

# MA3211 MA3211S MA4247 MA5217 Complex Analysis Notes

Thang Pang Ern

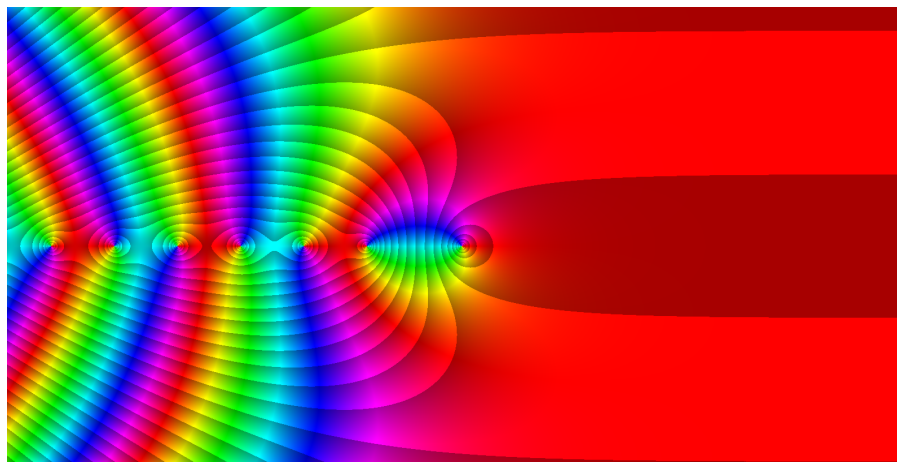
## Reference books:

- (1) Sasane, S. M. and Sasane, A. (2014). *A Friendly Approach to Complex Analysis*. World Scientific.
- (2) Needham, T. (2023). *Visual Complex Analysis: 25th Anniversary Edition*. Oxford.
- (3) Conway, J. B. (1973). *Functions of One Complex Variable I*. Springer.

## Contents

<b>1. Complex Numbers</b>	<b>3</b>
1.1. Some Background Knowledge	3
1.2. Complex-Valued Functions	7
<b>2. Holomorphic and Analytic Functions</b>	<b>8</b>
2.1. Holomorphic Functions	8
2.2. The Cauchy-Riemann Equations	11
2.3. Analytic Functions and Entire Functions	13
2.4. Harmonic Functions	17
<b>3. Line Integrals</b>	<b>19</b>
3.1. Fundamental Results	19
<b>4. The Cauchy-Goursat Theorem</b>	<b>20</b>
4.1. Some Results in Topology	20
4.2. Cauchy-Goursat Theorem	21
4.3. Cauchy's Integral Formula	22
4.4. Applications of Cauchy's Integral Formula	25
<b>5. Series</b>	<b>28</b>
5.1. Laurent Series	28
<b>6. Residue Theory</b>	<b>30</b>
6.1. Introduction	30
6.2. Residue Computation Methods	32
<b>7. Further Properties of Holomorphic Functions</b>	<b>43</b>
7.1. Properties of Holomorphic and Harmonic Functions	43
7.2. The Argument Principle and Rouché's Theorem	48
7.3. Open Mapping Theorem and the Maximum Modulus Principle	51
7.4. Winding Numbers	54
<b>8. Conformal Mappings and Möbius Transformations</b>	<b>56</b>
8.1. Univalent Functions	56
8.2. Automorphisms of the Complex Plane $\mathbb{C}$	56
8.3. Automorphisms of the Unit Disc $\mathbb{D}$	57
8.4. Maps from the Upper Half-Plane $\mathbb{H}$ to the Unit Disc $\mathbb{D}$	59
8.5. Automorphisms of the Upper Half-Plane $\mathbb{H}$	60
8.6. Möbius Transformations	62

8.7. Cross Ratio	65
<b>9. Harmonic Functions</b>	<b>67</b>
9.1. Basic Properties of Harmonic Functions	67
9.2. Dirichlet Problem and Poisson Kernel	68
<b>10. Analytic Continuation</b>	<b>69</b>
10.1. Analytic Continuation	69
10.2. Schwarz Reflection Principle	69



# 1. Complex Numbers

## 1.1. Some Background Knowledge

First, we define

$\mathbb{R}[x]$  to be the set of polynomials with real coefficients.

The polynomial  $x^2 + 1 \in \mathbb{R}[x]$  of degree 2 over  $\mathbb{R}$  has no solution in  $\mathbb{R}$  since for all  $\alpha \in \mathbb{R}$ , we have  $\alpha^2 + 1 > 0$ , so  $x^2 + 1$  is irreducible over  $\mathbb{R}[x]$ . For those who have prior knowledge on Abstract Algebra, since  $\mathbb{R}[x]$  is a principal ideal domain (PID)<sup>†</sup>, then

$$(x^2 + 1)\mathbb{R}[x] \subseteq \mathbb{R}[x] \text{ is a maximal ideal.}$$

As such, we are now in position to define the complex numbers  $\mathbb{C}$ .

**Definition 1.1 (complex numbers).** Define

$$\mathbb{C} = \mathbb{R}[x] / (x^2 + 1)\mathbb{R}[x]$$

to be the quotient ring of  $\mathbb{R}[x]$  modulo the maximal ideal  $(x^2 + 1)\mathbb{R}[x]$ . This is a field, known as the field of complex numbers.

**Proposition 1.1.** The image of

$$x \in \mathbb{R}[x] \text{ in } \mathbb{C} \text{ is denoted by } i \in \mathbb{C},$$

called the imaginary unit.  $i$  has the property that  $i^2 = -1$ .

**Proposition 1.2 (field extension).** The composite of the canonical ring homomorphisms

$$\mathbb{R} \hookrightarrow \mathbb{R}[x] \twoheadrightarrow \mathbb{C} \text{ where } x \mapsto i$$

is an inclusion of fields  $\mathbb{R} \hookrightarrow \mathbb{C}$  so  $\mathbb{C}$  is a field extension of  $\mathbb{R}$ .

**Proposition 1.3.** As an  $\mathbb{R}$ -vector space,  $\mathbb{C}$  has dimension 2 with standard ordered  $\mathbb{R}$ -basis  $\{1, i\}$ .

**Definition 1.2.** The  $\mathbb{R}$ -linear projection maps

$$\operatorname{Re} : \mathbb{C} \rightarrow \mathbb{R} \text{ where } z \mapsto x \text{ and } \operatorname{Im} : \mathbb{C} \rightarrow \mathbb{R} \text{ where } z \mapsto y$$

are called the real part and imaginary part of  $z \in \mathbb{C}$ . So,

$$\text{for all } z \in \mathbb{C} \text{ one has } z = \operatorname{Re} z + i \operatorname{Im} z \text{ in } \mathbb{C}.$$

<sup>†</sup>Recall from MA3201 that if  $F$  is a field, then  $F[x]$  is a Euclidean domain. In fact, recall the chain of inclusions  $\text{ED} \subseteq \text{PID} \subseteq \text{UFD}$ , where ED and UFD denote Euclidean domain and unique factorisation domain respectively. I recall in one of Sadhukhan's MA2101S that one student asked whether  $F$  is a field implies  $F[x]$  is also a field. Clearly, this is wrong and Sadhukhan mentioned that  $F[x]$  is a UFD. It was only when I crashed one of Bao Haunchen's MA4203 lectures (first lecture actually) where I learnt that the stronger statement  $F[x]$  is an ED holds.

**Proposition 1.4 (field operations).** The field operations of  $\mathbb{C}$ , expressed in terms of the real/imaginary parts, are:

(i) **Addition/Subtraction:**

$$(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$$

(ii) **Multiplication:**

$$(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$$

(iii) **Division:**

$$\frac{(a + ib)}{(c + id)} = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

(iv) **Multiplicative inverse:**

$$(c + id)^{-1} = \frac{c - id}{c^2 + d^2}$$

**Definition 1.3 (complex conjugation).** The  $\mathbb{R}$ -linear map

$$(\cdot) : \mathbb{C} \rightarrow \mathbb{C} \quad \text{where} \quad z = x + iy \mapsto \bar{z} = x - iy$$

is called complex conjugation.

**Proposition 1.5.** We say that complex conjugation is an automorphism of  $\mathbb{C}$  as a field over  $\mathbb{R}$ . The automorphism group  $\text{Aut}(\mathbb{C}/\mathbb{R})$  is of order 2. That is to say,

$$\overline{\bar{z}} = z.$$

**Proposition 1.6.** The following properties hold for all  $z, w \in \mathbb{C}$ :

- (i)  $\overline{z + w} = \bar{z} + \bar{w}$  and  $\overline{zw} = \bar{z}\bar{w}$
- (ii)  $\text{Re } z = \frac{1}{2}(z + \bar{z})$  and  $\text{Im } z = \frac{1}{2i}(z - \bar{z})$

**Definition 1.4 (absolute value).** The absolute value of a complex number is the map

$$|\cdot|_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0} \quad \text{where} \quad z \mapsto |z|_{\mathbb{C}} \quad \text{given by} \quad |z|_{\mathbb{C}} = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2} = \sqrt{z\bar{z}}.$$

As such, if  $z = x + iy$  (where  $x, y \in \mathbb{R}$ ), we have

$$|z|_{\mathbb{C}}^2 = x^2 + y^2 = z\bar{z}.$$

**Proposition 1.7.** For any  $a \in \mathbb{R} \subseteq \mathbb{C}$ , we have  $|a|_{\mathbb{C}} = |a|_{\mathbb{R}}$ .

**Lemma 1.1.** For any  $z, w \in \mathbb{C}$ , we have

- (i) **Positive-definiteness:**  $|z|_{\mathbb{C}} = 0$  in  $\mathbb{R}_{\geq 0}$  if and only if  $z = 0$  in  $\mathbb{C}$
- (ii)  $|\bar{z}|_{\mathbb{C}} = |z|_{\mathbb{C}}$  in  $\mathbb{R}_{\geq 0}$
- (iii) **Multiplicativity:**  $|zw|_{\mathbb{C}} = |z|_{\mathbb{C}} |w|_{\mathbb{C}}$  in  $\mathbb{R}_{\geq 0}$
- (iv)  $|\text{Re } z|_{\mathbb{R}}, |\text{Im } z|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$  in  $\mathbb{R}_{\geq 0}$

*Proof.* (i) and (ii) are trivial. To prove (iii), we have

$$|zw|_{\mathbb{C}}^2 = zw\overline{zw} = z\bar{z} \cdot w\bar{w} = |z|_{\mathbb{C}}^2 |w|_{\mathbb{C}}^2.$$

Taking square roots on both sides, (iii) follows.

For (iv), let  $z = x + iy$ , where  $x, y \in \mathbb{R}$ . Then,  $x^2, y^2 \leq x^2 + y^2$ , so  $|x|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$  and  $|y|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$ . □

**Lemma 1.2 (triangle inequality).** For any  $z, w \in \mathbb{C}$ , we have

$$|z + w|_{\mathbb{C}} \leq |z|_{\mathbb{C}} + |w|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\geq 0}.$$

*Proof.* We have

$$\begin{aligned} |z + w|_{\mathbb{C}}^2 &= (z + w)(\overline{z + w}) = z\bar{z} + w\bar{w} + (z\bar{w} + \bar{z}w) \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2\operatorname{Re}(z\bar{w}) \\ &\leq |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2|z\bar{w}|_{\mathbb{C}} \quad \text{by (iv) of Lemma 1.1} \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2|z|_{\mathbb{C}}|w|_{\mathbb{C}} \\ &= (|z|_{\mathbb{C}} + |w|_{\mathbb{C}})^2 \end{aligned}$$

Taking square roots on both sides, the result follows. □

By (i) and (iii) of Lemma 1.1 on the positive-definiteness and multiplicativity, as well as Lemma 1.2 on the triangle inequality, we infer that

$|\cdot|_{\mathbb{C}}$  is an absolute value of  $\mathbb{C}$  in the abstract sense.

**Corollary 1.1.** We say that

$\mathbb{C}$  equipped with the absolute value function  $|\cdot|_{\mathbb{C}}$  as a normed  $\mathbb{R}$ -vector space is isomorphic to  $\mathbb{R}^2$  with the standard Euclidean norm  $\|\cdot\|_2$ , so  $\mathbb{C}$  is said to be *Cauchy complete*.

**Corollary 1.2 (generalised triangle inequality).** For any  $z_1, z_2, \dots, z_n \in \mathbb{C}$ , we have

$$|z_1 + \dots + z_n|_{\mathbb{C}} \leq |z_1|_{\mathbb{C}} + \dots + |z_n|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\geq 0}.$$

*Proof.* Consider the triangle inequality (Lemma 1.2) and use induction. □

**Theorem 1.1 (Cauchy-Schwarz inequality for  $\mathbb{R}^2$ ).** For any  $z, w \in \mathbb{C}$ , we have

$$|\langle z, w \rangle_{\mathbb{R}^2}|_{\mathbb{R}} \leq |z|_{\mathbb{C}} |w|_{\mathbb{C}} \quad \text{with equality if and only if } z \text{ and } w \text{ are } \mathbb{R}\text{-linearly dependent.}$$

Here,  $\langle \cdot, \cdot \rangle$  denotes the inner product of the two inputs. That is to say,

$$z = x + iy \text{ and } w = u + iv \quad \text{implies} \quad \langle z, w \rangle_{\mathbb{R}^2} = xu + yv.$$

*Proof.* The trick is as follows:

$$\begin{aligned}
 \langle z, w \rangle_{\mathbb{R}^2}^2 + \langle iz, w \rangle_{\mathbb{R}^2}^2 &= (xu + yv)^2 + (-yu + xv)^2 \\
 &= x^2u^2 + y^2v^2 + 2xuyv + y^2u^2 + x^2v^2 - 2yuxv \\
 &= (x^2 + y^2)(u^2 + v^2) \\
 &= |z|_{\mathbb{C}}^2 |w|_{\mathbb{C}}^2
 \end{aligned}$$

which implies  $\langle z, w \rangle_{\mathbb{R}^2} \leq |z|_{\mathbb{C}} |w|_{\mathbb{C}}$ . Equality holds if and only if  $\langle iz, w \rangle_{\mathbb{R}^2} = 0$ , or equivalently,  $-yu + xv = 0$ , i.e.  $z$  and  $w$  are  $\mathbb{R}$ -linearly dependent. Well, to be more explicit, we recall that

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{as vectors in } \mathbb{R}^2.$$

If  $z$  and  $w$  are linearly dependent, there exists  $k \in \mathbb{R}$  such that  $(x, y) = k(u, v)$ , so  $x = ku$  and  $y = kv$ . As such,  $-yu + xv = 0$ .  $\square$

We can generalise Theorem 1.1 to the Cauchy-Schwarz inequality for  $\mathbb{C}^n$  (Theorem 1.2).

**Theorem 1.2 (Cauchy-Schwarz inequality for  $\mathbb{C}^n$ ).** For any  $z_1, \dots, z_n, w_1, \dots, w_n \in \mathbb{C}$ , we have

$$|z_1 w_1 + \dots + z_n w_n|_{\mathbb{C}}^2 \leq (|z_1|_{\mathbb{C}}^2 + \dots + |z_n|_{\mathbb{C}}^2) (|w_1|_{\mathbb{C}}^2 + \dots + |w_n|_{\mathbb{C}}^2)$$

and

$$\text{equality holds if and only if } \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \text{ and } \begin{bmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{bmatrix} \text{ are } \mathbb{C}\text{-linearly dependent over } \mathbb{C}^n.$$

Equivalently, this means that there exist  $\lambda, \mu \in \mathbb{C}$  which are both non-zero such that

$$\text{for all } 1 \leq j \leq n \quad \text{we have} \quad \lambda z_j = \mu \overline{w_j} \quad \text{in } \mathbb{C}.$$

*Proof.* We have

$$\begin{aligned}
 0 &\leq \sum_{i < j} |z_i \overline{w_j} - z_j \overline{w_i}|_{\mathbb{C}}^2 \\
 &= \sum_{i < j} (z_i \overline{w_j} - z_j \overline{w_i})(\overline{z_i \overline{w_j} - z_j \overline{w_i}}) \\
 &= \sum_{i < j} |z_i|^2 |w_j|^2 + |z_j|^2 |w_i|^2 - 2 \operatorname{Re}(z_i \overline{z_j} \overline{w_i} w_j)
 \end{aligned}$$

We now add the following term to both sides of the inequality:

$$\left| \sum_{i=1}^n z_i w_i \right|^2 = \sum_{i=1}^n |z_i|^2 |w_i|^2 + \sum_{i < j} (z_i w_i \overline{z_j w_j} + \overline{z_i w_i} z_j w_j)$$

for which it follows that

$$\begin{aligned}
 \left| \sum_{i=1}^n z_i w_i \right|^2 &\leq \sum_{i=1}^n |z_i|^2 |w_i|^2 + \sum_{i < j} (|z_i|^2 |w_j|^2 + |z_j|^2 |w_i|^2) \\
 &= \left( \sum_{i=1}^n |z_i|^2 \right) \left( \sum_{i=1}^n |w_i|^2 \right)
 \end{aligned}$$

Equality holds if and only if

$$\sum_{i < j} |z_i \overline{w_j} - z_j \overline{w_i}|^2 = 0.$$

This holds if and only if for all  $i < j$ , one has  $z_i \overline{w_j} = z_j \overline{w_i}$ . □

**Theorem 1.3 (de Moivre's theorem).** For  $n \in \mathbb{Z}$ ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

*Proof.* By Euler's formula,  $e^{i\theta} = \cos \theta + i \sin \theta$ . In de Moivre's theorem, the left side of the equation is  $e^{in\theta}$  by raising both sides to the power of  $n$ . The result follows by using Euler's formula on  $e^{in\theta}$ . □

**Example 1.1 (MA5217 AY24/25 Sem 1 Homework 1).** Find all solutions of the equation  $e^z = 1$ .

*Solution.* Note that  $1 = e^{2k\pi i}$  for all  $k \in \mathbb{Z}$ . Since the exponential function is injective, we have  $e^z = 2k\pi i$ . Hence,  $z = \ln|2k\pi| + i\pi/2$ . □

## 1.2. Complex-Valued Functions

Let  $X$  be any set. Then, we have the following:

$\text{Maps}(X, \mathbb{R}) = \{\text{all } \mathbb{R}\text{-valued functions on } X\}$  is an  $\mathbb{R}$ -vector space

$\text{Maps}(X, \mathbb{C}) = \{\text{all } \mathbb{C}\text{-valued functions on } X\}$  is a  $\mathbb{C}$ -vector space

**Proposition 1.8.** The  $\mathbb{R}$ -basis  $\{1, i\}$  of  $\mathbb{C}$  gives an  $\mathbb{R}$ -linear decomposition:

$$\text{Maps}(X, \mathbb{C}) \cong \text{Maps}(X, \mathbb{R}) \oplus i \cdot \text{Maps}(X, \mathbb{R}) \quad \text{where} \quad f \mapsto \text{Re } f + i \cdot \text{Im } f.$$

This is such that for any  $x \in X$ ,

$$\text{Re}(f)(x) = \text{Re}(f) \in \mathbb{R}, \quad \text{Im}(f)(x) = \text{Im}(f) \in \mathbb{R}.$$

**Proposition 1.9.** The  $\mathbb{R}$ -automorphism  $(\bar{\cdot})$  of  $\mathbb{C}$  also gives an  $\mathbb{R}$ -linear automorphism:

$$(\bar{\cdot}) : \text{Maps}(X, \mathbb{C}) \rightarrow \text{Maps}(X, \mathbb{C}) \quad \text{where} \quad f \mapsto \bar{f}.$$

This is such that for any  $x \in X$ ,

$$\bar{f}(x) = \overline{f(x)} \quad \text{in } \mathbb{C}.$$

**Proposition 1.10.** One has the following decomposition:

$$\text{Re } f = \frac{f + \bar{f}}{2}, \quad \text{Im } f = \frac{f - \bar{f}}{2i}.$$

## 2. Holomorphic and Analytic Functions

### 2.1. Holomorphic Functions

**Definition 2.1.** Let

$\Omega \subseteq \mathbb{C}$  be an open and connected set in  $\mathbb{C}$

$H(\Omega)$  be the set of holomorphic functions in  $\Omega$

**Definition 2.2 (holomorphic function).** Let  $\Omega \subseteq \mathbb{C}$  be an open set. A function  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic at  $a$  or  $\mathbb{C}$ -differentiable at  $a$  (Proposition 2.3) if and only if

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists in } \mathbb{C}.$$

In this case, the limit, which is uniquely determined by  $f$  and  $a$ , is called the holomorphic derivative of  $f$  at  $a$ , denoted by

$$\frac{df}{dz}(a) = f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ in } \mathbb{C}.$$

As such,

$f : \Omega \rightarrow \mathbb{C}$  is holomorphic on  $G$  if and only if for all  $a \in G$ ,  $f$  is holomorphic at  $a$ .

**Proposition 2.1.** Let  $\Omega \subseteq \mathbb{C}$  be an open set and  $f, g : \Omega \rightarrow \mathbb{C}$  be functions holomorphic at  $a$ . Then, the following hold:

(i)  **$\mathbb{C}$ -linearity:** for all  $c, d \in \mathbb{C}$ ,

the function  $cf + dg : \Omega \rightarrow \mathbb{C}$  is also holomorphic at  $a$

equipped with

its holomorphic derivative  $(cf + dg)'(a) = c \cdot f'(a) + d \cdot g'(a)$  in  $\mathbb{C}$

(ii) **Product rule:** the function  $f \cdot g : \Omega \rightarrow \mathbb{C}$  is also holomorphic at  $a$  equipped with its holomorphic derivative

$$(fg)'(a) = f'(a)g(a) + g'(a)f(a) \text{ in } \mathbb{C}$$

**Remark 2.1.** Recall Definition 2.1, which mentioned that  $H(\Omega)$  denotes the set of all functions  $f : \Omega \rightarrow \mathbb{C}$  which are holomorphic on  $\Omega$ . We say that

$H(\Omega)$  is a  $\mathbb{C}$ -algebra under pointwise  $\pm, \times$  of functions.

Note that for any point  $a \in \Omega$ , we have the evaluation at  $a$  map, i.e.

$$\text{ev}_a : H(\Omega) \rightarrow \mathbb{C} \text{ where } f \mapsto f(a),$$

which is a  $\mathbb{C}$ -algebra homomorphism.



Also, Proposition 2.1 says that the derivative at  $a$  map

$$H(\Omega) \rightarrow \mathbb{C} \quad \text{where} \quad f \mapsto f'(a)$$

is a  $\mathbb{C}$ -linear derivative of  $H(\Omega)$  to the  $H(\Omega)$ -module  $\mathbb{C}$  via  $\text{ev}_a$ .

**Example 2.1 (identity map).** For any open  $\Omega \subset \mathbb{C}$ , the identity map  $\text{id}$  is holomorphic with derivative

$$\text{id}'(a) = \frac{dz}{dz}(a) = 1 \quad \text{for all } a \in G.$$

Hence,  $z \in H(\Omega)$ . In fact, for any polynomial  $f \in \mathbb{C}[z]^\dagger$ , the function  $z \mapsto f(z)$  is also  $H(G)$ .

**Example 2.2.** For any open  $G = \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ , the reciprocal function  $z^{-1}$  is holomorphic with derivative

$$\frac{dz^{-1}}{dz}(a) = -\frac{1}{a^2} \quad \text{for all } a \in G.$$

Hence,  $z^{-1} \in H(\Omega)$ . Moreover, for any Laurent polynomial  $f \in \mathbb{C}[z, z^{-1}]$  (we will only discuss this when formally defining Laurent series/polynomials in Theorem 5.1), the function  $z \mapsto f(z)$  is also in  $H(\Omega)$ .

**Proposition 2.2 (chain rule).** Let  $\Omega_1, \Omega_2 \subseteq \mathbb{C}$  be open sets. Let

$$f : \Omega_1 \rightarrow \mathbb{C} \text{ and } g : \Omega_2 \rightarrow \mathbb{C} \quad \text{such that} \quad f(\Omega_1) \subseteq \Omega_2$$

so  $g \circ f : \Omega_1 \rightarrow \mathbb{C}$  is defined. If  $f$  is holomorphic at  $a$  and  $g$  is holomorphic at  $f(a)$ , then  $g \circ f$  is holomorphic at  $a$ , equipped with its holomorphic derivative

$$(g \circ f)'(a) = g'(f(a)) f'(a).$$

*Proof.* Let  $b = f(a) \in \Omega_2$ . Define the functions  $\xi : \Omega_1 \rightarrow \mathbb{C}$  and  $\eta : \Omega_2 \rightarrow \mathbb{C}$  by setting

$$\xi(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} - f'(a) & \text{if } z \in \Omega_1 \setminus \{a\} \\ \text{any value} & \text{if } z = a \end{cases} \quad \text{and} \quad \eta(w) = \begin{cases} \frac{g(w) - g(b)}{w - b} - g'(b) & \text{if } w \in \Omega_2 \setminus \{b\} \\ \text{any value} & \text{if } w = b. \end{cases}$$

Then, for all  $z \in \Omega_1$  and  $w \in \Omega_2$ , we have the following in  $\mathbb{C}$ :

$$\begin{aligned} f(z) - f(a) &= [f'(a) + \xi(z)](z - a) \\ g(w) - g(b) &= [g'(b) + \eta(w)](w - b) \end{aligned}$$

Thus, for all  $z \in \Omega_1$ , we have

$$\begin{aligned} g(f(z)) - g(f(a)) &= (g'(f(a)) + \eta(f(z)))(f(z) - f(a)) \\ &= (g'(f(a)) + \eta(f(z)))(f'(a) + \xi(z))(z - a) \end{aligned}$$

so for all  $z \in \Omega_1 \setminus \{a\}$ , we have

$$\frac{g(f(z)) - g(f(a))}{z - a} = (g'(f(a)) + \eta(f(z)))(f'(a) + \xi(z)).$$

<sup>†</sup>Here, one should perhaps recall from MA3201 that  $\mathbb{C}[z]$  denotes the set of all polynomials in  $z$  with complex coefficients. That is,  $\mathbb{C}[z] \ni f(z) = a_0 + a_1 z + \dots + a_n z^n$  where  $a_0, a_1, \dots, a_n \in \mathbb{C}$ .

Since

$$\begin{aligned} f & \text{ is holomorphic at } a \in \Omega_1 \quad \text{and} \\ g & \text{ is holomorphic at } b \in \Omega_2 \end{aligned}$$

then

$$\lim_{z \rightarrow a} \xi(z) = 0 \quad \text{and} \quad \lim_{w \rightarrow b} \eta(w) = 0.$$

Also,

$$f \text{ is continuous at } a \quad \text{implies} \quad \lim_{z \rightarrow a} f(z) = f(a) = b.$$

Hence,

$$\lim_{z \rightarrow a} \frac{g(f(z)) - g(f(a))}{z - a} \text{ exists in } \mathbb{C} \quad \text{and} \quad \text{equals } g'(f(a))f'(a).$$

□

Next, recall Definition 2.3 on  $\mathbb{R}$ -differentiability from MA3210.

**Definition 2.3 ( $\mathbb{R}$ -differentiability).** We say that  $f$  is  $\mathbb{R}$ -differentiable at  $a$  if and only if there exists an  $\mathbb{R}$ -linear map  $(Df)(a) : \mathbb{C} \rightarrow \mathbb{C}$  such that

$$\text{for all } \varepsilon \in \mathbb{R}_{>0}, \text{ there exists } \delta \in \mathbb{R}_{>0}$$

such that

$$\text{for all } z \in G \text{ with } 0 \leq \|z - a\| < \delta \quad \text{we have} \quad \|f(z) - f(a) - (Df)(a)(z - a)\| \leq \varepsilon \cdot \|z - a\|.$$

When this holds, the  $\mathbb{R}$ -linear map  $(Df)(a)$  is uniquely determined by  $f$  and  $a$  and we call this the derivative of  $f$  at  $a$ .

**Proposition 2.3 ( $\mathbb{C}$ -differentiability).** If  $f$  is holomorphic at  $a$  ( $\mathbb{C}$ -differentiable at  $a$ ), then  $f$  is  $\mathbb{R}$ -differentiable at  $a$  and

$$(Df)(a) \in \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{is the image of} \quad f'(a) \in \text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

under the following canonical inclusion:

$$\text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \hookrightarrow \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{where} \quad z \mapsto \text{multiplication by } z.$$

**Corollary 2.1.** Suppose  $f$  is holomorphic on  $\Omega$  and for all  $a \in G$ , we have  $f'(a) = 0$  in  $\mathbb{C}$ . Then,  $f$  is locally constant on  $\Omega$ .

*Proof.* Let  $a \in \Omega$  be an arbitrary point. Choose  $r \in \mathbb{R}_{>0}$  be sufficiently small such that  $B(a, r) \subseteq G$ , where

$$B(a, r) \quad \text{is the open ball in } \mathbb{C} \text{ centred at } a \text{ of radius } r.$$

By the mean-value inequality, for any  $z \in B(a, r)$ , there exists  $\xi \in [a, z] \subseteq B(a, r)$  such that

$$\|f(z) - f(a)\| \leq \|f'(\xi)\| \|z - a\|$$

Since  $f'(\xi) = 0$ , then  $f$  is constant of value  $f(a)$  on  $B(a, r)$ .

□

Now, identify  $\mathbb{C}$  with the standard  $\mathbb{R}$ -basis  $\{1, i\}$ . Then, consider the following comparison:

$$\mathbb{R}^2 \xleftarrow{1, i} \mathbb{C} \xrightarrow{z \mapsto \text{multiplication by } z} \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \xleftarrow{1, i} \mathcal{M}_{2 \times 2}(\mathbb{R})$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + bi \mapsto (x + yi \mapsto (a + bi)(x + yi) = (ax - by) + i(bx + ay)) \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We infer that via 1 and  $i$ , the matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R})$$

corresponds to the  $\mathbb{R}$ -linear map  $\mathbb{C} \rightarrow \mathbb{C}$  given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad x + yi \mapsto (px + qy) + i(rx + sy).$$

This  $\mathbb{R}$ -linear map is  $\mathbb{C}$ -linear if and only if  $p = s$  and  $q = -r$  in  $\mathbb{R}$ . As such, we can set  $a = p$  and  $q = -b$ .

Now, again via 1 and  $i$ , write

$$f : \Omega \rightarrow \mathbb{C} \quad \text{as} \quad x + iy \mapsto f(x + iy) = u(x, y) + iv(x, y).$$

Suppose  $f$  is  $\mathbb{R}$ -differentiable at  $a$ . Then,

$$(Df)(a) \in \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{corresponds to} \quad \begin{bmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a) \end{bmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{R}).$$

Hence,  $(Df)(a)$  lies in the image of  $\mathbb{C} \hookrightarrow \text{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$  if and only if

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \text{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

This is precisely the Cauchy-Riemann equations (will formally introduce in Theorem 2.1).

## 2.2. The Cauchy-Riemann Equations

**Theorem 2.1 (Cauchy-Riemann equations).** Let  $\Omega \subseteq \mathbb{C}$  be an open set. Let  $f : \Omega \rightarrow \mathbb{C}$  be a function written as

$$x + iy \mapsto f(x + iy) = u(x, y) + iv(x, y).$$

Suppose  $f$  is  $\mathbb{R}$ -differentiable at  $a$ . Then,

$$f \text{ is holomorphic at } a \quad \text{if and only if} \quad \frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \text{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a) \quad \text{are satisfied.}$$

**Theorem 2.2 (polar form of CR equations).** If  $u$  and  $v$  are expressed in terms of polar coordinates  $(r, \theta)$ , then

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

*Proof.* Using the substitution  $z = re^{i\theta}$ , we have  $x = r \cos \theta$  and  $y = r \sin \theta$ . Since  $f(z) = u(x, y) + iv(x, y)$ , we will now perform change of variables from  $(x, y)$  to  $(r, \theta)$ . By the chain rule for partial derivatives, to compute  $\partial u / \partial r$ ,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta.$$

By the CR equations (Theorem 2.1),

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial v}{\partial x} \sin \theta.$$

To compute  $\partial v / \partial \theta$ ,

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x} (-r \sin \theta) + \frac{\partial v}{\partial y} (r \cos \theta).$$

It is thus clear that the first equation of the theorem holds true. The proof of the second theorem is left as an exercise.  $\square$

**Theorem 2.3.** Let  $f(z) = u(x, y) + iv(x, y)$ . Suppose the first-order partial derivatives of  $u$  and  $v$  ( $u_x, u_y, v_x$  and  $v_y$ ) exist in a neighbourhood of  $z$ . If they are continuous at  $z$  and the CR equations hold, then  $f$  is differentiable at  $z$ .

**Example 2.3.** Suppose

$$f(z) = \begin{cases} (\bar{z})^2 / z & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at the point  $z = 0$  but the derivative of  $f$  fails to exist at  $z = 0$ .

*Solution.* We let  $z = x + iy$ , where  $x, y \in \mathbb{R}$ . Then, for  $z \neq 0$ ,

$$f(z) = \frac{(x - iy)^2}{x + iy} = \frac{(x - iy)^3}{x^2 + y^2} = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \left( \frac{-3x^2y + y^3}{x^2 + y^2} \right)$$

which is of the form  $f(z) = u(x, y) + iv(x, y)$ . The reader can check that at  $(0, 0)$ ,  $u_x, u_y, v_x, v_y$  are all zero, so the CR equations are satisfied. Next, we consider the following limit:

$$L = \lim_{h \rightarrow 0} \frac{(\bar{h})^2 / h - 0}{h} = \lim_{h \rightarrow 0} \left( \frac{\bar{h}}{h} \right)^2 = \lim_{(x, y) \rightarrow (0, 0)} \left( \frac{x - iy}{x + iy} \right)^2.$$

Say we approach along the real axis. Then,  $L = 1$ . However, if we approach along the line  $y = x$ ,

$$L = \lim_{(x, x) \rightarrow (0, 0)} \left[ \frac{x(1 - i)}{x(1 + i)} \right]^2 = -1$$

so we conclude that  $f'(0)$  does not exist.  $\square$

**Example 2.4.** Let

$$f(z) = f(x, y) = \begin{cases} \frac{xy(x + iy)}{x^2 + y^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at  $z = 0$  but  $f$  is not differentiable at  $z = 0$ .

*Solution.* We let the reader verify that the CR equations are satisfied at  $z = 0$ . As for differentiability, let  $h = a + ib$ , where  $a, b \in \mathbb{R}$ . Then consider

$$\frac{f(h) - f(0)}{h} = \frac{ab(a + ib)}{(a^2 + b^2)(a + ib)} = \frac{ab}{a^2 + b^2}.$$

We need to prove that as  $(a, b) \rightarrow (0, 0)$ , the limit  $L$  does not exist. Suppose we approach along the  $x$ -axis, then  $L = 0$ . However, if we approach along the line  $y = x$ , we have

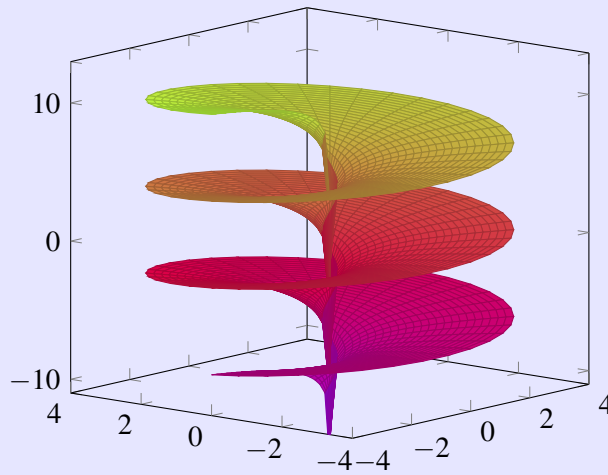
$$L = \lim_{a \rightarrow 0} \frac{a^2}{a^2 + a^2} = \frac{1}{2}.$$

As such, the limit  $L$  does not exist so we can conclude that  $f'(0)$  does not exist.  $\square$

**Definition 2.4 (principal logarithm).** Define

$$\text{Log } z = \ln |z| + i \text{Arg } z.$$

Note that  $\text{Log } z$  is a single-valued function defined on  $\mathbb{C} \setminus \{0\}$ .



### 2.3. Analytic Functions and Entire Functions

**Definition 2.5 (power series).** A power series over  $\mathbb{C}$  in the variable  $z$  centred at  $a \in \mathbb{C}$  is a formal sum

$$\sum_{n=0}^{\infty} a_n (z - a)^n \quad \text{for all } n \in \mathbb{N} \text{ and } a_n \in \mathbb{C}.$$

**Definition 2.6 (different types of convergence).** Let

$$\sum_{n=0}^{\infty} a_n (z - a)^n \quad \text{with } z \in \mathbb{C} \quad \text{be a power series over } \mathbb{C}.$$

We say that

(i) the series converges at  $z \in \mathbb{C}$  if and only if

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n (z - a)^n \quad \text{exists in } \mathbb{C};$$

(ii) the series converges absolutely at  $z \in \mathbb{C}$  if and only if

$$\sum_{n=0}^{\infty} |a_n (z - a)^n| < \infty \quad \text{in } \mathbb{R}_{\geq 0};$$

(iii) the series converges normally on some compact  $D \subseteq \mathbb{C}$  if and only if

$$\sum_{n=0}^{\infty} \sup_{z \in D} |a_n (z - a)^n| < \infty \quad \text{in } \mathbb{R}_{\geq 0};$$

(iv) the series converges locally normally on some open  $U \subseteq \mathbb{C}$  if and only if for all  $a \in U$ , there exists a neighbourhood  $D \subseteq U$  such that

$$\sum_{n=0}^{\infty} a_n (z - a)^n \quad \text{converges normally on } D$$

**Example 2.5.** We have the classic example of the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad \text{in } \mathbb{C}.$$

this series converges absolutely for all  $z \in \mathbb{C}$  with  $|z| < 1$  to  $1/(1 - z) \in \mathbb{C}$  and it does not converge for all  $z \in \mathbb{C}$  with  $|z| > 1$ . Also, for all  $r \in (0, 1)$ , the series converges normally on  $\overline{B}(0, r)$  and it converges locally normally on  $B(0, 1)$ .

**Lemma 2.1.** Let

$$S = \sum_{n=0}^{\infty} a_n (z - a)^n \quad \text{with } z \in \mathbb{C} \quad \text{be a power series over } \mathbb{C}.$$

Then, the following hold:

- (i) If  $S$  converges absolutely at  $z_0 \in \mathbb{C}$ , then it converges normally on the compact set  $\overline{B}(a, |z_0 - a|)$
- (ii) If  $S$  converges at  $z_0 \in \mathbb{C}$ , then it converges locally normally on the open set  $B(a, |z_0 - a|)$

**Definition 2.7 (radius of convergence).** The radius of convergence of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$$

is given by

$$\begin{aligned} R &= \sup \{ r \in \mathbb{R}_{\geq 0} : f(z) \text{ converges at all points in } B(a, r) \} \\ &= \sup \left\{ r \in \mathbb{R}_{\geq 0} : f(z) \text{ converges absolutely at all points in } \overline{B}(a, r) \right\} \end{aligned}$$

We note that  $R \in \mathbb{R}_{\geq 0}$ .

**Proposition 2.4 (Cauchy-Hadamard formula).** There is a nice formula on the radius of convergence of a power series over  $\mathbb{C}$  which is given by

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}.$$

One notes that the Cauchy-Hadamard formula in Proposition 2.4 can be easily deduced from the root test.

**Definition 2.8 (analytic function).** Let  $U \subseteq \mathbb{C}$  be an open set and  $a \in U$  be a point. A  $\mathbb{C}$ -valued function  $\varphi : U \rightarrow \mathbb{C}$  on  $U$  is analytic at  $a \in U$  if and only if there exists a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ centred at } a \text{ with positive radius of convergence } R$$

such that for all  $z \in U \cap B(a, R)$ , one has

$$\varphi(z) = f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{in } \mathbb{C}.$$

Then,

$\varphi : U \rightarrow \mathbb{C}$  is an analytic function (on  $U$ ) if and only if for all  $a \in U$ ,  $\varphi$  is analytic at  $a$ .

**Proposition 2.5.** Let

$$\sum_{n=0}^{\infty} c_n z^n \text{ be a power series centred at } 0 \text{ with positive radius of convergence } R.$$

Write  $f : B(0, R) \rightarrow \mathbb{C}$  for the  $\mathbb{C}$ -valued function it represents. Then,  $f$  is analytic on  $B(0, R)$ .

We will see an alternative and more rigorous way of formulating Proposition 2.5 in Proposition 2.6<sup>†</sup>.

**Example 2.6.** Show that there are no analytic functions  $f = u + iv$  such that  $u(x, y) = x^2 + y^2$ .

*Solution.* Suppose on the contrary that there exists some analytic function  $f$ . Then,  $u_x = 2x$  and  $u_y = 2y$ , so by the CR equations,  $v_y = 2x$  and  $v_x = -2y$ .  $v_y = 2x$  implies that  $v(x, y) = 2xy + g(x)$ . Taking the partial with respect to  $x$  and substituting it into  $v_x = -2y$ , we have  $2y + g'(x) = -2y$ . As such,  $g'(x) = -4y$ , so  $g(x) = -4xy + c$ , where  $c$  is an arbitrary constant. Putting everything together,

$$f(x, y) = x^2 + y^2 + i(-2xy + c).$$

However, this does not satisfy  $u_x = v_y$  in the CR equations. So, such an  $f$  does not exist. □

**Example 2.7.** Suppose  $f$  is analytic and real-valued in a domain  $D$ . Prove that  $f$  is constant in  $D$ .

*Solution.* Suppose  $f(z) = u + iv$ . We have  $\text{Im}(f) = 0$  so by the CR equations,  $u_x = 0$  and  $u_y = -v_x = 0$ . This implies that  $f'(z) = u_x + iv_x = 0$  so  $f$  is constant in  $D$ . □

**Example 2.8.** Suppose  $f$  and  $\bar{f}$  are analytic in a domain  $D$ . Show that  $f$  is constant in  $D$ .

*Solution.* Observe that  $\text{Re}(f) = (f + \bar{f})/2$  which is real-valued and analytic if both  $f$  and  $\bar{f}$  are analytic. By Example 2.7,  $\text{Re}(f)$  is constant, so  $f$  is constant. □

**Proposition 2.6.** For any  $a \in B(0, R)$  and  $k \in \mathbb{N}$ , define

$$d_k = \sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k}.$$

Then, the following properties hold:

- (i) For all  $k \in \mathbb{N}$ , the series  $d_k$  converges absolutely in  $\mathbb{C}$

<sup>†</sup>As you will see in Proposition 2.6, the latter is indeed more rigorous. Also, I think Prof. Chin Chee Whye set *something related* for an iteration of his MA2108S finals.

(ii) The power series

$$g(z) = \sum_{k=0}^{\infty} d_k (z-a)^k \quad \text{has positive radius of convergence } r \geq R - |a| > 0$$

(iii) For all  $z \in B(0, R) \cap B(a, r)$ , we have  $f(z) = g(z)$

*Proof.* We first prove (i). Fix  $\rho \in \mathbb{R}_{\geq 0}$  with  $|a| < \rho < R$ . Then,

$$\begin{aligned} \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} |c_n| |a|^{n-k} \right) (\rho - |a|)^k &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} |c_n| |a|^{n-k} (\rho - |a|)^k \\ &= \sum_{n=0}^{\infty} |c_n| \left[ \sum_{k=0}^n \binom{n}{k} |a|^{n-k} (\rho - |a|)^k \right] \\ &= \sum_{n=0}^{\infty} |c_n| (|a| + \rho - |a|)^n \\ &= \sum_{n=0}^{\infty} |c_n| \rho^n \end{aligned}$$

which is  $< \infty$  by the choice of  $\rho$ . Hence, the series defining  $d_k$  converges absolutely, proving (i).

Next, we take a look at (ii). As the power series

$$\sum_{k=0}^{\infty} |d_k| (\rho - |a|)^k \quad \text{is finite,}$$

then the power series  $g(z)$  converges normally on the compact set  $\overline{B(a, \rho - |a|)}$  so it has a radius of convergence  $r$  with  $r \geq \rho - |a|$  for any  $|a| < \rho < R$ . As such,  $r \geq R - |a|$ , which is positive. This proves (ii).

Lastly, we prove (iii). For all  $z \in B(0, R) \cap B(a, r)$ , we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n (a + z - a)^n \\ &= \sum_{n=0}^{\infty} c_n \left[ \sum_{k=0}^n \binom{n}{k} a^{n-k} (z-a)^k \right] \quad \text{by the binomial theorem} \\ &= \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k} \right) (z-a)^k \\ &= g(z) \end{aligned}$$

and the result follows. □

**Definition 2.9** (entire function). A function  $f$  which is analytic on the whole of  $\mathbb{C}$  is entire.

**Example 2.9.** Let

$$f(z) = x^3 - 3xy^2 + x^2 - y^2 + x + 1 + i(3x^2y - y^3 + 2xy + y).$$

(a) Show that  $f(z)$  is entire.

(b) Express  $f(z)$  as a function of  $z$ .

*Solution.*

(a) This is a very simple exercise using the CR equations.



(b) Recall the binomial theorem and see that

$$\begin{aligned}
 f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3) + x^2 - y^2 + x + 1 + i(2xy + y) \\
 &= x^3 - 3xy^2 + i(3x^2y - y^3) + x^2 - y^2 + 2ixy + x + iy + 1 \\
 &= (x + iy)^3 + (x + iy)^2 + x + iy + 1 \\
 &= z^3 + z^2 + z + 1
 \end{aligned}$$

So,  $f(z) = z^3 + z^2 + z + 1$ . □

**Example 2.10.** Find an entire function  $f$  such that  $\operatorname{Re}(f) = x^2 - 3x - y^2$  or explain why there is no such function.

*Solution.* Write  $f = u + iv$ , where  $u$  and  $v$  are real-valued functions. Given that  $u = x^2 - 3x - y^2$ , we apply the CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 3 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y.$$

Solving the first equation yields  $v = 2xy + g(y)$ , where  $g(y)$  is a function in terms of  $y$ . Then,  $2x + g'(y) = 2x - 3$ , which implies that  $g(y) = -3y + c$  for some constant  $c$ .

Now, we have  $v = 2xy - 3y + c$ . We conclude that the following function satisfies the hypotheses:

$$\begin{aligned}
 f(z) &= x^2 - 3x - y^2 + i(2xy - 3y + c) \\
 &= x^2 - y^2 + 2ixy - 3x - 3iy + ic \\
 &= z^2 - 3z + ci
 \end{aligned}$$

So,  $f(z) = z^2 - 3z + ci$ . □

**Example 2.11 (Dinh's 70 problems).** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function such that

$$f(0) = f'(0) = 0 \quad \text{and} \quad \operatorname{Re}(f') = x^2 - y^2 + 6xy.$$

Find  $f$ .

*Solution.* Let  $z = x + iy$ , so  $z^2 = x^2 - y^2 + 2xyi$ . As such,

$$x^2 - y^2 + 6xy = \operatorname{Re}(z^2 - 3iz^2)$$

Since  $f'(0) = 0$ , then  $f'(z) = z^2 - 3iz^2$ . It follows that  $f(z) = z^3/3 - iz^3$  as  $f(0) = 0$ . □

## 2.4. Harmonic Functions

**Definition 2.10 (harmonic function).** A real-valued function  $h(x, y)$  is said to be harmonic if it is twice continuously differentiable and satisfies Laplace's equation. That is,

$$h_{xx} + h_{yy} = 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

**Example 2.12.** Show that  $u^2$  cannot be harmonic for any non-constant harmonic function  $u$ .

*Solution.* Let  $u$  be a non-constant harmonic function. Then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Also, we have

$$\frac{\partial^2 (u^2)}{\partial x^2} = 2u \frac{\partial^2 u}{\partial x^2} + 2 \left( \frac{\partial u}{\partial x} \right)^2 \quad \text{and} \quad \frac{\partial^2 (u^2)}{\partial y^2} = 2u \frac{\partial^2 u}{\partial y^2} + 2 \left( \frac{\partial u}{\partial y} \right)^2.$$

However,

$$\frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2} = 2u \frac{\partial^2 u}{\partial x^2} + 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2u \frac{\partial^2 u}{\partial y^2} + 2 \left( \frac{\partial u}{\partial y} \right)^2 = 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial u}{\partial y} \right)^2 \neq 0,$$

which concludes the proof.  $\square$

**Definition 2.11 (harmonic conjugate).** Let  $u$  be a harmonic function. If  $v$  is a harmonic function satisfying the Cauchy-Riemann equations, then  $v$  is a harmonic conjugate of  $u$ .

**Example 2.13 (MA5217 AY24/25 Sem 1 Homework 1).** Show that the function

$$u(x, y) = e^{x-y} \cos(x+y) + e^{x+y} \cos(x-y)$$

is harmonic in  $\mathbb{C}$  and find a harmonic conjugate of  $u$ .

*Solution.* By definition, we need to show that  $u$  satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Let  $s = x + y$  and  $t = x - y$ , so

$$u\left(\frac{s+t}{2}, \frac{s-t}{2}\right) = e^t \cos s + e^s \cos t$$

Hence,

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = e^s \cos t - e^t \cos s + e^t \cos s - e^s \cos t = 0$$

so  $u$  is harmonic. Finding a harmonic conjugate is trivial.  $\square$

**Example 2.14 (Dinh's 70 problems).** Find all harmonic functions  $u(x, y)$  in  $\mathbb{C}$  such that

$$(x^2 - y^2) u(x, y) \quad \text{is harmonic in } \mathbb{C}.$$

*Solution.* Let  $f(x, y) = (x^2 - y^2)u(x, y)$ . Then,

$$f_{xx} = (x^2 - y^2)u_{xx} + 4xu_x + 2u \quad \text{and} \quad f_{yy} = (x^2 - y^2)u_{yy} - 4yu_y - 2u.$$

As such,

$$f_{xx} + f_{yy} = 4(xu_x - yu_y),$$

where we used the fact that  $u$  is harmonic (i.e.  $u_{xx} + u_{yy} = 0$ ). For  $f$  to be harmonic,  $xu_x = yu_y$ . One can use techniques taught to solve partial differential equations to deduce that  $u(x, y) = g(xy)$ , where  $g : \mathbb{R} \rightarrow \mathbb{R}$ . Therefore,  $g''(xy) = 0$ , so  $g(t) = at + b$ , where  $a, b \in \mathbb{R}$ . Hence,  $u(x, y) = axy + b$ .  $\square$

### 3. Line Integrals

#### 3.1. Fundamental Results

**Theorem 3.1** (fundamental theorem of line integrals). Suppose  $C$  is a smooth curve given by  $z(t)$  :  $a \leq t \leq b$  and  $F'(z) = f(z)$ . Then,

$$\int_C f(z) dz = F(z(b)) - F(z(a)).$$

**Lemma 3.1** (triangle inequality). Suppose  $f$  is a continuous complex-valued function of  $t$ . Then,

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt.$$

**Lemma 3.2** (ML inequality/estimation lemma). If  $C$  is a smooth curve of length  $L$ ,  $f$  is continuous on  $C$  and  $|f| \leq M$  on  $C$ , then

$$\left| \int_C f(z) dz \right| \leq ML \quad \text{where} \quad M = \sup_{z \in C} |f(z)| \quad \text{alternatively.}$$

**Example 3.1.** Let  $\gamma$  be the contour given by  $\gamma(t) = 3e^{it}$ , where  $0 \leq t \leq \pi$ . Prove that

$$\left| \int_{\gamma} \frac{\overline{z} e^{iz}}{z^2 - 11z + 30} dz \right| \leq 5.$$

*Solution.* Obviously,  $L = 3\pi$  since  $\gamma(t) = 3e^{it}$ , where  $0 \leq t \leq \pi$  is the equation of the upper half of a circle of radius 3 centred at the origin, so its arc length is  $3\pi$ . Now, we need to justify that  $M \leq 5/3\pi$ . Let  $z = x + iy$ .

We have

$$\left| \frac{\overline{z} e^{iz}}{z^2 - 11z + 30} \right| = \left| \frac{\overline{z} \cdot \overline{e^{iz}}}{(z-5)(z-6)} \right| = \frac{|\overline{z}| e^{-y}}{|z-5||z-6|} = \frac{|z| e^{-y}}{|z-5||z-6|}.$$

Since  $|z| \leq 3$  and applying the triangle inequality, we see that

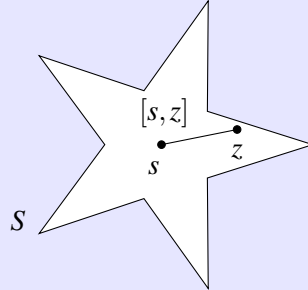
$$\frac{|z| e^{-y}}{|z-5||z-6|} \leq \frac{3 \cdot 1}{||z|-5||z|-6|} \leq \frac{3}{|3-5||3-6|} = \frac{1}{2}$$

so  $M = 1/2$ . It is clear that  $1/2 < 5/3\pi$  so we conclude that  $M \leq 5/3\pi$ . □

## 4. The Cauchy-Goursat Theorem

### 4.1. Some Results in Topology

**Definition 4.1 (star-shaped set).** A set  $S$  is star-shaped if it has a point  $s$ , known as the star centre, so that for each  $z \in S$ , the segment  $[s, z]$  lies in  $S$ .



**Remark 4.1.** Note that a star domain is not necessarily convex.

**Example 4.1.** A cross-shaped figure is a star domain but is not convex.

**Theorem 4.1.** Let  $S$  be an open star-shaped region and  $f$  continuous on  $S$ . Let  $T$  be a closed triangular region and  $\partial T$  be the boundary of the triangle traversed in the anticlockwise direction. Suppose

$$\int_{\partial T} f(z) dz = 0$$

for every  $T$  in  $S$ , then  $f$  has an anti-derivative,  $F$ , in  $S$ .

**Definition 4.2 (open ball).** Define  $B_r(w)$  or  $B(w, r)$  to be

the open ball of radius  $r$  and centre  $w$ .

Throughout this set of notes, we will be using these two representations interchangeably.

**Definition 4.3 (boundary point).** A point  $w \in \mathbb{C}$  is a boundary point of  $S$  if

for every  $r \in \mathbb{R}^+$  we have  $B_r(w) \cap S \neq \emptyset$ .

**Definition 4.4 (closure).** Denote the set of boundary points by  $\partial S$ . Given a set  $S$ , the closure of  $S$ , denoted by  $\bar{S}$ , is defined by

$$\bar{S} = S \cup \partial S.$$

**Theorem 4.2.** A set  $G$  is closed if and only if  $G = \bar{G}$ .

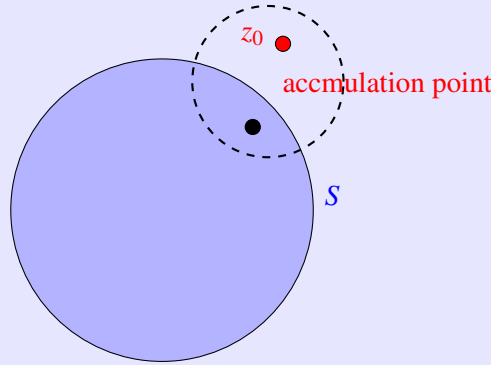
*Proof.* For the forward direction, suppose  $G$  is closed. We wish to prove that  $G = G \cup \partial G$ , or equivalently,  $\partial G \subseteq G$ . Suppose on the contrary that  $\partial G \not\subseteq G$ . Then, there exists  $w \in \partial G \setminus G$ . For every  $\varepsilon > 0$ , we have

$$B_\varepsilon(w) \cap G \neq \emptyset \text{ and } B_\varepsilon(w) \cap G' \neq \emptyset \text{ which implies } B_\varepsilon(w) \cap G \neq \emptyset.$$

However,  $w \notin G$ , so  $w \in G'$ . As  $G$  is closed, then  $G'$  is open, so there exists  $\varepsilon' > 0$  such that  $B(w, \varepsilon') \subseteq G'$ . Hence,  $B(w, \varepsilon') \cap G = \emptyset$  and this is a contradiction, so  $\partial G \subseteq G$ .

We then prove the reverse direction. Suppose  $G = G \cup \partial G$ . We wish to prove that  $G'$  is open. Let  $x \in G'$ . As  $\partial G \subseteq G$ , then  $G' \cap \partial G = \emptyset$ . There exists  $\varepsilon > 0$  such that  $B(x, \varepsilon) \cap G = \emptyset$  or  $B(x, \varepsilon) \cap G = \emptyset$ . As  $x \in G'$ , then  $B(x, \varepsilon) \cap G' \neq \emptyset$ . Therefore,  $B(x, \varepsilon) \cap G = \emptyset$  or  $B(x, \varepsilon) \subseteq G'$ , which is the definition of  $G'$  being open.  $\square$

**Definition 4.5 (accumulation point).** A point  $z_0$  is an accumulation point of a set  $S$  if each neighbourhood of  $z_0$  contains at least one point of  $S$  distinct from  $z_0$ .



**Remark 4.2.** The accumulation point of a set  $S$  does not have to be an element of that set.

**Example 4.2.** Prove that a set  $S$  is closed if and only if  $S$  contains all its accumulation points.

*Solution.* For the forward direction, we proceed with contradiction. Let  $y$  be an accumulation of  $S$  which is not in  $S$ . Then,  $y \in S'$ . As  $S'$  is an open set, there exists  $\delta > 0$  such that  $B_\delta(y) \subseteq S'$ . As such,  $B_\delta(y) \cap S = \emptyset$ , contradicting the assumption that  $y$  is an accumulation point for  $S$ .

For the reverse direction, suppose  $S$  contains all its accumulation points. We need to show that  $S$  is closed. It suffices to show that  $S'$  is open. Let  $x \in S'$ . Then,  $x$  is not an accumulation of  $S$  since  $S$  already contains all its accumulation points. So, there exists  $\delta > 0$  such that

$$B_\delta(x) \setminus (\{x\} \cap S) = B_\delta(x) \cap S = \emptyset.$$

We conclude that  $B_\delta(x) \subseteq S'$ , so  $S'$  is open.  $\square$

#### 4.2. Cauchy-Goursat Theorem

**Theorem 4.3 (Cauchy-Goursat theorem/Cauchy integral theorem).** Suppose  $f$  is analytic on a star-shaped region  $S$ . Then, for every simple closed path  $C$  in  $S$  traversed in the anticlockwise direction,

$$\int_C f(z) dz = 0.$$

**Example 4.3.** Let  $f(z) = \text{Log}(z+2)$  and the contour  $\gamma$  be the circle  $|z| = 1$  oriented in the anticlockwise direction. Use the Cauchy-Goursat theorem to prove that

$$\int_\gamma f(z) dz = 0.$$

*Solution.* Recall that  $\text{Log } z$  is analytic on  $\mathbb{C} \setminus (-\infty, 0]$ . Thus,  $f(z) = \text{Log}(z+2)$  is analytic on  $\mathbb{C} \setminus (-\infty, -2]$ . However,  $(-\infty, -2]$  lies outside the circle  $|z| = 1$ . Thus,  $f(z)$  is analytic inside and on the circle  $|z| = 1$ , which is a simple closed contour. The result follows by the Cauchy-Goursat theorem.  $\square$

**Theorem 4.4.** Let  $f$  be continuous on a star-shaped region  $S$ , and analytic on  $S \setminus \{z_0\}$ , i.e. the set  $S$  but excluding the point  $z_0$ . Then,  $f$  has an anti-derivative on  $S$ , and consequently,

$$\int_C f(z) dz = 0 \quad \text{for every simple closed curve } C \in S \text{ traversed anticlockwise.}$$

#### 4.3. Cauchy's Integral Formula

**Theorem 4.5 (Cauchy's integral formula).** Let  $f$  be analytic everywhere within and on a simple closed contour  $C$  traversed in the anticlockwise direction. If  $a$  is interior to  $C$ , then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

*Proof.* Define

$$g(z) = \frac{f(z) - f(a)}{z-a} \quad \text{which is analytic everywhere except at } z = a.$$

Since the derivative of  $f$  exists at  $a$ , then by the first principles of differentiation,

$$\lim_{z \rightarrow a} g(z) = f'(a).$$

Using the Cauchy-Goursat formula applied to a star-shaped region excluding  $a$  (since  $g$  is analytic everywhere except  $a$ ), then

$$\int_C g(z) dz = 0 \quad \text{which implies} \quad \int_C \frac{f(z) - f(a)}{z-a} dz = 0.$$

So,

$$\int_C \frac{f(z)}{z-a} dz = f(a) \int_C \frac{1}{z-a} dz = f(a) \cdot 2\pi i,$$

where the last equality follows since we are taking the contour integral on a loop around  $a$ .  $\square$

**Example 4.4.** Let  $z_0 \in \mathbb{C}$  and  $\gamma$  be a simple closed contour enclosing  $z_0$  with positive orientation. Without using Cauchy's integral formula, and using only the fact that

$$\int_{\gamma} \frac{1}{z-z_0} dz = 2\pi i,$$

show that

$$\text{if } p(z) = z_0 + z_1 z + \dots + a_{n-1} z^{n-1} + a_n z^n \text{ is a polynomial then } \int_{\gamma} \frac{p(z)}{z-z_0} dz = p(z_0) \cdot 2\pi i.$$

*Solution.* By the division algorithm for polynomials, there exist polynomials  $f(z)$  and  $r$  such that  $p(z) = (z-z_0)f(z) + r$ . So,  $p(z_0) = r$ .

Hence,  $p(z) = (z-z_0)f(z) + p(z_0)$  and we have

$$\int_{\gamma} \frac{p(z)}{z-z_0} dz = \int_{\gamma} f(z) + \frac{p(z_0)}{z-z_0} dz = \int_{\gamma} f(z) dz + p(z_0) \int_{\gamma} \frac{1}{z-z_0} dz = p(z_0) \cdot 2\pi i.$$

Note that the integral

$$\int_{\gamma} f(z) dz = 0$$

by the Cauchy-Goursat theorem.  $\square$

**Example 4.5.** Let  $C$  be the circle  $|z| = 2$  oriented in the anticlockwise direction. Evaluate

$$\int_C \frac{1}{|z-i|^2} dz.$$

*Solution.* We use the identity  $|z|^2 = z\bar{z}$ , so  $|z-i|^2 = (z-i)(\bar{z}+i)$ . Since  $|z| = 2$ , then  $\bar{z} = 4/z$ , so

$$(z-i)(\bar{z}+i) = z\bar{z} + i(z-\bar{z}) + 1 = 5 + i\left(z - \frac{4i}{z}\right) = \frac{iz^2 + 5z + 4}{z} = \frac{(iz+1)(z-4i)}{z}.$$

Hence, the contour integral is equivalent to

$$\int_C \frac{z}{(iz+1)(z-4i)} dz = \int_C \frac{f(z)}{z-i} dz \quad \text{where } f(z) = -\frac{iz}{z-4i}.$$

By Cauchy's integral formula, the integral is equivalent to  $2\pi i f(i) = -2\pi/3$ . □

**Corollary 4.1 (Cauchy's differentiation formula).** If  $f$  is analytic at a point  $a$ , then  $f^{(n)}(a)$  exists for  $n = 1, 2, \dots$  and are also analytic  $z_0$ , and

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz,$$

where  $C$  is a simple closed curve traversed in the anticlockwise direction that encloses  $a$ .

*Proof.* Apply induction on Cauchy's integral formula. □

**Theorem 4.6 (Liouville's theorem).** If  $f$  is a bounded and entire function, then  $f$  is a constant.

*Proof.* Since  $f$  is entire, we can represent it using a Taylor series about  $z = 0$ , so

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By Cauchy's integral formula,

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz,$$

where  $C$  is a circle of radius  $r$  centred at the origin. Since  $f$  is bounded, then  $|f(z)| \leq M$  for some constant  $M$  and for all  $z \in \mathbb{C}$ . We have

$$|a_n| = \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_C \left| \frac{f(z)}{z^{n+1}} \right| |dz| \leq \frac{1}{2\pi} \int_C \frac{M}{|z|^{n+1}} |dz| \leq \frac{M}{2\pi r^{n+1}} \int_C |dz| = \frac{M}{2\pi r^{n+1}} \cdot 2\pi r = \frac{M}{r^n}$$

Now, as  $|z| = r$  on the circle  $C$ , by setting  $r > 0$  to be arbitrary, as  $r$  tends to infinity,  $a_n = 0$  for all  $n \geq 1$ . This is because  $f$  is entire. Hence,  $f(z) = a_0 = M/r$  which is a constant. □

**Example 4.6.** Find all entire functions  $f(z)$  with  $f(0) = 2$  and  $|f(z) - e^z| \geq 1$  for all  $z \in \mathbb{C}$ .

*Solution.* We note that

$$\frac{1}{|f(z) - e^z|} \leq 1 \quad \text{where } f(z) - e^z \neq 0.$$

So,  $1/(f(z) - e^z)$  is bounded and entire. By Liouville's theorem,

$$\frac{1}{f(z) - e^z} = c,$$

where  $c$  is a constant. Since  $f(0) = 2$ , then  $c = 1$ . As such,  $f(z) = e^z + 1$ . □

**Example 4.7.** Let  $g$  be an entire function such that  $|g'(z)| < |g'(z) + i|$  for all complex numbers  $z$ . Show that there exist  $\alpha, \beta \in \mathbb{C}$  such that  $g(z) = \alpha z + \beta$  for all  $z \in \mathbb{C}$ .

*Solution.* Since  $g$  is entire, then  $g'$  is also entire. Let

$$h(z) = \frac{g'(z)}{g'(z) + i}.$$

Then  $h$  is the quotient of two entire functions such that the denominator is not equal to zero at each  $z \in \mathbb{C}$ , hence  $h$  is entire. It is clear that for all  $z \in \mathbb{C}$ ,  $|h(z)| < 1$ , so  $h$  is bounded on  $\mathbb{C}$ . By Liouville's theorem,  $h(z) = c$ , where  $c$  is a constant, so  $g'(z) = cg'(z) + ci$ . We have

$$g'(z) = \frac{ic}{1-c} = \alpha.$$

Hence,  $g(z) = \alpha z + \beta$ . □

**Example 4.8.** Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be an entire function such that

$$\lim_{z \rightarrow \infty} f(z) = \infty.$$

Show that  $f$  has at least one zero in  $\mathbb{C}$ .

*Solution.* Suppose on the contrary  $f$  has no zeros in  $\mathbb{C}$ . Consider

$$g(z) = \frac{1}{f(z)}.$$

Note that  $g$  is entire. Using the given limit, there exists  $R > 0$  such that for all  $|z| > R$ ,  $|f(z)| > 1$ . This implies that  $|g(z)| < 1$  but since  $g$  is continuous, it obtains a maximum  $M$  on the compact set  $\overline{D(0, R)}$ . Hence, for all  $z \in \mathbb{C}$ ,  $|g(z)| \leq \max\{1, M\}$ , so by Liouville's theorem,  $g$  is a constant, implying that  $f$  is a constant, which is a contradiction. □

**Example 4.9.** Find all entire functions  $f(z)$  such that

$$|f(z)| \leq \frac{1}{1+x^2+2y^2} \quad \text{for all } z = x + iy \in \mathbb{C}.$$

*Solution.* Since  $x^2, y^2 \geq 0$ , then  $|f(z)| \leq 1$ . By Liouville's theorem,  $f$  is a constant, say  $c$ . Then,

$$c \leq \frac{1}{1+|z|^2+y^2}.$$

It is clear that

$$\lim_{z \rightarrow \infty} f(z) = 0$$

so  $c = 0$ . Hence, the only function satisfying the hypothesis is  $f(z) = 0$ . □

**Example 4.10 (MA5217 AY24/25 Sem 1 Homework 1).** Find all entire functions  $f$  satisfying  $f(z+1) = f(z)$  and  $f(z+i) = f(z)$  for every  $z \in \mathbb{C}$ .

*Solution.* By an inductive argument, for all  $n \in \mathbb{Z}$ , we have

$$f(z+n) = f(z) \quad \text{and} \quad f(z+ni) = f(z).$$

Hence, it suffices to consider the behaviour of  $f$  on the unit square  $[0, 1] \times [0, 1]$ . Since the unit square is a compact set, it is bounded by the Heine-Borel theorem. Hence,  $f(x+iy)$  is bounded for all  $x, y \in \mathbb{R}$ . Since  $f$  is a bounded function, it is constant (follows by Liouville's theorem where we assumed that  $f$  is entire). So,  $f(z) = c$  for some  $c \in \mathbb{R}$ . □



**Example 4.11 (Dinh's 70 problems).** Let  $f = u + iv$  be an entire function. Show that if  $u^2(z) \geq v^2(z)$  for all  $z \in \mathbb{C}$ , then  $f$  must be a constant.

*Solution.* We have  $f^2 = u^2 - v^2 + 2uvi$ . Consider

$$g = e^{-f^2} = e^{v^2 - u^2} e^{-2uvi} \quad \text{which is entire} \quad \text{and} \quad |g| \leq \frac{1}{e}.$$

By Liouville's theorem,  $g$  is a constant. So,  $e^{-f^2} = k$  for some constant  $k$ . Thus,  $f$  is a constant.  $\square$

**Theorem 4.7 (fundamental theorem of algebra).** Every non-constant polynomial with complex coefficients has a zero in  $\mathbb{C}$ .

#### 4.4. Applications of Cauchy's Integral Formula

**Theorem 4.8 (Morera's theorem).** Let  $f$  be a continuous function on  $D$ . Let  $T$  be a closed triangle in  $D$  and  $\partial T$  be the boundary of  $T$  traversed in the anticlockwise direction. Then,

$$\int_{\partial T} f(z) dz = 0.$$

**Theorem 4.9.** Let  $f$  be an entire function. Define  $g(z) = f'(a)$  if  $z = a$  and

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

if  $z \neq a$ . Then,  $g$  is also entire.

**Theorem 4.10 (extended Liouville's theorem).** If  $f$  is entire and if for some  $k \in \mathbb{N}$ , there exists constants  $A, B > 0$  such that

$$|f(z)| \leq A + B|z|^k,$$

then  $f$  is a polynomial of degree at most  $k$ .

**Example 4.12 (Dinh's 70 problems).** Let  $u$  be a real-valued harmonic function in the complex plane such that

$$u(z) \leq a|\ln|z|| + b$$

for all  $z$ , where  $a$  and  $b$  are positive constants. Prove that  $u$  is constant.

*Solution.* By Liouville's theorem, since  $u$  is harmonic, it suffices to show that  $u$  is bounded. Let  $f(z) = a|\ln|z|| + b$ . Then, by Cauchy's integral formula,

$$|u'(k)| = \left| \frac{1}{2\pi i} \int_{\gamma: |z|=R} \frac{f(z)}{(z-k)^2} dz \right| \leq R \cdot \frac{a|\ln R| + b}{|R - |k||^2},$$

where we have considered  $\gamma$  to be the circle of radius  $R$  centred at the origin and naturally, the path is taken to be positively-oriented. To establish the upper bound for  $|u'(k)|$ , the triangle inequality and reverse triangle inequality are used. Now, note that

$$\lim_{R \rightarrow \infty} R \cdot \frac{a|\ln R| + b}{|R - |k||^2} = 0$$

which implies that  $|u'(k)| = 0$ , or rather,  $u'(k) = 0$ . So,  $u(k)$  is a constant for all  $k \in \mathbb{R}$ .  $\square$

**Theorem 4.11 (Gauss' mean value theorem).** If  $f$  is analytic in  $D$  and  $\alpha \in D$ , then

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta.$$

*Proof.* By Cauchy's integral formula, for  $a \in D$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz.$$

Let  $C$  be a circle of radius  $r$  centred at  $a$ . Then, our parameterisation is  $z = a + re^{i\theta}$ , so  $dz/d\theta = ire^{i\theta}$ . Hence,

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{a + re^{i\theta} - a} \cdot ire^{i\theta} d\theta$$

and the result follows with some simple cancellation.  $\square$

**Theorem 4.12 (maximum modulus theorem for open balls).** Suppose  $f(z)$  is analytic throughout a neighbourhood  $|z - z_0| < R$  of a point  $z_0$ . If  $|f(z)| \leq |f(z_0)|$  for each  $z$  in the neighbourhood, then  $f(z)$  attains a constant value  $f(z_0)$  throughout the neighbourhood.

**Theorem 4.13 (maximum modulus principle).** If  $f$  is analytic in  $D$  and

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in D \text{ then } f(z) \text{ is a constant.}$$

**Example 4.13 (Dinh's 70 problems).** Let  $f(z) = a_0 + a_1z + \dots + a_nz^n$  be a complex polynomial of degree  $n > 0$ . Prove that

$$\frac{1}{2\pi i} \int_{|z|=R} z^{n-1} |f(z)|^2 dz = a_0 \bar{a}_n R^{2n}.$$

*Solution.* Note that  $|f(z)|^2 = f(z) \cdot \overline{f(z)}$ . Setting  $z = Re^{i\theta}$ , the integral becomes

$$\frac{1}{2\pi} \int_0^{2\pi} R^n e^{in\theta} \left( a_0 + a_1 R e^{i\theta} + \dots + a_n R^n e^{in\theta} \right) \left( \bar{a}_0 + \bar{a}_1 R e^{-i\theta} + \dots + \bar{a}_n R^n e^{-in\theta} \right) d\theta.$$

Since

$$\int_0^{2\pi} e^{ik\theta} d\theta = 0 \text{ for all } k \neq 0,$$

upon multiplying the polynomials  $a_0 + a_1 R e^{i\theta} + \dots + a_n R^n e^{in\theta}$  and  $\bar{a}_0 + \bar{a}_1 R e^{-i\theta} + \dots + \bar{a}_n R^n e^{-in\theta}$ , we wish to extract the coefficient of  $e^{-in\theta}$ . So, the integral becomes

$$\frac{1}{2\pi i} \int_0^{2\pi} R^n e^{in\theta} a_0 \bar{a}_n R^n e^{-in\theta} d\theta$$

and the result follows.  $\square$

**Example 4.14 (Dinh's 70 problems).** Suppose  $u(z)$  is harmonic on  $D(0, r)$ , where  $r > 1$ . Prove that

$$\int_0^{2\pi} u(e^{it}) \cos^2\left(\frac{t}{2}\right) dt = \pi u(0) + \frac{\pi}{2} u'(0) \quad \text{and} \quad \int_0^{2\pi} u(e^{it}) \sin^2\left(\frac{t}{2}\right) dt = \pi u(0) - \frac{\pi}{2} u'(0),$$

where  $u'(0) = u_x(0)$ .

*Solution.* Let  $I_1$  and  $I_2$  denote the two integrals respectively. We have

$$I_1 + I_2 = \int_0^{2\pi} u(e^{it}) dt \text{ and } I_1 - I_2 = \int_0^{2\pi} u(e^{it}) \cos t dt.$$

We parametrise each integral using  $z = e^{it}$  so  $dz/dt = ie^{it}$ . Also, recall that  $\cos t = (z + z^{-1})/2$ . So,

$$I_1 + I_2 = \frac{1}{i} \int_{|z|=1} \frac{u(z)}{z} dz = \pi u(0),$$

where we used Cauchy's integral formula. Also,

$$I_1 - I_2 = \frac{1}{2i} \int_{|z|=1} u(z) + \frac{u(z)}{z^2} dz = \frac{1}{2i} \int_{|z|=1} \frac{u(z)}{z^2} dz = \pi u'(0),$$

where we used Cauchy's integral formula and the fact that  $u(z)$  is analytic on  $D(0, r)$  (since  $u(z)$  is harmonic on  $D(0, r)$ ).  $\square$

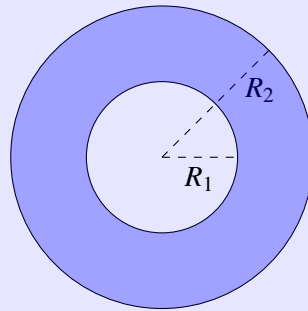
## 5. Series

### 5.1. Laurent Series

**Definition 5.1 (annulus).** Define

$$\text{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$$

to be the shaded region as follows:



**Theorem 5.1 (Laurent expansion).** If  $f$  is analytic in the annulus

$$\text{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\},$$

then it has a Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz.$$

Here,  $C$  is a circle of radius  $R$  centred at the origin with  $R_1 < R < R_2$ .

#### Example 5.1.

(a) Consider the function

$$f(z) = \frac{5z-3}{(z+1)(z-3)}.$$

Find the Laurent series of  $f(z)$  for the annular domain  $1 < |z| < 3$ .

(b) Find the value of the contour integral

$$\int_C \frac{5z-3}{z^5(z+1)(z-3)} dz,$$

where  $C$  denotes the circle  $|z| = 2$  oriented in the anticlockwise direction.

(c) Find the Laurent series of the function

$$\frac{10z^6 - 6z^4}{(z^2 + 1)(z^2 - 3)}$$

in the annular domain  $1 < |z| < \sqrt{3}$ .

*Solution.*

(a) We see that

$$\begin{aligned}\frac{5z-3}{(z+1)(z-3)} &= \frac{2}{z+1} + \frac{3}{z-3} \\ &= \frac{2}{z} \cdot \frac{1}{1+1/z} - \frac{1}{1-z/3} \\ &= \frac{2}{z} \sum_{n=0}^{\infty} (-1)^n (-z)^n - \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n\end{aligned}$$

We note that the first summation is valid for  $|1/z| < 1$  while the second summation is valid for  $|z/3| < 1$ .

(b) We see that the contour integral is equivalent to

$$\int_C \frac{f(z)}{z^5} dz = 2\pi i \left(-\frac{1}{3^4}\right) = -\frac{2\pi i}{81}.$$

(c) Let us make a comparison. Perhaps we can consider  $f(z^2)$ . Note that

$$f(z^2) = \frac{5z^2-3}{(z^2+1)(z-3)}.$$

Hence, it is clear that the function in (c) is  $2z^4 f(z^2)$ . Recall that the Laurent series of  $f$  in the annulus  $1 < |z| < 3$  is

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} - \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n$$

so the required answer is

$$4z^4 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} - 2z^4 \sum_{n=0}^{\infty} \left(\frac{z^2}{3}\right)^n = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n-2}} - 2 \sum_{n=0}^{\infty} \frac{z^{2n+4}}{3^n}$$

in the annular domain  $1 < |z| < \sqrt{3}$ . □

**Example 5.2.** Suppose  $f(z)$  is entire and  $|f(z)| > 1$  when  $|z| > 1$ . Prove that  $f(z)$  is a polynomial.

*Solution.* Since  $f$  is entire, then in the closed unit disk, it has a finite number of zeros. Say the zeros are  $z_1, \dots, z_m$ . So, we can write

$$f(z) = (z - z_1) \dots (z - z_m) g(z) = p(z) g(z),$$

where  $g$  is entire with no zeros and  $p(z)$  is a polynomial of degree  $m$ . It suffices to show that  $g$  is a constant. Let  $h(z) = 1/g(z)$  so we shall write  $h$  as the following Laurent series:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{h(z)}{z^{n+1}} dz$$

Here, we let  $\gamma$  be  $|z| = R$ , i.e. the circle of radius  $R$  centred at the origin. Letting  $z = Re^{i\theta}$ , the contour integral becomes

$$\int_0^{2\pi} \frac{iRe^{i\theta} h(Re^{i\theta})}{R^{n+1} e^{i(n+1)\theta}} d\theta.$$

Let  $h(Re^{i\theta}) \leq kR^m$  so it is clear that for all  $n > m$ ,

$$\lim_{R \rightarrow \infty} |a_n| \leq \lim_{R \rightarrow \infty} \frac{kR^m}{R^n} = 0.$$

As such,  $h(z)$  is a constant, and  $g(z)$  is a constant. □

## 6. Residue Theory

### 6.1. Introduction

We adopt an alternative representation for the annulus  $\text{Ann}(z_0, R_1, R_2)$ , so if  $f(z)$  is analytic in this annulus,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{-n+1}} ds$$

and  $C$  is any positively oriented simple closed contour around  $z_0$  lying inside  $\text{Ann}(z_0, R_1, R_2)$ .

**Definition 6.1** (principal part of Laurent series). The sum

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{is the principal part of } f(z) \text{ at } z_0.$$

**Theorem 6.1** (removable singularity). If  $b_n = 0$  for all  $n \in \mathbb{N}$ , then  $z_0$  is a point of removable singularity of  $f(z)$ . Thus, the Laurent series of  $f(z)$  is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where } 0 < |z - z_0| < R.$$

**Example 6.1.** The singular point  $z = 0$  of  $\sin z/z$  is a removable singularity. We have

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

where  $0 < |z| < \infty$ . This asserts that our claim is true.

**Example 6.2** (Dinh's 70 problems). Let  $f(z)$  be holomorphic in  $\mathbb{C} \setminus \{0\}$  and suppose that

$$\int_{|z|=1} z^n f(z) dz = 0 \quad \text{for any } n \in \mathbb{Z}_{\geq 0}.$$

Show that  $f$  has a removable singularity at  $z = 0$ .

*Solution.*  $f$  has a Laurent series representation around  $z = 0$ . Write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$$

so the integral becomes

$$\begin{aligned} \int_{|z|=1} z^n \sum_{k=-\infty}^{\infty} a_k z^k dz &= 0 \\ \sum_{k=-\infty}^{\infty} \int_{|z|=1} a_k z^{n+k} dz &= 0 \end{aligned}$$

since that the series converges uniformly on compact sets away from the singularity. Note that

$$\int_{|z|=1} z^k dz = 0 \quad \text{for all } k \neq -1.$$

As such,  $n + k = -1$ . Since  $n \geq 0$ , it forces the inequality  $k \leq -1$ , which implies that  $a_k = 0$  for all  $k \leq -1$ , i.e.

$$\sum_{k=-1}^{\infty} \int_{|z|=1} a_k z^{n+k} dz = 0.$$

It is clear that  $a_{-1} = 0$ . With all coefficients of negative powers being zero, it shows that  $f(z)$  has a removable singularity at  $z = 0$ .  $\square$

**Definition 6.2 (essential singularity).** If  $b_n \neq 0$  for infinitely many  $n$ , then  $z_0$  is a point of essential singularity of  $f(z)$ . In this case, some of the  $b_n$ 's may be zero.

**Example 6.3.** The point  $z = 0$  of  $\exp(1/z)$  is an essential singularity as

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \dots$$

where  $0 < |z| < \infty$ .

**Definition 6.3 (pole).** If there exists  $m \in \mathbb{N}$  such that  $b_m \neq 0$  but  $b_n = 0$  for all  $n > m$  so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^m \frac{b_n}{(z - z_0)^n},$$

then  $z_0$  is a pole of order  $m$  of  $f(z)$ . If  $m = 1$ ,  $z_0$  is a simple pole of  $f(z)$ ; if  $m = 2$ ,  $z_0$  is a double pole of  $f(z)$ .

**Example 6.4.** Consider the point  $z = 1$  of

$$f(z) = \frac{1}{(z-1)^2} + z.$$

We can rewrite it as

$$f(z) = \frac{1}{(z-1)^2} + 1 + (z-1)$$

Hence,  $z = 1$  is a double pole.

**Example 6.5 (MA5217 AY24/25 Sem 1 Homework 1).** Find all the singularities in  $\mathbb{C}$  of the following function  $f(z)$  and their types where

$$f(z) = \frac{z^2 + 3z + 2}{z(z^4 - 1)} e^{1/z^2}.$$

*Solution.* Consider the term  $z^4 - 1$  in the denominator of  $f(z)$ . Then,  $z^4 - 1 = (z^2 + 1)(z^2 - 1) = (z^2 + 1)(z + 1)(z - 1)$ . Also, the numerator can be factorised as  $(z + 2)(z + 1)$ . Also, consider

$$\frac{e^{1/z^2}}{z} = \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n \frac{1}{n!} \cdot \frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1} n!}.$$

So,  $f(z)$  has simple poles at  $z = 1, z = i, z = -i$ , a removable singularity at  $z = -1$ , and an essential singularity at  $z = 0$ .  $\square$

**Theorem 6.2 (residue theorem).** Let  $C$  be a positively oriented simple closed contour within and on which a function  $f$  is analytic except for a finite number of singular points  $z_1, z_2, \dots, z_n$  interior to  $C$ . Let

$\text{Res}(f, a_k)$  denote the residue of  $f$  at  $a_k$ , for all  $1 \leq k \leq n$ . Then,

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, a_k).$$

**Theorem 6.3.** If  $f$  is analytic everywhere on the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour  $C$ , then

$$\int_C f(z) dz = 2\pi i \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

**Example 6.6 (Dinh's 70 problems).** Evaluate the integral

$$\int_{C^+(0,2)} e^{1/z} dz.$$

*Solution.* Let  $w = 1/z$  so  $dw/dz = -w^2$ . The integral becomes

$$\int_{C^+(0,1/2)} e^{e^w} \cdot \frac{dw}{w^2}.$$

Let  $f(w) = e^{e^w}$ . By the residue theorem,

$$\int_{C^+(0,1/2)} \frac{f(w)}{w^2} dw = 2\pi i \text{Res}(f(w), 0) = 2\pi i e$$

and we are done. □

## 6.2. Residue Computation Methods

There are three methods for computing residues.

**Theorem 6.4 (method 1).** Suppose for  $z$  near  $z_0$ ,  $f(z)$  can be written as

$$f(z) = \frac{\phi(z)}{z - z_0},$$

where  $\phi(z)$  is analytic at  $z_0$  and  $f$  has a simple pole or a removable singularity at  $z_0$ . Then,

$$\text{Res}_{z=z_0} f(z) = \phi(z_0).$$

*Proof.* Since  $\phi(z)$  is analytic at  $z_0$ , then by Taylor's theorem, for  $z$  near  $z_0$ ,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \dots$$

so the Laurent series of  $f(z)$  at  $z_0$  is

$$f(z) = \frac{\phi(z)}{z - z_0} = \frac{\phi(z_0) + \phi'(z_0)(z - z_0) + \dots}{z - z_0} = \frac{\phi(z_0)}{z - z_0} + \phi'(z_0) + \dots$$

and the result follows. □



**Theorem 6.5 (method 2).** Suppose for  $z$  near  $z_0$ ,  $f(z)$  can be written as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

where  $\phi(z)$  is analytic at  $z_0$  and  $m \geq 1$ . Then,

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

*Proof.* It is inferred that  $f$  has a pole of order less than or equal to  $m$  or a removable point of singularity at  $z_0$ . Observe that when  $m = 1$ , it is just method 1 (recall Theorem 6.4). Using Taylor's theorem again, the series expansion of  $\phi(z)$  is the same as before. That is,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \dots$$

so

$$\begin{aligned} f(z) &= \frac{\phi(z)}{(z - z_0)^m} \\ &= \frac{1}{(z - z_0)^m} \left[ \phi(z_0) + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} (z - z_0)^{m-1} + \dots \right] \\ &= \frac{\phi(z_0)}{(z - z_0)^m} + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \cdot \frac{1}{z - z_0} + \dots \end{aligned}$$

The result follows. □

**Theorem 6.6 (method 3).** If  $p(z)$  and  $q(z)$  are analytic at  $z_0$  and  $q(z)$  has a simple zero at  $z_0$  (i.e.  $q(z_0) = 0$  but  $q'(z_0) \neq 0$ ), then

$$\operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

**Theorem 6.7 (method 4).** If all the above methods fail, use the Laurent series of  $f(z)$  and read  $b_1$ .

**Example 6.7.** For the following function  $f(z)$ , find all of its singularities in  $\mathbb{C}$ , their types and residues at these points:

$$f(z) = \frac{z^2 + 1}{z^6 + 1}.$$

*Solution.* The singularities of  $f(z)$  are the zeros of the denominator  $z^6 + 1$ , that is the 6 points

$$z_k = \exp\left(\frac{i\pi}{6} + \frac{k\pi i}{3}\right),$$

where  $0 \leq k \leq 5$ . These points are simple zeros of  $z^6 + 1 = 0$ . The points  $z_1 = i$  and  $z_4 = -i$  are the roots of the equation  $z^2 + 1 = 0$  (refer to the numerator). Thus,  $z_1, z_4$  are removable and  $z_0, z_2, z_3, z_5$  are simple poles of  $f$ .

So, the residues of  $f$  at  $z_1, z_4$  are 0, whereas the residue of  $f$  at  $z_k$  for  $k = 0, 2, 3, 5$  is equal to  $(z_k^2 + 1)/6z_k^5$ . □

**Example 6.8 (classic result).** Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx = \frac{\pi}{e}.$$

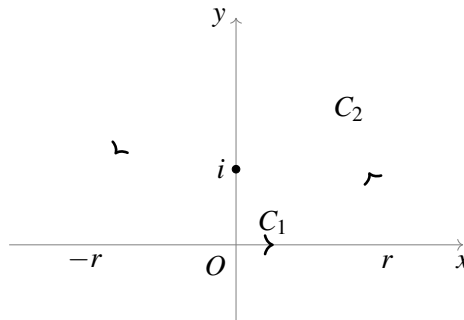
*Solution.* We consider

$$f(z) = \frac{e^{iz}}{z^2 + 1}$$

so the integral  $\operatorname{Re}(f(z))$  over the real numbers is the required answer. Let  $C$  be the path  $C_1 + C_2$ , where  $C_1$  and  $C_2$  are parametrised as follows:

$$C_1(t) = t, \text{ where } t \in [-r, r]$$

$$C_2(t) = re^{it}, \text{ where } t \in [0, \pi]$$



By Cauchy's residue theorem,

$$\sum \operatorname{Res}(f(z)) = \frac{1}{2\pi i} \int_C f(z) dz.$$

Only one of the two poles of  $f(z)$ ,  $z = i$ , is inside  $C$  as we are considering the upper half of the circle centred at the origin. We have

$$\int_C f(z) dz = \int_{C_1} \frac{e^{iz}}{z^2 + 1} dz + \int_{C_2} \frac{e^{iz}}{z^2 + 1} dz.$$

For the integral over  $C_1$ , applying the parametrisation,

$$\int_{C_1} \frac{e^{iz}}{z^2 + 1} dz = \int_{-r}^r \frac{e^{it}}{t^2 + 1} dt = \int_{-r}^r \frac{\cos t}{t^2 + 1} dt + i \int_{-r}^r \frac{\sin t}{t^2 + 1} dt.$$

Since  $\sin t$  is an odd function, then the integral of  $\sin t / (t^2 + 1)$  is zero. Hence,

$$\int_{C_1} \frac{e^{iz}}{z^2 + 1} dz = \int_{-r}^r \frac{\cos t}{t^2 + 1} dt.$$

As for the integral over  $C_2$ , applying the parametrisation,

$$\int_{C_2} \frac{e^{iz}}{z^2 + 1} dz = \int_0^\pi \frac{\exp(ire^{it})}{r^2 e^{i2t} + 1} \cdot ire^{it} dt.$$

By applying Euler's Formula,

$$\begin{aligned} \int_0^\pi \frac{\exp(ire^{it})}{r^2 e^{i2t} + 1} \cdot ire^{it} dt &= ir \int_0^\pi \frac{e^{i(t+r\cos t)} e^{-r\sin t}}{r^2 e^{i2t} + 1} dt \\ \left| \int_0^\pi \frac{\exp(ire^{it})}{r^2 e^{i2t} + 1} \cdot ire^{it} dt \right| &= r \int_0^\pi \frac{e^{-r\sin t}}{|r^2 e^{i2t} + 1|} dt \\ &\leq \frac{r}{r^2 - 1} \int_0^\pi e^{-r\sin t} dt \end{aligned}$$

Let the radius  $r$  of the semicircle tend to infinity so it is then clear that

$$\int_{C_2} \frac{e^{iz}}{z^2 + 1} dz = 0.$$

Therefore, by Cauchy's residue theorem (rearrange the equation at the start of our solution),

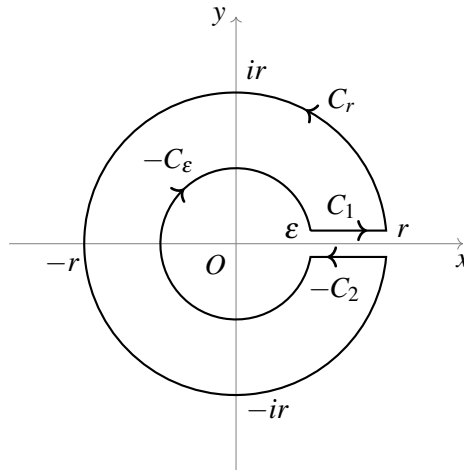
$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + 1} dt = 2\pi i \cdot \frac{e^{i^2}}{2i} = \frac{\pi}{e}.$$

□

**Example 6.9 (branch cut).** Prove that

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 5x + 6} dx = \pi(\sqrt{3} - \sqrt{2}).$$

*Solution.* Note that 0 is a branch point of  $\sqrt{z}$ . So,  $\sqrt{z}$  has a branch cut along the positive real axis, i.e.  $[0, \infty)$ . Hence,  $\sqrt{z}$  is analytic on  $\mathbb{C} \setminus [0, \infty)$ . We adopt the following keyhole contour.



One should think of the above contour as having  $\epsilon$  so small that  $C_1$  and  $C_2$  are essentially on the  $x$ -, or rather, real axis. Let the region the contour encloses be  $D$ . Then, we shall consider the integral over the boundary (this is denoted by  $\partial D$ ). That is,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2 + 5z + 6} dz.$$

By Cauchy's residue theorem,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2 + 5z + 6} dz = 2\pi i \left[ \frac{\sqrt{z}}{2z + 5} \Big|_{z=-3} + \frac{\sqrt{z}}{2z + 5} \Big|_{z=-2} \right] = 2\pi (\sqrt{3} - \sqrt{2}).$$

We now evaluate the contour integral by considering the different *pieces*.

$$\begin{aligned} \int_{\partial D} \frac{\sqrt{z}}{z^2 + 5z + 6} dz &= \int_{C_r} - \int_{C_\epsilon} + \int_{C_1} - \int_{C_2} \\ &= \int_{C_r} - \int_{C_\epsilon} + 2 \int_{\epsilon}^r \frac{\sqrt{x}}{x^2 + 5x + 6} dx \end{aligned}$$

By the estimation lemma,

$$\left| \int_{C_r} f(z) dz \right| \leq 2\pi r \cdot \frac{\sqrt{r}}{r^2 - 5r - 6}$$

which tends to 0 as  $r$  tends to infinity. In a similar fashion, one can prove that

$$\left| \int_{C_\epsilon} f(z) dz \right| \leq 2\pi \epsilon \cdot \frac{\sqrt{\epsilon}}{6 - 5\epsilon - \epsilon^2}$$

which tends to 0 too as  $\epsilon$  tends to 0. As such,

$$2\pi (\sqrt{3} - \sqrt{2}) = 2 \int_0^{\infty} \frac{\sqrt{x}}{x^2 + 5x + 6} dx$$

and the result follows.

□

**Example 6.10 (pizza contour).** Prove that for  $n \geq 2$ ,

$$\int_0^\infty \frac{1}{x^n + 1} dx = \frac{\pi}{n \sin\left(\frac{\pi}{n}\right)}.$$

*Solution.* Let

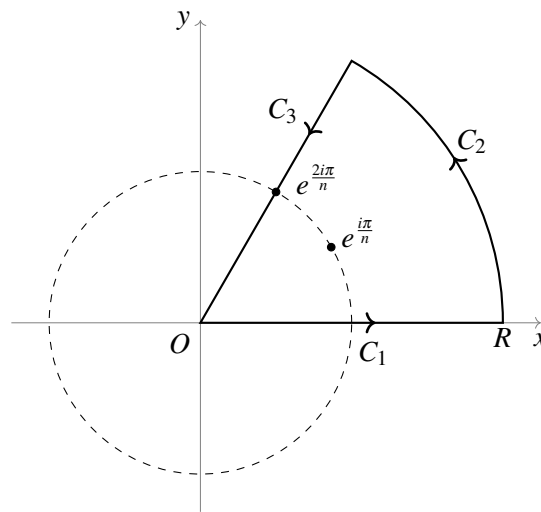
$$f(z) = \frac{1}{z^n + 1}$$

and the required integral to be  $I$ . Consider the following parametrisation (informally known as the *pizza contour*):

$$C_1(t) := t, \text{ where } 0 \leq t \leq R$$

$$C_2 : |z| = R \text{ (note that the angle subtended by the arc does not matter)}$$

$$C_3(t) := (R-t) \exp\left(\frac{2\pi i}{n}\right), \text{ where } 0 \leq t \leq R$$



By defining  $C$  to be the contour, it is clear that

$$\int_C \frac{1}{z^n + 1} dz = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

Only the pole  $z = e^{i\pi/n}$  is in  $C$  so by the residue theorem,

$$\int_C \frac{1}{z^n + 1} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/n}} f(z) = -\frac{2\pi i}{n} \exp\left(\frac{i\pi}{n}\right).$$

We focus on  $C_1$ .

$$\int_{C_1} \frac{1}{z^n + 1} dz = \int_0^R \frac{1}{t^n + 1} dt.$$

Letting  $R$  tend to infinity, and since  $t$  is a dummy variable, it is easy to see that

$$\int_{C_1} \frac{1}{z^n + 1} dz = \int_0^\infty \frac{1}{x^n + 1} dx = I.$$

For  $C_2$ , by the triangle inequality,  $|z^n + 1| \geq ||z^n| - |-1|| = |R^n - 1|$ . Hence,

$$\left| \int_{C_2} \frac{1}{z^n + 1} dz \right| \leq \int_{C_2} \frac{1}{R^n - 1} dz = \frac{c\pi R}{R^n - 1}.$$

Letting  $R$  tend to infinity, we see that the integral over  $C_2$  is zero. Earlier, we mentioned that the angle subtended by the arc does not matter and we affirm this statement here.

The integral over  $C_3$  is more complicated. Using the substitution

$$z = (R - t) \exp\left(\frac{2\pi i}{n}\right),$$

we see that

$$\int_{C_3} \frac{1}{z^n + 1} dz = -\exp\left(\frac{2\pi i}{n}\right) \int_0^R \frac{1}{(R - t)^n + 1} dt.$$

This calls for a substitution, say  $u = R - t$ . Hence, the integral over  $C_3$  becomes

$$-\exp\left(\frac{2\pi i}{n}\right) \int_0^R \frac{1}{u^n + 1} du \xrightarrow{R \rightarrow \infty} -\exp\left(\frac{2\pi i}{n}\right) \int_0^\infty \frac{1}{x^n + 1} dx = -I \exp\left(\frac{2\pi i}{n}\right).$$

To conclude,

$$\begin{aligned} -\frac{2\pi i}{n} \exp\left(\frac{i\pi}{n}\right) &= I \left[ 1 - \exp\left(\frac{2\pi i}{n}\right) \right] \\ I &= -\frac{2\pi i}{n} \cdot \frac{\exp\left(\frac{i\pi}{n}\right)}{1 - \exp\left(\frac{2\pi i}{n}\right)} \\ &= -\frac{2\pi i}{n} \cdot \frac{\exp\left(\frac{i\pi}{n}\right)}{\exp\left(\frac{i\pi}{n}\right) \exp\left(-\frac{i\pi}{n}\right) - \exp\left(\frac{i\pi}{n}\right) \exp\left(\frac{i\pi}{n}\right)} \\ &= \frac{\pi}{n \sin\left(\frac{\pi}{n}\right)} \end{aligned}$$

so we have finally derived this beautiful result. □

**Example 6.11.** Prove that

$$\int_0^{2\pi} \frac{1}{5 + 3 \sin \theta} d\theta = \frac{\pi}{2}.$$

*Solution.* Set  $z = e^{i\theta}$  so  $\sin \theta = (z - z^{-1})/2i$ . The integral becomes

$$\int_{|z|=1} \frac{1}{5 + 3 \left( \frac{z - z^{-1}}{2i} \right)} \cdot \left( -\frac{i}{z} \right) dz = 2 \int_{|z|=1} \frac{1}{3z^2 + 10iz - 3} dz.$$

Let

$$f(z) = \frac{1}{3z^2 + 10iz - 3}.$$

It has two simple poles  $z_1 = -i/3$  and  $z_2 = -3i$ . The first one is interior to the circle  $|z| = 1$  so we shall consider this. By the residue theorem, the answer is

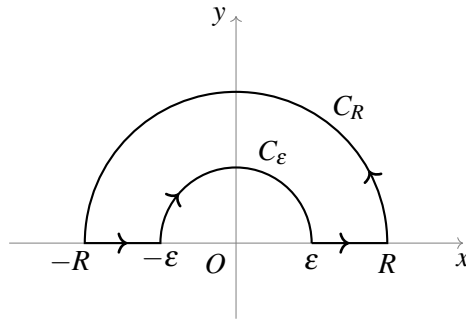
$$2 \cdot 2\pi i \cdot \frac{1}{3(z_1 + 3i)} = \frac{\pi}{2}.$$

□

**Example 6.12.** Prove that

$$\int_0^\infty \frac{(\log x)^2}{x^2 + 1} dx = \frac{\pi^3}{8}.$$

*Solution.* We consider the following contour.



Define

$$f(z) = \frac{(\log z)^2}{z^2 + 1}$$

and in our contour, say  $C$ , we let  $0 < \epsilon < 1 < R$ .  $\log z$  denotes the branch of the logarithm function defined on  $\{z \in \mathbb{C} : -\pi/2 < \arg z < 3\pi/2\}$ . Hence, it is clear that

$$\int_C = \int_{C_R} + \int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_{\epsilon}^R.$$

By the residue theorem,

$$\int_C \frac{(\log z)^2}{z^2 + 1} dz = \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}.$$

Now, let us focus on  $C_R$ . We use the estimation lemma to help us.

$$\left| \int_{C_R} \right| \leq \pi R \cdot \frac{(\log R + i\theta)^2}{R^2 - 1}$$

which tends to 0 as  $R$  tends to infinity. In a similar fashion, one can show that

$$\lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} = 0.$$

As such,

$$\begin{aligned} \int_C \frac{(\log z)^2}{z^2 + 1} dz &= \int_{-R}^{-\epsilon} \frac{(\log z)^2}{z^2 + 1} dz + \int_{\epsilon}^R \frac{(\log z)^2}{z^2 + 1} dz \\ &= \int_{\epsilon}^R \frac{(\log(-z))^2}{z^2 + 1} dz + \int_{\epsilon}^R \frac{(\log z)^2}{z^2 + 1} dz \\ &= \int_{\epsilon}^R \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2 + 1} dz \end{aligned}$$

Now we set  $R$  to tend to infinity and  $\epsilon$  to tend to 0. Also, we computed the value of the integral over  $C$  earlier so putting everything together,

$$\begin{aligned} -\frac{\pi^3}{4} &= \int_0^\infty \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2 + 1} dz \\ &= -\pi^2 \int_0^\infty \frac{1}{z^2 + 1} dz + 2i\pi \int_0^\infty \frac{\log z}{z^2 + 1} dz + 2 \int_0^\infty \frac{(\log z)^2}{z^2 + 1} dz \\ &= -\frac{\pi^3}{2} + 2i\pi \int_0^\infty \frac{\log z}{z^2 + 1} dz + 2 \int_0^\infty \frac{(\log z)^2}{z^2 + 1} dz \end{aligned}$$

Lastly, we will show that

$$\int_0^\infty \frac{\log x}{x^2 + 1} dx = 0.$$

Using the substitution  $u = 1/x$ ,

$$\int_0^\infty \frac{\log x}{x^2 + 1} dx = \int_0^\infty \frac{-\log u}{(1/u)^2 + 1} \cdot \left(-\frac{1}{u}\right)^2 du = -\int_0^\infty \frac{\log u}{u^2 + 1} du$$

and the result follows. □

We have the following beautiful corollary:

**Corollary 6.1.** Let

$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx.$$

Then for all  $n \geq 1$ ,  $I_{2n}$  satisfies the recurrence relation

$$I_{2n} = \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}} - \frac{1}{2} \sum_{k=1}^n \binom{2n}{2k} (-1)^k \pi^{2k} I_{2n-2k}.$$

It is not surprising that we only discuss the integrals  $I_{2n}$  instead of  $I_{2n+1}$  because

$$\int_0^\infty \frac{(\log x)^{2n+1}}{x^2 + 1} dx = 0$$

for all  $n \geq 0$  by performing the substitution  $u = 1/x$ .

The above formula is also equivalent to the following by using the Dirichlet beta function:

**Definition 6.4 (Dirichlet beta function).** Define the Dirichlet beta function to be

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

**Corollary 6.2.** Let

$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx.$$

Then for all  $n \geq 1$ ,  $I_{2n} = 2(2n)! \beta(2n+1)$ .

*Proof.*

$$\begin{aligned} \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx &= \int_0^1 \frac{(\log u)^{2n}}{u^2 + 1} du \quad \text{using } u = \frac{1}{x} \\ \int_0^\infty \frac{(\log x)^{2n}}{x^2 + 1} dx &= 2 \int_0^1 \frac{(\log x)^{2n}}{x^2 + 1} dx \\ &= 2 \int_0^1 (-1)^k \sum_{k=0}^{\infty} (\log x)^{2n} x^{2k} dx \quad \text{using integration by parts} \\ &= 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}} \end{aligned}$$

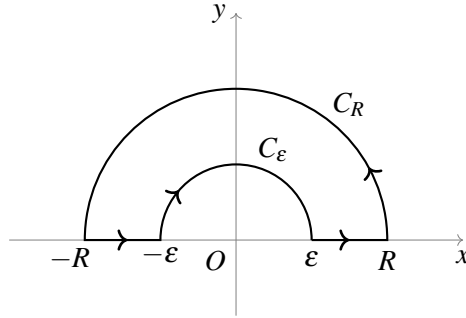
and the result follows. □

**Example 6.13 (Dinh's 70 problems).** Show that

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)^2} dx = \frac{\pi(1-\alpha)}{2 \cos\left(\frac{\pi\alpha}{2}\right)}$$

for  $-1 < \alpha < 3$ ,  $\alpha \neq 1$ . What happens if  $\alpha = 1$ ?

*Solution.* We consider the following contour.



Here,  $0 < \epsilon < R$  and  $C_R$  and  $C_\epsilon$  denote the upper-half of the semicircle of radius  $R$  and  $\epsilon$  respectively. So,

$$\int_C f(z) dz = \int_{C_R} + \int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_{\epsilon}^R.$$

By the residue theorem, it is clear that

$$\int_C f(z) dz = \frac{i\pi e^{ia\pi/2}}{2}.$$

It is clear that

$$\lim_{R \rightarrow \infty} \int_{C_R} = 0 \text{ and } \lim_{\epsilon \rightarrow 0} \int_{C_\epsilon} = 0.$$

So,

$$\int_{-R}^{-\epsilon} + \int_{\epsilon}^R = \int_C f(z) dz$$

and the result follows.  $\square$

**Example 6.14** (MA5217 AY24/25 Sem 1 Homework 1). Compute the following integrals using the residue formula:

$$\int_{-\infty}^{\infty} \frac{x-4}{(x^2-4x+5)(x^2+4)} dx \quad \text{and} \quad \int_0^{\infty} \frac{x^2}{(x^2+4)^2} dx$$

*Solution.* We deal with the first integral. Note that  $z = 2+i, z = 2-i, z = 2i, z = -2i$  are simple poles of the integral. Let  $C = C_1 + C_2$  be the upper half of the semicircle of radius  $R$  centred at the origin on the complex plane, where  $C_1$  is the diameter and  $C_2$  is the arc.

So,

$$C_1 = \{z = x + iy \in \mathbb{C} : -R \leq x \leq R\}$$

$$C_2 = \left\{z = x + iy \in \mathbb{C} : z = Re^{i\theta}, 0 \leq \theta \leq \pi\right\}$$

Let  $f(z)$  denote the integrand. We are only interested in the poles interior and on the boundary of  $C$ . By the residue theorem,

$$\int_C f(z) dz = 2\pi i \sum \text{Res}(f(z), z = z_k) = 2\pi i \left(\frac{2i}{13}\right) = -\frac{4\pi}{13}$$

Hence,

$$\int_{C_1} f(z) dz = \int_{-R}^R \frac{x-4}{(x^2-4x+5)(x^2+4)} dx.$$



Letting  $R \rightarrow \infty$ , we see that we obtain the original integral. Also,

$$\begin{aligned} \left| \int_{C_2} f(z) dz \right| &= \left| \int_0^\pi \frac{Re^{i\theta} \cdot iRe^{i\theta}}{(R^2 e^{2i\theta} - 4Re^{i\theta} + 5)(R^2 e^{2i\theta} + 4)} d\theta \right| \\ &= \left| \int_0^\pi \frac{R^2}{(R^2 e^{2i\theta} - 4Re^{i\theta} + 5)(R^2 e^{2i\theta} + 4)} d\theta \right| \end{aligned}$$

which is equal to 0 by the triangle inequality. Hence, the answer is  $-4\pi/13$ .

For the second integral, we note that the function is even. Letting  $g$  denote the integrand, we have

$$\int_0^\infty g(z) dz = \frac{1}{2} \int_{-\infty}^\infty g(z) dz.$$

We consider the same contour as the previous part, acknowledging that  $z = \pm 2i$  are double poles of  $g$ . So, it follows that the sum of residues is  $-\pi/8$ , and by some tedious computation, the integral evaluates to  $\pi/8$ .

To compute the residue of the double pole  $z = 2i$ , we use the formula

$$\lim_{z \rightarrow 2i} \frac{d}{dz} ((z - 2i)^2 g(z))$$

which is quite easy. □

**Example 6.15** (Dinh's 70 problems). Evaluate

$$\int_{-\infty}^\infty \frac{x \sin x}{(1+x^2)^2} dx.$$

*Solution.* Let  $f(z) = \frac{ze^{iz}}{(1+z^2)^2}$ . Define  $C_1$  to be the upper half of the semicircle of radius  $R$  centred at the origin and  $C_2$  to be the real axis bounded by  $\pm R$ . So,  $C_1$  can be parametrised using  $z = Re^{it}$  for  $t \in [0, \pi]$ , whereas  $C_2$  can be parametrised using  $z = t$  for  $t \in [-R, R]$ . Let  $C = C_1 \cup C_2$ . By the residue theorem,

$$\int_C f(z) dz = 2\pi i \operatorname{Res}(f(z), i).$$

Note that

$$\operatorname{Res}(f(z), i) = \lim_{z \rightarrow i} \frac{d}{dz} \left( \frac{ze^{iz}}{(z+i)^2} \right) = \frac{1}{4e}.$$

Hence,

$$\int_C f(z) dz = \frac{i\pi}{2e}.$$

Now,

$$\lim_{R \rightarrow \infty} \left| \int_{C_1} f(z) dz \right| = \lim_{R \rightarrow \infty} \left| R^2 \int_0^\pi \frac{1}{(1+R^2 e^{2i\theta})^2} d\theta \right| = 0.$$

Lastly, we work with  $C_2$ . So, we have

$$\lim_{R \rightarrow \infty} \int_{C_2} f(z) dz = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{t \sin t}{(1+t^2)^2} dt = \int_{-\infty}^\infty \frac{x \sin x}{(1+x^2)^2} dx.$$

It follows that the answer is  $\pi/2e$ . □

**Example 6.16** (Dinh's 70 problems). Show that for any  $0 < a < 1$ ,

$$\int_0^\infty \frac{x^a}{x(1+x)} dx = \frac{\pi}{\sin(a\pi)}.$$

*Solution.* Let  $t = x/(1+x)$ , so

$$x = \frac{t}{1-t} \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{(1-t)^2}.$$

The integral becomes

$$\begin{aligned} \int_0^1 t^{a-1} (1-t)^{-a} dt &= B(a, 1-a) \quad \text{by definition of beta function} \\ &= \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)} \quad \text{by relationship with gamma function} \\ &= \Gamma(a)\Gamma(1-a) \end{aligned}$$

and the result follows by Euler's reflection formula. □

## 7. Further Properties of Holomorphic Functions

### 7.1. Properties of Holomorphic and Harmonic Functions

**Definition 7.1** (extended complex plane). Define

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\} \quad \text{to be the extended complex plane.}$$

**Theorem 7.1** (Cauchy's estimate). Let  $f \in H(\Omega)$  and let  $\overline{D}(z_0, r) \subseteq \Omega$ . Then, for all  $n = 0, 1, 2, \dots$ ,

$$|a_n| \leq r^{-n} \sup_{|z-z_0|=r} |f(z)|.$$

**Example 7.1** (Dinh's 70 problems). Suppose  $f(z)$  is an odd function and holomorphic in  $\mathbb{C} \setminus \{0\}$  and satisfies

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^2} \quad \text{for all } z \neq 0.$$

Prove that

$$f(z) = \frac{a_{-1}}{z} + a_1 z \quad \text{for all } z \in \mathbb{C} \setminus \{0\} \text{ where } a_{-1}, a_1 \in \mathbb{C}.$$

*Solution.* Since  $f$  is holomorphic in  $\mathbb{C} \setminus \{0\}$ , its Laurent series representation about  $z = 0$  is

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

$f$  is odd implies  $f(-z) = -f(z)$ , so

$$f(z) = \dots + \frac{a_{-3}}{z^3} + \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \dots$$

Note that for  $|z| \leq 1$ , we have  $|z^2 f(z)| \leq |z|^4 + 1 \leq 2$  and

$$z^2 f(z) = \dots + \frac{a_{-3}}{z} + a_{-1} z + a_1 z^3 + a_3 z^5 + \dots$$

so it forces  $a_{-3}, a_{-5}, \dots, a_5, a_7, \dots = 0$ . The result follows.  $\square$

Here is an alternative solution.

*Solution.* Again, write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

By Cauchy's estimate (Theorem 7.1), if  $|f(z)| \leq M$ , we have

$$|f^{(k)}(a)| \leq \frac{k! M}{R^k}.$$

So, for  $|z| \leq R$ , we have

$$|a_k| \leq \frac{1}{R^k} \left( \frac{1}{R^2} + R^2 \right).$$

For  $k \geq 3$ ,  $\lim_{r \rightarrow \infty} |a_k| = 0$  and for  $k \leq -3$ ,  $\lim_{r \rightarrow 0} |a_k| = 0$ . So,  $a_k = 0$  for all  $|k| \geq 3$ . Hence,

$$f(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2.$$

Using the fact that  $f$  is odd, the result follows.  $\square$

**Example 7.2.** Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be holomorphic in  $D(0, 1)$  and assume that the integral

$$A := \iint_{D(0,1)} |f'(z)|^2 dx dy < \infty.$$

(a) Express  $A$  in terms of the coefficients  $a_n$ .

(b) Prove that

$$|f(z) - f(0)| \leq \sqrt{\frac{A}{\pi} \ln \left( \frac{1}{1 - |z|^2} \right)}$$

for all  $z \in D(0, 1)$ .

*Solution.*

(a) Note that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

We shall parametrise  $z$  using polar coordinates. Let  $z = r e^{i\theta}$ . As such,

$$\begin{aligned} \iint_{D(0,1)} |f'(z)|^2 dx dy &= \int_0^1 \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 r dr d\theta \\ &= \int_0^1 \int_0^{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m n a_m a_n r^{m+n-1} e^{i(m+n-2)\theta} dr d\theta \\ &= 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} dr \\ &= \pi \sum_{n=1}^{\infty} n |a_n|^2 \end{aligned}$$

(b) Note that  $f(0) = 0$  and the RHS can be written as

$$\sqrt{\ln \left( \frac{1}{1 - |z|^2} \right) \sum_{n=1}^{\infty} n |a_n|^2}.$$

Starting with the LHS,

$$|f(z)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} (\sqrt{n} a_n) \left( \frac{z^n}{\sqrt{n}} \right) \right| \leq \sqrt{\left( \sum_{n=1}^{\infty} n |a_n|^2 \right) \left( \sum_{n=1}^{\infty} \frac{|z|^{2n}}{n} \right)}$$

where we applied the Cauchy-Schwarz inequality at the end. The result follows.  $\square$

**Theorem 7.2 (identity theorem).** If two holomorphic functions  $f$  and  $g$  coincide on some set  $E \subseteq D$  containing at least one limit point in  $D$ , then  $f(z) = g(z)$  everywhere in  $D$ .

**Example 7.3.** Does there exist an entire function with the property that for  $n \in \mathbb{N}$ ,

$$f\left(\frac{1}{n}\right) = \frac{n^4}{1 + n^4}?$$

*Solution.* Replacing  $n$  with  $1/z$ , we consider the function

$$g(z) = \frac{1}{z^4 + 1}.$$

Note that the roots of the equation  $z^4 + 1 = 0$  can be found as follows. As  $z^4 = -1 = e^{i\pi+2k\pi i}$ , then

$$z = \exp\left(i\pi \cdot \frac{2k+1}{4}\right),$$

where  $k = 0, 1, 2, 3$ . We denote the roots by  $p_n$ , where  $0 \leq n \leq 3$ . Obviously,  $g(z)$  is holomorphic outside the 4 points  $p_n$ . By our hypothesis,  $f(z) = g(z)$  for  $z = 1, 1/2, 1/3, \dots$  and both  $f$  and  $g$  are defined on  $\Omega = \mathbb{C} \setminus \{p_0, p_1, p_2, p_3\}$ . The sequence  $1, 1/2, 1/3, \dots$  converges to 0 which is inside  $\Omega$ , so this sequence is not discrete in  $\Omega$ . We conclude that  $f = g$  in  $\Omega$  by the identity theorem.

On the other hand, the function  $f$  is entire and bounded near  $p_n$  but  $g$  is not bounded near these points. We have obtained a contradiction so such a function  $f$  does not exist.  $\square$

**Example 7.4.** Do there exist functions  $f$  and  $g$  that are holomorphic at  $z = 0$  and that satisfy

- (a)  $f(1/n) = f(-1/n) = 1/n^2$ , where  $n \in \mathbb{N}$ ;
- (b)  $g(1/n) = g(-1/n) = 1/n^3$ , where  $n \in \mathbb{N}$ ?

*Solution.*

- (a) Yes,  $f(z) = z^2$ .
- (b) We prove that such a function  $g$  does not exist in a neighbourhood of 0. Suppose on the contrary that  $g$  exists. Define  $h(z) = z^3$  and  $l(z) = -z^3$ . We have  $g(z) = h(z)$  on a non-discrete sequence  $z = 1, 1/2, 1/3, \dots$  which converges to 0, and 0 is in the domain of  $g$ . By the identity theorem,  $g(z) = h(z)$ . In a similar fashion, by considering the sequence  $z = -1, -1/2, -1/3, \dots$ , we obtain  $g(z) = l(z)$ . Hence,  $h(z) = l(z)$ , implying that  $z^3 = -z^3$ , so  $z^3 = 0$ . However, this is a contradiction.  $\square$

**Example 7.5.** Show that there is no holomorphic function  $f$  in  $\mathbb{C}$  such that

$$f\left(\frac{1}{n}\right) = \frac{ne^{-2/n}}{n+1} \text{ for all } n \in \mathbb{N}.$$

*Solution.* Suppose on the contrary that such a function exists. Consider

$$g(z) = \frac{e^{-2z}}{z+1}.$$

This function is defined for all  $z \in \mathbb{C}$  except at  $z = -1$ . By the hypothesis, this function is equal to  $f$  on the sequence  $1/n$  which is not discrete on  $\mathbb{C} \setminus \{-1\}$  and so,  $f = g$  on  $\mathbb{C} \setminus \{-1\}$ . However, this is a contradiction.  $\square$

**Example 7.6 (MA5217 AY24/25 Sem 1 Homework 1).** Show that the function  $h(z) = \sin(\sin z) + \sin|z|^2$  is not holomorphic in any domain of  $\mathbb{C}$ .

*Solution.* Note that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

We let  $z = x + iy, x, y \in \mathbb{R}$  and note that  $|z|^2 = x^2 + y^2$ . Hence,

$$h(z) = \sin\left(\frac{e^{iz}}{2i}\right) \cos\left(\frac{e^{-iz}}{2i}\right) - \cos\left(\frac{e^{iz}}{2i}\right) \sin\left(\frac{e^{-iz}}{2i}\right) + \sin(|z|^2)$$

By the Looman-Menchoff theorem, it suffices to prove that  $h$  does not satisfy the Cauchy-Riemann equations, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

The computation is tedious so we skip the details.  $\square$

**Example 7.7** (MA5217 AY24/25 Sem 1 Homework 1). Find all holomorphic functions  $f(z)$  in  $\mathbb{C} \setminus \{1\}$  such that

$$\operatorname{Res}(f, 1) = 1, \quad \lim_{z \rightarrow \infty} (f(z) - z) = 2, \quad \lim_{z \rightarrow 1} |z - 1|^{4/3} f(z) = 0.$$

*Solution.* We claim that

$$f(z) = z + 2 + \frac{1}{z - 1}.$$

By the second condition, we infer that

$$f(z) = z + 2 + \sum_{n=1}^{\infty} \frac{1}{(az + b)^n}.$$

By the third condition, we infer that

$$\lim_{z \rightarrow 1} (z - 1)^{4/3} \sum_{n=1}^{\infty} \frac{1}{(az + b)^n} = 0$$

which implies we have to restrict the index of the infinite sum to  $n = 1$  instead of  $n \in \mathbb{N}$ . Hence,

$$f(z) = z + 2 + \frac{1}{az + b}.$$

We see that  $1/(az + b)$  has a simple pole at  $z = -b/a$  but the first condition implies that  $z = 1$  is a pole, so  $a = -b$ . Since the value of the residue at  $z = 1$  is 1, then  $a = 1$ , so

$$f(z) = z + 2 + \frac{1}{z - 1}.$$

□

**Example 7.8.** Let  $f$  and  $g$  be entire functions and suppose that  $|f(z)| \leq |g(z)|$  for all  $z \in \mathbb{C}$ . Show that  $f(z) = cg(z)$  for some constant  $c \in \mathbb{C}$ .

*Solution.* First, we assume that  $g$  is identically equal to zero. Then, the result immediately follows. Now, we consider the case where  $g$  is not identically equal to zero. Define  $h(z) = f(z)/g(z)$  on  $\mathbb{C}$  excluding the set of zeros of  $g$ . As such,  $h$  is holomorphic outside the zeros of  $g$  and  $|h(z)| \leq 1$ . As  $h$  is bounded and entire, the result follows by Liouville's theorem. □

**Definition 7.2** (analytic function).  $f : \Omega \rightarrow \mathbb{C}$  is analytic if for any  $z_0 \in \Omega$ , we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where  $a_n \in \mathbb{C}$  for all  $n \in \mathbb{Z}_{\geq 0}$  and the series is convergent to  $f(z)$  for  $z$  in a neighbourhood of  $z_0$ .

**Example 7.9** (Dinh's 70 problems). Suppose  $f$  is entire and  $f(z)$  is real iff  $z$  is real. Prove that  $f$  has at most one zero.

*Solution.* Suppose  $f(z) = 0$ , then it implies that  $z$  is real. Suppose on the contrary that  $f$  has a zero  $x_0 \in \mathbb{R}$  with a multiplicity  $m \geq 2$ . Then, we can write  $f(z)$  as the following power series:

$$f(z) = (z - x_0)^m (a_0 + a_1(z - x_0) + a_2(z - x_0)^2 + \dots)$$

Here,  $a_0 \neq 0$ . Note that

$$a_0 = \lim_{z \rightarrow x_0} \frac{f(z)}{(z - x_0)^m}$$

so for any  $z \in \mathbb{R} \setminus \{x_0\}$ , we have  $\frac{f(z)}{(z-x_0)^m} \in \mathbb{R}$ . Hence,  $a_0 \in \mathbb{R}$ . Now, write  $z = x_0 + \varepsilon e^{i\theta}$ , so we are considering the general case when  $z \in \mathbb{C}$ . It is clear that

$$f(z) = \varepsilon^m e^{mi\theta} (a_0 + a_1 \varepsilon e^{i\theta} + a_2 \varepsilon^2 e^{2i\theta} + \dots).$$

Define  $g(\theta) = \text{Im}(e^{mi\theta}(a_0 + \varepsilon u(\theta, \varepsilon) + i\varepsilon v(\theta, \varepsilon)))$ , where  $u, v$  are real and continuous functions and  $\varepsilon$  is sufficiently small. Note that  $g(\pi/2m)g(3\pi/2m) < 0$  so by the intermediate value theorem, there exists  $\theta' \in (\pi/2m, 3\pi/2m)$  such that  $g(\theta_0) = 0$ . So,  $f(x_0 + \varepsilon e^{i\theta_0}) \in \mathbb{R}$ . But because  $m \geq 2$ , it implies that  $x_0 + \varepsilon e^{i\theta_0} \notin \mathbb{R}$ , so we reached a contradiction. The result follows.  $\square$

The next theorem summarises a list of important properties regarding holomorphic functions.

**Theorem 7.3.** Let  $\Omega$  be an open and simply-connected domain in  $\mathbb{C}$  and let  $f \in H(\Omega)$ . Then,

- $f \in C^\infty(\Omega)$  and  $f$  satisfies the Cauchy-Riemann equations on  $\Omega$ .
- **Cauchy-Goursat theorem:** If  $\gamma$  is a piecewise differentiable simple closed curve in  $\Omega$ , then

$$\int_{\gamma} f(z) dz = 0.$$

- **Cauchy's integral formula:** If  $\gamma$  is an anticlockwise oriented and piecewise differentiable simple closed curve in  $\Omega$ , then for any  $a$  interior to  $\gamma$ ,

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz.$$

Moreover, Cauchy's differentiation formula applies here.

- $f$  is analytic, i.e. for any  $z_0 \in \Omega$ , one can write for  $z \in D(z_0, r)$  with  $\overline{D(z_0, r)} \subseteq \Omega$ ,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad n = 0, 1, 2, \dots$$

**Example 7.10.** Let  $f$  be a holomorphic function in the unit disc  $\mathbb{D}$  such that  $|f(z)| < 1$  for  $z \in \mathbb{D}$ . Show that  $|f''(0)| \leq 2$ . Give an example of such a map with  $f''(0) = 2$ .

*Solution.* We use Cauchy's Differentiation Formula. Note that

$$f''(0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{z^3} dz.$$

Here, we let  $C$  be such that  $|z| = r$ , where  $r < 1$  (i.e.  $C$  contains all points interior to the circle of radius 1 centred at the origin). Using the parametrisation  $z = re^{i\theta}$  for  $0 \leq \theta \leq 2\pi$ , we see that

$$|f''(0)| \leq \frac{1}{\pi} \left| \int_0^{2\pi} \frac{f(re^{i\theta})}{r^3 e^{3i\theta}} \cdot ire^{i\theta} d\theta \right| \leq \frac{1}{\pi r^2} \left| \int_0^{2\pi} f(re^{i\theta}) d\theta \right| \leq \frac{2}{r^2}.$$

Hence, letting  $r$  tend to 1, the result follows.

For the later part of the question, we need to find a map such that  $f''(0) = 2$ . Well, consider

$$\int_{|z|=r} \frac{f(z)}{z^3} dz = 2\pi i$$

for which an obvious answer is  $f(z) = z^2$ .  $\square$

**Example 7.11 (Dinh's 70 problems).** Determine all complex holomorphic functions  $f$  defined on the unit disk which satisfy

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for  $n = 2, 3, 4, \dots$

*Solution.* Let  $g(z) = f''(z) + f(z)$ , so  $g$  is holomorphic on  $\mathbb{D}$ . We have  $g(1/n) = 0$  for all  $n = 2, 3, 4, \dots$  and since  $\lim_{n \rightarrow \infty} 1/n = 0 \in \mathbb{D}$ , it follows that  $g(z) = 0$  on  $\mathbb{D}$ . As such,  $f(z) = -f''(z)$  on  $z \in \mathbb{D}$ . One can use Maclaurin Series to deduce that  $f(z) = f(0) \cos z + f'(0) \sin z$ .  $\square$

**Theorem 7.4 (Casorati-Weierstrass theorem).** Let  $f$  have an isolated essential singularity at  $z_0$ . Then, for any  $w \in \mathbb{C}$ ,  $f(z)$  comes arbitrarily close to  $w$  in every deleted neighbourhood of  $z_0$ . That is, for any  $\delta > 0$ ,  $f(D'(z_0, \delta))$  is a dense subset of  $\mathbb{C}$ .

*Proof.* Suppose on the contrary that for some  $\delta > 0$ ,  $f(D'(z_0, \delta))$  is not dense in  $\mathbb{C}$ . Then, there exists  $w \in \mathbb{C}$  and  $\varepsilon > 0$  such that

$$D(w, \varepsilon) \cap f(D'(z_0, \delta)) = \emptyset.$$

For  $z \in D'(z_0, \delta)$ , write

$$g(z) = \frac{1}{f(z) - w}.$$

Then,  $g$  is bounded and holomorphic on  $D'(z_0, \delta)$ , so  $g$  has a removable singularity at  $z_0$ . Let  $m$  be the order of the zero of  $g$  at  $z_0$ . If  $g(z_0) \neq 0$ , set  $m = 0$ . Otherwise, write  $g(z) = (z - z_0)^m g_1(z)$ , where  $g_1$  is holomorphic and does not vanish on  $D(z_0, \delta)$ . Hence,

$$(z - z_0)^m g_1(z) = \frac{1}{f(z) - w}.$$

Thus, we can write  $f(z)$  as

$$f(z) = w + \frac{g_2(z)}{(z - z_0)^m},$$

where  $g_2(z) = 1/g_1(z)$  is a holomorphic function on  $D(z_0, \delta)$ . Thus,  $f$  has a removable singularity ( $m = 0$ ) or a pole ( $m \neq 0$ ) at  $z_0$ , which is a contradiction.  $\square$

**Definition 7.3.** A meromorphic function in  $D$  is holomorphic on all  $D$ , except on a set of isolated points which are poles. Also, they can be written in the form  $f = u/v$ , where  $u, v \in H(D)$  and  $v \neq 0$ , and they do not have a common zero.

## 7.2. The Argument Principle and Rouché's Theorem

**Theorem 7.5 (argument principle).** Let  $f \in H(\Omega)$  and  $\gamma$  be a positively oriented, piecewise differentiable, simple closed contour in  $\Omega$  such that all points interior to  $\gamma$  belong to  $\Omega$ . Suppose  $f$  has no zero on  $\gamma$ . The zeros of  $f$  inside  $\gamma$  are  $a_1, a_2, \dots, a_n$  and  $\alpha_1, \alpha_2, \dots, \alpha_n$  are their respective multiplicities. Then,

$$\sum_{j=1}^n \alpha_j = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz.$$

**Example 7.12 (Dinh's 70 problems).** Evaluate the integral

$$\int_{|z|=2} \frac{f'(z)}{f(z)} dz,$$



where  $f(z) = \frac{\sin z \cos z}{z^7 - z^5 + z^3 - z}$ , and  $|z| = 2$  is positively oriented.

*Solution.* We use the argument principle. The answer is  $2\pi i(Z - P)$ , where  $Z$  and  $P$  are to be determined. Here,  $Z$  and  $P$  refer to the respective number of zeros and poles in the circle  $|z| = 2$ . To calculate  $Z$ , set  $\sin z \cos z = 0$ , so  $z = -\pi/2, 0, \pi/2$ . Hence,  $Z = 3$ . To calculate  $P$ , set  $z^7 - z^5 + z^3 - z = 0$ , so  $z(z^4 + 1)(z + 1)(z - 1) = 0$ . The solutions to  $z^4 + 1 = 0$  are  $z = e^{i\pi/4}, e^{-i\pi/4}, e^{3i\pi/4}, e^{-3i\pi/4}$ . As such,  $P = 7$ , so the required answer is  $-8\pi i$ .  $\square$

**Theorem 7.6 (Rouché's theorem).** Let  $f, g \in H(\Omega)$  and  $\gamma$  be a piecewise differentiable simple closed curve such that all the points interior to  $\gamma$  are contained in  $\Omega$ . Assume that

$$|f(z) - g(z)| < |f(z)|$$

for all  $z \in \gamma$ . Then,  $f$  and  $g$  have the same number of zeros (counting multiplicity) inside  $\gamma$ .

**Example 7.13 (Dinh's 70 problems).** Determine the number of zeros of  $e^{z^2} - 3z^4$  in the unit disk.

*Solution.* For  $|z| = 1$  (i.e. on the boundary of the unit disk),  $|e^{z^2}| \leq e \leq 3 = 3|z^4|$  so it follows by Rouché's theorem that there are 4 zeros.  $\square$

**Example 7.14 (Dinh's 70 problems).** Let  $N_k$  be the number of roots (counting multiplicity) in the disk  $D(0, k) = \{|z| < k\}$  of the equation

$$z^6 - 5z^2 + 10 = 0.$$

For each positive integer  $k$ , determine  $N_k$ .

*Solution.*  $N_1 = 0$ ; now consider the case when  $k \geq 2$ . On  $|z| = 2$ ,  $|5z^2 - 10| \leq 5|z|^2 + 10 = 30 \leq 2^6 = |z|^6$ , so by Rouché's theorem,  $N_k = 6$  for  $k \geq 2$ .  $\square$

**Example 7.15.** Let  $r > 0$ . Prove that for  $n$  sufficiently large, the polynomial

$$1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}$$

has no root in  $D(0, r)$ .

*Solution.* Fix an  $r > 0$ . Define  $f(z) = e^z$  and  $g_n(z)$  to be the polynomial above. For  $z$  on  $\overline{D(0, r)}$ , i.e.  $|z| = r$ ,

$$|f(z) - g_n(z)| = \left| \sum_{k \geq n+1} \frac{z^k}{k!} \right| \leq \sum_{k \geq n+1} \frac{r^k}{k!}.$$

We note that as  $n \rightarrow \infty$ , the sum on the right tends to zero. For  $n$  large enough, the last sum on the right is smaller than  $e^{-r}$ . On the other hand, by setting  $z = x + iy$ , where  $x, y \in \mathbb{C}$ , we see that  $|f(z)| = e^x \geq e^{-|z|} = e^{-r}$ . Therefore, for  $z$  on  $\overline{D(0, r)}$ , we have

$$|f(z) - g_n(z)| < |f(z)|.$$

By Rouché's theorem,  $f$  and  $g_n$  have the same number of zeros inside  $D(0, r)$ . However,  $f$  vanishes nowhere so we can conclude that  $g_n$  does not vanish in  $D(0, r)$ .  $\square$

**Example 7.16.** Find the number of zeros (counting multiplicity) of the function  $z^5 + 6z^3 + 11$  in the annulus  $2 < |z| < 3$ .

*Solution.* Let  $f(z) = z^5 + 6z^3 + 11$ . On the circle  $|z| = 3$ ,  $|f(z) - z^5| = |6z^3 + 11| \leq 6|z^3| + 11 = 173 < 243 = |z^5|$  so by Rouché's theorem, the number of zeros of  $f(z)$  in the region  $0 < |z| < 3$  is equal to that of  $z^5$ , which is 5.

On the circle  $|z| = 2$ , we have  $|f(z) - 6z^3| = |z^5 + 11| \leq 2^5 + 11 = 43 < 48 = |6z^3|$  so the number of zeros of  $f(z)$  in the region  $0 < |z| < 2$  is equal to that of  $6z^3$ , which is 3. Therefore,  $f$  has exactly  $5 - 3 = 2$  zeros in the annulus  $2 < |z| < 3$ .  $\square$

**Example 7.17 (Dinh's 70 problems).**

- (a) For each integer  $n \geq 1$ , find the number of zeros (counting multiplicity) in the disk  $D(0, n)$  of the polynomial  $z^7 + 5z^3 - z - 2$ .  
 (b) Prove that the function  $u(x, y) = \sinh x \sin y$  is harmonic and find its harmonic conjugates.

*Solution.*

- (a) Let  $N_n$  be the number of zeros. We first show that  $N_1 = 3$ . Note that on  $|z| = 1$ , we have

$$|z^7 + 5z^3 - z - 2 - 5z^3| = |z^7 - z - 2| \leq |z|^7 + |z| + 2 = 4 \leq 5 = 5|z|^3.$$

By Rouché's theorem,  $N_1 = 3$ .

For  $n \geq 2$ , we show that  $N_n = 7$ . Note that on the boundary  $|z| = n$ , we have

$$|z^7 + 5z^3 - z - 2 - z^7| \leq 5|z|^3 + |z| + 2 = 5n^3 + n + 2 \leq n^7 = |z|^7.$$

The result follows by Rouché's theorem.

- (b) Trivial. Show that  $u$  satisfies Laplace's equation, then to find its harmonic conjugates, use the Cauchy-Riemann equations.  $\square$

**Example 7.18.** Let  $a_1, \dots, a_n \in D(0, 1)$  and

$$f(z) = \prod_{k=1}^n \frac{a_k - z}{1 - \overline{a_k}z}.$$

Prove that for each  $b \in D(0, 1)$ ,  $f(z) = b$  has exactly  $n$  roots in  $D(0, 1)$ , counting multiplicity.

*Solution.* For  $|z| = 1$ , we have  $\bar{z} = 1/z$ . Hence,

$$\left| \frac{a_k - z}{1 - \overline{a_k}z} \right| = \frac{|a_k - z|}{|z| |1/z - \overline{a_k}|} = \frac{|a_k - z|}{|\bar{z} - \overline{a_k}|} = 1.$$

We infer that for  $|z| = 1$ ,  $|f(z)| = 1$ . We deduce that for  $b \in D(0, 1)$ ,

$$|(f(z) - b) - f(z)| = |b| < 1 = |f(z)|.$$

By Rouché's theorem,  $f(z) - b$  and  $f(z)$  have the same number of zeros in  $D(0, 1)$ . The roots of  $f(z) = 0$  are  $a_1, \dots, a_n$  (so there are  $n$  roots), as such, the result follows.  $\square$

**Example 7.19 (Dinh's 70 problems).** Show that if the integer  $n$  is sufficiently large, the equation

$$z = 1 + \left(\frac{z}{2}\right)^n$$

has exactly one solution in the disk  $|z| < 2$ .

*Solution.* Let

$$f_n(z) = z - 1 - \left(\frac{z}{2}\right)^n \quad \text{and} \quad f(z) = z - 1.$$

For arbitrary  $\varepsilon > 0$ , consider the boundary of  $C(0, 2 - \varepsilon)$ , we have

$$|f_n(z) - f(z)| = \left|\frac{z}{2}\right|^n = \left(\frac{2 - \varepsilon}{2}\right)^n = \left(1 - \frac{\varepsilon}{2}\right)^n.$$

Also,

$$|f(z)| = |z - 1| \geq |z| - 1 = 1 - \varepsilon \quad \text{by the reverse triangle inequality.}$$

By Rouché's theorem, we need  $|f_n - f| \leq |f|$ , i.e.

$$\left(1 - \frac{\varepsilon}{2}\right)^n \leq 1 - \varepsilon.$$

So, we choose

$$n \geq \frac{\ln(1 - \varepsilon)}{\ln(1 - \varepsilon/2)} \quad \text{where } n \in \mathbb{N} \text{ and } 0 < \varepsilon \leq \frac{1}{2}.$$

The number of zeros of  $f_n$  in  $D(0, 2 - \varepsilon)$  is 1. Letting  $\varepsilon \rightarrow 0$ , the result follows.  $\square$

**Theorem 7.7 (Hurwitz's theorem).** Let  $f_n : \Omega \rightarrow \mathbb{C}$ , where  $n \in \mathbb{N}$ , be a sequence of holomorphic functions that converges locally uniformly to a function  $f : \Omega \rightarrow \mathbb{C}$ . Let  $\gamma$  be a piecewise differentiable, simple closed contour in  $\Omega$  such that all points interior to  $\gamma$  are contained in  $\Omega$ . Assume that  $f$  has no zero on  $\gamma$ . Then,

there exists  $N \in \mathbb{N}$  such that for all  $n > N$   $f_n$  and  $f$  have the same number of zeros inside  $\gamma$ .

**Example 7.20.** Assume that  $f$  is holomorphic in a neighbourhood of  $\overline{D(0, 1)}$  and that  $f'(z)$  has no zero on  $\partial D(0, 1)$ . Prove that for  $n$  sufficiently large,

$$F_n(z) = f\left(z + \frac{1}{n}\right) - f(z)$$

has the same number of zeros in  $D(0, 1)$  as  $f'(z)$ .

*Solution.* We consider the function  $g_n(z) = nF_n(z)$ . Note that

$$g_n(z) = n \left[ f\left(z + \frac{1}{n}\right) - f(z) \right],$$

so

$$\lim_{n \rightarrow \infty} g_n(z) = \lim_{n \rightarrow \infty} \frac{f(z + 1/n) - f(z)}{1/n} = f'(z).$$

By the Fundamental Theorem of Calculus,

$$g_n(z) = \frac{f(z + 1/n) - f(z)}{1/n} = \int_0^1 f'\left(z + \frac{t}{n}\right) dt.$$

Hence,  $g_n$  converges locally and uniformly in a neighbourhood to  $f'$ . By Hurwitz's Theorem,  $g_n$  has the same number of zeros as  $f'$  in  $D(0, 1)$  when  $n$  is sufficiently large. Therefore,  $F_n$  satisfies the same property.  $\square$

### 7.3. Open Mapping Theorem and the Maximum Modulus Principle

**Theorem 7.8 (open mapping theorem).** Let  $f$  be a non-constant holomorphic function on an open connected set  $\Omega$ . Then,  $f$  is open, i.e. for any open set  $U \subseteq \Omega$ , we have  $f(U)$  is open.

**Theorem 7.9 (maximum modulus principle).** Suppose  $f$  is a non-constant holomorphic function defined on a domain  $\Omega$ . Then,  $|f|$  does not attain the maximum value in  $\Omega$ .

**Example 7.21.** Suppose  $f$  is holomorphic on a neighbourhood of the unit disc  $\overline{D(0, 1)}$  and satisfies  $f(0) = 3 + 4i$ ,  $|f(z)| \leq 5$  if  $|z| = 1$ . Find  $f'(0)$ .

*Solution.* We prove that  $f$  is constant. Suppose on the contrary that  $f$  is not constant, then by the maximum modulus principle,

$$5 = f(0) < \max_{|z|=1} |f(z)| \leq 5.$$

This is a contradiction, so  $f'(0) = 0$ . □

**Example 7.22.** Let  $f$  be a continuous function on  $\overline{A} = \{1 \leq |z| \leq 4\}$  and holomorphic on  $A = \{1 < |z| < 4\}$ . Assume that

$$\max_{|z|=1} |f(z)| = 5 \text{ and } \max_{|z|=4} |f(z)| = 20.$$

- (i) Show that  $|f(2)| \leq 10$ .
- (ii) Find all functions  $f$  such that  $f(2) = 10$ .

*Solution.* Let us discuss the solutions.

- (i) Define  $g(z) = f(z)/z$ . Then,

$$\max_{|z|=1} |g(z)| = \max_{|z|=4} |g(z)| = 5.$$

By the maximum modulus principle,  $|g(z)| \leq 5$  for  $z \in A$ . Setting  $z = 2$ , we have  $f(2) \leq 10$ .

- (ii)  $g(2) = 5$ . By the maximum modulus principle,  $g$  is a constant, so  $g(z) = 5$ . Hence,  $f(z) = 5z$ . □

**Corollary 7.1.** Let  $\Omega$  be a domain in  $\mathbb{C}$  and  $f$  be holomorphic in  $\Omega$ .

- (i) If  $|f|$  assumes a local maximum at some point in  $\Omega$ , then  $f$  is constant in  $\Omega$ .
- (ii) If  $\Omega$  is bounded and  $f$  is continuous up to the boundary  $b\Omega$ , then,

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in b\Omega} |f(z)|.$$

We obtain the next corollary on the minimum modulus principle by switching to the reciprocal  $1/f(z)$ .

**Corollary 7.2 (minimum modulus principle).** Let  $\Omega$  be a domain in  $\mathbb{C}$  and  $f$  be holomorphic but never zero in  $\Omega$ .

- If  $|f|$  assumes a local minimum at some point in  $\Omega$ , then  $f$  is constant on  $\Omega$ .
- If  $\Omega$  is bounded and  $f$  is continuous up to the boundary of  $\Omega$  and never vanishes in  $\overline{\Omega}$ , then

$$\min_{z \in \overline{\Omega}} |f(z)| = \min_{z \in b\Omega} |f(z)|.$$

**Example 7.23.** Suppose  $f$  is holomorphic on a neighbourhood of  $\overline{D(0, 1)}$ ,  $f(0) = i$  and  $|f(z)| > 1$  whenever  $|z| = 1$ . Prove that  $f$  has a zero in  $D(0, 1)$ .

*Solution.* Suppose on the contrary that  $f$  does not have a zero in  $D(0,1)$ . Then,  $g(z) = 1/f(z)$  would be holomorphic in a neighbourhood  $\overline{D(0,1)}$ . Moreover, we have  $|g(0)| = 1$  and  $|g(z)| < 1$  when  $|z| = 1$ . This contradicts the maximum modulus principle.  $\square$

**Theorem 7.10 (maximum and minimum principle for harmonic functions).** Let  $\Omega$  be a domain in  $\mathbb{C}$  and  $u$  be a real-valued harmonic in  $\Omega$ .

- (i) If  $u$  has either a local maximum or a local minimum at some point of  $\Omega$ , then  $u$  is a constant on  $\Omega$ .
- (ii) If  $\Omega$  is bounded and  $f$  is continuous up to the boundary of  $\Omega$ , then

$$\max_{z \in \overline{\Omega}} u(z) = \max_{z \in \partial\Omega} u(z) \text{ and } \min_{z \in \overline{\Omega}} u(z) = \min_{z \in \partial\Omega} u(z).$$

**Example 7.24.** Find the maximal value of  $\operatorname{Re}(z^3)$  for  $z \in [0,1] \times [0,1]$ .

*Solution.* Note that  $\operatorname{Re}(z^3)$  is harmonic as it is the real part of a holomorphic function. Hence, it achieves its maximal value on the boundary of the unit square. Throughout this problem,  $a \in \mathbb{R}$  and  $a \in [0,1]$ .

- **Case 1 (bottom edge of square):**  $z = a$ . Then,  $\operatorname{Re}(z^3) = a^3$ , whose maximum is 1.
- **Case 2 (top edge of square):**  $z = a + i$ . Then,  $\operatorname{Re}(z^3) = a^3 - 3a$ . The maximum here is 0.
- **Case 3 (left edge of square):**  $z = ai$ . Then,  $\operatorname{Re}(z^3) = 0$ .
- **Case 4 (right edge of square):**  $z = 1 + ai$ . Then,  $\operatorname{Re}(z^3) = 1 - 3a^2$ . The maximum here is 1.

Overall, the maximum value is 1 which is achieved when  $z = 1$ .  $\square$

**Example 7.25 (Dinh's 70 problems).** Let  $a \in \mathbb{C}$ ,  $|a| \leq 1$ , and consider the polynomial

$$P(z) = \frac{a}{2} + (1 - |a|^2)z - \frac{\bar{a}}{2}z^2.$$

Prove that  $|P(z)| \leq 1$  whenever  $|z| \leq 1$ .

*Solution.* Note that  $z\bar{z} = 1$  on  $|z| = 1$ . Consider

$$\frac{P(z)}{z} = \frac{a}{2z} - \frac{\bar{a}z}{2} + 1 - |a|^2.$$

We have

$$\frac{a}{2z} - \frac{\bar{a}z}{2} = \frac{1}{2} \left( \frac{a}{z} - \overline{a/z} \right).$$

Let  $\lambda = a/z \in \mathbb{C}$ . Then,  $\lambda - \bar{\lambda} = 2i\operatorname{Im}(\lambda)$ , so

$$\frac{a}{2z} - \frac{\bar{a}z}{2} = i\operatorname{Im}\left(\frac{a}{z}\right) = i\operatorname{Im}(a\bar{z}).$$

Hence,

$$\begin{aligned} \left| \frac{P(z)}{z} \right| &\leq \left| i\operatorname{Im}(a\bar{z}) + 1 - |a|^2 \right| \\ |P(z)| &\leq \left| i\operatorname{Im}(a\bar{z}) + 1 - |a|^2 \right| \text{ since } |z| \leq 1 \\ |P(z)|^2 &\leq |\operatorname{Im}(a\bar{z})|^2 + (1 - |a|^2)^2 \end{aligned}$$

We bluntly state that  $|\operatorname{Im}(a\bar{z})|^2 \leq |a|^2$ , so  $|P(z)|^2 \leq 1 - |a|^2 + |a|^4 \leq 1$  since  $|a| \leq 1$ . By the maximum modulus principle, whenever  $|z| \leq 1$ , we have  $|P(z)| \leq 1$ .

Now, we justify that  $|\operatorname{Im}(a\bar{z})|^2 \leq |a|^2$ . Let  $z = x + iy$  and  $a = \alpha + i\beta$ , where  $x, y, \alpha, \beta \in \mathbb{R}$  such that  $x^2 + y^2 \leq 1$  and  $\alpha^2 + \beta^2 \leq 1$ . We have  $a\bar{z} = (\alpha + i\beta)(x - iy) = \alpha x - \beta y + i(\beta x - \alpha y)$  so  $\operatorname{Im}(a\bar{z}) = \beta x - \alpha y$ . It suffices to prove that  $(\beta x - \alpha y)^2 \leq \alpha^2 + \beta^2$ . In other words,  $\alpha^2(1 - y^2) + \beta^2(1 - x^2) + 2\alpha\beta xy \geq 0$ . Let  $x = \cos \theta$  and  $y = \sin \theta$  so  $\alpha^2 \sin^2 \theta + \beta^2 \cos^2 \theta + 2\alpha\beta \cos \theta \sin \theta \geq 0$ . This inequality is obviously true since  $(\alpha \sin \theta + \beta \cos \theta)^2 \geq 0$ , or equivalently  $(\alpha y + \beta x)^2 \geq 0$ .  $\square$

We then introduce the Schwarz-Pick lemma (Lemma 7.1), which is also known as the Schwarz lemma.

**Lemma 7.1 (Schwarz-Pick Lemma).** Let  $f : D(0, 1) \rightarrow \mathbb{C}$  be a holomorphic function with  $f(0) = 0$  and  $|f(z)| \leq 1$  for each  $z \in D(0, 1)$ . Then,

$$|f(z)| \leq |z| \text{ and } |f'(0)| \leq 1.$$

Moreover, if  $|f(z)| = |z|$  for some  $z \in D(0, 1) \setminus \{0\}$  or if  $|f'(0)| = 1$ , then  $f$  is a rotation of  $D(0, 1)$ ; that is, there exists a constant  $\theta \in \mathbb{R}$  such that

$$f(z) = e^{i\theta} z \text{ for all } z \in D(0, 1).$$

**Example 7.26 (Dinh's 70 problems).** Does there exist a holomorphic function  $f : \mathbb{D} \rightarrow \mathbb{D}$  with  $f(1/2) = 3/4$  and  $f'(1/2) = 2/3$ ?

*Solution.* Yes, this is simply proven using the Schwarz-Pick lemma since  $f'(1/2) \leq 7/12 < 2/3$ .  $\square$

**Example 7.27 (Dinh's 70 problems).** Let  $f$  be a holomorphic function from the unit disk  $D(0, 1)$  to itself. Assume that there is a point  $z_0 \in D(0, 1)$  such that  $f(z_0) = z_0$ . Prove that  $|f'(z_0)| \leq 1$ .

*Solution.* We use the Schwarz-Pick Lemma, which says that for  $a, b \in \mathbb{D}$ , a holomorphic function  $f : \mathbb{D} \rightarrow \mathbb{D}$  satisfies  $f(a) = b$  and  $|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$ . So, we set  $a = b = z_0$ . The result follows.  $\square$

**Example 7.28.** Is there a holomorphic function of  $D(0, 1)$  onto itself such that  $f(0) = 0$  and  $f(i/4) = i/3$ ? Justify.

*Solution.* We will show that there is no such function. Suppose on the contrary that there exist such a function. By the Schwarz Lemma, as  $|f(z)| \leq |z|$  for  $z \in \mathbb{D}$ , we have  $|f(i/4)| \leq |i/4| = 1/4$ , which is a contradiction.  $\square$

#### 7.4. Winding Numbers

**Definition 7.4 (winding number).** Let  $\gamma : [a, b] \rightarrow \mathbb{C} \setminus \{z_0\}$  be a closed curve that does not pass through  $z_0$ . Given an argument  $\theta_a$  for  $\gamma(a) - z_0$ ,

there exists a unique continuous function  $\theta : [a, b] \rightarrow \mathbb{R}$

such that for each  $t \in [a, b]$ ,  $\theta(t)$  is an argument of  $\gamma(t) - z_0$  and such that  $\theta(a) = \theta_a$ . Define

$$n(\gamma, z_0) = \frac{\theta(b) - \theta(a)}{2\pi} \quad \text{to be the winding number of } \gamma \text{ around } z_0.$$

Sometimes, we also refer it to the index of  $z_0$  with respect to  $\gamma$ .

**Theorem 7.11.**  $n(\gamma, z_0) \in \mathbb{Z}$

**Theorem 7.12.** Let  $\gamma$  be a closed contour piecewise differentiable and  $z_0 \in \gamma$ . Then,

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz.$$

**Corollary 7.3.** Let  $f$  be holomorphic on an open set  $\Omega$  containing  $\gamma$  and  $z_0 \in f(\gamma)$ . Then,

$$n(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz.$$

**Example 7.29 (Dinh's 70 problems).** Let  $C$  be the unit circle  $|z| = 1$ , anti-clockwise oriented, and let  $f(z) = z^3$ . How many times does the curve  $f(C)$  wind around the origin? Explain.

*Solution.* We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_C \frac{3}{z} dz = 3.$$

□

**Example 7.30 (Dinh's 70 problems).** Let  $C$  be the unit circle  $|z| = 1$ , anti-clockwise oriented, and let  $f(z) = (z^2 + 2)/z^3$ . How many times does the curve  $f(C)$  wind around the origin? Explain.

*Solution.* We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = -\frac{1}{2\pi i} \int_C \frac{z^2 + 6}{z(z^2 + 2)} dz.$$

The residue at  $z = 0$  is 3, so by Cauchy's residue theorem, the answer is  $-3$ .

□

**Theorem 7.13 (generalised Cauchy's integral formula).** Suppose  $f$  is a holomorphic function in a simply connected domain  $\Omega$ . Then for any piecewise differentiable closed contour  $\gamma$  in  $\Omega$ , if  $a \notin \gamma$ ,

$$n(\gamma, a)f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz$$

**Theorem 7.14 (generalised residue theorem).** Let  $\Omega$  be a simply connected domain in  $\mathbb{C}$ . Suppose  $f$  is holomorphic outside a finite number of points  $z_1, \dots, z_N$  in  $\Omega$ . Then, for any piecewise differentiable closed contour  $\gamma$  in  $\Omega$  which does not pass through  $z_1, \dots, z_N$ ,

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^N n(\gamma, z_k) \text{Res}(f, z_k).$$

## 8. Conformal Mappings and Möbius Transformations

### 8.1. Univalent Functions

**Definition 8.1 (univalent function).** Let  $f : \Omega \rightarrow \mathbb{C}$  be a holomorphic function. Then,  $f$  is univalent if it is injective, i.e.

$$f(z_1) = f(z_2) \quad \text{implies} \quad z_1 = z_2.$$

$f$  is locally univalent if for each  $z_0 \in \Omega$ , there exists a neighbourhood  $U$  of  $z_0$  such that  $f|_U \rightarrow \mathbb{C}$  is injective.

**Theorem 8.1.** A holomorphic function  $f : \Omega \rightarrow \mathbb{C}$

$$\text{locally univalent at } z_0 \quad \text{if and only if} \quad f'(z_0) \neq 0.$$

**Corollary 8.1 (inverse function theorem).** If  $f : \Omega \rightarrow \mathbb{C}$  is a univalent holomorphic function, then its inverse  $f^{-1}$  is also holomorphic defined on  $f(\Omega)$ . Moreover, for each  $z \in \Omega$ ,

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

**Definition 8.2.** Suppose two curves  $\gamma$  and  $\eta$  intersect at  $z_0$  and  $\alpha$  is the oriented angle between the tangent vectors to these curves at  $z_0$ . A holomorphic map  $f$  preserves angles at  $z_0$  if the image curves  $f \circ \gamma$  and  $f \circ \eta$  intersect at  $f(z_0)$  and their tangent vectors at  $f(z_0)$  form an angle equal to  $\alpha$ .

**Theorem 8.2.** Suppose  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and  $f'(z_0) \neq 0$ . Then,  $f$  preserves angles at  $z_0$ .

**Definition 8.3 (conformal map and automorphism group).** A bijective holomorphic function  $f : U \rightarrow V$  is a conformal map or a biholomorphism. A conformal map from a domain  $\Omega \rightarrow \Omega$  is a conformal automorphism of  $\Omega$ . Define  $\text{Aut}(\Omega)$  to be the set of conformal automorphisms of  $\Omega$ .

**Theorem 8.3.** If  $f$  and  $g$  are automorphisms of  $\Omega$ , then  $f \circ g$  is also an automorphism.

### 8.2. Automorphisms of the Complex Plane $\mathbb{C}$

**Example 8.1.** Translations, rotations, and dilations are examples of automorphisms of the complex plane. We only discuss translations here. Suppose  $h \in \mathbb{C}$ . Then, the translation

$$z \mapsto z + h \text{ is a conformal map } \mathbb{C} \rightarrow \mathbb{C} \quad \text{whose inverse is} \quad w \mapsto w - h.$$

Moreover, if  $h \in \mathbb{R}$ , this translation is also a conformal map from the upper half-plane  $\mathbb{H}$  to itself.

**Theorem 8.4.** Let  $f$  be a conformal map from  $\mathbb{C}$  to itself. Then, there exist  $a, b \in \mathbb{C}$  with  $a \neq 0$  such that  $f(z) = az + b$  for  $z \in \mathbb{C}$ . In particular, we have

$$\text{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C}, a \neq 0\}.$$



8.3. Automorphisms of the Unit Disc  $\mathbb{D}$ 

**Definition 8.4** (unit disc). Define  $\mathbb{D}$  to be the unit disc. This is sometimes denoted by  $D(0, 1)$  which represents

the open disc of radius 1 centred at 0.

**Example 8.2.** Any rotation by an angle  $\theta \in \mathbb{R}$ , i.e.  $\rho_\theta(z) = e^{i\theta}z$ , is an automorphism of  $\mathbb{D}$  whose inverse is  $e^{-i\theta}z$ .

We can generalise the previous example to the following lemma:

**Lemma 8.1** (Blaschke factor). For any  $a \in \mathbb{D}$ , the map

$$\phi_a(z) = \frac{a-z}{1-\bar{a}z} \text{ is a conformal automorphism of } \mathbb{D} \text{ with inverse } \phi_a^{-1} = \phi_a.$$

The transformation  $\phi_a$  is known as the Blaschke factor.

**Theorem 8.5.** If  $f : \mathbb{D} \rightarrow \mathbb{D}$  is a conformal automorphism and  $f^{-1}(0) = a$ , then there exists  $\theta \in \mathbb{R}$  such that

$$f(z) = e^{i\theta} \frac{a-z}{1-\bar{a}z}.$$

Hence,

$$\text{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \frac{a-z}{1-\bar{a}z} : \theta \in \mathbb{R}, a \in \mathbb{D} \right\}.$$

**Example 8.3.** Let  $f$  be a holomorphic function on  $\mathbb{D}$  such that  $|f(z)| \leq 1$  when  $|z| < 1$ . Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|} \text{ for all } |z| < 1.$$

*Solution.* We first consider the case where  $|f(z)| = 1$  for some  $z \in \mathbb{D}$ . By the maximum modulus principle,  $f$  is constant and so  $|f(z)| = 1$  for all  $z \in \mathbb{D}$ . The above inequality is equivalent to

$$|f(0)| - |z| \leq 1 + |f(0)||z| \text{ and } 1 - |f(0)||z| \leq |f(0)| + |z|,$$

so  $1 - |z| \leq 1 + |z|$ , which holds.

Now, consider the case where  $|f(z)| < 1$  for all  $z \in \mathbb{D}$ . Let  $f(0) = a \in \mathbb{D}$ . Note that

$$\phi(z) = \frac{a-z}{1-\bar{a}z} \in \text{Aut}(\mathbb{D}).$$

As such,  $g = \phi \circ f$  is a holomorphic function from  $\mathbb{D}$  to itself. Moreover,  $g(0) = \phi(f(0)) = \phi(a) = 0$ . By the Schwarz Lemma,  $|g(z)| \leq |z|$  for all  $z \in \mathbb{D}$ . Since  $\phi^{-1} = \phi$ , then

$$f(z) = (\phi^{-1} \circ g)(z) = \frac{a-g(z)}{1-\bar{a}g(z)}.$$

As such,

$$\left| \frac{a-g(z)}{1-\bar{a}g(z)} \right| \leq 1 \Rightarrow 1 - |f(0)||z| \leq 1 - |\bar{a}||g(z)| \leq |a-g(z)| \leq |1-\bar{a}g(z)| \leq 1 + |\bar{a}||g(z)| \leq 1 + |f(0)||z|$$

and in a similar fashion, we can deduce that

$$|f(0) - |z|| \leq |a| - |g(z)| \leq |a-g(z)| \leq |a| + |g(z)| \leq |f(0)| + |z|.$$

Hence, we have shown that

$$1 - |f(0)|z| \leq |1 - \bar{a}g(z)| \leq 1 + |f(0)||z| \text{ and } |f(0) - |z|| \leq |a - g(z)| \leq |f(0)| + |z|.$$

The desired inequality is thus proven.  $\square$

**Example 8.4.** Find a conformal map  $T : D(0, 1) \rightarrow D(1, 2)$  such that  $T(0) = 1 + i$  and  $T(1) = 1 - 2i$ . Is the transformation unique?

*Solution.* Let  $S(z) = 2z + 1$ , which maps  $D(0, 1)$  to  $D(1, 2)$  conformally. Define  $f = S^{-1} \circ T$ , which is an automorphism of the unit disc. We have  $S^{-1}(z) = (z - 1)/2$ . So, the conditions  $T(0) = 1 + i$  and  $T(1) = 1 - 2i$  are equivalent to  $f(0) = i/2$  and  $f(1) = -i$ . To find such a map  $f$ , consider

$$g(z) = -i \cdot \frac{\frac{i}{2} - z}{1 + \frac{i}{2}z}$$

which is a conformal automorphism of  $D(0, 1)$  such that  $g(i/2) = 0$  and  $g(-i) = 1$ . Thus,

$$f(z) = \frac{i(1 - 2z)}{2 - z}.$$

We conclude that

$$T(z) = \frac{2(1 + i) - (1 + 4i)z}{2 - z}$$

is the required conformal map satisfying the conditions.

Suppose  $\tilde{T}$  also satisfies the requirements. Then,  $R = T^{-1} \circ \tilde{T}$  is a conformal automorphism of  $D(0, 1)$  satisfying  $R(0) = 0$  and  $R(1) = 1$ . It is known that all automorphisms of the unit disc which fix 0 are rotations. Hence,  $R$  is the identity function so we conclude that  $\tilde{T} = T$ .  $\square$

**Example 8.5.** Let  $f : \mathbb{D} \rightarrow \mathbb{C}$  be a holomorphic function. Suppose  $f(0) = 0$  and there exists a constant  $A > 0$  such that  $\operatorname{Re}(f(z)) \leq A$  for  $z \in \mathbb{D}$ . Prove that for  $z \in \mathbb{D}$ ,

$$|f(z)| \leq \frac{2A|z|}{1 - |z|}.$$

*Solution.* Since  $f(0) = 0$ , then  $f$  is identically 0 or  $f$  is not identically constant. If  $f(z) = 0$  for all  $z \in \mathbb{D}$ , the inequality is obvious. Suppose  $f$  is not identically constant. Consider

$$\phi_1(z) = -\frac{z}{A} + 1, \quad \phi_2(z) = \frac{1 - z}{1 + z} \text{ and } \phi(z) = (\phi_2 \circ \phi_1)(z) = \frac{z}{2A - z}.$$

Note that  $\phi_1$  is a conformal map from  $\{\operatorname{Re}(z) < A\}$  to  $\{\operatorname{Re}(z) > 0\}$  and sends 0 to 1;  $\phi_2$  is a conformal map from  $\{\operatorname{Re}(z) > 0\}$  to the unit disc and sends 1 to 0. Hence,  $\phi$  is a conformal map from  $\{\operatorname{Re}(z) < A\}$  to the unit disc and sends 0 to 0. As such,  $F = \phi \circ f$  is a holomorphic map from  $\mathbb{D}$  to itself and  $F(0) = 0$ .

By the Schwarz Lemma, note that the conditions  $F(0) = 0$  and  $|F(z)| \leq 1$  are satisfied since  $z \in \mathbb{D}$ . Hence,  $|F(z)| \leq |z|$ . That is to say,

$$|z| \geq |\phi(f(z))| = \left| \frac{f(z)}{2A - f(z)} \right|.$$

The desired inequality follows with some simple algebraic manipulation.  $\square$

**Example 8.6 (Dinh's 70 problems).** Suppose that  $f$  is holomorphic on the open set containing  $\mathbb{D}$ ,  $|f(z)| \leq 4$  if  $|z| = 1$  and  $f(i/2) = 0$ . Show that for all  $|z| \leq 1$ ,

$$|f(z)| \leq 4 \left| \frac{z - i/2}{1 + i/2 \cdot z} \right|.$$

*Solution.* Note that  $g(z) = \frac{a-z}{1-\bar{a}z}$  is an automorphism of  $\mathbb{D}$ , so we set  $f(z) = 4g(z)$  and  $a = i/2$ . The result follows.  $\square$

**Example 8.7 (Dinh's 70 problems).** Show that if  $D(0, R) \rightarrow \mathbb{C}$  is holomorphic with  $|f(z)| < M$  for some  $M > 0$ , then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)} \right| \leq \frac{|z|}{MR}.$$

*Solution.* Let  $a = f(0)$ . We wish to prove

$$\left| \frac{a - f(z)}{M^2 - \bar{a}f(z)} \right| \leq \frac{|z|}{MR}.$$

Define  $\phi : \mathbb{D} \rightarrow \mathbb{D}$  via

$$\phi(z) = \frac{a/M - z}{1 - (\bar{a}/M)z}.$$

Note that  $\overline{a/M} = \bar{a}/M$  since  $M \in \mathbb{R}$ . So, define  $g = \phi \circ \frac{f(Rz)}{M}$ . It is clear that  $g(0) = 0$  and  $g : \mathbb{D} \rightarrow \mathbb{D}$ . By the Schwarz Lemma,  $|g(z)| \leq |z|$  for all  $z \in \mathbb{D}$ . Hence,

$$\begin{aligned} |g(z)| &\leq |z| \\ \frac{1}{M} \left| \frac{a - f(Rz)}{1 - \bar{a}f(Rz)/M^2} \right| &\leq |z| \\ \frac{M^2}{M} \left| \frac{a - f(Rz)}{M^2 - \bar{a}f(Rz)} \right| &\leq |z| \\ \left| \frac{a - f(z)}{M^2 - \bar{a}f(z)} \right| &\leq \frac{|z|}{MR} \end{aligned}$$

and we are done.  $\square$

**Lemma 8.2 (Schwarz-Pick lemma).** Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic function,  $a \in \mathbb{D}$  and  $f(a) = b$ .

Then,

(i) for each  $z \in \mathbb{D}$ ,  $|\phi_b(f(z))| \leq |\phi_a(z)|$

(ii)  $|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$

If equality holds in (ii) or if we have equality in (i) for some  $z \neq a$ , then  $f \in \text{Aut}(\mathbb{D})$ .

#### 8.4. Maps from the Upper Half-Plane $\mathbb{H}$ to the Unit Disc $\mathbb{D}$

**Definition 8.5 (upper half-plane).** Define  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$  to be the upper half-plane.

**Lemma 8.3.** Let

$$F(z) = \frac{i-z}{i+z} \quad \text{and} \quad G(w) = i \cdot \frac{1-w}{1+w}.$$

Then,  $F : \mathbb{H} \rightarrow \mathbb{D}$  is a conformal map with inverse  $G : \mathbb{D} \rightarrow \mathbb{H}$ .

**Theorem 8.6.** All conformal mappings from  $\mathbb{H}$  to  $\mathbb{D}$  take the form

$$\left\{ e^{i\theta} \frac{z - \beta}{z - \bar{\beta}} : \theta \in \mathbb{R}, \beta \in \mathbb{H} \right\}.$$

### 8.5. Automorphisms of the Upper Half-Plane $\mathbb{H}$

**Theorem 8.7.**

$$\text{Aut}(\mathbb{H}) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$$

*Proof.* Let  $a, b, c, d \in \mathbb{R}$  and  $ad - bc > 0$ . Define  $a', b', c', d'$  to be as follows:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \sqrt{ad - bc},$$

where  $a', b', c', d' \in \mathbb{R}$  and  $a'd' - b'c' = 1$ . As such,

$$\mathcal{G} = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\} = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}.$$

We shall prove that  $\mathcal{G} \subseteq \text{Aut}(\mathbb{H})$ . Let

$$f(z) = \frac{az + b}{cz + d} \in \mathcal{G}.$$

Then,  $f : \mathbb{R} \rightarrow \mathbb{R}$ . If we let  $z = x + iy$ , where  $x, y \in \mathbb{R}$ , Since  $a, b, c, d \in \mathbb{R}$ , then

$$\begin{aligned} \text{Im}(f(z)) &= \text{Im} \left[ \frac{a(x + iy) + b}{c(x + iy) + d} \right] = \text{Im} \left[ \frac{ax + b + i(ay)}{cx + d + i(cy)} \cdot \frac{cx + d - i(cy)}{cx + d - i(cy)} \right] \\ &= \text{Im} \left[ \frac{ac(x^2 + y^2) + bcx + adx + bd + iy(ad - bc)}{c^2(x^2 + y^2) + 2cdx + d^2} \right] \\ &= \text{Im}(z) \cdot \frac{ad - bc}{c^2|z|^2 + 2cdx + d^2} \\ &= \text{Im}(z) \cdot \frac{ad - bc}{|cz + d|^2} \end{aligned}$$

which shows  $f : \mathbb{H} \rightarrow \mathbb{H}$ . It is clear that

$$g(z) = \frac{-dz + b}{cz - a} \in \mathcal{G}$$

and  $g \circ f = \text{id}$ . Hence,  $f \in \text{Aut}(\mathbb{H})$  and  $\mathcal{G} \subseteq \text{Aut}(\mathbb{H})$ .

Conversely, let  $f$  be an arbitrary map in  $\text{Aut}(\mathbb{H})$ . We will show that  $f \in \mathcal{G}$ . Define

$$F(z) = \frac{i - z}{i + z}$$

which is a conformal map from  $\mathbb{H}$  to  $\mathbb{D}$  with inverse

$$F^{-1}(z) = i \cdot \frac{1 - z}{1 + z}$$

and this maps from  $\mathbb{D}$  to  $\mathbb{H}$ . Hence,  $h = F \circ f$  is a conformal map from  $\mathbb{H}$  to  $\mathbb{D}$ . All such a map  $h$  must be of the form

$$e^{2i\theta} \frac{z - \beta}{z - \bar{\beta}}$$

with  $\beta \in \mathbb{H}$  and  $\theta \in \mathbb{R}$ . We let the reader prove that

$$f(z) = F^{-1} \left( e^{2i\theta} \frac{z - \beta}{z - \overline{\beta}} \right) = \frac{az + b}{cz + d}$$

and  $ad - bc = \text{Im}(\beta) > 0$  which would show that  $f \in \mathcal{G}$ , so  $\text{Aut}(\mathbb{H}) \subseteq \mathcal{G}$ .  $\square$

**Example 8.8 (Dinh's 70 problems).** Find a conformal map from

$$H = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$$

onto

$$A = \{z \in \mathbb{C} : |z - 2| < 3, |z| > 1\}.$$

You may leave your answer as a composition of conformal mappings.

*Solution.* We find a conformal map from  $A$  to  $H$  first.

- Let  $\phi_1(z) = 1/(z + 1)$ . Let us figure out what  $A$  gets mapped to via  $\phi_1$ . So, if we let  $w = 1/(z + 1)$ , we have  $z = 1/w - 1$ . Consider the annulus  $|z - 2| < 3$ , so after the transformation, we have  $w > 1/6$ . For the region  $|z| > 1$ , we have  $1/w > 2$ . So,  $\phi_1$  maps  $A$  to  $A_1$ , where  $A_1 = \{z \in \mathbb{C} : 1/6 < \text{Re}(z) < 1/2\}$ .
- Let  $\phi_2(z) = z - \frac{1}{6}$ . So,  $\phi_2$  maps  $A_1$  to  $A_2 = \{z \in \mathbb{C} : 0 < \text{Re}(z) < 1/3\}$ .
- Let  $\phi_3(z) = \tan\left(\frac{3\pi z}{2}\right)$ , which maps  $A_2$  to  $H$ .

As such, the required conformal map from  $H$  to  $A$  is  $\phi_3^{-1} \circ \phi_2^{-1} \circ \phi_1^{-1}$ .  $\square$

**Example 8.9 (Dinh's 70 problems).** Let  $f : D(0, 1) \rightarrow \mathbb{C}$  be a holomorphic function such that  $\text{Re}(f(z)) > 0$  for each  $z \in D(0, 1)$  and such that  $f(0) = 1$ .

- Prove that  $|f'(0)| \leq 2$ .
- Assume that  $|f'(0)| = 2$ . Determine all possible forms of  $f$ .

*Solution.*

- We first find a holomorphic map from  $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$  to  $\mathbb{D}$ . To do this, let  $\phi_1(z) = iz$ , which maps the right half of the complex plane to the upper half,  $\mathbb{H}$ . Then, recall that  $\phi_2(z) = \frac{z - i}{z + i}$  maps  $\mathbb{H}$  to  $\mathbb{D}$ . So,  $\phi = \phi_2 \circ \phi_1$  is the required holomorphic map. We have

$$\phi(z) = \frac{iz - i}{iz + i} = \frac{z - 1}{z + 1}.$$

Define  $F = \phi \circ f$  so  $F : \mathbb{D} \rightarrow \mathbb{D}$ , i.e.  $F$  is an automorphism of the unit disk and  $F(0) = 0$ . By the Schwarz Lemma,  $|F'(0)| \leq 1$ , so  $|\phi'(1)f'(0)| \leq 1$ . Since  $\phi'(1) = 1/2$ , the result follows.

- Suppose equality holds. Then,  $F'(0) = 1$ , where

$$F(z) = \frac{f(z) - 1}{f(z) + 1}.$$

Then,  $F(z) = ze^{i\theta}$  (recall that this is just rotating some point in the unit disk about the origin) for  $\theta \in \mathbb{R}$ . One can work out that  $f = \phi^{-1} \circ F$  and find an explicit expression for it.  $\square$

## 8.6. Möbius Transformations

**Definition 8.6** (Möbius transformation). A transformation of the form

$$T(z) = \frac{az + b}{cz + d},$$

where  $a, b, c, d \in \mathbb{C}$ , is a linear fractional transformation (LFT). If  $ad - bc \neq 0$ , then  $T$  is a Möbius transformation.

Note that the condition  $ad - bc \neq 0$  is equivalent to saying  $T$  is not constant. Consider

$$T(z) = \frac{az + b}{cz + d},$$

where  $a, b, c, d \in \mathbb{C}$  and  $ad - bc \neq 0$ . If  $c = 0$ , then  $T : \mathbb{C} \rightarrow \mathbb{C}$ . If  $c \neq 0$ , then  $T : \mathbb{C} \setminus \{-d/c\} \rightarrow \mathbb{C}$ .

**Definition 8.7.** The Möbius transformation  $T : \mathbb{C}^* \rightarrow \mathbb{C}^*$  associated with  $a, b, c, d$  is defined by

$$T(z) = \begin{cases} \frac{az + b}{cz + d} & z \neq \infty, z \neq -d/c; \\ a/c & z = \infty; \\ \infty & z = -d/c. \end{cases}$$

Moreover, if  $c = 0$ , then  $a \neq 0$  and  $d \neq 0$  so that the usual agreements regarding  $\infty$  can be made. That is,  $T(\infty) = \infty$ .

The map  $T : \mathbb{C}^* \rightarrow \mathbb{C}^*$  is injective.

**Remark 8.1.**  $T$  is holomorphic on  $\mathbb{C}^* \setminus \{-d/c\}$  with a simple pole at the point  $\{-d/c\}$ .

**Proposition 8.1.** A Möbius transformation is a composition of transformations of the following forms:

- (i) **translation:**  $z \mapsto z + b$ ,  $b \in \mathbb{C}$ ;
- (ii) **rotation and dilation:**  $z \mapsto \lambda z$ ,  $\lambda \in \mathbb{C} \setminus \{0\}$ ;
- (iii) **reciprocation:**  $z \mapsto 1/z$

**Definition 8.8.** Let  $\text{Aut}(\mathbb{C}^*)$  be the set of meromorphic automorphisms of  $\mathbb{C}^*$ .

**Theorem 8.8.** A Möbius transformation  $T(z) = \frac{az + b}{cz + d}$  is such that  $T \in \text{Aut}(\mathbb{C}^*)$  with

$$T^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Conversely,

$$\text{Aut}(\mathbb{C}^*) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0 \right\}.$$

**Definition 8.9.** Define a line  $l$  in  $\mathbb{C}^*$  to be the union of a line in  $\mathbb{C}$  with  $\{\infty\}$ .

**Lemma 8.4.** Let

$$L = \left\{ z \in \mathbb{C}^* : \alpha z \bar{z} + \beta z + \bar{\beta} \bar{z} + \gamma = 0, \text{ where } \alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C} \text{ and } \beta \bar{\beta} - \alpha \gamma > 0 \right\}.$$

(i) If  $\alpha \neq 0$ , then  $L$  is a circle;

(ii) if  $\alpha = 0$ , then  $L$  is a line

Conversely, each line or circle can be expressed as one of the set  $L$  for appropriate  $\alpha, \beta, \gamma$ .

**Theorem 8.9.** Suppose  $L$  is a line or a circle and  $T$  is a Möbius transformation. Then,  $T(L)$  is a line or a circle.

Note that a Möbius transformation does not necessarily map circles to circles and lines to lines; even if it maps a circle to another circle, it does not necessarily map the first circle's centre to the second circle's centre.

**Example 8.10.** For  $a \in \mathbb{D}$ ,

$$T_a : \mathbb{D} \rightarrow \mathbb{D} \quad \text{where} \quad T_a(z) = \frac{-z + a}{1 - \bar{a}z} \text{ and } T_a(0) = a.$$

To see why, note that  $T$  is a holomorphic function on  $\mathbb{C} \setminus \{1/\bar{a}\}$ , so it is defined in a neighbourhood of  $\bar{\mathbb{D}}$ . For  $z$  in the boundary of  $\mathbb{D}$ , we have  $|z| = 1$  and  $z\bar{z} = 1$ . It is easy to see that  $|T_a(z)| = 1$ . By the maximum modulus principle, when  $|z| < 1$ , we have  $|T_a(z)| < 1$ . Hence,  $T_a(z)$  is a conformal automorphism of  $\mathbb{D}$ .

Also,  $T_a(0) = a$  is obvious.

**Example 8.11.**

$$T(z) = i \cdot \frac{z-1}{z+1}$$

maps the real line to the imaginary line and  $T(-1) = \infty$ .

To see why, let  $z = a$ , where  $a \in \mathbb{R}$ . Then,

$$T(z) = \frac{i(a-1)}{a+1},$$

which is purely imaginary. It is also clear that  $T(-1) = \infty$ .

**Example 8.12.**

$$T(z) = \frac{i-z}{i+z}$$

maps the real line to the unit circle and  $T(\infty) = -1$ .

To see why, let  $z = a$ , where  $a \in \mathbb{R}$ . It suffices to show that  $|T(a)| = 1$ , i.e.

$$\left| \frac{i-a}{i+a} \right| = 1.$$

This is obvious.

**Example 8.13.**

$$T(z) = i \cdot \frac{1-z}{1+z}$$

maps the unit circle to the real line and  $T(-1) = \infty$ .

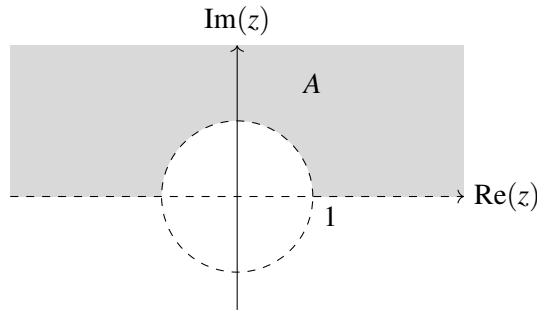
To see why, let  $z = e^{i\theta}$ . Then, we need to show that  $T(z) \in \mathbb{R}$ .

$$T(e^{i\theta}) = \frac{i(1 - e^{i\theta})}{1 + e^{i\theta}} = \tan\left(\frac{\theta}{2}\right),$$

which is real. Also,  $T(-1)$  can be attained by setting  $\theta = (2k+1)\pi$  for  $k \in \mathbb{Z}$ , which implies  $\tan(\theta/2) = \infty$ .

**Example 8.14.** Find a conformal map  $f$  from  $A = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0, |z| > 1\}$  onto the unit disc.

*Solution.* The locus  $A$  represents the intersection of the upper half-plane and the points exterior to the circle of radius 1 centred at the origin.



Consider the Cayley transform given by  $f(z) = (z - i)/(z + i)$ . □

**Example 8.15.** Let

$$A = \left\{ z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im}(z) > \frac{1}{2} \right\} \text{ and } B = \left\{ \frac{2\pi}{3} < \arg(z) < \pi \right\}.$$

Find a conformal map  $f$  from  $A$  to  $B$ .

*Solution.* We first find the intersection of  $|z| = 1$  and  $\operatorname{Im}(z) = 1/2$ . Consider  $x^2 + y^2 = 1$  and  $y = 1/2$ . Solving yields  $x = -\sqrt{3}/2$ . Hence,  $z = e^{i\pi/6}$  or  $z = e^{5\pi i/6}$ .

Note that

$$f(z) = \frac{z - e^{i\pi/6}}{z - e^{5\pi i/6}}$$

is an example of such a conformal map.

To see why, we note that it is a Möbius transformation so it sends lines and circles to lines and circles. Note that  $f(e^{5\pi i/6}) = \infty$  and  $f(e^{i\pi/6}) = 0$ . Hence, the boundary of  $A$  is sent to the union of two half-lines which form an angle at the origin. For  $z$  in the interval joining  $e^{5\pi i/6}$  and  $e^{i\pi/6}$  (along the line  $\operatorname{Im}(z) = 1/2$ ), note that  $z - e^{i\pi/6} \in \mathbb{R}_{<0}$  and  $z - e^{5\pi i/6} \in \mathbb{R}_{>0}$ , so  $f(z) \in \mathbb{R}_{<0}$ .

The angle at  $e^{i\pi/6}$  between this interval and the rest of the boundary of  $A$  forms an angle of  $-\pi/3$ . Since  $f$  is conformal at  $e^{i\pi/6}$ , we conclude that the boundary of  $A$  is sent to the union of  $\mathbb{R}_{<0}$  with the half line  $e^{2\pi i/3}\mathbb{R}_{>0}$ . We deduce that  $A$  is sent to  $B$ . □

**Example 8.16.** Find a Möbius transformation mapping the upper half-plane onto the unit disc and mapping a given point  $z_0$  in the upper half-plane to 0.

*Solution.* Note that  $T$  maps the real line to the unit disc. Since  $z_0$  and  $\overline{z_0}$  are symmetric about the real axis, then  $T(\overline{z_0})$  and  $T(z_0) = 0$  are symmetric with respect to the unit circle. Hence,  $T(\overline{z_0}) = \infty$ . As such,

$$T(z) = \lambda \cdot \frac{z - z_0}{z - \overline{z_0}}$$



for some  $\lambda \in \mathbb{C} \setminus \{0\}$ . Since  $|T(0)| = 1$ , then  $|\lambda| = 1$ . Hence,

$$T(z) = e^{i\theta} \cdot \frac{z - z_0}{z - \bar{z}_0}$$

for some  $\theta \in \mathbb{R}$ . □

**Example 8.17.** Find a Möbius transformation that maps from

$$D = \{z : |z| > 1, |z - 1| < 2\} \text{ to } G = \{w : 0 < \operatorname{Re}(w) < 1\}.$$

*Solution.* Observe that the region  $D$  is bounded by two circles  $x^2 + y^2 = 1$  and  $(x - 1)^2 + y^2 = 4$ . The tangent to these circles is  $x = -1$ . We consider the conformal map  $T(z) = 1/(z + 1)$ . Since  $T(\mathbb{R}) = \mathbb{R}$  and  $C_1$  and  $C_2$  are perpendicular to  $\mathbb{R}$ , it follows that  $T(C_1)$  and  $T(C_2)$  are perpendicular to  $\mathbb{R}$ .

Hence,  $T(C_1) = \{z : \operatorname{Re}(z) = 1/2\}$  and  $T(C_2) = \{z : \operatorname{Re}(z) = 1/4\}$ . So,  $T(D)$  is bounded by these lines. Let  $S(w) = 4w - 1$ . Then,  $S \circ T = (3 - z)/(1 - z)$  maps  $D$  onto  $G$  conformally. □

**Example 8.18 (Dinh's 70 problems).** Let  $T(z) = \frac{az+b}{cz+d}$  be a Möbius transformation.

- (i) Assume that  $z_1, z_2 \in \mathbb{C}$  are two distinct fixed points for  $T$ , i.e.  $T(z_j) = z_j$ ,  $j = 1, 2$ . Show that there exists a constant  $\lambda$  such that

$$\frac{T(z) - z_1}{T(z) - z_2} = \lambda \cdot \frac{z - z_1}{z - z_2}.$$

- (ii) Let  $T^1(z) := T(z)$ ,  $T^{n+1}(z) := T(T^n(z))$ ,  $n = 1, 2, 3, \dots$ . Use (i) to find an expression for  $T^n$ ,  $n = 1, 2, 3, \dots$ , if

$$T(z) = \frac{1 - 3z}{z - 3}.$$

*Solution.*

- (i) We have

$$\begin{aligned} \frac{(T(z) - z_1)(z - z_2)}{(T(z) - z_2)(z - z_1)} &= \frac{\left(\frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d}\right)(z - z_2)}{\left(\frac{az+b}{cz+d} - \frac{az_2+b}{cz_2+d}\right)(z - z_1)} \\ &= \frac{((az+b)(cz_1+d) - (az_1+b)(cz+d))(cz_2+d)(z - z_2)}{((az+b)(cz_2+d) - (az_2+b)(cz+d))(cz_1+d)(z - z_1)} \\ &= \frac{cz_2+d}{cz_1+d} \end{aligned}$$

$$\text{so } \lambda = \frac{cz_2+d}{cz_1+d}.$$

- (ii) We first find the fixed points of  $T$ . Set  $\frac{-3z+1}{z-3} = z$ , so  $z = \pm 1$ . We can take  $z_1 = -1$  and  $z_2 = 1$ , so by repeatedly applying (i), we have

$$\frac{T^n(z) + 1}{T^n(z) - 1} = \left(\frac{1}{2}\right)^n \cdot \frac{z + 1}{z - 1}.$$

□

## 8.7. Cross Ratio

**Definition 8.10 (cross ratio).** The cross ratio of a 4-tuple of points  $z_0, z_1, z_2, z_3 \in \mathbb{C}^*$  is defined to be

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

When one of the  $z_j$  is  $\infty$ , the RHS is understood as the limit as  $z \rightarrow \infty$ .

**Example 8.19.**

$$(\infty, z_1, z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}.$$

**Proposition 8.2.** A Möbius transformation  $T$  preserves cross ratios. That is,

$$(T(z_0), T(z_1), T(z_2), T(z_3)) = (z_0, z_1, z_2, z_3)$$

**Lemma 8.5.** Given three distinct points  $z_1, z_2, z_3 \in \mathbb{C}^*$ , let  $T(z) = (z, z_1, z_2, z_3)$ . Then,  $T$  is a Möbius transformation and

$$T(z_1) = 1, T(z_2) = 0 \text{ and } T(z_3) = \infty.$$

In fact,  $T$  is the unique Möbius transformation such that the above holds.

**Theorem 8.10.** Given two sets of three distinct points  $\{z_1, z_2, z_3\}$  and  $\{w_1, w_2, w_3\}$ , there exists a unique Möbius transformation  $T$  such that  $T(z_j) = w_j$  for  $j = 1, 2, 3$ .

**Corollary 8.2.** Let  $z_0, z_1, z_2, z_3$  be distinct points in  $\mathbb{C}^*$ . Then, they lie in a circle or a line in  $\mathbb{C}^*$  if and only if  $(z_0, z_1, z_2, z_3) \in \mathbb{R}$ .

**Example 8.20.** Find a Möbius transformation  $f$  that maps  $\mathbb{H}$  bijectively to the disc  $D(0, 2)$  such that  $f(i) = 1$  and  $f(1) = -2$ .

*Solution.* A Möbius transformation preserves points of symmetry so  $f(-i)$  is symmetric to  $f(i) = 1$  with respect to  $C(0, 2)$ . Hence,  $f(-i) = 4$ . Since the Möbius transformation  $f$  preserves cross ratios, then

$$\begin{aligned} (f(z), f(1), f(i), f(-i)) &= (z, 1, i, -i) \\ (f(z), -2, 1, 4) &= (z, 1, i, -i) \\ \frac{f(z) - 1}{f(z) - 4} \cdot \frac{-6}{-3} &= \frac{z - i}{z + i} \cdot \frac{1 + i}{1 - i} \end{aligned}$$

Finding  $f(z)$  is left as a simple algebraic exercise. Note that  $f(-1) = 2$ . □

## 9. Harmonic Functions

### 9.1. Basic Properties of Harmonic Functions

Recall that a real-valued function  $u$  is defined on a domain  $\Omega \subseteq \mathbb{C}$  is harmonic if it belongs to  $\mathcal{C}^2$  (second derivative of  $f$  is continuous on  $\Omega$ ) and  $\Delta u = 0$ . The real and imaginary parts of a holomorphic function are harmonic.

**Proposition 9.1.** Let  $\Omega$  be a simply connected domain in  $\mathbb{C}$ . A function  $u : \Omega \rightarrow \mathbb{R}$  is harmonic if and only if  $u$  is the real part of some holomorphic function on  $\Omega$ .

The above proposition implies that for any domain  $\Omega$ ,  $u$  is harmonic if and only if it is locally the real (or imaginary) part of a holomorphic function. In particular, harmonic functions belong to  $\mathcal{C}^\infty$ .

**Example 9.1.** Consider the function

$$u(x, y) = \frac{1}{2} \log(x^2 + y^2)$$

on the annulus  $\Omega = \{0 < r < |z| < R\}$ . This is not a simply connected domain, which means that not all simple closed curves in  $\Omega$  can be shrunk to a point while remaining in  $\Omega$ . One can establish that  $u$  is harmonic but there is no holomorphic function on  $\Omega$  whose real part is equal to  $u$ .

Showing that  $u$  is harmonic, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is simple

**Example 9.2.** Prove that the function

$$u(x, y) = \frac{\sin x}{\cos x + \cosh y}$$

is harmonic in

$$\Omega = \{x + iy : -\pi < x < \pi \text{ and } y \in \mathbb{R}\}.$$

*Solution.* One can see that  $\cosh y = \cos(iy)$ , so by setting  $z = x + iy$ , where  $-\pi < x < \pi$  and  $y \in \mathbb{R}$ , it is clear that

$$u(x, y) = \operatorname{Re} \left( \tan \left( \frac{z}{2} \right) \right).$$

□

**Example 9.3 (Dinh's 70 problems).** Let  $f = u + iv$  be a holomorphic function in an open set  $\Omega$ . Define

$$U := e^{u^2 - v^2} \cos(2uv) \text{ and } V := e^{u^2 - v^2} \sin(2uv).$$

Show that  $U$  is harmonic and  $V$  is a harmonic conjugate of  $U$ .

*Solution.* To show that  $U$  is harmonic, we need to show that it satisfies Laplace's Equation, i.e.  $U_{uu} + U_{vv} = 0$ . This is trivial. Next, one of the Cauchy-Riemann Equations states that  $U_u = V_v$ , so

$$V_v = -2e^{u^2 - v^2} (v \sin(2uv) - u \cos(2uv)).$$

Using integration by parts or Euler's Formula, it can be shown that  $\int V_v dv = V + c$ , where  $c$  is an arbitrary constant. This shows that  $V$  is a harmonic conjugate of  $U$ . □

**Theorem 9.1 (maximum-minimum principle).** If  $u$  is a real-valued non-constant harmonic function on a domain  $\Omega$ , then  $u$  has no local maximum and no local minimum on  $\Omega$ .

## 9.2. Dirichlet Problem and Poisson Kernel

**Theorem 9.2 (Dirichlet problem).** Let  $\Omega$  be a bounded domain in  $\mathbb{C}$ . Given a function  $h : \partial\Omega \rightarrow \mathbb{R}$ , is there a unique continuous function  $u : \overline{\Omega} \rightarrow \mathbb{R}$  such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = h & \text{on } \partial\Omega? \end{cases}$$

In layman's terms, think of  $u$  being harmonic on the interior and  $u = h$  on the boundary.

**Definition 9.1 (Poisson kernel).** Define the Poisson kernel of the unit disc to be

$$P(a, e^{i\theta}) = \frac{1}{2\pi} \cdot \frac{1 - |a|^2}{|e^{i\theta} - a|^2}.$$

We shall consider the case where  $\Omega$  is the unit disc  $\mathbb{D}$ . The following theorem gives the uniqueness of the solution to the Dirichlet problem.

**Theorem 9.3.** Let  $u : \overline{\mathbb{D}} \rightarrow \mathbb{R}$  be a continuous function which is harmonic in  $\mathbb{D}$ . Then, for each  $a \in \mathbb{D}$ ,

$$u(a) = \int_0^{2\pi} P(a, e^{i\theta}) u(e^{i\theta}) d\theta.$$

*Proof.* Consider the automorphism of  $\mathbb{D}$ , which is

$$f(z) = \frac{a - z}{1 - \bar{a}z}.$$

Note that  $f(0) = a$  and  $f$  is self-inverse. Find  $f'$  and  $f'/f$ , then use Gauss' mean value theorem to prove the result.  $\square$

**Corollary 9.1 (Harnack's inequality).** Let  $u$  be a harmonic function in a neighborhood of  $\overline{\mathbb{D}}$ . Assume that  $u \geq 0$  on  $\{|z| = 1\}$ . Then,

$$\frac{1 - |z|}{1 + |z|} u(0) \leq u(z) \leq \frac{1 + |z|}{1 - |z|} u(0)$$

for  $|z| < 1$ .

*Proof.* Apply the Poisson kernel formula. Consider the region  $|z| < 1$  and the identity  $1 - |z|^2 = (1 + |z|)(1 - |z|)$ .  $\square$

## 10. Analytic Continuation

### 10.1. Analytic Continuation

**Definition 10.1 (analytic continuation).** Let  $f$  be a holomorphic function defined on a domain  $\Omega$ . If there exists a domain  $\Omega \subseteq \Omega'$  and a holomorphic function  $F : \Omega' \rightarrow \mathbb{C}$  such that  $F(z) = f(z)$  for each  $z \in \Omega$ , then  $F$  is an analytic continuation of  $f$  on  $\Omega'$ .

**Example 10.1.** The power series

$$f(z) = 1 + z + z^2 + \dots$$

has a radius of convergence  $R = 1$  and so  $f(z)$  is a holomorphic function on the unit disc  $\mathbb{D}$ . On the other hand, one can see that

$$f(z) = \frac{1}{1-z} \text{ for } |z| < 1$$

but  $g(z) = 1/(1-z)$  is holomorphic on  $\mathbb{C} \setminus \{1\}$ . Thus,  $g$  is an analytic continuation of  $f$  to the much bigger domain  $\mathbb{C} \setminus \{1\}$ .

**Lemma 10.1.** Let  $\Omega \subseteq \Omega'$  be domains in  $\mathbb{C}$ . Let  $F_1$  and  $F_2$  be analytic continuations of a holomorphic function  $f : \Omega \rightarrow \mathbb{C}$  to a domain  $\Omega'$ . Then,

$$F_1(z) = F_2(z) \quad \text{for all } z \in \Omega'.$$

**Lemma 10.2.** Let  $f_j : \Omega_j \rightarrow \mathbb{C}$  be holomorphic functions such that  $f_1(z) = f_2(z)$  for  $z \in \Omega_1 \cap \Omega_2$ . Then,

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in \Omega_1; \\ f_2(z) & \text{if } z \in \Omega_2 \setminus \Omega_1. \end{cases}$$

### 10.2. Schwarz Reflection Principle

We say that a region  $\Omega$  is symmetric with respect to the real axis if  $z \in \Omega$  implies  $\bar{z} \in \Omega$ . We consider here an important particular case of analytic continuation.

**Theorem 10.1 (reflection principle for holomorphic functions).** Define  $\Omega^+, \Omega^-, L$  as the intersections of  $\Omega$  with the upper half-plane, lower half-plane, and the real axis respectively. If  $f$  is a continuous complex-valued function on  $\Omega^+ \cup L$ , which is analytic on  $\Omega^+$  and real on  $L$ , then

$f$  admits a unique extension to a holomorphic function  $F$  on  $\Omega$ .

Moreover, the extension is given by

$$F(z) = \begin{cases} f(z) & \text{for } z \in \Omega^+ \cup L; \\ \overline{f(\bar{z})} & \text{for } z \in \Omega^-. \end{cases}$$

In particular,  $F(\bar{z}) = \overline{F(z)}$  for all  $z \in \Omega$ .

**Example 10.2 (MA5217 Lecture Notes).** Suppose  $f$  is holomorphic on  $\mathbb{H}$  and continuous on  $S = \mathbb{H} \cup (0, 1)$ . Assume  $f(x) = x^4 - 2x^2$  for all  $x \in (0, 1)$ . Find  $f(i)$ .

*Solution.* We have  $f(i) = i^4 - 2i^2 = 1 + 2 = 3$ .

□