MA2108 and MA3210 Mathematical Analysis I and II Notes

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Reference books:

(1) Rudin, W. (1953). '*Principles of Mathematical Analysis*'. McGraw Hill.

(2) Johnsonbaugh, R. and Pfaffenberger, W. E. (2010). '*Foundations of Mathematical Analysis*'. Dover Publications.

(3) Munkres, J. (2000). '*Topology 2nd Edition*'. Pearson.

Contents

1. The Real Numbers, $\mathbb{R}$

1.1. *Completeness, Supremum and Infimum*

We say that $\mathbb{R}$ is a complete ordered field. There are three big ideas to be discussed — completeness, ordering, and fields! We will first discuss the property on fields, and we say that $\mathbb{R}$ satisfies the field axioms (Definition 1.1)[†]. There are many properties which might be deemed *trivial* but we will still discuss them. For example, the trichotomy property of $\mathbb{R}$[‡] states that

$$\text{if } a,b\in\mathbb{R} \quad \text{then} \quad \text{either } a<b, a>b \text{ or } a=b.$$

This is intuitive!

Definition 1.1 (field axioms). A field consists of a set F satisfying the following properties:

(i) an additive map

$$+ : F\times F\to F \quad \text{where} \quad (x,y)\mapsto x+y$$

(ii) the existence of an additive identity $0\in F$

(iii) a negation map

$$- : F\times F\to F \quad \text{where} \quad x\mapsto -x$$

(iv) a multiplication map

$$\cdot : F\times F\to F \quad \text{where} \quad (x,y)\mapsto xy$$

(v) the existence of a multiplicative identity $1\in F$

(vi) a reciprocal map

$$(-)^{-1} : F\backslash\{0\}\to F\backslash\{0\} \quad \text{where} \quad x\mapsto x^{-1}$$

such that the following properties are satisfied:

(i) $+$ is commutative, i.e. for all $x,y\in F$, we have $x+y=y+x$

(ii) $+$ is associative, i.e. for all $x,y,z\in F$, we have $(x+y)+z=x+(y+z)$

(iii) 0 is the identity for $+$, i.e. for all $x\in F$, we have $x+0=x=0+x$

(iv) $-$ is the additive inverse of addition, i.e. for all $x\in F$, we have $x+(-x)=0=(-x)+x$

(v) $\cdot$ is commutative, i.e. for all $x,y\in F$, we have $xy=yx$

(vi) $\cdot$ is associative, i.e. for all $x,y,z\in F$, we have $(xy)z=x(yz)$

(vii) 1 is the identity for $\cdot$, i.e. for all $x\in F$, we have $x1=x=1x$

(viii) $(-1)^{-1}$ is the inverse of $\cdot$, i.e. for all $x\in F$, we have $xx^{-1}=1=x^{-1}x$

(ix) $1\neq 0$, i.e. F is not the zero (trivial) field

(x) $\cdot$ is distributive over $+$, i.e. for all $x,y,z\in F$, we have

$$x(y+z)=xy+xz \quad \text{and} \quad (x+y)z=xz+yz$$

[†] An abrupt introduction.

[‡] In fact, we can regard the trichotomy property of $\mathbb{R}$ as a combination of the reflexivity and antisymmetry properties in Definition 1.2 and the comparability property in Definition 1.3. Alternatively, one can refer to **(iii)** of Proposition 1.5.

Remark 1.1. When we were discussing the properties of a field in Definition 1.1, recall that multiplication is denoted by $\cdot$, and we can *condense* $x \cdot y$ as xy. For example, refer to **(v)**, which can also be written as $x \cdot y = y \cdot x$.

Example 1.1. The best known fields are those of

$$\begin{aligned}\mathbb{Q} &= \text{field of rational numbers}\\ \mathbb{R} &= \text{field of real numbers}\\ \mathbb{C} &= \text{field of complex numbers}\end{aligned}$$

Example 1.2. In Number Theory or Abstract Algebra in general,

$$\begin{aligned}\mathbb{Q}_p &= \text{field of } p\text{-adic numbers}\\ \mathbb{F}_p &= \text{finite field of } p \text{ elements}\end{aligned}$$

Example 1.3. Let k be a field. Then, define

$$K(t) \quad \text{to be the field of rational functions over } K.$$

We then discuss the general properties of fields.

Proposition 1.1. The axioms for addition in Definition 1.1 imply the following statements: for all $x, y, z \in F$,

(i) Cancellation for $+$: if $x + y = x + z$, then $y = z$;
(ii) Uniqueness of 0: if $x + y = x$, then $y = 0$;
(iii) Uniqueness of negative: if $x + y = 0$, then $y = -x$;
(iv) Negative of negative: $-(-x) = x$

We will only prove **(i)** and **(iv)**.

Proof. First, we prove **(i)**. Suppose $x, y, z \in F$ such that $x + y = x + z$. Then, as $-x \in F$, we have

$$\begin{aligned}((-x)) + x + y &= ((-x)) + x + z\\ ((-x) + x) + y &= ((-x) + x) + z \quad \text{by associativity of } +\\ 0 + y &= 0 + z \quad \text{since 0 is the additive identity in } F\end{aligned}$$

and we conclude that $y = z$. $\square$

We then prove **(iv)**.

Proof. Recall that $x + (-x) = 0$. The trick now is to consider

$$-(-x) + (-x) = 0 \quad \text{which again follows by the axiom for negation!}$$

As such,

$$\begin{aligned}x + (-x) &= -(-x) + (-x)\\ x &= -(-x) \quad \text{by the cancellation property in } \textbf{(i)}\end{aligned}$$

so **(iv)** holds. $\square$

Proposition 1.2. The axioms for multiplication in Definition 1.1 imply the following statements: for all $x, y, z \in F$,

- **(i) Cancellation for $\cdot$:** if $x \neq 0$ and $xy = xz$, then $y = z$;
- **(ii) Uniqueness of multiplicative identity:** if $x \neq 0$ and $xy = x$, then $y = 1$;
- **(iii) Uniqueness of reciprocal:** if $x \neq 0$ and $xy = 1$, then $y = 1/x$;
- **(iv) Reciprocal of reciprocal:** if $x \neq 0$, then $1/(1/x) = x$

Proposition 1.3. The field axioms (Definition 1.1) imply the following statements: for all $x, y \in F$,

- **(i)** $0x = 0$;
- **(ii)** if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$
- **(iii)** $(-x)y = -(xy) = x(-y)$
- **(iv)** $(-x)(-y) = xy$

We now discuss what it means for a set to be ordered.

Definition 1.2 (partial order). Let S be a set. A partial ordering relation on S is a relation $\leq$ on S satisfying the following properties:

- **(i) Reflexivity:** for all $x \in S$, we have $x \leq x$
- **(ii) Transitivity:** for all $x, y, z \in S$, we have $x \leq y$ and $y \leq z$ imply $x \leq z$
- **(iii) Antisymmetry:** for all $x, y \in S$, we have $x \leq y$ and $y \leq x$ implies $x = y$

Definition 1.3 (total order). A total ordering relation on S is partial ordering relation $\leq$ (Definition 1.2) on S which also satisies the following property that $\leq$ is comparable:

$$\text{for all } x, y \in S \quad \text{we have} \quad x \leq y \text{ or } y \leq x.$$

Example 1.4. Let S be a set. Then, the subset relation $\subseteq$ on $\mathcal{P}(S)$ is a partial ordering but not a total ordering when $|S| > 1$.

Definition 1.4 (ordered field). An ordered field consists of a field F and a total ordering $\leq$ on F saitsyfing the following properties:

- **(i) $\leq$ is compatible with $+$:** for all $x, y, z \in F$, we have

$$x \leq y \quad \text{implies} \quad x + z \leq y + z$$

- **(ii) $\leq$ is compatible with $\cdot$:** for all $x, y, z \in F$, we have

$$x \leq y \text{ and } z > 0 \quad \text{implies} \quad xz \leq yz$$

Definition 1.5. If

$$x > 0 \text{ we call } x \text{ positive} \quad \text{and if} \quad x < 0 \text{ we call } x \text{ negative} \quad \text{and if}$$
$$x \geq 0 \text{ we call } x \text{ non-negative} \quad \text{and if} \quad x \leq 0 \text{ we call } x \text{ non-positive}$$

Definition 1.6. We have

$$\begin{aligned}
F_{>0} &= \{x \in F : x > 0\} \\
F_{<0} &= \{x \in F : x < 0\} \\
F_{\geq 0} &= \{x \in F : x \geq 0\} = F_{>0} \cup \{0\} \\
F_{\leq 0} &= \{x \in F : x \leq 0\} = F_{<0} \cup \{0\}
\end{aligned}$$

Example 1.5. $\mathbb{Q}$ given with the usual ordering $\leq$ is an ordered field. We will eventually construct $\mathbb{R}$ as an ordered field.

Proposition 1.4. Let F be an ordered field. Then,

$$\text{for all } x, y \in F \quad \text{we have} \quad x \leq y \text{ if and only if } -x \geq -y.$$

In particular, $F_{<0} = -F_{>0}$ and $F_{\leq 0} = -F_{\geq 0}$.

Proof. We first prove the forward direction. If $x \leq y$, we take $z = (-x) + (-y)$ in F. As such, $x + z \leq y + z$, which implies $-y \leq -x$. For the reverse direction, we apply the same idea to $(x, y) = (-y, -x)$ to obtain $-(-x) \leq -(-y)$. As such, $x \leq y$. □

Proposition 1.5 (closure properties and trichotomy)**.** For any ordered field F,

(i) $F_{>0}$ **is closed under addition:** $F_{>0} + F_{>0} \subseteq F_{>0}$

(ii) $F_{>0}$ **is closed under multiplication:** $F_{>0} \cdot F_{>0} \subseteq F_{>0}$

(iii) Trichotomy: $F = F_{>0} \sqcup \{0\} \cup (-F_{>0})$

Proposition 1.6. For any ordered field F, the following hold:

(i) for all $x \in F$, we have $x^2 \geq 0$

(ii) for all $x, y \in F$ such that $0 < x < y$, we have $0 < 1/y < 1/x$

Proof. We first prove **(i)**. Suppose $x \geq 0$. Then, $x^2 = x \cdot x \geq 0 \cdot x = 0$ by the compatibility of $\leq$ with $\cdot$ (recall **(ii)** of Definition 1.4). If $x < 0$, then $-x > 0$, so $x^2 = (-x)(-x) > 0 \cdot (-x) = 0$ again by **(ii)** of Definition 1.4.

We then prove **(ii)**. Suppose $x > 0$. If $x^{-1} \leq 0$, then $0 = x \cdot 0 \geq x \cdot x^{-1} = 1$, which is a contradiction. As such, we must have $x^{-1} > 0$. If $0 < x < y$, then $xy > 0$ since $F_{>0}$ is closed under multiplication (**(ii)** of Proposition 1.5). As such, $(xy)^{-1} > 0$. Hence,

$$0 < y^{-1} = x \cdot (xy)^{-1} < y \cdot (xy)^{-1} = x^{-1}$$

by the compatibility of $\leq$ with $\cdot$ as mentioned in **(ii)** of Definition 1.4. □

Proposition 1.7 (field characteristic)**.** Let F be an ordered field. Then,

$$\text{for all } n \in \mathbb{N} \quad \text{we have} \quad n \cdot 1 = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ terms}} \quad \text{in } F.$$

In Abstract Algebra, we say that ordered fields have characteristic zero.

Proof. We shall induct on n. The base case $n = 1$ is trivial as $1 > 0$ in F. Next, for any $n \in \mathbb{N}$, if $n \cdot 1 > 0$ in F, then $(n+1) \cdot 1 = n \cdot 1 + 1 > 0$ because $n \cdot 1 > 0$ by the inductive hypothesis and $1 > 0$ trivially. As such, the proposition holds. □

For those who are interested in Abstract Algebra, Definition 1.7 would appeal to you.

Definition 1.7. Let F be an ordered field. Then, F is of characteristic zero. Also, there exists a unique *homomorphism* of fields

$$\iota : \mathbb{Q} \hookrightarrow F \quad \text{called} \quad \text{the canonical inclusion of } \mathbb{Q} \text{ into } F.$$

Moreover, ι is injective and order-preserving.

Via the canonical inclusion $\iota : \mathbb{Q} \hookrightarrow F$ of $\mathbb{Q}$ into F, we will identify $\mathbb{Q}$ with $\iota(\mathbb{Q}) \subseteq F$ and regard $\mathbb{Q}$ as a subfield of F. All these will be covered in MA3201.

Remark 1.2. It follows that ordered fields must be infinite. Also, ordered fields cannot be algebraically closed. To see why, we note that $x^2 + 1 = 0$ has no solution in the ordered field F.

Definition 1.8 (upper and lower bound)**.** Let S be an ordered set, i.e. a set given with a total ordering. We say that a subset $R \subseteq S$ is

$$\begin{aligned} &\text{bounded above} && \text{if and only if} && \text{there exists } B \in S \text{ such that for all } x \in E && \text{we have } x \leq B \\ &\text{bounded below} && \text{if and only if} && \text{there exists } A \in S \text{ such that for all } x \in E && \text{we have } A \leq x \\ &\text{bounded} && \text{if and only if} && \text{it is bounded above and bounded below} \end{aligned}$$

We say that

$$\begin{aligned} A \in S &\quad \text{is a lower bound of } E \text{ in } S \\ B \in S &\quad \text{is an upper bound of } E \text{ in } S \end{aligned}$$

Definition 1.9 (supremum and infimum)**.** Let S be an ordered set and $E \subseteq S$ be any subset. A real number α is the supremum (least upper bound or LUB) of E if

$$\alpha \text{ is an upper bound of } E \quad \text{and} \quad \alpha \leq u \text{ for every upper bound } u \in E, \text{ i.e. } \alpha = \sup(E).$$

A real number β is the infimum (greatest lower bound or GLB) of E if

$$\beta \text{ is a lower bound of } E \quad \text{and} \quad \beta \geq u \text{ for every lower bound } u \in E, \text{ i.e. } \beta = \inf(E).$$

Proposition 1.8. For an ordered set S, let $E \subseteq S$. Then, the set of upper bounds of E in S is always a subset of S. However, it may be empty. We remark that

$$\text{the set of upper bounds} = \emptyset \quad \text{if and only if} \quad E \text{ is not bounded above in } S.$$

Example 1.6. Take $S = \mathbb{Q}$ and $E = \mathbb{Z}$. Then, $E \subseteq S$, and we note that the set of upper bounds of $\mathbb{Z}$ in $\mathbb{Q}$ is $\emptyset$ as the $\sup(\mathbb{Z})$ does not exist.

Remark 1.3. The supremum and infimum of a set may or may not be elements of the set.

Example 1.7. Consider

$$E = \{x \in \mathbb{R} : 0 < x < 1\} \quad \text{where } \inf(E) = 0 \notin E \text{ and } \sup(E) = 1 \notin E.$$

Lemma 1.1 (supremum is unique)**.** Let S be an ordered set. Given $E \subseteq S$,

$$\text{if there exists a least upper bound of } E \text{ in } S \quad \text{then} \quad \sup(E) \text{ is unique.}$$

As mentioned, we write $\sup(E) \in S$ for the unique least upper bound of E in S if it exists.

Proof. The proof is very straightforward. Suppose both α and α' are least upper bounds of E in S. Then, one can show that $\alpha \le \alpha'$ and $\alpha' \le \alpha$ by using the two conditions mentioned in Definition 1.9. □

At this juncture, we note that a number of properties of the infimum, or greatest lower bound of a set, have not been discussed. These draw parallelisms with the definition of the supremum (both in Definition 1.9).

Definition 1.10 (least upper bound property)**.** An ordered set S has the least upper bound property if and only if for any non-empty subset $E \subseteq S$ which is bounded above, there exists a least upper bound $\sup(E) \in S$ of E in S.

Example 1.8. Let S_0 be an ordered set[†], and let $S \subseteq S_0$ be any finite subset. Then, S, which is regarded as an ordered set, has the least upper bound property (Definition 1.10). In fact, for any non-empty subset $E \subseteq S$ (which is necessarily finite since any subset of a finite set is also finite),

$$\sup(E) = \max(E) \quad \text{exists in } S \text{ (in fact in } E).$$

Lemma 1.2. $\mathbb{Z}$, as an ordered set, has the least upper bound property.

Proof. Suppose $E \subseteq \mathbb{Z}$ is any non-empty subset which is bounded above by $b_0 \in \mathbb{Z}$. Then, the set

$$b_0 \setminus E = \{b_0 - x \in \mathbb{Z} : x \in E\} = \{k \in \mathbb{Z} : b_0 - k \in E\} \quad \text{is a non-empty subset of } \mathbb{Z}_{\ge 0}.$$

Note that $b_0 \setminus E$ is indeed non-empty as $E \neq \emptyset$. By the well-ordering property of $\mathbb{Z}_{\ge 0}$, there exists a smallest element $k_0 \in b_0 \setminus E$. As such,

$$\sup(E) = \max(E) = b_0 - k_0 \quad \text{exists in } S.$$

□

Example 1.9 ($\mathbb{Q}$ does not have the least upper bound property)**.** One would know that the equation

$$p^2 = 2 \quad \text{is not satisfied by any } p \in \mathbb{Q}.$$

[†]If you are unable to appreciate this example well, always make reference to sets, or number systems, that you already know which would be applicable here. For example, we can take $S_0 = \mathbb{Q}$. Consequently as we would see later, $S \subseteq S_0$ is a finite subset of the rationals. Suppose $S = \{-1/2, 3, 10/7\}$ and $E = \{-1/2, 10/7\}$. Then, $\sup(E)$ exists and it is equal to $\max(E) = 10/7$.

This shows that $\sqrt{2}$ is irrational, and consequently, $\mathbb{Q}$ does not have the least upper bound property. Anyway, the proof using the unique factorisation of $\mathbb{Z}$ is as follows:

$$p = \frac{a}{b} \quad \text{for some } a, b \in \mathbb{Z} \text{ and } b \neq 0.$$

Then, consider the prime factorisations of a and b to obtain

$$p = \frac{p_1^{\alpha_1} \dots p_r^{\alpha_r}}{q_1^{\beta_1} \dots q_s^{\beta_s}} \quad \text{so} \quad p_1^{2\alpha_1} \dots p_r^{\alpha_r} = 2 \cdot q_1^{\beta_1} \dots q_s^{\beta_s}.$$

The exponent of 2 on the LHS is even but it is odd on the RHS, resulting in a contradiction.

Now, let

$$A = \left\{ p \in \mathbb{Q}^+ : p^2 < 2 \right\}.$$

Note that A is non-empty and bounded above in $\mathbb{Q}$ since $1 \in A$ and for all $p \in A$, we have $p > 0$ and $p^2 < 2$, so we must have $p < 2$. We shall prove that A contains no largest number. More explicitly,

$$\text{for every } p \in A \quad \text{there exists } q \in A \text{ such that } p < q.$$

Now, for every $p \in A$, we construct q as follows:

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \in \mathbb{Q}^+.$$

Also,

$$q^2 - 2 = \left(\frac{2p + 2}{p + 2} \right)^2 - 2 = \frac{2\left(p^2 - 2\right)}{(p + 2)^2}.$$

Since $p \in A$, then $p^2 - 2 < 0$, so $q > p$ and $q^2 - 2 < 0$. Hence, $q \in A$ and is $> p$. As such, A contains no largest number. Anyway, here is a geometrical interpretation of the relationship between p and q (Figure 1). By constructing the line segment joining $\left(p, p^2 - 2\right)$ and $(2, 2)$ and defining $(q, 0)$ to be the point where this line intersects the x-axis, one can indeed deduce that

$$q = p - \frac{p^2 - 2}{p + 2}.$$

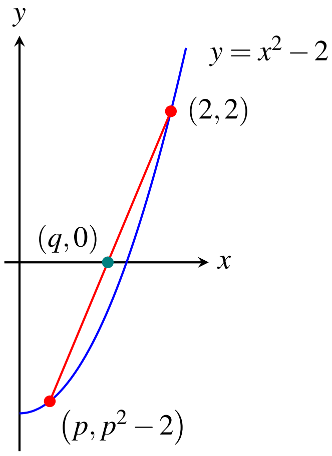

Figure 1: Graph of $y = x^2 - 2$

Lemma 1.3. Let S be an ordered set, and suppose $E \subseteq S$. Let u be an upper bound of E. Then,

$$u = \sup(E) \quad \text{if and only if} \quad \text{for all } \varepsilon > 0 \text{ there exists } x \in E \text{ such that } u - \varepsilon < x.$$

We refer to Figure 2 for an illustration of Lemma 1.3.

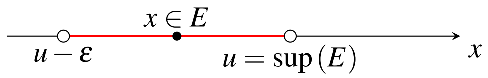

Figure 2: Illustration of the supremum condition in Lemma 1.3

Axiom 1.1 (completeness axiom). Every non-empty subset of $\mathbb{R}$ which is

$$\begin{matrix} \text{bounded above} \\ \text{bounded below} \end{matrix} \text{ has a } \begin{matrix} \text{supremum;} \\ \text{infimum.} \end{matrix}$$

Example 1.10 (MA2108 AY19/20 Sem 1 Tutorial 1). Let $a, b \in \mathbb{R}$. Show that

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|) \quad \text{and} \quad \min(a,b) = \frac{1}{2}(a + b - |a - b|).$$

Solution. We consider two cases, namely $a \geq b$ and $a < b$. If $a \geq b$, then $a - b \geq 0$, then

$$\frac{1}{2}(a + b + |a - b|) = \frac{1}{2}(a + b + a - b) = a = \max(a,b).$$

Similarly,

$$\frac{1}{2}(a + b - |a - b| = \frac{1}{2}(a + b - (a - b) = b = \min(a,b).$$

The case where $a < b$ has similar working. $\square$

Proposition 1.9 (Archimedean property). For any $x, y \in \mathbb{R}$, where $x > 0$,

$$\text{there exists } n \in \mathbb{N} \quad \text{such that} \quad nx > y.$$

The real numbers satisfy the *completeness axiom*[†].

[†]Definition 1.11 is rather intuitive and simple. In fact, this was coined by Dedekind.

Definition 1.11 (completeness of $\mathbb{R}$). There are no gaps or missing points in $\mathbb{R}$.

Corollary 1.1. $\mathbb{N}$ is not bounded above.

Proof. For any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$. This is justified by setting $x = \varepsilon$ and $y = 1$. □

Example 1.11 (MA2108 AY19/20 Sem 1 Tutorial 1). Let

$$S = \left\{ \frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N} \right\}.$$

Prove that $\sup S = 1$ and $\inf S = -1$.

Solution. Note that

$$\frac{1}{n} - \frac{1}{m} < \frac{1}{n}.$$

Since $n_{\min} = 1$, then 1 is an upper bound for S. Fixing $n = 1$, consider the following sequence of numbers:

$$1 - \frac{1}{1},\ 1 - \frac{1}{2},\ 1 - \frac{1}{3}, \ldots,\ 1 - \frac{1}{m}$$

For large m, it appears that the sequence tends to 1, so $\sup S = 1$. To justify this, by the Archimedean property (Proposition 1.9), for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \varepsilon \quad \text{so} \quad 1 - \frac{1}{m} > 1 - \varepsilon.$$

As such, $\sup S = 1$.

Next, note that

$$\frac{1}{n} - \frac{1}{m} > \frac{1}{n} - 1 > -1,$$

which implies that -1 is a lower bound for S. Consider the following sequence of numbers:

$$\frac{1}{1} - 1,\ \frac{1}{2} - 1,\ \frac{1}{3} - 1, \ldots,\ \frac{1}{n} - 1$$

For large n, it appears that the sequence tends to -1, which asserts that $\inf S = -1$. Again, to justify this, by the Archimedean property (Proposition 1.9), for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon \quad \text{so} \quad \frac{1}{n} - 1 < \varepsilon - 1.$$

This asserts that $\inf S = -1$. □

Definition 1.12. Let $a > 0$ and $n \in \mathbb{N}$. There exists a unique positive real number u such that $u^n = a$. The number u is known as the positive n^{th} root of a and thus,

$$u = \sqrt[n]{a} = a^{1/n}.$$

Theorem 1.1. If n is non-square, then $\sqrt{n}$ is irrational.

Proof. Suppose on the contrary that $\sqrt{n}$ is rational, where n is non-square. Then,

$$\sqrt{n} = p/q \text{ implies } nq^2 = p^2 \quad \text{where } p, q \in \mathbb{N}, q \neq 0 \text{ and } \gcd(p, q) = 1.$$

We consider the prime factorisations of p^2 and q^2, each one of them having an even number of primes. Thus, n must also have an even number of primes. As n is non-square, there exists at least a prime with an odd multiplicity, which is a contradiction. □

Theorem 1.2 (density theorem). The rational numbers are dense in $\mathbb{R}$, i.e.

$$\text{if } a,b\in\mathbb{R} \text{ such that } a<b \quad \text{then} \quad \text{there exists } r\in\mathbb{Q} \text{ such that } a<r<b.$$

In short, we are always able to find another rational number that lies between two real numbers.

Corollary 1.2. The irrational are dense in $\mathbb{R}$, i.e.

$$\text{if } a,b\in\mathbb{R} \text{ such that } a<b \quad \text{then} \quad \text{there exists } x\in\mathbb{Q}' \text{ such that } a<x<b.$$

Theorem 1.3. Every non-empty interval $I\subseteq\mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

1.2. *Important Inequalities*

Bernoulli's inequality (named after Jacob Bernoulli) is an inequality that approximates exponentiations of $1+x$. We discuss a widely-used version of this result.

Theorem 1.4 (Bernoulli's inequality). For every $r\in\mathbb{Z}_{\geq 0}$ and $x\geq -1$, we have

$$(1+x)^n \geq 1+nx.$$

The inequality is strict if $x\neq 0$ and $r\geq 2$.

One can use induction to prove Theorem 1.4.

Example 1.12 (MA2108 AY19/20 Sem 1 Tutorial 1). Use Bernoulli's inequality to deduce that for any integer $n>1$, the following hold:

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{1}{n} \quad \text{and} \quad \left(1+\frac{1}{n-1}\right)^{n-1} < \left(1+\frac{1}{n}\right)^n$$

Solution. The first result is obvious by setting $x=-1/n^2$ in Theorem 1.4. For the second result, we wish to prove that

$$\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^{n-1}} > 1.$$

Using some algebraic manipulation, we have

$$1+\frac{1}{n-1} = \frac{n}{n-1} = \frac{1}{1-\frac{1}{n}}.$$

Hence,

$$\begin{aligned}
\text{LHS} &= \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{n-1} \\
&= \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^{-1} \\
&= \left(1-\frac{1}{n^2}\right)^n \left(1-\frac{1}{n}\right)^{-1} \\
&> 1
\end{aligned}$$

where the inequality follows from the first result. □

Theorem 1.5 (QM-AM-GM-HM inequality). Let $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. Let

$$\begin{aligned} Q(n) &= \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2} && \text{denote} && \text{the quadratic mean} \\ A(n) &= \frac{1}{n}\sum_{i=1}^{n} x_i && \text{denote} && \text{the arithmetic mean} \\ G(n) &= \sqrt[n]{\prod_{i=1}^{n} x_i} && \text{denote} && \text{the geometric mean} \\ H(n) &= n\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} && \text{denote} && \text{the harmonic mean} \end{aligned}$$

Then, $Q(n) \geq A(n) \geq G(n) \geq H(n)$. Equality is attained if and only if $x_1 = \ldots = x_n$.

Remark 1.4. The quadratic mean $Q(n)$ is also referred to as root mean square or RMS.

We first prove that $Q(n) \geq A(n)$.

Proof. By the Cauchy-Schwarz inequality,

$$n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2 \quad \text{which implies} \quad \frac{n[Q(n)]^2}{n} \geq [nA(n)]^2.$$

With some simple rearrangement, the result follows. □

Example 1.13 (MA2108S AY16/17 Sem 2 Homework 5). For each $n \in \mathbb{Z}^+$, let

$$a_n = \left(1+\frac{1}{n}\right)^n \quad \text{and} \quad b_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

(a) Show that a_n is strictly monotonically increasing.
(b) Show that b_n is strictly monotonically decreasing.
Hint: Use the GM-HM Inequality.
(c) Show that for each $n \in \mathbb{Z}^+$, one has $a_n < b_n$.

Solution.

(a) A special form of the AM-GM inequality states that

$$\frac{x+ny}{n+1} \geq (xy^n)^{1/(n+1)}.$$

Setting $x = 1$ and $y = 1 + 1/n$, we have

$$1+\frac{1}{n+1} > \left(1+\frac{1}{n}\right)^{n/(n+1)} \quad \text{so} \quad \left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^n$$

which shows that $a_{n+1} > a_n$. Note that the inequality is strict since $x \neq y$.

(b) Similar to **(a)**.

(c) Let $(1+1/n) = u$. Then, $b_n - a_n = u^{n+1} - u^n = u^n/n$, which is positive. □

In fact, both sequences a_n and b_n in Example 1.13 converge in $\mathbb{R}$ and to the same limit. Justifying this requires the use of the monotone convergence theorem (Theorem 2.8) which will be covered in due course. The real number which is the common limit of the sequences a_n and b_n is called Euler's number[†] and it is denoted by e.

[†]Not to be confused with Euler's constant as this typically denotes the Euler-Mascheroni constant γ.

We now return to the proof of the QM-AM-GM-HM inequality (Theorem 1.5). There are numerous proofs of the AM-GM inequality like using backward-forward induction (Cauchy), considering e^x (Pólya), Lagrange Multipliers (MA2104) etc. This proof hinges on Jensen's inequality.

Theorem 1.6 (Jensen's inequality)**.** For a concave function $f(x)$,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \le f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Proof. Consider the logarithmic function $f(x) = \ln x$, where $x \in \mathbb{R}^+$. It can be easily verified that $f(x)$ is concave as $f''(x) = -1/x^2 < 0$ (this is a simple exercise using knowledge from MA2002). We wish to prove

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right).$$

Using Jensen's inequality (Theorem 1.6),

$$\frac{1}{n}\sum_{i=1}^{n} \ln(x_i) \le \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Note that

$$\sum_{i=1}^{n} \ln(x_i) = \ln(x_1) + \ln(x_2) + \ldots + \ln(x_n) = \ln\left(\prod_{i=1}^{n} x_i\right).$$

As such, the inequality becomes

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1}{n}\ln\left(\prod_{i=1}^{n} x_i\right).$$

With some simple rearrangement, the AM-GM inequality follows. □

Lastly, we will prove the GM-HM Inequality using the AM-GM Inequality.

Proof. Note that

$$\prod_{i=1}^{n} \frac{1}{x_i} = \left(\prod_{i=1}^{n} x_i\right)^{-1},$$

so we have

$$\frac{n/H(n)}{n} \ge \frac{1}{G(n)}.$$

Upon rearranging, we are done. □

Theorem 1.7 (triangle inequality)**.** For $x, y \in \mathbb{R}$, $|x+y| \le |x| + |y|$ and equality is attained if and only if $xy \ge 0$.

Corollary 1.3. The following hold:

(i) $|x-y| \le |x| + |y|$

(ii) Reverse triangle inequality: $||x| - |y|| \le |x-y|$

Proof. **(i)** can be easily proven by replacing $-y$ with y. We now prove **(ii)**. Write x as $x-y+y$ and y as $y-x+x$. Hence,

$$\begin{aligned}|x| &= |x-y+y| \leq |x-y| + |y| \\ |y| &= |y-x+x| \leq |y-x| + |x| = |x-y| + |x|\end{aligned}$$

As such, $|x|-|y| \leq |x-y|$ and $|x|-|y| \leq -|y-x|$, and taking the absolute value of $|x|-|y|$, the result follows. ☐

Corollary 1.4 (generalised triangle inequality). For $x_1, \ldots, x_n \in \mathbb{R}$,

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|.$$

Proof. Repeatedly apply the triangle inequality (Theorem 1.7). ☐

Example 1.14 (MA2108 AY19/20 Sem 1 Tutorial 1). Prove that if $x, y \in \mathbb{R}$, $y \neq 0$ and $|x| \leq \frac{|y|}{2}$, then

$$\frac{|x|}{|x-y|} \leq 1.$$

Solution. We wish to prove that $|x| \leq |x-y|$. Using the given inequality, we apply the triangle inequality, so

$$|x| \leq |y|/2 = \frac{|y+x-x|}{2} \leq \frac{|y-x|+|x|}{2}.$$

The result follows with some simple rearrangement and using the property that $|x-y| = |y-x|$. ☐

Example 1.15 (MA2108S AY16/17 Sem 2 Homework 5; Chebyshev's sum inequality). Let $n \in \mathbb{N}$. Show that for any elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}$ with $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$, one has Chebyshev's inequality, i.e.

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a_i b_i.$$

Solution. For any $1 \leq i, j \leq n$, we have

$$\begin{aligned}(a_i - a_j)(b_i - b_j) &\geq 0 \\ a_i b_i + a_j b_j &\geq a_i b_j + a_j b_i\end{aligned}$$

Taking the double sum over all i and j on both sides,

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_i + a_j b_j &\geq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j + a_j b_i \\ \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_i + \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_j &\geq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j + \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_i \\ n\sum_{i=1}^{n} a_i b_i + n\sum_{j=1}^{n} a_j b_j &\geq \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j + \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_j \\ 2n\sum_{i=1}^{n} a_i b_i &\geq 2\left(\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j\right) \\ \sum_{i=1}^{n} a_i b_i &\geq \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j\end{aligned}$$

Changing the right sum of b_j's to run from $i=1$ to $i=n$, and dividing both sides by n, the result follows. ☐

Example 1.16 (MA2108S AY16/17 Sem 2 Homework 5; Hölder's inequality)**.** Let $n \in \mathbb{N}$. Show that for any $a_1, \ldots, a_n$ in $\mathbb{R}$ with $a_i \geq 0$ for each $1 \leq i \leq n$ and for any $p \in \mathbb{N}$, one has the inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} a_i^p.$$

Solution. We use Hölder's inequality, which states that for $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}^+$ and $p, q > 1$ such that $1/p + 1/q = 1$,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

Set $q = p/(p-1)$ so

$$\begin{aligned}
\sum_{i=1}^{n} a_i b_i &\leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^{p/(p-1)}\right)^{(p-1)/p} \\
\left(\sum_{i=1}^{n} a_i b_i\right)^p &\leq \left(\sum_{i=1}^{n} a_i^p\right) \left(\sum_{i=1}^{n} b_i^{p/(p-1)}\right)^{p-1}
\end{aligned}$$

We can set $b_i = 1$ for all $1 \leq i \leq n$ so the inequality becomes

$$\left(\sum_{i=1}^{n} a_i\right)^p \leq \left(\sum_{i=1}^{n} a_i^p\right) n^{p-1}$$

and with some simple algebraic manipulation, the result follows. $\square$

2. Sequences

2.1. *Limit of a Sequence*

Definition 2.1 (sequence)**.** A sequence in $\mathbb{R}$ is

$$\text{a real-valued function } X \text{ with domain } \mathbb{N} \quad \text{i.e. } X : \mathbb{N} \to \mathbb{R}.$$

The sequence is usually denoted by x_n or $X(n)$.

Definition 2.2 (neighbourhood)**.** For any $a \in \mathbb{R}$ and $\varepsilon > 0$, the ε-neighbourhood of a is

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon) \text{ Figure (3).}$$

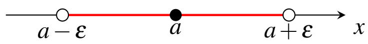

Figure 3: ε-neighbourhood of a

Definition 2.3 (formal definition of limit)**.** For a sequence of numbers x_n, we say that L is the limit of the sequence if

$$\text{for every } \varepsilon > 0 \text{ there exists } K \in \mathbb{N} \quad \text{such that} \quad \text{for all } n \geq K \text{ we have } |x_n - L| < \varepsilon.$$

If L exists, then x_n is convergent; x_n diverges otherwise.

Theorem 2.1 (uniqueness of limit)**.** The limit of a sequence x_n is unique, i.e.

$$\text{if } \lim_{n\to\infty} x_n = L \text{ and } \lim_{n\to\infty} x_n = L' \quad \text{then} \quad L = L'.$$

Proof. Suppose on the contrary that L and L' are two distinct limits of x_n. Let $\varepsilon > 0$ be arbitrary. The trick is to define $\varepsilon' = \varepsilon/2$. Since $x_n \to L$, there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$, the inequality

$$|x_n - L| < \varepsilon' = \frac{\varepsilon}{2} \quad \text{holds.}$$

Similarly, as $x_n \to L'$, then there exists $K_2 \in \mathbb{N}$ such that for all $n \geq K_2$, the inequality

$$|x_n - L'| < \varepsilon' = \frac{\varepsilon}{2} \quad \text{holds.}$$

We define $K = \max\{K_1, K_2\}$, which is also $\in \mathbb{N}$. Then, for all $n \geq K$, we have

$$|L - L'| = |L - x_n + x_n - L'| \leq |x_n - L| + |x_n - L'| < 2\varepsilon' = \varepsilon.$$

Here, the first inequality follows from the triangle inequality. Since ε is arbitrary, we can set $|L - L'| = 0$, resulting in $L = L'$, contradicting the earlier assumption that L and L' are distinct. □

Remark 2.1. The triangle inequality is a helpful tool when finding limits. Note that changing a finite number of terms in a sequence does not affect its convergence or its limit.

Same as the formal definition of a limit in MA2002, to prove that a given sequence x_n converges to L, we first express $|x_n - L|$ in terms of n, and find a *simple* upper bound, L, for it. Then, let $\varepsilon > 0$ be arbitrary. We find $K \in \mathbb{N}$ such that

$$\text{for all } n \geq K \text{ we have } L < \varepsilon \quad \text{or equivalently} \quad |x_n - L| < \varepsilon.$$

Example 2.1. Prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Solution. Let $\varepsilon > 0$. By the Archimedean property (Proposition 1.9), there exists $K \in \mathbb{N}$ such that $K > 1/\varepsilon$. So, if $n \geq K$, then $n > 1/\varepsilon$. As such, $1/n < \varepsilon$. We conclude that for all $n \geq K$, $|1/n - 0| < \varepsilon$. □

Example 2.2. Prove that

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+3n} = 2.$$

Solution. We have

$$\left|\frac{2n^2+1}{n^2+3n} - 2\right| = \left|\frac{1-6n}{n^2+3n}\right| \leq \frac{1+6n}{n^2+3n} < \frac{1+6n}{n^2} < \frac{n+6n}{n^2} = \frac{7}{n}.$$

Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ such that $K > 7/\varepsilon$. Then, for all $n \geq K$, we have

$$\left|\frac{2n^2+1}{n^2+3n} - 2\right| < \frac{7}{n} \leq \frac{7}{K} < \varepsilon$$

and the result follows. □

Example 2.3 (MA2108S AY16/17 Sem 2 Homework 6)**.** Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{x_n} = \infty.$$

Solution. We first prove the forward direction. Suppose

$$\lim_{n\to\infty} x_n = 0.$$

Let $\varepsilon > 0$ be arbitrary and set $M = 1/\varepsilon$. Then, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n| < \varepsilon = 1/M$. Thus, for $n \geq N$, we have $1/x_n > M$, and the result follows.

Now, we prove the reverse direction. Suppose

$$\lim_{n\to\infty} \frac{1}{x_n} = \infty.$$

Then, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $1/x_n > M$. Let $\varepsilon > 0$ be arbitrary and set $M = 1/\varepsilon$. Then, $1/x_n > 1/\varepsilon$, so $|x_n| < \varepsilon$. The result follows. □

Theorem 2.2 (limit theorems)**.** The following hold:

(i) Every convergent sequence is bounded, i.e. if

$$\lim_{n\to\infty} x_n = L \quad \text{then} \quad |x_n| \leq M \text{ for some } M \in \mathbb{R}.$$

(ii) **Linearity:** Just like how linear operators (i.e. derivatives and integrals) work, we have a similar result for limits. Suppose $\alpha, \beta \in \mathbb{R}$ and

$$\lim_{n\to\infty} x_n = L_1 \text{ and } \lim_{n\to\infty} y_n = L_2 \quad \text{then} \quad \lim_{n\to\infty} (\alpha x_n \pm \beta y_n) = \alpha L_1 \pm \beta L_2.$$

(iii) Product and quotient: Considering the sequences x_n and y_n as mentioned in **(ii)**,

$$\lim_{n\to\infty} x_n y_n = L_1 L_2 \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{L_1}{L_2} \quad \text{provided that } y_n, y \neq 0 \text{ for all } n \in \mathbb{N}.$$

The converse of Theorem 2.2 is not true as not all bounded sequences are convergent. As an example, the sequence $x_n = (-1)^n$ is bounded by -1 and 1 and it oscillates about only these two values. However, as $n \to \infty$, the limit does not exist!

We first prove **(i)**.

Proof. Suppose

$$\lim_{n\to\infty} x_n = L.$$

Set $\varepsilon = 1$ and take $K \in \mathbb{N}$ such that $|x_n - L| < 1$ for all $n \geq K$. Then, $L - 1 < x_n < L + 1$. Let $|x_n| = \max\{|L-1|, |L+1|\}$ for all $n \geq K$. Since $\{|x_1|, \ldots, |x_{K-1}|\}$ is a finite set of real numbers, it is bounded, so it contains a maximum. As such, for $1 \leq n \leq K-1$, $|x_n| \leq A$ for some $A \in \mathbb{R}$. Define

$$M = \max\{|L-1|, |L+1|, A\}$$

so $|x_n| \leq M$ and the result follows. □

We then prove part of **(ii)**.

Proof. We shall prove that

$$\lim_{n\to\infty} (x_n + y_n) = L_1 + L_2.$$

We know that there exist $K_1, K_2 \in \mathbb{N}$ such that

$$|x_n - L_1| < \frac{\varepsilon}{2} \text{ for all } n \geq K_1 \quad \text{and} \quad |y_n - L_2| < \frac{\varepsilon}{2} \text{ for all } n \geq K_2.$$

Set $K = \max\{K_1, K_2\}$. By the triangle inequality (Theorem 1.7),

$$|x_n - L_1 + y_n - L_2| < |x_n - L_1| + |y_n - L_2| < \varepsilon$$

and the result follows. □

Lastly, for **(iii)**, we only prove the result involving the product of two sequences.

Proof. Since $|x_n|$ is convergent, then it is bounded by **(i)** of Theorem 2.2, i.e.

$$|x_n| \leq M_1 \text{ for all } n \in \mathbb{N}.$$

We have

$$\begin{aligned}
|x_n y_n - L_1 L_2| &= |x_n y_n - x_n L_2 + x_n L_2 - L_1 L_2| \\
&\leq |x_n y_n - x_n L_2| + |x_n L_2 - L_1 L_2| \\
&= |x_n||y_n - L_2| + |L_2||x_n - L_1| \\
&\leq M_1 |y_n - L_2| + |L_2||x_n - L_1|
\end{aligned}$$

Set $M = \max\{M_1, |L_2|\} > 0$. So,

$$M_1 |y_n - L_2| + |L_2||x_n - L_1| \leq M(|y_n - L_2| + |x_n - L_1|).$$

Let $\varepsilon > 0$ be arbitrary. Then, there exist $K_1, K_2 \in \mathbb{N}$ such that

$$|x_n - L_1| < \varepsilon/2M \text{ for all } n \geq K_1$$
$$|y_n - L_2| < \varepsilon/2M \text{ for all } n \geq K_2$$

Let $K = \max\{K_1, K_2\}$. Hence,

$$|x_n y_n - L_1 L_2| < M\left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}\right) < \varepsilon.$$

and we are done. □

Example 2.4 (MA2108S AY16/17 Sem 2 Homework 6)**.** Let x_n and y_n be sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0.$$

(a) Show that if

$$\lim_{n\to\infty} x_n = \infty \quad \text{then} \quad \lim_{n\to\infty} y_n = \infty.$$

(b) Show that if y_n is bounded, then

$$\lim_{n\to\infty} x_n = 0.$$

Solution.

(a) Since

$$\lim_{n\to\infty} x_n = \infty \quad \text{then} \quad \text{there exists } K \in \mathbb{N} \text{ such that for all } n \geq K \text{ we have } x_n > 1.$$

Since

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0 \quad \text{then} \quad \text{there exists } N \in \mathbb{N} \text{ such that for all } n \geq N \text{ we have } \left|\frac{x_n}{y_n}\right| < \varepsilon.$$

Thus,

$$\frac{1}{|y_n|} < \left|\frac{x_n}{y_n}\right| < \varepsilon \quad \text{for all } n \geq \max\{K, N\},$$

which shows that

$$\lim_{n\to\infty} \frac{1}{y_n} = 0.$$

By Example 2.3, the result follows.

(b) Since y_n is bounded, then there exists $M > 0$ such that $0 < |y_n| \leq M$. We wish to prove that there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n| < \varepsilon$. We have $|x_n/y_n| < \varepsilon/M$ so $|x_n| < \varepsilon/M \cdot M = \varepsilon$. □

Corollary 2.1. If x_n converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Theorem 2.3 (limit theorems)**.** The following hold:

(i) If

$$\lim_{n\to\infty} |x_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} x_n = 0.$$

(ii) If $0 < b < 1$, then

$$\lim_{n\to\infty} b^n = 0.$$

(iii) If $c > 0$, then

$$\lim_{n\to\infty} c^{1/n} = 1.$$

If $c = n$, the limit is still the same.

(iv) If

$$\lim_{n\to\infty} x_n = L \quad \text{then} \quad \lim_{n\to\infty} |x_n| = |L|.$$

(v) Suppose $x_n \geq 0$ for all $n \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} x_n = L \quad \text{implies} \quad \lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}.$$

(vi) If $x_n \geq 0$ for all $n \in \mathbb{N}$ and x_n converges, then

$$\lim_{n\to\infty} x_n \geq 0.$$

We only prove **(ii)** and **(iii)**.

Proof. To prove **(ii)**, write $a = 1/b - 1$, so we have $b = 1/(1+a)$. By Bernoulli's inequality (Theorem 1.4), we have $(1+a)^n \geq 1 + na$. Hence,

$$\frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

By the Archimedean property (Proposition 1.9), $1/na < \varepsilon$ and we are done.

To prove **(iii)**, consider $0 < c \leq 1$ and $c > 1$. For the first case, write $d_n = c^{1/n} - 1$ and use Bernoulli's inequality. □

Corollary 2.2. If x_n and y_n are convergent sequences and $x_n \geq y_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n.$$

Corollary 2.3. If $a, b \in \mathbb{R}$ and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ and x_n is convergent, then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

Example 2.5. Suppose we wish to evaluate the following limit:

$$\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}}$$

Solution. Recognise that for $0 \leq a < 1$, then $a^n \to 0$ as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}} &= \lim_{n\to\infty} \frac{2^n + 3\left(3^n\right) + 25\left(5^n\right)}{4\left(2^n\right) + 3^n + 5\left(5^n\right)} \\
&= 5 - \lim_{n\to\infty} \frac{19\left(2^n\right) + 2\left(3^n\right)}{4\left(2^n\right) + 3^n + 5\left(5^n\right)} \\
&= 5 - \lim_{n\to\infty} \frac{19\left(\frac{2}{5}\right)^n + 2\left(\frac{3}{5}\right)^n}{4\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n + 5} \\
&= 5
\end{aligned}$$

□

Example 2.6 (MA2108S AY16/17 Sem 2 Homework 4)**.** Let $X = x_n$ and $Y = y_n$ be given sequences, and let the "shuffled" sequence $Z = z_n$ be defined by

$$z_1 = x_1, z_2 = y_1, \ldots, z_{2n-1} = x_n, z_{2n} = y_n.$$

Show that

$$Z \text{ is convergent} \quad \text{if and only if} \quad X \text{ and } Y \text{ are convergent and } \lim_{n\to\infty} X = \lim_{n\to\infty} Y.^{\dagger}$$

Solution. We first prove the reverse direction. Suppose X and Y are convergent and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L.$$

By the definition of a limit of a sequence, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \text{ whenever } n \geq N_1 \quad \text{and} \quad |y_n - L| < \varepsilon \text{ whenever } n \geq N_2.$$

Set $N = \max\{2N_1, 2N_2\}$. Then, whenever $n \geq N$, $|z_n - L| < \varepsilon$ and we are done.

Now, we prove the reverse direction. Suppose Z is convergent. That is

$$\lim_{n\to\infty} z_n = L.$$

By the definition of the limit of a sequence, there exists $N \in \mathbb{N}$ such that

$$|z_n - L| < \varepsilon \text{ whenever } n \geq N.$$

We need to show that

$$|x_n - L| < \varepsilon \text{ and } |y_n - L| < \varepsilon \text{ whenever } n \geq N,$$

which are

$$|z_{2n-1} - L| < \varepsilon \text{ and } |z_{2n} - L| < \varepsilon \quad \text{equivalently}.$$

Thus, we need $2n - 1 \geq N$ and $2n \geq N$, which are obviously true. Hence, the result follows. □

Theorem 2.4 (squeeze theorem)**.** Let x_n, y_n and z_n be sequences of numbers such that for all $n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L \quad \text{then} \quad \lim_{n\to\infty} y_n = L.$$

Proof. Let $\varepsilon > 0$. Since $x_n \to L$ and $z_n \to L$, then there exists $K \in \mathbb{N}$ such that for all $n \geq K$,

$$|x_n - a| < \varepsilon \text{ and } |z_n - a| < \varepsilon.$$

Working with the modulus,

$$-\varepsilon < x_n - a < \varepsilon \text{ and } -\varepsilon < z_n - a < \varepsilon.$$

Thus,

$$-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon,$$

which implies that $|y_n - a| < \varepsilon$, and the result follows. □

Example 2.7. Evaluate the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}$$

† Refer to this problem on StackExchange here.

Even though one might think that the Riemann sum comes into play, it actually does not work in this case because

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+k/n^2}}$$

and setting

$$f\left(\frac{k}{n}\right)=\sqrt{1+\frac{k}{n^2}},$$

it is impossible to obtain an explicit expression for $f(x)$.

Solution. We use the squeeze theorem to help us. As

$$\frac{n}{\sqrt{n^2+n}}\le\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\le\sum_{k=1}^{n}\frac{1}{\sqrt{n^2}},$$

then

$$\begin{aligned}\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}&\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\le\frac{n}{\sqrt{n^2}}\\ \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}&\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\le 1\\ 1&\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\le 1\end{aligned}$$

By the squeeze theorem, the required limit is 1. □

Example 2.8 (MA2108 AY19/20 Sem 1). Let $f:(a,\infty)\to\mathbb{R}$ be a function such that it is bounded in any interval (a,b) and

$$\lim_{x\to\infty}(f(x+1)-f(x))=A.$$

Prove that

$$\lim_{x\to\infty}\frac{f(x)}{x}=A.$$

Solution. Let $\varepsilon>0$ be arbitrary. By the given limit, there exists $M>0$ such that for all $x>M$,

$$|f(x+1)-f(x)-A|<\varepsilon.$$

So,

$$A-\varepsilon<f(x+1)-f(x)<A+\varepsilon.$$

Since f is locally bounded, then for $M<x\le M+1$, $-B<f(x)<B$ for some $B\in\mathbb{R}$. Hence,

$$-B+(A+\varepsilon)\cdot\lfloor x-M\rfloor<f(x)<B+(A+\varepsilon)\cdot\lceil x-M\rceil.$$

Dividing by x on both sides, since ε is made arbitrarily small, by the squeeze theorem, $f(x)/x$ tends to A as $x\to\infty$. □

Theorem 2.5 (L'Hôpital's Rule). If f and g are differentiable functions such that $g'(x)\neq 0$ on an open interval I containing a,

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\pm\infty\quad\text{or}\quad\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$$

and

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ exists} \quad \text{then} \quad \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Theorem 2.6 (Stolz-Cesàro theorem)**.** Let x_n and y_n be two sequences of real numbers. If y_n is strictly monotone and divergent and

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=L \text{ exists} \quad \text{then} \quad \lim_{n\to\infty}\frac{x_n}{y_n}=L.$$

Theorem 2.7 (Stolz-Cesàro theorem, alt.)**.** If

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=0$$

where y_n is strictly decreasing and

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=L \quad \text{then} \quad \lim_{n\to\infty}\frac{x_n}{y_n}=L.$$

Example 2.9 (MA2108 AY19/20 Sem 1)**.** Let a_n be a sequence in $\mathbb{R}$.

(i) Prove that if

$$\lim_{n\to\infty}a_n=a \quad \text{then} \quad \lim_{n\to\infty}\frac{a_1+a_2+\ldots+a_n}{n}=a.$$

(ii) Suppose the sequence

$$\frac{a_1+a_2+\ldots+a_n}{n}$$

converges. Can we deduce that a_n converges? Justify your answer.

Solution.

(i) Let $\varepsilon>0$ be arbitrary. There exists $K_1\in\mathbb{N}$ such that $|a_j-a|\le\varepsilon/2$ for all $j\ge K_1$. Then,

$$\left|\frac{a_1+a_2+\ldots+a_n}{n}-a\right|=\left|\frac{1}{n}\sum_{j=1}^{n}(a_j-a)\right|.$$

We can bound this sum accordingly. For $n\ge K_1$,

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=1}^{n}(a_j-a)\right| &\le \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j-a)\right|+\frac{1}{n}\left|\sum_{j=K_1+1}^{n}(a_j-a)\right| \quad \text{by triangle inequality}\\ &\le \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j-a)\right|+\frac{n-K_1}{n}\cdot\frac{\varepsilon}{2}\\ &< \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j-a)\right|+\frac{\varepsilon}{2}\\ &= \frac{C}{n}+\frac{\varepsilon}{2}\end{aligned}$$

Here, we let C be the sum of $a_j - a$ from $j=1$ to $j=K_1$. Next, for $K \in \mathbb{N}$, where $K > \max\{K_1, 2C/\varepsilon\}$, it is now easy to see that

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=1}^{n}(a_j - a)\right| &< \frac{C}{n} + \frac{\varepsilon}{2} \\ &\le \frac{C}{K} + \frac{\varepsilon}{2} \quad \text{since } n \ge K \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{since } K > \frac{2C}{\varepsilon} \\ &= \varepsilon\end{aligned}$$

(ii) No. Define $s_n = (a_1 + a_2 + \ldots + a_n)/n$. Setting $a_n = (-1)^n$,

$$s_n = \begin{cases} -1/n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

By the squeeze theorem, as $n \to \infty$, $s_n \to 0$, so it converges. However, a_n diverges. □

2.2. *Monotone Sequences*

Theorem 2.8 (monotone convergence theorem). If x_n is a monotone and bounded sequence, then we can find an expression for

$$\lim_{n\to\infty} x_n.$$

If

$$x_n \text{ is increasing then } \lim_{n\to\infty} x_n = \sup x_n \quad \text{and if} \quad x_n \text{ is decreasing then } \lim_{n\to\infty} x_n = \inf x_n.$$

Example 2.10 (MA2108S AY16/17 Sem 2 Homework 4). Let

$$x_n = \frac{1}{1^2} + \ldots + \frac{1}{n^2} \quad \text{for each } n \in \mathbb{N}.$$

Prove that x_n is increasing and bounded, and hence converges.
Hint: Note that if $k \ge 2$, then

$$\frac{1}{k^2} \le \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Solution. We show that x_n is increasing. Consider

$$x_{n+1} - x_n = \frac{1}{n+1} > 0 \quad \text{which implies} \quad x_{n+1} > x_n$$

So, x_n is a strictly increasing sequence. Next, we can bound x_n in the following manner.

$$\begin{aligned} 1 + \sum_{n=2}^{\infty}\frac{1}{n^2} &\le 1 + \sum_{n=2}^{\infty}\frac{1}{n(n-1)} \\ \sum_{n=1}^{\infty}\frac{1}{n^2} &\le 1 + \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) \\ \sum_{n=1}^{\infty}\frac{1}{n^2} &\le 2 \end{aligned}$$

So, x_n is bounded above by 2. By the monotone convergence theorem (Theorem 2.8), it follows that x_n converges. □

Example 2.11 (MA2108S AY16/17 Sem 2 Homework 4). Let $x_1 = 1$ and $x_{n+1} = \sqrt{2+x_n}$ for $n \in \mathbb{N}$. Show that x_n converges and find the limit.

Solution. It is clear that x_n is bounded above by 2. Given that $x_1 = 1$, we show that x_n is strictly increasing. That is, for $n \in \mathbb{N}$, $x_{n+1} > x_n$.

$$x_{n+1} - x_n = \sqrt{2+x_n} - x_n = \frac{2+x_n-x_n^2}{\sqrt{2+x_n}+x_n} = \frac{(x_n-2)(x_n+1)}{\sqrt{2+x_n}+x_n}.$$

It is clear that x_n is a sequence of positive terms so we consider the numerator of $x_{n+1} - x_n$, which is $(2-x_n)(x_n+1)$. For $1 \le x_n \le 2$, this product is always positive, and hence $x_{n+1} - x_n \ge 0$. By the monotone convergence theorem (Theorem 2.8), x_n converges.

Suppose

$$\lim_{n\to\infty} x_n = L.$$

Then, $L = \sqrt{2+L}$, but since $L > 0$, then $L = 2$. □

Methods of computing square roots are numerical analysis algorithms for approximating the principal, or non-negative, square root of a real number, say S.

Theorem 2.9 (Babylonian method). We start with an initial value somewhere near $\sqrt{S}$. That is $x_0 \approx \sqrt{S}$. We then use the following recurrence relation to find a better estimate for $\sqrt{S}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right) \quad \text{where} \quad \lim_{n\to\infty} x_n = \sqrt{S}$$

Proof. Suppose

$$\lim_{n\to\infty} x_n = L.$$

Substituting this into the recurrence relation yields

$$L = \frac{1}{2}\left(L + \frac{S}{L}\right).$$

Rearranging and the result follows. □

Theorem 2.10 (nested interval theorem). Let $I_n = [a_n, b_n]$, where $n \in \mathbb{N}$, be a nested sequence of closed and bounded sequences. That is, $I_n \supseteq I_{n+1}$. Then, the intersection

$$\bigcap_{n=1}^{\infty} I_n = \{x : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

is non-empty. In addition, if $b_n - a_n \to 0$ (i.e. length of I_n tends to 0), then the intersection contains exactly one point.

Definition 2.4 (harmonic numbers). The harmonic numbers, H_n, are defined to be

$$\sum_{k=1}^{n} \frac{1}{k}.$$

Definition 2.5 (harmonic series)**.** The Harmonic series is defined to be the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} H_n$$

Note that the harmonic numbers are increasing (since $H_{n+1} - H_n > 0$) and

$$\lim_{n\to\infty} H_n = 0.$$

However, the harmonic series is divergent! Another interesting property is that other than H_1, the harmonic numbers are never integers, whose proof hinges on some elementary Number Theory.

2.3. *Euler's Number, e*

Definition 2.6 (Euler's number)**.** Euler's number, $e \approx 2.71828$, is defined to be

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Theorem 2.11. The sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n \quad \text{is strictly increasing.}$$

That is, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

Proof. It is easier to prove $x_n > x_{n-1}$, so we wish to prove

$$\left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

First, we write $1 + 1/n$ as

$$1 + \frac{1}{n-1} = \frac{n}{n-1} = \frac{1}{1 - 1/n}.$$

Hence,

$$\begin{aligned}
\frac{(1+1/n)^n}{(1+1/(n-1))^{n-1}} &= \left(1 + \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{n-1} \\
&= \left(1 + \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{-1} \\
&= \left(1 - \frac{1}{n^2}\right)^n \left(1 - \frac{1}{n}\right)^{-1}
\end{aligned}$$

By Bernoulli's inequality (Theorem 1.4), this is greater than 1, and so $x_n > x_{n-1}$. □

Theorem 2.12. $2 \le e \le 3$

Proof. We use the series expansion of x_n.

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \ldots + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 \\
&= 1 + 1 + \frac{n-1}{(2!)n} + \frac{(n-1)(n-2)}{3!(n^2)} + \ldots
\end{aligned}$$

It is clear that $e \geq 2$. To prove that $e \leq 3$, we consider the infinite series, but starting from the third term of the expansion of x_n. It suffices to show that

$$\frac{n-1}{2n}+\frac{(n-1)(n-2)}{6n^2}+\frac{(n-1)(n-2)(n-3)}{24n^3}+\ldots \leq 1.$$

Observe that the r^{th} term can be written as

$$\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r}=\frac{1}{(r+1)!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\left(1-\frac{r}{n}\right)\leq\frac{1}{(r+1)!}.$$

It is clear that

$$\frac{1}{(r+1)!}\leq\frac{1}{2^r},$$

since the factorial grows much faster than the geometric series, and so taking the reciprocal, the result follows. To conclude,

$$\sum_{r=1}^{\infty}\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r}\leq\sum_{r=1}^{\infty}\frac{1}{2^r}=1,$$

and we are done. □

Though the incredible constant is named after the Swiss mathematician Leonhard Euler, its discovery is actually accredited to another Swiss mathematician, Jacob Bernoulli. Just like π, e is also irrational (Theorem 2.13), which can be proven by contradiction.

Theorem 2.13. e is irrational

Proof. Suppose on the contrary that e is rational. Then, $e = p/q$, where $p,q \in \mathbb{Z}$ but $q \neq 0$. Then, as e can be expressed as the following infinite series

$$\sum_{k=0}^{\infty}\frac{1}{k!},$$

we have

$$\begin{aligned} e=\frac{p}{q}&=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots+\frac{1}{m!}+\frac{1}{(m+1)!}+\ldots \\ m!e&=\frac{m!}{0!}+\frac{m!}{1!}+\frac{m!}{2!}+\frac{m!}{3!}+\frac{m!}{4!}+\ldots+\frac{m!}{m!}+\frac{m!}{(m+1)!}+\ldots \end{aligned}$$

By setting $q = m!$, we see that $m!e \in \mathbb{Z}$. Next, we take a look at the RHS. Observe that

$$\frac{m!}{0!}+\frac{m!}{1!}+\frac{m!}{2!}+\frac{m!}{3!}+\frac{m!}{4!}+\ldots+\frac{m!}{m!}$$

is an integer but

$$\frac{m!}{(m+1)!}+\frac{m!}{(m+2)!}+\frac{m!}{(m+3)!}+\ldots=\frac{1}{m+1}+\frac{1}{(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}+\ldots$$

is not an integer, which is a contradiction. □

2.4. *Euler-Mascheroni Constant*

Definition 2.7 (Euler-Mascheroni constant)**.** The Euler-Mascheroni constant, $\gamma \approx 0.5772$, is the limiting difference between the harmonic series and the natural logarithm. That is,

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right).$$

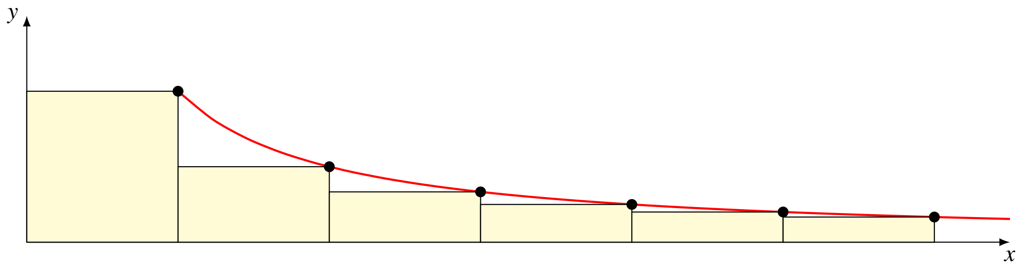

Figure 4: The graph of $y = 1/x$ and an approximation for the area under the curve

γ is an epic constant. From Figure 4, the Euler-Mascheroni constant can be regarded as the sum of areas of the yellow rectangles minus the area under the curve $y = 1/x$ for $x \geq 1$.
It is interesting to note that the Euler-Mascheroni constant converges even though the harmonic series diverges and $\ln n$ tends to infinity as n tends to infinity. Let us prove this result using the monotone convergence theorem. We make the following claims. First, we define x_n to be the following sequence:

$$x_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

We shall prove that x_n is a decreasing sequence.

Proof. We wish to prove that $x_n > x_{n+1}$. Consider

$$x_n - x_{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n+1) = \ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1}.$$

Consider the graph of $f(x) = 1/x$, for $n \leq x \leq n+1$. We can regard

$$\begin{aligned} \ln\left((n+1)/n\right) \text{ as } & \quad \text{the area under the curve from } x = n \text{ to } x = n+1 \quad \text{and} \\ 1/(n+1) \text{ as } & \quad \text{the area of a rectangle bounded by } x = n, x = n+1 \text{ and } y = 1/n \end{aligned}$$

Since f is strictly decreasing and concave up, then the area under the curve is less than the area of the rectangle. Hence, $x_n - x_{n+1} > 0$ and the result follows. □

We then prove that $0 < x_n \leq 1$, i.e. x_n is bounded.

Proof. Note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then

$$1 = x_1 > x_2 > x_3 > \ldots$$

and so x_n is bounded above by 1.

Write x_n as

$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, dx.$$

Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up. □

With these two facts, by the monotone convergence theorem (Theorem 2.8), x_n converges, and it converges to γ. It is still unknown whether γ is rational or irrational. This remains an open problem.

Example 2.12 (MA2108 AY21/22 Sem 1 Midterm).

(i) Let $n \in \mathbb{N}$. Prove that
$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$

(ii) Use the above inequalities to prove that
$$x_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n$$
has a limit as $n \to \infty$.

Solution.

(i) Let
$$f(n) = \frac{1}{n+1}, \quad g(n) = \ln\left(1+\frac{1}{n}\right) \quad \text{and} \quad h(n) = \frac{1}{n}.$$
Note that f, g and h are concave up on $(0,\infty)$. If we establish that $f'(n) > g'(n) > h'(n)$ for all $n \in (0,\infty)$, then we are done.

Consider
$$g'(n) - f'(n) = \frac{1}{(n+1)^2} - \frac{1}{n^2+n} = -\frac{1}{n(n+1)^2}$$
and since $n > 0$, then $g'(n) < f'(n)$.

Next, consider
$$g'(n) - h'(n) = -\frac{1}{n^2+n} + \frac{1}{n^2} = \frac{1}{n^2(n+1)}$$
and in a similar fashion, $g'(n) > h'(n)$. We are done.

(ii) It suffices to show that x_n is decreasing and bounded.

To show x_n is decreasing, consider
$$x_n - x_{n+1} = -\frac{1}{n+1} + \ln\left(1+\frac{1}{n}\right) > 0$$
by **(i)**.

To show x_n is bounded, note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then
$$1 = x_1 > x_2 > x_3 > \ldots$$
and so x_n is bounded above by 1.

Write x_n as
$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, dx.$$
Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up.

Since x_n is decreasing and between 0 and 1, its limit exists. $\square$

2.5. *Subsequences*

Definition 2.8 (subsequence)**.** Let a_n be a sequence. Then, a subsequence of a_n can be formed by deleting elements of a_n. The subsequence is usually written as a_{n_k}, where n_k is an increasing sequence of positive integers.

Theorem 2.14. If x_n converges to a limit, L, then any subsequence x_{n_k} also converges to L.

Corollary 2.4. If x_n has two convergent subsequences whose limits are not equal, then x_n is divergent.

Theorem 2.15 (squeeze theorem)**.** Every sequence has a monotone subsequence.

Theorem 2.16 (Bolzano-Weierstrass theorem)**.** Every bounded sequence has a convergent subsequence.

Example 2.13 (MA2108S AY16/17 Sem 2 Homework 4)**.** Suppose that every subsequence of x_n has a subsequence that converges to 0. Show that

$$\lim_{n\to\infty} x_n = 0.^{\dagger}$$

Solution. We first show that x_n is bounded. Suppose on the contrary that it is not. Then, for every $M,N>0$, there exists $n \geq N$ such that $|x_n| \geq M$. Then, we can find a subsequence x_{n_k} such that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

However, this subsequence does not have a subsequence that converges to 0, which is a contradiction. Hence, x_n is bounded.

Next, x_n must have only one limit point. Let L be a limit point of x_n. Then, x_{n_k} converges to L. Also, any subsequence of x_{n_k} converges to L. By the uniqueness of the limit, as every subsequence of x_{n_k} converges to 0, then x_{n_k} converges to 0, we establish the required result. □

Example 2.14 (MA2108S AY16/17 Sem 2 Homework 4)**.** Let x_n be a bounded sequence and for each $n \in \mathbb{N}$, let $s_n = \sup\{x_k : k \geq n\}$ and $S = \inf\{s_n\}$. Show that there exists a subsequence of x_n that converges to S.

Solution. We first prove that if $A \subseteq B$, then $\sup A \leq \sup B$. This is clear because $A \subseteq B$ implies that $\sup B$ is an upper bound for A. By definition, $\sup A$ is the least upper bound for A, so the result follows. This result will help us in understanding the monotonicity of s_n.

From here, it is easy to see that s_n is monotonically decreasing. As x_n is bounded, then s_n is bounded and it converges to the infimum. The result follows. □

2.6. *Cluster Point, Limit Superior and Limit Inferior*

Definition 2.9 (cluster point)**.** Let x_n be a sequence. A point x is a cluster point, or accumulation point, of x_n if x_n has a subsequence x_{n_k} which converges to x. That is, $x_{n_k} \to x$.

†See here for a reference.

Definition 2.10. Let $C(x_n)$ be the set of all cluster points of x_n.

Proposition 2.1. $C(x_n)$ is non-empty.

Proof. Suppose x_n is bounded. Then, by the Bolzano-Weierstrass theorem (Theorem 2.16), x_n has a convergent subsequence, so the result follows. □

Definition 2.11 (limit superior and limit inferior)**.** We define the limit superior and limit inferior of x_n to be the following:

$$\limsup(x_n) = \sup C(x_n) \text{ and } \liminf(x_n) = \inf C(x_n)$$

Theorem 2.17. Let x_n be a bounded sequence and let $M = \limsup x_n$. Then,

(i) For every $\varepsilon > 0$, there are at most finitely many n's such that $x_n \geq M + \varepsilon$. Equivalently,

$$\text{there exists } k \in \mathbb{N} \text{ such that for all } n \geq K \quad \text{implies} \quad x_n < M + \varepsilon.$$

(ii) For each $\varepsilon > 0$, there are infinitely many n's such that $x_n > M - \varepsilon$.

We have similar results related to the limit inferior.

Corollary 2.5. If x_n is a bounded sequence, then,

$$x_n \text{ converges} \quad \text{if and only if} \quad \limsup x_n = \liminf x_n.$$

Lemma 2.1. Suppose x_n and y_n are bounded sequences such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Then,

$$\limsup x_n \leq \limsup y_n \quad \text{and} \quad \liminf x_n \leq \liminf y_n.$$

Example 2.15 (MA2108 AY18/19 Sem 1 Midterm)**.** For each $n \in \mathbb{N}$, let

$$y_n = \frac{2n - \sqrt{n} + 1}{n + 2\sqrt{n} + 1} \cos\left(\frac{(n-1)\pi}{4}\right).$$

(i) Find $\limsup y_n$ and $\liminf y_n$.

(ii) Is the sequence y_n convergent? Justify your answer.

Solution.

(i) We first find $\sup y_n$. Since cosine is bounded above by 1, then

$$y_n \leq \frac{2n - \sqrt{n} + 1}{n + 2\sqrt{n} + 1} = 2 - \frac{5\sqrt{n} + 1}{n + 2\sqrt{n} + 1}.$$

On the right side of the inequality, the denominator grows much faster than the numerator, so $\sup y_n = 2$. Now, we show that $\limsup y_n = 2$. Define

$$a_n = \cos\left(\frac{(n-1)\pi}{4}\right).$$

so that $a_{8n+1} = 1$ for all $n \in \mathbb{N}$. The result follows. Use the same method to find $\liminf y_n$.

(ii) No, since $\liminf y_n \neq \limsup y_n$. □

Example 2.16 (MA2108 AY18/19 Sem 1 Midterm)**.** Let a_n and b_n be bounded sequences, and let

$$c_n = \max\{a_n, b_n\} \quad \text{for all } n \in \mathbb{N}.$$

Prove that

$$\limsup c_n = \max\{\limsup a_n, \limsup b_n\}.$$

Solution. Note that $a_n, b_n \le c_n$. Define $M_1 = \limsup a_n$, $M_2 = \limsup b_n$ and $M = \max\{M_1, M_2\}$. So, $M_1 \le \limsup c_n$ and $M_2 \le \limsup c_n$. Thus, $M \le \limsup c_n$. Now, we prove that $M = \limsup c_n$.

Let c be a cluster point of c_{n_k} and $c_{n_k} \to c$. For any arbitrary $\varepsilon > 0$, there exists $K_1, K_2 \in \mathbb{N}$ such that for all $n > K_1$ and $n > K_2$, we have

$$|a_n - M_1| < \varepsilon \quad \text{and} \quad |b_n - M_2| < \varepsilon \text{ respectively.}$$

The expansion of these two inequalities yields $a_n < M_1 + \varepsilon$ and $b_n < M_2 + \varepsilon$. We'll now relate this to $c_n = \max\{a_n, b_n\}$. Let $K = \max\{K_1, K_2\}$. Then, for all $n > K$,

$$a_n < M_1 + \varepsilon < M + \varepsilon \quad \text{and} \quad b_n < M_2 + \varepsilon < M + \varepsilon.$$

Hence, $c_n < M + \varepsilon$. As mentioned, c is a cluster point of c_{n_k}, so $c_{n_k} < M + \varepsilon$. As $k \to \infty$, it is clear that $c < M + \varepsilon$. Hence, M is an upper bound for the cluster points of c_n, and so $\limsup c_n \le M$. Combining the purple inequalities yields the result. □

Example 2.17 (MA2108 AY19/20 Sem 1)**.** Let x_n and y_n be two bounded sequences in $\mathbb{R}$.

(i) Prove that

$$\liminf x_n + \liminf y_n \le \liminf (x_n + y_n).$$

(ii) Suppose there exists an $N \in \mathbb{N}$ such that when $n > N$, one has $x_n \le y_n$. Prove that

$$\liminf x_n \le \liminf y_n.$$

Solution.

(i) Let u be a subsequential limit of $x_n + y_n$. Then, there exists a subsequence $x_{n_k} + y_{n_k}$ of $x_n + y_n$ which converges to u. Let $\varepsilon > 0$. Then, there exists $K_1, K_2 \in \mathbb{N}$ such that

$$x_n \ge K_1 \text{ implies } x_n > \liminf x_n + \frac{\varepsilon}{2} \quad \text{and} \quad y_n \ge K_2 \text{ implies } y_n > \liminf y_n + \frac{\varepsilon}{2}.$$

Define $K = \max\{K_1, K_2\}$. Since $n_k \ge k$, then for all $k \ge K$, we have

$$u = \lim_{k \to \infty} (x_{n_k} + y_{n_k}) \ge \liminf x_n + \liminf y_n + \varepsilon.$$

Since ε is some arbitrary small positive number, it follows that $\liminf x_n + \liminf y_n$ is a lower bound for $x_n + y_n$.

As there exists a subsequence $x_{n_k} + y_{n_k}$ converging to $\inf(x_n + y_n)$, then we conclude with the following statement:

$$\liminf(x_n + y_n) = \inf(x_n + y_n) \ge \liminf x_n + \liminf y_n$$

(ii) Define

$$a_n = \inf\{x_k : k \ge n\} \text{ and } b_n = \inf\{y_k : k \ge n\}.$$

As each $x_n \le y_n$, then $\inf x_k \le \inf y_k$ for all $1 \le k \le n$. That is, $a_n \le b_n$. To conclude,

$$\liminf x_n = \inf a_n \le \inf b_n = \liminf y_n.$$

□

2.7. *Cauchy Sequences*

Definition 2.12 (Cauchy sequence). A sequence x_n is Cauchy if

$$\text{for every } \varepsilon > 0 \text{ there exists } K \in \mathbb{N} \quad \text{such that} \quad \text{for all } m,n \geq K \text{ we have } |x_n - x_m| < \varepsilon.$$

Intuitively, what Definition 2.12 means is that for large n, the x_n's are very close to each other.

Theorem 2.18. The following hold:

(i) A sequence is a Cauchy sequence if and only if it is convergent

(ii) Every Cauchy sequence is bounded

We take a look at an example where **(i)** of Theorem 2.18 can be applied.

Example 2.18. The sequence $x_n = n$ is not Cauchy since it is not convergent. However, $y_n = 2^{-n}$ and $z_n = 1/n^2$ are Cauchy sequences. y_n is a geometric sequence and z_n is related to one of the most famous mathematical constants, known as $\zeta(2) = \pi^2/6$, which is the solution to the Basel problem.

Definition 2.13 (contractive sequence). A sequence x_n is contractive if

$$\text{there exists } 0 < C < 1 \quad \text{such that} \quad |x_{n+2} - x_{n+1}| \leq C\,|x_{n+1} - x_n| \text{ for all } n \in \mathbb{N}.$$

Lemma 2.2 (contractive implies convergent). Every contractive sequence is convergent, and hence Cauchy.

Lemma 2.3. A sequence x_n is contractive if

$$\text{there exists } 0 < C < 1 \quad \text{such that} \quad |x_{n+2} - x_{n+1}| \leq C^{n-1}\,|x_2 - x_1| \text{ for all } n \in \mathbb{N}.$$

Proof. Repeatedly apply the inequality in Definition 2.13. □

Example 2.19 (MA2108 AY19/20 Sem 1). Let $a_1 \geq 0$ and for $n \geq 1$, define

$$a_{n+1} = \frac{3\,(1+a_n)}{3+a_n}$$

(a) Prove that a_n converges.

(b) Find the limit.

Solution.

(a) We have

$$|a_{n+2} - a_{n+1}| = \left|\frac{3(1+a_{n+1}) - 3a_{n+1} - a_{n+1}^2}{3+a_{n+1}}\right| = \left|\frac{3-a_{n+1}^2}{3+a_{n+1}}\right|,$$

which simplifies to

$$\left|\frac{3-a_n^2}{(3+a_n)\,(2+a_n)}\right| = \left|\frac{3-a_n^2}{3+a_n}\right| \cdot \frac{1}{|2+a_n|} < \left|\frac{3-a_n^2}{3+a_n}\right| = |a_{n+1} - a_n|$$

so a_n is a contractive sequence. By Lemma 2.2, a_n converges.

(b) Suppose $\lim_{n\to\infty} a_n = L$. Thus,

$$L = \frac{3(1+L)}{3+L}.$$

Since $L > 0$, then $L = \sqrt{3}$. $\square$

A sequence x_n tends to ∞ if for every $M > 0$, then there exists $K \in \mathbb{N}$ such that

$$x_n > M \ \forall n \geq K.$$

We write

$$\lim_{n\to\infty} x_n = \infty.$$

Similarly, a sequence x_n tends to $-\infty$ if for every $M < 0$, then there exists $K \in \mathbb{N}$ such that

$$x_n < M \ \forall n \geq K.$$

We write

$$\lim_{n\to\infty} x_n = -\infty.$$

To conclude, a sequence x_n is properly divergent if either $x_n \to \infty$ or $x_n \to -\infty$.

3. Infinite Series

3.1. *Geometric Series*

Definition 3.1 (partial sum, sum to infinity)**.** A sequence, a_n, has sum to n terms, or partial sum,

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n.$$

As $n \to \infty$, we are able to find the sum to infinity (if it exists). The sum to infinity is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$$

Definition 3.2 (geometric sequence)**.** A geometric sequence, u_n, has first term a and common ratio r. The first few terms are

$$u_1 = a,\ u_2 = ar,\ u_3 = ar^2,\ u_4 = ar^3.$$

The general term, u_n, is $u_n = ar^{n-1}$, where $n \in \mathbb{N}$.

Proposition 3.1. The sum to n terms of a geometric sequence is denoted by S_n. We establish the formula

$$S_n = \frac{a(1-r^n)}{1-r}.$$

For the sum to infinity, S_∞, we impose a restriction on r for the sum to exist. That is, $|r| < 1$. Hence, S_∞ is

$$S_\infty = \frac{a}{1-r}.$$

Remark 3.1. If $r = -1$, we obtain the famous Grandi's series $1 - 1 + 1 - 1 + \ldots$.

Definition 3.3 (telescoping series)**.** A telescoping series is a series whose general term can be written in the form $a_n - a_{n-1}$.

Let $b_n = a_n - a_{n-1}$. Then,

$$\sum_{k=1}^{n} b_k = a_n - a_0.$$

This process is known as the method of differences. There are times when the partial fraction decomposition method has to be used on b_n.

3.2. *Properties of Convergence and Divergence*

Some properties on convergence are regarded as trivial. For instance, if two series are convergent, then their sum is also convergent.

Theorem 3.1. The following hold:

(i) If

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

(ii) If $a_n \geq 0$ for all n, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{if and only if} \quad \text{the sequence of partial sums } s_n \text{ is bounded,}$$

where $s_{n+1} - s_n = a_n$

Theorem 3.2 (Cauchy criterion).

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{if and only if}$$

for every $\varepsilon > 0$, then there exists $K \in \mathbb{N}$ such that

$$|a_{n+1} + a_{n+2} + \ldots + a_m| < \varepsilon \text{ for all } m > n \geq K.$$

Example 3.1 (MA2108 AY18/19 Sem 1). Let

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

be two series with the property that there exists $K \in \mathbb{N}$ such that

$$a_n = b_n \text{ for all } n \geq K.$$

Prove that

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} b_n \text{ is convergent.}$$

Solution. Just use Cauchy criterion. I believe the working is pretty short. □

3.3. *Tests for Convergence*

Definition 3.4 (p-series). The p-series is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Theorem 3.3 (p-series test). If $p > 1$, the p-series converges. If $0 < p \leq 1$, the p-series diverges.

Example 3.2 (MA2108S AY16/17 Sem 2 Homework 6). Show that

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent.

Solution. As $|\cos n| \leq 1$, the above sum is bounded above by

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{which is convergent}$$

since it is the p-series when $p = 2$, or rather the Basel problem, which has a value of $\pi^2/6$. □

Theorem 3.4 (direct comparison test)**.** Suppose there exists $K \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq K$. Then,

$$\sum_{n=1}^{\infty} b_n \text{ converges implies } \sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \text{ diverges implies } \sum_{n=1}^{\infty} b_n \text{ diverges}$$

Example 3.3 (MA2108S AY16/17 Sem 2 Homework 6)**.** If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum a_n^2$ always convergent? Either prove it or give a counterexample.

Solution. Observe that

$$\left(\sum_{n=1}^{N} a_n\right)^2 = \sum_{n=1}^{N} a_n^2 + 2\sum_{i<j} a_i a_j$$

so

$$\left(\sum_{n=1}^{N} a_n\right)^2 \geq \sum_{n=1}^{N} a_n^2.$$

Since $\sum a_n$ converges, then $\sum a_n^2$ converges too. □

Example 3.4 (MA2108S AY16/17 Sem 2 Homework 6)**.** If $\sum a_n$ with $a_n > 0$ is convergent, then is $\sum \sqrt{a_n}$ always convergent? Either prove it or give a counterexample.

Solution. Not always convergent. Consider $a_n = 1/n^2$. □

Theorem 3.5 (limit comparison test)**.** Let

$$\sum_{i=n}^{\infty} a_n \text{ and } \sum_{i=n}^{\infty} b_n \quad \text{be} \quad \text{series of positive terms.}$$

Define

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

(i) If $L > 0$, then the series are either both convergent or both divergent.

(ii) If $L = 0$ and

$$\sum_{i=1}^{\infty} b_i \text{ converges} \quad \text{then} \quad \sum_{i=1}^{\infty} a_i \text{ converges.}$$

Definition 3.5 (alternating series)**.** An alternating series is a series of the form

$$\sum_{n=1}^{\infty} a_n(-1)^n = a_1 - a_2 + a_3 - a_4 + \ldots,$$

where all $a_n > 0$ or all $a_n < 0$.

Theorem 3.6 (alternating series test)**.** If a_n is an alternating series with

$$\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \quad \text{for } n \geq 1, \text{ i.e. } a_n \text{ decreases monotonically} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0,$$

then a_n converges.

Example 3.5 (MA2108 AY21/22 Sem 1 Midterm)**.** Consider the following alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2/3}}.$$

Is it convergent? Prove your conclusion.

Solution. Let

$$a_n = \frac{1}{n^{2/3}} = n^{-2/3}.$$

We verify if

$$\lim_{n\to\infty} a_n = 0$$

and a_n is monotonically decreasing. The limit property is obviously true.

Consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{-2/3}}{n^{-2/3}}\right| = \left|\left(1+\frac{1}{n}\right)^{-2/3}\right| < 1$$

and so $a_{n+1} < a_n$. Thus, the series is convergent. □

Definition 3.6 (absolute convergence)**.**

$$\sum_{n=1}^{\infty} a_n \text{ converges absolutely} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} |a_n| \text{ converges implies } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Theorem 3.7 (D'Alembert's ratio test)**.** Let

$$\sum_{i=1}^{\infty} a_n \quad \text{be} \quad \text{a series of positive terms.}$$

Define

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $L < 1$ the series converges;
(ii) if $L > 1$ the series diverges;
(iii) if $L = 1$, the test is inconclusive

Theorem 3.8 (Cauchy's root test)**.** We wish to determine if the series

$$\sum_{i=1}^{\infty} a_n \text{ of positive terms} \quad \text{is absolutely convergent.}$$

Define

$$L = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

(i) If $L < 1$, the series is absolutely convergent;
(ii) if $L > 1$, the series diverges;
(iii) if $L = 1$, the test is inconclusive

Theorem 3.9 (Cauchy's condensation test). For a non-increasing sequence of non-negative real numbers $f(n)$,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \quad \text{if and only if} \quad \text{the condensed series } \sum_{n=0}^{\infty} 2^n f(2^n) \text{ converges.}$$

Observe the difference in the lower indices — one of them is 1 and another is 0.

Proof. Suppose the original series converges. We wish to prove that the condensed series converges. Consider twice the original series.

$$\begin{aligned} 2\sum_{n=1}^{\infty} f(n) &= (f(1)+f(1)) + (f(2)+f(2)+f(3)+f(3)) + \dots \\ &\geq (f(1)+f(2)) + (f(2)+f(4)+f(4)+f(4)) + \dots \\ &= f(1) + (f(2)+f(2)) + (f(4)+f(4)+f(4)+f(4)) + \dots \\ &= \sum_{n=0}^{\infty} 2^n f(2^n) \end{aligned}$$

Dividing both sides by 2, the condensed series converges.

Now, suppose the condensed series converges. We wish to prove the original series converges.

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n f(2^n) &= f(1)+f(2)+f(2)+f(4)+f(4)+f(4)+f(4)+\dots \\ &\geq f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+\dots \\ &= \sum_{n=1}^{\infty} f(n) \end{aligned}$$

This concludes the proof. □

Corollary 3.1. If both series converge, the sum of the condensed series is no more than twice as large as the sum of the original. We have the inequality

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2\sum_{n=1}^{\infty} f(n).$$

Corollary 3.2. Consider a variant of the p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

If $p > 1$, the series converges. If $p \leq 1$, the series diverges.

Proof. We use Cauchy's condensation test (Theorem 3.9). Note that

$$f(n) = \frac{1}{n(\ln n)^p},$$

so

$$2^n f(2^n) = \frac{2^n}{2^n(\ln(2^n))^p} = \frac{1}{n^p(\ln 2)^p}.$$

We have

$$\frac{1}{(\ln 2)^p}\sum_{n=2}^{\infty}\frac{1}{n^p}$$

so the result follows by the conventional p-series test. □

3.4. *Grouping and Rearrangement of Series*

Theorem 3.10. If

$$\sum_{n=1}^{\infty} a_n \text{ converges,}$$

then any series obtained by

$$\text{grouping the terms of } \sum_{n=1}^{\infty} a_n \text{ is also convergent} \quad \text{and} \quad \text{has the same value as } \sum_{n=1}^{\infty} a_n.$$

Definition 3.7 (rearrangement)**.** A series

$$\sum_{n=1}^{\infty} b_n \quad \text{is} \quad \text{a rearrangement of the series } \sum_{n=1}^{\infty} a_n$$

if there is a bijection $f:\mathbb{N}\to\mathbb{N}$ such that $b_n = a_{f(n)}$ for all $n\in\mathbb{N}$.

Example 3.6 (paradoxical?)**.** The series

$$1+2+3+4+\ldots$$

is an interesting one. Although it is a divergent series, by certain methods such as rearrangement of the original series or by Ramanujan summation, we obtain the formula

$$1+2+3+4+\ldots = -\frac{1}{12}.$$

Example 3.7 (MA2108S AY16/17 Sem 2 Homework 4)**.** For x_n given by the following formulae, establish either the convergence or the divergence of the series

$$\sum_{n=1}^{\infty} x_n.$$

(a) $x_n = \dfrac{n}{n+1}$ **(b)** $x_n = \dfrac{(-1)^n n}{n+1}$ **(c)** $x_n = \dfrac{n^2}{n+1}$ **(d)** $x_n = \dfrac{2n^2+3}{n^2+1}$

Solution.

(a) Note that

$$x_n = 1-\frac{1}{n+1}$$

so

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty}\left(1-\frac{1}{n+1}\right)$$

which diverges.

(b) Pairing the terms,

$$\begin{aligned}\sum_{n=1}^{\infty} x_n &= (x_1+x_2)+(x_3+x_4)+(x_5+x_6)+\dots\\ &= \frac{1}{6}+\frac{1}{20}+\frac{1}{42}+\frac{1}{72}+\dots\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n)(2n+1)}\\ &= \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\dots\end{aligned}$$

The above alternating sum motivates us to use the infinite series representation of $\ln 2$, so the sum of x_n is $1-\ln 2$, implying that x_n converges.

(c) x_n can be written as

$$x_n = n-1+\frac{1}{n+1}$$

so the sum is divergent.

(d) x_n can be written as

$$x_n = 2+\frac{1}{n^2+1}$$

so the sum is divergent. □

Example 3.8 (MA2108 AY18/19 Sem 1). Determine whether each of the following sequences is convergent. Justify your answers.

(i)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}+\sqrt{n^2+1}}$$

(ii)

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{2n^2-\cos n}$$

(iii)

$$\sum_{n=1}^{\infty} x_n,$$

where x_n is defined to be the following:

$$x_n = \frac{3^{n+1}}{(n+1)!} \text{ if } n \text{ is odd}, \; x_n = -\frac{3^{n-1}}{(n-1)!} \text{ if } n \text{ is even.}$$

Solution.

(i) We use the alternating series test as $(-1)^{n+1}$ is present here. Define

$$a_n = \frac{1}{\sqrt{n}+\sqrt{n^2+1}}.$$

We prove that a_n is monotonically decreasing. Note that

$$a_{n+1} = \frac{1}{\sqrt{n+1}+\sqrt{(n+1)^2+1}}.$$

It is clear that $a_n > a_{n+1}$ because $\sqrt{n} < \sqrt{n+1}$ and $\sqrt{n^2+1} < \sqrt{(n+1)^2+1}$, thus the sequence is decreasing. Lastly,

$$\lim_{n\to\infty} a_n = 0 \quad \text{so} \quad \text{the series converges by the alternating series test.}$$

(ii) We use the limit comparison test. Let

$$a_n = \frac{\sqrt{n}+1}{2n^2-\cos n} \text{ and } b_n = \frac{1}{n^{3/2}}.$$

Note that b_n is the p-series, where $p = 3/2$, so b_n converges. Consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{2n^2-\cos n} = \lim_{n\to\infty} \frac{1+n^{-1/2}}{2-\cos n/n^2} = \frac{1}{2}.$$

As this limit is finite, the series converges by the limit comparison test.

(iii) The sum of x_n is a rearrangement of the following alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$$

Use the ratio test to prove that this alternating series converges. □

4. Limits of Functions

4.1. *Limit Theorems*

We will not discuss the formal definition of limits that have already been discussed in MA2002 and the early 2 and the front section of this set of notes.

Theorem 4.1 (sequential criterion). We have

$$\lim_{x \to a} f(x) = L$$

if and only if x_n is any sequence in the domain of f such that $x_n \neq a$ for all n and

$$\lim_{n \to \infty} x_n = a \quad \text{implies} \quad \lim_{n \to \infty} f(x_n) =$$

Theorem 4.2 (divergent criteria). Say we wish to prove that

$$\lim_{x \to a} f(x) \quad \text{does not exist.}$$

(i) Method 1: find a sequence x_n such that $x_n \neq a$ for all $n \in \mathbb{N}$ with

$$\lim_{n \to \infty} x_n = a \quad \text{but} \quad \lim_{n \to \infty} f(x_n) \text{ diverges}$$

(ii) Method 2: Find two sequences x_n and y_n such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = a \quad \text{but} \quad \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$$

Lemma 4.1. Let $c \in \mathbb{R}$. Then, there exist sequences $x_n \in \mathbb{Q}, y_n \in \mathbb{Q}'$ such that

$$x_n, y_n \neq c \quad \text{and} \quad \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c.$$

Example 4.1. It is a well-known fact that

$$\lim_{x \to 0} \cos\left(\frac{1}{x^2}\right)$$

does not exist, but how do we prove it?

Solution. We make use of the fact that $\cos(n\pi) = (-1)^n$ for all $n \in \mathbb{N}$ to establish the result. We set $f(x) = \cos(1/x^2)$ and $x_n = 1/\sqrt{n\pi}$. Note that $x_n \neq 0$. Hence, $f(x_n) = \cos(n\pi) = (-1)^n$ but $f(x_n)$ is divergent. By Method 1 of the divergent criteria (Theorem 4.2), the limit as $x \to 0$ does not exist. $\square$

4.2. *One-Sided Limits*

Proposition 4.1. We have

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

Example 4.2 (signum function)**.** The signum function, or $\operatorname{sgn}(x)$, is defined by the following piece-wise function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0. \end{cases}$$

Note that

$$\lim_{x \to 0^+} \operatorname{sgn}(x) = 1 \text{ but } \lim_{x \to 0^-} \operatorname{sgn}(x) = -1 \quad \text{so} \quad \lim_{x \to 0} \operatorname{sgn}(x) \text{ does not exist.}$$

Definition 4.1 (floor function)**.** The floor function of a number x, which is denoted by $\lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lfloor x \rfloor = n \quad \text{if } x \in [n, n+1).$$

Example 4.3. $\lfloor \pi \rfloor = 3$ and $\lfloor -4.8 \rfloor = -5$

Definition 4.2 (ceiling function)**.** The ceiling function of a number x, which is denoted by $\lceil x \rceil$, is defined to be the least integer greater than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lceil x \rceil = n \quad \text{if } x \in (n, n+1].$$

Example 4.4. $\lceil 6.1 \rceil = 7$ and $\lceil -7.8 \rceil = -7$

Two important inequalities in relation to the floor and ceiling function respectively are

$$n \leq \lfloor x \rfloor < n+1 \text{ and } n < \lceil x \rceil \leq n+1 \quad \text{for } n \in \mathbb{Z},$$

which can be used to solve equations, inequalities and limits involving them.

Definition 4.3 (fractional part)**.** For any number x, the fractional part of it is defined by $\{x\}$. So, for any $x > 0$, we have

$$\{x\} = x - \lfloor x \rfloor.$$

5. Continuous Functions

5.1. *Types of Discontinuity*

These are covered in MA2002 so we shall not emphasise much here.

Definition 5.1 (continuity). A function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Definition 5.2 (removable discontinuity). A removable discontinuity is a point on the graph that is undefined or does not fit the rest of the graph.

Example 5.1. The graph of $f(x) = x^2/x$ is discontinuous at $x = 0$ even though the right side can be simplified to $f(x) = x$. However, based on the original domain of the function, if $x = 0$, then the denominator will be 0 as well, which is impossible!

Example 5.2 (infinite discontinuity). Consider the graph of $g(x) = 1/x$, where

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x) = \infty.$$

g is said to have infinite discontinuity at $x = 0$.

Example 5.3 (jump discontinuity). In relation to the signum function discussed in Example 4.2, there is a jump discontinuity at $x = 0$.

Definition 5.3 (oscillating discontinuity). An oscillating discontinuity exists when the values of the function appear to be approaching two or more values simultaneously.

Example 5.4. Consider the graph of $h(x) = \sin(1/x)$, where $x = 0$ is regarded as a point of oscillating discontinuity.

Here is the formal definition for continuity at a point.

Definition 5.4 (formal definition of continuity). A function f is continuous at $x = a$ if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that} \quad |f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

5.2. *Special Functions*

Definition 5.5 (Dirichlet function). Named after mathematician Peter Gustav Lejeune Dirichlet, the Dirichlet function, $f(x)$, is defined to be the following:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is an example of a function that is nowhere continuous.

Theorem 5.1. The Dirichlet function is nowhere continuous.

Proof. Suppose $x \in \mathbb{Q}$, so $f(x) = 1$. We show that f is discontinuous at x. Let $\delta > 0$ be arbitrary and $y \in \mathbb{Q}$ such that $|x - y| < \delta$. Choose $\varepsilon = 1/2$. Without a loss of generality, assume $x < y$. Since there exists $z \in \mathbb{Q}'$ such that $x < z < y$ (due to the density of the irrationals in the reals), then

$$|f(x) - f(z)| = |1 - 0| = 1 > \frac{1}{2} = \varepsilon.$$

In a similar fashion, we now consider the case where $x > y$. There exists $z' \in \mathbb{Q}'$ such that $y < z' < x$, so

$$|f(x) - f(z')| = |1 - 0| = 1 > 1/2 = \varepsilon.$$

Therefore, if $x \in \mathbb{Q}$, f is discontinuous at x. For the case where $x \in \mathbb{Q}'$, the proof is very similar. □

Lemma 5.1. The Dirichlet function can be constructed as the double limit of a sequence of continuous function. That is,

$$f(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Definition 5.6 (Thomae's function)**.** Thomae's function maps all real numbers to the unit interval $[0,1]$. The function can be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}; \\ 1/q & \text{if } x = p/q,\ p,q \in \mathbb{N} \text{ and } \gcd(p,q) = 1 \end{cases}$$

It is named after Carl Johannes Thomae, and the function is also known as the popcorn function due to its nature.

It is a well-known fact that Thomae's function is not continuous at all rational points but continuous at all irrational points.

5.3. *Properties of Continuous Functions*

Proposition 5.1. The following hold:

(i) Combinations: Suppose f and g are continuous at $x = a$. Then,

$$\text{for any } \alpha \in \mathbb{R} \quad f \pm g, fg, \alpha f \text{ are also continuous at } x = a.$$

If $g(a) \neq 0$, then f/g is also continuous at $x = a$.

(ii) Composite functions: Suppose f and g are such that $g \circ f$ is defined. If

$$f \text{ is continuous at } a \text{ and } g \text{ is continuous at } f(a) \quad \text{then} \quad g \circ f \text{ is continuous at } a.$$

Moreover, suppose $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$ so that $g \circ f$ is defined. If f is continuous on A and g is continuous on B, then $g \circ f$ is continuous on A.

Theorem 5.2 (extreme value theorem)**.** If f is continuous on $[a,b]$, then there exist $x_1, x_2 \in [a,b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \text{ for all } x \in [a,b].$$

In short, if f is continuous on a closed and bounded interval, then its range is also bounded.

Theorem 5.3 (intermediate value theorem). If f is continuous on $[a,b]$, and $f(a) < k < f(b)$, then there exists a point $c \in (a,b)$ such that $f(c) = k$.

Corollary 5.1. If f is continuous on $[a,b]$, then

$$f([a,b]) = [m,M] \quad \text{where } m = \inf f([a,b]) \text{ and } M = \sup f([a,b]).$$

Corollary 5.2 (location of roots). If f is continuous on $[a,b]$, and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

Example 5.5 (MA2108 AY19/20 Sem 1). Let f be continuous on $[0,1]$ and $f(0) = f(1)$. Prove that for any positive integer n, there exists a $\zeta \in [0,1]$ such that

$$f\left(\zeta + \frac{1}{n}\right) = f(\zeta).$$

Solution. Define

$$g(x) = f\left(x + \frac{1}{n}\right) - f(x).$$

By the intermediate value theorem, g does not experience a change in its polarity for all $x \in [0,1]$. Suppose on the contrary that this claim is false. Then, by the method of differences,

$$\sum_{i=1}^{n} g\left(1 - \frac{i}{n}\right) = f\left(\frac{1}{n}\right) - f(0)$$
$$g(0) = f\left(\frac{1}{n}\right) - f(0)$$

Without a loss of generality, assume that $g(x) > 0$ for all $x \in [0,1]$. Then, setting $n = 1$, it implies that $f(1) - f(0) > 0$, which is a contradiction! $\square$

5.4. *Monotone and Inverse Functions*

Definition 5.7. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $x_1, x_2 \in A$. Then, the following hold:

(i) f is increasing on A if $x_1 \le x_2$ implies $f(x_1) \le f(x_2)$
(ii) f is strictly increasing on A if $x_1 < x_2$ implies $f(x_1) < f(x_2)$
(iii) f is decreasing on A if $x_1 \le x_2$ implies $f(x_1) \ge f(x_2)$
(iv) f is strictly decreasing on A if $x_1 < x_2$ implies $f(x_1) > f(x_2)$
(v) f is monotone if it is either increasing or decreasing
(vi) f is strictly monotone if it either strictly increasing or strictly decreasing

Proposition 5.2. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function. If $c \in I$ is not an endpoint of I, then

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x \in I, x < c\} \quad \text{and} \quad \lim_{x \to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}.$$

Theorem 5.4 (continuous inverse theorem). Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If f is continuous on I and $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is strictly monotone and continuous on J.

5.5. *Uniform Continuity*

Definition 5.8 (uniform continuity). Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. f is uniformly continuous on I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for any } x, y \in I \quad |x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.$$

Corollary 5.3. If a function is uniformly continuous on I, then it is continuous on I.

Example 5.6. We claim that $f(x) = x^2$ is uniformly continuous on $[0,1]$. To see why, let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/2$. For $x, y \in [0,1]$, suppose $|x - y| < \delta$. Then,

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 2 \cdot \delta = \varepsilon$$

and we are done.

Theorem 5.5. A function f is uniformly continuous on I if and only if f' is bounded.

It is worth noting that $f(x) = x^2$ is uniformly continuous on $[a,b]$ in general, where $a, b \in \mathbb{R}$, but it is not uniformly continuous on $\mathbb{R}$!

Example 5.7 (MA2108 AY19/20 Sem 1). Prove that the function $f(x) = \sqrt{x^2 - x + 1}$ is uniformly continuous on $[1, \infty)$[†].

Solution. Since

$$f'(x) = \frac{1}{2}\left(x^2 - x + 1\right)^{-1/2} \cdot (2x - 1) = \frac{2x - 1}{2\sqrt{x^2 - x + 1}},$$

and noting that $x^2 - x + 1 > 0$ for all $x \in [1, \infty)$, as well as $|f'(x)| < 1$, by Theorem 5.5, the result follows. $\square$

Theorem 5.6 (sequential criterion for uniform continuity). $f : I \to \mathbb{R}$ is uniformly continuous on I if and only if for any two sequences $x_n, y_n \in I$ such that

$$\text{if } \lim_{n \to \infty} (x_n - y_n) = 0 \quad \text{then} \quad \lim_{n \to \infty} [f(x_n) - f(y_n)] = 0.$$

Definition 5.9 (Lipschitz continuity). Let I be an interval and $f : I \to \mathbb{R}$ satisfies the Lipschitz condition on I. Then, there is $K > 0$ such that

$$|f(x) - f(y)| \le K|x - y|, \quad \text{for all } x, y \in I.$$

Theorem 5.7. If a function is Lipschitz continuous on I, then it is uniformly continuous on I.

Example 5.8. We verify that $f(x) = x^2$, in the interval $[0,1]$, satisfies the Lipschitz condition.

Solution. Since $f(x) - f(y) = x^2 - y^2$, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{x^2 - y^2}{x - y}\right| = |x + y| \le 2,$$

[†]The original question had an error. It used $f(x) = \sqrt{x(x-1)}$, which is undefined at $x = 1$. Confirmed the change with one of the students.

and since $2>0$, $f(x)=x^2$, in $[0,1]$, is said to satisfy the Lipschitz condition. In other words, f is Lipschitz continuous. $\square$

Theorem 5.8. If $f:I\to\mathbb{R}$ is uniformly continuous on I and x_n is Cauchy, then $f(x_n)$ is Cauchy.

If the function $f:(a,b)\to\mathbb{R}$ is uniformly continuous on (a,b), then $f(a)$ and $f(b)$ can be defined so that the extended function is continuous on $[a,b]$.

6. The Topology of the Real Numbers

6.1. *Open and Closed Sets in $\mathbb{R}$*

Recall that a neighbourhood of a point $x \in \mathbb{R}$ is any set V that

$$\text{contains an } \varepsilon\text{-neighbourhood } V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \text{ of } x \quad \text{for some } \varepsilon > 0.$$

Definition 6.1 (open set). A subset G of $\mathbb{R}$ is open in $\mathbb{R}$ if for each $x \in G$, there exists a neighbourhood V of x such that $V \subseteq G$.

To show that $G \subseteq \mathbb{R}$ is open, it suffices to show that each point in G has an ε-neighbourhood contained in G. In fact, G is open if and only if for each $x \in G$, there exists $\varepsilon_x > 0$ such that $(x-\varepsilon_x, x+\varepsilon_x)$ is contained in G.

Definition 6.2 (closed set). A subset F of $\mathbb{R}$ is closed in $\mathbb{R}$ if the complement $\mathbb{R} \backslash F$ is open in $\mathbb{R}$.

To show that F is closed, it suffices to show that each point $y \notin F$ has an ε-neighbourhood disjoint from F. In fact, F is closed if and only if for each $y \notin F$, there exists $\varepsilon_y > 0$ such that $F \cap (y-\varepsilon_y, y+\varepsilon_y) = \emptyset$.

Proposition 6.1. Any open interval $I = (a,b)$ is an open set.

Proof. For $x \in I$, take $\varepsilon_x = \min\{x-a, b-x\}$. Then, $x+\varepsilon_x = b$ and $x-\varepsilon_x = a$, so $(x-\varepsilon_x, x+\varepsilon_x) \subseteq I$. □

Example 6.1. We can show that the set $[0,1]$ is not open but closed. Note that in English, the terms 'open' and 'closed' are antonyms, but in Topology, these are not the opposite of each other.

We first show that $[0,1]$ is not open. This is clear since every ε-neighbourhood of $0 \in I$ contains points not in I. In a similar fashion, instead of 0, we can take the other endpoint, which is 1. Note that $(0-\varepsilon_x, 0+\varepsilon_x)$ is the same as $(-\varepsilon_x, \varepsilon_x)$, which clearly shows that this interval contains points not in I.

To show that $[0,1]$ is closed, let $y \notin I$. Then, $y < 0$ or $y > 1$. If $y < 0$, take $\varepsilon_y = |y|$, and if $y > 1$, take $\varepsilon_y = y-1$. For the former, $[0,1] \cap (y-|y|, y+|y|)$ is equivalent to $(2y, 0) \cap [0,1]$, which is clearly $\emptyset$. For the latter, $[0,1] \cap (1, 2y-1)$ is equivalent to $\emptyset$ too.

7. Differentiable Functions

7.1. *First Principles*

Definition 7.1 (differentiable function). A function f is differentiable at a point a if f is defined in some open interval containing a and the limit

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad \text{exists.}$$

In this case, $f'(a)$ is the derivative of f at $x=a$.

Geometrically, $f'(a)$ is the slope of the tangent to the curve $y=f(x)$ at $x=a$. The formula in Definition 7.1 is similar to one that students have learnt in H2 Mathematics, which is the derivative of $f(x)$, denoted by $f'(x)$, can be expressed as

$$\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}.$$

Example 7.1 (MA2108 AY22/23 Sem 1 Warmup). Let $y=f(x)$ be continuous everywhere for $x\in(-\infty,\infty)$ and satisfy

$$f(0)=1, f(1)=e \text{ and } f(x+y)=f(x)f(y).$$

Prove that $f(x)=e^x$ for $x\in(-\infty,\infty)$.

Solution. It is clear that

$$f\left(\sum_{i=1}^n x_i\right)=\prod_{i=1}^n f(x_i).$$

By first principles,

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=\lim_{\delta x\to 0}\frac{f(x)f(\delta x)-f(x)}{\delta x}=f(x)\lim_{\delta x\to 0}\frac{f(\delta x)-1}{\delta x}=f(x)f'(0).$$

We set $c=f'(0)$ so $f'(x)=cf(x)$. Integrating, we have $\ln|f(x)|=cx+d$. Since $f(0)=1$, then $d=0$. Since $f(1)=e$, then $c=1$. Thus, $\ln|f(x)|=x$. As such, we conclude that $f(x)=e^x$. □

Definition 7.2. If f is differentiable at every point in (a,b), then f is differentiable on (a,b).

Proposition 7.1. If the function $f:[a,b]\to\mathbb{R}$ is such that f is differentiable on (a,b) and the one sided limits

$$L_1=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} \quad\text{and}\quad L_2=\lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$$

exist, then f is differentiable on $[a,b]$. In this case, $f'(a)=L_1$ and $f'(b)=L_2$.

7.2. *Continuity and Differentiability*

Definition 7.3. f is continuously differentiable on I if f is differentiable on I and f' is continuous on I.

Definition 7.4. The collection of all functions which are continuously differentiable on I is denoted by $\mathcal{C}^1(I)$.

Proposition 7.2. If f is differentiable at a, then it is continuous at a.

Proof. We have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right) + f(a)\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a} x-a\right) + f(a)\\ &= f'(a)\cdot 0 + f(a)\end{aligned}$$

which is just $f(a)$. □

The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere. It is an example of a fractal curve named after its discoverer German mathematician Karl Weierstrass[†].

Definition 7.5 (Weierstrass function)**.** In Weierstrass's original paper, the function was defined as the following Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is a positive odd integer and $ab > 1 + 3\pi/2$.

7.3. *Derivative Rules and Theorems*

Theorem 7.1 (Carathéodory's theorem)**.** Let I be an interval, $f : I \to \mathbb{R}$ and $c \in I$. Then $f'(c)$ exists if and only if there exists a function ϕ on I such that ϕ is continuous at c and

$$f(x) - f(c) = \phi(x)(x-c) \quad \text{for all } x \in I.$$

Proposition 7.3 (chain rule)**.** Let I and J be intervals, and let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be such that $f(J) \subseteq I$. If $a \in J$, f is differentiable at a and g is differentiable at $f(a)$, then $h = g \circ f$ is differentiable at a, and

$$h'(a) = g'(f(a))f'(a).$$

Theorem 7.2 (inverse function theorem)**.** If f is a continuously differentiable function with non-zero derivative at a; then f is invertible in a neighbourhood of a, the inverse is continuously differentiable, and the derivative of the inverse function at $b = f(a)$ is the reciprocal of the derivative of f at a. As an equation, we have

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

[†]This link provides an analysis of the Weierstrass function involving its uniform convergence (this term will be studied in due course) and it being nowhere differentiable. This involves the Weierstrass M-test.

7.4. *Mean Value Theorem and Applications*

Definition 7.6. Let I be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$.

(i) If $f(x_0) \geq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute maximum of f on I

(ii) If $f(x_0) \leq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute minimum of f on I

(iii) If there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f

(iv) If there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f

(v) If there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative minimum of f

(vi) If $f(x_0)$ is either a relative minimum or relative maximum of f, then $f(x_0)$ is a relative extremum of f

Remark 7.1. A relative extremum can only occur at an interior point, but an absolute extremum may occur at one of the end points of the interval. So if a function has an absolute maximum at a point x_0, it may not have a relative maximum at x_0. If f has an absolute maximum at an interior point x_0 of I, then $f(x_0)$ is also a relative maximum of f.

Lemma 7.1. Let $f : (a,b) \to \mathbb{R}$ and $f'(c)$ exists for some $c \in (a,b)$.

(i) If $f'(c) > 0$, then there exists $\delta > 0$ such that

$$f(x) < f(c) \text{ for every } x \in (c - \delta, c) \text{ and } f(x) > f(c) \text{ for every } x \in (c, c + \delta).$$

(ii) If $f'(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) > f(c) \text{ for every } x \in (c - \delta, c) \text{ and } f(x) < f(c) \text{ for every } x \in (c, c + \delta).$$

Theorem 7.3 (Fermat's extremum theorem)**.** Suppose c is an interior point of an interval I and $f : I \to \mathbb{R}$ is differentiable at c. If f has a relative extremum at c, then $f'(c) = 0$.

Proof. Without a loss of generality, assume that f has a relative maximum at c (the proof if f has a relative minimum is similar). Suppose on the contrary that either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then by the lemma above, there exists $\delta > 0$ such that $f(x) < f(c)$ for every $x \in (c - \delta, c)$ and $f(x) > f(c)$ for every $x \in (c, c + \delta)$. This contradicts the assumption that f has a relative maximum at c. The proofs for other cases are similar. □

Remark 7.2. A function f may have a relative extremum at x_0, but $f'(x_0)$ does not exist.

Example 7.2. Consider $f(x) = |x|$. There is a relative (absolute) minimum at $x = 0$, but $f'(0)$ does not exist.

The converse of Fermat's theorem is false. For example, consider $f(x) = x^3$, where $f'(0) = 0$ but $x = 0$ is not a relative extremum point of f. It is merely a point of inflection.

Theorem 7.4 (Rolle's theorem)**.** If f is continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Proof. The proof where $f(x)$ is a constant will not be discussed since it is trivial. For the more meaningful cases, we have $f(x) > f(a)$ or $f(x) < f(a)$ for some $x \in (a,b)$. Without a loss of generality, we shall prove the former case since the proof for the latter is similar.

By the extreme value theorem (Theorem 5.2), we know that $f(x)$ has a maximum, M in the closed interval $[a,b]$. As $f(a) = f(b)$, the maximum value is attained at $x = c$. That is, $f(c) = M$. So, f has a local maximum at c. Since f is differentiable, the result follows. □

Theorem 7.5 (mean value theorem)**.** If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof. We wish to construct a function $g : [a,b] \to \mathbb{R}$ such that $g(a) = g(b) = 0$, with a point $c \in (a,b)$ such that $g'(c) = 0$. Suppose

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a} \cdot (x-a).$$

It is clear that g is continuous on $[a,b]$ and differentiable on (a,b), and $g(a) = g(b) = 0$. By Rolle's theroem, there exists $c \in (a,b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} = 0.$$

Rearranging the equation, we are done. □

Corollary 7.1. If f is continuous on $[a,b]$, differentiable on (a,b) and $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Example 7.3 (Berkeley Problems in Mathematics P1.1.25)**.** Let the function f from $[0,1]$ to $[0,1]$ have the following properties:

(i) f is C^1 (i.e. f is differentiable and its derivative is continuous)

(ii) $f(0) = f(1) = 0$

(iii) f' is non-increasing (i.e. f is concave down)

Prove that the arc length of the graph of f does not exceed 3.

Solution. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, by Rolle's theorem, there exists $c \in (0,1)$ such that $f'(c) = 0$. Since f is concave down, then f is increasing on $(0,c)$ and decreasing on $(c,1)$. On $[0,c]$, the arc length of f is given by

$$\int_0^c \sqrt{1 + [f'(x)]^2}\, dx.$$

We shall partition $(0,c)$ into n equally sized subintervals. Hence, each interval has width c/n. So,

$$\int_0^c \sqrt{1 + [f'(x)]^2}\, dx = \lim_{n\to\infty} \frac{c}{n} \sum_{k=1}^{n} \sqrt{1 + [f'(\zeta_k)]^2} \quad \text{where } \zeta_k = \left(\frac{c(k-1)}{n}, \frac{ck}{n} \right).$$

By the mean value theorem, each ζ_k satisfies

$$f'(\zeta_k) = \frac{f(ck/n) - f(c(k-1)/n)}{c/n}$$

so

$$\begin{aligned}
\int_0^c \sqrt{1+[f'(x)]^2}\,dx &\le \lim_{n\to\infty} \frac{c}{n}\sum_{k=1}^{n}\sqrt{1+\left[\frac{f(ck/n)-f(c(k-1)/n)}{c/n}\right]^2} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\left(\frac{c}{n}\right)^2+\left[f\left(\frac{ck}{n}\right)-f\left(\frac{c(k-1)}{n}\right)\right]^2} \\
&\le \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{c}{n}+f\left(\frac{ck}{n}\right)-f\left(\frac{c(k-1)}{n}\right)\right] \quad \text{since } \sqrt{a^2+b^2}\le a+b \\
&= c+f(c) \quad \text{by method of difference.}
\end{aligned}$$

In a similar fashion, one can prove that

$$\int_c^1 \sqrt{1+[f'(x)]^2}\,dx \le 1-c+f(c).$$

so

$$\int_0^1 \sqrt{1+[f'(x)]^2}\,dx \le c+f(c)+1-c+f(c)=1+2f(c)\le 1+2=3$$

where we used the fact that the range of f is $[0,1]$. $\square$

Proposition 7.4 (increasing and decreasing functions). Let f be differentiable on (a,b).

(i) If $f'(x)\ge 0$ for all $x\in(a,b)$, then f is increasing on (a,b)

(ii) If $f'(x)\le 0$ for all $x\in(a,b)$, then f is decreasing on (a,b)

Theorem 7.6 (first derivative test). Let f be a continuous function on $[a,b]$ and $c\in(a,b)$. Suppose f is differentiable on (a,b) except possibly at c.

(i) If there is a neighbourhood $(c-\delta,c+\delta)\subseteq I$ of c such that $f'(x)\ge 0$ for $x\in(c-\delta,c)$ and $f'(x)\le 0$ for $x\in(c,c+\delta)$, then $f(c)\ge f(x)$ $\forall x\in(c-\delta,c+\delta)$. Hence, f has a relative maximum at c

(ii) If there is a neighbourhood $(c-\delta,c+\delta)\subseteq I$ of c such that $f'(x)\le 0$ for $x\in(c-\delta,c)$ and $f'(x)\ge 0$ for $x\in(c,c+\delta)$, then $f(c)\le f(x)$ $\forall x\in(c-\delta,c+\delta)$. Hence, f has a relative minimum at c

Consider a function $f(x)$. Its first derivative is denoted by $f'(x)$, second derivative is denoted by $f''(x)=f^{(2)}(x)$, and so on. In general, for $n\in\mathbb{N}$, the n^{th} derivative of f at c is defined as

$$f^{(n)}(c)=(f^{n-1})'(c).$$

Let I be an interval. Then, for $n\in\mathbb{N}$, $\mathcal{C}^n(I)$ is defined to be the set of functions f such that $f^{(n)}$ exists and is continuous on I. Note that

$$\mathcal{C}^\infty(I)=\bigcap_{n=1}^{\infty}\mathcal{C}^n(I).$$

If $\in\mathcal{C}^\infty(I)$, then f is infinitely differentiable on I.

Proposition 7.5. For $m>n\ge 1$, where $m,n\in\mathbb{Z}$,

$$\mathcal{C}^\infty(I)\subseteq\mathcal{C}^m(I)\subseteq\mathcal{C}^n(I)\subseteq\mathcal{C}(I).$$

Theorem 7.7 (second derivative test). Let f be defined on an interval I and let its derivative f' exist on I. Suppose c is an interior point of f such that $f'(c) = 0$ and $f''(c)$ exists.

(i) If $f''(c) > 0$, then f has a relative minimum at c

(ii) If $f''(c) < 0$, then f has a relative maximum at c

(iii) If $f''(c) = 0$, then the test is inconclusive. Hence, we have to use the first derivative test to prove whether c is a relative minimum, relative maximum, or a point of inflection

Theorem 7.8 (Cauchy's mean value theorem). Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. We first claim that $g(a) \neq g(b)$. Suppose otherwise, then $g(a) = g(b)$, so by Rolle's theorem, there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$, contradicting the assumption that $g'(x) \neq 0$ for all $x \in (a,b)$. Next, define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)} \cdot ((g(x)-g(a)) - (f(x)-f(a)),$$

where $x \in [a,b]$. Since h is continuous on $[a,b]$, differentiable on (a,b) and $h(a) = h(b) = 0$, by Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. The result follows. $\square$

Theorem 7.9 (Taylor's theorem). Let f be a function such that $f \in C^n([a,b])$ and $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$, then for any $x \in [a,b]$, there exists a point c between x and x_0 such that

$$f(x) = \sum_{k=0}^{n+1} \frac{f^k(c)}{k!}(x-x_0)^k.$$

Corollary 7.2. If $n = 0$, then $f(x) = f(x_0) + f'(c)(x-x_0)$, which is the mean value theorem.

The polynomial $P_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

is the n^{th} Taylor polynomial for f at x_0.

By Taylor's theorem, as $f(x) = P_n(x) + R_n(x)$, then

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1}$$

for some point c_n between x and x_0. This formula for R_n is the Lagrange form of the remainder.

Let f be infinitely differentiable on $I = (x_0 - r, x_0 + r)$ and $x \in I$. Then, recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

if and only if

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1} = 0,$$

where each c_n is between x and x_0.

8. The Riemann–Stieltjes Integral

8.1. *Definition and Existence*

Let $I = [a,b]$. A finite set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where

$$a < x_0 < x_1 < x_2 < \ldots < x_n < b$$

is a partition of I. It divides I into the subintervals as

$$I = [x_0,x_1] \cup [x_1,x_2] \cup [x_2,x_3] \cup \ldots \cup [x_{n-1},x_n] = \bigcup_{i=1}^{n} [x_{i-1},x_i].$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$. For each $1 \le i \le n$, let

$$M_i = M_i(f,P) = \sup\{f(x) : x \in [x_{i-1},x_i]\},$$
$$m_i = m_i(f,P) = \inf\{f(x) : x \in [x_{i-1},x_i]\} \text{ and}$$

$\Delta x_i = x_i - x_{i-1}$. Define the upper sum and lower sum of f with respect to P to be

$$U(f,p) = \sum_{i=1}^{n} M_i \Delta x_i \text{ and } L(f,p) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Note that each partition may not be of uniform length.

By setting $m = \inf\{f(x) : x \in [a,b]\}$ and $M = \sup\{f(x) : x \in [a,b]\}$, then

$$m(b-a) \le L(f,p) \le U(f,p) \le M(b-a).$$

Furthermore,

$$m(b-a) \le \int_a^b f \le M(b-a)$$

and if $f(x) \ge 0$ for all $x \in [a,b]$, then

$$\int_a^b f \ge 0.$$

Definition 8.1 (Darboux integral)**.** The upper Darboux integral of f on $[a,b]$ is defined to be

$$U(f) = \overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

and the lower Darboux integral of f on $[a,b]$ is defined to be

$$L(f) = \underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

Lemma 8.1. $L(f) \le U(f)$

Proof. We prove by contradiction. Suppose $U(f) < L(f)$. Then, there exists a partition P_1 of $[a,b]$ such that

$$U(f) \le U(f,P_1) < L(f).$$

Also, there exists a partition P_2 of $[a,b]$ such that

$$U(f) \le U(f,P_1) < L(f,P_2) \le L(f).$$

However, $L(f,P_2) \le U(f,P_1)$, which is a contradiction. □

From Lemma 8.1, it is clear that for partitions P and Q of $[a,b]$,

$$L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,Q),$$

and consequently,

$$L(f) = \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f).$$

Definition 8.2. If P and Q are partitions of $[a,b]$, then Q is a refinement of P if $P \subseteq Q$.

Proposition 8.1. If P and Q are partitions of $[a,b]$ with Q a refinement of P, then

$$L(f,P) \le L(f,Q) \quad \text{and} \quad U(f,Q) \le U(f,P).$$

Definition 8.3. A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

The Riemann integral is only defined for bounded functions (i.e. if f is unbounded on $[a,b]$, it is not integrable on $[a,b]$) .

Example 8.1.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

is not integrable since $\lim_{x\to 0} 1/x^2 = \infty$, implying that the function is unbounded on $[-1,1]$.

Consider the Dirichlet function and denote it by $f(x)$. Since the rational and irrational numbers both form dense subsets of $\mathbb{R}$, then f takes on the value of 0 and 1 on every sub-interval of any partition. Thus for any partition P, $U(f,P) = 1$ and $L(f,P) = 0$. By noting that the upper and lower Darboux integrals are unequal, we conclude that f is not Riemann integrable on $[0,1]$[†].

8.2. *Riemann Integrability Criterion and Consequences*

Theorem 8.1 (Riemann integrability criterion). For a bounded function $f : [a,b] \to \mathbb{R}$, then f is integrable on $[a,b]$ if and only if for any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. We first prove that if $U(f,P) - L(f,P) < \varepsilon$, then f is integrable on $[a,b]$. Note that $\varepsilon > 0$ is arbitrary. Recall that

$$L(f,P) \le L(f) \le U(f) \le U(f,P).$$

Hence,

$$U(f) - L(f) \le U(f,P) - L(f,P) < \varepsilon,$$

and we are done.

Now, suppose f is integrable on $[a,b]$. We wish to prove that $U(f,P) - L(f,P) < \varepsilon$. Note that there exists

[†] A fun fact is that the Dirichlet function is actually Lebesgue integrable (covered in MA4262).

a partition P_1 on $[a,b]$ such that $U(f,P_1) < U(f)$ so

$$U(f,P_1) < U(f) + \frac{\varepsilon}{2}.$$

In a similar fashion, there exists a partition P_2 such that

$$L(f,P_2) > L(f) - \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ be the common refinement of the previous two partitions. Since

$$0 \le U(f,P) - L(f,P),$$

then

$$0 \le U(f,P) - L(f,P) < U(f) - L(f) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

$\square$

Corollary 8.1. If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then f is integrable on $[a,b]$.

Corollary 8.2. If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Corollary 8.3. Let $f,g : [a,b] \to \mathbb{R}$ be bounded functions, P be a partition of $[a,b]$ and $c \in \mathbb{R}$. Then,

(i)
$$L(cf,P) = \begin{cases} cL(f,P) & \text{if } c > 0 \\ cU(f,P) & \text{if } c < 0 \end{cases}$$

(ii)
$$U(cf,P) = \begin{cases} cU(f,P) & \text{if } c > 0 \\ cL(f,P) & \text{if } c < 0 \end{cases}$$

(iii)
$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$$

Proposition 8.2. Let $f,g : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and $c \in \mathbb{R}$. Then,

(i) Just like linear transformations, the function $cf+g$ is integrable on $[a,b]$ and
$$\int_a^b (cf+g) = c\int_a^b f + \int_a^b g.$$

(ii) If $f(x) \le g(x)$ for all $x \in [a,b]$, then
$$\int_a^b f \le \int_a^b g.$$

(iii) $|f|$ is integrable on $[a,b]$ and
$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

(iv) fg is integrable on $[a,b]$.

Proposition 8.3. If f is integrable on $[a,b]$, then for any $c \in (a,b)$, f is integrable on $[a,c]$ and $[c,b]$. The converse is true and we have the following result:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Let f be a continuous function on $[a,b]$. If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a,b]$, define

$$L(f,P) = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$

where the supremum is taken over all possible partitions $a = x_0 < x_1 < x_2, \ldots < x_n = b$. This definition as the supremum of the all possible partition sums is also valid if f is merely continuous, not differentiable.

8.3. *Fundamental Theorems of Calculus*

Theorem 8.2 (First Fundamental Theorem of Calculus)**.** Let f be integrable on $[a,b]$ and for $x \in [a,b]$, let

$$F(x) = \int_a^x f.$$

If f is continuous at a point $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Remark 8.1. Not all functions have an elementary antiderivative. That is, for example, there do not exist elementary functions $F(x)$ and $G(x)$ such that

$$F(x) = \int e^{-x^2}\, dx \text{ and } G(x) = \int \frac{1}{\ln x}.$$

Theorem 8.3 (Second Fundamental Theorem of Calculus)**.** Let g be a differentiable function on $[a,b]$ and assume that g' is continuous on $[a,b]$. Then,

$$\int_a^b g' = g(b) - g(a).$$

Theorem 8.4 (Cauchy's Fundamental Theorem of Calculus)**.** Let g be a differentiable function on $[a,b]$ and assume that g' is integrable on $[a,b]$. Then,

$$\int_a^b g' = g(b) - g(a).$$

Example 8.2. It is possible for the derivative of a function to not be integrable. Consider the following function:

$$f(x) = \begin{cases} x^2 \sin\left(1/x^2\right) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

For $x \neq 0$,

$$f'(x) = -\frac{2}{x}\cos\left(\frac{1}{x^2}\right) + 2x\sin\left(\frac{1}{x^2}\right)$$

but $f'(x)$ is not integrable on $[-1,1]$ as this is a region of oscillating discontinuity!

8.4. *Riemann Sum*

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$ and let $\Delta x = x_i - x_{i-1}$ for $1 \leq i \leq n$. Then, the norm of P, denoted by $\|P\|$, is defined by

$$\|P\| = \max\{\Delta x_i : 1 \leq i \leq n\}.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$, $\|P\| < \delta$ implies that

$$U(f,P) < \overline{\int_a^b} f + \varepsilon \text{ and } L(f,P) > \underline{\int_a^b} f - \varepsilon.$$

We are now ready to define the Riemann sum of f with respect to P.

Definition 8.4 (Riemann sum)**.** Let ξ_i be a point in the i^{th} sub-interval $[x_{i-1}, x_i]$ for $1 \leq i \leq n$. The sum

$$S(f,P)(\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is the Riemann sum of f with respect to P and $\xi = (\xi_1, \ldots, \xi_n)$.

If there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$ and any choice of $\xi = (\xi_1, \ldots, \xi_n)$,

$$\|P\| < \delta \quad \text{implies} \quad |S(f,P)(\xi) - A| < \varepsilon,$$

then

$$\lim_{\|P\| \to 0} S(f,P)(\xi) = A.$$

Note that

$$L(f,P) \leq S(f,P)(\xi) \leq U(f,P).$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then,

$$\lim_{\|P\| \to 0} U(f,P) = \overline{\int_a^b} f \quad \text{and} \quad \lim_{\|P\| \to 0} L(f,P) = \underline{\int_a^b} f.$$

Hence, f is integrable on $[a,b]$ and $\int_a^b f = A$ if and only if

$$\lim_{\|P\| \to 0} S(f,P)(\xi) = A.$$

Corollary 8.4. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. For each $n \in \mathbb{N}$, let $P_n = \left\{x_0^{(n)}, x_1^{(n)}, \ldots, x_{m_n}^{(n)}\right\}$ be a partition of $[a,b]$ and let $\xi^{(n)} = \left(\xi_1^{(n)}, \ldots, \xi_{m_n}^{(n)}\right)$ be such that $\xi_i^{(n)} \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right]$ for all $1 \leq i \leq m_n$. Define the sequence y_n as follows:

$$y_n = S(f,p)(\xi^{(n)})$$

If $\lim_{n\to\infty} \|P_n\| = 0$, then

$$\lim_{n\to\infty} y_n = \int_a^b f.$$

8.5. *Improper Integrals*

Definition 8.5 (improper integral)**.** An improper integral is one such that either the integrand, f, is unbounded on (a,b) or the interval of integration is unbounded.

Proposition 8.4. Suppose f is defined on $[a,b)$ and f is integrable on $[a,c]$ for every $c \in (a,b)$. If the limit

$$L = \lim_{c \to b^-} \int_a^c f(x)\, dx$$

exists, then

$$\text{the improper integral } \int_a^b f(x)\, dx \text{ converges} \quad \text{and} \quad \int_a^b f(x)\, dx = L.$$

If the limit does not exist, then the improper integral diverges.

Similarly, if f is defined on $(a,b]$ and f is integrable on $[c,b]$ for every $c \in (a,b)$, then

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} f(x)\, dx \quad \text{provided that the limit exists.}$$

9. Sequences and Series of Functions

9.1. *Pointwise and Uniform Convergence*

An example of a sequence of functions $f_n(x)$, where $n \in \mathbb{N}$ is

$$f_n(x) = \frac{x+x^n}{2+x^n}$$

for $x \in [0,1]$. Then, consider the integral

$$\int_0^{\frac{1}{2}} f_n(x).$$

As $n \to \infty$, what can be deduced?

This section deals with questions like these. To start off, we need to introduce the ideas of pointwise convergence and uniform convergence.

Definition 9.1 (pointwise convergence). Let E be a non-empty subset of $\mathbb{R}$. Suppose for each $n \in \mathbb{N}$, we have a function $f_n : E \to \mathbb{R}$. Then, f_n is a sequence of functions on E. For each $x \in E$, the sequence $f_n(x)$ of real numbers converges. Define the function $f : E \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in E.$$

Then, f_n converges to f pointwise on E, and so $f_n \to f$ pointwise on E.

Definition 9.2 (pointwise convergence). $f_n \to f$ pointwise on E if and only if for every $x \in E$ and for every $\varepsilon > 0$, there exists $K = K(\varepsilon, x) \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon.$$

Remark 9.1. If $f_n \to f$ pointwise on I and each f_n is continuous on I, then f is not necessarily continuous on I.

Example 9.1. Consider $f_n(x) = x^n$ for $x \in [0,1]$. Note that each f_n is continuous on $[0,1]$. However, f is not continuous at $x = 1$ since for $x \in [0,1)$, then

$$\lim_{n\to\infty} f_n(x) = 0$$

but for $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = 1.$$

Remark 9.2. If $f_n \to f$ pointwise on $[a,b]$ and each f_n is integrable on $[a,b]$, then

(1). f is not necessarily integrable on $[a,b]$

(2). the pointwise convergence

$$\int_a^b g_n \to \int_a^b g \quad \text{is not necessarily true.}$$

Remark 9.3. If $f_n \to f$ pointwise on $[a,b]$ and each f_n and f are differentiable on $[a,b]$, then $f_n' \to f'$ not necessarily pointwise on $[a,b]$.

Example 9.2. Consider $f_n(x) = \sin(nx)/\sqrt{n}, x \in \mathbb{R}$. $f(x) = 0$ for all $x \in \mathbb{R}$, and thus $f_n \to f$ pointwise on $\mathbb{R}$. As $f'(x) = \sqrt{n}\cos(nx)$, for each $n \in \mathbb{N}$, $f' = 0$, but $f_n' \to f'$ pointwise on $\mathbb{R}$. Then, $f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$, but $f'(0) = 0$.

Definition 9.3 (uniform convergence). A sequence of functions f_n converges uniformly to f on E if for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. In this case, $f_n \to f$ uniformly on E. We say that the sequence f_n of functions converges uniformly on E if there exists a function f such that f_n converges to f uniformly on E.

Definition 9.4 (uniform norm). Let $E \subseteq \mathbb{R}$ and let $\phi : E \to \mathbb{R}$ be a bounded function. The uniform norm of ϕ on E is defined as

$$\|\phi\|_E = \sup\{|\phi(x)| : x \in E\}.$$

Then, $|\phi(x)| \leq \|\phi\|_E$ for all $x \in E$.

Lemma 9.1. A sequence of functions f_n converges to f uniformly on E if and only if $\|f_n - f\|_E \to 0$.

Proposition 9.1 (Cauchy criterion). A sequence of functions f_n converges uniformly on E if and only if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_E < \varepsilon \quad \text{for all } m, n \geq K.$$

Proposition 9.2. The following hold:

(i) If f_n converges uniformly on E, then f_n converges pointwise on E

(ii) If f_n converges uniformly on E and $F \subseteq E$, then f_n converges uniformly on F

Proposition 9.3. If f_n converges uniformly to f on an interval I and each f_n is continuous at $x_0 \in I$, then f is continuous at x_0.

Corollary 9.1. If f_n converges uniformly to f on I and each f_n is continuous on I, then f is continuous on I. Hence,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} f_n(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x),$$

implying that we can interchange the order of the two limit operations.

Proposition 9.4. Suppose $f_n \to f$ uniformly on $[a,b]$ and each f_n is integrable on $[a,b]$. Then,
(i): f is integrable on $[a,b]$ and

(ii): for each $x_0 \in [a,b]$, the sequence of functions

$$F_n(x) = \int_{x_0}^{x} f_n(t)\, dt$$

converges uniformly to the function

$$F(x) = \int_{x_0}^{x} f(t)\, dt$$

on $[a,b]$. Hence,

$$\lim_{n\to\infty} \int_{x_0}^{x} f_n(t)\, dt = \int_{x_0}^{x} \lim_{n\to\infty} f_n(t)\, dt$$

and in particular,

$$\lim_{n\to\infty} \int_{a}^{b} f_n(t)\, dt = \int_{a}^{b} f(t)\, dt.$$

Proposition 9.5. Suppose f_n is a sequence of differentiable functions on $[a,b]$ such that

$$f_n(x_0) \text{ converges for some } x_0 \in [a,b] \quad \text{and} f_n' \text{ converges uniformly on } [a,b].$$

Then, f_n converges uniformly on $[a,b]$ to a differentiable function f and for $a \le x \le b$,

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

9.2. *Infinite Series of Functions*

If f_n is a sequence of functions on E, then

$$S = \sum_{n=1}^{\infty} f_n \quad \text{is} \quad \text{an infinite series of functions.}$$

For each $n \in \mathbb{N}$ and $x \in E$, the n^{th} partial sum of S is the function

$$S_n(x) = \sum_{i=1}^{n} f_i(x).$$

Proposition 9.6. The following hold:

- **(i)** S converges pointwise to a function S on E if the sequence S_n of functions converges pointwise to S on E
- **(ii)** S converges uniformly to a function S on E if the sequence S_n of functions converges uniformly to S on E
- **(iii)** S converges absolutely on E if the series

$$\sum_{n=1}^{\infty} |f_n| \quad \text{converges pointwise on } E$$

Proposition 9.7 (Cauchy criterion)**.** Let f_n be a sequence of functions on E. Then,

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } E$$

if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\text{for all } n > m \geq K \quad \text{we have} \quad \left\| \sum_{i=m+1}^{n} f_i \right\|_E < E.$$

Corollary 9.2. If

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } E \quad \text{then} \quad f_n \to 0 \text{ uniformly on } E.$$

Theorem 9.1 (Weierstrass M-test)**.** Let f_n be a sequence of functions on E and M_n be a sequence of positive real numbers such that $\|f_n\|_E \leq M_n$ for all $n \in \mathbb{N}$. If

$$\sum_{n=1}^{\infty} M_n \text{ converges} \quad \text{then} \quad \sum_{n=1}^{\infty} f_n \text{ converges uniformly and absolutely on } E.$$

Example 9.3. We can prove that the series expansion of the exponential function can be uniformly convergent on any bounded subset $S \subseteq \mathbb{C}$.

Solution. Let $z \in \mathbb{C}$. Note that the series expansion of the complex exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Any bounded subset is a subset of some disc D_R of radius R centred on the origin on the complex plane. The Weierstrass M-test requires us to find an upper bound, M_n, on the terms of the series, with M_n independent of the position in the disc. Observe that

$$\left| \frac{z^n}{n!} \right| \leq \frac{|z|^n}{n!} \leq \frac{R^n}{n!}$$

so by setting $M = R^n/n!$, we are done. □

Proposition 9.8. If

$$\sum_{n=1}^{\infty} f_n \to f \quad \text{uniformly on } I$$

and each f_n is continuous on each $x_0 \in I$, then f is continuous at x_0.

We now state some properties related to differentiability and integrability.

Proposition 9.9. If

$$\sum_{n=1}^{\infty} f_n \to f \text{ uniformly on } [a,b] \quad \text{and} \quad \text{each } f_n \text{ is integrable on } [a,b],$$

(1). f is integrable on $[a,b]$

(2). for every $x \in [a,b]$,

$$\sum_{n=1}^{\infty} \int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt = \int_a^x \sum_{n=1}^{\infty} f_n(t)\, dt$$

where the convergence is uniform on $[a,b]$

Proposition 9.10. Suppose f_n is a sequence of differentiable functions on $[a,b]$ such that

$$\sum_{n=1}^{\infty} f_n(x_0) \text{ converges for some } x_0 \in [a,b] \quad \text{and} \quad \sum_{n=1}^{\infty} f_n' \text{ converges uniformly on } [a,b].$$

Then,

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } [a,b] \text{ to a differentiable function } f \quad \text{and}$$
$$\sum_{n=1}^{\infty} f_n'(x) = f'(x) \text{ for } a \le x \le b$$

10. Power Series

10.1. *Introduction*

Definition 10.1 (power series)**.** A series of functions of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

where $x_0, a_1, a_2, \ldots$ are constants, is a power series in $x - x_0$. So,

$$\sum_{n=0}^{\infty} (x-x_0)^n = \sum_{n=0}^{\infty} f_n(x)$$

where for each n, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = a_n(x-x_0)^n$.

If $x_0 = 0$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Proposition 10.1. Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{be} \quad \text{a power series.}$$

(i) If it converges at $x = x_1$, then it is absolutely convergent for all values of x for which $|x - x_0| < |x_1 - x_0|$

(ii) If it diverges for $x = x_2$, then it diverges for all values of x such that $|x - x_0| > |x_2 - x_0|$

10.2. *Radius of Convergence*

Definition 10.2 (radius of convergence)**.** Given a power series, let

$$S = \left\{ |x - x_0| : x \in \mathbb{R} \quad \text{and} \quad \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges} \right\}$$

The radius of convergence of the series, R, is defined as follows:

(i) $R = 0$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges only for } x = x_0$$

(ii) $R = \infty$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for all } x \in \mathbb{R}$$

(iii) $R = \sup S$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for some } x \text{ and diverges for others}$$

Example 10.1. The series

$$\sum_{n=0}^{\infty} n! x^n$$

converges only at $x = 0$, implying that $R = 0$.

Example 10.2. The exponential function e^x converges at every point of $\mathbb{R}$, and so $R = \infty$.

Example 10.3. Consider the geometric series $1+x+x^2+x^3+\ldots$, which converges for all $x \in \mathbb{R}$ and diverges for all oher x's. Hence, $R = 1$.

Definition 10.3 (absolute convergence).

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges absolutely}$$

for all $x \in (x_0 - R, x_0 + R)$ and diverges for all x with $|x - x_0| > R$.

Theorem 10.1 (ratio test). Suppose a_n is non-zero for all n. Let

$$\rho = \lim_{n=0} \left| \frac{a_{n+1}}{a_n} \right|.$$

(i) If the limit ρ exists, then the radius of convergence, R, of the power series is

$$R = \begin{cases} 1/\rho & \text{if } \rho > 0; \\ \infty & \text{if } \rho = 0 \end{cases}$$

(ii) If $\rho = \infty$, then $R = 0$

The ratio test is frequently easier to apply than the root test since it is usually easier to evaluate ratios than n^{th} roots. However, the root test is a *stronger* test for convergence. This means that whenever the ratio test shows convergence, the root test does too and whenever the root test is inconclusive, the ratio test is too (merely the contrapositive statement).

For any sequence x_n of positive numbers,

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \liminf_{n\to\infty} \sqrt[n]{x_n} \text{ and } \limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Theorem 10.2 (Cauchy-Hadamard formula). Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ be a power series} \quad \text{and} \quad \rho = \limsup |a_n|^{1/n}.$$

The radius of convergence, R, is

$$R = \begin{cases} 0 & \text{if } \rho = \infty; \\ 1/\rho & \text{if } 0 < \rho < \infty; \\ \infty & \text{if } \rho = 0 \end{cases}$$

10.3. *Properties of Power Series*

Proposition 10.2. Suppose

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

has a radius of convergence $R>0$. Then, f is infinitely differentiable on (x_0-R,x_0+R), i.e.

$$f'(x)=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}\quad\text{where } x\in(x_0-R,x_0+R).$$

Proposition 10.3. For every $k\in\mathbb{N}$, we have the following result:

$$f^{(k)}(x)=\sum_{n=k}^{\infty}n(n-1)(n-2)\ldots(n-k+1)a_n(x-x_0)^{n-k}\quad\text{where } x\in(x_0-R,x_0+R)$$

and the radius of convergence of each of these derived series is also R.

Although a power series and its derived series have the same values of R, they may converge on different sets.

Example 10.4. Consider the power series

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

By the ratio test, $R=1$, so the series converges in $(-1,1)$. The series also converges at $x=\pm1$. In fact, when $x=1$, we obtain the famous p-series for which $p=2$, and it is also known as the Basel problem. When $x=-1$, we obtain a variant of the Basel problem which can be evaluated as well. Hence, the series converges in $[-1,1]$.

Differentiating both sides of the power series gives

$$f'(x)=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n},$$

where $x\in(-1,1)$. $f'(x)$ converges at $x=-1$ but diverges at $x=1$, which is the harmonic series. Hence, $f'(x)$ converges on $[-1,1)$.

Corollary 10.1. If there exists $r>0$ such that

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n\quad\text{for all } x\in(x_0-r,x_0+r),$$

then

$$a_k=\frac{f^{(k)}(x_0)}{k!}\quad\text{for all } k\in\mathbb{Z}_{\geq0}.$$

Corollary 10.2 (uniqueness of power series). If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

for all $x\in(x_0-r,x_0+r)$ for some $r>0$, then $a_n=b_n$ for all $n\in\mathbb{Z}_{\geq0}$.

Corollary 10.3. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{have} \quad \text{a non-zero radius of convergence } R.$$

Then, for any a and b for which $x_0 - R < a < b < x_0 + R$,

$$\int_a^b f(x)\, dx = \int_a^b \sum_{n=0}^{\infty} a_n (x - x_0)^n\, dx = \sum_{n=0}^{\infty} \int_a^b a_n (x - x_0)^n\, dx.$$

In other words, a power series can be integrated term-by-term over any closed interval $[a,b]$ contained in the interval of convergence.

Theorem 10.3 (Abel summation formula)**.** Let b_n and c_n be sequences of real numbers, and for each pair of integers $n \geq m \geq 1$, set

$$B_{n,m} = \sum_{k=m}^{n} b_k.$$

Then,

$$\sum_{k=m}^{n} b_k c_k = B_{n,m} c_n - \sum_{k=m}^{n-1} B_{k,m} (c_{k+1} - c_k).$$

for all $n > m \geq 1$, $n, m \in \mathbb{N}$.

Theorem 10.4 (Abel's theorem)**.** Suppose

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{has} \quad \text{a finite non-zero radius of convergence } R.$$

(i) If the series converges at $x = x_0 + R$, then it converges uniformly on $[x_0, x_0 + R]$

(ii) If the series converges at $x = x_0 - R$, then it converges uniformly on $[x_0 - R, x_0]$

10.4. *Taylor Series*

A function f is infinitely differentiable on (a,b) if $f^{(n)}(x)$ exists for all $x \in (a,b)$ and for all $n \in \mathbb{N}$. This class of functions is denoted by $\mathcal{C}^{\infty}$.

Definition 10.4 (Taylor series)**.** Let f be infinitely differentiable on $(x_0 - r, x_0 + r)$ for some $r > 0$. The power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad \text{is} \quad \text{the Taylor series of } f \text{ about } x_0.$$

Definition 10.5 (Taylor series)**.** Considering the Taylor series, set $x_0 = 0$. We then obtain the Maclaurin Series of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 10.6 (analytic function)**.** A function f is analytic on (a,b) if f is infinitely differentiable on (a,b) and for any $x_0 \in (a,b)$, the Taylor Series of f about x_0 converges to f in a neighbourhood of x_0.

Example 10.5. The functions e^x, $\sin x$ and $\cos x$ are analytic on $\mathbb{R}$ and the infinite geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is analytic on $(-1,1)$.

10.5. *Arithmetic Operations with Power Series*

Definition 10.7 (Cauchy product). The Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where for each $n \in \mathbb{N}$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0.$$

Proposition 10.4. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n,\ |x-x_0| < R_1 \text{ and } g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n,\ |x-x_0| < R_2.$$

For $\alpha, \beta \in \mathbb{R}$, we have the following:

(i)

$$\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n \text{ for } |x-x_0| < \min(R_1, R_2)$$

(ii)

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n,\ |x-x_0| < \min(R_1, R_2) \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Theorem 10.5 (Merten's theorem). If

$$\sum_{n=0}^{\infty} a_n \text{ converges absolutely and } \sum_{n=0}^{\infty} b_n \text{ converges} \quad \text{then} \quad \text{the Cauchy product } \sum_{n=0}^{\infty} c_n \text{ converges.}$$

Also,

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Recall that for Merten's theorem, we just need at least one of the series to converge absolutely.

Definition 10.8 (conditional convergence). A series is conditionally convergent if it converges but does not converge absolutely.

Remark 10.1. If

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n \quad \text{converge conditionally,}$$

then their Cauchy product may not converge.

Example 10.6. Set

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}},$$

where $n \geq 0$. It is clear that both series are conditionally convergent (but not absolutely convergent) by the alternating series test. The Cauchy product of these two series is

$$c_n = \sum_{k=0}^{\infty} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}} \quad \text{for all } n \in \mathbb{N}.$$

Note that $n \geq k$ so $n+1 \geq k+1$ and $n+1 \geq n-k+1$ so we are able to obtain a lower bound for $|c_n|$. Hence,

$$|c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1 \quad \text{which implies} \quad \sum_{n=0}^{\infty} c_n \text{ diverges.}$$

Theorem 10.6 (Riemann rearrangement theorem)**.** Suppose a_n is a sequence of real numbers, and that

$$\sum_{n=1}^{\infty} a_n \quad \text{is} \quad \text{conditionally convergent.}$$

Let $M \in \mathbb{R}$. Then, there exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

There also exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \infty.$$

The sum can also be rearranged to diverge to $-\infty$ or to fail to approach any limit, finite or infinite.

10.6. *Some Special Functions*

Definition 10.9 (exponential function)**.** The function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for all $x \in \mathbb{R}$, is the exponential function.

Proposition 10.5. $E : \mathbb{R} \to \mathbb{R}$ has the following properties:

- **(i)** $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$
- **(ii)** $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$
- **(iii)** $E(x) > 0$ for all $x \in \mathbb{R}$
- **(iv)** E is strictly increasing (i.e. $E'(x) > 0$ for all $x \in \mathbb{R}$)
- **(v)**

$$\lim_{x \to \infty} E(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} E(x) = 0$$

For **(i)**, any function $f(x)$ that has this property is invariant under successive levels of differentiation. Actually, one can verify that the exponential function is indeed the only function that is invariant under the differential operator by treating the differential equation $f'(x) = f(x)$ as a separable one.

Proposition 10.6. The functional equation

$$f(x+y) = f(x)f(y)$$

holds true only for the exponential function.

Euler's number, e (≈ 2.71828459045) is defined as the following limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Proposition 10.7. By considering the Maclaurin series of e^x, setting $x = 1$ gives the expansion

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

Proposition 10.8. In relation to sequences, for $x \in \mathbb{R}$, e^x is defined as

$$e^x = \lim_{n\to\infty} e^{r_n},$$

where r_n is an increasing rational sequence which converges to x.

Proposition 10.9. For $x \in \mathbb{R}$, e^x is continuous on $\mathbb{R}$.

Since the exponential function E is strictly increasing on $\mathbb{R}$ and $E(\mathbb{R}) = (0,\infty)$, then it implies that E is injective and thus has an inverse function $L : (0,\infty) \to \mathbb{R}$, which is also strictly increasing.

We have the following composition of functions

$$L(E(x)) = x \;\forall x \in \mathbb{R}$$

and

$$E(L(y)) = y \;\forall y > 0.$$

Definition 10.10 (natural logarithm)**.** By the Fundamental Theorem of Calculus, we define $L(y)$ to be the following integral:

$$L(y) = \int_1^y \frac{1}{t}\,dt$$

The function $L : (0,\infty) \to \mathbb{R}$ is the natural logarithm, $\ln(x)$.

Proposition 10.10. The natural logarithm $\ln : (0,\infty) \to \mathbb{R}$ has the following properties:

(i)

$$\frac{d}{dy}\ln y = \frac{1}{y} \quad \text{for all } y > 0$$

(ii)

$$\ln y = \int_1^y \frac{1}{t}\,dt \quad \text{for all } y > 0$$

(iii) $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y, > 0$

(iv) $\ln(1) = 0$ and $\ln(e) = 1$

(v) For $x > 0$ and $\alpha \in \mathbb{R}$, $x^\alpha = e^{\alpha \ln x}$

Proposition 10.11. The functional equation

$$f(xy) = f(x) + f(y)$$

holds true only for the logarithmic function.

Corollary 10.4. Let $\alpha \in \mathbb{R}$. Then, the function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$f(x) = x^\alpha$$

for all $x > 0$ is differentiable on $(0,\infty)$ and

$$f'(x) = \alpha x^{\alpha-1}$$

for all $x > 0$ as well.

Definition 10.11 (cosine)**.** The function

$$C(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$, is the cosine function.

Definition 10.12 (sine)**.** The function

$$S(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all $x \in \mathbb{R}$, is the sine function.

These two trigonometric functions have the following relationship, that for all $x \in \mathbb{R}$,

$$C'(x) = -S(x) \text{ and } S'(x) = C(x).$$

Differentiating both sides of each equation will yield

$$C''(x) = -C(x) \text{ and } S''(x) = -S(x),$$

which are second order linear homogeneous differential equations which constant coefficients.

Thus, we make the claim that if $g : \mathbb{R} \to \mathbb{R}$ has the property that $g''(x) = -g(x)$ for all $x \in \mathbb{R}$, then

$$g(x) = \alpha C(x) + \beta S(x)$$

for all $x \in \mathbb{R}$ too, where $\alpha = g(0)$ and $\beta = g'(0)$. The two functions satisfy the identity $(C(x))^2 + (S(x))^2 = 1$ for all $x \in \mathbb{R}$, which is also known as the Pythagorean identity.

The cosine function is even. That is, $C(-x) = C(x)$ (i.e. the graph is symmetrical about the y-axis). It satisfies the following addition formula:

$$C(x+y) = C(x)C(y) - S(x)S(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The sine function is odd. That is, $S(-x) = -S(x)$ (i.e. the graph is symmetrical about the origin). It satisfies the following addition formula:

$$S(x+y) = S(x)C(y) + C(x)S(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The four other trigonometric functions, as well as all the inverse trigonometric functions, will not be discussed. Moreover, respective small angle approximations will not be discussed too.

Definition 10.13 (gamma function)**.** The gamma function is one commonly used extension of the factorial function to complex numbers. Denoted by $\Gamma(z)$, it is defined to be the following convergent improper integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad \text{where } \mathrm{Re}(z) > 0.$$

Theorem 10.7. For $z \geq 0$, we have the following relationship:

$$\Gamma(z+1) = z\Gamma(z),$$

which has some semblance to the functional equation $f(x+1) = xf(x)$. Hence,

$$\Gamma(n) = (n-1)! \quad \text{where } n \in \mathbb{N}.$$

Proof. Using integration by parts,

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z\, dt = -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t} t^{z-1}\, dt = z\Gamma(z)$$

and we are done.

Next, to prove the closed form for $\Gamma(n)$, as $\Gamma(1) = 1$, so

$$\prod_{i=1}^{n-1} \frac{\Gamma(i+1)}{\Gamma(i)} = \prod_{i=1}^{n-1} i$$
$$\frac{\Gamma(n)}{\Gamma(1)} = (n-1)!$$

and the result follows by the telescoping product. □

Theorem 10.8. $\ln\Gamma(z)$ is convex on $(0,\infty)$

Theorem 10.9 (Bohr-Mollerup theorem)**.** The gamma function is the only function satisfying $f(1) = 1$, $f(x+1) = xf(x)$ and f is logarithmically convex.

The Bohr-Mullerup theorem characterises the gamma function.

There are two types of Euler integral. The gamma function is also known as the Euler integral of the first kind and the beta function (discussed in the next section) is also known as the Euler integral of the second kind.

Euler's reflection formula and Legendre's duplication formula are examples of functional equations closely related to the gamma function.

Theorem 10.10 (Euler's reflection formula)**.** For $z \notin \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z).$$

Theorem 10.11 (Legendre duplication formula)**.**

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

Definition 10.14 (beta function)**.** For $x, y \in \mathbb{C}$, where $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, the Beta Function $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Proposition 10.12. $B(x,y)$ is symmetric.

Proof. Use the substitution $x = y$ so $B(x,y) = B(y,x)$, □

Theorem 10.12. The beta function is closely related to the gamma function and the binomial coefficients by the following equation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

The proof of Theorem 10.12 hinges on writing $\Gamma(x)\Gamma(y)$ as a double integral and using the technique of change of variables.

11. Functions of Several Variables