MA3211 MA3211S MA4247 MA5217 Complex Analysis

Thang Pang Ern

Contents

References

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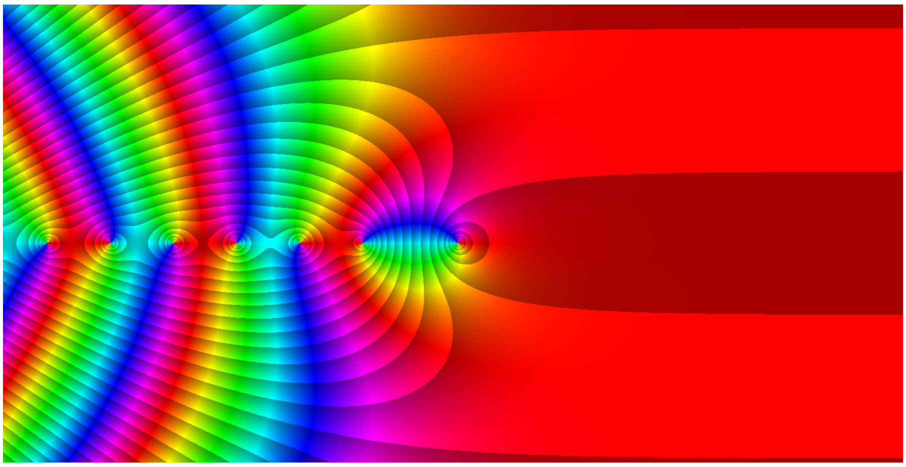


Chapter 1
Complex Numbers

1.1
Some Background Knowledge

First, we define

$$\mathbb{R}[x] \quad \text{to be} \quad \text{the set of polynomials with real coefficients.}$$

The polynomial $x^2+1 \in \mathbb{R}[x]$ of degree 2 over $\mathbb{R}$ has no solution in $\mathbb{R}$ since for all $\alpha \in \mathbb{R}$, we have $\alpha^2+1>0$, so x^2+1 is irreducible over $\mathbb{R}[x]$. For those who have prior knowledge on Abstract Algebra, since $\mathbb{R}[x]$ is a principal ideal domain (PID)†, then

$$\left(x^2+1\right)\mathbb{R}[x] \subseteq \mathbb{R}[x] \quad \text{is} \quad \text{a maximal ideal.}$$

As such, we are now in position to define the complex numbers $\mathbb{C}$.

Definition 1.1 (complex numbers). Define

$$\mathbb{C} = \mathbb{R}[x] / \left(x^2+1\right)\mathbb{R}[x]$$

to be the quotient ring of $\mathbb{R}[x]$ modulo the maximal ideal $\left(x^2+1\right)\mathbb{R}[x]$. This is a field, known as the field of complex numbers.

Proposition 1.1. The image of

$$x \in \mathbb{R}[x] \text{ in } \mathbb{C} \quad \text{is denoted by} \quad i \in \mathbb{C},$$

called the imaginary unit. i has the property that $i^2=-1$.

Proposition 1.2 (field extension). The composite of the canonical ring homomorphisms

$$\mathbb{R} \hookrightarrow \mathbb{R}[x] \twoheadrightarrow \mathbb{C} \quad \text{where} \quad x \mapsto i$$

is an inclusion of fields $\mathbb{R} \hookrightarrow \mathbb{C}$ so $\mathbb{C}$ is a field extension of $\mathbb{R}$.

Proposition 1.3. As an $\mathbb{R}$-vector space, $\mathbb{C}$ has dimension 2 with standard ordered $\mathbb{R}$-basis $\{1, i\}$.

Definition 1.2. The $\mathbb{R}$-linear projection maps

$$\mathrm{Re}: \mathbb{C} \to \mathbb{R} \text{ where } z \mapsto x \quad \text{and} \quad \mathrm{Im}: \mathbb{C} \to \mathbb{R} \text{ where } z \mapsto y$$

†Recall from MA3201 that if F is a field, then $F[x]$ is a Euclidean domain. In fact, recall the chain of inclusions ED $\subseteq$ PID $\subseteq$ UFD, where ED and UFD denote Euclidean domain and unique factorisation domain respectively. I recall in one of Sadhukhan's MA2101S that one student asked whether F is a field implies $F[x]$ is also a field. Clearly, this is wrong and Sadhukhan mentioned that $F[x]$ is a UFD. It was only when I crashed one of Bao Haunchen's MA4203 lectures (first lecture actually) where I learnt that the stronger statement $F[x]$ is an ED holds.

are called the real part and imaginary part of $z \in \mathbb{C}$. So,

$$\text{for all } z \in \mathbb{C} \quad \text{one has } z = \operatorname{Re} z + i \operatorname{Im} z \text{ in } \mathbb{C}.$$

Proposition 1.4 (field operations). The field operations of $\mathbb{C}$, expressed in terms of the real/imaginary parts, are:

(i) Addition/Subtraction:

$$(a+ib) \pm (c+id) = (a \pm c) + i(b \pm d)$$

(ii) Multiplication:

$$(a+ib) \cdot (c+id) = (ac-bd) + i(ad+bc)$$

(iii) Division:

$$\frac{(a+ib)}{(c+id)} = \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

(iv) Multiplicative inverse:

$$(c+id)^{-1} = \frac{c-id}{c^2+d^2}$$

Definition 1.3 (complex conjugation). The $\mathbb{R}$-linear map

$$\overline{(\cdot)} : \mathbb{C} \to \mathbb{C} \quad \text{where} \quad z = x+iy \mapsto \overline{z} = x-iy$$

is called complex conjugation.

Proposition 1.5. We say that complex conjugation is an automorphism of $\mathbb{C}$ as a field over $\mathbb{R}$. The automorphism group $\operatorname{Aut}(\mathbb{C}/\mathbb{R})$ is of order 2. That is to say,

$$\overline{\overline{z}} = z.$$

Proposition 1.6. The following properties hold for all $z, w \in \mathbb{C}$:

(i) $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$

(ii) $\operatorname{Re} z = \frac{1}{2}(z+\overline{z})$ and $\operatorname{Im} z = \frac{1}{2i}(z-\overline{z})$

Definition 1.4 (absolute value). The absolute value of a complex number is the map

$$|\cdot|_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}_{\geq 0} \text{ where } z \mapsto |z|_{\mathbb{C}} \quad \text{given by} \quad |z|_{\mathbb{C}} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{z\overline{z}}.$$

As such, if $z = x+iy$ (where $x, y \in \mathbb{R}$), we have

$$|z|_{\mathbb{C}}^2 = x^2 + y^2 = z\overline{z}.$$

Proposition 1.7. For any $a \in \mathbb{R} \subseteq \mathbb{C}$, we have $|a|_{\mathbb{C}} = |a|_{\mathbb{R}}$.

Lemma 1.1. For any $z, w \in \mathbb{C}$, we have

- **(i) Positive-definiteness:** $|z|_{\mathbb{C}} = 0$ in $\mathbb{R}_{\geq 0}$ if and only if $z = 0$ in $\mathbb{C}$
- **(ii)** $|\overline{z}|_{\mathbb{C}} = |z|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$
- **(iii) Multiplicativity:** $|zw|_{\mathbb{C}} = |z|_{\mathbb{C}} |w|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$
- **(iv)** $|\operatorname{Re} z|_{\mathbb{R}}, |\operatorname{Im}(z)|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$

Proof. **(i)** and **(ii)** are trivial. To prove **(iii)**, we have

$$|zw|_{\mathbb{C}}^2 = zw\overline{zw} = z\overline{z} \cdot w\overline{w} = |z|_{\mathbb{C}}^2 |w|_{\mathbb{C}}^2 .$$

Taking square roots on both sides, **(iii)** follows.

For **(iv)**, let $z = x + iy$, where $x, y \in \mathbb{R}$. Then, $x^2, y^2 \leq x^2 + y^2$, so $|x|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$ and $|y|_{\mathbb{R}} \leq |z|_{\mathbb{C}}$. □

Lemma 1.2 (triangle inequality). For any $z, w \in \mathbb{C}$, we have

$$|z + w|_{\mathbb{C}} \leq |z|_{\mathbb{C}} + |w|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\geq 0}.$$

Proof. We have

$$\begin{aligned} |z+w|_{\mathbb{C}}^2 = (z+w)\left(\overline{z+w}\right) &= z\overline{z} + w\overline{w} + (z\overline{w} + \overline{z}w) \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2\operatorname{Re}(z\overline{w}) \\ &\leq |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2|z\overline{w}|_{\mathbb{C}} \quad \text{by } \textbf{(iv)} \text{ of Lemma 1.1} \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2|z|_{\mathbb{C}} |w|_{\mathbb{C}} \\ &= (|z|_{\mathbb{C}} + |w|_{\mathbb{C}})^2 \end{aligned}$$

Taking square roots on both sides, the result follows. □

By **(i)** and **(iii)** of Lemma 1.1 on the positive-definiteness and multiplicativity, as well as Lemma 1.2 on the triangle inequality, we infer that

$$|\cdot|_{\mathbb{C}} \text{ is an absolute value of } \mathbb{C} \quad \text{in the abstract sense.}$$

Corollary 1.1. We say that

$$\mathbb{C} \text{ equipped with the absolute value function } |\cdot|_{\mathbb{C}} \quad \text{as a normed } \mathbb{R}\text{-vector space}$$

is isomorphic to $\mathbb{R}^2$ with the standard Euclidean norm $\|\cdot\|_2$, so $\mathbb{C}$ is said to be *Cauchy complete*.

Corollary 1.2 (generalised triangle inequality). For any $z_1, z_2, \ldots, z_n \in \mathbb{C}$, we have

$$|z_1 + \ldots + z_n|_{\mathbb{C}} \leq |z_1|_{\mathbb{C}} + \ldots + |z_n|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\geq 0}.$$

Proof. Consider the triangle inequality (Lemma 1.2) and use induction. □

Theorem 1.1 (Cauchy-Schwarz inequality for $\mathbb{R}^2$). For any $z, w \in \mathbb{C}$, we have

$$|\langle z, w\rangle_{\mathbb{R}^2}|_{\mathbb{R}} \leq |z|_{\mathbb{C}} |w|_{\mathbb{C}} \quad \text{with equality} \quad \text{if and only if} z \text{ and } w \text{ are } \mathbb{R}\text{-linearly dependent.}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product of the two inputs. That is to say,

$$z = x + iy \text{ and } w = u + iv \quad \text{implies} \quad \langle z, w \rangle_{\mathbb{R}^2} = xu + yv.$$

Proof. The trick is as follows:

$$\begin{aligned}
\langle z, w \rangle_{\mathbb{R}^2}^2 + \langle iz, w \rangle_{\mathbb{R}^2}^2 &= (xu + yv)^2 + (-yu + xv)^2 \\
&= x^2u^2 + y^2v^2 + 2xuyv + y^2u^2 + x^2v^2 - 2yuxv \\
&= \left(x^2 + y^2\right)\left(u^2 + v^2\right) \\
&= |z|_{\mathbb{C}}^2 |w|_{\mathbb{C}}^2
\end{aligned}$$

which implies $\langle z, w \rangle_{\mathbb{R}^2} \leq |z|_{\mathbb{C}} |w|_{\mathbb{C}}$. Equality holds if and only if $\langle iz, w \rangle_{\mathbb{R}^2} = 0$, or equivalently, $-yu + xv = 0$, i.e. z and w are $\mathbb{R}$-linearly dependent. Well, to be more explicit, we recall that

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{as } \textit{vectors} \text{ in } \mathbb{R}^2.$$

If z and w are linearly dependent, there exists $k \in \mathbb{R}$ such that $(x, y) = k(u, v)$, so $x = ku$ and $y = kv$. As such, $-yu + xv = 0$. □

We can generalise Theorem 1.1 to the Cauchy-Schwarz inequality for $\mathbb{C}^n$ (Theorem 1.2).

Theorem 1.2 (Cauchy-Schwarz inequality for $\mathbb{C}^n$). For any $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$, we have

$$|z_1w_1 + \ldots + z_nw_n|_{\mathbb{C}}^2 \leq \left(|z_1|_{\mathbb{C}}^2 + \ldots + |z_n|_{\mathbb{C}}^2\right)\left(|w_1|_{\mathbb{C}}^2 + \ldots + |w_n|_{\mathbb{C}}^2\right)$$

and

$$\text{equality holds} \quad \text{if and only if} \quad \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \text{ and } \begin{bmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{bmatrix} \text{ are } \mathbb{C}\text{-linearly dependent over } \mathbb{C}^n.$$

Equivalently, this means that there exist $\lambda, \mu \in \mathbb{C}$ which are both non-zero such that

$$\text{for all } 1 \leq j \leq n \quad \text{we have} \quad \lambda z_j = \mu \overline{w_j} \quad \text{in } \mathbb{C}.$$

Proof. We have

$$\begin{aligned}
0 &\leq \sum_{i<j} \left|z_i\overline{w_j} - z_j\overline{w_i}\right|_{\mathbb{C}}^2 \\
&= \sum_{i<j} \left(z_i\overline{w_j} - z_j\overline{w_i}\right)\left(\overline{z_i\overline{w_j} - z_j\overline{w_i}}\right) \\
&= \sum_{i<j} |z_i|^2 \left|w_j\right|^2 + \left|z_j\right|^2 |w_i|^2 - 2\operatorname{Re}\left(z_i\overline{z_j}w_i\overline{w_j}\right)
\end{aligned}$$

We now add the following term to both sides of the inequality:

$$\left|\sum_{i=1}^n z_iw_i\right|^2 = \sum_{i=1}^n |z_i|^2 |w_i|^2 + \sum_{i<j} \left(z_iw_i\overline{z_jw_j} + \overline{z_iw_i}z_jw_j\right)$$

for which it follows that

$$\left|\sum_{i=1}^{n} z_i w_i\right|^2 \le \sum_{i=1}^{n} |z_i|^2 |w_i|^2 + \sum_{i<j} \left(|z_i|^2 |w_j|^2 + |z_j|^2 |w_i|^2\right) = \left(\sum_{i=1}^{n} |z_i|^2\right)\left(\sum_{i=1}^{n} |w_i|^2\right)$$

Equality holds if and only if

$$\sum_{i<j} \left|z_i\overline{w_j} - z_j\overline{w_i}\right|_{\mathbb{C}}^2 = 0.$$

This holds if and only if for all $i < j$, one has $z_i\overline{w_j} = z_j\overline{w_i}$. □

Example 1.1 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all solutions of the equation $e^{e^z} = 1$.

Solution. Note that $1 = e^{2k\pi i}$ for all $k \in \mathbb{Z}$. Since the exponential function is injective, we have $e^z = 2k\pi i$. Hence, $z = \ln|2k\pi| + i\pi/2$. □

1.2 Complex-Valued Functions

Let X be any set. Then, we have the following:

$$\begin{aligned} \operatorname{Maps}(X,\mathbb{R}) &= \{\text{all } \mathbb{R}\text{-valued functions on } X\} \quad &&\text{is an } \mathbb{R}\text{-vector space} \\ \operatorname{Maps}(X,\mathbb{C}) &= \{\text{all } \mathbb{C}\text{-valued functions on } X\} \quad &&\text{is a } \mathbb{C}\text{-vector space} \end{aligned}$$

Proposition 1.8. The $\mathbb{R}$-basis $\{1,i\}$ of $\mathbb{C}$ gives an $\mathbb{R}$-linear decomposition:

$$\operatorname{Maps}(X,\mathbb{C}) \cong \operatorname{Maps}(X,\mathbb{R}) \oplus i\cdot \operatorname{Maps}(X,\mathbb{R}) \quad \text{where} \quad f \mapsto \operatorname{Re} f + i\cdot \operatorname{Im} f.$$

This is such that for any $x \in X$,

$$\operatorname{Re}(f)(x) = \operatorname{Re}(f) \in \mathbb{R}, \quad \operatorname{Im}(f)(x) = \operatorname{Im}(f) \in \mathbb{R}.$$

Proposition 1.9. The $\mathbb{R}$-automorphism $(\bar{\cdot})$ of $\mathbb{C}$ also gives an $\mathbb{R}$-linear automorphism:

$$(\bar{\cdot}) : \operatorname{Maps}(X,\mathbb{C}) \to \operatorname{Maps}(X,\mathbb{C}) \quad \text{where} \quad f \mapsto \overline{f}.$$

This is such that for any $x \in X$,

$$\overline{f}(x) = \overline{f(x)} \quad \text{in } \mathbb{C}.$$

Proposition 1.10. One has the following decomposition:

$$\operatorname{Re} f = \frac{f+\overline{f}}{2}, \quad \operatorname{Im} f = \frac{f-\overline{f}}{2i}.$$

Chapter 2
Holomorphic and Analytic Functions

2.1
Holomorphic Functions

Definition 2.1. Let

$$\begin{aligned} \Omega \subseteq \mathbb{C} \quad & \text{be an open and connected set in } \mathbb{C} \\ H(\Omega) \quad & \text{be the set of holomorphic functions in } \Omega \end{aligned}$$

Definition 2.2 (holomorphic function)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set. A function $f : \Omega \to \mathbb{C}$ is holomorphic at a or $\mathbb{C}$-differentiable at a (Proposition 2.3) if and only if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists in } \mathbb{C}.$$

In this case, the limit, which is uniquely determined by f and a, is called the holomorphic derivative of f at a, denoted by

$$\frac{df}{dz}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{in } \mathbb{C}.$$

As such,

$$f : \Omega \to \mathbb{C} \text{ is holomorphic on } G \quad \text{if and only if} \quad \text{for all } a \in G,\ f \text{ is holomorphic at } a.$$

Proposition 2.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f, g : \Omega \to \mathbb{C}$ be functions holomorphic at a. Then, the following hold:

(i) **$\mathbb{C}$-linearity:** for all $c, d \in \mathbb{C}$,

$$\text{the function } cf + dg : \Omega \to \mathbb{C} \quad \text{is} \quad \text{also holomorphic at } a$$

equipped with

$$\text{its holomorphic derivative} \quad (cf + dg)'(a) = c \cdot f'(a) + d \cdot g'(a) \text{ in } \mathbb{C}$$

(ii) **Product rule:** the function $f \cdot g : \mathbb{C} \to \mathbb{C}$ is also holomorphic at a equipped with its holomorphic derivative

$$(fg)'(a) = f'(a)\, g(a) + g'(a)\, f(a) \quad \text{in } \mathbb{C}$$

Remark 2.1. Recall Definition 2.1, which mentioned that $H(\Omega)$ denotes the set of all functions $f : \Omega \to \mathbb{C}$ which are holomorphic on Ω. We say that

$$H(\Omega) \quad \text{is} \quad \text{a } \mathbb{C}\text{-algebra under pointwise } \pm, \times \text{ of functions.}$$

Note that for any point $a \in \Omega$, we have the evaluation at a map, i.e.

$$\mathrm{ev}_a : H(\Omega) \to \mathbb{C} \quad \text{where} \quad f \mapsto f(a),$$

which is a $\mathbb{C}$-algebra homomorphism.

Also, Proposition 2.1 says that the derivative at a map

$$H(\Omega) \to \mathbb{C} \quad \text{where} \quad f \mapsto f'(a)$$

is a $\mathbb{C}$-linear derivative of $H(\Omega)$ to the $H(\Omega)$-module $\mathbb{C}$ via ev_a.

Example 2.1 (identity map). For any open $\Omega \subset \mathbb{C}$, the identity map id is holomorphic with derivative

$$\mathrm{id}'(a) = \frac{dz}{dz}(a) = 1 \quad \text{for all } a \in G.$$

Hence, $z \in H(\Omega)$. In fact, for any polynomial $f \in \mathbb{C}[z]^{\dagger}$, the function $z \mapsto f(z)$ is also $H(G)$.

Example 2.2. For any open $G = \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$, the reciprocal function z^{-1} is holomorphic with derivative

$$\frac{dz^{-1}}{dz}(a) = -\frac{1}{a^2} \quad \text{for all } a \in G.$$

Hence, $z^{-1} \in H(\Omega)$. Moreover, for any Laurent polynomial $f \in \mathbb{C}\left[z, z^{-1}\right]$ (we will only discuss this when formally defining Laurent series/polynomials in Theorem 4.1), the function $z \mapsto f(z)$ is also in $H(\Omega)$.

Proposition 2.2 (chain rule). Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open sets. Let

$$f : \Omega_1 \to \mathbb{C} \text{ and } g : \Omega_2 \to \mathbb{C} \quad \text{such that} \quad f(\Omega_1) \subseteq \Omega_2$$

so $g \circ f : \Omega_1 \to \mathbb{C}$ is defined. If f is holomorphic at a and g is holomorphic at $f(a)$, then $g \circ f$ is holomorphic at a, equipped with its holomorphic derivative

$$(g \circ f)'(a) = g'(f(a))\, f'(a).$$

Proof. Let $b = f(a) \in \Omega_2$. Define the functions $\xi : \Omega_1 \to \mathbb{C}$ and $\eta : \Omega_2 \to \mathbb{C}$ by setting

$$\xi(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} - f'(a) & \text{if } z \in \Omega_1 \setminus \{a\} \\ \text{any value} & \text{if } z = a \end{cases} \quad \text{and} \quad \eta(w) = \begin{cases} \dfrac{g(w) - g(b)}{w - b} - g'(b) & \text{if } w \in \Omega_2 \setminus \{b\} \\ \text{any value} & \text{if } w = b. \end{cases}$$

Then, for all $z \in \Omega_1$ and $w \in \Omega_2$, we have the following in $\mathbb{C}$:

$$f(z) - f(a) = \left[f'(a) + \xi(z)\right](z - a)$$
$$g(w) - g(b) = \left[g'(b) + \eta(w)\right](w - b)$$

Thus, for all $z \in \Omega_1$, we have

$$\begin{aligned} g(f(z)) - g(f(a)) &= \left(g'(f(a)) + \eta(f(z))\right)(f(z) - f(a)) \\ &= \left(g'(f(a)) + \eta(f(z))\right)\left(f'(a) + \xi(z)\right)(z - a) \end{aligned}$$

†Here, one should perhaps recall from MA3201 that $\mathbb{C}[z]$ denotes the set of all polynomials in z with complex coefficients. That is, $\mathbb{C}[z] \ni f(z) = a_0 + a_1 z + \ldots + a_n z^n \quad \text{where } a_0, a_1, \ldots, a_n \in \mathbb{C}$.

so for all $z \in \Omega_1 \setminus \{a\}$, we have

$$\frac{g(f(z)) - g(f(a))}{z-a} = \left(g'(f(a)) + \eta(f(z))\right)\left(f'(a) + \xi(z)\right).$$

Since

$$\begin{aligned} &f \quad \text{is holomorphic at } a \in \Omega_1 \quad \text{and} \\ &g \quad \text{is holomorphic at } b \in \Omega_2 \end{aligned}$$

then

$$\lim_{z \to a} \xi(z) = 0 \text{ and } \lim_{w \to b} \eta(w) = 0.$$

Also,

$$f \text{ is continuous at } a \quad \text{implies} \quad \lim_{z \to a} f(z) = f(a) = b.$$

Hence,

$$\lim_{z \to a} \frac{g(f(z)) - g(f(a))}{z-a} \text{ exists in } \mathbb{C} \quad \text{and} \quad \text{equals } g'(f(a))\, f'(a).$$

$\square$

Next, recall Definition 2.3 on $\mathbb{R}$-differentiability from MA3210.

Definition 2.3 ($\mathbb{R}$-differentiability). We say that f is $\mathbb{R}$-differentiable at a if and only if there exists an $\mathbb{R}$-linear map $(Df)(a) : \mathbb{C} \to \mathbb{C}$ such that

$$\text{for all } \varepsilon \in \mathbb{R}_{>0}, \text{ there exists } \delta \in \mathbb{R}_{>0}$$

such that

$$\text{for all } z \in G \text{ with } 0 \leq \|z-a\| < \delta \quad \text{we have} \quad \|f(z) - f(a) - (Df)(a)(z-a)\| \leq \varepsilon \cdot \|z-a\|.$$

When this holds, the $\mathbb{R}$-linear map $(Df)(a)$ is uniquely determined by f and a and we call this the derivative of f at a.

Proposition 2.3 ($\mathbb{C}$-differentiability). If f is holomorphic at a ($\mathbb{C}$-differentiable at a), then f is $\mathbb{R}$-differentiable at a and

$$(Df)(a) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{is the image of} \quad f'(a) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

under the following canonical inclusion:

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{where} \quad z \mapsto \text{multiplication by } x.$$

Corollary 2.1. Suppose f is holomorphic on Ω and for all $a \in G$, we have $f'(a) = 0$ in $\mathbb{C}$. Then, f is locally constant on Ω.

Proof. Let $a \in \Omega$ be an arbitrary point. Choose $r \in \mathbb{R}_{>0}$ be sufficiently small such that $B(a,r) \subseteq \Omega$, where

$$B(a,r) \quad \text{is} \quad \text{the open ball in } \mathbb{C} \text{ centred at } a \text{ of radius } r.$$

By the mean-value inequality, for any $z \in B(a,r)$, there exists $\xi \in [a,z] \subseteq B(a,r)$ such that

$$\|f(z) - f(a)\| \leq \|f'(\xi)\| \, \|z-a\|$$

Since $f'(\xi) = 0$, then f is constant of value $f(a)$ on $B(a,r)$. □

Remark 2.2. Throughout this set of notes, we will generally use the terms open ball $B(a,r)$ and open disc $D(a,r)$ interchangeably. Also, the same can be said for closed balls and closed discs.

Now, identify $\mathbb{C}$ with the standard $\mathbb{R}$-basis $\{1,i\}$. Then, consider the following comparison:

$$\mathbb{R}^2 \xleftrightarrow{1,i} \mathbb{C} \xrightarrow{z \mapsto \text{multiplication by } z} \operatorname{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \xleftrightarrow{1,i} \mathcal{M}_{2\times 2}(\mathbb{R})$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto a + bi \mapsto (x + yi \mapsto (a+bi)(x+yi) = (ax - by) + i(bx + ay)) \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We infer that via 1 and i, the matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$$

corresponds to the $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$ given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad x + yi \mapsto (px + qy) + i(rx + sy).$$

This $\mathbb{R}$-linear map is $\mathbb{C}$-linear if and only if $p = s$ and $q = -r$ in $\mathbb{R}$. As such, we can set $a = p$ and $q = -b$.

Now, again via 1 and i, write

$$f : \Omega \to \mathbb{C} \quad \text{as} \quad x + iy \mapsto f(x+iy) = u(x,y) + iv(x,y).$$

Suppose f is $\mathbb{R}$-differentiable at a. Then,

$$(Df)(a) \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \quad \text{corresponds to} \quad \begin{bmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a) \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}).$$

Hence, $(Df)(a)$ lies in the image of $\mathbb{C} \hookrightarrow \operatorname{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ if and only if

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \text{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

This is precisely the Cauchy-Riemann equations (will formally introduce in Theorem 2.1).

2.2
The Cauchy-Riemann Equations

Theorem 2.1 (Cauchy-Riemann equations). Let $\Omega \subseteq \mathbb{C}$ be an open set. Let $f : \Omega \to \mathbb{C}$ be a function written as

$$x+iy \mapsto f(x+iy) = u(x,y) + iv(x,y).$$

Suppose f is $\mathbb{R}$-differentiable at a. Then,

$$f \text{ is holomorphic at } a \quad \text{if and only if} \quad \frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \text{ and } \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a) \text{ are satisfied.}$$

Theorem 2.2 (polar form of CR equations). If u and v are expressed in terms of polar coordinates (r,θ), then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Proof. Using the substitution $z = re^{i\theta}$, we have $x = r\cos\theta$ and $y = r\sin\theta$. Since $f(z) = u(x,y) + iv(x,y)$, we will now perform change of variables from (x,y) to (r,θ). By the chain rule for partial derivatives, to compute $\partial u/\partial r$,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta.$$

By the CR equations (Theorem 2.1),

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta.$$

To compute $\partial v/\partial \theta$,

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta).$$

It is thus clear that the first equation of the theorem holds true. The proof of the second theorem is left as an exercise. □

Theorem 2.3. Let $f(z) = u(x,y) + iv(x,y)$. Suppose the first-order partial derivatives of u and v (u_x, u_y, v_x and v_y) exist in a neighbourhood of z. If they are continuous at z and the CR equations hold, then f is differentiable at z.

Example 2.3. Suppose

$$f(z) = \begin{cases} (\overline{z})^2/z & \text{if } z \neq 0; \\ 0 & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at the point $z = 0$ but the derivative of f fails to exist at $z = 0$.

Solution. We let $z = x+iy$, where $x,y \in \mathbb{R}$. Then, for $z \neq 0$,

$$f(z) = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2} = \frac{x^3-3xy^2}{x^2+y^2} + i\left(\frac{-3x^2y+y^3}{x^2+y^2}\right)$$

which is of the form $f(z) = u(x,y) + iv(x,y)$. The reader can check that at $(0,0)$, u_x, u_y, v_x, v_y are all zero, so the CR equations are satisfied. Next, we consider the following limit:

$$L = \lim_{h\to 0}\frac{(\overline{h})^2/h - 0}{h} = \lim_{h\to 0}\left(\frac{\overline{h}}{h}\right)^2 = \lim_{(x,y)\to(0,0)}\left(\frac{x-iy}{x+iy}\right)^2.$$

Say we approach along the real axis. Then, $L = 1$. However, if we approach along the line $y = x$,

$$L = \lim_{(x,x)\to(0,0)} \left[\frac{x(1-i)}{x(1+i)}\right]^2 = -1$$

so we conclude that $f'(0)$ does not exist. □

Example 2.4. Let

$$f(z) = f(x,y) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at $z = 0$ but f is not differentiable at $z = 0$.

Solution. We let the reader verify that the CR equations are satisfied at $z = 0$. As for differentiability, let $h = a + ib$, where $a, b \in \mathbb{R}$. Then consider

$$\frac{f(h) - f(0)}{h} = \frac{ab(a+ib)}{(a^2+b^2)(a+ib)} = \frac{ab}{a^2+b^2}.$$

We need to prove that as $(a,b) \to (0,0)$, the limit L does not exist. Suppose we approach along the x-axis, then $L = 0$. However, if we approach along the line $y = x$, we have

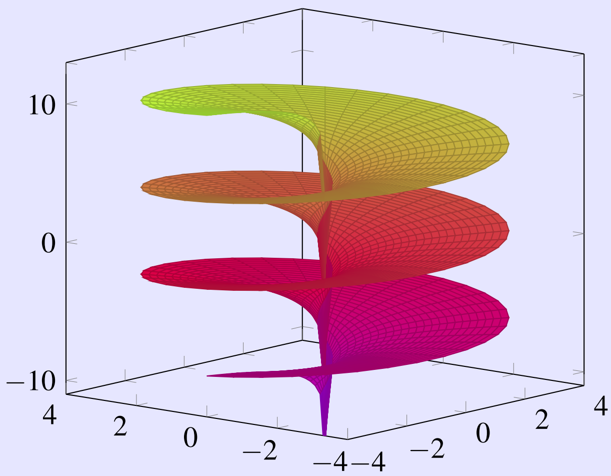
$$L = \lim_{a\to 0} \frac{a^2}{a^2+a^2} = \frac{1}{2}.$$

As such, the limit L does not exist so we can conclude that $f'(0)$ does not exist. □

Definition 2.4 (principal logarithm)**.** Define

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

Note that $\operatorname{Log} z$ is a single-valued function defined on $\mathbb{C} \setminus \{0\}$.



2.3
Analytic Functions and Entire Functions

Definition 2.5 (power series)**.** A power series over $\mathbb{C}$ in the variable z centred at $a \in \mathbb{C}$ is a formal sum

$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for all } n \in \mathbb{N} \text{ and } a_n \in \mathbb{C}.$$

Definition 2.6 (different types of convergence)**.** Let

$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{with } z \in \mathbb{C} \quad \text{be a power series over } \mathbb{C}.$$

We say that

(i) the series converges at $z \in \mathbb{C}$ if and only if

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (z-a)^n \quad \text{exists in } \mathbb{C};$$

(ii) the series converges absolutely at $z \in \mathbb{C}$ if and only if

$$\sum_{n=0}^{\infty} |a_n (z-a)^n| < \infty \quad \text{in } \mathbb{R}_{\geq 0};$$

(iii) the series converges normally on some compact $D \subseteq \mathbb{C}$ if and only if

$$\sum_{n=0}^{\infty} \sup_{z \in D} |a_n (z-a)^n| < \infty \quad \text{in } \mathbb{R}_{\geq 0};$$

(iv) the series converges locally normally on some open $U \subseteq \mathbb{C}$ if and only if for all $a \in U$, there exists a neighbourhood $D \subseteq U$ such t hat

$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{converges normally on } D$$

Example 2.5. We have the classic example of the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad \text{in } \mathbb{C}.$$

this series converges absolutely for all $z \in \mathbb{C}$ with $|z| < 1$ to $1/(1-z) \in \mathbb{C}$ and it does not converge for all $z \in \mathbb{C}$ with $|z| > 1$. Also, for all $r \in (0,1)$, the series converges normally on $\overline{B(0,r)}$ and it converges locally normally on $B(0,1)$.

Lemma 2.1. Let

$$S = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{with } z \in \mathbb{C} \quad \text{be a power series over } \mathbb{C}.$$

Then, the following hold:

(i) If S converges absolutely at $z_0 \in \mathbb{C}$, then it converges normally on the compact set $\overline{B}(a, |z_0 - a|)$

(ii) If S converges at $z_0 \in \mathbb{C}$, then it converges locally normally on the open set $B(a, |z_0 - a|)$

Definition 2.7 (radius of convergence). The radius of convergence of a power series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$$

is given by

$$\begin{aligned}R&=\sup\{r\in\mathbb{R}_{\geq 0}:f(z)\text{ converges at all points in }B(a,r)\}\\&=\sup\left\{r\in\mathbb{R}_{\geq 0}:f(z)\text{ converges absolutely at all points in }\overline{B(a,r)}\right\}\end{aligned}$$

We note that $R\in\mathbb{R}_{\geq 0}$.

Proposition 2.4 (Cauchy-Hadamard formula). There is a nice formula on the radius of convergence of a power series over $\mathbb{C}$ which is given by

$$\frac{1}{R}=\limsup_{n}|a_n|^{1/n}.$$

One notes that the Cauchy-Hadamard formula in Proposition 2.4 can be easily deduced from the root test.

Definition 2.8 (analytic function). Let $U\subseteq\mathbb{C}$ be an open set and $a\in U$ be a point. A $\mathbb{C}$-valued function $\varphi:U\to\mathbb{C}$ on U is analytic at $a\in U$ if and only if there exists a power series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n\text{ centred at }a\text{ with positive radius of convergence }R$$

such that for all $z\in U\cap B(a,R)$, one has

$$\varphi(z)=f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n\quad\text{in }\mathbb{C}.$$

Then,

$$\varphi:U\to\mathbb{C}\text{ is an analytic function (on }U)\quad\text{if and only if}\quad\text{for all }a\in U,\ \varphi\text{ is analytic at }a.$$

Proposition 2.5. Let

$$\sum_{n=0}^{\infty}c_nz^n\quad\text{be a power series centred at 0}\quad\text{with positive radius of convergence }R.$$

Write $f:B(0,R)\to\mathbb{C}$ for the $\mathbb{C}$-valued function it represents. Then, f is analytic on $B(0,R)$.

We will see an alternative and more rigorous way of formulating Proposition 2.5 in Proposition 2.6[†].

Example 2.6. Show that there are no analytic functions $f=u+iv$ such that $u(x,y)=x^2+y^2$.

Solution. Suppose on the contrary that there exists some analytic function f. Then, $u_x=2x$ and $u_y=2y$, so by the CR equations, $v_y=2x$ and $v_x=-2y$. $v_y=2x$ implies that $v(x,y)=2xy+g(x)$. Taking the partial with respect to x and substituting it into $v_x=-2y$, we have $2y+g'(x)=-2y$. As such, $g'(x)=-4y$, so $g(x)=-4xy+c$,

[†]As you will see in Proposition 2.6, the latter is indeed more rigorous. Also, I think Prof. Chin Chee Whye set *something related* for an iteration of his MA2108S finals.

where c is an arbitrary constant. Putting everything together,

$$f(x,y) = x^2 + y^2 + i(-2xy + c).$$

However, this does not satisfy $u_x = v_y$ in the CR equations. So, such an f does not exist. □

Example 2.7. Suppose f is analytic and real-valued in a domain D. Prove that f is constant in D.

Solution. Suppose $f(z) = u + iv$. We have $\operatorname{Im}(f) = 0$ so by the CR equations, $u_x = 0$ and $u_y = -v_x = 0$. This implies that $f'(z) = u_x + iv_x = 0$ so f is constant in D. □

Example 2.8. Suppose f and $\overline{f}$ are analytic in a domain D. Show that f is constant in D.

Solution. Observe that $\operatorname{Re}(f) = \left(f + \overline{f}\right)/2$ which is real-valued and analytic if both f and $\overline{f}$ are analytic. By Example 2.7, $\operatorname{Re}(f)$ is constant, so f is constant. □

Proposition 2.6. For any $a \in B(0,R)$ and $k \in \mathbb{N}$, define

$$d_k = \sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k}.$$

Then, the following properties hold:

(i) For all $k \in \mathbb{N}$, the series d_k converges absolutely in $\mathbb{C}$

(ii) The power series

$$g(z) = \sum_{k=0}^{\infty} d_k (z-a)^k \quad \text{has positive radius of convergence } r \geq R - |a| > 0$$

(iii) For all $z \in B(0,R) \cap B(a,r)$, we have $f(z) = g(z)$

Proof. We first prove **(i)**. Fix $\rho \in \mathbb{R}_{\geq 0}$ with $|a| < \rho < R$. Then,

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} |c_n| \, |a|^{n-k} \right) (\rho - |a|)^k &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} |c_n| \, |a|^{n-k} (\rho - |a|)^k \\
&= \sum_{n=0}^{\infty} |c_n| \left[\sum_{k=0}^{n} \binom{n}{k} |a|^{n-k} (\rho - |a|)^k \right] \\
&= \sum_{n=0} |c_n| \, (|a| + \rho - |a|)^n \\
&= \sum_{n=0}^{\infty} |c_n| \, \rho^n
\end{aligned}$$

which is $< \infty$ by the choice of ρ. Hence, the series defining d_k converges absolutely, proving **(i)**.

Next, we take a look at **(ii)**. As the power series

$$\sum_{k=0}^{\infty} |d_k| (\rho - |a|)^k \quad \text{is finite,}$$

then the power series $g(z)$ converges normally on the compact set $\overline{B(a, \rho - |a|)}$ so it has a radius of convergence r with $r \geq \rho - |a|$ for any $|a| < \rho < R$. As such, $r \geq R - |a|$, which is positive. This proves **(ii)**.

Lastly, we prove **(iii)**. For all $z \in B(0,R) \cap B(a,r)$, we have

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} c_n z^n &= \sum_{n=0}^{\infty} c_n (a+z-a)^n \\ &= \sum_{n=0}^{\infty} c_n \left[\sum_{k=0}^{n} \binom{n}{k} a^{n-k} (z-a)^k \right] \quad \text{by the biomial theorem} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k} \right) (z-a)^k \\ &= g(z) \end{aligned}$$

and the result follows. □

Definition 2.9 (convolution of series)**.** Let

$$\sum_{n\in\mathbb{Z}} a_n \text{ and } \sum_{n\in\mathbb{Z}} b_n \quad \text{be} \quad \text{two series in } \mathbb{C} \text{ indexed by } \mathbb{Z}.$$

Their convolution is the double series

$$\sum_{n\in\mathbb{Z}} c_n \quad \text{defined by} \quad \text{for all } n \in \mathbb{Z} \text{ we have } c_n = \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=n}} a_k b_l = \sum_{k\in\mathbb{Z}} a_k b_{n-k}.$$

In Definition 2.9, we can also write

$$\sum_{k+l=n} a_k b_l \quad \text{in place of} \quad \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=n}} a_k b_l.$$

Proposition 2.7 (convolution)**.** Suppose

$$\sum_{n\in\mathbb{Z}} a_n \text{ and } \sum_{n\in\mathbb{Z}} b_n \quad \text{are} \quad \text{absolutely convergent series in } \mathbb{C}.$$

Also, we define

$$c_n = \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=n}} a_k b_l.$$

Then, the following hold:

(i) For all $n \in \mathbb{Z}$, the series c_n converges absolutely in $\mathbb{C}$

(ii) The series $\sum_{n\in\mathbb{Z}} c_n$ converges absolutely in $\mathbb{C}$

(iii) We have

$$\left(\sum_{n\in\mathbb{Z}} a_n \right) \left(\sum_{n\in\mathbb{Z}} b_n \right) = \sum_{n\in\mathbb{Z}} c_n = \sum_{n\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=n}} a_k b_l \quad \text{in } \mathbb{C}$$

Proof. We first prove **(i)**. Consider the double series

$$\sum_{n\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=n}} |a_k| |b_l| = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} |a_k| |b_l| = \sum_{k\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} |a_k| |b_l| \right) = \left(\sum_{k\in\mathbb{Z}} |a_k| \right) \left(\sum_{l\in\mathbb{Z}} |b_l| \right)$$

which is the product of two series with finite value. Hence, c_n converges absolutely in $\mathbb{C}$. This proves **(i)**. As a consequence, **(ii)** follows from the triangle inequality for series (see it as an application of Corollary 1.2).

To prove **(iii)**, we start with the RHS. So,

$$\sum_{n\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} a_k b_l = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} a_k b_l = \sum_{k\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} a_k b_l \right) = \left(\sum_{k\in\mathbb{Z}} a_k \right) \left(\sum_{l\in\mathbb{Z}} b_l \right).$$

Since k and l are dummy variables, the result follows. □

Theorem 2.4 (ℂ-differentiability of analytic functions)**.** Let $a \in \mathbb{C}$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{be} \quad \text{a power series with strictly positive radius of convergence } R.$$

Then, the following hold:

(i) The termwise differentiated power series

$$\sum_{n=1}^{\infty} n a_n (z-a)^{n-1} \quad \text{has} \quad \text{the same radius of convergence } R$$

(ii) The $\mathbb{C}$-valued function $f : B(a,R) \to \mathbb{C}$ represented by the power series is $\mathbb{C}$-differentiable on $B(a,R)$

(iii) The $\mathbb{C}$-derivative $f' : B(a,R)$ is represented by the power series

$$g(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

We will only prove **(i)** as the proofs of **(ii)** and **(iii)** are pretty long.

Proof. Without loss of generality, we may assume that $a = 0$ throughout the proof. For **(i)**, by the Cauchy-Hadamard formula (Proposition 2.4), it suffices to show that

$$\limsup_{n\to\infty} (n \cdot |a_n|)^{1/(n-1)} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

We will prove that

$$\lim_{n\to\infty} (n+1)^{1/n} = 1.$$

For $n \geq 1$, we can write $(n+1)^{1/n} = 1 + \delta_n$ for some $\delta n > 0$. Then,

$$\begin{aligned} n+1 = (1+\delta_n)^n &= 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \ldots + \delta_n^n \\ &> 1 + \frac{n(n-1)}{2}\delta_n^2 \quad \text{when } n \geq 2 \end{aligned}$$

so

$$\delta_n^2 < \frac{2}{n-1} \quad \text{which implies} \quad \lim_{n\to\infty} \delta_n^2 = 0.$$

This proves **(i)**. □

For any open set $U \subseteq \mathbb{C}$, we let

$$\begin{array}{llll} \mathcal{C}^{\omega}(U) & \text{denote} & \text{the set of analytic functions on } U & \text{and} \\ \mathcal{C}^{\infty}(U) & \text{denote} & \text{the set of smooth functions on } U & \end{array}$$

We note that $\mathcal{C}^{\omega}(U) \subseteq \mathcal{C}^{\infty}(U)$, i.e. analytic functions are smooth, with derivatives of all orders.

Corollary 2.2 (Taylor's theorem). Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathbb{C}^{\omega}(U)$ be an analytic function on U. Let $a \in U$ and

$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{be a power series with positive radius of convergence.}$$

Then, for all $n \in \mathbb{N}$, we have

$$a_n = \frac{1}{n!} f^{(n)}(a) \quad \text{in } \mathbb{C}.$$

In particular, the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \quad \text{must have positive radius of convergence.}$$

Corollary 2.3 (uniqueness of power series). If two power series with the same centre a converge to the same function on a disc of positive radius centred at a, then the two power series are the same, i.e. have the same coefficients.

Definition 2.10 (entire function). A function f which is analytic on the whole of $\mathbb{C}$ is entire.

Example 2.9. Let

$$f(z) = x^3 - 3xy^2 + x^2 - y^2 + x + 1 + i(3x^2y - y^3 + 2xy + y).$$

(a) Show that $f(z)$ is entire.

(b) Express $f(z)$ as a function of z.

Solution.

(a) This is a very simple exercise using the CR equations.

(b) Recall the binomial theorem and see that

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i\left(3x^2y - y^3\right) + x^2 - y^2 + x + 1 + i(2xy + y) \\ &= x^3 - 3xy^2 + i\left(3x^2y - y^3\right) + x^2 - y^2 + 2ixy + x + iy + 1 \\ &= (x+iy)^3 + (x+iy)^2 + x + iy + 1 \\ &= z^3 + z^2 + z + 1 \end{aligned}$$

So, $f(z) = z^3 + z^2 + z + 1$. □

Example 2.10. Find an entire function f such that $\mathrm{Re}(f) = x^2 - 3x - y^2$ or explain why there is no such function.

Solution. Write $f = u + iv$, where u and v are real-valued functions. Given that $u = x^2 - 3x - y^2$, we apply the CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 3 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y.$$

Solving the first equation yields $v = 2xy + g(y)$, where $g(y)$ is a function in terms of y. Then, $2x + g'(y) = 2x - 3$, which implies that $g(y) = -3y + c$ for some constant c.

Now, we have $v = 2xy - 3y + c$. We conclude that the following function satisfies the hypotheses:

$$\begin{aligned} f(z) &= x^2 - 3x - y^2 + i(2xy - 3y + c) \\ &= x^2 - y^2 + 2ixy - 3x - 3iy + ic \\ &= z^2 - 3z + ci \end{aligned}$$

So, $f(z) = z^2 - 3z + ci$. □

Example 2.11 (Dinh's 70 problems). Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(0) = f'(0) = 0 \quad \text{and} \quad \operatorname{Re}(f') = x^2 - y^2 + 6xy.$$

Find f.

Solution. Let $z = x + iy$, so $z^2 = x^2 - y^2 + 2xyi$. As such,

$$x^2 - y^2 + 6xy = \operatorname{Re}\left(z^2 - 3iz^2\right)$$

Since $f'(0) = 0$, then $f'(z) = z^2 - 3iz^2$. It follows that $f(z) = z^3/3 - iz^3$ as $f(0) = 0$. □

2.4 The Exponential Function

Recall from Real Analysis (MA2108) that e can be defined to be the following infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

This can be deduced from the Maclaurin expansion of e^x, which is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{which has radius of convergence } R = \infty.$$

Definition 2.11 (complex exponential function). The complex exponential function is the function $\exp : \mathbb{C} \to \mathbb{C}$ defined by the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By the ratio test, the power series representing the complex exponential function converges absolutely for all $z \in \mathbb{C}$, which implies that the radius of convergence R is ∞. This implies

$$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0.$$

Alternatively, one can directly deduce the value of this lim sup using Stirling's formula, which states that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} \quad \text{or} \quad \text{the alternative asymptotic relation } n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n.$$

Proposition 2.8. For any $z, w \in \mathbb{C}$, we have

$$\exp(z+w) = \exp(z) \cdot \exp(w) \quad \text{in } \mathbb{C}.$$

Proof. The power seires for $\exp(z)$ and $\exp(w)$ converge absolutely, so by Proposition 2.7 (an important proposition on convolution), we have the following:

(i) For all $n \in \mathbb{Z}_{\geq 0}$, the series

$$c_n = \sum_{\substack{k,\ell \in \mathbb{Z}_{\geq 0} \\ k+\ell=n}} \frac{z^k}{k!} \cdot \frac{w^\ell}{\ell!} \quad \text{converges absolutely in } \mathbb{C}$$

(ii) The series

$$\sum_{n \in \mathbb{Z}_{\geq 0}} c_n = \sum_{n \in \mathbb{Z}} \frac{(z+w)^n}{n!} \quad \text{converges in } \mathbb{C}$$

(iii) One has $\exp(z) \cdot \exp(w) = \exp(z+w)$ in $\mathbb{C}$

By considering **(iii)**, we see that the result follows. $\square$

From the lens of Group Theory, we say that the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a continuous group homomorphism from

$$\text{the additive group } \mathbb{C} \quad \text{to} \quad \text{the multiplicative group } \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

To see why, we have $\exp(0) = 0^0/0! = 1$. Then, for all $z \in \mathbb{C}$, we have

$$1 = \exp(0) = \exp(z) \cdot \exp(-z) \quad \text{so} \quad \exp(z) \in \mathbb{C}^\times.$$

Proposition 2.8 shows that exp is a group homomorphism from $\mathbb{C}$ to $\mathbb{C}^\times$. Since exp is a function defined by a convergent power series, we conclude that it is continuous.

Remark 2.3. $\mathbb{C} = \mathbb{R} \times i\mathbb{R}$ as groups.

Theorem 2.5. exp restricts to an isomorphism $\exp : \mathbb{R} \to \mathbb{R}^\times_{>0}$.

Proof. It is clear from the power series definition that $\exp : \mathbb{R} \to \mathbb{R}$. Also, note that

$$\exp : \mathbb{R} \to \mathbb{C}^\times \cap \mathbb{R} = \mathbb{R}^\times = \mathbb{R}^\times_{>0} \sqcup \mathbb{R}_{<0}.$$

Since exp is continuous and $\mathbb{R}$ is connected, we must have $\exp(\mathbb{R})$ being connected in $\mathbb{R}^\times$, where $\exp(\mathbb{R}) \subseteq \mathbb{R}^\times_{>0}$. For $x \in \mathbb{R}_{\geq 0}$, we have $\exp(x) \geq 1 + x$ is not bounded above so $[1, \infty) \subseteq \exp(\mathbb{R})$ and $\ker(\exp) \cap \mathbb{R}_{\geq 0} = \{0\}$. Then from $\exp(-x) = [\exp(x)]^{-1}$, we have $(0, 1] \subseteq \exp(\mathbb{R})$ and $\ker(\exp) \cap \mathbb{R}_{\geq 0} = \{0\}$. $\square$

Lemma 2.2. For $z \in \mathbb{C}$, we have $\exp(\overline{z}) = \overline{\exp(z)}$.

Proof. We note that

$$\exp(\overline{z}) = \sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \overline{\exp(z)}.$$

$\square$

Definition 2.12 (circle group)**.** Let

$$\mathbb{T} = \left\{z \in \mathbb{C}^\times : |z|_{\mathbb{C}} = 1\right\} \leq \mathbb{C}^\times \quad \text{denote} \quad \text{the circle group.}$$

Proposition 2.9. For any $t \in \mathbb{R}$, we have $|\exp(it)|_{\mathbb{C}} = 1$. In other words, exp maps $iR \subseteq \mathbb{C}$ into $\mathbb{T} \subseteq \mathbb{C}^\times$.

Proof. We have

$$\begin{aligned} |\exp(it)|_{\mathbb{C}}^2 &= \exp(it)\overline{\exp(it)} \\ &= \exp(it)\exp\left(\overline{it}\right) \quad \text{by Lemma 2.2} \\ &= \exp(it)\exp(-it) \end{aligned}$$

which is equal to $\exp 0 = 1$. □

Corollary 2.4. For any $z \in \mathbb{C}$, we have

$$|\exp(z)|_{\mathbb{C}} = \exp(\mathrm{Re}(z)) \quad \text{in } \mathbb{R}_{>0}.$$

Proof. We have $z = \mathrm{Re}(z) + i\,\mathrm{Im}(\mathrm{z})$ implies $\exp(z) = \exp(\mathrm{Re}(z)) \cdot \exp(i\,\mathrm{Im}(z))$. □

Theorem 2.6. For any $z \in \mathbb{C}$, we have

$$\exp(z) \in \mathbb{T} \quad \text{if and only if} \quad z \in i\mathbb{R}.$$

Proof. We have $\exp(z) \in \mathbb{T}$ if and only if $\exp(\mathrm{Re}(z)) = 1$, or equivalently $\mathrm{Re}(z) = 0$. □

For this set of notes, we let

$$\mathbb{D} = B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$$

denote the open unit ball centred at 0 in $\mathbb{C}$.

Definition 2.13 (logarithmic function)**.** The logarithmic series $\lambda : \mathbb{D} \to \mathbb{C}$ is the power series

$$\log(1+z) = \lambda(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{z^n}{n!} = z - \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

Proposition 2.10. For any $z \in \mathbb{D}$, one has $\exp(\lambda(z)) = 1 + z$.

Lemma 2.3. The series defining $\lambda(z)$ has radius of convergence 1.

Proof. As $z \in \mathbb{D}$ (open unit disc centred at 0), the series converges absolutely by the ratio test, i.e.

$$\left|\frac{z^{n+1}/(n+1)}{z^n/n}\right| = \frac{n}{n+1}|z| \quad \text{which is } < 1.$$

□

Theorem 2.7. The function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is surjective.

Theorem 2.8. $\ker(\exp) \subseteq \mathbb{C}$ is a non-trivial, discrete subgroup contained in $i\mathbb{R} \subseteq \mathbb{C}$.

Proof. By surjectivity (Theorem 2.7), there exists $z \in \mathbb{C}$ such that $\exp(z) = -1$ in $\mathbb{C}^\times$. Then, $z \neq 0$ in $\mathbb{C}$ since $\exp(0) = 1 \neq -1$ so $2z \neq 0$ in $\mathbb{C}$. However,

$$\exp(2z) = \exp(z+z) = [\exp(z)]^2 = (-1)^2 = 1$$

so $\ker(\exp)$ is a non-trivial subgroup of $\mathbb{C}$.

We then prove that $\ker(\exp)$ is contained in $i\mathbb{R}$. Note that

$$\begin{aligned}\ker(\exp) &= \{z \in \mathbb{C} : \exp(z) = 1\} \\ &\subseteq \{z \in \mathbb{C} : |\exp(z)|_{\mathbb{C}} = 1\}\end{aligned}$$

which is equal to $\exp^{-1}(\mathbb{T}) = i\mathbb{R}$.

Lastly, we prove that $\ker(\exp)$ is a discrete subgroup of $\mathbb{C}$. Note that for $z \in \mathbb{C} \setminus \{0\}$, we have the following†:

$$\frac{\exp(z)-1}{z} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^n}{(n+1)!} \quad \text{so} \quad \lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$

Thus, the function

$$g : \mathbb{C} \to \mathbb{C} \quad \text{where} \quad g(z) = \begin{cases} \dfrac{\exp(z)-1}{z} & \text{if } z \neq 0; \\ 1 & \text{if } z = 0 \end{cases} \quad \text{is continuous.}$$

As such, there exists an open subset $U \subseteq \mathbb{C}$ with $0 \in U$ such that $0 \notin g(U)$. Equivalently, $g^{-1}(0) \cap U \neq \emptyset$. Then, for all $z \in U$, $\exp(z) = 1$ if and only if $z = 0$, so $\ker(\exp) \cap U = \{0\}$. As such, for all $w \in \ker(\exp)$, we have $\ker(\exp) \cap (w+U) = \{w\}$, so every point in $\ker(\exp)$ is isolated. $\square$

Now, *we will define* π!

Definition 2.14. We define π to be the following:

$$\begin{aligned}\pi &= \inf\{t \in \mathbb{R}_{>0} : \exp(2it) = 1\} \\ &= \inf\left\{\frac{1}{2i}\ker(\exp) \cap \mathbb{R}_{>0}\right\}\end{aligned}$$

In Theorem 2.8, we mentioned that

$$\frac{1}{2\pi}\ker(\exp) \cap \mathbb{R}_{>0} \quad \text{is} \quad \text{non-empty and discrete.}$$

As such, π is a positive real number!

Proposition 2.11. $\ker(\exp) = 2\pi i\mathbb{Z} \subseteq i\mathbb{R}$

Proof. The reverse inclusion $\supseteq$ is obvious. For the forward inclusion, suppose $z \in \ker(\exp)$. Then, write

$$z = 2i\pi(n\pi + t) \quad \text{where } n \in \mathbb{Z}, 0 \leq t < \pi.$$

So, $\exp(2it) = 1$ and the result follows. $\square$

†Here is an interesting fact: the function $x/(e^x - 1)$ appears in the definition of Bernoulli numbers. This pops up in Combinatorics and Analytic Number Theory.

Corollary 2.5 (Euler's identity). $e^{\pi i}+1=0$

Proof. Note that $w=e^{\pi i}$ in $\mathbb{C}$ satisfies $w^2=e^{2\pi i}=1$. So, $w=\pm 1$ in $\mathbb{C}$. Since $\pi i\notin 2\pi i\mathbb{Z}$, then $w\neq 1$, so $w=-1$. □

Theorem 2.9 (de Moivre's theorem). For $n\in\mathbb{Z}$,

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin\theta.$$

Proof. By Euler's formula, $e^{i\theta}=\cos\theta+i\sin\theta$. In de Moivre's theorem, the left side of the equation is $e^{in\theta}$ by raising both sides to the power of n. The result follows by using Euler's formula on $e^{in\theta}$. □

Definition 2.15 (topological group). A topological group is a group G equipped with a topology such that we have the following:

(i) G is a topological space
(ii) The group operation $\cdot : G\times G\to G$, given by $(g,h)\mapsto g\cdot h$, is continuous with respect to the product topology on $G\times G$
(iii) The inverse function $(\cdot)^{-1}: G\to G$ given by $g\mapsto g^{-1}$ is continuous

In summary

$\exp:\mathbb{C}\to\mathbb{C}^\times$ is a continuous, surjective homomorphism of topological groups (Definition 2.15).

Its kernel is $\ker(\exp)=2\pi i\mathbb{Z}\subseteq\mathbb{C}$. Hence, it induces an isomorphism of topological groups

$$\mathbb{C}/2\pi i\mathbb{Z}\xrightarrow{\sim}\mathbb{C}^\times \quad \text{where} \quad z+2\pi i\mathbb{Z}\mapsto e^z.$$

Restricting to the real axis yields

$$\mathbb{R}\xrightarrow{\sim}\mathbb{R}_{>0} \quad \text{where} \quad x\mapsto e^x,$$

while restricting to purely imaginary parts modulo $2\pi i$ yields

$$i\mathbb{R}/2\pi i\mathbb{Z}\xrightarrow{\sim}\mathbb{T} \quad \text{where} \quad iy+2\pi i\mathbb{Z}\mapsto e^{iy}.$$

Using polar coordinates, we obtain an isomorphism

$$\mathbb{R}_{>0}\times\mathbb{T}\xrightarrow{\sim}\mathbb{C}^\times,\quad (r,\theta)\mapsto r\theta \quad \text{whose inverse is} \quad z\mapsto\left(|z|,\frac{z}{|z|}\right).$$

On the additive side, we note that

$$\mathbb{R}\oplus i\mathbb{R}\cong\mathbb{C} \quad \text{which is given by the map} \quad (x,iy)\mapsto x+iy.$$

2.5 Harmonic Functions

Definition 2.16 (harmonic function)**.** A real-valued function $h(x,y)$ is said to be harmonic if it is twice continuously differentiable and satisfies Laplace's equation. That is,

$$h_{xx}+h_{yy}=0 \quad \text{or} \quad \frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=0.$$

Example 2.12. Show that u^2 cannot be harmonic for any non-constant harmonic function u.

Solution. Let u be a non-constant harmonic function. Then,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Also, we have

$$\frac{\partial^2\left(u^2\right)}{\partial x^2}=2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial x}\right)^2 \quad \text{and} \quad \frac{\partial^2\left(u^2\right)}{\partial y^2}=2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2.$$

However,

$$\frac{\partial^2\left(u^2\right)}{\partial x^2}+\frac{\partial^2\left(u^2\right)}{\partial y^2}=2u\frac{\partial^2 u}{\partial x^2}+2\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\right)^2=2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}\right)^2\neq 0,$$

which concludes the proof. □

Definition 2.17 (harmonic conjugate)**.** Let u be a harmonic function. If v is a harmonic function satisfying the Cauchy-Riemann equations, then v is a harmonic conjugate of u.

Example 2.13 (MA5217 AY24/25 Sem 1 Homework 1)**.** Show that the function

$$u(x,y)=e^{x-y}\cos(x+y)+e^{x+y}\cos(x-y)$$

is harmonic in $\mathbb{C}$ and find a harmonic conjugate of u.

Solution. By definition, we need to show that u satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Let $s=x+y$ and $t=x-y$, so

$$u\left(\frac{s+t}{2},\frac{s-t}{2}\right)=e^t\cos s+e^s\cos t$$

Hence,

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=e^s\cos t-e^t\cos s+e^t\cos s-e^s\cos t=0$$

so u is harmonic. Finding a harmonic conjugate is trivial. □

Example 2.14 (Dinh's 70 problems)**.** Find all harmonic functions $u(x,y)$ in $\mathbb{C}$ such that

$$\left(x^2-y^2\right)u(x,y) \quad \text{is harmonic in } \mathbb{C}.$$

Solution. Let $f(x,y) = (x^2 - y^2)u(x,y)$. Then,

$$f_{xx} = (x^2 - y^2)u_{xx} + 4xu_x + 2u \quad \text{and} \quad f_{yy} = (x^2 - y^2)u_{yy} - 4yu_y - 2u.$$

As such,

$$f_{xx} + f_{yy} = 4(xu_x - yu_y),$$

where we used the fact that u is harmonic (i.e. $u_{xx} + u_{yy} = 0$). For f to be harmonic, $xu_x = yu_y$. One can use techniques taught to solve partial differential equations to deduce that $u(x,y) = g(xy)$, where $g : \mathbb{R} \to \mathbb{R}$. Therefore, $g''(xy) = 0$, so $g(t) = at + b$, where $a, b \in \mathbb{R}$. Hence, $u(x,y) = axy + b$. $\square$

Chapter 3
Complex Integration

3.1
Riemann-Stieltjes Integrals

In this section, we are interested in integration over paths in $\mathbb{C}$.

Definition 3.1 (continuous map). Let X be a Euclidean space (take for example $X = \mathbb{C}$) and let $[a,b] \subseteq \mathbb{R}$ be a compact interval. A piecewise-$\mathcal{C}^1$ path in X parametrized by $[a,b]$ such as a contour, arc, etc. is a continuous map $\gamma\colon [a,b] \to X$ such that there exists a partition $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a,b]$ and for all $1 \leq j \leq m$, the map

$$\gamma|_{[t_{j-1},t_j]}\colon [t_{j-1},t_j] \to X \quad \text{is} \quad \text{continuously differentiable, i.e. } \mathcal{C}^1.$$

What Definition 3.1 really means is that γ' exists no (t_{j-1},t_j) and is continuous, and both the limits

$$\lim_{t \to t_{j-1}^+} \gamma'(t) \text{ and } \lim_{t \to t_j^-} \gamma'(t) \quad \text{exist in } X.$$

Definition 3.2 (closed path). A path γ in X is closed if and only if $\gamma(a) = \gamma(b)$ in X, where

$$\gamma(a) \text{ is the initial point} \quad \text{and} \quad \gamma(b) \text{ is the endpoint.}$$

Definition 3.3 (variation). For any map $\gamma : [a,b] \to X$ and any partition (necessarily finite) $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ of $[a,b]$, define

$$v(\gamma;P) = \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \text{ in } \mathbb{R}_{\geq 0} \quad \text{to be} \quad \text{the variation of } \gamma \text{ with respect to } P.$$

Set

$$V(\gamma) = \sup\{v(\gamma;P) : P \text{ a partition of } [a,b]\} \text{ in } \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{to be} \quad \text{the total variation of } \gamma.$$

Definition 3.4 (rectifiable path). A path γ is said to be rectifiable or a function of bounded variation if and only if $V(\gamma) < \infty$.

Theorem 3.1 (fundamental theorem of line integrals). Suppose C is a smooth curve given by $z(t) : a \leq z \leq b$ and $F'(z) = f(z)$. Then,

$$\int_C f(z)\, dz = F(z(b)) - F(z(a)).$$

Lemma 3.1 (triangle inequality). Suppose f is a continuous complex-valued function of t. Then,

$$\left|\int_a^b f(t)\ dt\right| \leq \int_a^b |f(t)|\ dt.$$

Proposition 3.1. If $\gamma : [a,b] \to \mathbb{C}$ is piecewise $\mathcal{C}^1$, then γ is of bounded variation and

$$V(\gamma) = \int_a^b |\gamma'(t)|\ dt.$$

In other words, the length of γ is equal to $V(\gamma)$.

Proof. We assume that γ is $\mathcal{C}^1$. Let $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ be any partition of $[a,b]$. Then, for each $1 \leq k \leq m$, we have

$$\begin{aligned} |\gamma(t_k) - \gamma(t_{k-1})| &= \left|\int_{t_{k-1}}^{t_k} \gamma'(t)\ dt\right| && \text{by the Fundamental Theorem of Calculus (Theorem 3.1)} \\ &\leq \int_{t_{k-1}}^{t_k} |\gamma'(t)\ dt| && \text{by the triangle inequality (Lemma 3.1)} \end{aligned}$$

As such,

$$v(\gamma;P) = \sum_{k=1}^m |\gamma(t_k) - \gamma(t_{k-1})| \leq \sum_{k=1}^m \int_{t_{k-1}}^{t_k} |\gamma'(t)|\ dt = \int_a^b |\gamma'(t)|\ dt.$$

This implies $V(t)$ is bounded by the integral on the RHS, which is finite. Hence, γ is of bounded variation. We then show that

$$\int_a^b |\gamma'(t)|\ dt \leq V(\gamma) = \sup_P v(\gamma;P) \text{ in } \mathbb{R}_{\geq 0}.$$

It suffices to show that for any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$\int_a^b |\gamma'(t)|\ dt - \varepsilon \cdot \text{constant} < v(\gamma;P).$$

Let $\varepsilon > 0$ be arbitrary. Since γ is $\mathcal{C}^1$ on $[a,b]$, a compact interval, then γ' is uniformly continuous on $[a,b]$. As such, there exists $\delta > 0$ such that for any $s,t \in [a,b]$ with $|s-t| < \delta$, we have $|\gamma'(s) - \gamma'(t)| < \varepsilon$. We choose any partition $P = \{a = t_0 < t_1 < \ldots < t_m = b\}$ such that

$$\|P\| = \max\{(t_k - t_{k-1}) : 1 \leq k \leq m\} \quad \text{is} \quad < \delta.$$

Then, for all $t_{k-1} \leq t \leq t_k$, one has

$$|\gamma'(t) - \gamma'(t_k)| < \varepsilon \quad \text{so} \quad |\gamma'(t)| \leq |\gamma'(t_k)| + \varepsilon.$$

Hence,

$$\begin{aligned} \int_{t_{k-1}}^{t_k} |\gamma'(t)|\ dt &\leq |\gamma'(t_k)|(t_k - t_{k-1}) + \varepsilon(t_k - t_{k-1}) \\ &= \left|\int_{t_{k-1}}^{t_k} (\gamma'(t) - (\gamma'(t) - \gamma'(t_k)))\ dt\right| + \varepsilon(t_k - t_{k-1}) \\ &\leq \left|\int_{t_{k-1}}^{t_k} \gamma'(t)\ dt\right| + \int_{t_{k-1}}^{t_k} |\gamma'(t) - \gamma'(t_k)|\ dt + \varepsilon(t_k - t_{k-1}) && \text{by the triangle inequality (Lemma 3.1)} \\ &\leq |\gamma'(t_k) - \gamma(t_{k-1})| + 2\varepsilon(t_k - t_{k-1}) \end{aligned}$$

Hence,

$$\int_a^b |\gamma'(t)|\ dt \le v(\gamma;\P) + 2\varepsilon(b-a)$$

so the result follows. □

Example 3.1 (line segment in $\mathbb{C}$). For any $w,z \in \mathbb{C}$, the line segment $[w,z] \subseteq \mathbb{C}$ parametrized by

$$\gamma : [0,1] \to \mathbb{C} \quad \text{where} \quad \gamma(t) = w + t(z-w) \quad \text{is rectifiable.}$$

Its length is

$$V(\gamma) = \int_0^1 |\gamma'(t)|\ dt = |z-w|(1-0) = |z-w|.$$

Example 3.2 (circles in $\mathbb{C}$). For any $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$, the circle $C(a,r) = \partial B(a,r)$ parametrized by

$$\gamma : [0,2\pi] \to \mathbb{C} \quad \text{where} \quad \gamma(t) = a + re^{it} \quad \text{is rectifiable.}$$

Its length is

$$v(\gamma) = \int_0^{2\pi} |\gamma'(t)|\ dt = \left|rie^{it}\right|(2\pi - 0) = 2\pi r.$$

In layman's terms, we say that the circumference of a circle of radius a (with an arbitrary centre) is $2\pi r$.

Example 3.3 (space-filling curves). A continuous space-filling curve is continuous but not rectifiable. A space-filling curve is a continuous mapping from a one-dimensional interval (often $[0,1]$) onto a higher-dimensional region (for example, the unit square $[0,1] \times [0,1]$). Such curves are famous because they challenge our usual intuition that *a 1-dimensional object cannot fill up an area (2-dimensional) or volume (3-dimensional)*.

A non-rectifiable curve is one that has infinite total length by this definition. For example, consider the Hilbert curve in Figure 1. In other words, if one tries to approximate the curve by successively finer polygonal chains, the total length of those polygonal approximations grows without bound.

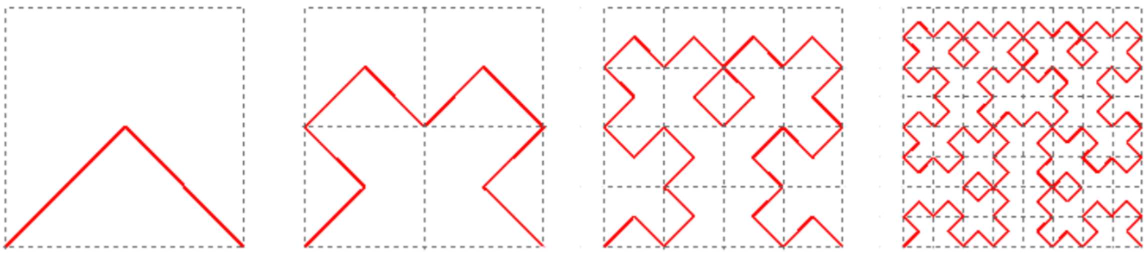


Figure 1: Hilbert curve

Definition 3.5. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise $\mathcal{C}^1$ path and let $f : [a,b] \to \mathbb{C}$ be a continuous function on $[a,b]$. We set

$$\int_a^b f\ d\gamma = \int_a^b f(t)\gamma'(t)\ dt \quad \text{in } \mathbb{C}.$$

Definition 3.6 (path integral)**.** Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth path and let $f : \{\gamma\} \to \mathbb{C}$ be a continuous function on the trace of γ. The path integral of f along γ is

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\ d\gamma = \int_a^b f(\gamma(t))\gamma'(t)\ dt.$$

Example 3.4. For any $w, z \in \mathbb{C}$, parameterized the line segment $[w,z] \subseteq \mathbb{C}$ by

$$\gamma : [0,1] \to \mathbb{C} \quad \text{where} \quad \gamma(t) = w + t(z-w).$$

Then, for any $n \in \mathbb{Z}_{\geq 0}$, we have

$$\int_\gamma z^n\ dz = \int_0^1 (w + t(z-w))^n (z-w)\ dt = \frac{z^{n+1} - w^{n+1}}{n+1}.$$

Example 3.5. For any $r \in \mathbb{R}_{>0}$, parameterize the circle $C(0,r) \subseteq \mathbb{C}$ as follows:

$$\gamma : [0, 2\pi] \to \mathbb{C} \quad \text{where} \quad \gamma(t) = re^{it}$$

Then, for any $n \in \mathbb{Z}$, we have

$$\int_\gamma z^n\ dz = \int_0^{2\pi} \left(re^{it}\right)^n \cdot ire^{it}\ dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\ dt,$$

which is equal to 0 is $n \neq -1$; $2\pi i$ if $n = -1$.

Proposition 3.2 (reparametrization of paths)**.** Let

$$\begin{aligned} \gamma : [a,b] \to \mathbb{C} &\quad \text{be} \quad \text{a piecewise } \mathcal{C}^1 \text{ path} \quad \text{and} \\ \varphi : [c,d] \to [a,b] &\quad \text{be} \quad \text{a } \mathcal{C}^1 \text{ bijection with } \varphi'(s) \text{ for all } s \in [c,d] \end{aligned}$$

Then, $\gamma \circ \varphi : [c,d] \to \mathbb{C}$ is also a piecewise $\mathcal{C}^1$ path and for any continuous function $f : \{\gamma\} \to \mathbb{C}$ on the trace of γ, we have

$$\int_\gamma f\ dz = \int_{\gamma \circ \varphi} f\ dz.$$

Proof. It is clear that $\gamma \circ \varphi$ is a piecewise $\mathcal{C}^1$ path. Thus, we have

$$\begin{aligned} \int_{\gamma\circ\varphi} f\ dz &= \int_c^d f((\gamma\circ\varphi)(s)) \cdot (\gamma\circ\varphi)'(s)\ ds \quad \text{by definition} \\ &= \int_c^d f(\gamma(\varphi(s))) \cdot \gamma'(\varphi(s)) \cdot \varphi'(s)\ ds \quad \text{by the chain rule} \\ &= \int_a^b f(\gamma(t)) \cdot \gamma'(t)\ dt \quad \text{by performing a change of variables } t = \varphi(s) \\ &= \int_\gamma f\ dz \quad \text{by definition} \end{aligned}$$

So, the result follows. □

Definition 3.7 (equivalent paths)**.** Let

$$\sigma : [c,d] \to \mathbb{C} \quad \text{and} \gamma : [a,b] \to \mathbb{C} \quad \text{be piecewise } \mathcal{C}^1 \text{ paths.}$$

We say that the path σ is equivalent to γ if there exists a function

$$\varphi : [c,d] \to [a,b] \quad \text{which is} \quad \mathcal{C}^1, \text{ strictly increasing, and with } \varphi(c) = a \text{ and } \varphi(d) = b$$

such that $\sigma = \gamma \circ \varphi$. We call the function φ a change of parameter.

Proposition 3.3. Let

$$\begin{aligned} \gamma : [a,b] \to \mathbb{C} \quad &\text{be} \quad \text{a piecewise } \mathcal{C}^1 \text{ path} \quad \text{and} \\ f, g : \{\gamma\} \to \mathbb{C} \quad &\text{be} \quad \text{continuous functions on the trace of } \gamma \end{aligned}$$

Then, the following hold:

(i) Linearity with respect to integrand: For any $\alpha, \beta \in \mathbb{C}$, we have

$$\int_\gamma \alpha f + \beta g \, dz = \alpha \int_\gamma f \, dz + \beta \int_\gamma g \, dz$$

(ii) Reverse orientation of path: We have

$$\int_{-\gamma} f \, dz = -\int_\gamma f \, dz$$

(iii) Translation of path: For any $c \in \mathbb{C}$, we have

$$\int_{\gamma + c} f(z) \, dz = \int f(z + c) \, dz$$

Lemma 3.2 (ML inequality/estimation lemma)**.** Let

$$\begin{aligned} \gamma : [a,b] \to \mathbb{C} \quad &\text{be} \quad \text{a piecewise } \mathcal{C}^1 \text{ path} \quad \text{and} \\ f : \{\gamma\} \to \mathbb{C} \quad &\text{be} \quad \text{continuous functions on the trace of } \gamma \end{aligned}$$

Then,

$$\left| \int_\gamma f \, dz \right| \le ML.$$

Here,

$$\begin{aligned} M = \sup_{z \in \{\gamma\}} |f(z)| \quad &\text{denotes} \quad \text{the supremum norm of } f \text{ on } \{\gamma\} \quad \text{and} \\ L = V(\gamma) \quad &\text{denotes} \quad \text{the length of } \gamma \end{aligned}$$

Proof. We have

$$\left| \int_\gamma f \, dz \right| \le \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, dt \right| \le \int_a^b |f(\gamma(t))| \left| \gamma'(t) \right| \, dt \le ML.$$

□

Theorem 3.2 (analogue of the Fundamental Theorem of Calculus)**.** Let Ω be an open subset of $\mathbb{C}$ and let γ be a piecewise $\mathcal{C}^1$ path in Ω with initial and endpoints α and β respectively. If

$$f : \Omega \to \mathbb{C} \text{ is a continuous function with primitive } F : \Omega \to \mathbb{C} \quad \text{then} \quad \int_\gamma f \, dz = F(\beta) - F(\alpha).$$

Note that F is said to be a primitive/antiderivative of f when $F' = f$. This notation yields what is known as the holomorphic derivative.

Corollary 3.1. If γ is a closed curve in Ω and $f : \Omega \to \mathbb{C}$ is continuous with a primitive $F : \Omega \to \mathbb{C}$, we have

$$\int_\gamma f\, dz = 0.$$

Example 3.6. Let γ be the contour given by $\gamma(t) = 3e^{it}$, where $0 \le t \le \pi$. Prove that

$$\left| \int_\gamma \frac{\overline{ze^{iz}}}{z^2 - 11z + 30}\, dz \right| \le 5.$$

Solution. Obviously, $L = 3\pi$ since $\gamma(t) = 3e^{it}$, where $0 \le t \le \pi$ is the equation of the upper half of a circle of radius 3 centred at the origin, so its arc length is 3π. Now, we need to justify that $M \le 5/3\pi$. Let $z = x + iy$.

We have

$$\left| \frac{\overline{ze^{iz}}}{z^2 - 11z + 30} \right| = \left| \frac{\overline{z} \cdot \overline{e^{iz}}}{(z-5)(z-6)} \right| = \frac{|\overline{z}|\, e^{-y}}{|z-5|\,|z-6|} = \frac{|z|\, e^{-y}}{|z-5|\,|z-6|}.$$

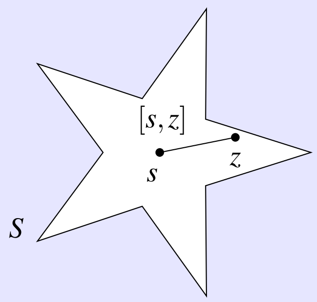
Since $|z| \le 3$ and applying the triangle inequality, we see that

$$\frac{|z|\, e^{-y}}{|z-5|\,|z-6|} \le \frac{3 \cdot 1}{||z|-5|\,||z|-6|} \le \frac{3}{|3-5|\,|3-6|} = \frac{1}{2}$$

so $M = 1/2$. It is clear that $1/2 < 5/3\pi$ so we conclude that $M \le 5/3\pi$. □

3.2 Some Results in Topology

Definition 3.8 (star-shaped set). A set S is star-shaped if it has a point s, known as the star centre, so that for each $z \in S$, the segment $[s, z]$ lies in S.



Remark 3.1. A star domain is not necessarily convex.

Example 3.7. A cross-shaped figure is a star domain but is not convex.

Theorem 3.3. Let S be an open star-shaped region and f continuous on S. Let T be a closed triangular region and ∂T be the boundary of the triangle traversed in the anticlockwise direction. Suppose

$$\int_{\partial T} f(z)\, dz = 0$$

for every T in S, then f has an antiderivative, F, in S.

Definition 3.9 (boundary point). A point $w \in \mathbb{C}$ is a boundary point of S if

$$\text{for every } r \in \mathbb{R}^+ \quad \text{we have} \quad B_r(w) \cap S \neq \emptyset.$$

Definition 3.10 (closure). Denote the set of boundary points by ∂S. Given a set S, the closure of S, denoted by $\overline{S}$, is defined by

$$\overline{S} = S \cup \partial S.$$

Theorem 3.4. A set G is closed if and only if $G = \overline{G}$.

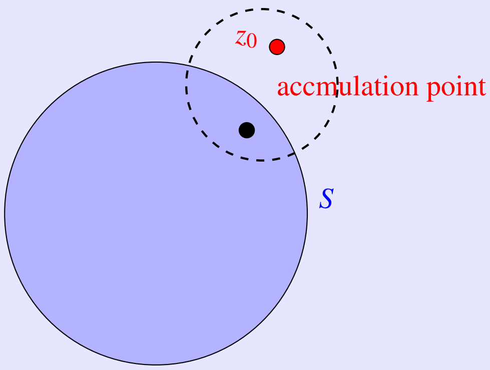
Proof. For the forward direction, suppose G is closed. We wish to prove that $G = G \cup \partial G$, or equivalently, $\partial G \subseteq G$. Suppose on the contrary that $\partial G \not\subseteq G$. Then, there exists $w \in \partial G \setminus G$. For every $\varepsilon > 0$, we have

$$B_\varepsilon(w) \cap G \neq \emptyset \text{ and } B_\varepsilon(w) \cap G' \neq \emptyset \quad \text{which implies} \quad B_\varepsilon(w) \cap G \neq \emptyset.$$

However, $w \notin G$, so $w \in G'$. As G is closed, then G' is open, so there exists $\varepsilon' > 0$ such that $B(w, \varepsilon') \subseteq G'$. Hence, $B(w, \varepsilon') \cap G = \emptyset$ and this is a contradiction, so $\partial G \subseteq G$.

We then prove the reverse direction. Suppose $G = G \cup \partial G$. We wish to prove that G' is open. Let $x \in G'$. As $\partial G \subseteq G$, then $G' \cap \partial G = \emptyset$. There exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap G = \emptyset$ or $B(x, \varepsilon) \cap G = \emptyset$. As $x \in G'$, then $B(x, \varepsilon) \cap G' \neq \emptyset$. Therefore, $B(x, \varepsilon) \cap G = \emptyset$ or $B(x, \varepsilon) \subseteq G'$, which is the definition of G' being open. □

Definition 3.11 (accumulation point). A point z_0 is an accumulation point of a set S if each neighbourhood of z_0 contains at least one point of S distinct from z_0.



Remark 3.2. The accumulation point of a set S does not have to be an element of that set.

Example 3.8. Prove that a set S is closed if and only if S contains all its accumulation points.

Solution. For the forward direction, we proceed with contradiction. Let y be an accumulation of S which is not in S. Then, $y \in S'$. As S' is an open set, there exists $\delta > 0$ such that $B_\delta(y) \subseteq S'$. As such, $B_\delta(y) \cap S = \emptyset$, contradicting the assumption that y is an accumulation point for S.

For the reverse direction, suppose S contains all its accumulation points. We need to show that S is closed. It suffices to show that S' is open. Let $x \in S'$. Then, x is not an accumulation of S since S already contains all its accumulation points. So, there exists $\delta > 0$ such that

$$B_\delta(x) \setminus (\{x\} \cap S) = B_\delta(x) \cap S = \emptyset.$$

We conclude that $B_\delta(x) \subseteq S'$, so S' is open. $\square$

3.3 The Cauchy-Goursat Theorem

Definition 3.12. Let

$$\gamma : [a,b] \to \mathbb{C} \text{ be a piecewise } C^1 \text{ path} \quad \text{and} \quad \varphi : \{\gamma\} \to \mathbb{C} \text{ be a continuous function.}$$

For any $z \in \mathbb{C} \setminus \{\gamma\}$, define

$$f(z) = \int_\gamma \frac{\varphi(w)}{w-z}\, dw = \int_a^b \frac{\varphi(\gamma(t))}{\gamma(t)-z} \cdot \gamma'(t)\, dt \quad \text{in } \mathbb{C}.$$

The resulting function

$$f : \mathbb{C} \setminus \{\gamma\} \to \mathbb{C} \text{ on } \mathbb{C} \setminus \{\gamma\} \quad \text{is said to be} \quad \text{Cauchy-integrally represented by } \gamma \text{ and } \varphi.$$

Note that the function $f(z)$ in Definition 3.12 is well-defined since the integrand

$$\frac{\varphi(w)}{w-z}$$

is a continuous function of w on $\{\gamma\}$.

Example 3.9 (classic example)**.** Fix some $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$. Take

$$\begin{aligned} \gamma : [0,2\pi] \to \mathbb{C} &\quad \text{to be} \quad \gamma(t) = a + re^{it} \text{ parametrizing } \{\gamma\} = C(a,r) \quad \text{and} \\ \varphi : \{\gamma\} \to \mathbb{C} &\quad \text{to be} \quad \text{the constant function } 1 \end{aligned}$$

Recall that $C(a,r)$ denotes the circle of radius r centred at a. Then, the function Cauchy-integrally represented by γ and φ is given as follows:

$$\text{for all } z \in \mathbb{C} \setminus \{\gamma\} \quad \text{we have} \quad \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\, dw = \begin{cases} 1 & \text{if } z \in B(a,r); \\ 0 & \text{if } z \in \mathbb{C} \setminus \overline{B(a,r)} \end{cases}$$

Naively, one would need to evaluate the following integral:

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\, dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ire^{it}}{re^{it}+a-z}\, dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{e^{it}+\frac{a-z}{r}}\, dt$$

However, how do we continue? Backtracking, consider the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\, dw.$$

We first compute at the centre $z = a$. For all $n \in \mathbb{Z}$, we have $\gamma(t) = a + re^{it}$. Hence, the integral becomes

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{(w-a)^{n+1}}\, dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{ire^{it}}{(re^{it})^{n+1}}\, dt = \frac{1}{2\pi r^n}\int_0^{2\pi} \left(re^{it}\right)^{-n} dt = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases}$$

Next, for $z \in B(a,r)$, we expand the integrand $1/(w-z)$ as a power series in terms of $z-a$ to obtain

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a} \cdot \frac{1}{1-\frac{z-a}{w-a}} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n.$$

For $w \in C(a,r)$, we have

$$\left|\frac{z-a}{w-a}\right| = \frac{|z-a|}{r} < 1$$

so the aforementioned series converges uniformly for $w \in C(a,r)$. As such,

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{1}{w-z}\, dw &= \frac{1}{2\pi i} \int_\gamma \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{1}{(w-a)^{n+1}}\, dw\right)(z-a)^n \end{aligned}$$

Since

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{(w-a)^{n+1}}\, dw = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{if } n \neq 0, \end{cases}$$

then

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\gamma \frac{1}{(w-a)^{n+1}}\, dw\right)(z-a)^n = 1.$$

Lastly, we fix $z \in \mathbb{C} \setminus \overline{B(a,r)}$, we expand the integrand $1/(w-z)$ as a power series in $(z-a)^{-1}$, so we obtain

$$\frac{1}{w-z} = \frac{-1}{(z-a)-(w-a)} = \frac{-1}{z-a} \cdot \frac{1}{1-\frac{w-a}{z-a}} = \frac{-1}{z-a} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a}\right)^n.$$

Hence, for $w \in C(a,r)$, we have

$$\left|\frac{w-a}{z-a}\right| = \frac{r}{|z-a|} < 1$$

so the last series above converges uniformly for $w \in C(a,r)$. Hence,

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{w-z}\, dw = \frac{1}{2\pi i} \cdot \frac{-1}{z-a} \int_\gamma \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a}\right)^n dw = \frac{1}{2\pi i} \cdot \frac{-1}{z-a} \sum_{n=0}^{\infty} \left(\int_\gamma (w-a)^n\, dw\right)(z-a)^{-n}\, dw$$

since

$$\int_\gamma (w-a)^n\, dw = 0.$$

Theorem 3.5. Let $\varphi : \{\gamma\} \to \mathbb{C}$ be a continuous function and suppose $f : \mathbb{C} \setminus \to \mathbb{C}$ be Cauchy-integrally represented by γ and φ. So, we have

$$\text{for all } z \in \mathbb{C} \setminus \{\gamma\} \quad \text{we have} \quad f(z) = \int_\gamma \frac{\varphi(w)}{w-z}\, dw \quad \text{in } \mathbb{C}.$$

Let $a \in \mathbb{C} \setminus \{\gamma\}$ be given. Then, for all $n \in \mathbb{N}$, define

$$c_{n,a} = \int_\gamma \frac{\varphi(w)}{(w-a)^{n+1}}\, dw \quad \text{in } \mathbb{C}.$$

Then, for any $r \in \mathbb{R}_{>0}$ such that $\overline{B(a,r)} \subseteq \mathbb{C} \setminus \{\gamma\}$,

$$\text{the power series } \sum_{n=0}^{\infty} c_{n,a} (z-a)^n \quad \text{converges uniformly to } f(z) \text{ on } \overline{B(a,r)}.$$

In particular, for all $z \in \overline{B(a,r)}$, we have

$$\int_\gamma \frac{\varphi(w)}{w-z}\, dw = f(z) = \sum_{n=0}^{\infty} c_{n,a}(z-a)^n .$$

Theorem 3.6 (Cauchy-Goursat theorem/Cauchy integral theorem)**.** Suppose f is analytic on a star-shaped region S. Then, for every simple closed path C in S traversed in the anticlockwise direction,

$$\int_C f(z)\, dz = 0.$$

Example 3.10. Let $f(z) = \operatorname{Log}(z+2)$ and the contour γ be the circle $|z| = 1$ oriented in the anticlockwise direction. Use the Cauchy-Goursat theorem to prove that

$$\int_\gamma f(z)\, dz = 0.$$

Solution. Recall that $\operatorname{Log} z$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$. Thus, $f(z) = \operatorname{Log}(z+2)$ is analytic on $\mathbb{C} \setminus (-\infty, -2]$. However, $(-\infty, -2]$ lies outside the circle $|z| = 1$. Thus, $f(z)$ is analytic inside and on the circle $|z| = 1$, which is a simple closed contour. The result follows by the Cauchy-Goursat theorem. □

Theorem 3.7. Let f be continuous on a star-shaped region S, and analytic on $S \setminus \{z_0\}$, i.e. the set S but excluding the point z_0. Then, f has an antiderivative on S, and consequently,

$$\int_C f(z)\, dz = 0 \quad \text{for every simple closed curve } \in S \text{ traversed anticlockwise.}$$

3.4 Cauchy's Integral Formula

Theorem 3.8 (Cauchy's integral formula)**.** Let f be analytic everywhere within and on a simple closed contour C traversed in the anticlockwise direction. If a is interior to C, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz.$$

Proof. Define

$$g(z) = \frac{f(z) - f(a)}{z-a} \quad \text{which is analytic everywhere except at } z = a.$$

Since the derivative of f exists at a, then by the first principles of differentiation,

$$\lim_{z \to a} g(z) = f'(a).$$

Using the Cauchy-Goursat formula applied to a star-shaped region excluding a (since g is analytic everywhere except a), then

$$\int_C g(z)\, dz = 0 \quad \text{which implies} \quad \int_C \frac{f(z) - f(a)}{z-a}\, dz = 0.$$

So,

$$\int_C \frac{f(z)}{z-a}\, dz = f(a) \int_C \frac{1}{z-a}\, dz = f(a) \cdot 2\pi i,$$

where the last equality follows since we are taking the contour integral on a loop around a. □

Example 3.11. Let $z_0 \in \mathbb{C}$ and γ be a simple closed contour enclosing z_0 with positive orientation. Without using Cauchy's integral formula, and using only the fact that

$$\int_\gamma \frac{1}{z-z_0}\,dz = 2\pi i,$$

show that

$$\text{if } p(z) = z_0 + z_1 z + \ldots + a_{n-1}z^{n-1} + a_n z^n \text{ is a polynomial} \quad \text{then} \quad \int_\gamma \frac{p(z)}{z-z_0} = p(z_0)\cdot 2\pi i.$$

Solution. By the division algorithm for polynomials, there exist polynomials $f(z)$ and r such that $p(z) = (z-z_0)f(z)+r$. So, $p(z_0) = r$.

Hence, $p(z) = (z-z_0)f(z) + p(z_0)$ and we have

$$\int_\gamma \frac{p(z)}{z-z_0}\,dz = \int_\gamma f(z) + \frac{p(z_0)}{z-z_0}\,dz = \int_\gamma f(z)\,dz + p(z_0)\int_\gamma \frac{1}{z-z_0}\,dz = p(z_0)\cdot 2\pi i.$$

Note that the integral

$$\int_\gamma f(z)\,dz = 0$$

by the Cauchy-Goursat theorem. □

Example 3.12. Let C be the circle $|z| = 2$ oriented in the anticlockwise direction. Evaluate

$$\int_C \frac{1}{|z-i|^2}\,dz.$$

Solution. We use the identity $|z|^2 = z\bar{z}$, so $|z-i|^2 = (z-i)(\bar{z}+i)$. Since $|z| = 2$, then $\bar{z} = 4/z$, so

$$(z-i)(\bar{z}+i) = z\bar{z} + i(z-\bar{z}) + 1 = 5 + i\left(z - \frac{4i}{z}\right) = \frac{iz^2+5z+4}{z} = \frac{(iz+1)(z-4i)}{z}.$$

Hence, the contour integral is equivalent to

$$\int_C \frac{z}{(iz+1)(z-4i)}\,dz = \int_C \frac{f(z)}{z-i}\,dz \quad \text{where } f(z) = -\frac{iz}{z-4i}.$$

By Cauchy's integral formula, the integral is equivalent to $2\pi i f(i) = -2\pi/3$. □

Corollary 3.2 (Cauchy's differentiation formula)**.** If f is analytic at a point a, then $f^{(n)}(a)$ exists for $n = 1, 2, \ldots$ and are also analytic z_0, and

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz,$$

where C is a simple closed curve traversed in the anticlockwise direction that encloses a.

Proof. Apply induction on Cauchy's integral formula. □

Theorem 3.9 (Liouville's theorem)**.** If f is a bounded and entire function, then f is a constant.

Proof. Since f is entire, we can represent it using a Taylor series about $z = 0$, so

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By Cauchy's integral formula,

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz,$$

where C is a circle of radius r centred at the origin. Since f is bounded, then $|f(z)| \le M$ for some constant M and for all $z \in \mathbb{C}$. We have

$$|a_n| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz\right| \le \frac{1}{2\pi}\int_C \left|\frac{f(z)}{z^{n+1}}\right||dz| \le \frac{1}{2\pi}\int_C \frac{M}{|z^{n+1}|}|dz| \le \frac{M}{2\pi r^{n+1}}\int_C |dz| = \frac{M}{2\pi r^{n+1}}\cdot 2\pi r = \frac{M}{r^n}$$

Now, as $|z| = r$ on the circle C, by setting $r > 0$ to be arbitrary, as r tends to infinity, $a_n = 0$ for all $n \ge 1$. This is because f is entire. Hence, $f(z) = a_0 = M/r$ which is a constant. □

Example 3.13. Find all entire functions $f(z)$ with $f(0) = 2$ and $|f(z) - e^z| \ge 1$ for all $z \in \mathbb{C}$.

Solution. We note that

$$\frac{1}{|f(z) - e^z|} \le 1 \quad \text{where} \quad f(z) - e^z \ne 0.$$

So, $1/(f(z) - e^z)$ is bounded and entire. By Liouville's theorem,

$$\frac{1}{f(z) - e^z} = c,$$

where c is a constant. Since $f(0) = 2$, then $c = 1$. As such, $f(z) = e^z + 1$. □

Example 3.14. Let g be an entire function such that $|g'(z)| < |g'(z) + i|$ for all complex numbers z. Show that there exist $\alpha, \beta \in \mathbb{C}$ such that $g(z) = \alpha z + \beta$ for all $z \in \mathbb{C}$.

Solution. Since g is entire, then g' is also entire. Let

$$h(z) = \frac{g'(z)}{g'(z) + i}.$$

Then h is the quotient of two entire functions such that the denominator is not equal to zero at each $z \in \mathbb{C}$, hence h is entire. It is clear that for all $z \in \mathbb{C}$, $|h(z)| < 1$, so h is bounded on $\mathbb{C}$. By Liouville's theorem, $h(z) = c$, where c is a constant, so $g'(z) = cg'(z) + ci$. We have

$$g'(z) = \frac{ic}{1 - c} = \alpha.$$

Hence, $g(z) = \alpha z + \beta$. □

Example 3.15. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$\lim_{z \to \infty} f(z) = \infty.$$

Show that f has at least one zero in $\mathbb{C}$.

Solution. Suppose on the contrary f has no zeros in $\mathbb{C}$. Consider

$$g(z) = \frac{1}{f(z)}.$$

Note that g is entire. Using the given limit, there exists $R > 0$ such that for all $|z| > R$, $|f(z)| > 1$. This implies that $|g(z)| < 1$ but since g is continuous, it obtains a maximum M on the compact set $\overline{D(0,R)}$. Hence, for all $z \in \mathbb{C}$, $|g(z)| \le \max\{1, M\}$, so by Liouville's theorem, g is a constant, implying that f is a constant, which is a contradiction. □

Example 3.16. Find all entire functions $f(z)$ such that

$$|f(z)| \leq \frac{1}{1+x^2+2y^2} \quad \text{for all } z = x+iy \in \mathbb{C}.$$

Solution. Since $x^2, y^2 \geq 0$, then $|f(z)| \leq 1$. By Liouville's theorem, f is a constant, say c. Then,

$$c \leq \frac{1}{1+|z|^2+y^2}.$$

It is clear that

$$\lim_{z\to\infty} f(z) = 0$$

so $c = 0$. Hence, the only function satisfying the hypothesis is $f(z) = 0$. □

Example 3.17 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all entire functions f satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z \in \mathbb{C}$.

Solution. By an inductive argument, for all $n \in \mathbb{Z}$, we have

$$f(z+n) = f(z) \quad \text{and} \quad f(z+ni) = f(z).$$

Hence, it suffices to consider the behaviour of f on the unit square $[0,1] \times [0,1]$. Since the unit square is a compact set, it is bounded by the Heine-Borel theorem. Hence, $f(x+iy)$ is bounded for all $x, y \in \mathbb{R}$. Since f is a bounded function, it is constant (follows by Liouville's theorem where we assumed that f is entire). So, $f(z) = c$ for some $c \in \mathbb{R}$. □

Example 3.18 (Dinh's 70 problems)**.** Let $f = u+iv$ be an entire function. Show that if $u^2(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, then f must be a constant.

Solution. We have $f^2 = u^2 - v^2 + 2uvi$. Consider

$$g = e^{-f^2} = e^{v^2-u^2} e^{-2uvi} \quad \text{which is entire} \quad \text{and} \quad |g| \leq \frac{1}{e}.$$

By Liouville's theorem, g is a constant. So, $e^{-f^2} = k$ for some constant k. Thus, f is a constant. □

Theorem 3.10 (fundamental theorem of algebra)**.** Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

3.5 Applications of Cauchy's Integral Formula

Theorem 3.11 (Morera's theorem)**.** Let f be a continuous function on D. Let T be a closed triangle in D and ∂T be the boundary of T traversed in the anticlockwise direction. Then,

$$\int_{\partial T} f(z)\, dz = 0.$$

Theorem 3.12. Let f be an entire function. Define $g(z) = f'(a)$ if $z = a$ and

$$g(z) = \frac{f(z)-f(a)}{z-a}$$

if $z \neq a$. Then, g is also entire.

Theorem 3.13 (extended Liouville's theorem). If f is entire and if for some $k \in \mathbb{N}$, there exists constants $A, B > 0$ such that

$$|f(z)| \leq A + B|z|^k,$$

then f is a polynomial of degree at most k.

Example 3.19 (Dinh's 70 problems). Let u be a real-valued harmonic function in the complex plane such that

$$u(z) \leq a|\ln|z|| + b$$

for all z, where a and b are positive constants. Prove that u is constant.

Solution. By Liouville's theorem, since u is harmonic, it suffices to show that u is bounded. Let $f(z) = a|\ln|z|| + b$. Then, by Cauchy's integral formula,

$$|u'(k)| = \left| \frac{1}{2\pi i} \int_{\gamma:|z|=R} \frac{f(z)}{(z-k)^2}\, dz \right| \leq R \cdot \frac{a|\ln R| + b}{|R - |k||^2},$$

where we have considered γ to be the circle of radius R centred at the origin and naturally, the path is taken to be positively-oriented. To establish the upper bound for $|u'(k)|$, the triangle inequality and reverse triangle inequality are used. Now, note that

$$\lim_{R \to \infty} R \cdot \frac{a|\ln R| + b}{|R - |k||^2}, = 0$$

which implies that $|u'(k)| = 0$, or rather, $u'(k) = 0$. So, $u(k)$ is a constant for all $k \in \mathbb{R}$. □

Theorem 3.14 (Gauss' mean value theorem). If f is analytic in D and $\alpha \in D$, then

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta.$$

Proof. By Cauchy's integral formula, for $a \in D$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz.$$

Let C be a circle of radius r centred at a. Then, our parameterisation is $z = a + re^{i\theta}$, so $dz/d\theta = ire^{i\theta}$. Hence,

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{a + re^{i\theta} - a} \cdot ire^{i\theta}\, d\theta$$

and the result follows with some simple cancellation. □

Theorem 3.15 (maximum modulus theorem for open balls). Suppose $f(z)$ is analytic throughout a neighbourhood $|z - z_0| < R$ of a point z_0. If $|f(z)| \leq |f(z_0)|$ for each z in the neighbourhood, then $f(z)$ attains a constant value $f(z_0)$ throughout the neighbourhood.

Theorem 3.16 (maximum modulus principle). If f is analytic in D and

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in D \quad \text{then} \quad f(z) \text{ is a constant.}$$

Example 3.20 (Dinh's 70 problems). Let $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a complex polynomial of degree $n > 0$. Prove that

$$\frac{1}{2\pi i}\int_{|z|=R} z^{n-1}|f(z)|^2\,dz = a_0\overline{a}_n R^{2n}.$$

Solution. Note that $|f(z)|^2 = f(z)\cdot\overline{f(z)}$. Setting $z = Re^{i\theta}$, the integral becomes

$$\frac{1}{2\pi}\int_0^{2\pi} R^n e^{in\theta}\left(a_0 + a_1 Re^{i\theta} + \ldots + a_n R^n e^{in\theta}\right)\left(\overline{a_0} + \overline{a_1} Re^{-i\theta} + \ldots + \overline{a_n} R^n e^{-in\theta}\right)\,d\theta.$$

Since

$$\int_0^{2\pi} e^{ik\theta}\,d\theta = 0 \quad \text{for all } k \neq 0,$$

upon multiplying the polynomials $a_0 + a_1 Re^{i\theta} + \ldots + a_n R^n e^{in\theta}$ and $\overline{a_0} + \overline{a_1} Re^{-i\theta} + \ldots + \overline{a_n} R^n e^{-in\theta}$, we wish to extract the coefficient of $e^{-in\theta}$. So, the integral becomes

$$\frac{1}{2\pi i}\int_0^{2\pi} R^n e^{in\theta} a_0\overline{a_n} R^n e^{-in\theta}\,d\theta$$

and the result follows. □

Example 3.21 (Dinh's 70 problems). Suppose $u(z)$ is harmonic on $D(0,r)$, where $r > 1$. Prove that

$$\int_0^{2\pi} u(e^{it})\cos^2\left(\frac{t}{2}\right)\,dt = \pi u(0) + \frac{\pi}{2}u'(0) \quad \text{and} \quad \int_0^{2\pi} u(e^{it})\sin^2\left(\frac{t}{2}\right)\,dt = \pi u(0) - \frac{\pi}{2}u'(0),$$

where $u'(0) = u_x(0)$.

Solution. Let I_1 and I_2 denote the two integrals respectively. We have

$$I_1 + I_2 = \int_0^{2\pi} u(e^{it})\,dt \text{ and } I_1 - I_2 = \int_0^{2\pi} u(e^{it})\cos t\,dt.$$

We parametrise each integral using $z = e^{it}$ so $dz/dt = ie^{it}$. Also, recall that $\cos t = \left(z + z^{-1}\right)/2$. So,

$$I_1 + I_2 = \frac{1}{i}\int_{|z|=1}\frac{u(z)}{z}\,dz = \pi u(0),$$

where we used Cauchy's integral formula. Also,

$$I_1 - I_2 = \frac{1}{2i}\int_{|z|=1} u(z) + \frac{u(z)}{z^2}\,dz = \frac{1}{2i}\int_{|z|=1}\frac{u(z)}{z^2}\,dz = \pi u'(0),$$

where we used Cauchy's integral formula and the fact that $u(z)$ is analytic on $D(0,r)$ (since $u(z)$ is harmonic on $D(0,r)$). □

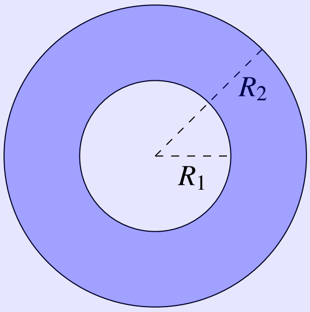
Chapter 4
Series

4.1
Laurent Series

Definition 4.1 (annulus). Define

$$\mathrm{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$$

to be the shaded region as follows:



Theorem 4.1 (Laurent expansion). If f is analytic in the annulus

$$\mathrm{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\},$$

then it has a Laurent expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz.$$

Here, C is a circle of radius R centred at the origin with $R_1 < R < R_2$.

Example 4.1.

(a) Consider the function

$$f(z) = \frac{5z-3}{(z+1)(z-3)}.$$

Find the Laurent series of $f(z)$ for the annular domain $1 < |z| < 3$.

(b) Find the value of the contour integral

$$\int_C \frac{5z-3}{z^5(z+1)(z-3)}\,dz,$$

where C denotes the circle $|z| = 2$ oriented in the anticlockwise direction.

(c) Find the Laurent series of the function

$$\frac{10z^6-6z^4}{(z^2+1)(z^2-3)}$$

in the annular domain $1 < |z| < \sqrt{3}$.

Solution.

(a) We see that

$$\begin{aligned}\frac{5z-3}{(z+1)(z-3)}&=\frac{2}{z+1}+\frac{3}{z-3}\\&=\frac{2}{z}\cdot\frac{1}{1+1/z}-\frac{1}{1-z/3}\\&=\frac{2}{z}\sum_{n=0}^{\infty}(-1)^n(-z)^n-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\\&=2\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\end{aligned}$$

We note that the first summation is valid for $|1/z|<1$ while the second summation is valid for $|z/3|<1$.

(b) We see that the contour integral is equivalent to

$$\int_C \frac{f(z)}{z^5}\,dz=2\pi i\left(-\frac{1}{3^4}\right)=-\frac{2\pi i}{81}.$$

(c) Let us make a comparison. Perhaps we can consider $f(z^2)$. Note that

$$f(z^2)=\frac{5z^2-3}{(z^2+1)(z-3)}.$$

Hence, it is clear that the function in **(c)** is $2z^4f(z^2)$. Recall that the Laurent series of f in the annulus $1<|z|<3$ is

$$2\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

so the required answer is

$$4z^4\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}-2z^4\sum_{n=0}^{\infty}\left(\frac{z^2}{3}\right)^n=4\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n-2}}-2\sum_{n=0}^{\infty}\frac{z^{2n+4}}{3^n}$$

in the annular domain $1<|z|<\sqrt{3}$. □

Example 4.2. Suppose $f(z)$ is entire and $|f(z)|>1$ when $|z|>1$. Prove that $f(z)$ is a polynomial.

Solution. Since f is entire, then in the closed unit disk, it has a finite number of zeros. Say the zeros are $z_1,\ldots,z_m$. So, we can write

$$f(z)=(z-z_1)\ldots(z-z_m)g(z)=p(z)g(z),$$

where g is entire with no zeros and $p(z)$ is a polynomial of degree m. It suffices to show that g is a constant. Let $h(z)=1/g(z)$ so we shall write h as the following Laurent series:

$$h(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{where } a_n=\frac{1}{2\pi i}\int_\gamma \frac{h(z)}{z^{n+1}}\,dz$$

Here, we let γ be $|z|=R$, i.e. the circle of radius R centred at the origin. Letting $z=Re^{i\theta}$, the contour integral becomes

$$\int_0^{2\pi}\frac{iRe^{i\theta}h\left(Re^{i\theta}\right)}{R^{n+1}e^{i(n+1)\theta}}\,d\theta.$$

Let $h\left(Re^{i\theta}\right)\le kR^m$ so it is clear that for all $n>m$,

$$\lim_{R\to\infty}|a_n|\le\lim_{R\to\infty}\frac{kR^m}{R^n}=0.$$

As such, $h(z)$ is a constant, and $g(z)$ is a constant. □

Chapter 5
Residue Theory

5.1 Introduction

We adopt an alternative representation for the annulus $\text{Ann}(z_0, R_1, R_2)$, so if $f(z)$ is analytic in this annulus,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{n+1}}\, ds \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z_0)^{-n+1}}\, ds$$

and C is any positively oriented simple closed contour around z_0 lying inside $\text{Ann}(z_0, R_1, R_2)$.

Definition 5.1 (principal part of Laurent series)**.** The sum

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{is} \quad \text{the principal part of } f(z) \text{ at } z_0.$$

Theorem 5.1 (removable singularity)**.** If $b_n = 0$ for all $n \in \mathbb{N}$, then z_0 is a point of removable singularity of $f(z)$. Thus, the Laurent series of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } 0 < |z-z_0| < R.$$

Example 5.1. The singular point $z = 0$ of $\sin z/z$ is a removable singularity. We have

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

where $0 < |z| < \infty$. This asserts that our claim is true.

Example 5.2 (Dinh's 70 problems)**.** Let $f(z)$ be holomorphic in $\mathbb{C} \setminus \{0\}$ and suppose that

$$\int_{|z|=1} z^n f(z)\, dz = 0 \quad \text{for any } n \in \mathbb{Z}_{\geq 0}.$$

Show that f has a removable singularity at $z = 0$.

Solution. f has a Laurent series representation around $z = 0$. Write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$$

so the integral becomes

$$\int_{|z|=1} z^n \sum_{k=-\infty}^{\infty} a_k z^k\, dz = 0$$

$$\sum_{k=-\infty}^{\infty} \int_{|z|=1} a_k z^{n+k}\, dz = 0$$

since that the series converges uniformly on compact sets away from the singularity. Note that

$$\int_{|z|=1} z^k \, dz = 0 \quad \text{for all } k \neq -1.$$

As such, $n+k=-1$. Since $n \geq 0$, it forces the inequality $k \leq -1$, which implies that $a_k = 0$ for all $k \leq -1$, i.e.

$$\sum_{k=-1}^{\infty} \int_{|z|=1} a_k z^{n+k} \, dz = 0.$$

It is clear that $a_{-1} = 0$. With all coefficients of negative powers being zero, it shows that $f(z)$ has a removable singularity at $z=0$. □

Definition 5.2 (essential singularity)**.** If $b_n \neq 0$ for infinitely many n, then z_0 is a point of essential singularity of $f(z)$. In this case, some of the b_n's may be zero.

Example 5.3. The point $z=0$ of $\exp(1/z)$ is an essential singularity as

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \ldots$$

where $0 < |z| < \infty$.

Definition 5.3 (pole)**.** If there exists $m \in \mathbb{N}$ such that $b_m \neq 0$ but $b_n = 0$ for all $n > m$ so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n},$$

then z_0 is a pole of order m of $f(z)$. If $m=1$, z_0 is a simple pole of $f(z)$; if $m=2$, z_0 is a double pole of $f(z)$.

Example 5.4. Consider the point $z=1$ of

$$f(z) = \frac{1}{(z-1)^2} + z.$$

We can rewrite it as

$$f(z) = \frac{1}{(z-1)^2} + 1 + (z-1)$$

Hence, $z=1$ is a double pole.

Example 5.5 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all the singularities in $\mathbb{C}$ of the following function $f(z)$ and their types where

$$f(z) = \frac{z^2+3z+2}{z(z^4-1)} e^{1/z^2}.$$

Solution. Consider the term z^4-1 in the denominator of $f(z)$. Then, $z^4-1 = (z^2+1)(z^2-1) = (z^2+1)(z+1)(z-1)$. Also, the numerator can be factorised as $(z+2)(z+1)$. Also, consider

$$\frac{e^{1/z^2}}{z} = \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n \frac{1}{n!} \cdot \frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1} n!}.$$

So, $f(z)$ has simple poles at $z=1, z=i, z=-i$, a removable singularity at $z=-1$, and an essential singularity at $z=0$. □

Theorem 5.2 (residue theorem). Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ interior to C. Let $\text{Res}(f, a_k)$ denote the residue of f at a_k, for all $1 \le k \le n$. Then,

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, a_k).$$

Theorem 5.3. If f is analytic everywhere on the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)\, dz = 2\pi i \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Example 5.6 (Dinh's 70 problems). Evaluate the integral

$$\int_{C^+(0,2)} e^{e^{1/z}}\, dz.$$

Solution. Let $w = 1/z$ so $dw/dz = -w^2$. The integral becomes

$$\int_{C^+(0,1/2)} e^{e^w} \cdot \frac{dw}{w^2}.$$

Let $f(w) = e^{e^w}$. By the residue theorem,

$$\int_{C^+(0,1/2)} \frac{f(w)}{w^2}\, dw = 2\pi i \text{Res}(f(w), 0) = 2\pi i e$$

and we are done. □

5.2 Residue Computation Methods

There are three methods for computing residues.

Theorem 5.4 (method 1). Suppose for z near z_0, $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{z - z_0},$$

where $\phi(z)$ is analytic at z_0 and f has a simple pole or a removable singularity at z_0. Then,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0).$$

Proof. Since $\phi(z)$ is analytic at z_0, then by Taylor's theorem, for z near z_0,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \ldots$$

so the Laurent series of $f(z)$ at z_0 is

$$f(z) = \frac{\phi(z)}{z - z_0} = \frac{\phi(z_0) + \phi'(z_0)(z - z_0) + \ldots}{z - z_0} = \frac{\phi(z_0)}{z - z_0} + \phi'(z_0) + \ldots$$

and the result follows. □

Theorem 5.5 (method 2)**.** Suppose for z near z_0, $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $m \geq 1$. Then,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

Proof. It is inferred that f has a pole of order less than or equal to m or a removable point of singularity at z_0. Observe that when $m = 1$, it is just method 1 (recall Theorem 5.4). Using Taylor's theorem again, the series expansion of $\phi(z)$ is the same as before. That is,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z-z_0) + \dots$$

so

$$\begin{aligned}
f(z) &= \frac{\phi(z)}{(z-z_0)^m} \\
&= \frac{1}{(z-z_0)^m}\left[\phi(z_0) + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \dots\right] \\
&= \frac{\phi(z_0)}{(z-z_0)^m} + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \cdot \frac{1}{z-z_0} + \dots
\end{aligned}$$

The result follows. □

Theorem 5.6 (method 3)**.** If $p(z)$ and $q(z)$ are analytic at z_0 and $q(z)$ has a simple zero at z_0 (i.e. $q(z_0) = 0$ but $q'(z_0) \neq 0$), then

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Theorem 5.7 (method 4)**.** If all the above methods fail, use the Laurent series of $f(z)$ and read b_1.

Example 5.7. For the following function $f(z)$, find all of its singularities in $\mathbb{C}$, their types and residues at these points:

$$f(z) = \frac{z^2+1}{z^6+1}.$$

Solution. The singularities of $f(z)$ are the zeros of the denominator z^6+1, that is the 6 points

$$z_k = \exp\left(\frac{i\pi}{6} + \frac{k\pi i}{3}\right),$$

where $0 \leq k \leq 5$. These points are simple zeros of $z^6+1=0$. The points $z_1 = i$ and $z_4 = -i$ are the roots of the equation $z^2+1=0$ (refer to the numerator). Thus, z_1, z_4 are removable and z_0, z_2, z_3, z_5 are simple poles of f.

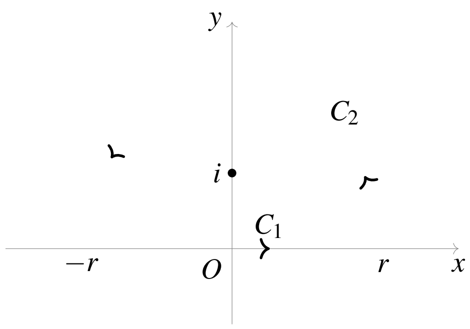
So, the residues of f at z_1, z_4 are 0, whereas the residue of f at z_k for $k = 0,2,3,5$ is equal to $(z_k^2+1)/6z_k^5$. □

Example 5.8 (classic result)**.** Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e}.$$

Solution. We consider
$$f(z) = \frac{e^{iz}}{z^2+1}$$
so the integral $\text{Re}(f(z))$ over the real numbers is the required answer. Let C be the path $C_1 + C_2$, where C_1 and C_2 are parametrised as follows:

$$C_1(t) = t, \text{ where } t \in [-r, r]$$
$$C_2(t) = re^{it}, \text{ where } t \in [0, \pi]$$



By Cauchy's residue theorem,
$$\sum \text{Res}(f(z)) = \frac{1}{2\pi i}\int_C f(z)\, dz.$$
Only one of the two poles of $f(z)$, $z = i$, is inside C as we are considering the upper half of the circle centred at the origin. We have
$$\int_C f(z)\, dz = \int_{C_1} \frac{e^{iz}}{z^2+1}\, dz + \int_{C_2} \frac{e^{iz}}{z^2+1}\, dz.$$
For the integral over C_1, applying the parametrisation,
$$\int_{C_1} \frac{e^{iz}}{z^2+1}\, dz = \int_{-r}^{r} \frac{e^{it}}{t^2+1}\, dt = \int_{-r}^{r} \frac{\cos t}{t^2+1}\, dt + i\int_{-r}^{r} \frac{\sin t}{t^2+1}\, dt.$$
Since $\sin t$ is an odd function, then the integral of $\sin t/(t^2+1)$ is zero. Hence,
$$\int_{C_1} \frac{e^{iz}}{z^2+1}\, dz = \int_{-r}^{r} \frac{\cos t}{t^2+1}\, dt.$$
As for the integral over C_2, applying the parametrisation,
$$\int_{C_2} \frac{e^{iz}}{z^2+1}\, dz = \int_0^{\pi} \frac{\exp\left(ire^{it}\right)}{r^2e^{i2t}+1} \cdot ire^{it}\, dt.$$
By applying Euler's Formula,
$$\begin{aligned}
\int_0^{\pi} \frac{\exp\left(ire^{it}\right)}{r^2e^{i2t}+1} \cdot ire^{it}\, dt &= ir\int_0^{\pi} \frac{e^{i(t+r\cos t)}e^{-r\sin t}}{r^2e^{i2t}+1}\, dt \\
\left|\int_0^{\pi} \frac{\exp\left(ire^{it}\right)}{r^2e^{i2t}+1} \cdot ire^{it}\, dt\right| &= r\int_0^{\pi} \frac{e^{-r\sin t}}{|r^2e^{i2t}+1|}\, dt \\
&\leq \frac{r}{r^2-1}\int_0^{\pi} e^{-r\sin t}\, dt
\end{aligned}$$
Let the radius r of the semicircle tend to infinity so it is then clear that
$$\int_{C_2} \frac{e^{iz}}{z^2+1}\, dz = 0.$$

Therefore, by Cauchy's residue theorem (rearrange the equation at the start of our solution),

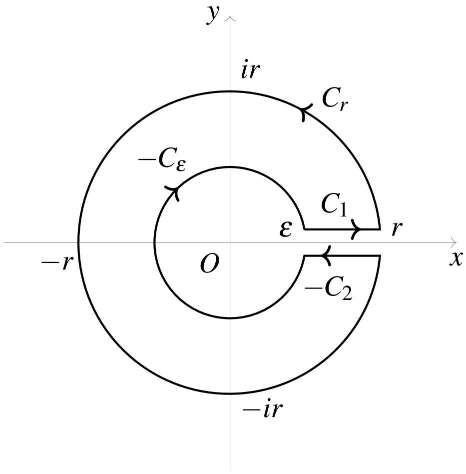
$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\, dt = 2\pi i \cdot \frac{e^{i^2}}{2i} = \frac{\pi}{e}.$$

□

Example 5.9 (branch cut). Prove that

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2+5x+6}\, dx = \pi(\sqrt{3}-\sqrt{2}).$$

Solution. Note that 0 is a branch point of $\sqrt{z}$. So, $\sqrt{z}$ has a branch cut along the positive real axis, i.e. $[0,\infty)$. Hence, $\sqrt{z}$ is analytic on $\mathbb{C}\setminus[0,\infty)$. We adopt the following keyhole contour.



One should think of the above contour as having ε so small that C_1 and C_2 are essentially on the x-, or rather, real axis. Let the region the contour encloses be D. Then, we shall consider the integral over the boundary (this is denoted by ∂D). That is,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2+5z+6}\, dz.$$

By Cauchy's residue theorem,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2+5z+6}\, dz = 2\pi i \left[\frac{\sqrt{z}}{2z+5}\bigg|_{z=-3} + \frac{\sqrt{z}}{2z+5}\bigg|_{z=-2} \right] = 2\pi\left(\sqrt{3}-\sqrt{2}\right).$$

We now evaluate the contour integral by considering the different *pieces*.

$$\begin{aligned}\int_{\partial D} \frac{\sqrt{z}}{z^2+5z+6}\, dz &= \int_{C_r} - \int_{C_\varepsilon} + \int_{C_1} - \int_{C_2} \\ &= \int_{C_r} - \int_{C_\varepsilon} + 2\int_{\varepsilon}^{r} \frac{\sqrt{x}}{x^2+5x+6}\, dx\end{aligned}$$

By the estimation lemma,

$$\left| \int_{C_r} f(z)\, dz \right| \le 2\pi r \cdot \frac{\sqrt{r}}{r^2-5r-6}$$

which tends to 0 as r tends to infinity. In a similar fashion, one can prove that

$$\left| \int_{C_\varepsilon} f(z)\, dz \right| \le 2\pi\varepsilon \cdot \frac{\sqrt{\varepsilon}}{6-5\varepsilon-\varepsilon^2}$$

which tends to 0 too as ε tends to 0. As such,

$$2\pi\left(\sqrt{3}-\sqrt{2}\right) = 2\int_0^{\infty} \frac{\sqrt{x}}{x^2+5x+6}\, dx$$

and the result follows. □

Example 5.10 (pizza contour). Prove that for $n \geq 2$,

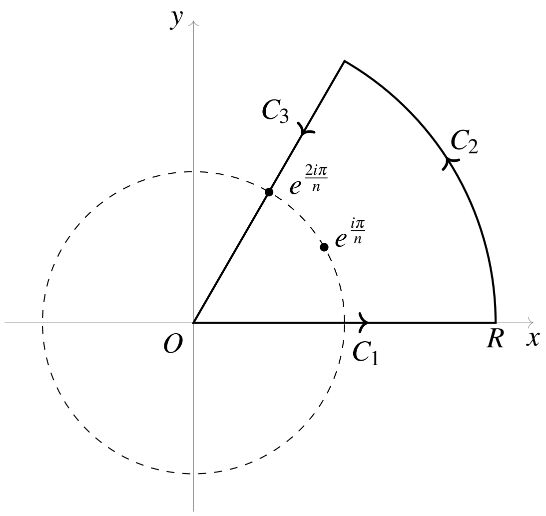
$$\int_0^\infty \frac{1}{x^n+1}\,dx = \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}.$$

Solution. Let

$$f(z) = \frac{1}{z^n+1}$$

and the required integral to be I. Consider the following parametrisation (informally known as the *pizza contour*):

$$\begin{aligned} C_1(t) &:= t, \text{ where } 0 \leq t \leq R \\ C_2 &: |z| = R \text{ (note that the angle subtended by the arc does not matter)} \\ C_3(t) &:= (R-t)\exp\left(\frac{2\pi i}{n}\right), \text{ where } 0 \leq t \leq R \end{aligned}$$



By defining C to be the contour, it is clear that

$$\int_C \frac{1}{z^n+1}\,dz = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

Only the pole $z = e^{i\pi/n}$ is in C so by the residue theorem,

$$\int_C \frac{1}{z^n+1}\,dz = 2\pi i \operatorname*{Res}_{z=e^{i\pi/n}} f(z) = -\frac{2\pi i}{n}\exp\left(\frac{i\pi}{n}\right).$$

We focus on C_1.

$$\int_{C_1} \frac{1}{z^n+1}\,dz = \int_0^R \frac{1}{t^n+1}\,dt.$$

Letting R tend to infinity, and since t is a dummy variable, it is easy to see that

$$\int_{C_1} \frac{1}{z^n+1}\,dz = \int_0^\infty \frac{1}{x^n+1}\,dx = I.$$

For C_2, by the triangle inequality, $|z^n+1| \geq ||z^n| - |-1|| = |R^n - 1|$. Hence,

$$\left|\int_{C_2} \frac{1}{z^n+1}\,dz\right| \leq \int_{C_2} \frac{1}{R^n-1}\,dz = \frac{c\pi R}{R^n-1}.$$

Letting R tend to infinity, we see that the integral over C_2 is zero. Earlier, we mentioned that the angle subtended by the arc does not matter and we affirm this statement here.

The integral over C_3 is more complicated. Using the substitution

$$z = (R-t)\exp\left(\frac{2\pi i}{n}\right),$$

we see that

$$\int_{C_3} \frac{1}{z^n+1}\,dz = -\exp\left(\frac{2\pi i}{n}\right)\int_0^R \frac{1}{(R-t)^n+1}\,dt.$$

This calls for a substitution, say $u = R-t$. Hence, the integral over C_3 becomes

$$-\exp\left(\frac{2\pi i}{n}\right)\int_0^R \frac{1}{u^n+1}\,du \xrightarrow{R\to\infty} -\exp\left(\frac{2\pi i}{n}\right)\int_0^\infty \frac{1}{x^n+1}\,dx = -I\exp\left(\frac{2\pi i}{n}\right).$$

To conclude,

$$\begin{aligned}
-\frac{2\pi i}{n}\exp\left(\frac{i\pi}{n}\right) &= I\left[1-\exp\left(\frac{2\pi i}{n}\right)\right] \\
I &= -\frac{2\pi i}{n}\cdot\frac{\exp\left(\frac{i\pi}{n}\right)}{1-\exp\left(\frac{2\pi i}{n}\right)} \\
&= -\frac{2\pi i}{n}\cdot\frac{\exp\left(\frac{i\pi}{n}\right)}{\exp\left(\frac{i\pi}{n}\right)\exp\left(-\frac{i\pi}{n}\right)-\exp\left(\frac{i\pi}{n}\right)\exp\left(\frac{i\pi}{n}\right)} \\
&= \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}
\end{aligned}$$

so we have finally derived this beautiful result. □

Example 5.11. Prove that

$$\int_0^{2\pi}\frac{1}{5+3\sin\theta}\,d\theta = \frac{\pi}{2}.$$

Solution. Set $z = e^{i\theta}$ so $\sin\theta = (z-z^{-1})/2i$. The integral becomes

$$\int_{|z|=1}\frac{1}{5+3\left(\frac{z-z^{-1}}{2i}\right)}\cdot\left(-\frac{i}{z}\right)\,dz = 2\int_{|z|=1}\frac{1}{3z^2+10iz-3}\,dz.$$

Let

$$f(z) = \frac{1}{3z^2+10iz-3}.$$

It has two simple poles $z_1 = -i/3$ and $z_2 = -3i$. The first one is interior to the circle $|z| = 1$ so we shall consider this. By the residue theorem, the answer is

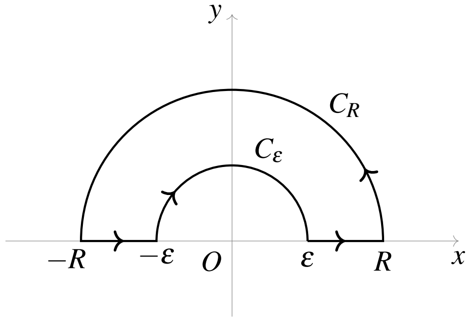
$$2\cdot 2\pi i\cdot\frac{1}{3(z_1+3i)} = \frac{\pi}{2}.$$

□

Example 5.12. Prove that

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}.$$

Solution. We consider the following contour.



Define

$$f(z) = \frac{(\log z)^2}{z^2+1}$$

and in our contour, say C, we let $0 < \varepsilon < 1 < R$. $\log z$ denotes the branch of the logarithm function defined on $\{z \in \mathbb{C} : -\pi/2 < \arg z < 3\pi/2\}$. Hence, it is clear that

$$\int_C = \int_{C_R} + \int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_\varepsilon^R .$$

By the residue theorem,

$$\int_C \frac{(\log z)^2}{z^2+1}\,dz = \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}.$$

Now, let us focus on C_R. We use the estimation lemma to help us.

$$\left|\int_{C_R}\right| \le \pi R \cdot \frac{(\log R + i\theta)^2}{R^2-1}$$

which tends to 0 as R tends to infinity. In a similar fashion, one can show that

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} = 0.$$

As such,

$$\begin{aligned}
\int_C \frac{(\log z)^2}{z^2+1}\,dz &= \int_{-R}^{-\varepsilon} \frac{(\log z)^2}{z^2+1}\,dz + \int_\varepsilon^R \frac{(\log z)^2}{z^2+1}\,dz \\
&= \int_\varepsilon^R \frac{(\log(-z))^2}{z^2+1}\,dz + \int_\varepsilon^R \frac{(\log z)^2}{z^2+1}\,dz \\
&= \int_\varepsilon^R \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2+1}\,dz
\end{aligned}$$

Now we set R to tend to infinity and ε to tend to 0. Also, we computed the value of the integral over C earlier so putting everything together,

$$\begin{aligned}
-\frac{\pi^3}{4} &= \int_0^\infty \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2+1}\,dz \\
&= -\pi^2 \int_0^\infty \frac{1}{z^2+1}\,dz + 2i\pi \int_0^\infty \frac{\log z}{z^2+1}\,dz + 2\int_0^\infty \frac{(\log z)^2}{z^2+1}\,dz \\
&= -\frac{\pi^3}{2} + 2i\pi \int_0^\infty \frac{\log z}{z^2+1}\,dz + 2\int_0^\infty \frac{(\log z)^2}{z^2+1}\,dz
\end{aligned}$$

Lastly, we will show that

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx = 0.$$

Using the substitution $u = 1/x$,

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx = \int_0^\infty \frac{-\log u}{(1/u)^2+1}\cdot\left(-\frac{1}{u}\right)^2 du = -\int_0^\infty \frac{\log u}{u^2+1}\,du$$

and the result follows. □

We have the following beautiful corollary:

Corollary 5.1. Let
$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx.$$
Then for all $n \geq 1$, I_{2n} satisfies the recurrence relation
$$I_{2n} = \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}} - \frac{1}{2}\sum_{k=1}^{n}\binom{2n}{2k}(-1)^k \pi^{2k} I_{2n-2k}.$$

It is not surprising that we only discuss the integrals I_{2n} instead of I_{2n+1} because
$$\int_0^\infty \frac{(\log x)^{2n+1}}{x^2+1}\,dx = 0$$
for all $n \geq 0$ by performing the substitution $u = 1/x$.

The above formula is also equivalent to the following by using the Dirichlet beta function:

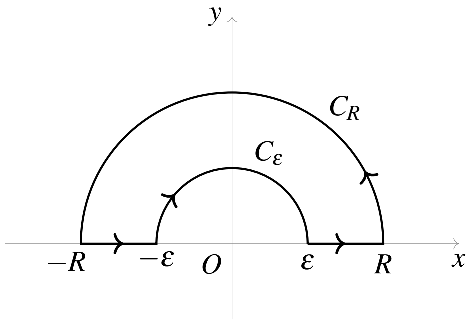
Definition 5.4 (Dirichlet beta function)**.** Define the Dirichlet beta function to be
$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

Corollary 5.2. Let
$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx.$$
Then for all $n \geq 1$, $I_{2n} = 2(2n)!\beta(2n+1)$.

Proof.
$$\begin{aligned}
\int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx &= \int_0^1 \frac{(\log u)^{2n}}{u^2+1}\,du \quad \text{using } u = \frac{1}{x} \\
\int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx &= 2\int_0^1 \frac{(\log x)^{2n}}{x^2+1}\,dx \\
&= 2\int_0^1 (-1)^k \sum_{k=0}^{\infty} (\log x)^{2n} x^{2k}\,dx \quad \text{using integration by parts} \\
&= 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}
\end{aligned}$$
and the result follows. □

Example 5.13 (Dinh's 70 problems)**.** Show that
$$\int_0^\infty \frac{x^\alpha}{(1+x^2)^2}\,dx = \frac{\pi(1-\alpha)}{2\cos\left(\frac{\pi\alpha}{2}\right)}$$
for $-1 < \alpha < 3$, $\alpha \neq 1$. What happens if $\alpha = 1$?

Solution. We consider the following contour.



Here, $0 < \varepsilon < R$ and C_R and C_ε denote the upper-half of the semicircle of radius R and ε respectively. So,

$$\int_C f(z)\,dz = \int_{C_R} + \int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_{\varepsilon}^{R}.$$

By the residue theorem, it is clear that

$$\int_C f(z)\,dz = \frac{i\pi e^{ia\pi/2}}{2}.$$

It is clear that

$$\lim_{R\to\infty} \int C_R = 0 \text{ and } \lim_{\varepsilon\to 0} \int C_\varepsilon = 0.$$

So,

$$\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} = \int_C f(z)\,dz$$

and the result follows. □

Example 5.14 (MA5217 AY24/25 Sem 1 Homework 1). Compute the following integrals using the residue formula:

$$\int_{-\infty}^{\infty} \frac{x-4}{(x^2-4x+5)(x^2+4)}\,dx \quad \text{and} \quad \int_0^{\infty} \frac{x^2}{(x^2+4)^2}\,dx$$

Solution. We deal with the first integral. Note that $z = 2+i, z = 2-i, z = 2i, z = -2i$ are simple poles of the integral. Let $C = C_1 + C_2$ be the upper half of the semicircle of radius R centred at the origin on the complex plane, where C_1 is the diameter and C_2 is the arc.

So,

$$C_1 = \{z = x+iy \in \mathbb{C} : -R \le x \le R\}$$
$$C_2 = \left\{z = x+iy \in \mathbb{C} : z = Re^{i\theta}, 0 \le \theta \le \pi\right\}$$

Let $f(z)$ denote the integrand. We are only interested in the poles interior and on the boundary of C. By the residue theorem,

$$\int_C f(z)\,dz = 2\pi i \sum \operatorname{Res}(f(z), z = z_k) = 2\pi i \left(\frac{2i}{13}\right) = -\frac{4\pi}{13}$$

Hence,

$$\int_{C_1} f(z)\,dz = \int_{-R}^{R} \frac{x-4}{(x^2-4x+5)(x^2+4)}\,dx.$$

Letting $R \to \infty$, we see that we obtain the original integral. Also,

$$\begin{aligned}\left|\int_{C_2} f(z)\, dz\right| &= \left|\int_0^\pi \frac{Re^{i\theta} \cdot iRe^{i\theta}}{\left(R^2e^{2i\theta} - 4Re^{i\theta} + 5\right)\left(R^2e^{2i\theta} + 4\right)}\, d\theta\right| \\ &= \left|\int_0^\pi \frac{R^2}{\left(R^2e^{2i\theta} - 4Re^{i\theta} + 5\right)\left(R^2e^{2i\theta} + 4\right)}\, d\theta\right|\end{aligned}$$

which is equal to 0 by the triangle inequality. Hence, the answer is $-4\pi/13$.

For the second integral, we note that the function is even. Letting g denote the integrand, we have

$$\int_0^\infty g(z)\, dz = \frac{1}{2}\int_{-\infty}^\infty g(z)\, dz.$$

We consider the same contour as the previous part, acknowledging that $z = \pm 2i$ are double poles of g. So, it follows that the sum of residues is $-\pi/8$, and by some tedious computation, the integral evaluates to $\pi/8$.

To compute the residue of the double pole $z = 2i$, we use the formula

$$\lim_{z \to 2i} \frac{d}{dz}\left((z-2i)^2 g(z)\right)$$

which is quite easy. □

Example 5.15 (Dinh's 70 problems)**.** Evaluate

$$\int_{-\infty}^\infty \frac{x \sin x}{(1+x^2)^2}\, dx.$$

Solution. Let $f(z) = \dfrac{ze^{iz}}{(1+z^2)^2}$. Define C_1 to be the upper half of the semicircle of radius R centred at the origin and C_2 to be the real axis bounded by $\pm R$. So, C_1 can be parametrised using $z = Re^{it}$ for $t \in [0,\pi]$, whereas C_2 can be parametrised using $z = t$ for $t \in [-R,R]$. Let $C = C_1 \cup C_2$. By the residue theorem,

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}(f(z), i).$$

Note that

$$\operatorname{Res}(f(z), i) = \lim_{z \to i} \frac{d}{dz}\left(\frac{ze^{iz}}{(z+i)^2}\right) = \frac{1}{4e}.$$

Hence,

$$\int_C f(z)\, dz = \frac{i\pi}{2e}.$$

Now,

$$\lim_{R\to\infty}\left|\int_{C_1} f(z)\, dz\right| = \lim_{R\to\infty}\left|R^2\int_0^\pi \frac{1}{\left(1+R^2e^{2i\theta}\right)^2}\, d\theta\right| = 0.$$

Lastly, we work with C_2. So, we have

$$\lim_{R\to\infty}\int_{C_2} f(z)\, dz = \lim_{R\to\infty}\int_{-R}^R \frac{t\sin t}{\left(1+t^2\right)^2}\, dt = \int_{-\infty}^\infty \frac{x\sin x}{\left(1+x^2\right)^2}\, dt.$$

It follows that the answer is $\pi/2e$. □

Example 5.16 (Dinh's 70 problems)**.** Show that for any $0 < a < 1$,

$$\int_0^\infty \frac{x^a}{x(1+x)}\, dx = \frac{\pi}{\sin(a\pi)}.$$

Solution. Let $t = x/(1+x)$, so

$$x = \frac{t}{1-t} \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{(1-t)^2}.$$

The integral becomes

$$\begin{aligned}
\int_0^1 t^{a-1}(1-t)^{-a}\,dt &= B(a, 1-a) \quad \text{by definition of beta function} \\
&= \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)} \quad \text{by relationship with gamma function} \\
&= \Gamma(a)\Gamma(1-a)
\end{aligned}$$

and the result follows by Euler's reflection formula. □

Chapter 6
Further Properties of Holomorphic Functions

6.1
Properties of Holomorphic and Harmonic Functions

Definition 6.1 (extended complex plane)**.** Define

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\} \quad \text{to be the extended complex plane.}$$

Theorem 6.1 (Cauchy's estimate)**.** Let $f \in H(\Omega)$ and let $\overline{D}(z_0, r) \subseteq \Omega$. Then, for all $n = 0, 1, 2, \ldots$,

$$|a_n| \leq r^{-n} \sup_{|z-z_0|=r} |f(z)|.$$

Example 6.1 (Dinh's 70 problems)**.** Suppose $f(z)$ is an odd function and holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfies

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^2} \text{ for all } z \neq 0.$$

Prove that

$$f(z) = \frac{a_{-1}}{z} + a_1 z \quad \text{for all } z \in \mathbb{C} \setminus \{0\} \text{ where } a_{-1}, a_1 \in \mathbb{C}.$$

Solution. Since f is holomorphic in $\mathbb{C} \setminus \{0\}$, its Laurent series representation about $z = 0$ is

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

f is odd implies $f(-z) = -f(z)$, so

$$f(z) = \ldots + \frac{a_{-3}}{z^3} + \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \ldots.$$

Note that for $|z| \leq 1$, we have $|z^2 f(z)| \leq |z|^4 + 1 \leq 2$ and

$$z^2 f(z) = \ldots + \frac{a_{-3}}{z} + a_{-1} z + a_1 z^3 + a_3 z^5 + \ldots$$

so it forces $a_{-3}, a_{-5}, \ldots, a_5, a_7, \ldots = 0$. The result follows. □

Here is an alternative solution.

Solution. Again, write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

By Cauchy's estimate (Theorem 6.1), if $|f(z)| \leq M$, we have

$$|f^{(k)}(a)| \leq \frac{k! M}{R^k}.$$

So, for $|z| \leq R$, we have

$$|a_k| \leq \frac{1}{R^k} \left(\frac{1}{R^2} + R^2 \right).$$

For $k \geq 3$, $\lim_{r\to\infty} |a_k| = 0$ and for $k \leq -3$, $\lim_{r\to 0} |a_k| = 0$. So, $a_k = 0$ for all $|k| \geq 3$. Hence,

$$f(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2.$$

Using the fact that f is odd, the result follows. □

Example 6.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in $D(0,1)$ and assume that the integral

$$A := \iint_{D(0,1)} |f'(z)|^2 \, dxdy < \infty.$$

(a) Express A in terms of the coefficients a_n.

(b) Prove that

$$|f(z) - f(0)| \leq \sqrt{\frac{A}{\pi} \ln\left(\frac{1}{1-|z|^2}\right)}$$

for all $z \in D(0,1)$.

Solution.

(a) Note that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

We shall parametrise z using polar coordinates. Let $z = re^{i\theta}$. As such,

$$\begin{aligned}
\iint_{D(0,1)} \left|f'(z)\right| \, dxdy &= \int_0^1 \int_0^{2\pi} \left| \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 r \, dr d\theta \\
&= \int_0^1 \int_0^{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m n a_m a_n r^{m+n-1} e^{i(m+n-2)\theta} \, dr d\theta \\
&= 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} \, dr \\
&= \pi \sum_{n=1}^{\infty} n |a_n|^2
\end{aligned}$$

(b) Note that $f(0) = 0$ and the RHS can be written as

$$\sqrt{\ln\left(\frac{1}{1-|z|^2}\right) \sum_{n=1}^{\infty} n |a_n|^2}.$$

Starting with the LHS,

$$|f(z)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| \sum_{n=1}^{\infty} \left(\sqrt{n} a_n\right) \left(\frac{z^n}{\sqrt{n}}\right) \right| \leq \sqrt{\left(\sum_{n=1}^{\infty} n |a_n^2|\right) \left(\sum_{n=1}^{\infty} \frac{z^{2n}}{n}\right)}$$

where we applied the Cauchy-Schwarz inequality at the end. The result follows. □

Theorem 6.2 (identity theorem)**.** If two holomorphic functions f and g coincide on some set $E \subseteq D$ containing at least one limit point in D, then $f(z) = g(z)$ everywhere in D.

Example 6.3. Does there exist an entire function with the property that for $n \in \mathbb{N}$,

$$f\left(\frac{1}{n}\right) = \frac{n^4}{1+n^4}?$$

Solution. Replacing n with $1/z$, we consider the function

$$g(z) = \frac{1}{z^4+1}.$$

Note that the roots of the equation $z^4+1=0$ can be found as follows. As $z^4=-1=e^{i\pi+2k\pi i}$, then

$$z = \exp\left(i\pi\cdot\frac{2k+1}{4}\right),$$

where $k=0,1,2,3$. We denote the roots by p_n, where $0\le n\le 3$. Obviously, $g(z)$ is holomorphic outside the 4 points p_n. By our hypothesis, $f(z)=g(z)$ for $z=1,1/2,1/3,\ldots$ and both f and g are defined on $\Omega=\mathbb{C}\setminus\{p_0,p_1,p_2,p_3\}$. The sequence $1,1/2,1/3,\ldots$ converges to 0 which is inside Ω, so this sequence is not discrete in Ω. We conclude that $f=g$ in Ω by the identity theorem.

On the other hand, the function f is entire and bounded near p_n but g is not bounded near these points. We have obtained a contradiction so such a function f does not exist. □

Example 6.4. Do there exist functions f and g that are holomorphic at $z=0$ and that satisfy

(a) $f(1/n)=f(-1/n)=1/n^2$, where $n\in\mathbb{N}$;

(b) $g(1/n)=g(-1/n)=1/n^3$, where $n\in\mathbb{N}$?

Solution.

(a) Yes, $f(z)=z^2$.

(b) We prove that such a function g does not exist in a neighbourhood of 0. Suppose on the contrary that g exists. Define $h(z)=z^3$ and $l(z)=-z^3$. We have $g(z)=h(z)$ on a non-discrete sequence $z=1,1/2,1/3,\ldots$ which converges to 0, and 0 is in the domain of g. By the identity theorem, $g(z)=h(z)$. In a similar fashion, by considering the sequence $z=-1,-1/2,-1/3,\ldots$, we obtain $g(z)=l(z)$. Hence, $h(z)=l(z)$, implying that $z^3=-z^3$, so $z^3=0$. However, this is a contradiction. □

Example 6.5. Show that there is no holomorphic function f in $\mathbb{C}$ such that

$$f\left(\frac{1}{n}\right)=\frac{ne^{-2/n}}{n+1} \text{ for all } n\in\mathbb{N}.$$

Solution. Suppose on the contrary that such a function exists. Consider

$$g(z)=\frac{e^{-2z}}{z+1}.$$

This function is defined for all $z\in\mathbb{C}$ except at $z=-1$. By the hypothesis, this function is equal to f on the sequence $1/n$ which is not discrete on $\mathbb{C}\setminus\{-1\}$ and so, $f=g$ on $\mathbb{C}\setminus\{-1\}$. However, this is a contradiction. □

Example 6.6 (MA5217 AY24/25 Sem 1 Homework 1). Show that the function $h(z)=\sin(\sin z)+\sin|z|^2$ is not holomorphic in any domain of $\mathbb{C}$.

Solution. Note that

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}.$$

We let $z=x+iy, x,y\in\mathbb{R}$ and note that $|z|^2=x^2+y^2$. Hence,

$$h(z)=\sin\left(\frac{e^{iz}}{2i}\right)\cos\left(\frac{e^{-iz}}{2i}\right)-\cos\left(\frac{e^{iz}}{2i}\right)\sin\left(\frac{e^{-iz}}{2i}\right)+\sin\left(|z|^2\right)$$

By the Looman-Menchoff theorem, it suffices to prove that h does not satisfy the Cauchy-Riemann equations, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

The computation is tedious so we skip the details. □

Example 6.7 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all holomorphic functions $f(z)$ in $\mathbb{C}\setminus\{1\}$ such that

$$\mathrm{Res}(f,1) = 1, \quad \lim_{z\to\infty}(f(z)-z) = 2, \quad \lim_{z\to 1}|z-1|^{4/3}f(z) = 0.$$

Solution. We claim that

$$f(z) = z+2+\frac{1}{z-1}.$$

By the second condition, we infer that

$$f(z) = z+2+\sum_{n=1}^{\infty}\frac{1}{(az+b)^n}.$$

By the third condition, we infer that

$$\lim_{z\to 1}(z-1)^{4/3}\sum_{n=1}^{\infty}\frac{1}{(az+b)^n} = 0$$

which implies we have to restrict the index of the infinite sum to $n=1$ instead of $nin\mathbb{N}$. Hence,

$$f(z) = z+2+\frac{1}{az+b}.$$

We see that $1/(az+b)$ has a simple pole at $z=-b/a$ but the first condition implies that $z=1$ is a pole, so $a=-b$. Since the value of the residue at $z=1$ is 1, then $a=1$, so

$$f(z) = z+2+\frac{1}{z-1}.$$

□

Example 6.8. Let f and g be entire functions and suppose that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that $f(z) = cg(z)$ for some constant $c \in \mathbb{C}$.

Solution. First, we assume that g is identically equal to zero. Then, the result immediately follows. Now, we consider the case where g is not identically equal to zero. Define $h(z) = f(z)/g(z)$ on $\mathbb{C}$ excluding the set of zeros of g. As such, h is holomorphic outside the zeros of g and $|h(z)| \leq 1$. As h is bounded and entire, the result follows by Liouville's theorem. □

Definition 6.2 (analytic function)**.** $f:\Omega\to\mathbb{C}$ is analytic if for any $z_0\in\Omega$, we can write

$$f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n,$$

where $a_n\in\mathbb{C}$ for all $n\in\mathbb{Z}_{\geq 0}$ and the series is convergent to $f(z)$ for z in a neighbourhood of z_0.

Example 6.9 (Dinh's 70 problems)**.** Suppose f is entire and $f(z)$ is real iff z is real. Prove that f has at most one zero.

Solution. Suppose $f(z) = 0$, then it implies that z is real. Suppose on the contrary that f has a zero $x_0 \in \mathbb{R}$ with a multiplicity $m \geq 2$. Then, we can write $f(z)$ as the following power series:

$$f(z) = (z - x_0)^m (a_0 + a_1(z - x_0) + a_2(z - x_0)^2 + \ldots)$$

Here, $a_0 \neq 0$. Note that

$$a_0 = \lim_{z \to x_0} \frac{f(z)}{(z - x_0)^m}$$

so for any $z \in \mathbb{R} \setminus \{x_0\}$, we have $\dfrac{f(z)}{(z-x_0)^m} \in \mathbb{R}$. Hence, $a_0 \in \mathbb{R}$. Now, write $z = x_0 + \varepsilon e^{i\theta}$, so we are considering the general case when $z \in \mathbb{C}$. It is clear that

$$f(z) = \varepsilon^m e^{mi\theta} (a_0 + a_1 \varepsilon e^{i\theta} + a_2 \varepsilon^2 e^{2i\theta} + \ldots).$$

Define $g(\theta) = \operatorname{Im}(e^{mi\theta}(a_0 + \varepsilon u(\theta, \varepsilon) + i\varepsilon v(\theta, \varepsilon)))$, where u, v are real and continuous functions and ε is sufficiently small. Note that $g(\pi/2m)g(3\pi/2m) < 0$ so by the intermediate value theorem, there exists $\theta' \in (\pi/2m, 3\pi/2m)$ such that $g(\theta_0) = 0$. So, $f(x_0 + \varepsilon e^{i\theta_0}) \in \mathbb{R}$. But because $m \geq 2$, it implies that $x_0 + \varepsilon e^{i\theta_0} \notin \mathbb{R}$, so we reached a contradiction. The result follows. □

The next theorem summarises a list of important properties regarding holomorphic functions.

Theorem 6.3. Let Ω be an open and simply-connected domain in $\mathbb{C}$ and let $f \in H(\Omega)$. Then,

- $f \in C^\infty(\Omega)$ and f satisfies the Cauchy-Riemann equations on Ω.
- **Cauchy-Goursat theorem:** If γ is a piecewise differentiable simple closed curve in Ω, then

$$\int_\gamma f(z)\, dz = 0.$$

- **Cauchy's integral formula:** If γ is an anticlockwise oriented and piecewise differentiable simple closed curve in Ω, then for any a interior to γ,

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz.$$

  Moreover, Cauchy's differentiation formula applies here.
- f is analytic, i.e. for any $z_0 \in \Omega$, one can write for $z \in D(z_0, r)$ with $\overline{D(z_0, r)} \subseteq \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

  where

$$c_n = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz,\ n = 0, 1, 2, \ldots$$

Example 6.10. Let f be a holomorphic function in the unit disc $\mathbb{D}$ such that $|f(z)| < 1$ for $z \in \mathbb{D}$. Show that $|f''(0)| \leq 2$. Give an example of such a map with $f''(0) = 2$.

Solution. We use Cauchy's Differentiation Formula. Note that

$$f''(0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{z^3}\, dz.$$

Here, we let C be such that $|z| = r$, where $r < 1$ (i.e. C contains all points interior to the circle of radius 1 centred at the origin). Using the parametrisation $z = re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we see that

$$|f''(0)| \leq \frac{1}{\pi} \left| \int_0^{2\pi} \frac{f(re^{i\theta})}{r^3 e^{3i\theta}} \cdot ire^{i\theta}\, d\theta \right| \leq \frac{1}{\pi r^2} \left| \int_0^{2\pi} f(re^{i\theta})\, d\theta \right| \leq \frac{2}{r^2}.$$

Hence, letting r tend to 1, the result follows.

For the later part of the question, we need to find a map such that $f''(0) = 2$. Well, consider

$$\int_{|z|=r} \frac{f(z)}{z^3}\,dz = 2\pi i$$

for which an obvious answer is $f(z) = z^2$. □

Example 6.11 (Dinh's 70 problems). Determine all complex holomorphic functions f defined on the unit disk which satisfy

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for $n = 2, 3, 4, \ldots$

Solution. Let $g(z) = f''(z) + f(z)$, so g is holomorphic on $\mathbb{D}$. We have $g(1/n) = 0$ for all $n = 2, 3, 4, \ldots$ and since $\lim_{n\to\infty} 1/n = 0 \in \mathbb{D}$, it follows that $g(z) = 0$ on $\mathbb{D}$. As such, $f(z) = -f''(z)$ on $z \in \mathbb{D}$. One can use Maclaurin Series to dedcue that $f(z) = f(0)\cos z + f'(0)\sin z$. □

Theorem 6.4 (Casorati-Weierstrass theorem). Let f have an isolated essential singularity at z_0. Then, for any $w \in \mathbb{C}$, $f(z)$ comes arbitrarily close to w in every deleted neighbourhood of z_0. That is, for any $\delta > 0$, $f(D'(z_0, \delta))$ is a dense subset of $\mathbb{C}$.

Proof. Suppose on the contrary that for some $\delta > 0$, $f(D'(z_0, \delta))$ is not dense in $\mathbb{C}$. Then, there exists $w \in \mathbb{C}$ and $\varepsilon > 0$ such that

$$D(w, \varepsilon) \cap f(D'(z_0, \delta)) = \varnothing.$$

For $z \in D'(z_0, \delta)$, write

$$g(z) = \frac{1}{f(z) - w}.$$

Then, g is bounded and holomorphic on $D'(z_0, \delta)$, so g has a removable singularity at z_0. Let m be the order of the zero of g at z_0. If $g(z_0) \neq 0$, set $m = 0$. Otherwise, write $g(z) = (z - z_0)^m g_1(z)$, where g_1 is holomorphic and does not vanish on $D(z_0, \delta)$. Hence,

$$(z - z_0)^m g_1(z) = \frac{1}{f(z) - w}.$$

Thus, we can write $f(z)$ as

$$f(z) = w + \frac{g_2(z)}{(z - z_0)^m},$$

where $g_2(z) = 1/g_1(z)$ is a holomorphic function on $D(z_0, \delta)$. Thus, f has a removable singularity $(m = 0)$ or a pole $(m \neq 0)$ at z_0, which is a contradiction. □

Definition 6.3. A meromorphic function in D is holomorphic on all D, except on a set of isolated points which are poles. Also, they can be written in the form $f = u/v$, where $u, v \in H(D)$ and $v \neq 0$, and they do not have a common zero.

6.2
The Argument Principle and Rouché's Theorem

Theorem 6.5 (argument principle). Let $f \in H(\Omega)$ and γ be a positively oriented, piecewise differentiable, simple closed contour in Ω such that all points interior to γ belong to Ω. Suppose f has no zero on γ. The zeros of f inside γ are $a_1, a_2, \ldots, a_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are their respective multiplicites. Then,

$$\sum_{j=1}^{n} \alpha_j = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Example 6.12 (Dinh's 70 problems). Evaluate the integral

$$\int_{|z|=2} \frac{f'(z)}{f(z)}\, dz,$$

where $f(z) = \dfrac{\sin z \cos z}{z^7 - z^5 + z^3 - z}$, and $|z| = 2$ is positively oriented.

Solution. We use the argument principle. The answer is $2\pi i(Z - P)$, where Z and P are to be determined. Here, Z and P refer to the respective number of zeros and poles in the circle $|z| = 2$. To calculate Z, set $\sin z \cos z = 0$, so $z = -\pi/2, 0, \pi/2$. Hence, $Z = 3$. To calculate P, set $z^7 - z^5 + z^3 - z = 0$, so $z(z^4 + 1)(z + 1)(z - 1) = 0$. The solutions to $z^4 + 1 = 0$ are $z = e^{i\pi/4}, e^{-i\pi/4}, e^{3i\pi/4}, e^{-3i\pi/4}$. As such, $P = 7$, so the required answer is $-8\pi i$. □

Theorem 6.6 (Rouché's theorem). Let $f, g \in H(\Omega)$ and γ be a piecewise differentiable simple closed curve such that all the points interior to γ are contained in Ω. Assume that

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \gamma$. Then, f and g have the same number of zeros (counting multiplicity) inside γ.

Example 6.13 (Dinh's 70 problems). Determine the number of zeros of $e^{z^2} - 3z^4$ in the unit disk.

Solution. For $|z| = 1$ (i.e. on the boundary of the unit disk), $|e^{z^2}| \leq e \leq 3 = 3|z^4|$ so it follows by Rouché's theorem that there are 4 zeros. □

Example 6.14 (Dinh's 70 problems). Let N_k be the number of roots (counting multiplicity) in the disk $D(0,k) = \{|z| < k\}$ of the equation

$$z^6 - 5z^2 + 10 = 0.$$

For each positive integer k, determine N_k.

Solution. $N_1 = 0$; now consider the case when $k \geq 2$. On $|z| = 2$, $|5z^2 - 10| \leq 5|z|^2 + 10 = 30 \leq 2^6 = |z|^6$, so by Rouché's's theorem, $N_k = 6$ for $k \geq 2$. □

Example 6.15. Let $r > 0$. Prove that for n sufficiently large, the polynomial

$$1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!}$$

has no root in $D(0,r)$.

Solution. Fix an $r > 0$. Define $f(z) = e^z$ and $g_n(z)$ to be the polynomial above. For z on $\overline{D(0,r)}$, i.e. $|z| = r$,

$$|f(z) - g_n(z)| = \left| \sum_{k \geq n+1} \frac{z^k}{k!} \right| \leq \sum_{k \geq n+1} \frac{r^k}{k!}.$$

We note that as $n \to \infty$, the sum on the right tends to zero. For n large enough, the last sum on the right is smaller than e^{-r}. On the other hand, by setting $z = x + iy$, where $x, y \in \mathbb{C}$, we see that $|f(z)| = e^x \geq e^{-|z|} = e^{-r}$. Therefore, for z on $\overline{D(0,r)}$, we have

$$f(z) - g_n(z)| < |f(z)|.$$

By Rouché's theorem, f and g_n have the same number of zeros inside $D(0,r)$. However, f vanishes nowhere so we can conclude that g_n does not vanish in $D(0,r)$. □

Example 6.16. Find the number of zeros (counting multiplicity) of the function $z^5 + 6z^3 + 11$ in the annulus $2 < |z| < 3$.

Solution. Let $f(z) = z^5 + 6z^3 + 11$. On the circle $|z| = 3$, $|f(z) - z^5| = |6z^3 + 11| \leq 6|z^3| + 11 = 173 < 243 = |z^5|$ so by Rouché's theorem, the number of zeros of $f(z)$ in the region $0 < |z| < 3$ is equal to that of z^5, which is 5.

On the circle $|z| = 2$, we have $|f(z) - 6z^3| = |z^5 + 11| \leq 2^5 + 11 = 43 < 48 = |6z^3|$ so the number of zeros of $f(z)$ in the region $0 < |z| < 2$ is equal to that of $6z^3$, which is 3. Therefore, f has exactly $5 - 3 = 2$ zeros in the annulus $2 < |z| < 3$. □

Example 6.17 (Dinh's 70 problems)**.**

(a) For each integer $n \geq 1$, find the number of zeros (counting multiplicity) in the disk $D(0,n)$ of the polynomial $z^7 + 5z^3 - z - 2$.

(b) Prove that the function $u(x,y) = \sinh x \sin y$ is harmonic and find its harmonic conjugates.

Solution.

(a) Let N_n be the number of zeros. We first show that $N_1 = 3$. Note that on $|z| = 1$, we have

$$|z^7 + 5z^3 - z - 2 - 5z^3| = |z^7 - z - 2| \leq |z|^7 + |z| + 2 = 4 \leq 5 = 5|z|^3.$$

By Rouché's's theorem, $N_1 = 3$.

For $n \geq 2$, we show that $N_n = 7$. Note that on the boundary $|z| = n$, we have

$$|z^7 + 5z^3 - z - 2 - z^7| \leq 5|z|^3 + |z| + 2 = 5n^3 + n + 2 \leq n^7 = |z|^7.$$

The result follows by Rouché's's theorem.

(b) Trivial. Show that u satisfies Laplace's equation, then to find its harmonic conjugates, use the Cauchy-Riemann equations. □

Example 6.18. Let $a_1, \ldots, a_n \in D(0,1)$ and

$$f(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a_k} z}.$$

Prove that for each $b \in D(0,1)$, $f(z) = b$ has exactly n roots in $D(0,1)$, counting multiplicity.

Solution. For $|z| = 1$, we have $\overline{z} = 1/z$. Hence,

$$\left| \frac{a_k - z}{1 - \overline{a_k} z} \right| = \frac{|a_k - z|}{|z| \, |1/z - \overline{a_k}|} = \frac{|a_k - z|}{|\overline{z} - \overline{a_k}|} = 1.$$

We infer that for $|z|=1$, $|f(z)|=1$. We deduce that for $b \in D(0,1)$,

$$|(f(z)-b)-f(z)| = |b| < 1 = |f(z)|.$$

By Rouché's theorem, $f(z)-b$ and $f(z)$ have the same number of zeros in $D(0,1)$. The roots of $f(z)=0$ are $a_1,\ldots,a_n$ (so there are n roots), as such, the result follows. □

Example 6.19 (Dinh's 70 problems). Show that if the integer n is sufficiently large, the equation

$$z = 1 + \left(\frac{z}{2}\right)^n$$

has exactly one solution in the disk $|z|<2$.

Solution. Let

$$f_n(z) = z - 1 - \left(\frac{z}{2}\right)^n \quad \text{and} \quad f(z) = z-1.$$

For arbitrary $\varepsilon > 0$, consider the boundary of $C(0, 2-\varepsilon)$, we have

$$|f_n(z) - f(z)| = \left|\frac{z}{2}\right|^n = \left(\frac{2-\varepsilon}{2}\right)^n = \left(1-\frac{\varepsilon}{2}\right)^n.$$

Also,

$$|f(z)| = |z-1| \geq |z| - 1 = 1-\varepsilon \quad \text{by the reverse triangle inequality.}$$

By Rouché's's theorem, we need $|f_n - f| \leq |f|$, i.e.

$$\left(1-\frac{\varepsilon}{2}\right)^n \leq 1-\varepsilon.$$

So, we choose

$$n \geq \frac{\ln(1-\varepsilon)}{\ln(1-\varepsilon/2)} \quad \text{where } n \in \mathbb{N} \text{ and } 0 < \varepsilon \leq \frac{1}{2}.$$

The number of zeros of f_n in $D(0, 2-\varepsilon)$ is 1. Letting $\varepsilon \to 0$, the result follows. □

Theorem 6.7 (Hurwitz's theorem). Let $f_n : \Omega \to \mathbb{C}$, where $n \in \mathbb{N}$, be a sequence of holomorphic functions that converges locally uniformly to a function $f : \Omega \to \mathbb{C}$. Let γ be a piecewise differentiable, simple closed contour in Ω such that all points interior to γ are contained in Ω. Assume that f has no zero on γ. Then,

$$\text{there exists } N \in \mathbb{N} \text{ such that for all } n > N \quad f_n \text{ and } f \text{ have the same number of zeros inside } \gamma.$$

Example 6.20. Assume that f is holomorphic in a neighbourhood of $\overline{D(0,1)}$ and that $f'(z)$ has no zero on $\partial D(0,1)$. Prove that for n sufficiently large,

$$F_n(z) = f\left(z+\frac{1}{n}\right) - f(z)$$

has the same number of zeros in $D(0,1)$ as $f'(z)$.

Solution. We consider the function $g_n(z) = nF_n(z)$. Note that

$$g_n(z) = n\left[f\left(z+\frac{1}{n}\right) - f(z)\right],$$

so

$$\lim_{n\to\infty} g_n(z) = \lim_{n\to\infty} \frac{f(z+1/n)-f(z)}{1/n} = f'(z).$$

By the Fundamental Theorem of Calculus,

$$g_n(z) = \frac{f(z+1/n) - f(z)}{1/n} = \int_0^1 f'\left(z + \frac{t}{n}\right)\, dt.$$

Hence, g_n converges locally and uniformly in a neighbourhood to f'. By Hurwitz's Theorem, g_n has the same number of zeros as f' in $D(0,1)$ when n is sufficiently large. Therefore, F_n satisfies the same property. □

6.3
Open Mapping Theorem and the Maximum Modulus Principle

Theorem 6.8 (open mapping theorem). Let f be a non-constant holomorphic function on an open connected set Ω. Then, f is open, i.e. for any open set $U \subseteq \Omega$, we have $f(U)$ is open.

Theorem 6.9 (maximum modulus principle). Suppose f is a non-constant holomorphic function defined on a domain Ω. Then, $|f|$ does not attain the maximum value in Ω.

Example 6.21. Suppose f is holomorphic on a neighbourhood of the unit disc $\overline{D(0,1)}$ and satisfies $f(0) = 3+4i$, $|f(z)| \leq 5$ if $|z| = 1$. Find $f'(0)$.

Solution. We prove that f is constant. Suppose on the contrary that f is not constant, then by the maximum modulus principle,

$$5 = f(0) < \max_{|z|=1} |f(z)| \leq 5.$$

This is a contradiction, so $f'(0) = 0$. □

Example 6.22. Let f be a continuous function on $\overline{A} = \{1 \leq |z| \leq 4\}$ and holomorphic on $A = \{1 < |z| < 4\}$. Assume that

$$\max_{|z|=1} |f(z)| = 5 \text{ and } \max_{|z|=4} |f(z)| = 20.$$

(i) Show that $|f(2)| \leq 10$.
(ii) Find all functions f such that $f(2) = 10$.

Solution. Let us discuss the solutions.

(i) Define $g(z) = f(z)/z$. Then,

$$\max_{|z|=1} |g(z)| = \max_{|z|=4} |g(z)| = 5.$$

By the maximum modulus principle, $|g(z)| \leq 5$ for $z \in A$. Setting $z = 2$, we have $f(2) \leq 10$.

(ii) $g(2) = 5$. By the maximum modulus principle, g is a constant, so $g(z) = 5$. Hence, $f(z) = 5z$.

□

Corollary 6.1. Let Ω be a domain in $\mathbb{C}$ and f be holomorphic in Ω.

(i) If $|f|$ assumes a local maximum at some point in Ω, then f is constant in Ω.
(ii) If Ω is bounded and f is continuous up to the boundary $b\Omega$, then,

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in b\Omega} |f(z)|.$$

We obtain the next corollary on the minimum modulus principle by switching to the reciprocal $1/f(z)$.

Corollary 6.2 (minimum modulus principle). Let Ω be a domain in $\mathbb{C}$ and f be holomorphic but never zero in Ω.

- If $|f|$ assumes a local minimum at some point in Ω, then f is constant on Ω.
- If Ω is bounded and f is continuous up to the boundary of Ω and never vanishes in $\overline{\Omega}$, then

$$\min_{z\in\overline{\Omega}} |f(z)| = \min_{z\in b\Omega} |f(z)|.$$

Example 6.23. Suppose f is holomorphic on a neighbourhood of $\overline{D(0,1)}$, $f(0) = i$ and $|f(z)| > 1$ whenever $|z| = 1$. Prove that f has a zero in $D(0,1)$.

Solution. Suppose on the contrary that f does not have a zero in $D(0,1)$. Then, $g(z) = 1/f(z)$ would be holomorphic in a neighbourhood $\overline{D(0,1)}$. Moreover, we have $|g(0)| = 1$ and $|g(z)| < 1$ when $|z| = 1$. This contradicts the maximum modulus principle. □

Theorem 6.10 (maximum and minimum principle for harmonic functions). Let Ω be a domain in $\mathbb{C}$ and u be a real-valued harmonic in Ω.

(i) If u has either a local maximum or a local minimum at some point of Ω, then u is a constant on Ω.

(ii) If Ω is bounded and f is continuous up to the boundary of Ω, then

$$\max_{z\in\overline{\Omega}} u(z) = \max_{z\in b\Omega} u(z) \text{ and } \min_{z\in\overline{\Omega}} u(z) = \min_{z\in b\Omega} u(z).$$

Example 6.24. Find the maximal value of $\operatorname{Re}(z^3)$ for $z \in [0,1]\times[0,1]$.

Solution. Note that $\operatorname{Re}(z^3)$ is harmonic as it is the real part of a holomorphic function. Hence, it achieves its maximal value on the boundary of the unit square. Throughout this problem, $a \in \mathbb{R}$ and $a \in [0,1]$.

- **Case 1 (bottom edge of square):** $z = a$. Then, $\operatorname{Re}(z^3) = a^3$, whose maximum is 1.
- **Case 2 (top edge of square):** $z = a + i$. Then, $\operatorname{Re}(z^3) = a^3 - 3a$. The maximum here is 0.
- **Case 3 (left edge of square):** $z = ai$. Then, $\operatorname{Re}(z^3) = 0$.
- **Case 4 (right edge of square):** $z = 1 + ai$. Then, $\operatorname{Re}(z^3) = 1 - 3a^2$. The maximum here is 1.

Overall, the maximum value is 1 which is achieved when $z = 1$. □

Example 6.25 (Dinh's 70 problems). Let $a \in \mathbb{C}$, $|a| \le 1$, and consider the polynomial

$$P(z) = \frac{a}{2} + (1-|a|^2)z - \frac{\overline{a}}{2}z^2.$$

Prove that $|P(z)| \le 1$ whenever $|z| \le 1$.

Solution. Note that $z\overline{z} = 1$ on $|z| = 1$. Consider

$$\frac{P(z)}{z} = \frac{a}{2z} - \frac{\overline{a}z}{2} + 1 - |a|^2.$$

We have

$$\frac{a}{2z} - \frac{\overline{a}z}{2} = \frac{1}{2}\left(\frac{a}{z} - \overline{a/z}\right).$$

Let $\lambda = a/z \in \mathbb{C}$. Then, $\lambda - \overline{\lambda} = 2i\operatorname{Im}(\lambda)$, so

$$\frac{a}{2z} - \frac{\overline{a}z}{2} = i\operatorname{Im}\left(\frac{a}{z}\right) = i\operatorname{Im}(a\overline{z}).$$

Hence,

$$\begin{aligned}\left|\frac{P(z)}{z}\right| &\le \left|i\operatorname{Im}(a\bar{z})+1-|a|^2\right| \\ |P(z)| &\le \left|i\operatorname{Im}(a\bar{z})+1-|a|^2\right| \text{ since } |z|\le 1 \\ |P(z)|^2 &\le |\operatorname{Im}(a\bar{z})|^2+\left(1-|a|^2\right)^2\end{aligned}$$

We bluntly state that $|\operatorname{Im}(a\bar{z})|^2 \le |a|^2$, so $|P(z)|^2 \le 1-|a|^2+|a|^4 \le 1$ since $|a|\le 1$. By the maximum modulus principle, whenever $|z|\le 1$, we have $|P(z)|\le 1$.

Now, we justify that $|\operatorname{Im}(a\bar{z})|^2 \le |a|^2$. Let $z=x+iy$ and $a=\alpha+i\beta$, where $x,y,\alpha,\beta\in\mathbb{R}$ such that $x^2+y^2\le 1$ and $\alpha^2+\beta^2\le 1$. We have $a\bar{z}=(\alpha+i\beta)(x-iy)=\alpha x-\beta y+i(\beta x-\alpha y)$ so $\operatorname{Im}(a\bar{z})=\beta x-\alpha y$. It suffices to prove that $(\beta x-\alpha y)^2\le\alpha^2+\beta^2$. In other words, $\alpha^2(1-y^2)+\beta^2(1-x^2)+2\alpha\beta xy\ge 0$. Let $x=\cos\theta$ and $y=\sin\theta$ so $\alpha^2\sin^2\theta+\beta^2\cos^2\theta+2\alpha\beta\cos\theta\sin\theta\ge 0$. This inequality is obviously true since $(\alpha\sin\theta+\beta\cos\theta)^2\ge 0$, or equivalently $(\alpha y+\beta x)^2\ge 0$. □

We then introduce the Schwarz-Pick lemma (Lemma 6.1), which is also known as the Schwarz lemma.

Lemma 6.1 (Schwarz-Pick Lemma)**.** Let $f: D(0,1)\to\mathbb{C}$ be a holomorphic function with $f(0)=0$ and $|f(z)|\le 1$ for each $z\in D(0,1)$. Then,

$$|f(z)|\le|z| \text{ and } |f'(0)|\le 1.$$

Moreover, if $|f(z)|=|z|$ for some $z\in D(0,1)\setminus\{0\}$ or if $|f'(0)|=1$, then f is a rotation of $D(0,1)$; that is, there exists a constant $\theta\in\mathbb{R}$ such that

$$f(z)=e^{i\theta}z \text{ for all } z\in D(0,1).$$

Example 6.26 (Dinh's 70 problems)**.** Does there exist a holomorphic function $f:\mathbb{D}\to\mathbb{D}$ with $f(1/2)=3/4$ and $f'(1/2)=2/3$?

Solution. Yes, this is simply proven using the Schwarz-Pick lemma since $f'(1/2)\le 7/12<2/3$. □

Example 6.27 (Dinh's 70 problems)**.** Let f be a holomorphic function from the unit disk $D(0,1)$ to itself. Assume that there is a point $z_0\in D(0,1)$ such that $f(z_0)=z_0$. Prove that $|f'(z_0)|\le 1$.

Solution. We use the Schwarz-Pick Lemma, which says that for $a,b\in\mathbb{D}$, a holomorphic function $f:\mathbb{D}\to\mathbb{D}$ satisfies $f(a)=b$ and $|f'(a)|\le\dfrac{1-|b|^2}{1-|a|^2}$. So, we set $a=b=z_0$. The result follows. □

Example 6.28. Is there a holomorphic function of $D(0,1)$ onto itself such that $f(0)=0$ and $f(i/4)=i/3$? Justify.

Solution. We will show that there is no such function. Suppose on the contrary that there exist such a function. By the Schwarz Lemma, as $|f(z)|\le|z|$ for $z\in\mathbb{D}$, we have $|f(i/4)|\le|i/4|=1/4$, which is a contradiction. □

6.4 Winding Numbers

Definition 6.4 (winding number)**.** Let $\gamma : [a,b] \to \mathbb{C} \setminus \{z_0\}$ be a closed curve that does not pass through z_0. Given an argument θ_a for $\gamma(a) - z_0$,

$$\text{there exists a unique continuous function } \theta : [a,b] \to \mathbb{R}$$

such that for each $t \in [a,b]$, $\theta(t)$ is an argument of $\gamma(t) - z_0$ and such that $\theta(a) = \theta_a$. Define

$$n(\gamma, z_0) = \frac{\theta(b) - \theta(a)}{2\pi} \quad \text{to be} \quad \text{the winding number of } \gamma \text{ around } z_0.$$

Sometimes, we also refer it to the index of z_0 with respect to γ.

Theorem 6.11. $n(\gamma, z_0) \in \mathbb{Z}$

Theorem 6.12. Let γ be a closed contour piecewise differentiable and $z_0 \in \gamma$. Then,

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, dz.$$

Corollary 6.3. Let f be holomorphic on an open set Ω containing γ and $z_0 \in f(\gamma)$. Then,

$$n(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - z_0} \, dz.$$

Example 6.29 (Dinh's 70 problems)**.** Let C be the unit circle $|z| = 1$, anti-clockwise oriented, and let $f(z) = z^3$. How many times does the curve $f(C)$ wind around the origin? Explain.

Solution. We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_C \frac{3}{z} \, dz = 3.$$

□

Example 6.30 (Dinh's 70 problems)**.** Let C be the unit circle $|z| = 1$, anti-clockwise oriented, and let $f(z) = \left(z^2 + 2\right) / z^3$. How many times does the curve $f(C)$ wind around the origin? Explain.

Solution. We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = -\frac{1}{2\pi i} \int_C \frac{z^2 + 6}{z(z^2 + 2)} \, dz.$$

The residue at $z = 0$ is 3, so by Cauchy's residue theorem, the answer is -3. □

Theorem 6.13 (generalised Cauchy's integral formula)**.** Suppose f is a holomorphic function in a simply connected domain Ω. Then for any piecewise differentiable closed contour γ in Ω, if $a \notin \gamma$,

$$n(\gamma, z) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} \, dz$$

Theorem 6.14 (generalised residue theorem)**.** Let Ω be a simply connected domain in $\mathbb{C}$. Suppose f is holomorphic outside a finite number of points $z_1, \ldots, z_N$ in Ω. Then, for any piecewise differentiable closed contour γ in Ω which does not pass through $z_1, \ldots, z_N$,

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{N} n(\gamma, z_k) \operatorname{Res}(f, z_k).$$

Chapter 7
Conformal Mappings and Möbius Transformations

7.1
Univalent Functions

Definition 7.1 (univalent function). Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, f is univalent if it is injective, i.e.

$$f(z_1) = f(z_2) \quad \text{implies} \quad z_1 = z_2.$$

f is locally univalent if for each $z_0 \in \Omega$, there exists a neighbourhood U of z_0 such that $f|_U \to \mathbb{C}$ is injective.

Theorem 7.1. A holomorphic function $f : \Omega \to \mathbb{C}$

$$\text{locally univalent at } z_0 \quad \text{if and only if} \quad f'(z_0) \neq 0.$$

Corollary 7.1 (inverse function theorem). If $f : \Omega \to \mathbb{C}$ is a univalent holomorphic function, then its inverse f^{-1} is also holomorphic defined on $f(\Omega)$. Moreover, for each $z \in \Omega$,

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

Definition 7.2. Suppose two curves γ and η intersect at z_0 and α is the oriented angle between the tangent vectors to these curves at z_0. A holomorphic map f preserves angles at z_0 if the image curves $f \circ \gamma$ and $f \circ \eta$ intersect at $f(z_0)$ and their tangent vectors at $f(z_0)$ form an angle equal to α.

Theorem 7.2. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$. Then, f preserves angles at z_0.

Definition 7.3 (conformal map and automorphism group). A bijective holomorphic function $f : U \to V$ is a conformal map or a biholomorphism. A conformal map from a domain $\Omega \to \Omega$ is a conformal automorphism of Ω. Define $\mathrm{Aut}(\Omega)$ to be the set of conformal automorphisms of Ω.

Theorem 7.3. If f and g are automorphisms of Ω, then $f \circ g$ is also an automorphism.

7.2
Automorphisms of the Complex Plane $\mathbb{C}$

Example 7.1. Translations, rotations, and dilations are examples of automorphims of the complex plane. We only discuss translations here. Suppose $h \in \mathbb{C}$. Then, the translation

$$z \mapsto z + h \text{ is a conformal map } \mathbb{C} \to \mathbb{C} \quad \text{whose inverse is} \quad w \mapsto w - h.$$

Moreover, if $h \in \mathbb{R}$, this translation is also a conformal map from the upper half-plane $\mathbb{H}$ to itself.

Theorem 7.4. Let f be a conformal map from $\mathbb{C}$ to itself. Then, there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $f(z) = az + b$ for $z \in \mathbb{C}$. In particular, we have

$$\mathrm{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C}, a \neq 0\}.$$

7.3 Automorphisms of the Unit Disc $\mathbb{D}$

Definition 7.4 (unit disc)**.** Define $\mathbb{D}$ to be the unit disc. This is sometimes denoted by $D(0,1)$ which represents

the open disc of radius 1 centred at 0.

Example 7.2. Any rotation by an angle $\theta \in \mathbb{R}$, i.e. $\rho_\theta(z) = e^{i\theta}z$, is an automorphism of $\mathbb{D}$ whose inverse is $e^{-i\theta}z$.

We can generalise the previous example to the following lemma:

Lemma 7.1 (Blaschke factor)**.** For any $a \in \mathbb{D}$, the map

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \quad \text{is a conformal automorphism of } \mathbb{D} \quad \text{with inverse } \varphi_a^{-1} = \varphi_a.$$

The transformation ϕ_a is known as the Blaschke factor.

Theorem 7.5. If $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism and $f^{-1}(0) = a$, then there exists $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta}\frac{a-z}{1-\bar{a}z}.$$

Hence,

$$\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta}\frac{a-z}{1-\bar{a}z} : \theta \in \mathbb{R}, a \in \mathbb{D} \right\}.$$

Example 7.3. Let f be a holomorphic function on $\mathbb{D}$ such that $|f(z)| \leq 1$ when $|z| < 1$. Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|} \quad \text{for all } |z| < 1.$$

Solution. We first consider the case where $|f(z)| = 1$ for some $z \in \mathbb{D}$. By the maximum modulus principle, f is constant and so $|f(z)| = 1$ for all $z \in \mathbb{D}$. The above inequality is equivalent to

$$|f(0)| - |z| \leq 1 + |f(0)||z| \text{ and } 1 - |f(0)||z| \leq |f(0)| + |z|,$$

so $1 - |z| \leq 1 + |z|$, which holds.

Now, consider the case where $|f(z)| < 1$ for all $z \in \mathbb{D}$. Let $f(0) = a \in \mathbb{D}$. Note that

$$\phi(z) = \frac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D}).$$

As such, $g = \phi \circ f$ is a holomorphic function from $\mathbb{D}$ to itself. Moreover, $g(0) = \phi(f(0)) = \phi(a) = 0$. By the Schwarz Lemma, $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Since $\phi^{-1} = \phi$, then

$$f(z) = (\phi^{-1} \circ g)(z) = \frac{a - g(z)}{1 - \bar{a}g(z)}.$$

As such,

$$\left| \frac{a-g(z)}{1-\bar{a}g(z)} \right| \leq 1 \Rightarrow 1 - |f(0)|\,|z| \leq 1 - |\bar{a}|\,|g(z)| \leq |a - g(z)| \leq |1 - \bar{a}g(z)| \leq 1 + |\bar{a}|\,|g(z)| \leq 1 + |f(0)|\,|z|$$

and in a similar fashion, we can deduce that

$$|f(0) - |z| \leq |a| - |g(z)| \leq |a - g(z)| \leq |a| + |g(z)| \leq |f(0)| + |z|.$$

Hence, we have shown that

$$1 - |f(0)|z| \leq |1 - \bar{a}g(z)| \leq 1 + |f(0)||z| \text{ and } |f(0) - |z| \leq |a - g(z)| \leq |f(0)| + |z|.$$

The desired inequality is thus proven. $\square$

Example 7.4. Find a conformal map $T : D(0,1) \to D(1,2)$ such that $T(0) = 1 + i$ and $T(1) = 1 - 2i$. Is the transformation unique?

Solution. Let $S(z) = 2z + 1$, which maps $D(0,1)$ to $D(1,2)$ conformally. Define $f = S^{-1} \circ T$, which is an automorphism of the unit disc. We have $S^{-1}(z) = (z-1)/2$. So, the conditions $T(0) = 1 + i$ and $T(1) = 1 - 2i$ are equivalent to $f(0) = i/2$ and $f(1) = -i$. To find such a map f, consider

$$g(z) = -i \cdot \frac{\frac{i}{2} - z}{1 + \frac{i}{2}z}$$

which is a conformal automorphism of $D(0,1)$ such that $g(i/2) = 0$ and $g(-i) = 1$. Thus,

$$f(z) = \frac{i(1-2z)}{2-z}.$$

We conclude that

$$T(z) = \frac{2(1+i) - (1+4i)z}{2-z}$$

is the required conformal map satisfying the conditions.

Suppose $\widetilde{T}$ also satisfies the requirements. Then, $R = T^{-1} \circ \widetilde{T}$ is a conformal automorphism of $D(0,1)$ satisfying $R(0) = 0$ and $R(1) = 1$. It is known that all automorphisms of the unit disc which fix 0 are rotations. Hence, R is the identity function so we conclude that $\widetilde{T} = T$. $\square$

Example 7.5. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Suppose $f(0) = 0$ and there exists a constant $A > 0$ such that $\mathrm{Re}(f(z)) \leq A$ for $z \in \mathbb{D}$. Prove that for $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{2A|z|}{1-|z|}.$$

Solution. Since $f(0)=0$, then f is identically 0 or f is not identically constant. If $f(z)=0$ for all $z\in\mathbb{D}$, the inequality is obvious. Suppose f is not identically constant. Consider

$$\phi_1(z)=-\frac{z}{A}+1,\ \phi_2(z)=\frac{1-z}{1+z}\text{ and }\phi(z)=(\phi_2\circ\phi_1)(z)=\frac{z}{2A-z}.$$

Note that ϕ_1 is a conformal map from $\{\mathrm{Re}(z)<A\}$ to $\{\mathrm{Re}(z)>0\}$ and sends 0 to 1; ϕ_2 is a conformal map from $\{\mathrm{Re}(z)>0\}$ to the unit disc and sends 1 to 0. Hence, ϕ is a conformal map from $\{\mathrm{Re}(z)<A\}$ to the unit disc and sends 0 to 0. As such, $F=\phi\circ f$ is a holomorphic map from $\mathbb{D}$ to itself and $F(0)=0$.

By the Schwarz Lemma, note that the conditions $F(0)=0$ and $|F(z)|\le 1$ are satisfied since $z\in\mathbb{D}$. Hence, $|F(z)|\le|z|$. That is to say,

$$|z|\ge|\phi(f(z))|=\left|\frac{f(z)}{2A-f(z)}\right|.$$

The desired inequality follows with some simple algebraic manipulation. □

Example 7.6 (Dinh's 70 problems)**.** Suppose that f is holomorphic on the open set containing $\mathbb{D}$, $|f(z)|\le 4$ if $|z|=1$ and $f(i/2)=0$. Show that for all $|z|\le 1$,

$$|f(z)|\le 4\left|\frac{z-i/2}{1+i/2\cdot z}\right|.$$

Solution. Note that $g(z)=\dfrac{a-z}{1-\overline{a}z}$ is an automorphism of $\mathbb{D}$, so we set $f(z)=4g(z)$ and $a=i/2$. The result follows. □

Example 7.7 (Dinh's 70 problems)**.** Show that if $D(0,R)\to\mathbb{C}$ is holomorphic with $|f(z)|<M$ for some $M>0$, then

$$\left|\frac{f(z)-f(0)}{M^2-\overline{f(0)}f(z)}\right|\le\frac{|z|}{MR}.$$

Solution. Let $a=f(0)$. We wish to prove

$$\left|\frac{a-f(z)}{M^2-\overline{a}f(z)}\right|\le\frac{|z|}{MR}.$$

Define $\phi:\mathbb{D}\to\mathbb{D}$ via

$$\phi(z)=\frac{a/M-z}{1-(\overline{a}/M)z}.$$

Note that $\overline{a/M}=\overline{a}/M$ since $M\in\mathbb{R}$. So, define $g=\phi\circ\dfrac{f(Rz)}{M}$. It is clear that $g(0)=0$ and $g:\mathbb{D}\to\mathbb{D}$. By the Schwarz Lemma, $|g(z)|\le|z|$ for all $z\in\mathbb{D}$. Hence,

$$\begin{aligned}|g(z)|&\le|z|\\ \frac{1}{M}\left|\frac{a-f(Rz)}{1-\overline{a}f(Rz)/M^2}\right|&\le|z|\\ \frac{M^2}{M}\left|\frac{a-f(Rz)}{M^2-\overline{a}f(Rz)}\right|&\le|z|\\ \left|\frac{a-f(z)}{M^2-\overline{a}f(z)}\right|&\le\frac{|z|}{MR}\end{aligned}$$

and we are done. □

Lemma 7.2 (Schwarz-Pick lemma)**.** Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, $a \in \mathbb{D}$ and $f(a) = b$. Then,

(i) for each $z \in \mathbb{D}$, $|\varphi_b(f(z))| \leq |\varphi_a(z)|$

(ii) $|f'(a)| \leq \dfrac{1-|b|^2}{1-|a|^2}$

If equality holds in **(ii)** or if we have equality in **(i)** for some $z \neq a$, then $f \in \mathrm{Aut}(\mathbb{D})$.

7.4 Maps from the Upper Half-Plane $\mathbb{H}$ to the Unit Disc $\mathbb{D}$

Definition 7.5 (upper half-plane)**.** Define $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to be the upper half-plane.

Lemma 7.3. Let

$$F(z) = \frac{i-z}{i+z} \quad \text{and} \quad G(w) = i \cdot \frac{1-w}{1+w}.$$

Then, $F : \mathbb{H} \to \mathbb{D}$ is a conformal map with inverse $G : \mathbb{D} \to \mathbb{H}$.

Theorem 7.6. All conformal mappings from $\mathbb{H}$ to $\mathbb{D}$ take the form

$$\left\{ e^{i\theta} \frac{z-\beta}{z-\overline{\beta}} : \theta \in \mathbb{R}, \beta \in \mathbb{H} \right\}.$$

7.5 Automorphisms of the Upper Half-Plane $\mathbb{H}$

Theorem 7.7.

$$\mathrm{Aut}(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc = 1 \right\}$$

Proof. Let $a,b,c,d \in \mathbb{R}$ and $ad-bc > 0$. Define a',b',c',d' to be as follows:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \sqrt{ad-bc},$$

where $a',b',c',d' \in \mathbb{R}$ and $a'd'-b'c' = 1$. As such,

$$\mathcal{G} = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc = 1 \right\} = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc > 0 \right\}.$$

We shall prove that $\mathcal{G} \subseteq \mathrm{Aut}(\mathbb{H})$. Let

$$f(z) = \frac{az+b}{cz+d} \in \mathcal{G}.$$

Then, $f:\mathbb{R}\to\mathbb{R}$. If we let $z=x+iy$, where $x,y\in\mathbb{R}$, Since $a,b,c,d\in\mathbb{R}$, then

$$\begin{aligned}\operatorname{Im}(f(z)) &= \operatorname{Im}\left[\frac{a(x+iy)+b}{c(x+iy)+d}\right] = \operatorname{Im}\left[\frac{ax+b+i(ay)}{cx+d+i(cy)}\cdot\frac{cx+d-i(cy)}{cx+d-i(cy)}\right] \\ &= \operatorname{Im}\left[\frac{ac\left(x^2+y^2\right)+bcx+adx+bd+iy(ad-bc)}{c^2\left(x^2+y^2\right)+2cdx+d^2}\right] \\ &= \operatorname{Im}(z)\cdot\frac{ad-bc}{c^2|z|^2+2cdx+d^2} \\ &= \operatorname{Im}(z)\cdot\frac{ad-bc}{|cz+d|^2}\end{aligned}$$

which shows $f:\mathbb{H}\to\mathbb{H}$. It is clear that

$$g(z)=\frac{-dz+b}{cz-a}\in\mathcal{G}$$

and $g\circ f=\mathrm{id}$. Hence, $f\in\operatorname{Aut}(\mathbb{H})$ and $\mathcal{G}\subseteq\operatorname{Aut}(\mathbb{H})$.

Conversely, let f be an arbitrary map in $\operatorname{Aut}(\mathbb{H})$. We will show that $f\in\mathcal{G}$. Define

$$F(z)=\frac{i-z}{i+z}$$

which is a conformal map from $\mathbb{H}$ to $\mathbb{D}$ with inverse

$$F^{-1}(z)=i\cdot\frac{1-z}{1+z}$$

and this maps from $\mathbb{D}$ to $\mathbb{H}$. Hence, $h=F\circ f$ is a conformal map from $\mathbb{H}$ to $\mathbb{D}$. All such a map h must be of the form

$$e^{2i\theta}\frac{z-\beta}{z-\overline{\beta}}$$

with $\beta\in\mathbb{H}$ and $\theta\in\mathbb{R}$. We let the reader prove that

$$f(z)=F^{-1}\left(e^{2i\theta}\frac{z-\beta}{z-\overline{\beta}}\right)=\frac{az+b}{cz+d}$$

and $ad-bc=\operatorname{Im}(\beta)>0$ which would show that $f\in\mathcal{G}$, so $\operatorname{Aut}(\mathbb{H})\subseteq\mathcal{G}$. □

Example 7.8 (Dinh's 70 problems). Find a conformal map from

$$H=\{z\in\mathbb{C}:\operatorname{Re}(z)>0\}$$

onto

$$A=\{z\in\mathbb{C}:|z-2|<3,|z|>1\}.$$

You may leave your answer as a composition of conformal mappings.

Solution. We find a conformal map from A to H first.

- Let $\phi_1(z)=1/(z+1)$. Let us figure out what A gets mapped to via ϕ_1. So, if we let $w=1/(z+1)$, we have $z=1/w-1$. Consider the annulus $|z-2|<3$, so after the transformation, we have $w>1/6$. For the region $|z|>1$, we have $1/w>2$. So, ϕ_1 maps A to A_1, where $A_1=\{z\in\mathbb{C}:1/6<\operatorname{Re}(z)<1/2\}$.
- Let $\phi_2(z)=z-\frac{1}{6}$. So, ϕ_2 maps A_1 to $A_2=\{z\in\mathbb{C}:0<\operatorname{Re}(z)<1/3\}$.
- Let $\phi_3(z)=\tan\left(\frac{3\pi z}{2}\right)$, which maps A_2 to H.

As such, the required conformal map from H to A is $\phi_3^{-1}\circ\phi_2^{-1}\circ\phi_1^{-1}$. □

Example 7.9 (Dinh's 70 problems). Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function such that $\operatorname{Re}(f(z)) > 0$ for each $z \in D(0,1)$ and such that $f(0) = 1$.

(a) Prove that $|f'(0)| \le 2$.

(b) Assume that $|f'(0)| = 2$. Determine all possible forms of f.

Solution.

(a) We first find a holomorphic map from $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ to $\mathbb{D}$. To do this, let $\phi_1(z) = iz$, which maps the right half of the complex plane to the upper half, $\mathbb{H}$. Then, recall that $\phi_2(z) = \dfrac{z-i}{z+i}$ maps $\mathbb{H}$ to $\mathbb{D}$. So, $\phi = \phi_2 \circ \phi_1$ is the required holomorphic map. We have

$$\phi(z) = \frac{iz - i}{iz + i} = \frac{z-1}{z+1}.$$

Define $F = \phi \circ f$ so $F : \mathbb{D} \to \mathbb{D}$, i.e. F is an automorphism of the unit disk and $F(0) = 0$. By the Schwarz Lemma, $|F'(0)| \le 1$, so $|\phi'(1) f'(0)| \le 1$. Since $\phi'(1) = 1/2$, the result follows.

(b) Suppose equality holds. Then, $F'(0) = 1$, where

$$F(z) = \frac{f(z) - 1}{f(z) + 1}.$$

Then, $F(z) = ze^{i\theta}$ (recall that this is just rotating some point in the unit disk about the origin) for $\theta \in \mathbb{R}$. One can work out that $f = \phi^{-1} \circ F$ and find an explicit expression for it. □

7.6 Möbius Transformations

Definition 7.6 (Möbius transformation). A transformation of the form

$$T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$, is a linear fractional transformation (LFT). If $ad - bc \neq 0$, then T is a Möbius transformation.

Note that the condition $ad - bc \neq 0$ is equivalent to saying T is not constant. Consider

$$T(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. If $c = 0$, then $T : \mathbb{C} \to \mathbb{C}$. If $c \neq 0$, then $T : \mathbb{C} \setminus \{-d/c\} \to \mathbb{C}$.

Definition 7.7. The Möbius transformation $T : \mathbb{C}^* \to \mathbb{C}^*$ associated with a, b, c, d is defined by

$$T(z) = \begin{cases} \dfrac{az+b}{cz+d} & z \neq \infty, z \neq -d/c; \\ a/c & z = \infty; \\ \infty & z = -d/c. \end{cases}$$

Moreover, if $c = 0$, then $a \neq 0$ and $d \neq 0$ so that the usual agreements regarding ∞ can be made. That is, $T(\infty) = \infty$.

The map $T : \mathbb{C}^* \to \mathbb{C}^*$ is injective.

Remark 7.1. T is holomorphic on $\mathbb{C}^* \setminus \{-d/c\}$ with a simple pole at the point $\{-d/c\}$.

Proposition 7.1. A Möbius transformation is a composition of transformations of the following forms:

- **(i) translation:** $z \mapsto z+b$, $b \in \mathbb{C}$;
- **(ii) rotation and dilation:** $z \mapsto \lambda z$, $\lambda \in \mathbb{C} \setminus \{0\}$;
- **(iii) reciprocation:** $z \mapsto 1/z$

Definition 7.8. Let $\mathrm{Aut}(\mathbb{C}^*)$ be the set of meromorphic automorphisms of $\mathbb{C}^*$.

Theorem 7.8. A Möbius transformation $T(z) = \dfrac{az+b}{cz+d}$ is such that $T \in \mathrm{Aut}(\mathbb{C}^*)$ with

$$T^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Conversely,

$$\mathrm{Aut}(\mathbb{C}^*) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0 \right\}.$$

Definition 7.9. Define a line l in $\mathbb{C}^*$ to be the union of a line in $\mathbb{C}$ with $\{\infty\}$.

Lemma 7.4. Let

$$L = \left\{ z \in \mathbb{C}^* : \alpha z\bar{z} + \beta z + \overline{\beta}\bar{z} + \gamma = 0, \text{ where } \alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C} \text{ and } \beta\overline{\beta} - \alpha\gamma > 0 \right\}.$$

- **(i)** If $\alpha \neq 0$, then L is a circle;
- **(ii)** if $\alpha = 0$, then L is a line

Conversely, each line or circle can be expressed as one of the set L for appropriate α, β, γ.

Theorem 7.9. Suppose L is a line or a circle and T is a Möbius transformation. Then, $T(L)$ is a line or a circle.

Note that a Möbius transformation does not necessarily map circles to circles and lines to lines; even if it maps a circle to another circle, it does not necessarily map the first circle's centre to the second circle's centre.

Example 7.10. For $a \in \mathbb{D}$,

$$T_a : \mathbb{D} \to \mathbb{D} \quad \text{where} \quad T_a(z) = \frac{-z+a}{1-\bar{a}z} \text{ and } T_a(0) = a.$$

To see why, note that T is a holomorphic function on $\mathbb{C} \setminus \{1/\bar{a}\}$, so it is defined in a neighbourhood of $\overline{\mathbb{D}}$. For z in the boundary of $\mathbb{D}$, we have $|z| = 1$ and $z\bar{z} = 1$. It is easy to see that $|T_a(z) = 1$. By the maximum modulus principle, when $|z| < 1$, we have $|T_a(z)| < 1$. Hence, $T_a(z)$ is a conformal automorphism of $\mathbb{D}$.

Also, $T_a(0) = a$ is obvious.

Example 7.11.

$$T(z) = i \cdot \frac{z-1}{z+1}$$

maps the real line to the imaginary line and $T(-1) = \infty$.

To see why, let $z = a$, where $a \in \mathbb{R}$. Then,

$$T(z) = \frac{i(a-1)}{a+1},$$

which is purely imaginary. It is also clear that $T(-1) = \infty$.

Example 7.12.

$$T(z) = \frac{i-z}{i+z}$$

maps the real line to the unit circle and $T(\infty) = -1$.

To see why, let $z = a$, where $a \in \mathbb{R}$. It suffices to show that $|T(a)| = 1$, i.e.

$$\left|\frac{i-a}{i+a}\right| = 1.$$

This is obvious.

Example 7.13.

$$T(z) = i \cdot \frac{1-z}{1+z}$$

maps the unit circle to the real line and $T(-1) = \infty$.

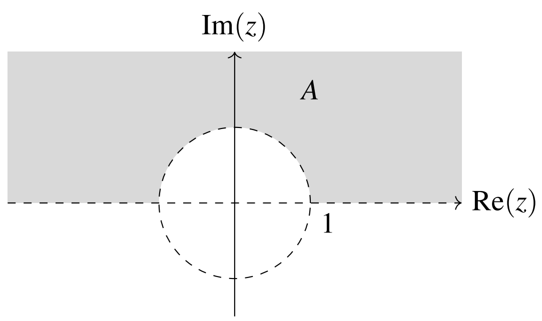
To see why, let $z = e^{i\theta}$. Then, we need to show that $T(z) \in \mathbb{R}$.

$$T(e^{i\theta}) = \frac{i(1-e^{i\theta})}{1+e^{i\theta}} = \tan\left(\frac{\theta}{2}\right),$$

which is real. Also, $T(-1)$ can be attained by setting $\theta = (2k+1)\pi$ for $k \in \mathbb{Z}$, which implies $\tan(\theta/2) = \infty$.

Example 7.14. Find a conformal map f from $A = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0, |z| > 1\}$ onto the unit disc.

Solution. The locus A represents the intersection of the upper half-plane and the points exterior to the circle of radius 1 centred at the origin.



Consider the Cayley transform given by $f(z) = (z-i)/(z+i)$. □

Example 7.15. Let

$$A = \left\{z \in \mathbb{C} : |z| < 1 \text{ and } \operatorname{Im}(z) > \frac{1}{2}\right\} \text{ and } B = \left\{\frac{2\pi}{3} < \arg(z) < \pi\right\}.$$

Find a conformal map f from A to B.

Solution. We first find the intersection of $|z| = 1$ and $\mathrm{Im}(z) = 1/2$. Consider $x^2 + y^2 = 1$ and $y = 1/2$. Solving yields $x = -\sqrt{3}/2$. Hence, $z = e^{i\pi/6}$ or $z = e^{5\pi i/6}$.

Note that
$$f(z) = \frac{z - e^{i\pi/6}}{z - e^{5\pi i/6}}$$
is an example of such a conformal map.

To see why, we note that it is a Möbius transformation so it sends lines and circles to lines and circles. Note that $f(e^{5\pi i/6}) = \infty$ and $f(e^{i\pi/6}) = 0$. Hence, the boundary of A is sent to the union of two half-lines which form an angle at the origin. For z in the interval joining $e^{5\pi i/6}$ and $e^{i\pi/6}$ (along the line $\mathrm{Im}(z) = 1/2$), note that $z - e^{i\pi/6} \in \mathbb{R}_{<0}$ and $z - e^{5\pi i/6} \in \mathbb{R}_{>0}$, so $f(z) \in \mathbb{R}_{<0}$.

The angle at $e^{i\pi/6}$ between this interval and the rest of the boundary of A forms an angle of $-\pi/3$. Since f is conformal at $e^{i\pi/6}$, we conclude that the boundary of A is sent to the union of $\mathbb{R}_{<0}$ with the half line $e^{2\pi i/3}\mathbb{R}_{>0}$. We deduce that A is sent to B. □

Example 7.16. Find a Möbius transformation mapping the upper half-plane onto the unit disc and mapping a given point z_0 in the upper half-plane to 0.

Solution. Note that T maps the real line to the unit disc. Since z_0 and $\overline{z_0}$ are symmetric about the real axis, then $T(\overline{z_0})$ and $T(z_0) = 0$ are symmetric with respect to the unit circle. Hence, $T(\overline{z_0}) = \infty$. As such,
$$T(z) = \lambda \cdot \frac{z - z_0}{z - \overline{z_0}}$$
for some $\lambda \in \mathbb{C} \setminus \{0\}$. Since $|T(0)| = 1$, then $|\lambda| = 1$. Hence,
$$T(z) = e^{i\theta} \cdot \frac{z - z_0}{z - \overline{z_0}}$$
for some $\theta \in \mathbb{R}$. □

Example 7.17. Find a Möbius transformation that maps from
$$D = \{z : |z| > 1, |z-1| < 2\} \text{ to } G = \{w : 0 < \mathrm{Re}(w) < 1\}.$$

Solution. Observe that the region D is bounded by two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 4$. The tangent to these circles is $x = -1$. We consider the conformal map $T(z) = 1/(z+1)$. Since $T(\mathbb{R}) = \mathbb{R}$ and C_1 and C_2 are perpendicular to $\mathbb{R}$, it follows that $T(C_1)$ and $T(C_2)$ are perpendicular to $\mathbb{R}$.

Hence, $T(C_1) = \{z : \mathrm{Re}(z) = 1/2\}$ and $T(C_2) = \{z : \mathrm{Re}(z) = 1/4\}$. So, $T(D)$ is bounded by these lines. Let $S(w) = 4w - 1$. Then, $S \circ T = (3-z)/(1-z)$ maps D onto G conformally. □

Example 7.18 (Dinh's 70 problems)**.** Let $T(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation.

(i) Assume that $z_1, z_2 \in \mathbb{C}$ are two distinct fixed points for T, i.e. $T(z_j) = z_j$, $j = 1, 2$. Show that there exists a constant λ such that
$$\frac{T(z) - z_1}{T(z) - z_2} = \lambda \cdot \frac{z - z_1}{z - z_2}.$$

(ii) Let $T^1(z) := T(z)$, $T^{n+1}(z) := T(T^n(z))$, $n = 1, 2, 3, \ldots$. Use **(i)** to find an expression for T^n, $n = 1, 2, 3, \ldots$, if
$$T(z) = \frac{1-3z}{z-3}.$$

Solution.

(i) We have

$$\begin{aligned}\frac{(T(z)-z_1)(z-z_2)}{(T(z)-z_2)(z-z_1)} &= \frac{\left(\frac{az+b}{cz+d}-\frac{az_1+b}{cz_1+d}\right)(z-z_2)}{\left(\frac{az+b}{cz+d}-\frac{az_2+b}{cz_2+d}\right)(z-z_1)} \\ &= \frac{((az+b)(cz_1+d)-(az_1+b)(cz+d))(cz_2+d)(z-z_2)}{((az+b)(cz_2+d)-(az_2+b)(cz+d))(cz_1+d)(z-z_1)} \\ &= \frac{cz_2+d}{cz_1+d}\end{aligned}$$

so $\lambda = \dfrac{cz_2+d}{cz_1+d}$.

(ii) We first find the fixed points of T. Set $\dfrac{-3z+1}{z-3} = z$, so $z = \pm 1$. We can take $z_1 = -1$ and $z_2 = 1$, so by repeatedly applying **(i)**, we have

$$\frac{T^n(z)+1}{T^n(z)-1} = \left(\frac{1}{2}\right)^n \cdot \frac{z+1}{z-1}.$$

□

7.7 Cross Ratio

Definition 7.10 (cross ratio)**.** The cross ratio of a 4-tuple of points $z_0, z_1, z_2, z_3 \in \mathbb{C}^*$ is defined to be

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

When one of the z_j is ∞, the RHS is understood as the limit as $z \to \infty$.

Example 7.19.

$$(\infty, z_1, z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}.$$

Proposition 7.2. A Möbius transformation T preserves cross ratios. That is,

$$(T(z_0), T(z_1), T(z_2), T(z_3)) = (z_0, z_1, z_2, z_3)$$

Lemma 7.5. Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}^*$, let $T(z) = (z, z_1, z_2, z_3)$. Then, T is a Möbius transformation and

$$T(z_1) = 1,\ T(z_2) = 0 \text{ and } T(z_3) = \infty.$$

In fact, T is the unique Möbius transformation such that the above holds.

Theorem 7.10. Given two sets of three distinct points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$, there exists a unique Möbius transformation T such that $T(z_j) = w_j$ for $j = 1, 2, 3$.

Corollary 7.2. Let z_0, z_1, z_2, z_3 be distinct points in $\mathbb{C}^*$. Then, they lie in a circle or a line in $\mathbb{C}^*$ if and only if $(z_0, z_1, z_2, z_3) \in \mathbb{R}$.

Example 7.20. Find a Möbius transformation f that maps $\mathbb{H}$ bijectively to the disc $D(0,2)$ such that $f(i)=1$ and $f(1)=-2$.

Solution. A Möbius transformation preserves points of symmetry so $f(-i)$ is symmetric to $f(i)=1$ with respect to $C(0,2)$. Hence, $f(-i)=4$. Since the Möbius transformation f preserves cross ratios, then

$$\begin{aligned}(f(z),f(1),f(i),f(-i)) &= (z,1,i,-i)\\ (f(z),-2,1,4) &= (z,1,i,-i)\\ \frac{f(z)-1}{f(z)-4}\cdot\frac{-6}{-3} &= \frac{z-i}{z+i}\cdot\frac{1+i}{1-i}\end{aligned}$$

Finding $f(z)$ is left as a simple algebraic exercise. Note that $f(-1)=2$. □

Chapter 8
Harmonic Functions

8.1
Basic Properties of Harmonic Functions

Recall that a real-valued function u is defined on a domain $\Omega \subseteq \mathbb{C}$ is harmonic if it belongs to $\mathcal{C}^2$ (second derivative of f is continuous on Ω) and $\Delta u = 0$. The real and imaginary parts of a holomorphic function are harmonic.

Proposition 8.1. Let Ω be a simply connected domain in $\mathbb{C}$. A function $u : \Omega \to \mathbb{R}$ is harmonic if and only if u is the real part of some holomorphic function on Ω.

The above proposition implies that for any domain Ω, u is harmonic if and only if it is locally the real (or imaginary) part of a holomorphic function. In particular, harmonic functions belong to $\mathcal{C}^\infty$.

Example 8.1. Consider the function

$$u(x,y) = \frac{1}{2}\log(x^2+y^2)$$

on the annulus $\Omega = \{0 < r < |z| < R\}$. This is not a simply connected domain, which means that not all simple closed curves in Ω can be shrunk to a point while remaining in Ω. One can establish that u is harmonic but there is no holomorphic function on Ω whose real part is equal to u.

Showing that u is harmonic, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is simple

Example 8.2. Prove that the function

$$u(x,y) = \frac{\sin x}{\cos x + \cosh y}$$

is harmonic in

$$\Omega = \{x+iy : -\pi < x < \pi \text{ and } y \in \mathbb{R}\}.$$

Solution. One can see that $\cosh y = \cos(iy)$, so by setting $z = x+iy$, where $-\pi < x < \pi$ and $y \in \mathbb{R}$, it is clear that

$$u(x,y) = \operatorname{Re}\left(\tan\left(\frac{z}{2}\right)\right).$$

□

Example 8.3 (Dinh's 70 problems)**.** Let $f = u+iv$ be a holomorphic function in an open set Ω. Define

$$U := e^{u^2-v^2}\cos(2uv) \text{ and } V := e^{u^2-v^2}\sin(2uv).$$

Show that U is harmonic and V is a harmonic conjugate of U.

Solution. To show that U is harmonic, we need to show that it satisfies Laplace's Equation, i.e. $U_{uu}+U_{vv}=0$. This is trivial. Next, one of the Cauchy-Riemann Equations states that $U_u = V_v$, so

$$V_v = -2e^{u^2-v^2}\left(v\sin(2uv)-u\cos(2uv)\right).$$

Using integration by parts or Euler's Formula, it can be shown that $\int V_v \, dv = V + c$, where c is an arbitrary constant. This shows that V is a harmonic conjugate of U. □

Theorem 8.1 (maximum-minimum principle)**.** If u is a real-valued non-constant harmonic function on a domain Ω, then u has no local maximum and no local minimum on Ω.

8.2
Dirichlet Problem and Poisson Kernel

Theorem 8.2 (Dirichlet problem)**.** Let Ω be a bounded domain in $\mathbb{C}$. Given a function $h : \partial\Omega \to \mathbb{R}$, is there a unique continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = h & \text{on } \partial\Omega? \end{cases}$$

In layman's terms, think of u being harmonic on the interior and $u = h$ on the boundary.

Definition 8.1 (Poisson kernel)**.** Define the Poisson kernel of the unit disc to be

$$P(a, e^{i\theta}) = \frac{1}{2\pi}\cdot\frac{1-|a|^2}{|e^{i\theta}-a|^2}.$$

We shall consider the case where Ω is the unit disc $\mathbb{D}$. The following theorem gives the uniqueness of the solution to the Dirichlet problem.

Theorem 8.3. Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be a continuous function which is harmonic in $\mathbb{D}$. Then, for each $a \in \mathbb{D}$,

$$u(a) = \int_0^{2\pi} P(a, e^{i\theta})u(e^{i\theta})\, d\theta.$$

Proof. Consider the automorphism of $\mathbb{D}$, which is

$$f(z) = \frac{a-z}{1-\overline{a}z.}$$

Note that $f(0) = a$ and f is self-inverse. Find f' and f'/f, then use Gauss' mean value theorem to prove the result. □

Corollary 8.1 (Harnack's inequality)**.** Let u be a harmonic function in a neighborhood of $\overline{\mathbb{D}}$. Assume that $u \geq 0$ on $\{|z| = 1\}$. Then,

$$\frac{1-|z|}{1+|z|}u(0) \leq u(z) \leq \frac{1+|z|}{1-|z|}u(0)$$

for $|z| < 1$.

Proof. Apply the Poisson kernel formula. Consider the region $|z| < 1$ and the identity $1 - |z|^2 = (1 + |z|)(1 - |z|)$. $\square$

Chapter 9
Analytic Continuation

9.1
Analytic Continuation

Definition 9.1 (analytic continuation)**.** Let f be a holomorphic function defined on a domain Ω. If there exists a domain $\Omega \subseteq \Omega'$ and a holomorphic function $F : \Omega' \to \mathbb{C}$ such that $F(z) = f(z)$ for each $z \in \Omega$, then F is an analytic continuation of f on Ω'.

Example 9.1. The power series

$$f(z) = 1 + z + z^2 + \ldots$$

has a radius of convergence $R = 1$ and so $f(z)$ is a holomorphic function on the unit disc $\mathbb{D}$. On the other hand, one can see that

$$f(z) = \frac{1}{1-z} \text{ for } |z| < 1$$

but $g(z) = 1/(1-z)$ is holomorphic on $\mathbb{C} \setminus \{1\}$. Thus, g is an analytic continuation of f to the much bigger domain $\mathbb{C} \setminus \{1\}$.

Lemma 9.1. Let $\Omega \subseteq \Omega'$ be domains in $\mathbb{C}$. Let F_1 and F_2 be analytic continuations of a holomorphic function $f : \Omega \to \mathbb{C}$ to a domain Ω'. Then,

$$F_1(z) = F_2(z) \quad \text{for all } z \in \Omega'.$$

Lemma 9.2. Let $f_j : \Omega_j \to \mathbb{C}$ be holomorphic functions such that $f_1(z) = f_2(z)$ for $z \in \Omega_1 \cap \Omega_2$. Then,

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in \Omega_1; \\ f_2(z) & \text{if } z \in \Omega_2 \setminus \Omega_1. \end{cases}$$

9.2
Schwarz Reflection Principle

We say that a region Ω is symmetric with respect to the real axis if $z \in \Omega$ implies $\bar{z} \in \Omega$. We consider here an important particular case of analytic continuation.

Theorem 9.1 (reflection principle for holomorphic functions)**.** Define Ω^+, Ω^-, L as the intersections of Ω with the upper half-plane, lower half-plane, and the real axis respectively. If f is a continuous complex-valued function on $\Omega^+ \cup L$, which is analytic on Ω^+ and real on L, then

$$f \text{ admits a unique extension} \quad \text{to a holomorphic function } F \text{ on } \Omega.$$

Moreover, the extension is given by

$$F(z) = \begin{cases} f(z) & \text{for } z \in \Omega^+ \cup L; \\ \overline{f(\overline{z})} & \text{for } z \in \Omega^-. \end{cases}$$

In particular, $F(\overline{z}) = \overline{F(z)}$ for all $z \in \Omega$.

Example 9.2 (MA5217 Lecture Notes)**.** Suppose f is holomorphic on $\mathbb{H}$ and continuous on $S = \mathbb{H} \cup (0,1)$. Assume $f(x) = x^4 - 2x^2$ for all $x \in (0,1)$. Find $f(i)$.

Solution. We have $f(i) = i^4 - 2i^2 = 1 + 2 = 3$. □