MA2104 Multivariable Calculus Notes

Thang Pang Ern

Reference books:

(1) Stewart, J. (2015). 'Calculus 8th Edition'. Cengage Learning.

Contents

1. The Dot and Cross Product

The relationship between points, lines, and planes (i.e. distance between two planes) is considered trivial for this discussion since these were already taught in H2 Mathematics.

1.1. *Dot Product*

Definition 1.1 (norm)**.** Let $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ be a vector in the Euclidean n-space. The norm of $\mathbf{v}$, denoted by $\|\mathbf{v}\|$, refers to its length, and we write

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \ldots + v_n^2}.$$

Definition 1.2. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Their dot product is defined to be

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \quad \text{where } 0 \leq \theta \leq \pi.$$

Definition 1.3 (orthogonal vectors)**.** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if the angle between them is 90°, or equivalently, $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 1.1 (triangle inequality)**.** For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have the following inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

and equality holds if and only if the triangle formed is degenerate, i.e. the vertices O, U, and V are collinear.

Proof. Expand the dot product $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$ and note that $|\cos\theta| \leq 1$. □

Corollary 1.1 (reverse triangle inequality)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have

$$\|\mathbf{u} - \mathbf{v}\| \geq \big|\|\mathbf{u}\| - \|\mathbf{v}\|\big|.$$

Proof. Write $\mathbf{u} = (\mathbf{u} - \mathbf{v}) + \mathbf{v}$. Applying the triangle inequality (Theorem 1.1) yields

$$\|\mathbf{u}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v}\| \quad \text{so} \quad \|\mathbf{u}\| - \|\mathbf{v}\| \leq \|\mathbf{u} - \mathbf{v}\|.$$

On the other hand, writing $\mathbf{v} = (\mathbf{v} - \mathbf{u}) + \mathbf{u}$ implies $\|\mathbf{v}\| - \|\mathbf{u}\| \leq \|\mathbf{v} - \mathbf{u}\|$. Since $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\|$, the result follows. □

Theorem 1.2. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Proof. Apply the definition of the dot product and recall that $|\cos\theta| \leq 1$. □

1.2. *Cross Product and Determinant*

We then introduce the cross product[†].

Definition 1.4 (cross product)**.** For two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, the cross product $\mathbf{u} \times \mathbf{v}$ is defined to be the determinant of the following matrix:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

We then discuss some properties of the scalar triple product for vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Proposition 1.1. The following properties hold:

(i) invariance under circular shifts:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

(ii) Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged, i.e.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

(iii) If any two vectors in the scalar triple product are equal, then its value is zero, i.e.

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{a}) = 0$$

Proof. The proofs of **(i)** and **(iii)** are straightforward; the proof of **(ii)** uses **(i)** and the commutativity of the dot product. □

Proposition 1.2 (Lagrange's formula)**.** $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Corollary 1.2 (Jacobi's identity)**.** $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

[†] Other than $\mathbb{R}^3$, the cross product is also and only valid in $\mathbb{R}^7$. This is tied to the algebraic and geometric framework of the Fano plane and the projective plane $\mathrm{PG}(2,2)$, where '2' in the second entry denotes the finite field of order 2.

2. Quadric Surfaces

2.1. *General Equation of a Quadric Surface*

A quadric surface is the graph of a second degree equation in x, y and z. That is,

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0.$$

Using translation and rotation, the equation can be expressed in one of the following two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{and} \quad Ax^2 + By^2 + Iz = 0$$

We will study a total of fifteen quadric surfaces in this section. Even though it may seem like there are many characteristics which we need to know such as the shape and critical points, it is crucial that we draw similarities between these quadric surfaces and the four conic sections - namely the circle, ellipse, parabola, and hyperbola. Along the way, we will provide certain techniques to remember them.

2.2. *Non-Degenerate Real Quadric Surfaces*

Definition 2.1 (ellipsoid)**.** The equation of an ellipsoid (Figure 1a) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Recall that the equation in Definition 2.1 is similar to the equation of an ellipse centered at the origin with semi-major axis a and semi-minor axis b, i.e. that is $a > b$.

Definition 2.2 (oblate and prolate spheroids)**.** If we have the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1,$$

we get either an oblate or a prolate spheroid (Figures 2b and 2a respectively).

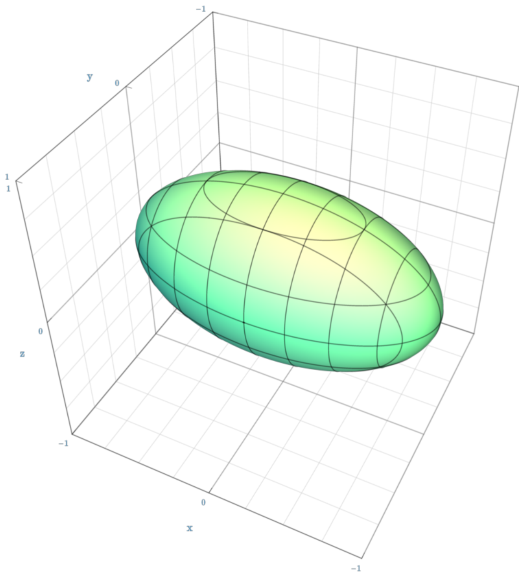
From Figures 2b and 2a, observe that the lines $z = k$, where $k \in \mathbb{R}$, cut the ellipsoid, forming concentric circles (i.e. circles sharing the same centre). This is true because each concentric circle is lying on the xy-plane and the vector with $\mathbf{k}$-component k is a normal vector to the plane.

Definition 2.3 (sphere)**.** If we have the equation

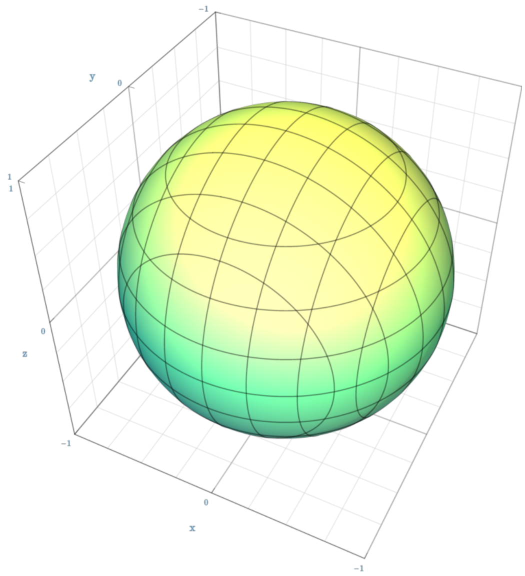
$$x^2 + y^2 + z^2 = a^2,$$

we have a sphere, which is also known as a spherical spheroid (Figure 1b).

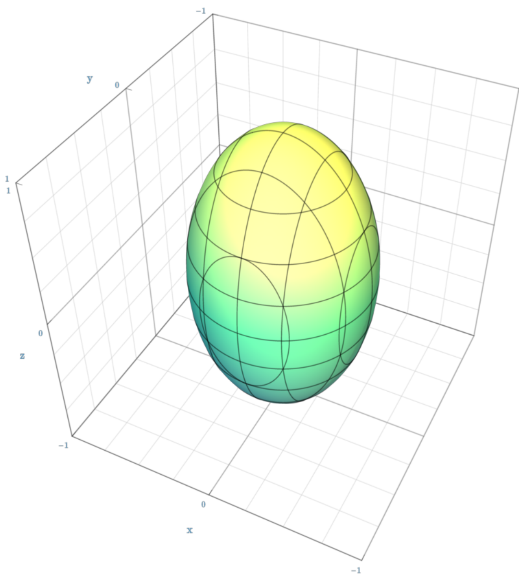
Definition 2.3 should be easy to remember as it as strong semblance to the equation of a circle.



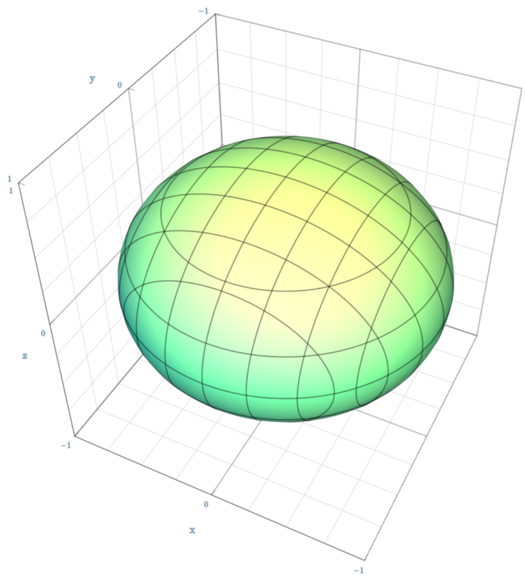
(a) Ellipsoid



(b) Sphere



(a) Prolate spheroid



(b) Oblate spheroid

Definition 2.4 (elliptic paraboloid). The equation of an elliptic paraboloid (Figure 3a) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0.$$

To remember Definition 2.4, make z the subject, which yields

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

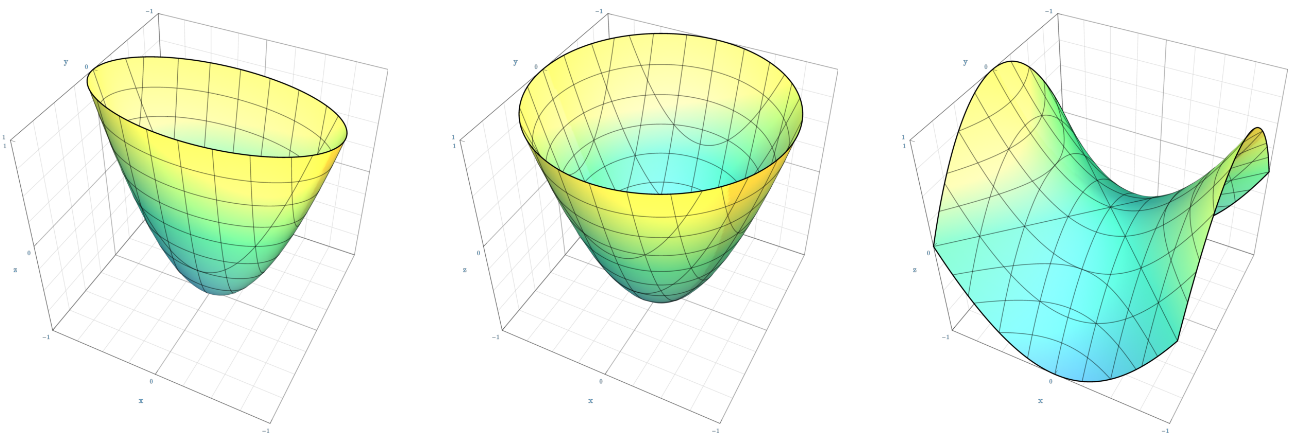
The RHS resembles that of an ellipse, which implies that any line $z=k$ that slices the elliptic paraboloid results in an ellipse to be obtained. As $x,y\geq 0$, it implies that when $x=y=0$, we obtain the minimum value of z, which is 0. Making reference to the xy-plane, as we go higher (i.e. z-coordinate increases), the semi-major axis and the semi-minor axis of the ellipses increase. This is a good way to remember how to sketch the elliptic paraboloid.

Definition 2.5 (circular paraboloid). A circular paraboloid (Figure 3b) is a special case of an elliptic paraboloid. It has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-z=0.$$

Definition 2.6 (hyperbolic paraboloid). The equation of a hyperbolic paraboloid (Figure 3c) is given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0.$$



(a) Elliptic paraboloid (b) Circular paraboloid (c) Hyperbolic paraboloid

A fun fact before we proceed with the properties of the hyperbolic paraboloid is that potato chips (I love sour cream and onion flavour) have such a shape. There is a saddle point (will be discussed in due course) on the quadric surface which allows easier stacking of chips.

We first observe that

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Without a loss of generality, we set

$$\begin{aligned} &\text{a line on the blue arrow} \quad \text{to be the } y\text{-axis} \quad \text{and} \\ &\text{a line on the red arrow} \quad \text{to be the } x\text{-axis} \end{aligned}$$

where these two lines are of course orthogonal to each other and their intersection is the origin. The intersection coincides with the saddle point. Observe that as x increases, z increases and as y increases, z increases. Regardless of the polarity of x and y, it would not affect z since x^2 and y^2 will always be non-negative. From here, observe that the loci of points traced by the red arrow and blue arrow form two hyperbolas opening in different directions.

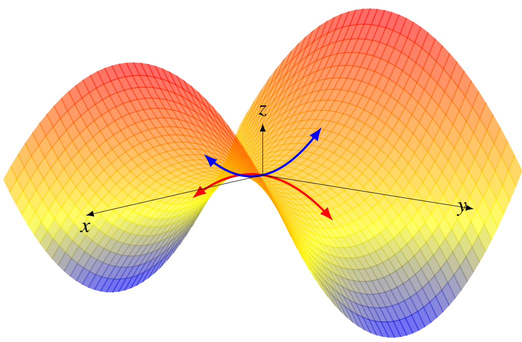


Figure 4: Analysing the saddle point $(0,0,0)$ of the hyperbolic paraboloid $z = x^2 - y^2$

Definition 2.7 (hyperboloid of one sheet). The equation of a hyperboloid of one sheet or a hyperbolic hyperboloid (Figure 5a) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Rearranging the equation in Definition 2.7, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1.$$

When $z = 0$, we obtain an ellipse on the xy-plane centered at the origin with semi-major axis a and semi-minor axis b (assuming that $a > b$). As z increases, then the semi-major and semi-minor axes increase as well, and as such, we obtain bigger ellipses. The same argument can be applied to the case where z decreases for $z < 0$ because z^2 will increase too.

Definition 2.8 (hyperboloid of two sheets). The equation of a hyperboloid of two sheets or an elliptic hyperboloid (Figure 5b) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$

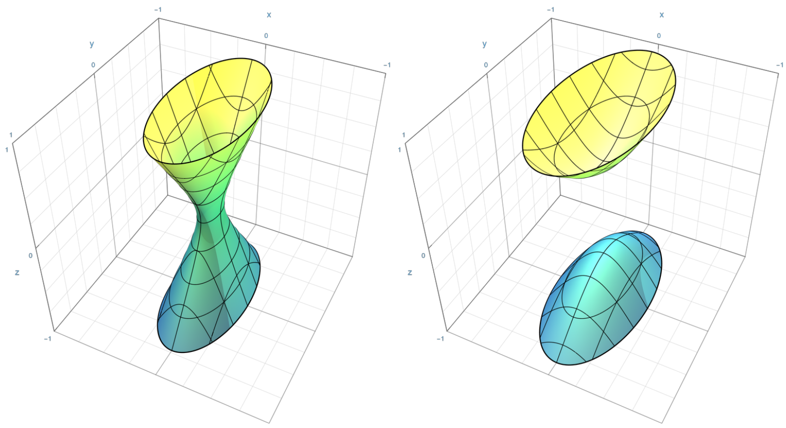
Rearranging,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}-1.$$

Note that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\geq 0 \quad \text{implies} \quad z^2\geq c^2 \quad \text{implies} \quad z\leq -c \text{ or } z\geq c.$$

Hence, we cannot sketch the quadric surface for $-c<z<c$. In a similar fashion to hyperboloids of one sheet, if we set the left side of the equation to be a constant k, then we obtain the equation of the ellipse, which semi-major and semi-minor axis increase for larger values of k. Note that k is affected by z^2 so as z^2 increases, k increases too.



(a) Hyperboloid of one sheet (b) Hyperboloid of two sheets

2.3. *Degenerate Real Quadric Surfaces*

Definition 2.9 (elliptic and circular cones)**.** The equation of an elliptic cone (Figure 6a) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

A circular cone with equation

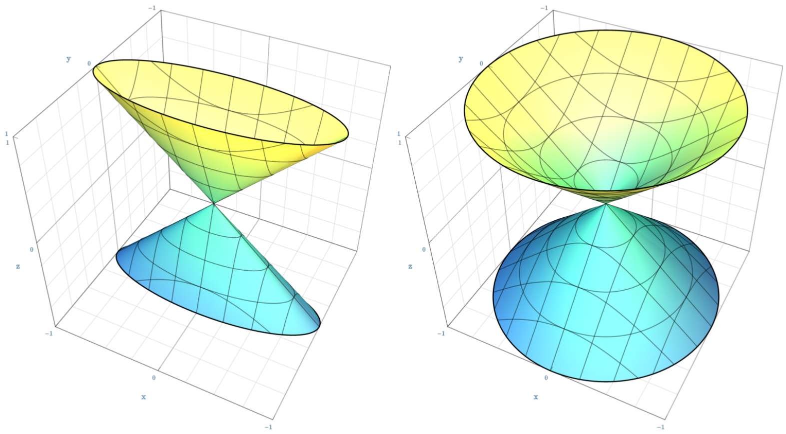
$$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$$

can be obtained from the above.

A good way to remember Definition 2.9 is by noting that the equation of an elliptic cone resembles that of an elliptic paraboloid.

Definition 2.10 (elliptic, circular, hyperbolic and parabolic cylinder)**.** The equation of an elliptic cylinder (Figure 7a) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$



(a) Elliptic cone (b) Circular cone

and that of a circular cylinder (Figure 7b) is

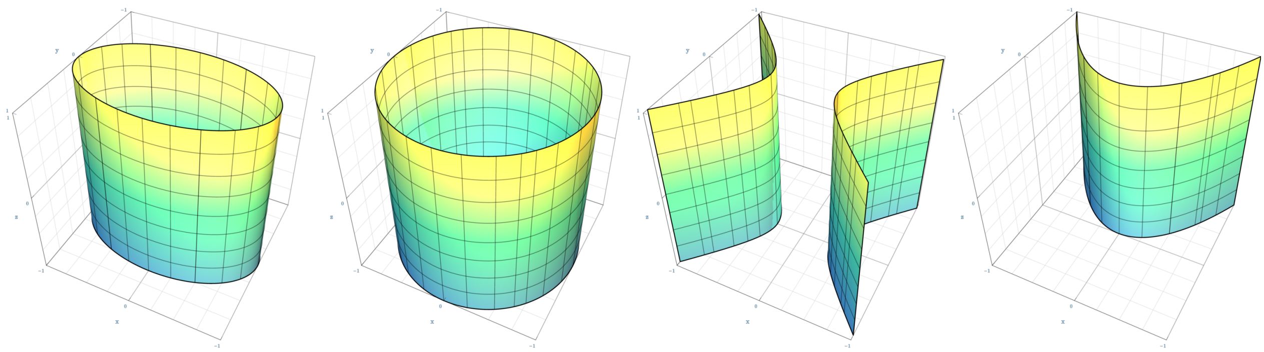
$$x^2 + y^2 = a^2.$$

The equation of a hyperbolic cylinder (Figure 7c) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

A parabolic cylinder (Figure 7d) is given by the equation

$$x^2 + 2ay = 0.$$



(a) Elliptic cylinder (b) Circular cylinder (c) Hyperbolic cylinder (d) Parabolic cylinder

Note that the first two equations equations in Definition 2.10 resemble the ellipse and circle respectively, just that now we are dealing in three-dimensional space.

3. Cylindrical and Spherical Coordinates

3.1. *Polar Coordinates*

Definition 3.1 (polar coordinates). The polar coordinates are given by (r,θ), where r is known as the radius vector and θ is known as the vectorial angle. The conversion formulae are

$$x = r\cos\theta \text{ and } y = r\sin\theta \quad \text{so equivalently} \quad x^2+y^2=r^2 \text{ and } \tan\theta = \frac{y}{x}$$

provided that $x \neq 0$.

Using a change of variables to polar coordinates, certain limits can be solved.

Example 3.1. We wish to compute the following limit by using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

Solution. Note that if we use Cartesian coordinates, we cannot obtain a solution since both the numerator and denominator tend to zero. Using polar coordinates, as $x,y \to 0$, then $r,\theta \to 0$. Hence,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} &= \lim_{(r,\theta)\to(0,0)} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \lim_{(r,\theta)\to(0,0)} (r\cos^3\theta + r\sin^3\theta) \\ &= 0\end{aligned}$$

□

3.2. *Cylindrical Coordinates*

Definition 3.2 (cylindrical coordinates). Suppose we want to convert Cartesian coordinates (x,y,z) to cylindrical coordinates (r,θ,z). For the x- and y- components, they are similar to polar coordinates. Thus, we have

$$x = r\cos\theta,\ y = r\sin\theta \text{ and } z = z.$$

3.3. *Spherical Coordinates*

For spherical coordinates, it is slightly more complicated. This time, we will convert from (x,y,z) to (ρ,θ,ϕ), where in a similar fashion, ρ is the radius and θ is the vectorial angle/inclination. The angle ϕ is known as the azimuthial angle, which can be commonly thought as the angle between the coordinate $P(x,y,z)$ and the z-axis.

Definition 3.3 (spherical coordinates). The conversion formulae state that

$$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi,$$

where $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

Observe that three equations in Definition 3.3 satisfy the identity $\rho^2 = x^2 + y^2 + z^2$.

One way how I like to remember this formula (but understand its derivation first) is to observe that x and y both contain the $\rho\cos\theta$ and $\rho\sin\theta$ components respectively, which resemble polar coordinates. Next, both x and y contain the term $\sin\phi$, which also appears in the calculation of its Jacobian determinant. If x and y contain $\sin\phi$, then z must contain $\cos\phi$!

Just to jump the gun, the Jacobian determinant for spherical coordinates, $\det(\mathbf{J})$, is given by $\det(\mathbf{J}) = \rho^2\sin\phi$. This idea will be formally introduced later.

Some limits can be solved by using a change of variables to spherical coordinates.

Example 3.2. Suppose we wish to find the following limit:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}.$$

Solution. As $x, y, z \to 0$, then it is clear that $\rho, \theta, \phi \to 0$ too.

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} &= \lim_{(\rho,\theta,\phi)\to(0,0,0)} \frac{\rho\sin\phi\cos\theta\cdot\rho\sin\phi\sin\theta\cdot\rho\cos\phi}{(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2+(\rho\cos\phi)^2} \\
&= \lim_{(\rho,\theta,\phi)\to(0,0,0)} \frac{\rho^3\cos\phi\sin^2\phi\cos\theta\sin\theta}{\rho^2} \\
&= \lim_{(\rho,\theta,\phi)\to(0,0,0)} \rho\cos\phi\sin^2\phi\cos\theta\sin\theta \\
&= 0
\end{aligned}$$

$\square$

4. Functions of Several Variables

4.1. *Vector Functions*

Definition 4.1 (vector function)**.** A vector function $\mathbf{r}(t)$ is a function whose domain is a set of real numbers and whose range is a set of vectors. It can be written as

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle .$$

Proposition 4.1. As t tends to a, then the limit of $\mathbf{r}(t)$ is defined by

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle .$$

Standard properties like linearity and scalar multiplicativity are considered *trivial* in this context. One important property in relation to the cross product is

$$\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}(t)\mathbf{v}'(t) + \mathbf{u}'(t)\mathbf{v}(t)$$

whose proof is rather rigorous. Note that this formula is analogous to the product rule.

Definition 4.2 (derivative of vector function)**.** For a vector function $\mathbf{r}(t)$, the derivative is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

which is a consequence of first principles.

Definition 4.3 (function of two variables)**.** A function f of 2 variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$. Here, D is the domain of f. The set of values that f takes on is the range of f. That is,

$$R_f = \{f(x,y) : (x,y) \in D\} .$$

The graph Γ of f is the set of all points $(x,y,z) \in \mathbb{R}^3$ such that $z = f(x,y)$.

Definition 4.4 (function of n variables)**.** Let

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R} \text{ where } f : (x_1, \ldots, x_n) \mapsto K \in \mathbb{R} \quad \text{be} \quad \text{a function of } n \text{ variables.}$$

We can describe f by examining its level surfaces (see Definition 4.5).

In Definition 4.4, we introduced what is called a function of n variables and mentioned that the function f can be described by examining its level surfaces. This is the more general case of what is known as a level curve (Definition 4.5).

Definition 4.5 (level curve). The level curves of a function of 2 variables are

$$\text{the curves in the } xy\text{-plane} \quad \text{with equation } f(x,y) = K \text{ where } K \in \mathbb{R}.$$

4.2. *Limits and Continuity*

Definition 4.6 (ε-δ definition of a limit). Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). We say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(x,y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{implies} \quad |f(x,y) - L| < \varepsilon.$$

Remark 4.1. Definition 4.6 can also be applied to a function that is defined parametrically, i.e. $f(x,y) = f(g(t),h(t))$.

If the limit L exists, then

(i) its value is unique and

(ii) L is independent of the choice of any path approaching (a,b)

The latter is related to a test (two path test) to determine if a limit exists.

To prove that the limit exists, we have to use the formal ε-δ definition. A common trick to simplify the inequality in a particular step would be to use the triangle inequality. On the other hand, to prove that the limit does not exist, we need to obtain a contradiction. This is where the two path test comes in handy. For example, the function might have two different values when approaching along the line $y = x$ and the curve $y = x^2$. This technique, though requires some intuition, is useful.

Definition 4.7 (continuity). A function f of 2 variables is said to be continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

f is continuous on $D \in \mathbb{R}^2$ if f is continuous at each point (a,b) in D.

Sidetrack to Real Analysis, the term *smoothness* is commonly associated with differentiability and continuity. Consider an open set (see Definition 7.9) on $\mathbb{R}$ and a function f defined on that set with real values. Let $k \in \mathbb{N}$. f is of class $\mathcal{C}^k$ if the derivatives $f', f'', \ldots, f^{(k)}$ exist and are continuous. f is

$$\text{infinitely differentiable, smooth or of class } \mathcal{C}^\infty \quad \text{if} \quad \text{it has derivatives of all orders.}$$

f is said to be of class C^ω, or *analytic*, if f is smooth and if its Taylor series expansion around any point in its domain converges to the function in some neighbourhood of the point. $\mathcal{C}^\omega$ is thus strictly contained in $\mathcal{C}^\infty$.

Simply said, $\mathcal{C}^0$ is the class of functions for which f is continuous, $\mathcal{C}^1$ is the class of functions for which f and f' are continuous, $\mathcal{C}^2$ is the class of functions for which f, f' and f'' are continuous and so on.

We shall address a common misconception in MA2002 on continuity and differentiability with the help of Figure 8. In Chinese, '可导函数' refers to a differentiable function, whereas '连续函数' refers to a continuous function. The foreground mentions '可倒一定连续' and the background mentions '连续不一定可倒', which translate to '*falling implies continuous*' and '*continuous does not imply that the bicycles would fall*'. For those who understand Chinese, this is precisely an example of a pun using homonyms because '可导' and '可倒' are pronounced the same but mean different things — '可导' means differentiable whereas '可倒' means able to fall!

Essentially, what Figure 8 is trying to address is that

differentiability implies continuity but the converse is not true in general.

One would know from MA2002 that $|x|$ is continuous on $\mathbb{R}$ but not differentiable at $x = 0$.

Figure 8: Bicycle meme

4.3. *Partial Derivatives*

Definition 4.8 (partial derivative). Let f be a function of 2 variables, namely x and y. The partial derivative of f with respect to x is denoted by $\partial f/\partial x$ or f_x and that with respect to y is denoted by $\partial f/\partial y$ or f_y. Using first principles, we have

$$\frac{\partial f(a,b)}{\partial x} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \quad \text{and} \quad \frac{\partial f(a,b)}{\partial y} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}.$$

In Definition 4.8, take f_x for instance. This means that we differentiate f with respect to x, while treating everything else as a constant. This is unlike the conventional differentiation problems that we have been exposed to in secondary school and junior college where we treat other letters like y, u, and v as variables. However, what is the geometric interpretation? It turns out that $f_x(a,b)$ measures the rate of change of f in the direction of $\mathbf{i}$ at the point (a,b).

We can also find expressions for higher order derivatives like $\partial^3 f/\partial x^3$. The following list shows how we can express them:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$
$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Some of us would have had exposure to ordinary differential equations (ODEs) in junior college. Here, partial derivatives have their own type of differential equations too, known as partial differential equations (PDEs)[†]. This imposes relations between the various partial derivatives of a multivariable function.

Example 4.1 (heat equation)**.** The heat equation is an example of a partial differential equation. We have

$$u_t = \kappa u_{xx} \quad \text{where } \kappa > 0 \text{ is the thermal conductivity of the material.}$$

It is known that the fundamental solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right).$$

Figure 9 provides a 3D surface plot of the solution $u(x,t)$ of the heat equation, where x is the horizontal axis, t is the axis going into the page (this denotes time), and $u(x,t)$ which denotes the temperature is the vertical axis. We see that as t increases, the solution $u(x,t)$ spreads out and becomes flatter, illustrating the diffusion of heat over time[‡].

[†]Take MA4221 Partial Differential Equations if you are interested in this topic. I strongly recommend the textbooks by Walter Strauss and Lawrence Evans.
[‡]I believe any person would find this fact on the diffusion of heat quite obvious.

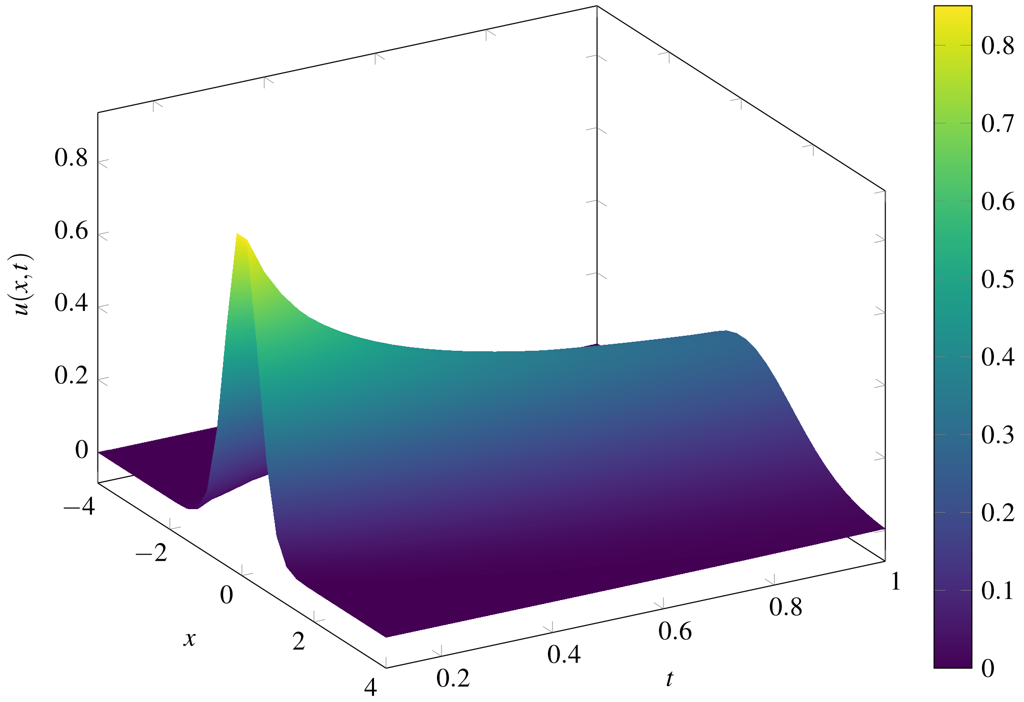


Figure 9: A graphical solution to the heat equation $u_t = u_{xx}$

Theorem 4.1 (Clairaut's theorem). Let f be defined on an open disk containing the point (a,b). If f_{xy} and f_{yx} are continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

It is also known as the symmetric property of the second derivatives.

We provide a proof of Clairaut's theorem using the mean value theorem. It can also be proven using iterated integrals which involves Fubini's theorem (will be discussed in one of the upcoming sections).

Proof. Assume that f_{xy} and f_{yx} are defined on a small open disk D centered at (a,b). Let (x,y) be a point in D. Fix x and consider the function

$$[f(x,y) - f(a,y)] - [f(x,b) - f(a,b)]$$

in y. Applying the mean value theorem with respect to y yields

$$[f(x,y) - f(a,y)] - [f(x,b) - f(a,b)] = [f_y(x,\zeta_1) - f_y(a,\zeta_2)]\,(y-b)$$

for some ζ_1 between y and b. Applying the mean value theorem to $f_y(x,\zeta_1)$ with respect to x,

$$[f(x,y) - f(a,y)] - [f(x,b) - f(a,b)] = f_{yx}(\zeta_2,\zeta_1)(x-a)(y-b)$$

for some ζ_2 between x and a. Note that

$$[f(x,y) - f(a,y)] - [f(x,b) - f(a,b)] = [f(x,y) - f(x,b)] - [f(a,y) - f(a,b).]$$

Applying the mean value theorem first with respect to x, then with respect to y, we have

$$\begin{aligned}[f(x,y)-f(a,y)]-[f(x,b)-f(a,b)] &= [f(x,y)-f(x,b)]-[f(a,y)-f(a,b)] \\ &= f_{xy}(\zeta_3,\zeta_4)(y-b)(x-a)\end{aligned}$$

where ζ_3 is between x and a, and ζ_4 is between y and b. Thus,

$$f_{xy}(\zeta_2,\zeta_1)=f_{yx}(\zeta_3,\zeta_4).$$

Since f_{xy} and f_{yx} are continuous on (a,b), by taking the limit as (x,y) tends to (a,b), we obtain Clairaut's theorem. □

4.4. *Tangent Plane*

Proposition 4.2 (tangent plane)**.** Let f be a function of two variables. The graph of f is a surface in $\mathbb{R}^3$ with equation $z=f(x,y)$. Let $P(x_0,y_0,z_0)$ be a point on this surface. Thus, $z_0=f(x_0,y_0)$. Assuming a tangent plane to the surface exists, its equation is given by

$$z=z_0+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

Proof. Any plane passing through $P(x_0,y_0,z_0)$ has the following equation:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

Assume that the plane is not vertical, i.e. $C\neq 0$. Then, the equation of the plane is

$$z-z_0=a(x-x_0)+b(y-y_0).$$

The tangent line to C_1 at P in the x-direction has a gradient of a, which by the geometric interpretation of the partial derivative, can be represented by $a=f_x(x_0,y_0)$. Similarly, $b=f_y(x_0,y_0)$. Substituting a and b into the equation of the plane yields the result. □

Note that the tangent plane to the surface $z=f(x,y)$ at $P(x,y,z)$ is very close to the surface at least when it is near P. Hence, we may use the function defining the tangent plane as a linear approximation to f. At the point $(x,y)=(a,b)$, note that P has coordinates $(a,b,f(a,b))$.

Definition 4.9 (linear approximation)**.** The linear function L whose graph is this tangent plane is given by

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

where L is known as the linearisation of f at (a,b). Therefore, the approximation

$$f(x,y)\approx L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

is called the linear approximation of f at (a,b).

The term differential refers to an infinitesimal change in some varying quantity. In single variable Calculus, we know that if y is a function of x, then the differential of y, known as dy, is related to dx by the following formula:

$$dy = \frac{dy}{dx} dx$$

Since we are now dealing with partial derivatives, we have a similar formula for the differential, dz, where $z = f(x,y)$.

Definition 4.10 (differential)**.** The differential dz is defined to be

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

The actual change Δz of z satisfies the equation $\Delta z \approx dz$ as a consequence of the tangent plane approximation.

Definition 4.11 (differentiability)**.** f is said to be differentiable at (a,b) if

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = 0 \quad \text{and} \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2 = 0.$$

If $f_x(x,y)$ and $f_y(x,y)$ exist in an open disk containing (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

4.5. *Chain Rule*

Proposition 4.3 (chain rule)**.** Suppose $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proposition 4.4 (implicit differentiation)**.** Suppose $F(x,y) = 0$ defines y implicitly as a function of x. That is $y = f(x)$. Then $F(x, f(x)) = 0$. We have the following result:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof. Using the chain rule to differentiate F with respect to x. □

4.6. *Directional Derivatives and Gradient Vector*

Definition 4.12 (directional derivative). Let f be a function of x and y. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \quad \text{provided that the limit exist.}$$

There are times where we are interested to compute the directional derivative of a function. Suppose f is a function in terms of x and y. Then, f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}.$$

Definition 4.13 (gradient vector). The gradient of a scalar-valued differentiable function f of several variables is the vector field ∇f whose value at a point P is the vector whose components are the partial derivatives of f at P. As such, it can be written as follows:

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

When we were talking about directional derivatives, observe that

$$\langle f_x(x, y), f_y(x, y) \rangle = \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j},$$

which implies that the gradient vector has an alternative (and useful) formula. That is,

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

Suppose $P \in D_f$. Then, note that

$$\nabla f(P) \cdot \mathbf{u} = \|\nabla f(P)\| \, \|\mathbf{u}\| \cos\theta = \|\nabla f(P)\| \cos\theta$$

where the second equality uses the fact that $\mathbf{u}$ is a unit vector. This equation implies that the maximum value of $D_{\mathbf{u}} f(P)$ is $|\nabla f(P)|$ and this occurs when $\mathbf{u}$ is acting in the same direction as the gradient vector $\nabla f(P)$. $\nabla f(P)$ can also be regarded as the direction and rate of fastest increase.

Here is a summary.

Proposition 4.5 (geometric properties of ∇f). Suppose f is a differentiable function at the point (a, b) such that $\nabla f(a, b) \neq 0$. Then, the following hold:

(i) The direction of $\nabla f(a, b)$ is perpendicular to the contour of f through (a, b) and it is in the direction of the maximum rate of increase of f

(ii) $\|\nabla f\|$ is the maximum rate of change of f at that point and is large when the contours are close together and small when they are far apart

5. Maxima and Minima

5.1. *Maxima, Minima and Saddle Points*

Definition 5.1 (local maximum and minimum). $f(x,y)$ has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some disk with centre (a,b). The value $f(a,b)$ is called a local maximum value. Similarly, if $f(x,y) \geq f(a,b)$ for all points (x,y) in some disk with centre (a,b), then $f(a,b)$ is a local minimum value.

Proposition 5.1. If f has a local maximum or a local minimum at (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$, i.e. $\nabla f(a,b) = \mathbf{0}$. (a,b) is a critical point of f if $f_x(a,b) = f_y(a,b) = 0$, or if one of these partial derivatives does not exist.

Definition 5.2 (saddle point). A function f is said to have a saddle point at (a,b) if there is a disk centred at (a,b) such that f assumes its maximum value on one diameter of the disk only at (a,b) and assumes its minimum value on another diameter of the disk only at (a,b).

To better understand Definition 5.2, at the saddle point, the derivatives in orthogonal directions are zero but the point is not regarded as a local extremum of the function.

Example 5.1. Refer to Figure 10 (or recall Figure 4) which shows the graph of $z = x^2 - y^2$. It is a hyperbolic paraboloid that passes through the origin. However, this point is not a local extremum so it is a saddle point. This is because along the path $z = x^2$, we observe that the origin *seems like* a minimum point. However, along the path $z = -y^2$, the origin *seems like* a maximum point.

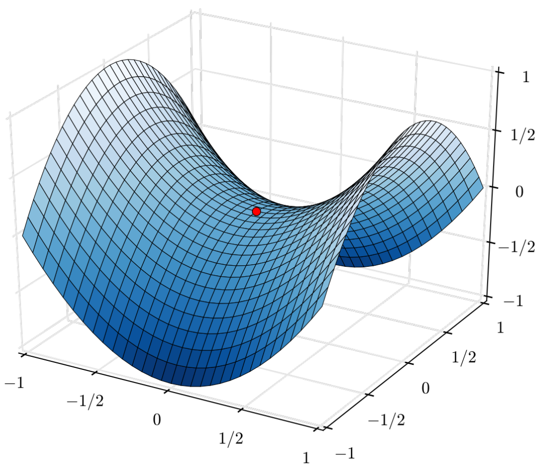


Figure 10: Graph of $z = x^2 - y^2$, which contains a saddle point

5.2. *Second Derivative Test*

Proposition 5.2 (second derivative test). Suppose the partial derivatives f_{xx}, f_{xy}, f_{yx} and f_{yy} are continuous on a disk with centre (a,b) and suppose $f_x(a,b) = f_y(a,b) = 0$. Let D (which is actually called the determinant) be defined as follows:

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

(1). $D > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b)
(2). $D > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b)
(3). $D < 0$, then f has a saddle point at (a,b)
(4). $D = 0$, then no conclusion can be drawn

To readers who are interested and have background knowledge on Linear Algebra, the Hessian matrix, denoted by $\mathbf{H}$ is of interest in this section. If the second partial derivatives of f exist and are continuous in its domain, then the Hessian Matrix of f is the following 2×2 (for simplicity sake) matrix:

$$\mathbf{H} = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$$

Observe that $\det(\mathbf{H}) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$, which is the same as what was stated previously in Proposition 5.2!

Example 5.2. Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disc centred at (a,b). If (a,b) is a critical point of f and $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, can you infer anything about $f(a,b)$? Give reasons for your answer.

Solution. Without loss of generality, suppose $f_{xx}(a,b) < 0$ and $f_{yy}(a,b) > 0$. By the second derivative test, we have

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2 < 0$$

so we conclude that (a,b) is a saddle point. □

5.3. *Some Analysis*

Definition 5.3 (bounded sets and closed sets). A bounded set in $\mathbb{R}^2$ is one that is contained in some disk. A closed set in $\mathbb{R}^2$ is one that contains all its boundary points.

Example 5.3. Figure 11 shows some bounded and closed sets.

Example 5.4. Figure 12 shows some non-examples. In particular, the left set is unbounded, whereas the middle and right sets are not closed.

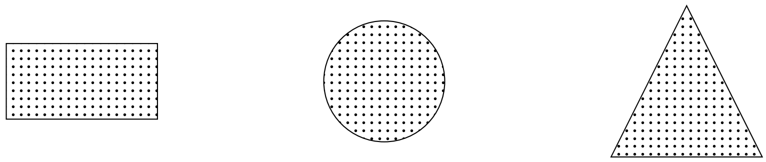


Figure 11: Bounded and closed sets

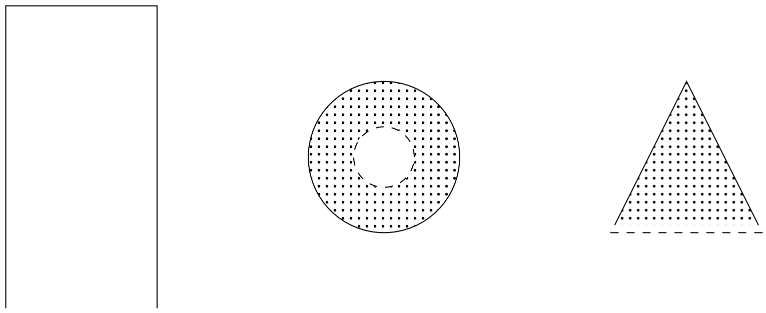


Figure 12: The left set is unbounded, whereas the middle and right sets are not closed

Theorem 5.1 (extreme value theorem). If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum/absolute minimum of a function defined on a closed and bounded set, firstly, we have to find the values of f at the critical points. That is, to find the coordinates (x, y) such that $f_x(x, y) = f_y(x, y) = 0$. We then find the extreme values of f on the boundary of D. Lastly, the largest of the values obtained from the earlier process yields the absolute maximum and vice versa.

5.4. *Method of Lagrange Multipliers*

The method of Lagrange multipliers involves maximising or minimising a function subject to a constraint. For example, the function could be $f(x, y)$ and the constraint could be $g(x, y) = 0$.

Figure 13 shows the level curves of $f(x, y) = \alpha$, where $8 \leq \alpha \leq 12$ and the constraint $g(x, y) = 0$. Suppose the extreme value of $f(x, y)$ subjected to the constraint is k and it is attained at the point (x_0, y_0). Firstly, the curve g must touch the level curve at $f(x, y) = k$ so as to allow one to move the point along $g(x, y) = 0$ in order to increase/decrease the value of f.
As such, observe that the gradients of f and g are parallel at (x_0, y_0).

Theorem 5.2 (method of Lagrange multipliers). If f attains an extreme value at (x_0, y_0),

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \quad \text{if } \nabla g(x_0, y_0) \neq \mathbf{0}$$

The method of Lagrange multipliers works for 2 constraints and 3 variables too! For such a case, suppose we want to maximise/minimise $f(x, y, z)$ subjected to the constraints $g(x, y, z) = h(x, y, z) = 0$.

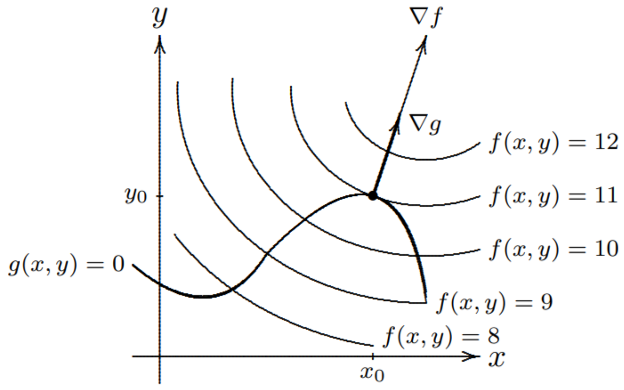


Figure 13: Geometric interpretation of the Lagrange multiplier

Theorem 5.3 (method of Lagrange multipliers)**.** If f attains an extreme value at (x_0,y_0,z_0),

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)+\mu\nabla h(x_0,y_0,z_0) \quad \text{if } \nabla g(x_0,y_0,z_0)\neq\mathbf{0} \text{ and } \nabla h(x_0,y_0,z_0)\neq\mathbf{0}.$$

To those who have substantial Olympiad experience, you would be familiar with some classical inequalities like Cauchy-Schwarz and AM-GM. Most extremum problems can be solved using these inequalities. However, there are some which cannot be solved using these methods. This is where the method of Lagrange multipliers comes in handy.

Example 5.5. Suppose $x,y,z\geq 0$ and they satisfy the equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

We wish to solve for x,y and z which can minimise

$$f(x,y,z)=x+y+z^2$$

and also, solve for the minimum of $f(x,y,z)$.

Solution. We let $f(x,y,z)$ and $g(x,y,z)$ denote the following:

$$f(x,y,z)=x+y+z^2 \quad \text{and} \quad g(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

Setting $\nabla f=\lambda\nabla g$, we have

$$\langle 1,1,2z\rangle=\lambda\left\langle -\frac{1}{x^2},-\frac{1}{y^2},-\frac{1}{z^2}\right\rangle,$$

which implies that $x^2=y^2=2z^2=-\lambda$. Note that $x^2=y^2$ so $x=y$ or $x=-y$. As $x,y\geq 0$, then the case where $x=-y$ does not make sense. We consider the case where $x=y$. Substituting into $g(x,y,z)=0$ yields

$$\frac{1}{x}+\frac{1}{x}+\frac{2^{\frac{1}{3}}}{x^{\frac{2}{3}}}=1.$$

Observe that $x=4$ satisfies the equation, implying that $y=4$ and $z=2$. Hence, $f_{\min}(4,4,2)=12$. $\square$

Example 5.6 (Stewart p. 1018 Question 49; AM-GM inequality)**.** The famous AM-GM Inequality, where AM stands for arithmetic mean and GM stands for geometric mean, can be proven using the Method of Lagrange multipliers too. Some other notable proofs are by using backward-forward induction (proven by Cauchy), using the exponential function e^x (proven by Pólya), as well as Jensen's inequality. The inequality states that

$$\frac{x_1+\ldots+x_n}{n} \geq (x_1\ldots x_n)^{\frac{1}{n}} \quad \text{where } x_1,\ldots,x_n \in \mathbb{R}_{\geq 0}$$

and equality is attained if and only if $x_1 = \ldots = x_n$. We shall prove this inequality.

Solution. The problem can be regarded as asking how do we maximise $f(x_1,x_2,\ldots,x_n) = x_1x_2\ldots x_n$ subject to the constraint $g(x_1,x_2,\ldots,x_n) = x_1+x_2+\ldots+x_n = c$. Using $\nabla f = \lambda\nabla g$,

$$\begin{aligned} x_2x_3\ldots x_n &= \lambda \\ x_1x_3\ldots x_n &= \lambda \\ \vdots &= \vdots \\ x_1x_2\ldots x_{n-1} &= \lambda \end{aligned}$$

This is a system of n equations. Multiplying all the left side of each equation together and equating them to the product of the right side of each equation,

$$(x_1x_2\ldots x_n)^{n-1} = \lambda^n.$$

Hence,

$$\begin{aligned} \frac{(x_1x_2\ldots x_n)^{n-1}}{(x_2x_3\ldots x_n)^{n-1}} &= \frac{\lambda^n}{\lambda^{n-1}} \\ x_1^{n-1} &= \lambda \\ x_1 &= \lambda^{\frac{1}{n-1}} \end{aligned}$$

It can be shown that

$$x_1 = x_2 = \ldots = x_n = \lambda^{\frac{1}{n-1}}$$

so

$$\begin{aligned} x_1+x_2+\ldots x_n = n\lambda^{\frac{1}{n-1}} &= c \\ \lambda &= \left(\frac{c}{n}\right)^{n-1} \\ x_1 = x_2 = \ldots = x_n &= \frac{c}{n} \end{aligned}$$

Now, note that $f(x_1,x_2,\ldots,x_n) = c^n/n^n$. As $g(x_1,x_2,\ldots,x_n) = c$, then

$$\frac{(x_1+x_2+\ldots+x_n)^n}{n^n} \geq x_1x_2\ldots x_n$$

so by taking the n^{th} root on both sides, we obtain the AM-GM inequality! □

6. Multiple Integrals

6.1. *Double Integrals*

We shall refer to Figure 14. Let f be a function of two variables defined over a rectangle $R = [a,b] \times [c,d]$. We would like to define the double integral of f over R as the volume of the solid under the graph of $z = f(x,y)$ over R. Without a loss of generality, we let $f(x,y) \geq 0$ here.

We subdivide R into mn small rectangles, each of area ΔA, namely R_{ij}, where $1 \leq i \leq m$ and $1 \leq j \leq n$. The volume of an infinitesimally small rectangular solid erected over R_{ij} is its height multiplied by its area, which is $f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$.

Using ideas of a Riemann sum, the double integral of f over R is defined by

$$\iint_R f(x,y)\ dA = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A \quad \text{if the limit exists,}$$

i.e. the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$\iint_R f(x,y)\ dA.$$

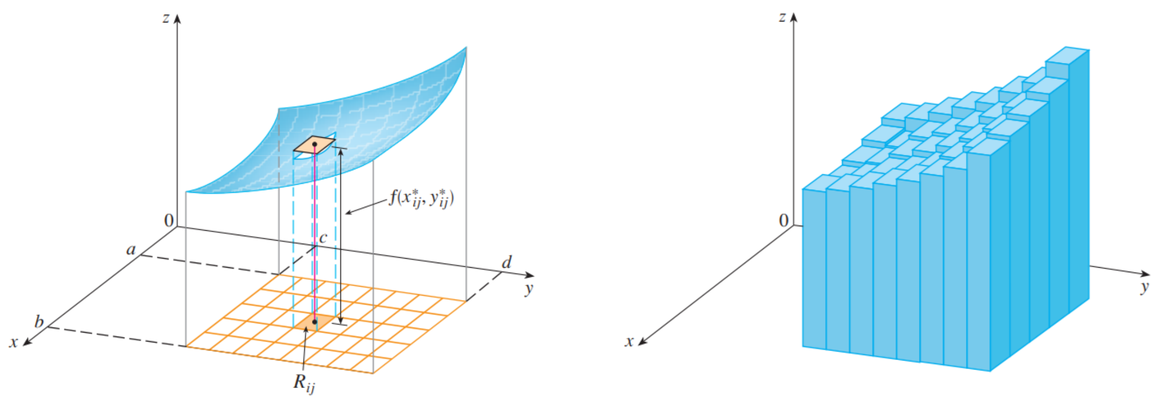


Figure 14: Geometric interpretation of the double integral

Proposition 6.1. If $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\ dA \quad \text{always exists.}$$

Proposition 6.2. Some properties of double integrals are as follows:

(i)

$$\iint_D \left(\alpha f(x,y) + g(x,y)\right)\ dA = \alpha \iint_D f(x,y)\ dA + \iint_D g(x,y)\ dA \quad \text{where } \alpha \in \mathbb{R}$$

This means that the integral operator is a linear transformation.

(ii) If $f(x,y) \geq g(x,y)$ for all $x \in D$, then

$$\iint_D f(x,y)\ dA \geq \iint_D g(x,y)\ dA.$$

(iii)

$$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA,$$

provided that $D = D_1 \cup D_2$ and D_1 and D_2 do not intersect except perhaps on their boundary.

(iv) If f, is bounded (i.e. $m \leq f(x,y) \leq M$), then

$$mA(D) \leq \iint_D f(x,y)\ dA \leq MA(D)$$

where $A(D)$ denotes the area of D.

Note that **(ii)** of Proposition 6.2 uses the fact that

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

which is a familiar concept from MA2002.

Example 6.1 (Stewart p. 1040 Question 38)**.** Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1,1] \times [1,2]$.

Solution. We have

$$V = \iint_R z\, dA = \int_{-1}^{1} \int_{1}^{2} 3y^2 - x^2 + 2\, dydx$$

which evaluates to $52/3$. □

Example 6.2 (Stewart p. 1040 Question 49)**.** Use symmetry to evaluate the double integral

$$\iint_R \frac{xy}{1+x^4}\, dA \quad \text{where } R = \{(x,y) : -1 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Solution. We have

$$\int_{-1}^{1} \int_0^1 \frac{xy}{1+x^4}\, dydx = \int_{-1}^{0} \int_0^1 \frac{xy}{1+x^4}\, dydx + \int_0^1 \int_0^1 \frac{xy}{1+x^4}\, dydx.$$

By symmetry (essentially the substitution $u = -x$),

$$\int_{-1}^{0} \int_0^1 \frac{xy}{1+x^4}\, dydx = -\int_0^1 \int_0^1 \frac{uy}{1+u^4} dydu,$$

so it follows that the integral over R evaluates to 0. □

6.2. *Iterated Integrals*

Definition 6.1 (iterated integral). Let $f(x,y)$ be a function defined on $R = [a,b] \times [c,d]$.

$$\int_c^d f(x,y)\ dy \quad \text{means that} \quad x \text{ is regarded as a constant}$$

and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$. Thus,

$$\int_c^d f(x,y)\ dy \quad \text{is a function of } x$$

and we can integrate it with respect to x from $x = a$ to $x = b$. The resulting integral

$$\int_a^b \int_c^d f(x,y)\ dydx$$

is known as an iterated integral.

Fubini's theorem allows the order of integration to be changed in certain iterated integrals.

Theorem 6.1 (Fubini's theorem). If $f(x,y)$ is *absolutely convergent* and continuous on $R = [a,b] \times [c,d]$,

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy.$$

As mentioned earlier, for Fubini's Theorem to be applied, f must be an absolutely convergent integral. Similar to the absolute convergence of series, we have a similar definition for integrals.

Definition 6.2 (absolute convergence). if an integral is said to be absolutely convergent, then

$$\int_R |f(x)|\ dx < \infty.$$

Example 6.3. One of the ways to evaluate the famous Gaussian integral, which is

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi} \quad \text{involves Fubini's theorem.}$$

Proof. We will use polar coordinates. Let I be the original integral. Then,

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} e^{-x^2}\ dx = \int_{-\infty}^{\infty} e^{-y^2}\ dy \\
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\ dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\ dxdy \quad \text{by Fubini's theorem} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dxdy
\end{aligned}$$

We will do a change of variables from Cartesian to polar coordinates. We will establish the following result

$$dxdy = rdrd\theta$$

using the Jacobian of a suitable matrix. That is,

$$\mathbf{J} = \begin{bmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{bmatrix}.$$

Since $dxdy = \det(\mathbf{J})\, drd\theta$, then the result follows. Hence, the integral can be transformed to

$$I^2 = \int_0^{2\pi} \int_0^{\infty} re^{-r^2}\, drd\theta = \pi$$

and we conclude that $I = \sqrt{\pi}$. $\square$

Proposition 6.3 (type 1 region). Consider a region D bounded by the vertical lines $x = a$ and $x = b$ and the curves $y = g_1(x)$ and $y = g_2(x)$, where $a < x < b$ and $g_1(x) < y < g_2(x)$ (Figure **??**). The double integral of f over D can be expressed as

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx.$$

Proposition 6.4 (type 2 region). Consider a region D bounded by the horizontal lines $y = c$ and $y = d$ and the curves $x = h_1(y)$ and $x = h_2(y)$, where $c < y < d$ and $h_1(y) < x < h_2(y)$ (Figure 16). The double integral of f over D can be expressed as

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dxdy.$$

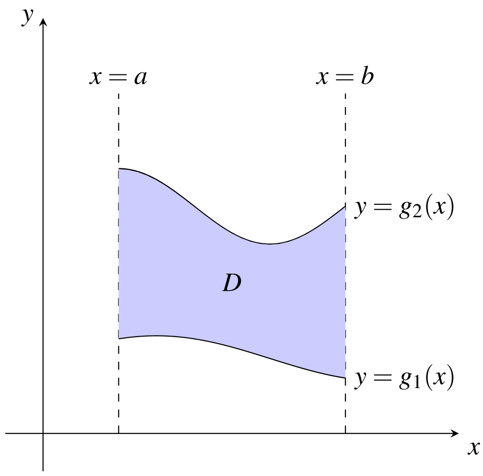


Figure 15: Type 1 region

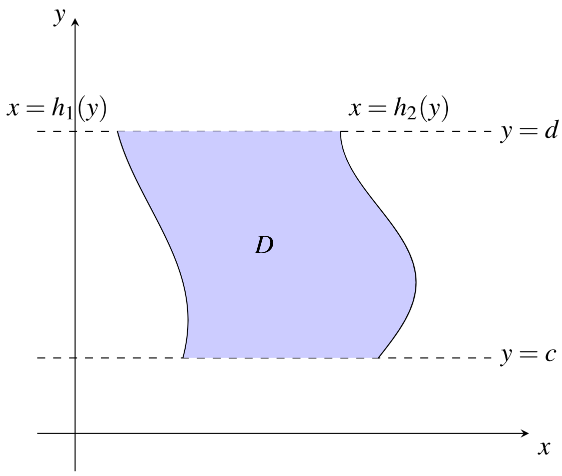


Figure 16: Type 2 region

Example 6.4. We can evaluate

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dxdy$$

by changing the order of integration because

$$\int e^{x^4}\, dx \quad \text{does not have an elementary anti-derivative.}$$

Solution. Consider the graph of $x = \sqrt[3]{y}$, or by making y the subject, $y = x^3$. Note that the desired region satisfies both $\sqrt[3]{y} \leq x \leq 2$ so $y \leq x^3 \leq 8$ and $0 \leq y \leq 8$. Thus, if we were to change the order of integration such that we integrate with respect to y first, then we have the inequalities $0 \leq y \leq x^3$ and $0 \leq x \leq 2$. Hence,

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dxdy = \int_0^2 \int_0^{x^3} e^{x^4}\, dydx = \int_0^2 \left[ye^{x^4}\right]_0^{x^3}\, dx = \frac{1}{4}\left(e^{16} - 1\right).$$

□

Example 6.5. We can also prove that

$$\int_0^1 \int_x^1 \sin\left(y^2\right)\, dydx = \frac{1}{2}(1 - \cos 1)$$

by changing the order of integration too.

Solution. Note that $x \leq y \leq 1$ and $0 \leq x \leq 1$. Hence, if we were to integrate with respect to x first, note that $0 \leq x \leq y$. Then, since $0 \leq y \leq 1$, we have

$$\int_0^1 \int_x^1 \sin\left(y^2\right)\, dydx = \int_0^1 \int_0^y \sin\left(y^2\right)\, dxdy = \int_0^1 \left[x \sin\left(y^2\right)\right]_0^y\, dy = \frac{1}{2}(1 - \cos 1).$$

□

6.3. *Double Integrals in Polar Coordinates*

We first introduce the area differential in polar coordinates. Consider a point (r, θ) on a plane. Small increments in r and θ are denoted by dr and $d\theta$ respectively. Hence, by considering the area of the new sector is $dA = rdrd\theta$, which is known as the area differential.

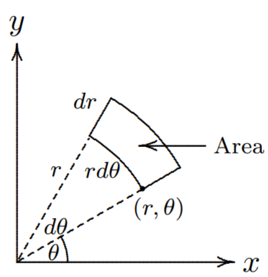


Figure 17: Area differential in polar coordinates

Let f be a continuous function defined on a polar rectangle R, where

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$$

and $0 \le \beta - \alpha \le 2\pi$. The double integral of f over R can be expressed as

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ rdrd\theta.$$

6.4. *Surface Area*

Let f be a differentiable function of 2 variables defined on a domain D. We wish to find the surface area, $A(S)$, of the graph of f over D, i.e.

$$\iint_D dS.$$

We wish to express the differential of the surface area, dS, in terms of the differential of the domain, dA. Note that $dA = |dxdy|$.

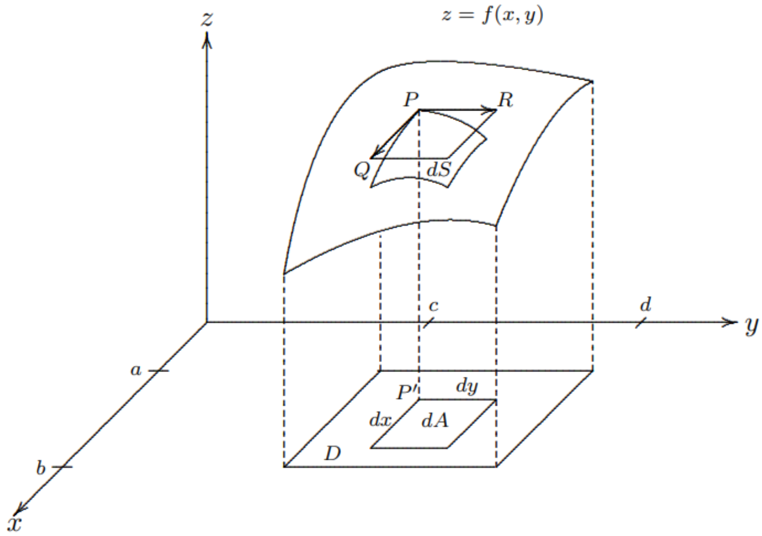


Figure 18: Derivation of the surface area formula

From Figure 18, let $\overrightarrow{PQ} = \langle dx, 0, f_x(x,y)\,dx\rangle$ and $\overrightarrow{PR} = \langle 0, y, f_y(x,y)\,dy\rangle$. Since $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are linearly independent and span the tangent plane, it is clear that the area of the tangent plane (which is in the shape of a parallelogram) formed by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ can be calculated by taking their cross product. That is,

$$\text{area of tangent plane} = \left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \left|\langle -f_x, -f_y, 1\rangle\, dxdy\right| = \sqrt{f_x^2 + f_y^2 + 1}\ dA$$

and hence, by the Fundamental Theorem of Calculus,

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\ dA.$$

Theorem 6.2 (surface area)**.** The area of the surface with equation $z = f(x,y)$, where $(x,y) \in D$ and f_x and f_y are continuous, is

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\ dA.$$

6.5. *Triple Integrals*

Similar to using a Riemann sum to explain double integrals, for triple integrals, we consider a continuous function $f : B \subseteq \mathbb{R}^3 \to \mathbb{R}$, where B is defined to be the following rectangular solid:

$$B = \{(x,y,z) : a \le x \le b, c \le y \le d, r \le z \le s\}$$

Dividing $[a,b],[c,d]$ and $[r,s]$ into l,m and n equal subintervals respectively, the triple integral of f over B is

$$\iiint_R f(x,y,z)\ dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\ \Delta V.$$

Note that Fubini's theorem still applies to triple integrals. There are $3! = 6$ such iterated integrals involved and they are all equal. The limits of integration are usually tricky to find. With sufficient practice, one should eventually be proficient at it.

Example 6.6 (Stewart p. 1040 Question 42). Find the volume of the solid in the first octant bounded by

$$\text{the graph of } z = 16 - x^2 \quad \text{and} \quad \text{the plane } y = 5.$$

Solution. The volume is

$$\int_0^4 \int_0^5 \int_0^{16-x^2} dzdydx = \frac{640}{3}.$$

□

Example 6.7. Suppose we wish to evaluate

$$\iiint_E xy\ dV,$$

where E is the solid tetrahedron with vertices $(0,0,0),(1,0,0),(0,2,0)$ and $(0,0,3)$.

Solution. We find the equation of the plane that is formed by the vertices $(1,0,0),(0,2,0)$ and $(0,0,3)$ since the tetrahedron is in the first octant where all the x,y and z values are positive. The equation of the plane is $6x+3y+2z=6$. Hence, E is the region that lies below the plane with x-, y-, and z-intercepts 1,2 and 3 respectively. Now, we find the bounds for x, y and z.

Suppose we integrate in the order $dzdydx$, then note that as $6x+3y+2z=6$, then $z = \frac{1}{2}(6-6x-3y)$, which implies that $0 \le z \le \frac{1}{2}(6-6x-3y)$. Next, $y = \frac{2}{3}(3-3x-z)$, so when $z=0$, we obtain an upper bound for y, which is $2-2x$. Hence, $0 \le y \le 2-2x$. It is clear that $0 \le x \le 1$. Combining everything together (I omitted the integration process since it is trivial),

$$\iiint_E xy\ dV = \int_0^1 \int_0^{2-2x} \int_0^{\frac{1}{2}(6-6x-3y)} xy\ dzdydx = \frac{1}{10}.$$

□

Definition 6.3 (Steinmetz solid). A Steinmetz solid is the solid obtained by the intersection of two or three cylinders of equal radius at right angles. In particular, a bicylinder is the intersection of 2 cylinders; a tricylinder is the intersection of 3 cylinders (Figure 19).

Proposition 6.5. The following properties hold:

(i) For a bicylinder of radius r,

$$V = \frac{16}{3}r^3 \quad \text{and} \quad A = 16r^2.$$

(ii) For a tricylinder of radius r,

$$V = V = 8\left(2-\sqrt{2}\right)r^3 \quad \text{and} \quad A = 24\left(2-\sqrt{2}\right)r^2.$$

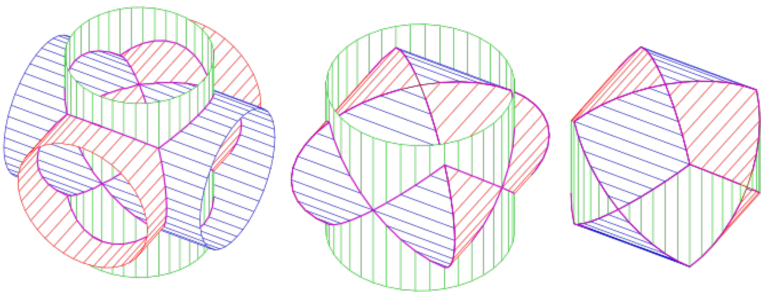


Figure 19: Forming a tricylinder

We shall prove a result in Proposition 6.5. That is, the volume of a bicylinder of radius r is $16r^3/3$.

Proof. Without a loss of generality, suppose that the two cylinders have equations $x^2+y^2=r^2$ and $x^2+z^2=r^2$. As $z=\pm\sqrt{r^2-x^2}$ and $y=\pm\sqrt{r^2-x^2}$, then plugging into the formula gives

$$\iiint_R 1\, dV = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 1\, dzdydx = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2}\, dydx$$

which is equal to

$$\int_{-r}^{r} 4\left(r^2-x^2\right)\, dx.$$

Evaluating this integral, the result follows. □

The formulae for the volumes of classical 3D shapes, which include the cuboid, circular cylinder, sphere and cone can be derived through integration. Conversion to cylindrical or spherical coordinates are more helpful, where appropriate[†].

[†]You may visit the following link which directs you to a summary page of the volumes of 3D shapes using integration created by the University of Washington.

Proposition 6.6 (type 1 region)**.** For a Type 1 region E (Figure 20), it is of the form

$$E = \{(x,y,z) : (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}.$$

We can write the triple integral as follows:

$$\iiint_E f(x,y,z)\ dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\right] dA$$

Proposition 6.7 (type 2 region)**.** For a Type 2 region E, it is of the form

$$E = \{(x,y,z) : (y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\}.$$

We can write the triple integral as follows:

$$\iiint_E f(x,y,z)\ dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx\right] dA$$

Proposition 6.8 (type 3 region)**.** For a Type 3 region E, it is of the form

$$E = \{(x,y,z) : (x,z) \in D, u_1(x,z) \le y \le u_2(x,z)\}.$$

We can write the triple integral as follows:

$$\iiint_E f(x,y,z)\ dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\right] dA$$

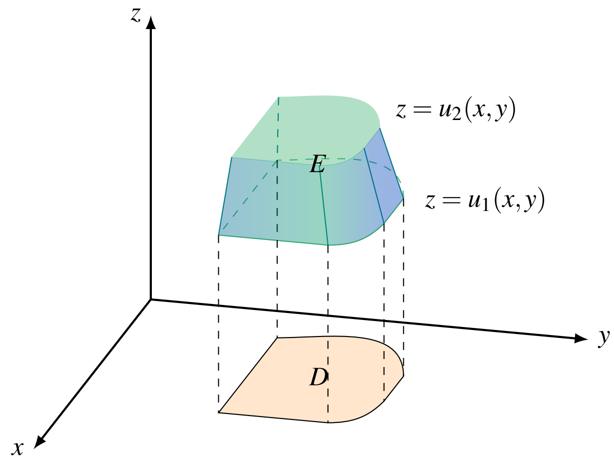


Figure 20: Type 1 region

For conversion to cylindrical coordinates, consider a region E defined by

$$E = \{(r,\theta,z) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta), u_1(r,\theta) \le z \le u_2(r,\theta)\}.$$

The triple integral of $f(x,y,z)$ over E can be expressed as

$$\iiint_E f(x,y,z)\ dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(\theta)}^{u_2(\theta)} f(r\cos\theta, r\sin\theta, z)\ rdzdrd\theta.$$

Similarly, for spherical coordinates, we consider a spherical wedge E where its spherical coordinates are bounded as follows:

$$E = \{(\rho,\theta,\phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

The triple integral of f over E can be expressed as

$$\iiint_E f(x,y,z)\ dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\ d\rho d\theta d\phi.$$

6.6. *Change of Variables and the Jacobian*

Definition 6.4 (bijective map). Consider two regions $S, R \subseteq \mathbb{R}^2$, where $(u_1,v_1) \in S$ and $(x_1,y_1) \in R$. Let $T : S \to R$ be a $\mathcal{C}^1$ transformation,

$$T : S \to R \text{ where } (u_1,v_1) \mapsto (x_1,y_1) \quad \text{be a } \mathcal{C}^1 \text{ transformation.}$$

We say that

$$T \text{ is bijective} \quad \text{if and only if} \quad T \text{ is injective and surjective.}$$

In Definition 21, assumming that T^{-1} exists, the point (x_1,y_1) in R will get mapped to (u_1,v_1) in S. This shows that T is injective. T is surjective too since (u_1,v_1) and (x_1,y_1) have pre-images, which are (x_1,y_1) and (u_1,v_1) respectively, implying that T is bijective.

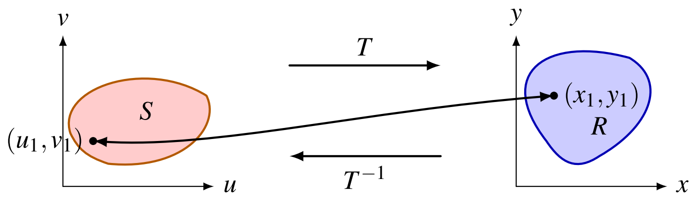


Figure 21: A bijective map T

The Jacobian of the transformation T given by $x = x(u,v), y = y(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The area differential, dA, can be written as

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

As such, we deduce the following important result (Theorem 6.3).

Theorem 6.3 (change of variables). Let $T(u,v)$ be a bijective $\mathcal{C}^1$-transformation whose Jacobian is non-zero except possibly at a finite number of points. Suppose T maps a region S in the uv-plane onto a region R of the xy-plane. Suppose f is continuous on R. Then,

$$\iint_R f(x,y)\ dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \ dudv.$$

For the case of triple integrals, we have a completely analogous formula for change of variables.

7. Introduction to Vector Calculus

7.1. *Vector Fields*

Definition 7.1 (vector field)**.** Let $D \subseteq \mathbb{R}^2$. A vector field on D is a function $\mathbf{F}$ that assigns to each point (x,y) in D a two dimensional vector $\mathbf{F}(x,y)$. $\mathbf{F}$ can be written in terms of its component functions. That is

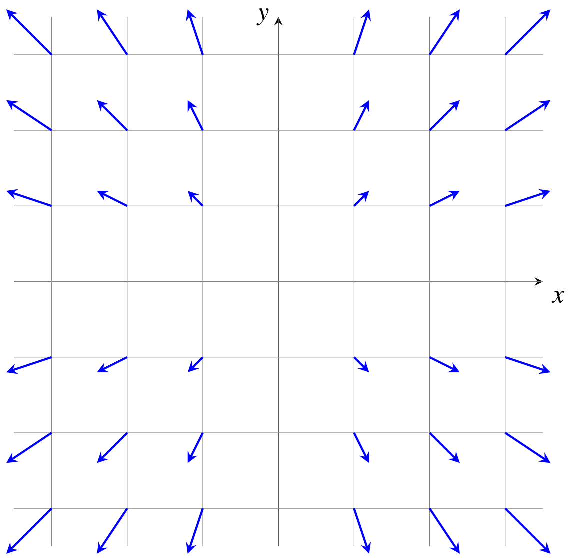
$$\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle \quad \text{or} \quad \mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

Remark 7.1. For the vector field to be defined on its domain, D, each of its component vectors must be continuous on D.

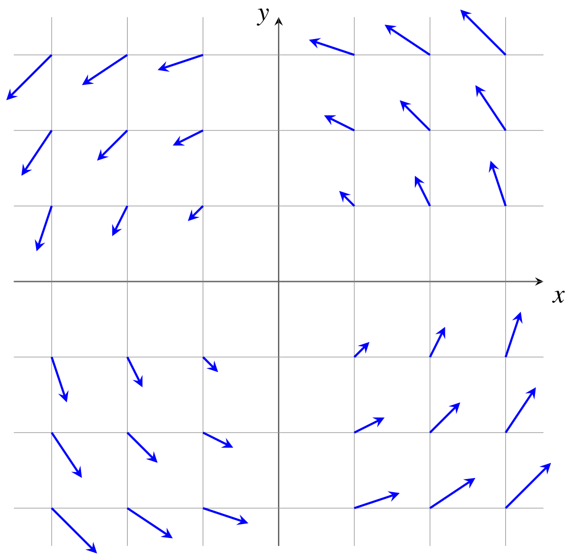
There is a variety of vector fields in $\mathbb{R}^2$. For example, there are radial and rotational vector fields.

Example 7.1 (radial field and rotational field)**.** A radial field is one where all the vectors point towards or away from the origin and is rotationally symmetric. An example would be $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ (Figure 22a).

An example of a rotational field in $\mathbb{R}^2$ is $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ (Figure 22b).



(a) Radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$



(b) Rotational field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

Figure 23 depicts the electric field of an electric dipole — two opposite charges, one positive $(+q)$ and one negative $(-q)$. The field lines originate from the positive charge and end on the negative charge. Electric field lines are a visual aid. Each line indicates the direction of the electric field at every point along it. The strength (magnitude) of the field at a given location is often inferred from how closely spaced the lines are in that region. Regions where the lines are packed densely indicate a stronger field, and where they are more spread out, the field is weaker.

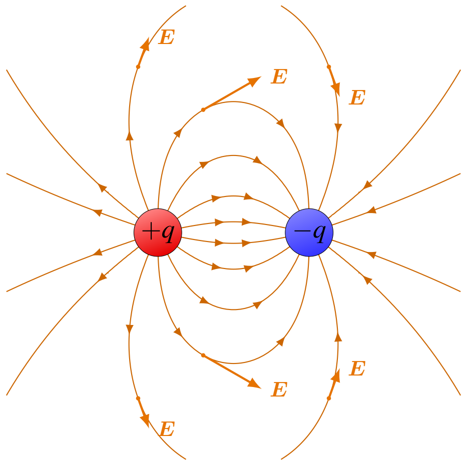


Figure 23: An electric field involving a positive and a negative point charge

We can also extend this idea to 3-dimensional vector fields.

Definition 7.2 (vector field). Let $E \subseteq \mathbb{R}^3$. A vector field on E is a function $\mathbf{F}$ that assigns to each point (x,y,z) in E a three dimensional vector $\mathbf{F}(x,y,z)$. In terms of its component functions, $\mathbf{F}$ can be written as

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle \quad \text{or} \quad \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

7.2. *Gradient Fields*

Definition 7.3 (gradient vector field). For any $n \geq 2$, if

$$f : \mathbb{R}^n \to \mathbb{R} \text{ is differentiable} \quad \text{then} \quad \nabla f \text{ is a vector field on } \mathbb{R}^2$$

and it is the gradient vector field of f.

Note that the gradient vectors are perpendicular to the level curves as proven using the chain rule.

Definition 7.4 (conservative vector field). A vector field $\mathbf{F}$ is conservative if it is the gradient of some scalar function, i.e.

$$\text{there exists a differentiable function } f \quad \text{such that} \quad \mathbf{F} = \nabla f.$$

In this situation, f is a potential function for $\mathbf{F}$.

7.3. *Introduction to Line Integrals*

Definition 7.5 (line integral). Consider a plane curve C with equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$. Assume that C is a smooth curve, i.e. $\mathbf{r}'(t) \neq 0$, and $\mathbf{r}'(t)$ is continuous. Let $f(x,y)$ be a continuous function defined in a domain containing C. The line integral of C is

$$\int_C f(x,y)\ ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

Recall from MA2002 that

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \quad \text{gives} \quad \text{the arc length of the curve from } a \text{ to } b.$$

A piecewise smooth curve C can be regarded as the union of a finite number of smooth curves, C_i, where $1 \le i \le n$, or in other words

$$C = \bigcup_{i=1}^n C_i \quad \text{where the initial point of } C_{i+1} \text{ is the terminal point of } C_i.$$

In Graph Theory terminologies, the above is the same as a *finite walk*. The line integral f along C can hence, be written as

$$\int_C f(x,y)\ ds = \sum_{i=1}^n \int_{C_i} f(x,y)\ ds.$$

The line integrals of f along C with respect to x and y respectively are

$$\int_C f(x,y)\ dx = \int_a^b f(x(t), y(t))\, x'(t)\ dt \text{ and } \int_C f(x,y)\ dx = \int_a^b f(x(t), y(t))\, y'(t)\ dt.$$

Definition 7.6 (line integral). For a smooth space curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{j}$ where $a \le t \le b$, the formula for a line integral of a scalar field in $\mathbb{R}^3$ is

$$\int_C f(x,y,z)\ ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt.$$

7.4. *Line Integrals of Vector Fields*

Proposition 7.1 (line integral). Let $\mathbf{F}$ be a continuous vector field defined on a domain containing a smooth curve C given by a vector function $\mathbf{r}(t)$, where $a \le t \le b$. The line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt.$$

We can write $\mathbf{F}$ and C in their component forms, where

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} \quad \text{and} \quad C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

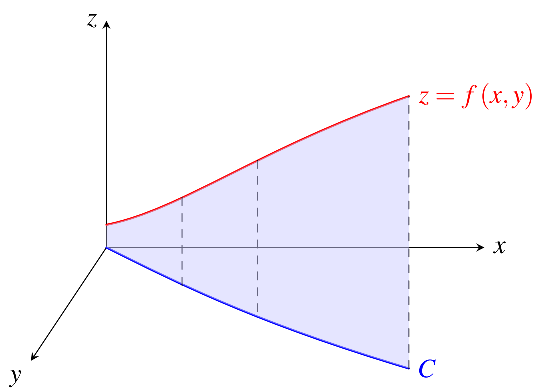


Figure 24: The line integral over a scalar field f is the area under the curve C along a surface $z = f(x,y)$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

Theorem 7.1 (fundamental theorem of line integrals). let C be a smooth curve given by $\mathbf{r}(t)$, where $a \leq t \leq b$. Let f be a function of 2 or 3 variables whose gradient, denoted by ∇f, is continuous. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

One should see the fundamental theorem of line integrals (Theorem 7.1) as a concept analogous to the second Fundamental Theorem of Calculus in MA2002 — now we apply this to any curve in a plane or space.

Definition 7.7 (path independence). Let $\mathbf{F}$ be a continuous vector field with domain D. The line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \text{ is independent of path in } D \quad \text{if} \quad \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any 2 paths C_1 and C_2 in D that have the same initial and terminal points.

Definition 7.8 (closed path). A path is closed if its terminal point coincides with its initial point. Also,

$$\int_C \mathbf{F} \cdot d\mathbf{r} \text{ is independent of path in } D \quad \text{if and only if} \quad \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for every closed path in } D.$$

In Definition 7.4, we said that a vector field $\mathbf{F}$ is conservative if there exists a differentiable function f such that $\mathbf{F} = \nabla f$.

Example 7.2 (gravitational field). In Physics, the gravitational field is conservative. Consider the gravitational force between two objects of masses m and M. By Newton's law of gravitation, we have the following result:

$$\mathbf{F} = -\frac{GMm}{|\mathbf{r}|^3}\mathbf{r} \quad \text{where } G \text{ is the universal gravitational constant and } r = \langle x, y, z \rangle .$$

A negative sign is present in the formula to indicate that the gravitational field is attractive. In this case, the potential function, f (known as the gravitational potential energy), is given by

$$f = \frac{GMm}{|\mathbf{r}|}.$$

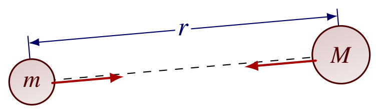


Figure 25: Gravitational force between two objects of masses m and M

Proposition 7.2. The following statements are equivalent:

(i) $\mathbf{F}$ is conservative on D

(ii) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D

(iii) $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path C in D

Definition 7.9 (open set and connected set)**.** A set D is open if for every point P in D, we can construct a disc about P such that it lies entirely within D. A connected set is one such that any two points in D can be joined by a path in D.

Example 7.3 (open set and connected set)**.** Figure 26a shows an open set D. To see why it is open, consider a point $P \in D$. We can construct a small disc around it such that the disc lies completely within the set.

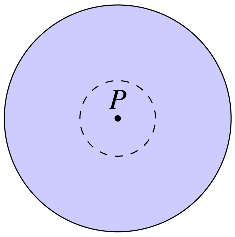
Figure 26b shows a connected set as any two points A and B can be joined by a path within the set.

Lastly, Figure 26c shows a disconnected set as there exist points J and K that cannot be joined by a path within the set.

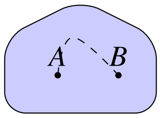
Definition 7.10 (simply-connected domain)**.** A simply-connected region D is one in which every simple closed curve in D encloses only points within D.

To show that a vector field $\mathbf{F}$ is non-conservative, it suffices to show that

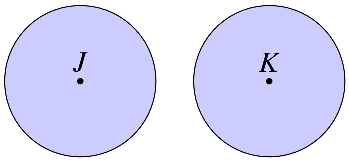
there exist two paths with the same initial and terminal points but their line integrals are different.



(a) An open set D



(b) A connected set



(c) A disconnected set

On the other hand, to show that a vector field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ in an open and simply-connected region $D \subseteq \mathbb{R}^2$ is conservative, where P and Q have continuous partial derivatives on D, it suffices to show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

In fact, this equation and the statement that $\mathbf{F}$ is conservative are equivalent!

8. Differential Operators

8.1. *Green's Theorem*

Green's theorem (Theorem 8.1) gives the relationship between a line integral along a simple closed curve C on the plane and the double integral over the plane region D that C bounds.

Theorem 8.1 (Green's theorem)**.** Let C is a positively oriented (single counterclockwise transversal), piecewise-smooth and simple closed curve on the plane and D is the region bounded by C (Figure 27). If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open simply connected region that contains D, then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

We say that $\partial D = C$, where ∂ denotes boundary. So, the boundary of the region D is the curve C!

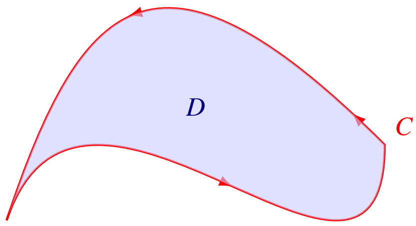


Figure 27: Illustration of the curve C and region D mentioned in Theorem 8.1

We can extend Theorem 8.1 to regions that are not simply-connected as they can be divided into regions which are simply connected. The line integrals along the divides end up being opposite in orientation and cancel out, and the result follows.

To analyse this, consider the region D in Figure 28, where $\partial D = C_1 + C_2$. We may cut the region D by 2 line segments L_1 and L_2 into 2 simply connected regions D' and D'' respectively. Observe that C_1 and C_2 are traversing in opposite directions.
As

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA - \left(-\iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA\right),$$

then it can be verified that

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_C Pdx + Qdy,$$

which correlates with what was mentioned regarding Green's Theorem.

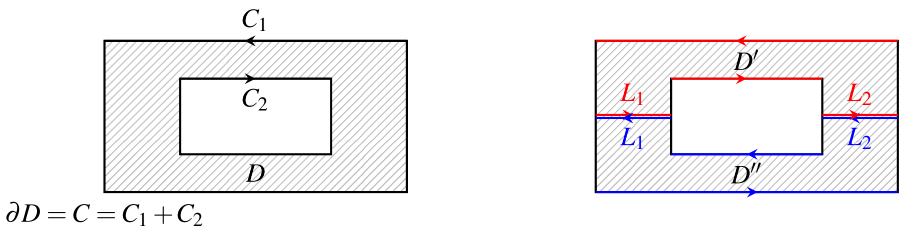


Figure 28: Green's theorem for a non simply-connected region

8.2. *Curl*

Definition 8.1 (curl). Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in $\mathbb{R}^3$. The curl of $\mathbf{F}$ is

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

This formula can be remembered easily by considering a suitable cross product. We first define the differential operator, ∇, by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Note that

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

and upon expansion of the determinant yields the result as mentioned.

Proposition 8.1. Some properties of the curl of $\mathbf{F}$ are as follows:

(i) If $f(x,y,z)$ has continuous second order partial derivatives, then $\operatorname{curl}(\nabla f) = \mathbf{0}$

(ii) If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$

(iii) If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative

We will only prove **(i)**.

Proof. By Clairaut's theorem,

$$\text{curl } (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}.$$

The result follows. □

8.3. *Divergence*

Definition 8.2 (divergence). Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in $\mathbb{R}^3$. The divergence of $\mathbf{F}$ is defined by

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Theorem 8.2. If P, Q and R have continuous second order partial derivatives, then $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

Proof. Use Clairaut's theorem. □

Theorem 8.3. A nice property related to divergence and curl is that if $\mathbf{F}$ and $\mathbf{G}$ are vector fields in $\mathbb{R}^3$, then

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G}\operatorname{curl}\mathbf{F} - \mathbf{F}\operatorname{curl}\mathbf{G}.$$

Proof. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and $\mathbf{G} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then,

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \operatorname{div}\left(\begin{bmatrix} P \\ Q \\ R \end{bmatrix} \times \begin{bmatrix} A \\ B \\ C \end{bmatrix}\right) = \operatorname{div}\begin{bmatrix} CQ - BR \\ AR - CP \\ BP - AQ \end{bmatrix}.$$

Expanding yields

$$\begin{aligned} \operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial(CQ)}{\partial x} - \frac{\partial(BR)}{\partial x} + \frac{\partial(AR)}{\partial x} - \frac{\partial(CP)}{\partial x} + \frac{\partial(BP)}{\partial x} - \frac{\partial(AQ)}{\partial x} \\ &= \underbrace{A\left(\frac{\partial R}{\partial z} - \frac{\partial Q}{\partial z}\right) - B\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + C\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\mathbf{G}\operatorname{curl}\mathbf{F}} \\ &\quad \underbrace{+P\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) - Q\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + R\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)}_{-\mathbf{F}\operatorname{curl}\mathbf{G}} \end{aligned}$$

and the result follows. □

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f.

Definition 8.3 (Laplacian). If f is a function of three variables, then

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

$\operatorname{div}(\nabla f)$ can also be written as $\nabla^2 f$, and we refer to this as the Laplace operator or the Laplacian.

Definition 8.3 has strong connections with Laplace's equation, an example of a partial differential equation. That is,

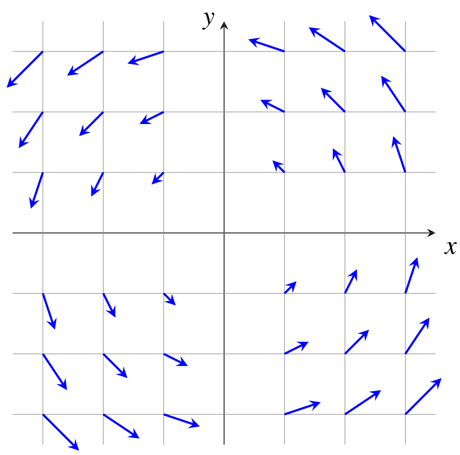
$$\nabla^2 f = 0 \quad \text{or equivalently} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Consider a velocity vector field $\mathbf{F}$. Divergence measures the amount of flow radiating at a point. The curl measures the rotational effect of the vector field. If the flow is uniform and without compression or expansion, then div $\mathbf{F} = 0$. As such, if div $\mathbf{F} > 0$, there is a net outflow; if div $\mathbf{F} < 0$, there is a net inflow.

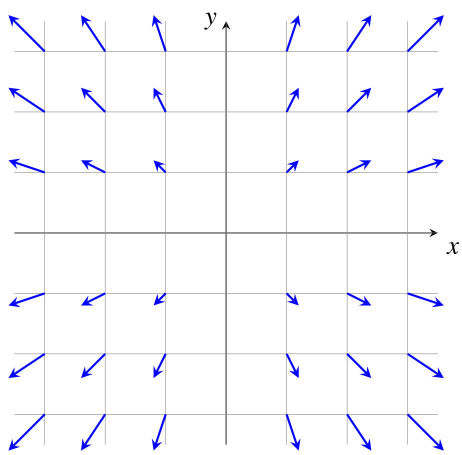
Definition 8.4 (incompressible and rotational vector fields)**.** For a velocity vector field $\mathbf{F}$,

$$\operatorname{div}\mathbf{F} = 0 \text{ implies } \mathbf{F} \text{ is incompressible} \quad \text{and} \quad \operatorname{curl}\mathbf{F} = \mathbf{0} \text{ implies } \mathbf{F} \text{ is irrotational.}$$

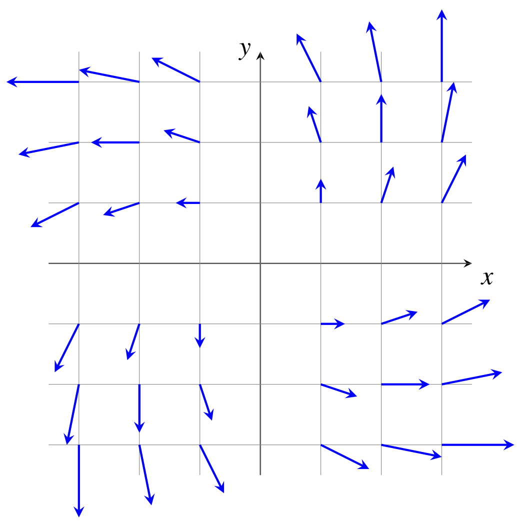
Example 8.1. Here are some examples of various vector fields.



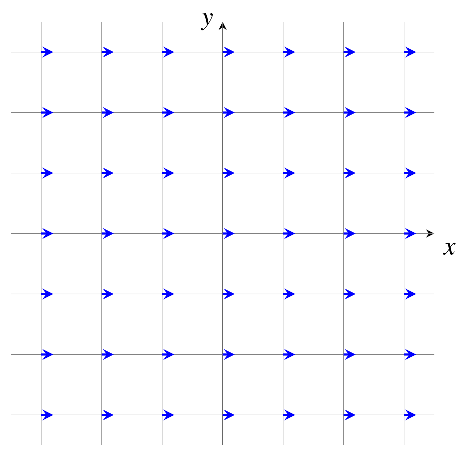
(a) Incompressible but not irrational



(b) Irrotational but not incompressible



(c) Neither irrotational nor incompressible



(d) Both irrotational and incompressible

Figure 29: Examples of velocity vector fields and their characteristics

8.4. *Green's Theorem in Vector Forms*

Theorem 8.4 (Green's theorem, alt.). We have three alternative representations of Green's theorem. They are as follows:

(i)

$$\int_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA \quad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k} \text{ is a vector field in } \mathbb{R}^3$$

(ii) Suppose ∂D can be parametrised by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, where $a \leq t \leq b$. Assuming the pararmetrisation gives the positive orientation of ∂D, the unit tangent vector is

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Then, Green's theorem can be also expressed as

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot \mathbf{T} \, ds \quad \text{where} \quad ds = \left\|\mathbf{r}'(t)\right\| dt \text{ is the arc length differential.}$$

(iii) We can also derive a formula involving the normal component of $\mathbf{F}$ along ∂D. So, Green's theorem will be stated in terms of $\operatorname{div} \mathbf{F}$. Using the parametrisation of C in **(ii)**, one can easily verify (by taking dot product with $\mathbf{T}$) that the outward unit normal vector to ∂D, $\mathbf{n}$, is

$$\mathbf{n}(t) = \left\langle \frac{y'(t)}{\|\mathbf{r}'(t)\|}, -\frac{x'(t)}{\|\mathbf{r}'(t)\|} \right\rangle \quad \text{where} \quad \mathbf{n} \text{ is pointing outwards.}$$

So,

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div } \mathbf{F} \, dA.$$

It only suffices to provide a brief explanation as to why **(i)** holds.

Proof. **(i)** uses the fact that

$$\text{curl } \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

(iii) follows from the first Green's theorem that we discussed in Theorem 8.1. □

In summary, we have the following 3 integrals:

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA \quad \text{and} \quad \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot \mathbf{T} \, ds \quad \text{and} \quad \int_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \text{div } \mathbf{F} \, dA$$

9. Parametric Surfaces, Oriented Surfaces and Integral Theorems

9.1. *Tangent Planes*

Definition 9.1 (parametric surface). Let $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ be a vector-valued function defined on a region D in the uv-plane. Then,

$$S = \{(x,y,z) : x = x(u,v), y = y(u,v), z = z(u,v), (u,v) \in D\}$$

is a parametric surface. $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ are the parametric equations of S.

Definition 9.2 (tangent plane). Let S be a parametric surface defined by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. The equation of the tangent plane to S at a point P_0 with position vector $\mathbf{r}_0 = \mathbf{r}(u_0,v_0)$ is

$$(\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{r}_u \times \mathbf{r}_v),$$

where

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right\rangle \text{ and } \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle.$$

It is not surprising that the formula in Definition 9.2 holds as $\mathbf{r}_u$ and $\mathbf{r}_v$ lie on the tangent plane to S at P_0, so their cross product gives the normal to the plane.

Definition 9.3 (smooth surface). S is said to be smooth if $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ for all $(x,y) \in D$.

9.2. *Surface Area*

Definition 9.4 (surface area). Suppose a smooth parametric surface S is injective except possibly on the boundary of D (Figure 30). The surface area of S over D

$$A(S) = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| \, dA.$$

Let S be a surface which is the graph of a function $f(x,y)$ defined on a domain $D \subseteq \mathbb{R}^2$. Recall that

$$A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1} \, dA.$$

We have the following corollary, which is already taught in MA2002.

Corollary 9.1. Consider a curve $y = f(x)$ where $a \leq x \leq b$, $f(x) \geq 0$ and $f'(x)$ is continuous. S is the surface obtained by rotating the curve 2π radians about the x-axis. Then, the area of the

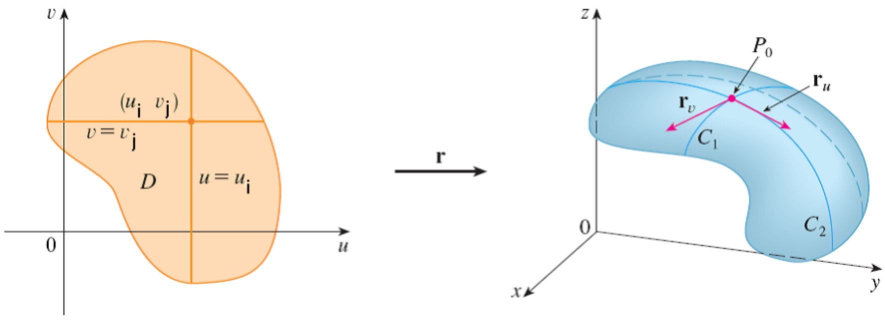


Figure 30: Geometric interpretation of the surface integral of a scalar field

surface of revolution is given by

$$2\pi \int_a^b f(x) \sqrt{1 + f'(x)}\, dx.$$

The case where the curve $x = g(y)$ (for $c \leq y \leq d$) is rotated about the y-axis yields a similar formula.

A surface integral is related to surface area much like how a line integral is related to arc length.

Theorem 9.1 (surface integral)**.** Let $f(x,y,z)$ be a continuous function defined on S. The surface integral of f over S is

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v)) \|\mathbf{r}_u \times \mathbf{r}_v\|\ dA.$$

If S is the graph of $z = g(x,y)$, then

$$\iint_S f(x,y,z)\ dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + g_x^2 + g_y^2}\ dA$$

9.3. *Oriented Surfaces*

Definition 9.5 (orientable surface)**.** A surface S is orientable if it is two-sided, otherwise it is non-orientable.

Example 9.1. Some examples of orientable surfaces include a sphere, plane, cylinder and elliptic paraboloid. On the other hand, some examples of non-orientable surfaces are the Möbius strip (Figure 31) and the Klein bottle[†].

If S is orientable, then it is possible to choose a unit normal vector $\mathbf{n}$ at every point S so that $\mathbf{n}$ varies continuously over S. In that case, S is an oriented surface and the choice of $\mathbf{n}$ is an orientation of S. There are only 2 orientations of S, namely one for each side of the surface which corresponds to the choice where all $\mathbf{n}$ point away from that side of the surface.

[†] The interesting surfaces, namely the Möbius strip and the Klein bottle, will be revisited in MA4266 — a nice blend of Abstract Algebra and Topology.

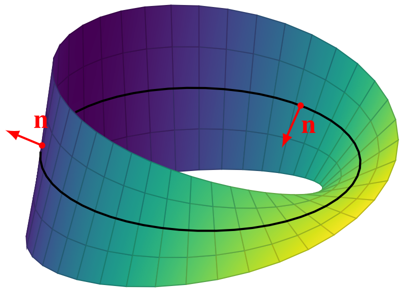


Figure 31: Möbius strip

If S is the graph of $z = g(x,y)$, then

$$\mathbf{n} = \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x^2 + g_y^2 + 1}}$$

is the upward orientation of S because the $\mathbf{k}$-component is positive. As such, the downward orientation is simply $-\mathbf{n}$.

If S is written in parametric form $\mathbf{r} = \mathbf{r}(u,v)$, then

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

9.4. *Surface Integrals of Vector Fields*

Definition 9.6 (surface integral/flux)**.** Let $\mathbf{F}$ be a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$. The surface integral of $\mathbf{F}$ over S (also known as flux of $\mathbf{F}$ over S) is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}\, dS.$$

If S is the graph of a function $z = g(x,y)$ over a region D in the xy-plane (assuming upward orientation of S), and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, the integral in Definition 9.6 can be written as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)\; dA.$$

Always check the following conditions before applying the formula in Definition 9.6:

(i) The surface S is traced out by $\mathbf{r}(u,v)$, $(u,v) \in D$, where D is the parameter domain

(ii) The orientation $\mathbf{n}$ given by the question is indeed the following expression:

$$\frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

9.5. *Integral Theorems*

Theorem 9.2 (Stokes' theorem). Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then,

$$\iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \iint_D -(R_y - Q_z)\, g_x - (P_z - R_x)\, g_y + (Q_x - P_y)\ dA.$$

Proposition 9.1. If S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' theorem, then

$$\iint_{S_1} (\text{curl } \mathbf{F}) \cdot d\mathbf{S} = \iint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} (\text{curl } \mathbf{F}) \cdot d\mathbf{S}.$$

If curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative.

Consider the special case where the surface S is flat and lies in the xy-plane with upward orientation. Then, the unit normal is $\mathbf{k}$ and the surface integral becomes a double integral. It can be shown that this is simply Green's theorem (Theorem 8.1).

Theorem 9.3 (divergence theorem). Let E be a solid region where the boundary surface S of E is piecewise smooth with positive orientation. Let $\mathbf{F}(x, y, z)$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F}\, dV.$$