MA4211 Functional Analysis

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Reference books:

(1) Kreyszig, E. (1989). '*Introductory Functional Analysis with Applications*'. Wiley.

(2) Sasane, A. (2017). '*A Friendly Approach to Functional Analysis*'. World Scientific.

Contents

1. Metric Spaces

1.1. *Metric Spaces*

Functional Analysis is essentially the study of infinite-dimensional Linear Algebra.

Example 1.1 (Euclidean metric/distance)**.** Recall the familiar metric in Euclidean space $\mathbb{R}$

$$d(x,y) = |x-y|.$$

We call this the Euclidean metric or Euclidean distance. Naturally, we can extend this to the Euclidean 2-space $\mathbb{R}^2$. Consider $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Then,

$$d(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

We give the definition of a metric space (Definition 1.1).

Definition 1.1 (metric space)**.** Let X be a set. A metric space is an ordered pair (X,d) equipped with a distance function $d : X \times X \to \mathbb{R}$ such that the following properties are satisifed:

(i) Non-negativity: $d(x,y) \geq 0$

(ii) Positive-definiteness: $x = y$ if and only if $d(x,y) = 0$

(iii) Symmetry: for all $x,y \in X$, we have $d(x,y) = d(y,x)$

(iv) Triangle inequality: for all x,y,z, we have $d(x,z) \leq d(x,y) + d(y,z)$

Example 1.2 (MA4211 AY24/25 Sem 2 Tutorial 1)**.** Given two non-empty subsets A,B of a metric space (X,d), their distance is defined as

$$D(A,B) = \inf_{a\in A, b\in B} d(a,b).$$

Consider the power set of X and the function D. Which of the axioms of a metric space does this pair satisfy?

Solution. We first claim that non-negativity is satisfied. For $A,B \subseteq X$, we have

$$d(a,b) \geq 0 \quad \text{which implies} \quad \inf_{a\in A, b\in B} d(a,b) \geq 0.$$

Next, symmetry is satisfied since

$$D(B,A) = \inf_{a\in A, b\in B} d(b,a) = \inf_{a\in A, b\in B} d(a,b)$$

which follows from the fact that d satisfies symmetry. Next, the triangle inequality is satisfied. To see why, let $A,B,C \subseteq X$. Then,

$$\begin{aligned} D(A,C) &= \inf_{a\in A, c\in C} d(a,c) \\ &\leq \inf_{a\in A, b\in B} d(a,b) + \inf_{b\in B, c\in C} d(b,c) \\ &= D(A,B) + D(B,C) \end{aligned}$$

We claim that the property $D(A,B) = 0$ if and only if $A = B$ is not satisfied. Recall from Definition 1.1 that this is called positive-definiteness. Anyway, to see why, let $A = \{x\}$ and $B = \{x_n\}_{n=1}^{\infty}$, where $x_n \to x$. Then, $D(A,B) = 0$ since x_n can be arbitrarily close to x for large n. However, $A \neq B$. $\square$

Definition 1.2 ($\mathbb{R}^\infty$). Define $\mathbb{R}^\infty$ to be the space of all infinite sequences of real numbers, i.e. $(x_1, x_2, \ldots)$ where $x_1, x_2, \ldots \in \mathbb{R}$.

Example 1.3 ($\mathbb{R}^\infty$). We have the infinite sequences $(0, 0, \ldots,)$ and $(1, 2, 3, \ldots, 100, \ldots)$ in $\mathbb{R}^\infty$.

Example 1.4. For $X = \mathbb{R}$, we can define

$$d(x, y) = \min\{|x - y|, 1\} \quad \text{such that it is a metric.}$$

Example 1.5. For $X = \mathbb{R}^\infty$, let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, where each element is in $\mathbb{R}$. Then, one can check that

$$d(\mathbf{x}, \mathbf{y}) = \sup d(x_i, y_i) \quad \text{is a metric.}$$

Example 1.6 (ℓ^∞). We give an introduction to the sequence space ℓ^∞. This example gives one an impression of how surprisingly general the concept of a metric space is. We can define

$$X = \{\text{bounded sequences of complex numbers}\}.$$

So, every element of X is a complex sequence ξ_j such that for all $j = 1, 2, \ldots$, we have

$$|\xi_j| \leq c_x \quad \text{where} \quad c_x \text{ is a real number which may depend on } x.$$

Then, the following is a metric:

$$d(\mathbf{x}, \mathbf{y}) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \quad \text{where} \quad y = (\eta_1, \eta_2, \ldots) \in X$$

Definition 1.3 (function space). A function space is a set of functions holding the properties of a vector space structure, norm, or inner product. In particular, it has either of the following properties:

- **Vector space structure:**
  - **Closure under addition:** for any $f, g \in \mathcal{C}[a, b]$, we have $f + g \in \mathcal{C}[a, b]$
  - **Closure under scalar multiplication:** for any $k \in \mathbb{R}$, $kf \in \mathcal{C}[a, b]$ for $f \in \mathcal{C}[a, b]$
- **Norm:** A norm $\|\cdot\|$ is a function $\|\cdot\| : \mathcal{C}[a, b] \to \mathbb{R}$ that satisfies the following:
  - **Non-negativity:** $\|f\| \geq 0$ for all $f \in \mathcal{C}[a, b]$, and $\|f\| = 0$ if and only if $f = 0$
  - **Scalar multiplication:** $\|kf\| = |k|\|f\|$ for any $k \in \mathbb{R}$ and $f \in \mathcal{C}[a, b]$
  - **Triangle inequality:** $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in \mathcal{C}[a, b]$
- **Inner product:** An inner product $\langle \cdot, \cdot \rangle$ is a function $\langle \cdot, \cdot \rangle : \mathcal{C}[a, b] \times \mathcal{C}[a, b] \to \mathbb{R}$ that satisfies the following:
  - **Conjugate symmetry:** $\langle f, g \rangle = \langle g, f \rangle$
  - **Linearity in the first argument:** $\langle kf + g, h \rangle = k\langle f, h \rangle + \langle g, h \rangle$ for any $k \in \mathbb{R}$
  - **Positive-definiteness:** $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{C}[a, b]$, and $\langle f, f \rangle = 0$ if and only if $f = 0$

Example 1.7. Let

$$\mathcal{C}[a, b] \quad \text{denote} \quad \text{the set of continuous functions on } [a, b].$$

Example 1.8 (function space). Let $X = \mathcal{C}[a, b]$, for which we recall that this refers to the set of continuous functions on $[a, b]$. Let $f, g \in \mathcal{C}[a, b]$. Then,

$$d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \quad \text{and} \quad d(f, g) = \sqrt{\int_0^L |f(x) - g(x)|^2 \, dx} \quad \text{are metrics.}$$

Example 1.9 (Hamming distance)**.** Consider the two English words 'word' and 'wind' of the same length for which the second and third letters differ. Since two letters differ, we say that their Hamming distance is 2. We write

$$d\,(\text{wind}, \text{word}) = 2.$$

In this case, d is a metric. The reader can read Kreyszig p. 9 Question 10 to prove that the Hamming distance is indeed a metric.

Example 1.10 (Hamming metric; Kreyszig p. 9 Question 10)**.** Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X defined by

$$d\,(x,y) = \text{number of places where } x \text{ and } y \text{ have different entries.}$$

Solution. First, non-negativity clearly holds as it is impossible for two words of the same length to differ by a negative number of letters. So, $d\,(x,y) \geq 0$. Symmetry is also obvious.

For positive-definiteness, suppose $x = y$. Then, x and y are the same word. Hence, they are two words of the same length which differ by 0 letters. By definition of d, we have $d\,(x,y) = 0$. Similarly, if $d\,(x,y) = 0$, then x and y are two words of the same length that are of Hamming distance 0, which implies that they differ by 0 letters. As such, $x = y$.

Lastly, we prove that d satisfies the triangle inequality. Let x, y and z be three words of length n. We can explicitly define d as follows:

$$d\,(x,y) = \sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq y_i\}} \quad \text{where} \quad \mathbf{1}_{\{x_i \neq y_i\}} = \begin{cases} 1 & \text{if } x_i \neq y_i; \\ 0 & \text{if } x_i = y_i \end{cases}$$

We note that for each position $1 \leq i \leq n$, the inequality

$$\mathbf{1}_{\{x_i \neq z_i\}} \leq \mathbf{1}_{\{x_i \neq y_i\}} + \mathbf{1}_{\{y_i \neq z_i\}}.$$

To see why, if $x_i = z_i$, then $\mathbf{1}_{\{x_i \neq z_i\}} = 0$ and the inequality holds trivially since $\mathbf{1}_{\{x_i \neq y_i\}}$ and $\mathbf{1}_{\{y_i \neq z_i\}}$ are non-negative. If $x_i \neq z_i$, then either $x_i \neq y_i$, or $y_i \neq z_i$, or both. Hence, at least one of $\mathbf{1}_{\{x_i \neq y_i\}}$ or $\mathbf{1}_{\{y_i \neq z_i\}}$ is 1, and the inequality holds. Hence,

$$\sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq z_i\}} \leq \sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq y_i\}} + \sum_{i=1}^{n} \mathbf{1}_{\{y_i \neq z_i\}} \quad \text{or equivalently} \quad d\,(x,z) \leq d\,(x,y) + d\,(y,z)\,.$$

so the triangle inequality is satisfied. □

Definition 1.4 (ℓ^p-space)**.** Let $p \geq 1$ be a fixed real number. Each element in the space ℓ^p is a sequence $(x_1, \ldots)$ such that $|x_1|^p + \ldots$ converges. So,

$$\ell^p = \left\{ \mathbf{x} \in \mathbb{R}^\infty : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

Definition 1.5 (p-norm)**.** Every element in ℓ^p-space is equipped with a norm, known as the p-norm. We define it as follows (will not be strict with the use of either x or $\mathbf{x}$):

$$\text{if } x \in \ell^p \quad \text{then} \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

Example 1.11 (MA4211 AY24/25 Sem 2 Tutorial 1)**.** Give an example of a sequence x_n such that

$$x_n \to 0 \quad \text{but} \quad x_n \notin \ell^p \text{ for any } 1 \le p < \infty.$$

Solution. Consider

$$x_n = \frac{1}{\log(n+1)} \quad \text{for which} \quad \lim_{n\to\infty} x_n = 0.$$

Then, we have

$$\|x_n\|_p = \frac{1}{[\log(n+1)]^p} \quad \text{so} \quad \sum_{n=1}^{\infty} \|x_n\|_p = \sum_{n=1}^{\infty} \frac{1}{[\log(n+1)]^p} \text{ diverges.}$$

□

Theorem 1.1 (Young's inequality)**.** Suppose $\alpha, \beta > 0$. Then,

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Example 1.12. Let $t = 1/p$. Then,

$$\begin{aligned}\ln\left(t\alpha^p + (1-t)\beta^q\right) &\ge t\ln(\alpha^p) + (1-t)\ln(\beta^q) \quad \text{since ln is concave down} \\ &= \frac{1}{p}\ln(\alpha^p) + \frac{1}{q}\ln(\beta^q) \\ &= \ln\alpha + \ln\beta \\ &= \ln\alpha\beta\end{aligned}$$

Taking exponentials on both sides yields the desired result.

Theorem 1.2 (Hölder's inequality)**.** We have

$$\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \|y\|_q .$$

Theorem 1.3 (Minkowski's inequality)**.** We have

$$\|x+y\|_p \le \|x\|_p + \|y\|_p .$$

Please refer to my MA4262 notes for proofs of Theorems 1.2 and 1.3.

Example 1.13 (MA4211 AY24/25 Sem 2 Tutorial 1)**.** Prove the reverse triangle inequality

$$|d(x,z) - d(z,y)| \le d(x,y) \quad \text{for all } x,y,z \text{ in a metric space } (X,d).$$

Solution. Recall the triangle inequality which states that

$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{for } x,y,z \text{ in a metric space } (X,d).$$

So,

$$d(x,z) - d(z,y) \leq d(x,y) \quad \text{where we used the fact that the metric } d \text{ is symmetric.}$$

If $d(x,z) - d(z,y) \geq 0$, then the result follows by taking absolute value; otherwise, we now consider the case where $d(x,z) < d(z,y)$. So,

$$d(x,y) \geq d(y,z) - d(x,z) > 0.$$

Taking absolute value again yields the desired result. □

1.2. *Convergence, Completeness, and Topology*

Definition 1.6 (convergence of sequence)**.** Let x_n be a sequence in $\mathbb{R}$. We say that

$$x_n \to x \quad \text{if} \quad \text{there exists } x \in \mathbb{R} \text{ such that } \lim_{n\to\infty} d(x_n, x) = 0.$$

One should recall from MA2108 that this is equivalent to saying that

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } n \geq N \text{ we have } d(x_n, x) < \varepsilon.$$

Definition 1.7 (continuous function)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We say that f is continuous at $x_0 \in X$ if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that } d_X(x, x_0) < \delta \text{ implies } d_Y(f(x), f(x_0)) < \varepsilon.$$

Theorem 1.4. Let (X, d_X) and (Y, d_Y) be metric spaces and $T : X \to Y$ be a map. Then,

$$T \text{ is continuous} \quad \text{if and only if} \quad x_n \to x \text{ implies } T(x_n) \to T(x).$$

Definition 1.8 (isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and $T : X \to Y$. We say that

$$T \text{ is an isometry} \quad \text{if} \quad d_X(x_1, x_2) = d_Y(T(x_1), T(x_2)).$$

Definition 1.9 (isometric spaces)**.** If there exists a bijective isometry T between two metric spaces X and Y, then we say that X and Y are isometric.

Definition 1.10 (open and closed intervals)**.** Let

$$(a,b) \text{ and } [a,b] \quad \text{denote} \quad \text{the open interval and closed interval in } \mathbb{R} \text{ respectively.}$$

Definition 1.11 (open and closed balls). Define

$$B(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\} \quad \text{denote} \quad \text{the open ball in } \mathbb{R}^n$$
$$\overline{B}(x,r) = \{y \in \mathbb{R}^n : d(x,y) \leq r\} \quad \text{denote} \quad \text{the closed ball in } \mathbb{R}^n$$

Here, each ball is centred at x and is of radius r.

Definition 1.12 (open set). A subset $A \subseteq X$ is open if

$$\text{for all } x \in A \text{ there exists } r > 0 \quad \text{such that} \quad B(x,r) \subseteq A.$$

Definition 1.13 (closed set). A subset of a metric space $S \subseteq X$ is closed if

$$\text{its complement } S^c \quad \text{is open.}$$

Definition 1.14 (topology). Given a set X, a topology $\mathcal{T}$ on X is a collection of subsets of X satisfying the following properties:

(i) $\emptyset, X \in \mathcal{T}$
(ii) $\mathcal{T}$ is closed under arbitrary unions
(iii) $\mathcal{T}$ is closed under finite intersection

Definition 1.15 (limit point). In a topological space, a point x is a limit point of a sequence x_n if for every neighbourhood of x, there exists $N \in \mathbb{N}$ such that for all $n \geq n$, x_n belongs to that neighbourhood.

Definition 1.16 (continuous function). If $f : X \to Y$ is a function between two topological spaces X and Y, then f is continuous if the pre-image of every open set in Y is open in X.

Definition 1.17 (closure). Let X be a topological space. For $A \subseteq X$, the closure of A, denoted by $\mathrm{cl}(A)$ or $\overline{A}$, is defined as follows:

$$\overline{A} = A \cup \{\text{limit points of } A\}$$

Definition 1.18 (dense set). Let X be a topological space. If $D \subseteq X$ such that

$$\overline{D} = X \quad \text{then} \quad D \text{ is dense in } X.$$

Definition 1.19 (separable space). A topological space is separable if it has a countable dense subset.

Theorem 1.5. Let $1 \leq p < \infty$. Then, ℓ^p is separable.

Proof. Define $X \subseteq \ell^p$ to be the collection of sequences of the form

$$(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \quad \text{where } x_i \in \mathbb{Q}.$$

As X is a countable union of countable sets, X is countable. Let $y \in \ell^p$ and $\varepsilon > 0$ be arbitrary. That is,

$$\sum_{i=1}^{\infty} |y_i|^p < \infty.$$

Also, there exists $n \in \mathbb{N}$ such that

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon^p}{2}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$. We can choose $x \in X$ such that

$$\sum_{i=1}^{n} |y_i - x_i|^p < \frac{\varepsilon^p}{2}.$$

Then,

$$\|y-x\|_p^p = \sum_{i=1}^{n} |y_i - x_i|^p + \sum_{i=n+1}^{\infty} |y_i|^p < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p.$$

Taking the p^{th} root, we obtain $d(x,y) = \|x-y\|_p < \varepsilon$. □

Theorem 1.6. ℓ^∞ is not separable.

Proof. Let $y \in \mathbb{R}^\infty$ be a sequence of 0s and 1s. Define $z \in \mathbb{R}$ to be as follows:

$$z = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \dots \quad \text{so} \quad \text{we infer that } 0 \le z \le 1.$$

Because there are uncountably many real numbers in $[0,1]$, it follows there are uncountably many distinct sequences $y \in \{0,1\}^{\mathbb{N}}$. Denote this uncountable family by $\mathcal{Y} \subset \ell^\infty$. Note that for any two distinct sequences

$$y = (y_1, y_2, y_3, \dots) \quad y' = (y_1', y_2', y_3', \dots) \quad \in \mathcal{Y},$$

there is at least one index i such that $y_i \neq y_i'$. Because each coordinate is either 0 or 1, at index i, we have$|y_i - y_i'| = 1$. Hence,

$$\|y-y'\|_\infty = \sup_{n \in \mathbb{N}} |y_n - y_n'| \ge 1.$$

In fact, it is exactly 1 if the two sequences differ in at least one place (and cannot exceed 1 because each coordinate difference is 0 or 1). Around each $y \in \mathcal{Y}$, consider the open ball $B(y, 1/3)$ of radius $1/3$. Since any two distinct y, y' are at distance $\|y-y'\|_\infty = 1$, their balls $B(y,1/3)$ and $B(y',1/3)$ cannot overlap. In other words, these balls are pairwise disjoint.

Suppose on the contrary that $D \subseteq \ell^\infty$ is a countable dense set. Then, for each $y \in \mathcal{Y}$, $B(y,1/3)$ must contain at least one point of D. However, there are uncountably many such disjoint balls $B(y,1/3)$ since $\mathcal{Y}$ is uncountable. A single countable set D cannot meet each of these uncountably many disjoint balls in a distinct point. This leads to a contradiction. □

Definition 1.20 (Cauchy sequence). A sequence x_n in a metric space X is Cauchy if

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } m,n \ge N \text{ we have } d(x_m, x_n) < \varepsilon.$$

Proposition 1.1. Every convergent sequence is Cauchy.

Remark 1.1. The converse of Proposition 1.1 is not true, i.e. not every Cauchy sequence is convergent.

Definition 1.21 (complete metric space)**.** A metric space is called complete if every Cauchy sequence in the space converges.

Example 1.14 (MA4211 AY24/25 Sem 2 Tutorial 1)**.** Show that the limits of Cauchy sequences in a complete metric space are unique.

Solution. Let x_n be a Cauchy sequence in a complete metric space. Suppose x_n has two limits x and y. Then, for all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\text{for all } n \geq N_1 \text{ we have } |x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad \text{for all } n \geq N_2 \text{ we have } |x_n - y| < \frac{\varepsilon}{2}.$$

By the triangle inequality,

$$|x - y| \leq |x_n - x| + |x_n - y| < \varepsilon.$$

Since ε can be made arbitrarily small, we conclude that $x = y$, i.e. the limits of the Cauchy sequence are the same. $\square$

2. Normed Spaces and Banach Spaces