{(k)}(a)| \leq \frac{k! M}{R^k}.$$

So, for $|z| \leq R$, we have

$$|a_k| \leq \frac{1}{R^k} \left(\frac{1}{R^2} + R^2 \right).$$

For $k \geq 3$, $\lim_{r \rightarrow \infty} |a_k| = 0$ and for $k \leq -3$, $\lim_{r \rightarrow 0} |a_k| = 0$. So, $a_k = 0$ for all $|k| \geq 3$. Hence,

$$f(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2.$$

Using the fact that f is odd, the result follows. \square

Example 7.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in $D(0, 1)$ and assume that the integral

$$A := \iint_{D(0,1)} |f'(z)|^2 dx dy < \infty.$$

(a) Express A in terms of the coefficients a_n .

(b) Prove that

$$|f(z) - f(0)| \leq \sqrt{\frac{A}{\pi} \ln \left(\frac{1}{1 - |z|^2} \right)}$$

for all $z \in D(0, 1)$.

Solution.

(a) Note that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

We shall parametrise z using polar coordinates. Let $z = re^{i\theta}$. As such,

$$\begin{aligned} \iint_{D(0,1)} |f'(z)|^2 dx dy &= \int_0^1 \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 r dr d\theta \\ &= \int_0^1 \int_0^{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m n a_m a_n r^{m+n-1} e^{i(m+n-2)\theta} dr d\theta \\ &= 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} dr \\ &= \pi \sum_{n=1}^{\infty} n |a_n|^2 \end{aligned}$$

(b) Note that $f(0) = 0$ and the RHS can be written as

$$\sqrt{\ln \left(\frac{1}{1 - |z|^2} \right) \sum_{n=1}^{\infty} n |a_n|^2}.$$

Starting with the LHS,

$$|f(z)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} (\sqrt{n} a_n) \left(\frac{z^n}{\sqrt{n}} \right) \right| \leq \sqrt{\left(\sum_{n=1}^{\infty} n |a_n|^2 \right) \left(\sum_{n=1}^{\infty} \frac{|z|^{2n}}{n} \right)}$$

where we applied the Cauchy-Schwarz inequality at the end. The result follows. \square

Theorem 7.2 (identity theorem). If two holomorphic functions f and g coincide on some set $E \subseteq D$ containing at least one limit point in D , then $f(z) = g(z)$ everywhere in D .

Example 7.3. Does there exist an entire function with the property that for $n \in \mathbb{N}$,

$$f\left(\frac{1}{n}\right) = \frac{n^4}{1+n^4}?$$

Solution. Replacing n with $1/z$, we consider the function

$$g(z) = \frac{1}{z^4 + 1}.$$

Note that the roots of the equation $z^4 + 1 = 0$ can be found as follows. As $z^4 = -1 = e^{i\pi+2k\pi i}$, then

$$z = \exp\left(i\pi \cdot \frac{2k+1}{4}\right),$$

where $k = 0, 1, 2, 3$. We denote the roots by p_n , where $0 \leq n \leq 3$. Obviously, $g(z)$ is holomorphic outside the 4 points p_n . By our hypothesis, $f(z) = g(z)$ for $z = 1, 1/2, 1/3, \dots$ and both f and g are defined on $\Omega = \mathbb{C} \setminus \{p_0, p_1, p_2, p_3\}$. The sequence $1, 1/2, 1/3, \dots$ converges to 0 which is inside Ω , so this sequence is not discrete in Ω . We conclude that $f = g$ in Ω by the identity theorem.

On the other hand, the function f is entire and bounded near p_n but g is not bounded near these points. We have obtained a contradiction so such a function f does not exist. \square

Example 7.4. Do there exist functions f and g that are holomorphic at $z = 0$ and that satisfy

- (a) $f(1/n) = f(-1/n) = 1/n^2$, where $n \in \mathbb{N}$;
- (b) $g(1/n) = g(-1/n) = 1/n^3$, where $n \in \mathbb{N}$?

Solution.

- (a) Yes, $f(z) = z^2$.
- (b) We prove that such a function g does not exist in a neighbourhood of 0. Suppose on the contrary that g exists. Define $h(z) = z^3$ and $l(z) = -z^3$. We have $g(z) = h(z)$ on a non-discrete sequence $z = 1, 1/2, 1/3, \dots$ which converges to 0, and 0 is in the domain of g . By the identity theorem, $g(z) = h(z)$. In a similar fashion, by considering the sequence $z = -1, -1/2, -1/3, \dots$, we obtain $g(z) = l(z)$. Hence, $h(z) = l(z)$, implying that $z^3 = -z^3$, so $z^3 = 0$. However, this is a contradiction. \square

Example 7.5. Show that there is no holomorphic function f in \mathbb{C} such that

$$f\left(\frac{1}{n}\right) = \frac{ne^{-2/n}}{n+1} \text{ for all } n \in \mathbb{N}.$$

Solution. Suppose on the contrary that such a function exists. Consider

$$g(z) = \frac{e^{-2z}}{z+1}.$$

This function is defined for all $z \in \mathbb{C}$ except at $z = -1$. By the hypothesis, this function is equal to f on the sequence $1/n$ which is not discrete on $\mathbb{C} \setminus \{-1\}$ and so, $f = g$ on $\mathbb{C} \setminus \{-1\}$. However, this is a contradiction. \square

Example 7.6 (MA5217 AY24/25 Sem 1 Homework 1). Show that the function $h(z) = \sin(\sin z) + \sin |z|^2$ is not holomorphic in any domain of \mathbb{C} .

Solution. Note that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

We let $z = x + iy, x, y \in \mathbb{R}$ and note that $|z|^2 = x^2 + y^2$. Hence,

$$h(z) = \sin\left(\frac{e^{iz}}{2i}\right) \cos\left(\frac{e^{-iz}}{2i}\right) - \cos\left(\frac{e^{iz}}{2i}\right) \sin\left(\frac{e^{-iz}}{2i}\right) + \sin(|z|^2)$$

By the Looman-Menchoff theorem, it suffices to prove that h does not satisfy the Cauchy-Riemann equations, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

The computation is tedious so we skip the details. \square

Example 7.7 (MA5217 AY24/25 Sem 1 Homework 1). Find all holomorphic functions $f(z)$ in $\mathbb{C} \setminus \{1\}$ such that

$$\text{Res}(f, 1) = 1, \quad \lim_{z \rightarrow \infty} (f(z) - z) = 2, \quad \lim_{z \rightarrow 1} |z - 1|^{4/3} f(z) = 0.$$

Solution. We claim that

$$f(z) = z + 2 + \frac{1}{z - 1}.$$

By the second condition, we infer that

$$f(z) = z + 2 + \sum_{n=1}^{\infty} \frac{1}{(az + b)^n}.$$

By the third condition, we infer that

$$\lim_{z \rightarrow 1} (z - 1)^{4/3} \sum_{n=1}^{\infty} \frac{1}{(az + b)^n} = 0$$

which implies we have to restrict the index of the infinite sum to $n = 1$ instead of $n \in \mathbb{N}$. Hence,

$$f(z) = z + 2 + \frac{1}{az + b}.$$

We see that $1/(az + b)$ has a simple pole at $z = -b/a$ but the first condition implies that $z = 1$ is a pole, so $a = -b$. Since the value of the residue at $z = 1$ is 1, then $a = 1$, so

$$f(z) = z + 2 + \frac{1}{z - 1}.$$

\square

Example 7.8. Let f and g be entire functions and suppose that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = cg(z)$ for some constant $c \in \mathbb{C}$.

Solution. First, we assume that g is identically equal to zero. Then, the result immediately follows. Now, we consider the case where g is not identically equal to zero. Define $h(z) = f(z)/g(z)$ on \mathbb{C} excluding the set of zeros of g . As such, h is holomorphic outside the zeros of g and $|h(z)| \leq 1$. As h is bounded and entire, the result follows by Liouville's theorem. \square

Definition 7.2 (analytic function). $f : \Omega \rightarrow \mathbb{C}$ is analytic if for any $z_0 \in \Omega$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}_{\geq 0}$ and the series is convergent to $f(z)$ for z in a neighbourhood of z_0 .

Example 7.9 (Dinh's 70 problems). Suppose f is entire and $f(z)$ is real iff z is real. Prove that f has at most one zero.

Solution. Suppose $f(z) = 0$, then it implies that z is real. Suppose on the contrary that f has a zero $x_0 \in \mathbb{R}$ with a multiplicity $m \geq 2$. Then, we can write $f(z)$ as the following power series:

$$f(z) = (z-x_0)^m(a_0 + a_1(z-x_0) + a_2(z-x_0)^2 + \dots)$$

Here, $a_0 \neq 0$. Note that

$$a_0 = \lim_{z \rightarrow x_0} \frac{f(z)}{(z-x_0)^m}$$

so for any $z \in \mathbb{R} \setminus \{x_0\}$, we have $\frac{f(z)}{(z-x_0)^m} \in \mathbb{R}$. Hence, $a_0 \in \mathbb{R}$. Now, write $z = x_0 + \varepsilon e^{i\theta}$, so we are considering the general case when $z \in \mathbb{C}$. It is clear that

$$f(z) = \varepsilon^m e^{mi\theta} (a_0 + a_1 \varepsilon e^{i\theta} + a_2 \varepsilon^2 e^{2i\theta} + \dots).$$

Define $g(\theta) = \text{Im}(e^{mi\theta}(a_0 + \varepsilon u(\theta, \varepsilon) + i\varepsilon v(\theta, \varepsilon)))$, where u, v are real and continuous functions and ε is sufficiently small. Note that $g(\pi/2m)g(3\pi/2m) < 0$ so by the intermediate value theorem, there exists $\theta' \in (\pi/2m, 3\pi/2m)$ such that $g(\theta_0) = 0$. So, $f(x_0 + \varepsilon e^{i\theta_0}) \in \mathbb{R}$. But because $m \geq 2$, it implies that $x_0 + \varepsilon e^{i\theta_0} \notin \mathbb{R}$, so we reached a contradiction. The result follows. \square

The next theorem summarises a list of important properties regarding holomorphic functions.

Theorem 7.3. Let Ω be an open and simply-connected domain in \mathbb{C} and let $f \in H(\Omega)$. Then,

- $f \in C^\infty(\Omega)$ and f satisfies the Cauchy-Riemann equations on Ω .
- **Cauchy-Goursat theorem:** If γ is a piecewise differentiable simple closed curve in Ω , then

$$\int_{\gamma} f(z) dz = 0.$$

- **Cauchy's integral formula:** If γ is an anticlockwise oriented and piecewise differentiable simple closed curve in Ω , then for any a interior to γ ,

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz.$$

Moreover, Cauchy's differentiation formula applies here.

- f is analytic, i.e. for any $z_0 \in \Omega$, one can write for $z \in D(z_0, r)$ with $\overline{D(z_0, r)} \subseteq \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n = 0, 1, 2, \dots$$

Example 7.10. Let f be a holomorphic function in the unit disc \mathbb{D} such that $|f(z)| < 1$ for $z \in \mathbb{D}$. Show that $|f''(0)| \leq 2$. Give an example of such a map with $f''(0) = 2$.

Solution. We use Cauchy's Differentiation Formula. Note that

$$f''(0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{z^3} dz.$$

Here, we let C be such that $|z| = r$, where $r < 1$ (i.e. C contains all points interior to the circle of radius 1 centred at the origin). Using the parametrisation $z = re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we see that

$$|f''(0)| \leq \frac{1}{\pi} \left| \int_0^{2\pi} \frac{f(re^{i\theta})}{r^3 e^{3i\theta}} \cdot ire^{i\theta} d\theta \right| \leq \frac{1}{\pi r^2} \left| \int_0^{2\pi} f(re^{i\theta}) d\theta \right| \leq \frac{2}{r^2}.$$

Hence, letting r tend to 1, the result follows.

For the later part of the question, we need to find a map such that $f''(0) = 2$. Well, consider

$$\int_{|z|=r} \frac{f(z)}{z^3} dz = 2\pi i$$

for which an obvious answer is $f(z) = z^2$. □

Example 7.11 (Dinh's 70 problems). Determine all complex holomorphic functions f defined on the unit disk which satisfy

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for $n = 2, 3, 4, \dots$

Solution. Let $g(z) = f''(z) + f(z)$, so g is holomorphic on \mathbb{D} . We have $g(1/n) = 0$ for all $n = 2, 3, 4, \dots$ and since $\lim_{n \rightarrow \infty} 1/n = 0 \in \mathbb{D}$, it follows that $g(z) = 0$ on \mathbb{D} . As such, $f(z) = -f''(z)$ on $z \in \mathbb{D}$. One can use Maclaurin Series to deduce that $f(z) = f(0) \cos z + f'(0) \sin z$. □

Theorem 7.4 (Casorati-Weierstrass theorem). Let f have an isolated essential singularity at z_0 . Then, for any $w \in \mathbb{C}$, $f(z)$ comes arbitrarily close to w in every deleted neighbourhood of z_0 . That is, for any $\delta > 0$, $f(D'(z_0, \delta))$ is a dense subset of \mathbb{C} .

Proof. Suppose on the contrary that for some $\delta > 0$, $f(D'(z_0, \delta))$ is not dense in \mathbb{C} . Then, there exists $w \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$D(w, \varepsilon) \cap f(D'(z_0, \delta)) = \emptyset.$$

For $z \in D'(z_0, \delta)$, write

$$g(z) = \frac{1}{f(z) - w}.$$

Then, g is bounded and holomorphic on $D'(z_0, \delta)$, so g has a removable singularity at z_0 . Let m be the order of the zero of g at z_0 . If $g(z_0) \neq 0$, set $m = 0$. Otherwise, write $g(z) = (z - z_0)^m g_1(z)$, where g_1 is holomorphic and does not vanish on $D(z_0, \delta)$. Hence,

$$(z - z_0)^m g_1(z) = \frac{1}{f(z) - w}.$$

Thus, we can write $f(z)$ as

$$f(z) = w + \frac{g_2(z)}{(z - z_0)^m},$$

where $g_2(z) = 1/g_1(z)$ is a holomorphic function on $D(z_0, \delta)$. Thus, f has a removable singularity ($m = 0$) or a pole ($m \neq 0$) at z_0 , which is a contradiction. \square

Definition 7.3. A meromorphic function in D is holomorphic on all D , except on a set of isolated points which are poles. Also, they can be written in the form $f = u/v$, where $u, v \in H(D)$ and $v \neq 0$, and they do not have a common zero.

7.2. The Argument Principle and Rouché's Theorem

Theorem 7.5 (argument principle). Let $f \in H(\Omega)$ and γ be a positively oriented, piecewise differentiable, simple closed contour in Ω such that all points interior to γ belong to Ω . Suppose f has no zero on γ . The zeros of f inside γ are a_1, a_2, \dots, a_n and $\alpha_1, \alpha_2, \dots, \alpha_n$ are their respective multiplicities. Then,

$$\sum_{j=1}^n \alpha_j = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz.$$

Example 7.12 (Dinh's 70 problems). Evaluate the integral

$$\int_{|z|=2} \frac{f'(z)}{f(z)} dz,$$

where $f(z) = \frac{\sin z \cos z}{z^7 - z^5 + z^3 - z}$, and $|z| = 2$ is positively oriented.

Solution. We use the argument principle. The answer is $2\pi i(Z - P)$, where Z and P are to be determined. Here, Z and P refer to the respective number of zeros and poles in the circle $|z| = 2$. To calculate Z , set $\sin z \cos z = 0$, so $z = -\pi/2, 0, \pi/2$. Hence, $Z = 3$. To calculate P , set $z^7 - z^5 + z^3 - z = 0$, so $z(z^4 + 1)(z + 1)(z - 1) = 0$. The solutions to $z^4 + 1 = 0$ are $z = e^{i\pi/4}, e^{-i\pi/4}, e^{3i\pi/4}, e^{-3i\pi/4}$. As such, $P = 7$, so the required answer is $-8\pi i$. \square

Theorem 7.6 (Rouché's theorem). Let $f, g \in H(\Omega)$ and γ be a piecewise differentiable simple closed curve such that all the points interior to γ are contained in Ω . Assume that

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \gamma$. Then, f and g have the same number of zeros (counting multiplicity) inside γ .

Example 7.13 (Dinh's 70 problems). Determine the number of zeros of $e^{z^2} - 3z^4$ in the unit disk.

Solution. For $|z| = 1$ (i.e. on the boundary of the unit disk), $|e^{z^2}| \leq e \leq 3 = 3|z^4|$ so it follows by Rouché's theorem that there are 4 zeros. \square

Example 7.14 (Dinh's 70 problems). Let N_k be the number of roots (counting multiplicity) in the disk $D(0, k) = \{|z| < k\}$ of the equation

$$z^6 - 5z^2 + 10 = 0.$$

For each positive integer k , determine N_k .

Solution. $N_1 = 0$; now consider the case when $k \geq 2$. On $|z| = 2$, $|5z^2 - 10| \leq 5|z|^2 + 10 = 30 \leq 2^6 = |z|^6$, so by Rouché's's theorem, $N_k = 6$ for $k \geq 2$. \square

Example 7.15. Let $r > 0$. Prove that for n sufficiently large, the polynomial

$$1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}$$

has no root in $D(0, r)$.

Solution. Fix an $r > 0$. Define $f(z) = e^z$ and $g_n(z)$ to be the polynomial above. For z on $\overline{D(0, r)}$, i.e. $|z| = r$,

$$|f(z) - g_n(z)| = \left| \sum_{k \geq n+1} \frac{z^k}{k!} \right| \leq \sum_{k \geq n+1} \frac{r^k}{k!}.$$

We note that as $n \rightarrow \infty$, the sum on the right tends to zero. For n large enough, the last sum on the right is smaller than e^{-r} . On the other hand, by setting $z = x + iy$, where $x, y \in \mathbb{C}$, we see that $|f(z)| = e^x \geq e^{-|z|} = e^{-r}$. Therefore, for z on $\overline{D(0, r)}$, we have

$$|f(z) - g_n(z)| < |f(z)|.$$

By Rouché's theorem, f and g_n have the same number of zeros inside $D(0, r)$. However, f vanishes nowhere so we can conclude that g_n does not vanish in $D(0, r)$. \square

Example 7.16. Find the number of zeros (counting multiplicity) of the function $z^5 + 6z^3 + 11$ in the annulus $2 < |z| < 3$.

Solution. Let $f(z) = z^5 + 6z^3 + 11$. On the circle $|z| = 3$, $|f(z) - z^5| = |6z^3 + 11| \leq 6|z^3| + 11 = 173 < 243 = |z^5|$ so by Rouché's theorem, the number of zeros of $f(z)$ in the region $0 < |z| < 3$ is equal to that of z^5 , which is 5.

On the circle $|z| = 2$, we have $|f(z) - 6z^3| = |z^5 + 11| \leq 2^5 + 11 = 43 < 48 = |6z^3|$ so the number of zeros of $f(z)$ in the region $0 < |z| < 2$ is equal to that of $6z^3$, which is 3. Therefore, f has exactly $5 - 3 = 2$ zeros in the annulus $2 < |z| < 3$. \square

Example 7.17 (Dinh's 70 problems).

- (a) For each integer $n \geq 1$, find the number of zeros (counting multiplicity) in the disk $D(0, n)$ of the polynomial $z^7 + 5z^3 - z - 2$.
- (b) Prove that the function $u(x, y) = \sinh x \sin y$ is harmonic and find its harmonic conjugates.

Solution.

- (a) Let N_n be the number of zeros. We first show that $N_1 = 3$. Note that on $|z| = 1$, we have

$$|z^7 + 5z^3 - z - 2 - 5z^3| = |z^7 - z - 2| \leq |z|^7 + |z| + 2 = 4 \leq 5 = 5|z|^3.$$

By Rouché's theorem, $N_1 = 3$.

For $n \geq 2$, we show that $N_n = 7$. Note that on the boundary $|z| = n$, we have

$$|z^7 + 5z^3 - z - 2 - z^7| \leq 5|z|^3 + |z| + 2 = 5n^3 + n + 2 \leq n^7 = |z|^7.$$

The result follows by Rouché's theorem.

- (b) Trivial. Show that u satisfies Laplace's equation, then to find its harmonic conjugates, use the Cauchy-Riemann equations. \square

Example 7.18. Let $a_1, \dots, a_n \in D(0, 1)$ and

$$f(z) = \prod_{k=1}^n \frac{a_k - z}{1 - \bar{a}_k z}.$$

Prove that for each $b \in D(0, 1)$, $f(z) = b$ has exactly n roots in $D(0, 1)$, counting multiplicity.

Solution. For $|z| = 1$, we have $\bar{z} = 1/z$. Hence,

$$\left| \frac{a_k - z}{1 - \bar{a}_k z} \right| = \frac{|a_k - z|}{|z| |1/z - \bar{a}_k|} = \frac{|a_k - z|}{|\bar{z} - \bar{a}_k|} = 1.$$

We infer that for $|z| = 1$, $|f(z)| = 1$. We deduce that for $b \in D(0, 1)$,

$$|(f(z) - b) - f(z)| = |b| < 1 = |f(z)|.$$

By Rouché's theorem, $f(z) - b$ and $f(z)$ have the same number of zeros in $D(0, 1)$. The roots of $f(z) = 0$ are a_1, \dots, a_n (so there are n roots), as such, the result follows. \square

Example 7.19 (Dinh's 70 problems). Show that if the integer n is sufficiently large, the equation

$$z = 1 + \left(\frac{z}{2}\right)^n$$

has exactly one solution in the disk $|z| < 2$.

Solution. Let

$$f_n(z) = z - 1 - \left(\frac{z}{2}\right)^n \quad \text{and} \quad f(z) = z - 1.$$

For arbitrary $\varepsilon > 0$, consider the boundary of $D(0, 2 - \varepsilon)$, we have

$$|f_n(z) - f(z)| = \left|\frac{z}{2}\right|^n = \left(\frac{2 - \varepsilon}{2}\right)^n = \left(1 - \frac{\varepsilon}{2}\right)^n.$$

Also,

$$|f(z)| = |z - 1| \geq |z| - 1 = 1 - \varepsilon \quad \text{by the reverse triangle inequality.}$$

By Rouché's theorem, we need $|f_n - f| \leq |f|$, i.e.

$$\left(1 - \frac{\varepsilon}{2}\right)^n \leq 1 - \varepsilon.$$

So, we choose

$$n \geq \frac{\ln(1 - \varepsilon)}{\ln(1 - \varepsilon/2)} \quad \text{where } n \in \mathbb{N} \text{ and } 0 < \varepsilon \leq \frac{1}{2}.$$

The number of zeros of f_n in $D(0, 2 - \varepsilon)$ is 1. Letting $\varepsilon \rightarrow 0$, the result follows. \square

Theorem 7.7 (Hurwitz's theorem). Let $f_n : \Omega \rightarrow \mathbb{C}$, where $n \in \mathbb{N}$, be a sequence of holomorphic functions that converges locally uniformly to a function $f : \Omega \rightarrow \mathbb{C}$. Let γ be a piecewise differentiable, simple closed contour in Ω such that all points interior to γ are contained in Ω . Assume that f has no zero on γ . Then,

there exists $N \in \mathbb{N}$ such that for all $n > N$ f_n and f have the same number of zeros inside γ .

Example 7.20. Assume that f is holomorphic in a neighbourhood of $\overline{D(0, 1)}$ and that $f'(z)$ has no zero on $\partial D(0, 1)$. Prove that for n sufficiently large,

$$F_n(z) = f\left(z + \frac{1}{n}\right) - f(z)$$

has the same number of zeros in $D(0, 1)$ as $f'(z)$.

Solution. We consider the function $g_n(z) = nF_n(z)$. Note that

$$g_n(z) = n \left[f\left(z + \frac{1}{n}\right) - f(z) \right],$$

so

$$\lim_{n \rightarrow \infty} g_n(z) = \lim_{n \rightarrow \infty} \frac{f(z + 1/n) - f(z)}{1/n} = f'(z).$$

By the Fundamental Theorem of Calculus,

$$g_n(z) = \frac{f(z + 1/n) - f(z)}{1/n} = \int_0^1 f' \left(z + \frac{t}{n} \right) dt.$$

Hence, g_n converges locally and uniformly in a neighbourhood to f' . By Hurwitz's Theorem, g_n has the same number of zeros as f' in $D(0, 1)$ when n is sufficiently large. Therefore, F_n satisfies the same property. \square

7.3. Open Mapping Theorem and the Maximum Modulus Principle

Theorem 7.8 (open mapping theorem). Let f be a non-constant holomorphic function on an open connected set Ω . Then, f is open, i.e. for any open set $U \subseteq \Omega$, we have $f(U)$ is open.

Theorem 7.9 (maximum modulus principle). Suppose f is a non-constant holomorphic function defined on a domain Ω . Then, $|f|$ does not attain the maximum value in Ω .

Example 7.21. Suppose f is holomorphic on a neighbourhood of the unit disc $\overline{D(0, 1)}$ and satisfies $f(0) = 3 + 4i$, $|f(z)| \leq 5$ if $|z| = 1$. Find $f'(0)$.

Solution. We prove that f is constant. Suppose on the contrary that f is not constant, then by the maximum modulus principle,

$$5 = f(0) < \max_{|z|=1} |f(z)| \leq 5.$$

This is a contradiction, so $f'(0) = 0$. \square

Example 7.22. Let f be a continuous function on $\bar{A} = \{1 \leq |z| \leq 4\}$ and holomorphic on $A = \{1 < |z| < 4\}$. Assume that

$$\max_{|z|=1} |f(z)| = 5 \text{ and } \max_{|z|=4} |f(z)| = 20.$$

- (i) Show that $|f(2)| \leq 10$.
- (ii) Find all functions f such that $f(2) = 10$.

Solution. Let us discuss the solutions.

- (i) Define $g(z) = f(z)/z$. Then,

$$\max_{|z|=1} |g(z)| = \max_{|z|=4} |g(z)| = 5.$$

By the maximum modulus principle, $|g(z)| \leq 5$ for $z \in A$. Setting $z = 2$, we have $|f(2)| \leq 10$.

- (ii) $g(2) = 5$. By the maximum modulus principle, g is a constant, so $g(z) = 5$. Hence, $f(z) = 5z$. \square

Corollary 7.1. Let Ω be a domain in \mathbb{C} and f be holomorphic in Ω .

- (i) If $|f|$ assumes a local maximum at some point in Ω , then f is constant in Ω .

(ii) If Ω is bounded and f is continuous up to the boundary $b\Omega$, then,

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in b\Omega} |f(z)|.$$

We obtain the next corollary on the minimum modulus principle by switching to the reciprocal $1/f(z)$.

Corollary 7.2 (minimum modulus principle). Let Ω be a domain in \mathbb{C} and f be holomorphic but never zero in Ω .

- If $|f|$ assumes a local minimum at some point in Ω , then f is constant on Ω .
- If Ω is bounded and f is continuous up to the boundary of Ω and never vanishes in $\overline{\Omega}$, then

$$\min_{z \in \overline{\Omega}} |f(z)| = \min_{z \in b\Omega} |f(z)|.$$

Example 7.23. Suppose f is holomorphic on a neighbourhood of $\overline{D(0,1)}$, $f(0) = i$ and $|f(z)| > 1$ whenever $|z| = 1$. Prove that f has a zero in $D(0,1)$.

Solution. Suppose on the contrary that f does not have a zero in $D(0,1)$. Then, $g(z) = 1/f(z)$ would be holomorphic in a neighbourhood $\overline{D(0,1)}$. Moreover, we have $|g(0)| = 1$ and $|g(z)| < 1$ when $|z| = 1$. This contradicts the maximum modulus principle. \square

Theorem 7.10 (maximum and minimum principle for harmonic functions). Let Ω be a domain in \mathbb{C} and u be a real-valued harmonic in Ω .

- (i) If u has either a local maximum or a local minimum at some point of Ω , then u is a constant on Ω .
- (ii) If Ω is bounded and f is continuous up to the boundary of Ω , then

$$\max_{z \in \overline{\Omega}} u(z) = \max_{z \in b\Omega} u(z) \text{ and } \min_{z \in \overline{\Omega}} u(z) = \min_{z \in b\Omega} u(z).$$

Example 7.24. Find the maximal value of $\operatorname{Re}(z^3)$ for $z \in [0,1] \times [0,1]$.

Solution. Note that $\operatorname{Re}(z^3)$ is harmonic as it is the real part of a holomorphic function. Hence, it achieves its maximal value on the boundary of the unit square. Throughout this problem, $a \in \mathbb{R}$ and $a \in [0,1]$.

- **Case 1 (bottom edge of square):** $z = a$. Then, $\operatorname{Re}(z^3) = a^3$, whose maximum is 1.
- **Case 2 (top edge of square):** $z = a + i$. Then, $\operatorname{Re}(z^3) = a^3 - 3a$. The maximum here is 0.
- **Case 3 (left edge of square):** $z = ai$. Then, $\operatorname{Re}(z^3) = 0$.
- **Case 4 (right edge of square):** $z = 1 + ai$. Then, $\operatorname{Re}(z^3) = 1 - 3a^2$. The maximum here is 1.

Overall, the maximum value is 1 which is achieved when $z = 1$. \square

Example 7.25 (Dinh's 70 problems). Let $a \in \mathbb{C}$, $|a| \leq 1$, and consider the polynomial

$$P(z) = \frac{a}{2} + (1 - |a|^2)z - \frac{\bar{a}}{2}z^2.$$

Prove that $|P(z)| \leq 1$ whenever $|z| \leq 1$.

Solution. Note that $z\bar{z} = 1$ on $|z| = 1$. Consider

$$\frac{P(z)}{z} = \frac{a}{2z} - \frac{\bar{a}z}{2} + 1 - |a|^2.$$

We have

$$\frac{a}{2z} - \frac{\bar{a}z}{2} = \frac{1}{2} \left(\frac{a}{z} - \overline{a/z} \right).$$

Let $\lambda = a/z \in \mathbb{C}$. Then, $\lambda - \bar{\lambda} = 2i\operatorname{Im}(\lambda)$, so

$$\frac{a}{2z} - \frac{\bar{a}z}{2} = i\operatorname{Im}\left(\frac{a}{z}\right) = i\operatorname{Im}(a\bar{z}).$$

Hence,

$$\begin{aligned} \left| \frac{P(z)}{z} \right| &\leq \left| i\operatorname{Im}(a\bar{z}) + 1 - |a|^2 \right| \\ |P(z)| &\leq \left| i\operatorname{Im}(a\bar{z}) + 1 - |a|^2 \right| \text{ since } |z| \leq 1 \\ |P(z)|^2 &\leq |\operatorname{Im}(a\bar{z})|^2 + (1 - |a|^2)^2 \end{aligned}$$

We bluntly state that $|\operatorname{Im}(a\bar{z})|^2 \leq |a|^2$, so $|P(z)|^2 \leq 1 - |a|^2 + |a|^4 \leq 1$ since $|a| \leq 1$. By the maximum modulus principle, whenever $|z| \leq 1$, we have $|P(z)| \leq 1$.

Now, we justify that $|\operatorname{Im}(a\bar{z})|^2 \leq |a|^2$. Let $z = x + iy$ and $a = \alpha + i\beta$, where $x, y, \alpha, \beta \in \mathbb{R}$ such that $x^2 + y^2 \leq 1$ and $\alpha^2 + \beta^2 \leq 1$. We have $a\bar{z} = (\alpha + i\beta)(x - iy) = \alpha x - \beta y + i(\beta x - \alpha y)$ so $\operatorname{Im}(a\bar{z}) = \beta x - \alpha y$. It suffices to prove that $(\beta x - \alpha y)^2 \leq \alpha^2 + \beta^2$. In other words, $\alpha^2(1 - y^2) + \beta^2(1 - x^2) + 2\alpha\beta xy \geq 0$. Let $x = \cos \theta$ and $y = \sin \theta$ so $\alpha^2 \sin^2 \theta + \beta^2 \cos^2 \theta + 2\alpha\beta \cos \theta \sin \theta \geq 0$. This inequality is obviously true since $(\alpha \sin \theta + \beta \cos \theta)^2 \geq 0$, or equivalently $(\alpha y + \beta x)^2 \geq 0$. \square

We then introduce the Schwarz-Pick lemma (Lemma 7.1), which is also known as the Schwarz lemma.

Lemma 7.1 (Schwarz-Pick Lemma). Let $f : D(0, 1) \rightarrow \mathbb{C}$ be a holomorphic function with $f(0) = 0$ and $|f(z)| \leq 1$ for each $z \in D(0, 1)$. Then,

$$|f(z)| \leq |z| \text{ and } |f'(0)| \leq 1.$$

Moreover, if $|f(z)| = |z|$ for some $z \in D(0, 1) \setminus \{0\}$ or if $|f'(0)| = 1$, then f is a rotation of $D(0, 1)$; that is, there exists a constant $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} z \text{ for all } z \in D(0, 1).$$

Example 7.26 (Dinh's 70 problems). Does there exist a holomorphic function $f : \mathbb{D} \rightarrow \mathbb{D}$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Solution. Yes, this is simply proven using the Schwarz-Pick lemma since $f'(1/2) \leq 7/12 < 2/3$. \square

Example 7.27 (Dinh's 70 problems). Let f be a holomorphic function from the unit disk $D(0, 1)$ to itself. Assume that there is a point $z_0 \in D(0, 1)$ such that $f(z_0) = z_0$. Prove that $|f'(z_0)| \leq 1$.

Solution. We use the Schwarz-Pick Lemma, which says that for $a, b \in \mathbb{D}$, a holomorphic function $f : \mathbb{D} \rightarrow \mathbb{D}$ satisfies $f(a) = b$ and $|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$. So, we set $a = b = z_0$. The result follows. \square

Example 7.28. Is there a holomorphic function of $D(0, 1)$ onto itself such that $f(0) = 0$ and $f(i/4) = i/3$? Justify.

Solution. We will show that there is no such function. Suppose on the contrary that there exist such a function. By the Schwarz Lemma, as $|f(z)| \leq |z|$ for $z \in \mathbb{D}$, we have $|f(i/4)| \leq |i/4| = 1/4$, which is a contradiction. \square

7.4. Winding Numbers

Definition 7.4 (winding number). Let $\gamma : [a, b] \rightarrow \mathbb{C} \setminus \{z_0\}$ be a closed curve that does not pass through z_0 . Given an argument θ_a for $\gamma(a) - z_0$,

there exists a unique continuous function $\theta : [a, b] \rightarrow \mathbb{R}$

such that for each $t \in [a, b]$, $\theta(t)$ is an argument of $\gamma(t) - z_0$ and such that $\theta(a) = \theta_a$. Define

$$n(\gamma, z_0) = \frac{\theta(b) - \theta(a)}{2\pi} \quad \text{to be the winding number of } \gamma \text{ around } z_0.$$

Sometimes, we also refer it to the index of z_0 with respect to γ .

Theorem 7.11. $n(\gamma, z_0) \in \mathbb{Z}$

Theorem 7.12. Let γ be a closed contour piecewise differentiable and $z_0 \in \gamma$. Then,

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz.$$

Corollary 7.3. Let f be holomorphic on an open set Ω containing γ and $z_0 \in f(\gamma)$. Then,

$$n(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz.$$

Example 7.29 (Dinh's 70 problems). Let C be the unit circle $|z| = 1$, anti-clockwise oriented, and let $f(z) = z^3$. How many times does the curve $f(C)$ wind around the origin? Explain.

Solution. We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_C \frac{3}{z} dz = 3.$$

□

Example 7.30 (Dinh's 70 problems). Let C be the unit circle $|z| = 1$, anti-clockwise oriented, and let $f(z) = (z^2 + 2)/z^3$. How many times does the curve $f(C)$ wind around the origin? Explain.

Solution. We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -\frac{1}{2\pi i} \int_C \frac{z^2 + 6}{z(z^2 + 2)} dz.$$

The residue at $z = 0$ is 3, so by Cauchy's residue theorem, the answer is -3 .

□

Theorem 7.13 (generalised Cauchy's integral formula). Suppose f is a holomorphic function in a simply connected domain Ω . Then for any piecewise differentiable closed contour γ in Ω , if $a \notin \gamma$,

$$n(\gamma, z)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz$$

Theorem 7.14 (generalised residue theorem). Let Ω be a simply connected domain in \mathbb{C} . Suppose f is holomorphic outside a finite number of points z_1, \dots, z_N in Ω . Then, for any piecewise differentiable closed contour γ in Ω which does not pass through z_1, \dots, z_N ,

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^N n(\gamma, z_k) \text{Res}(f, z_k).$$

8. Conformal Mappings and Möbius Transformations

8.1. Univalent Functions

Definition 8.1 (univalent function). Let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. Then, f is univalent if it is injective, i.e.

$$f(z_1) = f(z_2) \quad \text{implies} \quad z_1 = z_2.$$

f is locally univalent if for each $z_0 \in \Omega$, there exists a neighbourhood U of z_0 such that $f|_U \rightarrow \mathbb{C}$ is injective.

Theorem 8.1. A holomorphic function $f : \Omega \rightarrow \mathbb{C}$

$$\text{locally univalent at } z_0 \quad \text{if and only if} \quad f'(z_0) \neq 0.$$

Corollary 8.1 (inverse function theorem). If $f : \Omega \rightarrow \mathbb{C}$ is a univalent holomorphic function, then its inverse f^{-1} is also holomorphic defined on $f(\Omega)$. Moreover, for each $z \in \Omega$,

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

Definition 8.2. Suppose two curves γ and η intersect at z_0 and α is the oriented angle between the tangent vectors to these curves at z_0 . A holomorphic map f preserves angles at z_0 if the image curves $f \circ \gamma$ and $f \circ \eta$ intersect at $f(z_0)$ and their tangent vectors at $f(z_0)$ form an angle equal to α .

Theorem 8.2. Suppose $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$. Then, f preserves angles at z_0 .

Definition 8.3 (conformal map and automorphism group). A bijective holomorphic function $f : U \rightarrow V$ is a conformal map or a biholomorphism. A conformal map from a domain $\Omega \rightarrow \Omega$ is a conformal automorphism of Ω . Define $\text{Aut}(\Omega)$ to be the set of conformal automorphisms of Ω .

Theorem 8.3. If f and g are automorphisms of Ω , then $f \circ g$ is also an automorphism.

8.2. Automorphisms of the Complex Plane \mathbb{C}

Example 8.1. Translations, rotations, and dilations are examples of automorphisms of the complex plane. We only discuss translations here. Suppose $h \in \mathbb{C}$. Then, the translation

$$z \mapsto z + h \text{ is a conformal map } \mathbb{C} \rightarrow \mathbb{C} \quad \text{whose inverse is} \quad w \mapsto w - h.$$

Moreover, if $h \in \mathbb{R}$, this translation is also a conformal map from the upper half-plane \mathbb{H} to itself.

Theorem 8.4. Let f be a conformal map from \mathbb{C} to itself. Then, there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $f(z) = az + b$ for $z \in \mathbb{C}$. In particular, we have

$$\text{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C}, a \neq 0\}.$$

8.3. Automorphisms of the Unit Disc \mathbb{D}

Definition 8.4 (unit disc). Define \mathbb{D} to be the unit disc. This is sometimes denoted by $D(0, 1)$ which represents

the open disc of radius 1 centred at 0.

Example 8.2. Any rotation by an angle $\theta \in \mathbb{R}$, i.e. $\rho_\theta(z) = e^{i\theta}z$, is an automorphism of \mathbb{D} whose inverse is $e^{-i\theta}z$.

We can generalise the previous example to the following lemma:

Lemma 8.1 (Blaschke factor). For any $a \in \mathbb{D}$, the map

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad \text{is a conformal automorphism of } \mathbb{D} \quad \text{with inverse } \phi_a^{-1} = \phi_a.$$

The transformation ϕ_a is known as the Blaschke factor.

Theorem 8.5. If $f : \mathbb{D} \rightarrow \mathbb{D}$ is a conformal automorphism and $f^{-1}(0) = a$, then there exists $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}.$$

Hence,

$$\text{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \frac{a - z}{1 - \bar{a}z} : \theta \in \mathbb{R}, a \in \mathbb{D} \right\}.$$

Example 8.3. Let f be a holomorphic function on \mathbb{D} such that $|f(z)| \leq 1$ when $|z| < 1$. Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|} \quad \text{for all } |z| < 1.$$

Solution. We first consider the case where $|f(z)| = 1$ for some $z \in \mathbb{D}$. By the maximum modulus principle, f is constant and so $|f(z)| = 1$ for all $z \in \mathbb{D}$. The above inequality is equivalent to

$$|f(0)| - |z| \leq 1 + |f(0)||z| \quad \text{and} \quad 1 - |f(0)||z| \leq |f(0)| + |z|,$$

so $1 - |z| \leq 1 + |z|$, which holds.

Now, consider the case where $|f(z)| < 1$ for all $z \in \mathbb{D}$. Let $f(0) = a \in \mathbb{D}$. Note that

$$\phi(z) = \frac{a-z}{1-\bar{a}z} \in \text{Aut}(\mathbb{D}).$$

As such, $g = \phi \circ f$ is a holomorphic function from \mathbb{D} to itself. Moreover, $g(0) = \phi(f(0)) = \phi(a) = 0$. By the Schwarz Lemma, $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Since $\phi^{-1} = \phi$, then

$$f(z) = (\phi^{-1} \circ g)(z) = \frac{a-g(z)}{1-\bar{a}g(z)}.$$

As such,

$$\left| \frac{a-g(z)}{1-\bar{a}g(z)} \right| \leq 1 \Rightarrow 1 - |f(0)||z| \leq 1 - |\bar{a}||g(z)| \leq |a-g(z)| \leq |1-\bar{a}g(z)| \leq 1 + |\bar{a}||g(z)| \leq 1 + |f(0)||z|$$

and in a similar fashion, we can deduce that

$$|f(0) - |z|| \leq |a| - |g(z)| \leq |a-g(z)| \leq |a| + |g(z)| \leq |f(0)| + |z|.$$

Hence, we have shown that

$$1 - |f(0)||z| \leq |1 - \bar{a}g(z)| \leq 1 + |f(0)||z| \text{ and } |f(0) - |z|| \leq |a-g(z)| \leq |f(0)| + |z|.$$

The desired inequality is thus proven. \square

Example 8.4. Find a conformal map $T : D(0, 1) \rightarrow D(1, 2)$ such that $T(0) = 1 + i$ and $T(1) = 1 - 2i$. Is the transformation unique?

Solution. Let $S(z) = 2z + 1$, which maps $D(0, 1)$ to $D(1, 2)$ conformally. Define $f = S^{-1} \circ T$, which is an automorphism of the unit disc. We have $S^{-1}(z) = (z-1)/2$. So, the conditions $T(0) = 1 + i$ and $T(1) = 1 - 2i$ are equivalent to $f(0) = i/2$ and $f(1) = -i$. To find such a map f , consider

$$g(z) = -i \cdot \frac{\frac{i}{2} - z}{1 + \frac{i}{2}z}$$

which is a conformal automorphism of $D(0, 1)$ such that $g(i/2) = 0$ and $g(-i) = 1$. Thus,

$$f(z) = \frac{i(1-2z)}{2-z}.$$

We conclude that

$$T(z) = \frac{2(1+i) - (1+4i)z}{2-z}$$

is the required conformal map satisfying the conditions.

Suppose \tilde{T} also satisfies the requirements. Then, $R = T^{-1} \circ \tilde{T}$ is a conformal automorphism of $D(0, 1)$ satisfying $R(0) = 0$ and $R(1) = 1$. It is known that all automorphisms of the unit disc which fix 0 are rotations. Hence, R is the identity function so we conclude that $\tilde{T} = T$. \square

Example 8.5. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic function. Suppose $f(0) = 0$ and there exists a constant $A > 0$ such that $\operatorname{Re}(f(z)) \leq A$ for $z \in \mathbb{D}$. Prove that for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{2A|z|}{1-|z|}.$$

Solution. Since $f(0) = 0$, then f is identically 0 or f is not identically constant. If $f(z) = 0$ for all $z \in \mathbb{D}$, the inequality is obvious. Suppose f is not identically constant. Consider

$$\phi_1(z) = -\frac{z}{A} + 1, \quad \phi_2(z) = \frac{1-z}{1+z} \quad \text{and} \quad \phi(z) = (\phi_2 \circ \phi_1)(z) = \frac{z}{2A-z}.$$

Note that ϕ_1 is a conformal map from $\{\operatorname{Re}(z) < A\}$ to $\{\operatorname{Re}(z) > 0\}$ and sends 0 to 1; ϕ_2 is a conformal map from $\{\operatorname{Re}(z) > 0\}$ to the unit disc and sends 1 to 0. Hence, ϕ is a conformal map from $\{\operatorname{Re}(z) < A\}$ to the unit disc and sends 0 to 0. As such, $F = \phi \circ f$ is a holomorphic map from \mathbb{D} to itself and $F(0) = 0$.

By the Schwarz Lemma, note that the conditions $F(0) = 0$ and $|F(z)| \leq 1$ are satisfied since $z \in \mathbb{D}$. Hence, $|F(z)| \leq |z|$. That is to say,

$$|z| \geq |\phi(f(z))| = \left| \frac{f(z)}{2A-f(z)} \right|.$$

The desired inequality follows with some simple algebraic manipulation. □

Example 8.6 (Dinh's 70 problems). Suppose that f is holomorphic on the open set containing \mathbb{D} , $|f(z)| \leq 4$ if $|z| = 1$ and $f(i/2) = 0$. Show that for all $|z| \leq 1$,

$$|f(z)| \leq 4 \left| \frac{z - i/2}{1 + i/2 \cdot z} \right|.$$

Solution. Note that $g(z) = \frac{a-z}{1-\bar{a}z}$ is an automorphism of \mathbb{D} , so we set $f(z) = 4g(z)$ and $a = i/2$. The result follows. □

Example 8.7 (Dinh's 70 problems). Show that if $D(0, R) \rightarrow \mathbb{C}$ is holomorphic with $|f(z)| < M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)} \right| \leq \frac{|z|}{MR}.$$

Solution. Let $a = f(0)$. We wish to prove

$$\left| \frac{a - f(z)}{M^2 - \bar{a}f(z)} \right| \leq \frac{|z|}{MR}.$$

Define $\phi : \mathbb{D} \rightarrow \mathbb{D}$ via

$$\phi(z) = \frac{a/M - z}{1 - (\bar{a}/M)z}.$$

Note that $\overline{a/M} = \bar{a}/M$ since $M \in \mathbb{R}$. So, define $g = \phi \circ \frac{f(Rz)}{M}$. It is clear that $g(0) = 0$ and $g : \mathbb{D} \rightarrow \mathbb{D}$. By the Schwarz Lemma, $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Hence,

$$\begin{aligned} |g(z)| &\leq |z| \\ \frac{1}{M} \left| \frac{a - f(Rz)}{1 - \bar{a}f(Rz)/M^2} \right| &\leq |z| \\ \frac{M^2}{M} \left| \frac{a - f(Rz)}{M^2 - \bar{a}f(Rz)} \right| &\leq |z| \\ \left| \frac{a - f(z)}{M^2 - \bar{a}f(z)} \right| &\leq \frac{|z|}{MR} \end{aligned}$$

and we are done. \square

Lemma 8.2 (Schwarz-Pick lemma). Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic function, $a \in \mathbb{D}$ and $f(a) = b$. Then,

(i) for each $z \in \mathbb{D}$, $|\phi_b(f(z))| \leq |\phi_a(z)|$

(ii) $|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$

If equality holds in (ii) or if we have equality in (i) for some $z \neq a$, then $f \in \text{Aut}(\mathbb{D})$.

8.4. Maps from the Upper Half-Plane \mathbb{H} to the Unit Disc \mathbb{D}

Definition 8.5 (upper half-plane). Define $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ to be the upper half-plane.

Lemma 8.3. Let

$$F(z) = \frac{i - z}{i + z} \quad \text{and} \quad G(w) = i \cdot \frac{1 - w}{1 + w}.$$

Then, $F : \mathbb{H} \rightarrow \mathbb{D}$ is a conformal map with inverse $G : \mathbb{D} \rightarrow \mathbb{H}$.

Theorem 8.6. All conformal mappings from \mathbb{H} to \mathbb{D} take the form

$$\left\{ e^{i\theta} \frac{z - \beta}{z - \bar{\beta}} : \theta \in \mathbb{R}, \beta \in \mathbb{H} \right\}.$$

8.5. Automorphisms of the Upper Half-Plane \mathbb{H}

Theorem 8.7.

$$\text{Aut}(\mathbb{H}) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$$

Proof. Let $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Define a', b', c', d' to be as follows:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \sqrt{ad - bc},$$

where $a', b', c', d' \in \mathbb{R}$ and $a'd' - b'c' = 1$. As such,

$$\mathcal{G} = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\} = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}.$$

We shall prove that $\mathcal{G} \subseteq \text{Aut}(\mathbb{H})$. Let

$$f(z) = \frac{az+b}{cz+d} \in \mathcal{G}.$$

Then, $f : \mathbb{R} \rightarrow \mathbb{R}$. If we let $z = x + iy$, where $x, y \in \mathbb{R}$, Since $a, b, c, d \in \mathbb{R}$, then

$$\begin{aligned} \text{Im}(f(z)) &= \text{Im} \left[\frac{a(x+iy)+b}{c(x+iy)+d} \right] = \text{Im} \left[\frac{ax+b+i(ay)}{cx+d+i(cy)} \cdot \frac{cx+d-i(cy)}{cx+d-i(cy)} \right] \\ &= \text{Im} \left[\frac{ac(x^2+y^2) + bcx + adx + bd + iy(ad-bc)}{c^2(x^2+y^2) + 2cdx + d^2} \right] \\ &= \text{Im}(z) \cdot \frac{ad-bc}{c^2|z|^2 + 2cdx + d^2} \\ &= \text{Im}(z) \cdot \frac{ad-bc}{|cz+d|^2} \end{aligned}$$

which shows $f : \mathbb{H} \rightarrow \mathbb{H}$. It is clear that

$$g(z) = \frac{-dz+b}{cz-a} \in \mathcal{G}$$

and $g \circ f = \text{id}$. Hence, $f \in \text{Aut}(\mathbb{H})$ and $\mathcal{G} \subseteq \text{Aut}(\mathbb{H})$.

Conversely, let f be an arbitrary map in $\text{Aut}(\mathbb{H})$. We will show that $f \in \mathcal{G}$. Define

$$F(z) = \frac{i-z}{i+z}$$

which is a conformal map from \mathbb{H} to \mathbb{D} with inverse

$$F^{-1}(z) = i \cdot \frac{1-z}{1+z}$$

and this maps from \mathbb{D} to \mathbb{H} . Hence, $h = F \circ f$ is a conformal map from \mathbb{H} to \mathbb{D} . All such a map h must be of the form

$$e^{2i\theta} \frac{z-\beta}{z-\bar{\beta}}$$

with $\beta \in \mathbb{H}$ and $\theta \in \mathbb{R}$. We let the reader prove that

$$f(z) = F^{-1} \left(e^{2i\theta} \frac{z-\beta}{z-\bar{\beta}} \right) = \frac{az+b}{cz+d}$$

and $ad-bc = \text{Im}(\beta) > 0$ which would show that $f \in \mathcal{G}$, so $\text{Aut}(\mathbb{H}) \subseteq \mathcal{G}$. □

Example 8.8 (Dinh's 70 problems). Find a conformal map from

$$H = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$$

onto

$$A = \{z \in \mathbb{C} : |z-2| < 3, |z| > 1\}.$$

You may leave your answer as a composition of conformal mappings.

Solution. We find a conformal map from A to H first.

- Let $\phi_1(z) = 1/(z+1)$. Let us figure out what A gets mapped to via ϕ_1 . So, if we let $w = 1/(z+1)$, we have $z = 1/w - 1$. Consider the annulus $|z-2| < 3$, so after the transformation, we have $w > 1/6$. For the region $|z| > 1$, we have $1/w > 2$. So, ϕ_1 maps A to A_1 , where $A_1 = \{z \in \mathbb{C} : 1/6 < \operatorname{Re}(z) < 1/2\}$.
- Let $\phi_2(z) = z - \frac{1}{6}$. So, ϕ_2 maps A_1 to $A_2 = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1/3\}$.
- Let $\phi_3(z) = \tan\left(\frac{3\pi z}{2}\right)$, which maps A_2 to H .

As such, the required conformal map from H to A is $\phi_3^{-1} \circ \phi_2^{-1} \circ \phi_1^{-1}$. \square

Example 8.9 (Dinh's 70 problems). Let $f : D(0,1) \rightarrow \mathbb{C}$ be a holomorphic function such that $\operatorname{Re}(f(z)) > 0$ for each $z \in D(0,1)$ and such that $f(0) = 1$.

- Prove that $|f'(0)| \leq 2$.
- Assume that $|f'(0)| = 2$. Determine all possible forms of f .

Solution.

- We first find a holomorphic map from $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ to \mathbb{D} . To do this, let $\phi_1(z) = iz$, which maps the right half of the complex plane to the upper half, \mathbb{H} . Then, recall that $\phi_2(z) = \frac{z-i}{z+i}$ maps \mathbb{H} to \mathbb{D} . So, $\phi = \phi_2 \circ \phi_1$ is the required holomorphic map. We have

$$\phi(z) = \frac{iz-i}{iz+i} = \frac{z-1}{z+1}.$$

Define $F = \phi \circ f$ so $F : \mathbb{D} \rightarrow \mathbb{D}$, i.e. F is an automorphism of the unit disk and $F(0) = 0$. By the Schwarz Lemma, $|F'(0)| \leq 1$, so $|\phi'(1)f'(0)| \leq 1$. Since $\phi'(1) = 1/2$, the result follows.

- Suppose equality holds. Then, $F'(0) = 1$, where

$$F(z) = \frac{f(z)-1}{f(z)+1}.$$

Then, $F(z) = ze^{i\theta}$ (recall that this is just rotating some point in the unit disk about the origin) for $\theta \in \mathbb{R}$. One can work out that $f = \phi^{-1} \circ F$ and find an explicit expression for it. \square

8.6. Möbius Transformations

Definition 8.6 (Möbius transformation). A transformation of the form

$$T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$, is a linear fractional transformation (LFT). If $ad - bc \neq 0$, then T is a Möbius transformation.

Note that the condition $ad - bc \neq 0$ is equivalent to saying T is not constant. Consider

$$T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. If $c = 0$, then $T : \mathbb{C} \rightarrow \mathbb{C}$. If $c \neq 0$, then $T : \mathbb{C} \setminus \{-d/c\} \rightarrow \mathbb{C}$.

Definition 8.7. The Möbius transformation $T : \mathbb{C}^* \rightarrow \mathbb{C}^*$ associated with a, b, c, d is defined by

$$T(z) = \begin{cases} \frac{az+b}{cz+d} & z \neq \infty, z \neq -d/c; \\ a/c & z = \infty; \\ \infty & z = -d/c. \end{cases}$$

Moreover, if $c = 0$, then $a \neq 0$ and $d \neq 0$ so that the usual agreements regarding ∞ can be made. That is, $T(\infty) = \infty$.

The map $T : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is injective.

Remark 8.1. T is holomorphic on $\mathbb{C}^* \setminus \{-d/c\}$ with a simple pole at the point $\{-d/c\}$.

Proposition 8.1. A Möbius transformation is a composition of transformations of the following forms:

- (i) **translation:** $z \mapsto z + b$, $b \in \mathbb{C}$;
- (ii) **rotation and dilation:** $z \mapsto \lambda z$, $\lambda \in \mathbb{C} \setminus \{0\}$;
- (iii) **reciprocation:** $z \mapsto 1/z$

Definition 8.8. Let $\text{Aut}(\mathbb{C}^*)$ be the set of meromorphic automorphisms of \mathbb{C}^* .

Theorem 8.8. A Möbius transformation $T(z) = \frac{az+b}{cz+d}$ is such that $T \in \text{Aut}(\mathbb{C}^*)$ with

$$T^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Conversely,

$$\text{Aut}(\mathbb{C}^*) = \left\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}.$$

Definition 8.9. Define a line l in \mathbb{C}^* to be the union of a line in \mathbb{C} with $\{\infty\}$.

Lemma 8.4. Let

$$L = \left\{ z \in \mathbb{C}^* : \alpha z \bar{z} + \beta z + \bar{\beta} \bar{z} + \gamma = 0, \text{ where } \alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C} \text{ and } \beta \bar{\beta} - \alpha \gamma > 0 \right\}.$$

- (i) If $\alpha \neq 0$, then L is a circle;
- (ii) if $\alpha = 0$, then L is a line

Conversely, each line or circle can be expressed as one of the set L for appropriate α, β, γ .

Theorem 8.9. Suppose L is a line or a circle and T is a Möbius transformation. Then, $T(L)$ is a line or a circle.

Note that a Möbius transformation does not necessarily map circles to circles and lines to lines; even if it maps a circle to another circle, it does not necessarily map the first circle's centre to the second circle's centre.

Example 8.10. For $a \in \mathbb{D}$,

$$T_a : \mathbb{D} \rightarrow \mathbb{D} \quad \text{where} \quad T_a(z) = \frac{-z+a}{1-\bar{a}z} \quad \text{and} \quad T_a(0) = a.$$

To see why, note that T is a holomorphic function on $\mathbb{C} \setminus \{1/\bar{a}\}$, so it is defined in a neighbourhood of $\bar{\mathbb{D}}$. For z in the boundary of \mathbb{D} , we have $|z| = 1$ and $z\bar{z} = 1$. It is easy to see that $|T_a(z)| = 1$. By the maximum modulus principle, when $|z| < 1$, we have $|T_a(z)| < 1$. Hence, $T_a(z)$ is a conformal automorphism of \mathbb{D} .

Also, $T_a(0) = a$ is obvious.

Example 8.11.

$$T(z) = i \cdot \frac{z-1}{z+1}$$

maps the real line to the imaginary line and $T(-1) = \infty$.

To see why, let $z = a$, where $a \in \mathbb{R}$. Then,

$$T(z) = \frac{i(a-1)}{a+1},$$

which is purely imaginary. It is also clear that $T(-1) = \infty$.

Example 8.12.

$$T(z) = \frac{i-z}{i+z}$$

maps the real line to the unit circle and $T(\infty) = -1$.

To see why, let $z = a$, where $a \in \mathbb{R}$. It suffices to show that $|T(a)| = 1$, i.e.

$$\left| \frac{i-a}{i+a} \right| = 1.$$

This is obvious.

Example 8.13.

$$T(z) = i \cdot \frac{1-z}{1+z}$$

maps the unit circle to the real line and $T(-1) = \infty$.

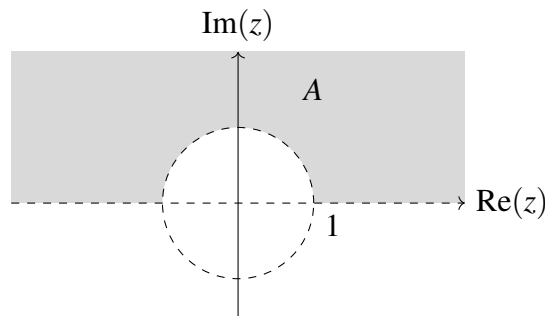
To see why, let $z = e^{i\theta}$. Then, we need to show that $T(z) \in \mathbb{R}$.

$$T(e^{i\theta}) = \frac{i(1 - e^{i\theta})}{1 + e^{i\theta}} = \tan\left(\frac{\theta}{2}\right),$$

which is real. Also, $T(-1)$ can be attained by setting $\theta = (2k+1)\pi$ for $k \in \mathbb{Z}$, which implies $\tan(\theta/2) = \infty$.

Example 8.14. Find a conformal map f from $A = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0, |z| > 1\}$ onto the unit disc.

Solution. The locus A represents the intersection of the upper half-plane and the points exterior to the circle of radius 1 centred at the origin.



Consider the Cayley transform given by $f(z) = (z - i)/(z + i)$. □

Example 8.15. Let

$$A = \left\{ z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im}(z) > \frac{1}{2} \right\} \text{ and } B = \left\{ \frac{2\pi}{3} < \arg(z) < \pi \right\}.$$

Find a conformal map f from A to B .

Solution. We first find the intersection of $|z| = 1$ and $\operatorname{Im}(z) = 1/2$. Consider $x^2 + y^2 = 1$ and $y = 1/2$. Solving yields $x = -\sqrt{3}/2$. Hence, $z = e^{i\pi/6}$ or $z = e^{5\pi i/6}$.

Note that

$$f(z) = \frac{z - e^{i\pi/6}}{z - e^{5\pi i/6}}$$

is an example of such a conformal map.

To see why, we note that it is a Möbius transformation so it sends lines and circles to lines and circles. Note that $f(e^{5\pi i/6}) = \infty$ and $f(e^{i\pi/6}) = 0$. Hence, the boundary of A is sent to the union of two half-lines which form an angle at the origin. For z in the interval joining $e^{5\pi i/6}$ and $e^{i\pi/6}$ (along the line $\operatorname{Im}(z) = 1/2$), note that $z - e^{i\pi/6} \in \mathbb{R}_{<0}$ and $z - e^{5\pi i/6} \in \mathbb{R}_{>0}$, so $f(z) \in \mathbb{R}_{<0}$.

The angle at $e^{i\pi/6}$ between this interval and the rest of the boundary of A forms an angle of $-\pi/3$. Since f is conformal at $e^{i\pi/6}$, we conclude that the boundary of A is sent to the union of $\mathbb{R}_{<0}$ with the half line $e^{2\pi i/3}\mathbb{R}_{>0}$. We deduce that A is sent to B . □

Example 8.16. Find a Möbius transformation mapping the upper half-plane onto the unit disc and mapping a given point z_0 in the upper half-plane to 0.

Solution. Note that T maps the real line to the unit disc. Since z_0 and $\overline{z_0}$ are symmetric about the real axis, then $T(\overline{z_0})$ and $T(z_0) = 0$ are symmetric with respect to the unit circle. Hence, $T(\overline{z_0}) = \infty$. As such,

$$T(z) = \lambda \cdot \frac{z - z_0}{z - \overline{z_0}}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. Since $|T(0)| = 1$, then $|\lambda| = 1$. Hence,

$$T(z) = e^{i\theta} \cdot \frac{z - z_0}{z - \overline{z_0}}$$

for some $\theta \in \mathbb{R}$. □

Example 8.17. Find a Möbius transformation that maps from

$$D = \{z : |z| > 1, |z - 1| < 2\} \text{ to } G = \{w : 0 < \operatorname{Re}(w) < 1\}.$$

Solution. Observe that the region D is bounded by two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 4$. The tangent to these circles is $x = -1$. We consider the conformal map $T(z) = 1/(z + 1)$. Since $T(\mathbb{R}) = \mathbb{R}$ and C_1 and C_2 are perpendicular to \mathbb{R} , it follows that $T(C_1)$ and $T(C_2)$ are perpendicular to \mathbb{R} .

Hence, $T(C_1) = \{z : \operatorname{Re}(z) = 1/2\}$ and $T(C_2) = \{z : \operatorname{Re}(z) = 1/4\}$. So, $T(D)$ is bounded by these lines. Let $S(w) = 4w - 1$. Then, $S \circ T = (3 - z)/(1 - z)$ maps D onto G conformally. □

Example 8.18 (Dinh's 70 problems). Let $T(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

- (i) Assume that $z_1, z_2 \in \mathbb{C}$ are two distinct fixed points for T , i.e. $T(z_j) = z_j$, $j = 1, 2$. Show that there exists a constant λ such that

$$\frac{T(z) - z_1}{T(z) - z_2} = \lambda \cdot \frac{z - z_1}{z - z_2}.$$

- (ii) Let $T^1(z) := T(z)$, $T^{n+1}(z) := T(T^n(z))$, $n = 1, 2, 3, \dots$. Use (i) to find an expression for T^n , $n = 1, 2, 3, \dots$, if

$$T(z) = \frac{1 - 3z}{z - 3}.$$

Solution.

- (i) We have

$$\begin{aligned} \frac{(T(z) - z_1)(z - z_2)}{(T(z) - z_2)(z - z_1)} &= \frac{\left(\frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d}\right)(z - z_2)}{\left(\frac{az+b}{cz+d} - \frac{az_2+b}{cz_2+d}\right)(z - z_1)} \\ &= \frac{((az+b)(cz_1+d) - (az_1+b)(cz+d))(cz_2+d)(z - z_2)}{((az+b)(cz_2+d) - (az_2+b)(cz+d))(cz_1+d)(z - z_1)} \\ &= \frac{cz_2 + d}{cz_1 + d} \end{aligned}$$

$$\text{so } \lambda = \frac{cz_2 + d}{cz_1 + d}.$$

- (ii) We first find the fixed points of T . Set $\frac{-3z+1}{z-3} = z$, so $z = \pm 1$. We can take $z_1 = -1$ and $z_2 = 1$, so by repeatedly applying (i), we have

$$\frac{T^n(z) + 1}{T^n(z) - 1} = \left(\frac{1}{2}\right)^n \cdot \frac{z + 1}{z - 1}.$$

□

8.7. Cross Ratio

Definition 8.10 (cross ratio). The cross ratio of a 4-tuple of points $z_0, z_1, z_2, z_3 \in \mathbb{C}^*$ is defined to be

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

When one of the z_j is ∞ , the RHS is understood as the limit as $z \rightarrow \infty$.

Example 8.19.

$$(\infty, z_1, z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}.$$

Proposition 8.2. A Möbius transformation T preserves cross ratios. That is,

$$(T(z_0), T(z_1), T(z_2), T(z_3)) = (z_0, z_1, z_2, z_3)$$

Lemma 8.5. Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}^*$, let $T(z) = (z, z_1, z_2, z_3)$. Then, T is a Möbius transformation and

$$T(z_1) = 1, T(z_2) = 0 \text{ and } T(z_3) = \infty.$$

In fact, T is the unique Möbius transformation such that the above holds.

Theorem 8.10. Given two sets of three distinct points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$, there exists a unique Möbius transformation T such that $T(z_j) = w_j$ for $j = 1, 2, 3$.

Corollary 8.2. Let z_0, z_1, z_2, z_3 be distinct points in \mathbb{C}^* . Then, they lie in a circle or a line in \mathbb{C}^* if and only if $(z_0, z_1, z_2, z_3) \in \mathbb{R}$.

Example 8.20. Find a Möbius transformation f that maps \mathbb{H} bijectively to the disc $D(0, 2)$ such that $f(i) = 1$ and $f(1) = -2$.

Solution. A Möbius transformation preserves points of symmetry so $f(-i)$ is symmetric to $f(i) = 1$ with respect to $C(0, 2)$. Hence, $f(-i) = 4$. Since the Möbius transformation f preserves cross ratios,

then

$$(f(z), f(1), f(i), f(-i)) = (z, 1, i, -i)$$

$$(f(z), -2, 1, 4) = (z, 1, i, -i)$$

$$\frac{f(z) - 1}{f(z) - 4} \cdot \frac{-6}{-3} = \frac{z - i}{z + i} \cdot \frac{1 + i}{1 - i}$$

Finding $f(z)$ is left as a simple algebraic exercise. Note that $f(-1) = 2$.

□

9. Harmonic Functions

9.1. Basic Properties of Harmonic Functions

Recall that a real-valued function u is defined on a domain $\Omega \subseteq \mathbb{C}$ is harmonic if it belongs to \mathcal{C}^2 (second derivative of f is continuous on Ω) and $\Delta u = 0$. The real and imaginary parts of a holomorphic function are harmonic.

Proposition 9.1. Let Ω be a simply connected domain in \mathbb{C} . A function $u : \Omega \rightarrow \mathbb{R}$ is harmonic if and only if u is the real part of some holomorphic function on Ω .

The above proposition implies that for any domain Ω , u is harmonic if and only if it is locally the real (or imaginary) part of a holomorphic function. In particular, harmonic functions belong to \mathcal{C}^∞ .

Example 9.1. Consider the function

$$u(x, y) = \frac{1}{2} \log(x^2 + y^2)$$

on the annulus $\Omega = \{0 < r < |z| < R\}$. This is not a simply connected domain, which means that not all simple closed curves in Ω can be shrunk to a point while remaining in Ω . One can establish that u is harmonic but there is no holomorphic function on Ω whose real part is equal to u .

Showing that u is harmonic, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is simple

Example 9.2. Prove that the function

$$u(x, y) = \frac{\sin x}{\cos x + \cosh y}$$

is harmonic in

$$\Omega = \{x + iy : -\pi < x < \pi \text{ and } y \in \mathbb{R}\}.$$

Solution. One can see that $\cosh y = \cos(iy)$, so by setting $z = x + iy$, where $-\pi < x < \pi$ and $y \in \mathbb{R}$, it is clear that

$$u(x, y) = \operatorname{Re} \left(\tan \left(\frac{z}{2} \right) \right).$$

□

Example 9.3 (Dinh's 70 problems). Let $f = u + iv$ be a holomorphic function in an open set Ω . Define

$$U := e^{u^2 - v^2} \cos(2uv) \text{ and } V := e^{u^2 - v^2} \sin(2uv).$$

Show that U is harmonic and V is a harmonic conjugate of U .

Solution. To show that U is harmonic, we need to show that it satisfies Laplace's Equation, i.e. $U_{uu} + U_{vv} = 0$. This is trivial. Next, one of the Cauchy-Riemann Equations states that $U_u = V_v$, so

$$V_v = -2e^{u^2-v^2} (v \sin(2uv) - u \cos(2uv)).$$

Using integration by parts or Euler's Formula, it can be shown that $\int V_v dv = V + c$, where c is an arbitrary constant. This shows that V is a harmonic conjugate of U . \square

Theorem 9.1 (maximum-minimum principle). If u is a real-valued non-constant harmonic function on a domain Ω , then u has no local maximum and no local minimum on Ω .

9.2. Dirichlet Problem and Poisson Kernel

Theorem 9.2 (Dirichlet problem). Let Ω be a bounded domain in \mathbb{C} . Given a function $h : \partial\Omega \rightarrow \mathbb{R}$, is there a unique continuous function $u : \overline{\Omega} \rightarrow \mathbb{R}$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = h & \text{on } \partial\Omega? \end{cases}$$

In layman's terms, think of u being harmonic on the interior and $u = h$ on the boundary.

Definition 9.1 (Poisson kernel). Define the Poisson kernel of the unit disc to be

$$P(a, e^{i\theta}) = \frac{1}{2\pi} \cdot \frac{1 - |a|^2}{|e^{i\theta} - a|^2}.$$

We shall consider the case where Ω is the unit disc \mathbb{D} . The following theorem gives the uniqueness of the solution to the Dirichlet problem.

Theorem 9.3. Let $u : \overline{\mathbb{D}} \rightarrow \mathbb{R}$ be a continuous function which is harmonic in \mathbb{D} . Then, for each $a \in \mathbb{D}$,

$$u(a) = \int_0^{2\pi} P(a, e^{i\theta}) u(e^{i\theta}) d\theta.$$

Proof. Consider the automorphism of \mathbb{D} , which is

$$f(z) = \frac{a - z}{1 - \bar{a}z}.$$

Note that $f(0) = a$ and f is self-inverse. Find f' and f'/f , then use Gauss' mean value theorem to prove the result. \square

Corollary 9.1 (Harnack's inequality). Let u be a harmonic function in a neighborhood of $\overline{\mathbb{D}}$. Assume that $u \geq 0$ on $\{|z| = 1\}$. Then,

$$\frac{1 - |z|}{1 + |z|} u(0) \leq u(z) \leq \frac{1 + |z|}{1 - |z|} u(0)$$

for $|z| < 1$.

Proof. Apply the Poisson kernel formula. Consider the region $|z| < 1$ and the identity $1 - |z|^2 = (1 + |z|)(1 - |z|)$. \square

10. Analytic Continuation

10.1. Analytic Continuation

Definition 10.1 (analytic continuation). Let f be a holomorphic function defined on a domain Ω . If there exists a domain $\Omega \subseteq \Omega'$ and a holomorphic function $F : \Omega' \rightarrow \mathbb{C}$ such that $F(z) = f(z)$ for each $z \in \Omega$, then F is an analytic continuation of f on Ω' .

Example 10.1. The power series

$$f(z) = 1 + z + z^2 + \dots$$

has a radius of convergence $R = 1$ and so $f(z)$ is a holomorphic function on the unit disc \mathbb{D} . On the other hand, one can see that

$$f(z) = \frac{1}{1-z} \text{ for } |z| < 1$$

but $g(z) = 1/(1-z)$ is holomorphic on $\mathbb{C} \setminus \{1\}$. Thus, g is an analytic continuation of f to the much bigger domain $\mathbb{C} \setminus \{1\}$.

Lemma 10.1. Let $\Omega \subseteq \Omega'$ be domains in \mathbb{C} . Let F_1 and F_2 be analytic continuations of a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ to a domain Ω' . Then,

$$F_1(z) = F_2(z) \quad \text{for all } z \in \Omega'.$$

Lemma 10.2. Let $f_j : \Omega_j \rightarrow \mathbb{C}$ be holomorphic functions such that $f_1(z) = f_2(z)$ for $z \in \Omega_1 \cap \Omega_2$. Then,

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in \Omega_1; \\ f_2(z) & \text{if } z \in \Omega_2 \setminus \Omega_1. \end{cases}$$

10.2. Schwarz Reflection Principle

We say that a region Ω is symmetric with respect to the real axis if $z \in \Omega$ implies $\bar{z} \in \Omega$. We consider here an important particular case of analytic continuation.

Theorem 10.1 (reflection principle for holomorphic functions). Define Ω^+, Ω^-, L as the intersections of Ω with the upper half-plane, lower half-plane, and the real axis respectively. If f is a continuous complex-valued function on $\Omega^+ \cup L$, which is analytic on Ω^+ and real on L , then

f admits a unique extension to a holomorphic function F on Ω .

Moreover, the extension is given by

$$F(z) = \begin{cases} f(z) & \text{for } z \in \Omega^+ \cup L; \\ \overline{f(\bar{z})} & \text{for } z \in \Omega^-. \end{cases}$$

In particular, $F(\bar{z}) = \overline{F(z)}$ for all $z \in \Omega$.

Example 10.2 (MA5217 Lecture Notes). Suppose f is holomorphic on \mathbb{H} and continuous on $S = \mathbb{H} \cup (0, 1)$. Assume $f(x) = x^4 - 2x^2$ for all $x \in (0, 1)$. Find $f(i)$.

Solution. We have $f(i) = i^4 - 2i^2 = 1 + 2 = 3$. □