MA4271 Differential Geometry of Curves and Surfaces

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These notes are based off **Prof. Loke Hung Yean's** MA3215 Three-Dimensional Differential Geometry materials. Additional references are cited in the bibliography. Note that the old course code is MA3215 and it has been upgraded to the current MA4271 Differential Geometry of Curves and Surfaces.

This set of notes was last updated on December 18, 2025. If you would like to contribute a nice discussion to the notes or point out a typo, please send me an email at thangpangern@u.nus.edu.

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CHAPTER 1

Smooth Functions

1.1 Open Subsets of $\mathbb{R}$

Recall from MA2108 Mathematical Analysis I that an open interval in $\mathbb{R}$ means any set of the form (a,b) or (a,∞) or $(-\infty,b)$ or $\mathbb{R}$. A subset $U \subseteq \mathbb{R}$ is open if it can be written as a (possibly infinite) union of open intervals. In practice, one can treat 'U open' as the hypothesis that every point of U sits inside some small interval still contained in U. This is the natural domain condition for differentiation.

Definition 1.1 (smoothness on an open set). Let $U \subseteq \mathbb{R}$ be open and let $f : U \to \mathbb{R}$. We say that f is smooth (or $\mathcal{C}^\infty$) on U if for every $n \in \mathbb{Z}_{\geq 0}$ and every $t \in U$, the n^{th} derivative

$$f^{(n)}(t) = \frac{d^n}{dt^n} f(t) \quad \text{exists.}$$

Proposition 1.1. Let $U \subseteq \mathbb{R}$ be open and let $f : U \to \mathbb{R}$. If f is smooth on U, then f is continuous on U.

Proof. If f is smooth, then in particular the first derivative exists for all $t \in U$. Differentiability implies continuity at each point of U, hence f is continuous on U. □

Recall that a smooth function need not equal to its Taylor series. That is to say, even if all derivatives exist, the Taylor series built from these derivatives may fail to represent the original function near the expansion point. Equivalently, $\mathcal{C}^\infty$ does not imply analytic. Take for example

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Then, f is smooth on $\mathbb{R}$ and $f^{(n)}(0) = 0$ for all $n \geq 0$. Hence, the Taylor series of f at 0 is identically 0, but $f(x) > 0$ for $x \neq 0$. So f does not have a Taylor series expansion at 0 (in the sense of being equal to its Taylor series in a neighbourhood of 0). Well, to make it more rigorous, for $x \neq 0$, repeated differentiation yields

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-1/x^2},$$

where P_n is a polynomial (depending on n). As $x \to 0$, the exponential decay of e^{-1/x^2} dominates any polynomial growth in $1/x$, so

$$\lim_{x\to 0} f^{(n)}(x) = 0.$$

Thus, defining $f^{(n)}(0) = 0$ makes each derivative continuous at 0, and inductively shows $f \in C^\infty(\mathbb{R})$ with all derivatives at 0 equal to 0.

In this course, as we would see eventually, *charts* and *transition maps* are required to be smooth. It is crucial to distinguish *smooth* from *analytic*: many geometric constructions live naturally in the C^∞ category without being representable by convergent power series.

1.2 Open Subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$

When we differentiate $f : U \to \mathbb{R}$ at a point $p \in U$, we take small perturbations of the input and compare values of f. Thus, we want the inputs to be able to wander freely in a small vicinity of p while staying inside U. This is precisely what 'U is open' formalises.

Recall from MA2108 Mathematical Analysis I or even MA3209 Metric and Topological Spaces the notion of open balls and open subsets of $\mathbb{R}^n$. We shall briefly state these definitions. Let $p \in \mathbb{R}^n$ and $\varepsilon > 0$. The open ball of radius ε centred at p is

$$B(p,\varepsilon) = \{q \in \mathbb{R}^n : \|q - p\| < \varepsilon\}.$$

Next, a subset $U \subseteq \mathbb{R}^n$ is open if for every $p \in U$ there exists $\varepsilon > 0$ such that $B(p,\varepsilon) \subseteq U$. By definition, every open ball $B(p,\varepsilon)$ is an open subset of $\mathbb{R}^n$. This notion is the Euclidean special case of the general definition of openness in metric/topological spaces.

Next, let $p \in \mathbb{R}^n$. A subset $N \subseteq \mathbb{R}^n$ is an open neighbourhood of p if $p \in N$ and N is open in $\mathbb{R}^n$. In particular, $B(p,\varepsilon)$ is an open neighbourhood of p for every $\varepsilon > 0$.

Example 1.1. We state some basic examples and non-examples.

(i) A unit square in $\mathbb{R}^2$ without its boundary is open; with its boundary it is not open

(ii) A unit cube in $\mathbb{R}^3$ without its boundary is open; with its boundary it is not open

(iii) The plane $\mathbb{R}^2$ is an open subset of itself, but the xy-plane $\left\{(x,y,z) \in \mathbb{R}^3 : z = 0\right\}$ is not open in $\mathbb{R}^3$

(iv) Note that the upper half-space

$$\left\{(x,y,z) \in \mathbb{R}^3 : z > 0\right\}$$

is open in $\mathbb{R}^3$, while $\left\{(x,y,z) \in \mathbb{R}^3 : z \geq 0\right\}$ is not open in $\mathbb{R}^3$

Lastly, a subset $C \subseteq \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus C$ is open in $\mathbb{R}^n$. Intuitively, a closed set contains its boundary points. For instance, an open ball is not closed because it excludes its boundary sphere.

1.3 Smooth Multivariable Functions

For $f : (a,b) \to \mathbb{R}$, the existence of derivatives of all orders forces those derivatives to be continuous. In several variables, one uses partial derivatives, but existence of higher partial derivatives alone does not guarantee continuity of those derivatives, so continuity must be built into the definition.

Definition 1.2 (higher-order partial derivatives). Let $V \subseteq \mathbb{R}^n$ be open, and let $f : V \to \mathbb{R}$. Write $f(x_1, \ldots, x_n)$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}$ and let

$$N = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

If the partial derivative

$$\frac{\partial^N f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x_1, \ldots, x_n)$$

exists for all $(x_1, \ldots, x_n) \in V$, then it is called a partial derivative of order N.

Definition 1.3 ($\mathcal{C}^N(V)$ and $\mathcal{C}^\infty(V)$). Let $V \subseteq \mathbb{R}^n$ be open. For $N \in \mathbb{N}$, denote by $\mathcal{C}^N(V)$ the set of continuous functions $f : V \to \mathbb{R}$ such that all partial derivatives of order not greater than N exist and are continuous on V. A function in $\mathcal{C}^N(V)$ is called differentiable of order N on V.

Define

$$\mathcal{C}^\infty(V) = \bigcap_{N=0}^{\infty} \mathcal{C}^N(V).$$

A function $f \in \mathcal{C}^\infty(V)$ is called a smooth function (in the sense of Calculus) on V.

Definition 1.4 (equivalent smoothness criterion). A function $f : V \to \mathbb{R}$ is smooth (in the sense of Calculus) if the following hold:

(i) f is continuous

(ii) all partial derivatives of any order exist

(iii) all higher derivatives are continuous functions on V

The continuity requirements in Definition 1.4 cannot be removed, otherwise one admits undesirable functions with many existing partial derivatives but poor regularity. For example, let $V = \mathbb{R}^2$ and define

$$f : \mathbb{R}^2 \to \mathbb{R} \quad \text{where} \quad f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then, f is not continuous[1] at $(0,0)$ even though certain partial derivatives at $(0,0)$ exist.

Definition 1.5 (smoothness at a point). Let $V \subseteq \mathbb{R}^n$ be open and $p \in V$. We say that $f : V \to \mathbb{R}$ is smooth at p if there exists an open subset $V' \subseteq V$ with $p \in V'$ such that the restriction

$$f|_{V'} : V' \to \mathbb{R}$$

is smooth. We call such a V' an open neighbourhood of p.

The pointwise notion 'smooth at p' is typically less useful than smoothness on an open domain; in Differential Geometry we usually specify a domain and ask for smoothness throughout it.

Lemma 1.1 (algebra of smooth functions). Let $V \subseteq \mathbb{R}^n$ be open and let $f, g : V \to \mathbb{R}$ be smooth. Then, the following functions are smooth on V:

$$f + g \quad \text{and} \quad f - g \quad \text{and} \quad fg \quad \text{and} \quad f^n \text{ for } n \in \mathbb{N}.$$

Moreover, the following hold:

(i) If $g(v) \neq 0$ for all $v \in V$, then $\frac{1}{g}$ is smooth on V.

(ii) If $g(v) > 0$ for all $v \in V$, then $\sqrt{g}$ is smooth on V.

We give a rough sketch of the proof.

Proof. Use the product rule to show closure under fg, and the quotient rule to show smoothness of $1/g$ when g is nowhere zero. The remaining statements follow by repeated applications of the chain rule. $\square$

Example 1.2 (smoothness away from the singular point). Let $V' = \mathbb{R}^2 \setminus \{(0,0)\}$. Then, V' is open in $\mathbb{R}^2$. Define

$$g(x,y) = x^2 + y^2.$$

On V', we have $g(x,y) > 0$, hence $\frac{1}{g}$ is smooth on V' by Lemma 1.1. As such,

$$(x,y) \mapsto \frac{xy}{x^2+y^2} = xy \cdot \frac{1}{g(x,y)}$$

is a smooth function on V'. In particular, if $p \in \mathbb{R}^2$ and $p \neq (0,0)$, then the original f is smooth at p.

[1] One can use the two path test taught in MA2104 Multivariable Calculus to show that f is not continuous at $(0,0)$ by approaching $(0,0)$ along two different paths.

Definition 1.6 (smooth maps into $\mathbb{R}^m$). Let $V \subseteq \mathbb{R}^n$ be open and let $f : V \to \mathbb{R}^m$ be given by

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

We say that f is smooth (in the sense of Calculus) if each component function $f_i : V \to \mathbb{R}$ is smooth for $i = 1, \ldots, m$.

Here is a course convention — unless otherwise stated, functions $f(x)$, $g(x,y)$, $h(x,y,z)$, etc. are assumed to be smooth. Also in this course, parametrisations, coordinate charts, and transition maps are required to be smooth maps between open subsets of Euclidean spaces. The definitions above are the Euclidean baseline that will be transplanted to smooth functions on a surface via charts.

Chapter 2

Curves

2.1 Curves in $\mathbb{R}^3$

In this chapter, we begin the study of curvature of a curve; much of the basic material overlaps with MA2104 Multivariable Calculus.

Definition 2.1 (parametrised smooth curve). A parametrised smooth curve in $\mathbb{R}^3$ is a map

$$\alpha : (a,b) \to \mathbb{R}^3 \quad \text{where} \quad \alpha(t) = (x(t), y(t), z(t)),$$

where x, y, z are smooth functions on (a,b). That is to say, they are differentiable as many times as we want.

Definition 2.2 (tangent/velocity vector and speed). The tangent vector (also called the velocity vector) of α at t is $\alpha'(t)$. The speed is $|\alpha'(t)|$.

Definition 2.3 (tangent line and acceleration). If $\alpha'(t) \neq (0,0,0)$, then the line through $\alpha(t)$ and parallel to $\alpha'(t)$ is called the tangent line at t. The second derivative $\alpha''(t)$ is called the acceleration.

Example 2.1 (helix). Let $a, b > 0$ and define

$$\alpha : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \alpha(t) = (a\cos t, a\sin t, bt).$$

Then,

$$\alpha'(t) = (-a\sin t, a\cos t, b) \quad \text{and} \quad \alpha''(t) = (-a\cos t, -a\sin t, 0),$$

and the speed is constant because $|\alpha'(t)| = \sqrt{a^2+b^2}$.

Example 2.2 (a smooth curve with zero velocity at an instant). Consider

$$\alpha : \mathbb{R} \to \mathbb{R}^2 \quad \text{where} \quad \alpha(t) = \left(t^3,\, t^2\right).$$

Then, $\alpha'(t) = \left(3t^2,\, 2t\right)$, $\alpha''(t) = (6t,\, 2)$, and $\alpha'(t) = |t|\sqrt{9t^2+4}$. In particular, $\alpha'(0) = (0,0)$, so the velocity vanishes at $t = 0$ even though α is smooth.

Definition 2.4 (non-singular curve). A smooth curve α is non-singular if $\alpha'(t) \neq \mathbf{0}$.

Example 2.3 (self-intersection). Let

$$\alpha : \mathbb{R} \to \mathbb{R}^2 \quad \text{where} \quad \alpha(t) = \left(t^3 - 4t,\, t^2 - 4\right).$$

Then, $\alpha(2) = \alpha(-2) = (0,0)$ so the curve may intersect itself.

In this course, we are mainly concerned with the shape of the curve rather than how fast the point moves along it; hence the same geometric curve may admit many different parametrisations. Take for example the parametrisations $\alpha(t) = (\cos t, \sin t)$, $\beta(t) = (\cos(2t), \sin(2t))$, and $\gamma(t) = (\cos(t-2), \sin(t-2))$. All three trace the unit circle, but with different *timings* — β moves with twice the speed of α, while γ is a time-shift of α.

Having said that, a natural choice is to reparametrise so that the point moves with constant unit speed, i.e. $|\alpha'(t)| = 1$ for all t; in particular, such a parametrisation is automatically non-singular. We will return to this in Chapter 2.2. The standing assumption from now on is that we will only consider non-singular smooth curves.

2.2 Arc Length

Definition 2.5 (arc length). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a (non-singular) smooth curve. The arc length of α from $t = a$ to $t = b$ is

$$L\left(\alpha|_{[a,b]}\right) = \int_a^b |\alpha'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

Fix a reference time (often 0). Define

$$s(t_0) = \int_0^{t_0} |\alpha'(t)|\, dt.$$

Then, the arc length from $t = a$ to $t = b$ is

$$\int_a^b |\alpha'(t)|\, dt = s(b) - s(a).$$

Example 2.4 (semicircle of radius r). Let $r > 0$ and

$$\gamma : (0,\pi) \to \mathbb{R}^2 \quad \text{where} \quad \gamma(t) = (r\cos t,\, r\sin t).$$

Then, $\gamma'(t) = (-r\sin t,\, r\cos t)$ so $|\gamma'| = r$, so the arc length from $t = 0$ to $t = \pi$ is

$$\int_0^\pi |\gamma'(t)|\, dt = \int_0^\pi r\, dt = r\pi.$$

Definition 2.6. Let $\alpha : (a,b) \to \mathbb{R}^3$ (or $\mathbb{R}^2$) be a curve. If $|\alpha'(t)| = 1$ for all t, then we say that α is parametrised by arc length (i.e. unit speed). In this case, the arc length from $t = a$ to $t = b$ is simply $b - a$. Moreover, we often switch notation and write the parameter as s instead of t.

Example 2.5. For the semicircle in Example 2.4, we have

$$s(t) = \int_0^t |\gamma'(u)|\, du = \int_0^t r\, du = rt,$$

so $t = \frac{s}{r}$. As such, define the new parametrisation

$$\alpha(s) = \gamma\left(\frac{s}{r}\right) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\right).$$

Then, $|\alpha'(s)| = 1$, so α is unit speed and traces the same semicircle.

Say we are given a non-singular smooth curve. Can we change variable from t to s so that the curve becomes parametrised by arc length? The answer is yes. Here is the general strategy.

(i) Define the distance-travelled function

$$s(t) = \int_{t_0}^t |\alpha'(u)|\, du.$$

(ii) Find the inverse function $t = t(s)$.

(iii) Define the reparametrised curve

$$\beta(s) = \alpha(t(s))$$

which will be parametrised by arc length.

Recall by the Fundamental Theorem of Calculus from MA2002 Calculus that

$$\frac{ds}{dt} = |\alpha'(t)|.$$

Since α is non-singular, then $|\alpha'(t)| > 0$, hence $s(t)$ is strictly increasing, and so it admits an inverse $t(s)$ (obtained by flipping the t- and s-axes).

Theorem 2.1 (inverse function theorem). Suppose s is continuous, smooth, and $s'(t) > 0$ for all t in an interval. Then, s is strictly increasing, the inverse $t = t(s)$ exists, and the inverse is smooth.

Theorem 2.2 (existence of arc-length parametrisation). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a non-singular smooth curve. Define

$$s(t) = \int_a^t |\alpha'(u)|\, du \quad \text{where } c = s(a) \text{ and } d = s(b).$$

Then, $s(t)$ is strictly increasing and has a smooth inverse $t = t(s)$ on (c,d). If we set $\beta(s) = \alpha(t(s))$, then β is parametrised by arc length, i.e. $|\beta'(s)| = 1$ for all $s \in (c,d)$.

Proof. Using the chain rule, we have $\beta'(s) = \alpha'(t(s)) \cdot t'(s)$. Since $\frac{ds}{dt} = |\alpha'(t)|$, we have

$$t'(s) = \frac{1}{|\alpha'(t(s))|}.$$

As such,

$$\left|\beta'(s)\right| = \left|\alpha'(t(s))\right| \cdot \left|t'(s)\right| = \left|\alpha'(t(s))\right| \cdot \frac{1}{|\alpha'(t(s))|} = 1.$$

□

2.3 Orientation

In $\mathbb{R}^3$, we write the vector product (cross product) as $\mathbf{v} \wedge \mathbf{w} = \mathbf{v} \times \mathbf{w}$.

Theorem 2.3 (product rules in $\mathbb{R}^3$). Let $\mathbf{w}(t), \mathbf{w}(t)$ be differentiable vector-valued functions in $\mathbb{R}^3$. Then, the following hold

$$\frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{w}(t)) = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$$

and

$$\frac{d}{dt}(\mathbf{v}(t) \wedge \mathbf{w}(t)) = \mathbf{v}'(t) \wedge \mathbf{w}(t) + \mathbf{v}(t) \wedge \mathbf{w}'(t).$$

Definition 2.7 (positive and negative orientation). Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis for $\mathbb{R}^3$.

(i) We say $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has positive orientation if $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (right-hand rule)

(ii) We say $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has negative orientation if $\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_3$ (left-hand rule)

Definition 2.8 (coordinate system in $\mathbb{R}^3$). A coordinate system in $\mathbb{R}^3$ is a choice of mutually perpendicular x-axis, y-axis and z-axis. There are two types: positively oriented and negatively oriented.

Theorem 2.4. We state the effects of rotations, translations, and reflections on orientation.

(i) **Rotations preserve orientation:** if we rotate the axes, the axes change but the orientation remains positive (respectively negative)

(ii) **Translations preserve orientation:** translating the axes by a vector does not change orientation

(iii) **Reflections reverse orientation:** reflection sends a positively oriented coordinate system to a negatively oriented one (right hand becomes left hand)

2.4 Curvature

How do we tell that one curve is more *curvy* than the other? Let us discuss curvature. Let $\alpha(s)$ be a curve parametrised by arc length, and write the unit tangent vector as

$$t(s) = \alpha'(s).$$

To measure how much the curve bends, we compare $t(s)$ and $t(s+\Delta s)$ and consider the rate of change

$$\frac{t(s+\Delta s) - t(s)}{\Delta s}.$$

Taking $\Delta s \to 0$ gives

$$\frac{dt(s)}{ds} = \frac{d}{ds}\left(\frac{d\alpha(s)}{ds}\right) = \alpha''(s).$$

Hence, curvature is governed by the acceleration $\alpha''(s)$. We give some physical intuition. If you travel at constant speed and the train goes around a bend, you feel a centripetal force; a sharper bend gives larger acceleration, hence larger curvature. Note that if the curve is not travelling at constant speed, i.e. not parametrised by arc length, then using acceleration to gauge the bend can be misleading.

Definition 2.9 (curvature and radius of curvature). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a curve parametrised by arc length. Define the curvature at s by

$$k(s) = \left|\alpha''(s)\right|$$

and define the radius of curvature by

$$\rho(s) = \frac{1}{k(s)}.$$

Example 2.6 (straight line has zero curvature). Let

$$\alpha(s) = (a_1, a_2, a_3) + s(t_1, t_2, t_3)$$

be a straight line parametrised by arc length. Then $\alpha'(s) = (t_1, t_2, t_3)$ is a unit vector and $\alpha''(s) = (0,0,0)$ so $k(s) = 0$. Thus, a straight line has zero curvature.

Example 2.7 (circle of radius r has curvature $1/r$). Let $\alpha(s)$ be a circle of radius r parametrised by arc length (for instance $\alpha(s) = (r\cos(s/r), r\sin(s/r))$). Then, one computes

$$k(s) = \left|\alpha''(s)\right| = \frac{1}{r} \quad \text{so} \quad \rho(s) = r.$$

Hence, the radius of curvature of a circle of radius r is r.

Lemma 2.1 (orthogonality lemma). Let $\alpha(s)$ be parametrised by arc length. Then,

$$\alpha''(s) \cdot \alpha'(s) = 0,$$

i.e. the acceleration is perpendicular to the unit tangent vector.

Proof. Since $\alpha(s)$ is parametrised by arc length, we have $|\alpha'(s)| = 1$, so

$$1 = |\alpha'(s)|^2 = \alpha'(s) \cdot \alpha'(s).$$

Differentiating both sides with respect to s, we have

$$0 = \frac{d}{ds}\left(\alpha'(s) \cdot \alpha'(s)\right) = \alpha''(s) \cdot \alpha'(s) + \alpha'(s) \cdot \alpha''(s) = 2\alpha'(s) \cdot \alpha''(s).$$

Hence, $\alpha''(s) \cdot \alpha'(s) = 0$. □

More generally, if $v(t)$ is a unit vector for all t, then $v'(t)$ is always perpendicular to $v(t)$.

Definition 2.10 (singular point of order 1). Suppose for some s_0, we have $\alpha''(s_0) = (0,0,0)$. Then, s_0 is called a singular point of order 1.

We will only consider curves such that

$$\alpha''(s) \neq (0,0,0) \quad \text{for all } s.$$

2.5 The Frenet Trihedron

For the rest of this chapter, we only consider curves $\alpha(s)$ parametrised by arc length such that

$$\alpha''(s) \neq (0,0,0) \quad \text{for all } s.$$

Definition 2.11 (unit tangent vector). If $\alpha(s)$ is parametrised by arc length, we define the unit tangent vector

$$t(s) = \alpha'(s).$$

Definition 2.12 (normal vector). Since $\alpha''(s) \neq 0$, the unit vector in the direction of $\alpha''(s)$ is well-defined. We call it the normal vector. That is,

$$n(s) = \frac{\alpha''(s)}{|\alpha''(s)|}.$$

Equivalently, $\alpha''(s) = k(s)\, n(s)$ and $k(s) = |\alpha''(s)|$.

Definition 2.13 (osculating plane). The plane spanned by $t(s)$ and $n(s)$ is called the osculating plane at s.

Definition 2.14 (binormal vector). Define the binormal vector by

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s).$$

Then $\mathbf{b}(s)$ is a unit vector, and $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ are mutually perpendicular.

Definition 2.15 (Frenet trihedron). The three unit vectors $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ are called the Frenet trihedron. They form a moving frame along the curve.

Example 2.8 (planar curves have constant binormal). Suppose $\alpha(s)$ lies in the xy-plane. Then, $\mathbf{t}(s)$ and $\mathbf{n}(s)$ lie in the xy-plane, so $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$ is perpendicular to the xy-plane, i.e. $\mathbf{b}(s) = (0,0,1)$ or $\mathbf{b}(s) = (0,0,-1)$.

Proposition 2.1. Let $\alpha(s)$ be parametrised by arc length in $\mathbb{R}^3$. Assume that for all $s \in \mathbb{R}$, we have

$$\alpha(0) = (0,0,0) \quad \text{and} \quad b(s) = (0,0,1).$$

Then, $\alpha(s)$ lies in the xy-plane.

Proof. Write $\alpha(s) = (x(s), y(s), z(s))$ and consider $\mathbf{z}(s) = \alpha(s) \cdot \mathbf{b}(s)$. Differentiating both sides yields

$$\mathbf{z}'(s) = \alpha'(s) \cdot \mathbf{b}(s) + \alpha(s) \cdot \mathbf{b}'(s).$$

Since $\mathbf{b}(s) \perp \mathbf{t}(s)$, where $\mathbf{t}(s) = \alpha'(s)$, and as we will see, $\mathbf{b}'(s)$ is parallel to $\mathbf{n}(s)$, we obtain $\mathbf{z}'(s) = \mathbf{0}$. As such, $\mathbf{z}(s)$ is constant. Since $\mathbf{z}(0) = \mathbf{0}$, then $\mathbf{z}(s) = \mathbf{0}$ for all s. $\square$

Lemma 2.2. Suppose $\alpha''(s) \neq 0$. Then, $\mathbf{b}'(s)$ is parallel to $\mathbf{n}(s)$.

Proof. Since $\mathbf{b}(s)$ is a unit vector, we have $\mathbf{b}(s) \cdot \mathbf{b}(s) = 1$, hence

$$0 = \frac{d}{ds}(\mathbf{b}(s) \cdot \mathbf{b}(s)) = 2\mathbf{b}'(s) \cdot \mathbf{b}(s),$$

so $\mathbf{b}'(s) \perp \mathbf{b}(s)$. Also, $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$, and differentiating shows that $\mathbf{b}'(s) \perp \mathbf{t}(s)$ as well. Since $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ is an orthonormal basis, a vector perpendicular to both t and b must be parallel to n. $\square$

Definition 2.16 (torsion). By Lemma 2.2, we may write

$$\mathbf{b}'(s) = \tau(s)\mathbf{n}(s),$$

where the scalar $\tau(s)$ is called the torsion of α at s.

Theorem 2.5 (Frenet-Serret formulae for arc length parametrisation). Let $\alpha(s)$ be parametrised by arc length with $\alpha''(s) \neq 0$. Then,

$$\mathbf{t}'(s) = \mathbf{k}(s)\mathbf{n}(s) \quad \text{and} \quad \mathbf{n}'(s) = -\mathbf{k}(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s) \quad \text{and} \quad \mathbf{b}'(s) = \tau(s)\mathbf{n}(s).$$

Equivalently,

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \mathbf{k} & 0 \\ -\mathbf{k} & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix},$$

where the coefficient matrix is skew-symmetric.

We now discuss the Frenet-formulae for curves not parametrised by arc length. In theory, a non-singular curve $\alpha(t)$ can be reparametrised by arc length $s = s(t)$, but in computations this can be tedious or impossible. Hence, we want formulae for $\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{k}, \tau$ directly in terms of t-derivatives. In this section, let $', '', ''', \ldots$ denote differentiation with respect to t (not s).

Theorem 2.6 (Frenet data for a general time parametrisation). Let $\alpha(t)$ be a non-singular curve in $\mathbb{R}^3$. Define

$$\mathbf{t}(t) = \frac{\alpha'(t)}{|\alpha'(t)|} \quad \text{and} \quad \mathbf{b}(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|} \quad \text{and} \quad \mathbf{n}(t) = \mathbf{b}(t) \wedge \mathbf{t}(t).$$

Then,

$$\mathbf{k}(t) = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3} \quad \text{and} \quad \tau(t) = -\frac{((\alpha'(t) \wedge \alpha''(t)) \cdot \alpha'''(t))}{|\alpha'(t) \wedge \alpha''(t)|^2}. \tag{2.1}$$

Moreover,

$$\alpha'(t) = \left|\alpha'(t)\right| \mathbf{t}(t),$$

and

$$\alpha''(t) = \frac{d}{dt}\left(\left|\alpha'(t)\right|\right) \mathbf{t}(t) + \mathbf{k}(t) \left|\alpha'(t)\right|^2 \mathbf{n}(t). \tag{2.2}$$

Finally, the Frenet-Serret equations become

$$\mathbf{t}'(t) = \mathbf{k}(t) \left|\alpha'(t)\right| \mathbf{n}(t) \tag{2.3}$$

$$\mathbf{n}'(t) = -k(t) \left|\alpha'(t)\right| \mathbf{t}(t) - \tau(t) \left|\alpha'(t)\right| \mathbf{b}(t) \tag{2.4}$$

$$\mathbf{b}'(t) = \tau(t) \left|\alpha'(t)\right| \mathbf{n}(t) \tag{2.5}$$

One might ask what changed compared to the s-version? When differentiating with respect to t, the extra factor $|\alpha'(t)| = \frac{ds}{dt}$ appears everywhere. In particular, the rates of turning measured per unit time are the arc length rates multiplied by speed.

Given a non-singular curve $\alpha(t)$, in theory we can convert it into an arc length parametrisation $\alpha(s)$ via $s = s(t)$. In practice, this conversion can be tedious (or impossible to do explicitly), so we want formulas for $\mathbf{t}, \mathbf{n}, \mathbf{b}, \mathbf{k}, \tau$ directly in terms of $\mathbf{t}$-derivatives. Now, we let $', '', ''', \ldots$ denote differentiation with respect to $\mathbf{t}$.

We now prove Theorem 2.6.

Proof. Let $\alpha(t) = \alpha(s(t))$, where s denotes arc length. Then,

$$\frac{ds}{dt} = \left|\alpha'(t)\right|.$$

Write $t(t) = t(s(t))$ etc. Using the chain rule, we have

$$\mathbf{t}'(t) = \left.\frac{dt}{ds}\right|_{s=s(t)} \cdot \frac{ds}{dt} = t'(s)\Big|_{s=s(t)} \cdot \left|\alpha'(t)\right| = \mathbf{k}(t) \left|\alpha'(t)\right| \mathbf{n}(t),$$

so (2.3) holds. We have similar results for $\mathbf{n}'$ and $\mathbf{b}'$, giving (2.4) and (2.5) respectively.

For the curvature formula, start from

$$\alpha'(t) = \left|\alpha'(t)\right|\mathbf{t}(t).$$

Then, differentiate both sides to obtain

$$\alpha''(t) = \frac{d}{dt}\left(\left|\alpha'(t)\right|\right)\mathbf{t}(t) + \left|\alpha'(t)\right|\mathbf{t}'(t) = \frac{d}{dt}\left(\left|\alpha'(t)\right|\right)\mathbf{t}(t) + \mathbf{k}(t)\left|\alpha'(t)\right|^2\mathbf{n}(t),$$

which is the displayed decomposition in (2.2). Now, wedge with $\alpha'(t)$ to obtain

$$\begin{aligned}\alpha'(t)\wedge\alpha''(t) &= \left(\left|\alpha'(t)\right|\mathbf{t}(t)\right)\wedge\left(\frac{d}{dt}\left(\left|\alpha'(t)\right|\right)t(t) + \mathbf{k}(t)\left|\alpha'(t)\right|^2\mathbf{n}(t)\right)\\ &= \mathbf{k}(t)\left|\alpha'(t)\right|^3(\mathbf{t}(t)\wedge\mathbf{n}(t))\end{aligned}$$

Since $\mathbf{t}\wedge\mathbf{n} = \mathbf{b}$, taking norms, we have

$$\left|\alpha'(t)\wedge\alpha''(t)\right| = \mathbf{k}(t)\left|\alpha'(t)\right|^3,$$

hence,

$$\mathbf{k}(t) = \frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3}.$$

We have proven that (2.1) holds. This also shows that

$$\mathbf{b}(t) = \frac{\alpha'(t)\wedge\alpha''(t)}{|\alpha'(t)\wedge\alpha''(t)|}.$$

For (2.1) on torsion, one differentiates the decomposition of $\alpha''(t)$ once more, dots with $\mathbf{b}(t)$, and uses that among the resulting terms, the only component not perpendicular to $\mathbf{b}$ arises from the $\mathbf{n}'$-term, which contains $\tau(t)|\alpha'(t)|\mathbf{b}(t)$. This yields

$$\tau(t) = -\frac{((\alpha'(t)\wedge\alpha''(t))\cdot\alpha'''(t))}{|\alpha'(t)\wedge\alpha''(t)|^2}.$$

□

2.6 The Fundamental Theorem of the Local Theory of Curves

In this chapter, we emphasise that we only care about the shape of a curve. Hence, two curves which differ by a rigid motion (translations and rotations) are regarded as the same curve. On the other hand, reflections are not considered rigid motions here.

> Definition 2.17 (rigid motion). A map $\mathbf{R}:\mathbb{R}^3\to\mathbb{R}^3$ is called a rigid motion if the following hold:

(i) it preserves distances, i.e. for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $|\mathbf{R}(\mathbf{v}) - \mathbf{R}(\mathbf{w})| = |\mathbf{v} - \mathbf{w}|$

(ii) it preserves orientation (it sends a positively oriented (x, y, z)-axes to a positively oriented axes)

Lemma 2.3. A rigid motion satisfies the following properties:

(i) sends straight lines to straight lines

(ii) preserves angles

(iii) preserves inner products in the sense that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, we have

$$(\mathbf{R}(\mathbf{v}) - \mathbf{R}(\mathbf{u})) \cdot (\mathbf{R}(\mathbf{w}) - \mathbf{R}(\mathbf{u})) = (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{w} - \mathbf{u}).$$

Theorem 2.7 (structure of rigid motions). Every rigid motion can be written as a rotation (about an axis through the origin) followed by a translation. Equivalently, every rigid motion has the form

$$\mathbf{R}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{v},$$

where $\mathbf{A}$ is an orthogonal matrix with $\det(\mathbf{A}) = 1$ and $\mathbf{v} \in \mathbb{R}^3$.

Note that rigid motions need not be linear since they

Proposition 2.2 (invariance of $\mathbf{k}(s)$ and $\tau(s)$). Let $\alpha(s)$ be a curve and let $\mathbf{R}$ be a rigid motion. Then, the curvature $\mathbf{k}(s)$ and torsion $\tau(s)$ of α are unchanged under the rigid motion of the curve.

Under a rigid motion, the Frenet trihedron $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ and their derivatives transform the same way, so their lengths are preserved. Since $\mathbf{k}(s)$ and $\tau(s)$ are defined via ratios of these lengths (Frenet-Serret formulas), they remain unchanged.

By a translation, we may assume that $\alpha(0) = (0,0,0)$. By rotations about the origin, we may further assume that the initial Frenet trihedron aligns with the standard basis $\mathbf{t}(0) = (1,0,0)$, $\mathbf{n}(0) = (0,1,0)$, and $\mathbf{b}(0) = (0,0,1)$.

Theorem 2.8 (uniqueness up to rigid motion). Suppose $\alpha(s)$ and $\beta(s)$ are two smooth curves parametrised by arc length, and they have the same curvature and torsion for all $s \in \mathbb{R}$. That is,

$$\mathbf{k}_\alpha(s) = \mathbf{k}_\beta(s) \quad \text{and} \quad \tau_\alpha(s) = \tau_\beta(s).$$

Then, α and β differ by a rigid motion.

Proof. Apply a rigid motion to each curve so that both satisfy the normalisation

$$\alpha(0) = \beta(0) = (0,0,0) \quad \text{and} \quad (t(0), n(0), b(0)) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3).$$

Let (t_1, n_1, b_1) be the Frenet trihedron of α and (t_2, n_2, b_2) that of β. Since $\mathbf{k}(s)$ and $\tau(s)$ agree, the Frenet-Serret equations give the same system of differential equations for both frames with the same initial condition at $s = 0$. By the uniqueness theorem for ordinary differential equations, we obtain for all s,

$$\mathbf{t}_1(s) = \mathbf{t}_2(s) \quad \text{and} \quad \mathbf{n}_1(s) = \mathbf{n}_2(s) \quad \text{and} \quad \mathbf{b}_1(s) = \mathbf{b}_2(s).$$

In particular,

$$\alpha'(s) = \mathbf{t}_1(s) = \mathbf{t}_2(s) = \beta'(s).$$

Integrating both sides, we have $\alpha(s) = \beta(s) + \mathbf{C}$ for some constant vector $\mathbf{C}$, and $\mathbf{C} = \mathbf{0}$ from $\alpha(0) = \beta(0)$. Undoing the normalisation gives that the original curves differ by a rigid motion. □

Theorem 2.9 (existence). Given any two smooth functions $\mathbf{k}(s)$ and $\tau(s)$ with $\mathbf{k}(s) > 0$, there exists a smooth curve $\alpha(s)$ whose curvature is $\mathbf{k}(s)$ and whose torsion is $\tau(s)$.

Chapter 3

Surfaces

3.1 Surfaces

Intuitively, a surface in $\mathbb{R}^3$ is something on which an ant can move with two degrees of freedom in a small vicinity of any point. Locally, the ant's neighbourhood should look more or less like a flat plane.

In this course, *nice* surfaces are expected to be smooth and non-self-intersecting (the surface should not cross itself). By smooth, intuitively, we cannot have sharp bends/edges so we exclude corners like a box, or a cone with a sharp tip. Some prototypical examples include a plane, a sphere, a disc, and a torus.

On a plane, we represent a point by coordinates (u,v) after choosing axes. On a sphere, longitude/latitude also give coordinates, but there are built-in defects. Some of which are as follows:

(i) crossing an international date line causes a sudden jump in the coordinate value

(ii) at the poles, longitude becomes ill-defined

These defects are fatal for Calculus, because differentiation depends on making arbitrarily small movements without sudden coordinate jumps.

> Definition 3.1 (local coordinate region). A local coordinate region on a surface is a small region of the surface equipped with a coordinate system (u,v) so that each point in the region is described by a pair of real numbers. Such a region is also called a local coordinate chart.

Example 3.1. Let $S=\mathbb{R}^2\subseteq\mathbb{R}^3$ as the $z=0$ plane. A chart is just $X(u,v)=(u,v,0)$ with $(u,v)\in U\subseteq\mathbb{R}^2$ open.

We insist that neighbouring regions overlap along borders, so that in the overlap, a point has (at least) two coordinate descriptions. To perform Calculus consistently, we must specify and control how coordinates change from one chart to another on overlaps.

Definition 3.2 (coordinate change/transition map). Let (U,φ) and (V,ψ) be two charts on a surface, where

$$\varphi : U \to \mathbb{R}^2 \quad \text{and} \quad \psi : V \to \mathbb{R}^2.$$

On the overlap $U \cap V$, the transition map from φ-coordinates to ψ-coordinates is

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

Definition 3.3 (smoothly compatible charts). Two charts (U,φ) and (V,ψ) are smoothly compatible if the transition maps

$$\psi \circ \varphi^{-1} \quad \text{and} \quad \varphi \circ \psi^{-1}$$

are smooth maps between open subsets of $\mathbb{R}^2$.

Note that if you differentiate a function using (u,v)-coordinates, and I differentiate the same function using (s,t)-coordinates, then the chain rule compares our derivatives using the transition map. Smooth transition maps guarantee that derivatives computed in different charts are consistent.

Definition 3.4 (atlas). An atlas on a set S is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that the following hold:

(i) the domains cover S, i.e. $S = \bigcup_{\alpha \in A} U_\alpha$

(ii) charts are pairwise smoothly compatible on overlaps

Note that one can reverse engineer a surface as follows. We start with many planar pieces (regions) with coordinates, then prescribe transition maps on overlaps, and lastly glue the regions along overlaps so that the coordinate identifications match the transition maps. This produces a surface intrinsically without first embedding it into $\mathbb{R}^3$.

What is the difference between intrinsic geometry and embedding? If we only care about what happens on the surface (distances, angles, curvature, differentiation along the surface), it is not always necessary to know how the surface sits in $\mathbb{R}^3$. This viewpoint allows exotic examples that may not embed nicely in $\mathbb{R}^3$. Take for example the Klein bottle.

We now introduce the concept of a manifold (Definition 3.5).

Definition 3.5 (manifold). A set M is an n-dimensional manifold if it can be covered by charts mapping into $\mathbb{R}^n$ with smooth transition maps on overlaps.

From Definition 3.5, we see that a surface is a 2-dimensional manifold. In this course, we treat surfaces as 2-manifolds (often embedded in $\mathbb{R}^3$), and we build Calculus on them by transporting Euclidean Calculus through charts. A surface $S \subseteq \mathbb{R}^3$ is a set which is locally 2-dimensional. That is to say, for each $p \in S$, if we take a sufficiently small open ball $V \subseteq \mathbb{R}^3$ centred at p, then the piece $V \cap S$ should look like an open disk in $\mathbb{R}^2$.

Definition 3.6 (open subsets of a surface). Let $S \subseteq \mathbb{R}^3$ be a surface and let $U \subseteq S$. We say that U is open in S if for every $p \in U$, there exists $\varepsilon > 0$ such that

$$B(p,\varepsilon) \cap S \subseteq U \quad \text{where} \quad B(p,\varepsilon) = \left\{ q \in \mathbb{R}^3 : |q-p| < \varepsilon \right\}.$$

Equivalently, U is open in S if there exists an open set $V \subseteq \mathbb{R}^3$ such that $U = S \cap V$. If $p \in S$, an open neighbourhood of p (in S) is a set $U \subseteq S$ such that $p \in U$ and U is open in S.

3.2 Regular Surfaces

Definition 3.7 (regular surface). A subset $S \subseteq \mathbb{R}^3$ is called a regular surface if for each $p \in S$, there exists an open neighbourhood $V \subseteq \mathbb{R}^3$ of p, an open set $U \subseteq \mathbb{R}^2$, and a map

$$\mathbf{x} : U \to V \cap S \quad \text{where} \quad \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

such that the following hold:

(i) The component functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are smooth on U

(ii) The map $\mathbf{x}$ is a bijective homeomorphism onto its image $V \cap S$, so the inverse $\mathbf{x}^{-1} : V \cap S \to U$ exists and is continuous

(iii) The differential

$$d\mathbf{x} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank 2 at every $\mathbf{q} = (u_0, v_0) \in U$, i.e. $d\mathbf{x}_q$ has rank 2 as a real 3×2 matrix.

Such a map x is called a coordinate function (or parametrisation), and the pair (U,x) is called a coordinate chart.

A standard way to prove that a subset $S \subset \mathbb{R}^3$ is a regular surface is to explicitly exhibit coordinate functions (charts) $x : U \to V \cap S$, where $U \subseteq \mathbb{R}^2$ and $V \subseteq \mathbb{R}^3$ are open, $\mathbf{x}$ is smooth, bijective onto $V \cap S$ with smooth inverse, and $d\mathbf{x}$ has rank 2 everywhere.

In practice, one often checks the smoothness and rank condition ((**i**) and (**iii**) respectively in Definition 3.7). In fact, for a regular surface, conditions (**i**) and (**iii**) already force the required topological behaviour in (**ii**). Also, the regularity condition is frequently stated as $dx_q : \mathbb{R}^2 \to \mathbb{R}^3$ being an injective linear transformation — this is equivalent to

$\operatorname{rank}\left(dx_q\right) = 2$.

We now interpret the conditions in Definition 3.7. One can think of $x : U \to S \cap V$ as assigning coordinates (u, v) to points on the surface patch $S \cap V$. Firstly, injective means a point on $S \cap V$ is not assigned two different coordinate pairs; surjective means every point of the patch $S \cap V$ has some coordinate pair. For **(i)**, we need the map to be smooth since we want to differentiate the parametrisation, i.e. perform some calculus on S. Lastly, with regards to **(iii)** on the rank 2 condition, fix $q = (u_0, v_0) \in U$ and put $p = x(u_0, v_0) \in S$. If we fix $v = v_0$ and vary u, we obtain a curve

$$\alpha(u) = x(u, v_0) \quad \text{with} \quad \alpha'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0).$$

If we fix $u = u_0$ and vary v, we obtain a curve

$$\beta(v) = x(u_0, v) \quad \text{with} \quad \beta'(v_0) = \frac{\partial x}{\partial v}(u_0, v_0).$$

These two vectors $\alpha'(u_0)$ and $\beta'(v_0)$ are precisely the two columns of $d\mathbf{x}_q$. Thus, $\operatorname{rank}\left(d\mathbf{x}_q\right) = 2$ means they are linearly independent, and hence they span a 2-dimensional plane in $\mathbb{R}^3$, which we interpret as the tangent plane to the surface at p.

Example 3.2. We now give a non-example illustrating condition **(iii)** on rank in the definition of a regular surface (Definition 3.7). Let

$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{where} \quad \mathbf{x}(u, v) = \left(u^2, u^3, v\right).$$

Then,

$$d\mathbf{x} = \begin{pmatrix} 2u & 0 \\ 3u^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{so} \quad d\mathbf{x}_q(0, v_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which has rank 1. Geometrically, for a fixed $v = v_0$, the curve $u \mapsto \left(u^2, u^3, v_0\right)$ is the standard planar cusp $(x, y) = \left(u^2, u^3\right)$ sitting at height $z = v_0$. Stacking these cusps over all v gives a cusp-edge along the z-axis, which is not a smooth surface.

Lemma 3.1. Suppose $\mathbf{x} : U \to V \cap S$ is a coordinate function. If $U' \subseteq U$ is open, then the restriction

$$\mathbf{x} : U' \to \mathbf{x}\left(U'\right)$$

is also a coordinate function (for an appropriate open set $V' \subseteq \mathbb{R}^3$ with $\mathbf{x}(U') = V' \cap S$).

Example 3.3 (the unit sphere S^2). Define the unit sphere

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}.$$

Let

$$U = \left\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\right\} \quad \text{and} \quad V = \left\{(x, y, z) \in \mathbb{R}^3 : z > 0\right\}.$$

Then, $V \cap S^2$ is the open northern hemisphere, and we define

$$\mathbf{x}_N : U \to V \cap S^2 \quad \text{where} \quad \mathbf{x}_N(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right). \tag{3.1}$$

This is a bijective map with inverse $\mathbf{x}_N^{-1}(x,y,z) = (x,y)$. Smoothness is clear for the first two coordinates; for the third coordinate, note that $1-u^2-v^2 > 0$ on U, so the square root is smooth on U.

Lastly, we verify that $\operatorname{rank}(d\mathbf{x}_N) = 2$. We shall compute

$$\frac{\partial \mathbf{x}_N}{\partial u}(u,v) = \left(1, 0, -\frac{u}{\sqrt{1-u^2-v^2}}\right) \quad \text{and} \quad \frac{\partial \mathbf{x}_N}{\partial v}(u,v) = \left(0, 1, -\frac{v}{\sqrt{1-u^2-v^2}}\right),$$

which are clearly linearly independent in $\mathbb{R}^3$. Hence, $\operatorname{rank}(d\mathbf{x}_N) = 2$ for all $(u,v) \in U$.

Using the same U and $V_- = \left\{(x,y,z) \in \mathbb{R}^3 : z < 0\right\}$, define

$$\mathbf{x}_S : U \to V_- \cap S^2 \quad \text{where} \quad \mathbf{x}_S(u,v) = \left(u, v, -\sqrt{1-u^2-v^2}\right), \tag{3.2}$$

with inverse $\mathbf{x}_S^{-1}(x,y,z) = (x,y)$. The rank check is identical. The two charts in (3.1) and (3.2) do not cover the equator though. One can add four further charts using open sets $\{y > 0\}$, $\{y < 0\}$, $\{x > 0\}$, and $\{x < 0\}$ to obtain a cover of S^2 by six coordinate functions.

From this example, we see that coordinate charts are not unique as there are many valid ways to cover the same surface. Next, redundant charts are allowed: extra charts do not invalidate anything; we simply ignore redundancy.

Example 3.4 (union of two planes that is not regular at the origin). Consider

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : \ yz = 0\right\}.$$

Equivalently, S is the union of the xy-plane $z = 0$ and the xz-plane $y = 0$. Then, S is not a regular surface at $p = (0,0,0)$ since near the origin, it looks like two smooth sheets crossing, so it fails to be locally parametrised by a single smooth coordinate patch with rank 2 everywhere. Intuitively, the tangent plane is not well-defined as a single plane there.

To make it more explicit, write $F(x,y,z) = yz$ and define $S = F^{-1}(0)$. So, $\nabla F(x,y,z) = (0,z,y)$. At $p = (0,0,0)$, we have $\nabla F(p) = \mathbf{0}$, so the rank condition fails here.

Proposition 3.1 (graphs are regular surfaces). Let $U \subseteq \mathbb{R}^2$ be open and let $f : U \to \mathbb{R}$ be smooth. Consider the graph

$$S := \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in U, z = f(x,y)\right\}.$$

Then, S is a regular surface.

Proof. Define

$$\mathbf{x} : U \to S \quad \text{where} \quad \mathbf{x}(u,v) = (u, v, f(u,v)).$$

Then, $\mathbf{x}$ is smooth, bijective, and the inverse is the projection $\mathbf{x}^{-1}(x,y,f(x,y)) = (x,y)$ which is smooth. Finally,

$$\frac{\partial x}{\partial u}(u,v) = \left(1, 0, \frac{\partial f}{\partial u}(u,v)\right) \quad \text{and} \quad \frac{\partial x}{\partial v}(u,v) = \left(0, 1, \frac{\partial f}{\partial v}(u,v)\right),$$

which are linearly independent. Hence, $\operatorname{rank}(d\mathbf{x}) = 2$ everywhere. □

Example 3.5 (surface of revolution). Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and assume $f(z) > 0$ for all $z \in \mathbb{R}$. Rotate the curve $x = f(z)$ in the xz-plane about the z-axis. The resulting surface is

$$S = \{(f(z)\cos u, f(z)\sin u, z) : u \in \mathbb{R}, z \in \mathbb{R}\}.$$

Let

$$U = (0, 2\pi) \times \mathbb{R} \quad \text{and} \quad \mathbf{x} : U \to S \text{ where } \mathbf{x}(u,v) = (f(v)\cos u, f(v)\sin u, v).$$

Then, $\mathbf{x}$ is smooth. For the rank condition, we compute

$$\frac{\partial x}{\partial u}(u,v) = (-f(v)\sin u, f(v)\cos u, 0) \quad \text{and} \quad \frac{\partial x}{\partial v}(u,v) = \left(f'(v)\cos u, f'(v)\sin u, 1\right).$$

The first vector has length $|f(v)|$, which is non-zero by assumption, so the two vectors are linearly independent. Hence, $\operatorname{rank}(d\mathbf{x}) = 2$.

Next, given $(x,y,z) \in S$, we can recover $v = z$. To recover u, we need an angle function $\theta(x,y)$ with values in $(0, 2\pi)$, but a single continuous choice forces us to remove a ray (a branch cut), e.g.

$$W = \mathbb{R}^2 \setminus \left\{(x,0) \in \mathbb{R}^2 : x \geq 0\right\} \quad \text{where} \quad \theta : W \to (0, 2\pi).$$

Then, on the corresponding subset of S, one may write $\mathbf{x}^{-1}(x,y,z) = (\theta(x,y), z)$. To cover the missing ray, one introduces an additional chart.

3.3 The Inverse Function Theorem

As we would see in Theorem 3.1 eventually, the inverse function theorem is the precise statement that $\det(d\mathbf{F}_p) \neq 0$ implies $\mathbf{F}$ is locally invertible near p with a smooth local inverse. In Differential Geometry, this is one of the key tools behind the local description of surfaces by coordinate charts.

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, where

$$\mathbf{F}(u_1, \ldots, u_n) = (\mathbf{F}_1(u_1, \ldots, u_n), \ldots, \mathbf{F}_m(u_1, \ldots, u_n)).$$

We define its differential $d\mathbf{F}$ to be the $m \times n$ matrix of first partial derivatives. That is,

$$d\mathbf{F} = \begin{pmatrix} \frac{\partial \mathbf{F}_1}{\partial u_1} & \cdots & \frac{\partial \mathbf{F}_m}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{F}_1}{\partial u_n} & \cdots & \frac{\partial \mathbf{F}_m}{\partial u_n} \end{pmatrix}.$$

When $n = m$, $d\mathbf{F}$ is an $n \times n$ matrix and we define the Jacobian determinant

$$\det(d\mathbf{F}) = \frac{\partial(\mathbf{F}_1, \ldots, \mathbf{F}_n)}{\partial(u_1, \ldots, u_n)}.$$

Definition 3.8 (critical point and regular point). Let $F : U \to \mathbb{R}^n$ be a smooth map, where $U \subseteq \mathbb{R}^n$ is open, and let $p \in U$.

(i) p is a critical point of $\mathbf{F}$ if $d\mathbf{F}_p$ is singular (i.e. not invertible)

(ii) p is a regular point of $\mathbf{F}$ if $d\mathbf{F}_p$ is invertible

Lemma 3.2. Let $F : U \to \mathbb{R}^n$ be a smooth map and $p \in U$. Then, the following are equivalent:

(i) p is a regular point of F

(ii) $\det(dF_p) \neq 0$

(iii) The $n \times n$ matrix dF_p is invertible

Proof. This is standard Linear Algebra from MA2001 — an $n \times n$ matrix is invertible if and only if its determinant is non-zero. □

Example 3.6 (the case $n = 1$). We perform a sanity check for the 1-dimensional case. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and write $y = f(u)$. Then,

$$df = \frac{\partial f}{\partial u} = \frac{df}{du} = f'(u).$$

So, a point $p \in \mathbb{R}$ is critical if $f'(p) = 0$, which is exactly the situation where f fails to be locally injective (e.g. local maxima/minima or other flattening behaviour). If instead p was a regular point, i.e. $f'(p) \neq 0$, then f is locally invertible near p. As such, one can solve $y = f(x)$ for x as a function of y in a neighbourhood).

From Example 3.6, we give a remark from MA2002 Calculus that even if $f'(p) \neq 0$ at some point p, this only guarantees local invertibility near p. It does not mean f is invertible on all of $\mathbb{R}$.

We now state the inverse function theorem (Theorem 3.1).

Theorem 3.1 (inverse function theorem). Let $U \subseteq \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^n$ be a smooth function. Suppose $p \in U$ is a regular point, i.e. $\det(dF_p) \neq 0$. Then, the following hold:

(i) There exists an open neighbourhood V of p in U and an open set $W \subseteq \mathbb{R}^n$ containing $F(p)$ such that $F : V \to W$ is a bijection

(ii) The inverse map $F^{-1} : W \to V$ exists and is smooth

The inverse function theorem (Theorem 3.1) is often summarised as follows. If $\det(d\mathbf{F}_p) \neq 0$, then F is a local diffeomorphism[1] near p. In particular, near a regular point, F has a well-behaved coordinate change. We shall give some applications of Theorem 3.1 in due course, but first, we shall give a nice interpretation of regular surfaces. That is, a very efficient way to prove that many subsets $S \subseteq \mathbb{R}^3$ are regular surfaces is to realise them as level sets

$$S = f^{-1}(a) = \{p \in V : f(p) = a\}$$

of a smooth function $f : V \to \mathbb{R}$. If the gradient never vanishes on the level set, then S is a regular surface.

We now discuss regular points of a scalar field. Let $V \subseteq \mathbb{R}^3$ be open and let $f : V \to \mathbb{R}$ be smooth.

Definition 3.9. Let $p \in V$. Define

$$d\mathbf{f}_p = \left(\frac{\partial \mathbf{f}}{\partial x}\Big|_p, \frac{\partial \mathbf{f}}{\partial y}\Big|_p, \frac{\partial \mathbf{f}}{\partial z}\Big|_p \right) \in \mathbb{R}^3.$$

We say that p is a regular point of $\mathbf{f}$ if $d\mathbf{f}_p \neq 0$. In MA2104 Multivariable Calculus, $d\mathbf{f}_p$ is the gradient vector and is denoted by $\operatorname{grad}\mathbf{f}(p)$ or $\nabla\mathbf{f}(p)$.

Proposition 3.2 (level sets as regular surfaces). Let $V \subseteq \mathbb{R}^3$ be open and let $\mathbf{f} : V \to \mathbb{R}$ be smooth. Fix $a \in \mathbb{R}$ and consider the level set

$$S = \mathbf{f}^{-1}(a) = \{p \in V : \mathbf{f}(p) = a\}.$$

If every point of S is a regular point of $\mathbf{f}$, i.e. $d\mathbf{f}_p \neq 0$ for all $p \in S$, then S is a regular surface in $\mathbb{R}^3$.

Here is a geometric remark with regards to Proposition 3.2 and MA2104 Multivariable Calculus. For a regular level set $S = \mathbf{f}^{-1}(a)$, the gradient $\operatorname{grad}\mathbf{f}(p)$ is always perpendicular to the surface at p.

Proof. Fix $p = (x_0, y_0, z_0) \in S$. Since $d\mathbf{f}_p \neq 0$, at least one partial derivative is non-zero. Without a loss of generality, assume that $\frac{\partial f}{\partial z}\big|_p \neq 0$. Then, define

$$\mathbf{F} : V \to \mathbb{R}^3 \quad \text{where} \quad \mathbf{F}(x,y,z) = (x, y, \mathbf{f}(x,y,z)).$$

[1] Just to jump the gun, we will formally what a diffeomorphism is in Definition 5.1. In this context, a map $F : V \to W$ is a diffeomorphism if F is smooth, F is bijective so F^{-1} exists, and F^{-1} is smooth.

Then,

$$d\mathbf{F} = \begin{pmatrix} 1 & 0 & \frac{\partial \mathbf{f}}{\partial x} \\ 0 & 1 & \frac{\partial \mathbf{f}}{\partial y} \\ 0 & 0 & \frac{\partial \mathbf{f}}{\partial z} \end{pmatrix}$$

so it implies that $\det(d\mathbf{F})$ evaluated at p is $\frac{\partial f}{\partial z}\mid_p \neq 0$. By the inverse function theorem (Theorem 3.1), there exist open sets $V_0 \subseteq V$ containing p and $W \subseteq \mathbb{R}^3$ containing $\mathbf{F}(p) = (x_0, y_0, a)$ such that $F : V_0 \to W$ is a bijection with smooth inverse F^{-1}.

Write

$$\mathbf{F}^{-1}(u,v,w) = (g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)).$$

Since $\mathbf{F}(x,y,z) = (x,y,\mathbf{f}(x,y,z))$, one obtains

$$g_1(u,v,w) = u \quad \text{and} \quad g_2(u,v,w) = v \quad \text{and} \quad \mathbf{f}(u,v,g_3(u,v,w)) = w.$$

Now, restrict to the slice $w = a$. Let

$$U = \{(u,v) : (u,v,a) \in W\} \subseteq \mathbb{R}^2,$$

and define the local parametrisation

$$\mathbf{x} : U \longrightarrow S \quad \text{where} \quad \mathbf{x}(u,v) = \mathbf{F}^{-1}(u,v,a) = (u,v,g_3(u,v,a)).$$

Then, $\mathbf{x}(U) \subseteq S$ and $\mathbf{x}$ is smooth. Moreover, the rank condition holds (it is a genuine chart), so S is a regular surface near p. □

Corollary 3.1 (local graph form). If $p \in S = \mathbf{f}^{-1}(a)$ and $\frac{\partial f}{\partial z}\mid_p \neq 0$, then S can be locally parametrised near p by (x,y)-coordinates. That is, there exists a smooth function h such that, near p,

$$S = \{(x,y,h(x,y))\}.$$

Equivalently, a local coordinate function is

$$\mathbf{x}(u,v) = (u,v,h(u,v)).$$

Example 3.7 (the unit sphere S^2 via a level set). Let $f(x,y,z) = x^2 + y^2 + z^2$, $V = \mathbb{R}^3$, and $a = 1$. Then,

$$f^{-1}(1) = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\} = S^2.$$

Next, $df = (2x, 2y, 2z)$. On S^2, we cannot have $(x,y,z) = (0,0,0)$, so $df \neq 0$ everywhere on S^2. Hence, S^2 is a regular surface.

Example 3.8 (A hyperboloid level set). Let $f(x,y,z) = -x^2 - y^2 + z^2 - 1$, $V = \mathbb{R}^3$, and $a = 0$. Then,

$$f^{-1}(0) = \left\{(x,y,z) : z^2 = x^2 + y^2 + 1\right\}.$$

Next, $df = (-2x, -2y, 2z)$. If $df = (0,0,0)$, then $x = y = z = 0$, but $(0,0,0) \notin f^{-1}(0)$ since $f(0,0,0) = -1 \neq 0$. Hence, $df \neq 0$ on $f^{-1}(0)$, so $f^{-1}(0)$ is a regular surface.

3.4 Tangent Spaces

Definition 3.10 (differential of a parametrisation). Let $S \subseteq \mathbb{R}^3$ be a regular surface, and let

$$x : U \to S \cap V \quad \text{where} \quad x(u,v) = (x(u,v), y(u,v), z(u,v))$$

be a coordinate function. The differential at $p = x(u_0, v_0)$ is the 3×2 matrix

$$(dx)_p = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}_p .$$

Its columns are the velocity vectors

$$(x_u)_p = \left.\frac{\partial x}{\partial u}\right|_p \quad \text{and} \quad (x_v)_p = \left.\frac{\partial x}{\partial v}\right|_p .$$

Definition 3.11 (tangent space). The tangent space of S at $p \in S$ is

$$T_pS = \operatorname{span}\left\{(x_u)_p, (x_v)_p\right\}.$$

Since S is regular, $(dx)_p$ has rank 2 by Definition 3.7, so $(x_u)_p$ and $(x_v)_p$ are linearly independent and T_pS is a 2-dimensional subspace of $\mathbb{R}^3$. Note that the cross product

$$(x_u)_p \times (x_v)_p \neq (0,0,0)$$

is perpendicular to T_pS, hence gives a normal direction to the tangent plane. Equivalently, one may view it in terms of Jacobians. That is,

$$x_u \wedge x_v = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right) \quad \text{and} \quad x_u \wedge x_v \neq (0,0,0).$$

If we translate coordinates so that p is the origin, the tangent plane can be written as

$$\left\{(X,Y,Z) \in \mathbb{R}^3 : \left((x_u)_p \times (x_v)_p\right) \cdot (X,Y,Z) = 0\right\}.$$

The tangent plane is said to be a first-order approximation to the surface near p. If we choose a different coordinate function, the definition via column space gives the same plane, i.e. T_pS is intrinsic.

Proposition 3.3 (local graph criterion). Let S be a regular surface and $x(u,v)$ a coordinate function. Suppose at $p = x(u_0, v_0)$ we have

$$\frac{\partial(x,y)}{\partial(u,v)}(u_0, v_0) \neq 0.$$

Then, there exists a neighbourhood W of p in S such that W is the graph of a smooth function $z = f(x,y)$.

The condition

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

in Proposition 3.3 geometrically means that the tangent plane at p (and nearby) is not perpendicular to the xy-plane.

Proof. Let $\pi : S \to \mathbb{R}^2$ denote the projection of S onto the xy-plane. Next, define

$$h = \pi \circ x \quad \text{where} \quad h(u,v) = (x(u,v), y(u,v)).$$

The hypothesis says that $\det(dh)_{(u_0,v_0)} \neq 0$, so by the inverse function theorem (Theorem 3.1), h has a local inverse h^{-1} near $(x(u_0,v_0), y(u_0,v_0))$. Write

$$h^{-1}(x,y) = (u(x,y), v(x,y)).$$

Then, define

$$\gamma(x,y) = x \circ h^{-1}(x,y) = (x, y, z(u(x,y), v(x,y))).$$

Hence, locally S is parametrised as $(x, y, f(x,y))$ with $f(x,y) = z(u(x,y), v(x,y))$, so the surface is a graph. □

Example 3.9 (sphere and the equator). On the unit sphere, at the equator it is not possible to write the surface locally as $z = f(x,y)$. The relevant Jacobian condition fails precisely on the equator so the local graph criterion (Proposition 3.3) does not apply there.

Away from the equator, we may write the upper and lower hemispheres as the following graphs:

$$z = \sqrt{1-x^2-y^2} \quad \text{and} \quad z = -\sqrt{1-x^2-y^2}$$

Example 3.10 (a surface that is globally a graph). Consider the parametrised surface

$$x(u,v) = (3u+4v, 4u+5v, \cos(uv)).$$

Compute

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} = -1 \neq 0,$$

so the graph criterion applies. Moreover, solving

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x+4y \\ 4x-3y \end{pmatrix},$$

we get

$$z = \cos(uv) = \cos((-5x+4y)(4x-3y)).$$

Hence the surface is the graph $z = \cos((-5x+4y)(4x-3y))$.

3.5 Change of Parameters

If $x : U \to S$ is a coordinate function (chart) of a regular surface $S \subseteq \mathbb{R}^3$, and $y : Y \to \mathbb{R}^3$ is a smooth parametrised patch whose image lies inside $x(U)$, then we can change parameters by passing from y-coordinates to x-coordinates via $x^{-1} \circ y : Y \to U$.. The key point is that this map is smooth.

Proposition 3.4 (change of parameters). Let S be a regular surface and let $x : U \to S$ be a coordinate function. Let $Y \subseteq \mathbb{R}^n$ be open and let $y : Y \to \mathbb{R}^3$ be a smooth map, i.e.

$$y(\xi_1, \ldots, \xi_n) = (x(\xi_1, \ldots, \xi_n), y(\xi_1, \ldots, \xi_n), z(\xi_1, \ldots, \xi_n))$$

with smooth component functions. Assume that $y(Y) \subseteq x(U)$. Then, the map $x^{-1} \circ y : Y \to U$ is smooth.

Proof. This is an application of the inverse function theorem (Theorem 3.1) to show that the local inverse x^{-1} behaves smoothly on the image[2]. □

Let $x : U \to S$ and $y : V \to S$ be two coordinate functions on the same regular surface S. Assume their images overlap. That is,

$$W = x(U) \cap y(V) \neq \emptyset \quad \text{where} \quad U' = x^{-1}(W) \subseteq U \text{ and } V' = y^{-1}(W) \subseteq V.$$

Then, we have a bijection (the change of coordinates)

$$h = x^{-1} \circ y : V' \to U'. \tag{3.3}$$

Proposition 3.5 (transition maps are diffeomorphisms). The bijection

$$h = x^{-1} \circ y : V' \to U'$$

in (3.3) is a diffeomorphism. Equivalently, both h and $h^{-1} = y^{-1} \circ x$ are smooth.

Proof. Smoothness of h follows from Proposition 3.4. Interchanging the roles of x and y yields the smoothness of h^{-1}. □

Example 3.11. Let $S^2 \subseteq \mathbb{R}^3$ denote the unit sphere and let

$$U = V = \left\{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\right\}.$$

We define two charts, namely the northern hemisphere

$$x(u,v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$

and the hemisphere pointing in the direction of the x-axis, which is

$$y(\xi, \eta) = \left(\sqrt{1 - \xi^2 - \eta^2}, \xi, \eta\right).$$

[2] We treat this as a technical tool; the detailed proof is omitted.

On the overlap $W = x(U) \cap y(V)$, the transition map is

$$h := x^{-1} \circ y : V' \to U' \quad \text{where} \quad h(\xi,\eta) = \left(\sqrt{1-\xi^2-\eta^2}, \xi\right),$$

and its inverse is

$$h^{-1}(u,v) = \left(v, \sqrt{1-u^2-v^2}\right).$$

Both are smooth on the appropriate restricted domains U', V'.

One can interpret h and h^{-1} as the precise gluing data that tells us how two flat parameter domains U' and V' fit together to form the surface. This viewpoint generalises to the abstract construction of manifolds by gluing open sets in $\mathbb{R}^n$.

We can define the tangent space T_pS intrinsically using smooth curves on the surface, and then show this definition agrees with the span of x_u and x_v from a coordinate chart. The punchline is that T_pS does not depend on which chart you use.

Let S be a regular surface and let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a (short) smooth curve on S. We do not assume arc length parametrisation; we even allow $\alpha'(t) = 0$ for some t. Fix $p = \alpha(0) \in S$, and choose a chart $x : U \to S$ such that the image $x(U)$ contains the whole curve. Then by change of parameters (Proposition 3.4), the map

$$h(t) = (h_1(t), h_2(t)) = \left(x^{-1} \circ \alpha\right)(t)$$

is smooth, and

$$\alpha(t) = x \circ h(t) = x(h_1(t), h_2(t)).$$

By the chain rule,

$$\alpha'(0) = \left.\frac{\partial x}{\partial u}\right|_{(0,0)} h_1'(0) + \left.\frac{\partial x}{\partial v}\right|_{(0,0)} h_2'(0).$$

Writing $a_1 = h_1'(0)$ and $a_2 := h_2'(0)$, we obtain the key formula

$$\alpha'(0) = a_1 x_u(0,0) + a_2 x_v(0,0),$$

so $\alpha'(0)$ lies in the tangent plane (column space of dx_p). Conversely, every tangent plane vector comes from a curve. We formally describe this in Proposition 3.6.

Proposition 3.6. Let v be any vector in the tangent plane at $p = x(0,0)$, say

$$v = a_1 x_u(0,0) + a_2 x_v(0,0).$$

Then, there exists a smooth curve α on S with $\alpha(0) = p$ and $\alpha'(0) = v$.

Proof. Define $\alpha(t) = x(a_1 t, a_2 t)$ be a parametrisation of a curve. Then, $\alpha(0) = x(0,0) = p$, and differentiating yields $\alpha'(0) = v$. □

Theorem 3.2 (intrinsic description of the tangent plane). The tangent plane at $p \in S$ is equal to the union of all tangent vectors $\alpha'(0)$ where α ranges over smooth curves on S with $\alpha(0) = p$.

Corollary 3.2 (independence of coordinates). The tangent plane T_pS (equivalently, the column space of dx_p) is independent of the choice of coordinate function x.

Example 3.12 (prescribing a tangent vector on S^2). Pick a chart $x(u,v)$ on S^2 and a point $p = x(u_0, v_0)$. Given a vector $v = a_1 x_u(u_0,v_0) + a_2 x_v(u_0,v_0)$, a curve with $\alpha(0) = p$ and $\alpha'(0) = v$ is

$$\alpha(t) = x(u_0 + a_1 t, v_0 + a_2 t).$$

3.6 Smooth Functions on Surfaces

We begin with a motivation on how to differentiate a function on a curved surface. Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $f : S \to \mathbb{R}$. Since S is curved, we do not differentiate f by moving in straight lines in $\mathbb{R}^3$. Instead, we differentiate along a parametrised patch $x(u,v)$, i.e. we pull back f to a function on an open set in $\mathbb{R}^2$.

Definition 3.12 (smoothness via coordinate functions). Let $f : S \to \mathbb{R}$ and let $p \in S$. Let $x : U \to S$ be a coordinate function with $p = x(u_0, v_0) \in x(U)$.

(i) We say that f is smooth on $x(U)$ if $f \circ x : U \to \mathbb{R}$ is smooth in the usual MA2104 Multivariable Calculus sense

(ii) We say that f is smooth at p if f is smooth on some open subset of S containing p

With $p = x(u_0, v_0)$, we define

$$\left.\frac{\partial f}{\partial u}\right|_p = \left.\frac{\partial (f \circ x)}{\partial u}\right|_{(u_0,v_0)} \quad \text{and} \quad \left.\frac{\partial^2 f}{\partial u \partial v}\right|_p = \left.\frac{\partial^2 (f \circ x)}{\partial u \partial v}\right|_{(u_0,v_0)}$$

and similarly for other mixed/second derivatives.

Example 3.13. Let $S = \left\{(x,y,z) : x^2 + y^2 + z^2 = 1\right\}$ and define $f(x,y,z) = xz^2$. We parametrise the northern hemisphere by

$$x(u,v) = \left(u,\ v,\ \sqrt{1 - u^2 - v^2}\right),$$

so that the north pole is $p = x(0,0)$. Then,

$$(f \circ x)(u,v) = u\left(1 - u^2 - v^2\right),$$

so

$$\frac{\partial (f \circ x)}{\partial u}(u,v) = 1 - 3u^2 - v^2 \quad \text{and} \quad \frac{\partial^2 (f \circ x)}{\partial u^2}(u,v) = -6u.$$

In particular, at $(u,v)=(0,0)$, we have $\frac{\partial f}{\partial u}\Big|_p = 1$ and $\frac{\partial^2 f}{\partial u^2}\Big|_p = 0$. Since $(f\circ x)(u,v) = u\left(1-u^2-v^2\right)$ is a polynomial (hence smooth) on the open unit disk in the uv-plane, it follows from the definition that f is smooth on the northern hemisphere (i.e. on $x(U)$).

Note that partial derivatives depend on the coordinate function. That is to say, if $y: V \to S$ is another coordinate function around p, then in general

$$\frac{\partial(f\circ x)}{\partial u}\bigg|_{(u_0,v_0)} \neq \frac{\partial(f\circ y)}{\partial \xi}\bigg|_{(\xi_0,\eta_0)}.$$

So the numerical value of $\partial f/\partial u$ depends on the chosen parameters.

Also, smoothness at a point is chart-independent. Although derivatives depend on the chart, the property of being smooth at p does not. That is, if $x: U \to S$ and $y: V \to S$ are coordinate functions with $p = x(u_0,v_0) = y(\xi_0,\eta_0)$, let $h = x^{-1}\circ y$ be defined near (ξ_0,η_0) (a change of parameters map). Then,

$$(f\circ y)(\xi,\eta) = (f\circ x\circ h)(\xi,\eta).$$

Since h is smooth (change of parameters) and $f\circ x$ is smooth, the composition $f\circ x\circ h$ is smooth. Hence, $f\circ y$ is smooth. Therefore, the definition of f being smooth at p is independent of the coordinate function.

Definition 3.13 (smooth function on a surface). A function $f: S \to \mathbb{R}$ is called a smooth function on S if it is smooth at every point of S.

Example 3.14 (checking smoothness on the whole sphere). For $f(x,y,z) = xz^2$ on the unit sphere, we can cover the sphere by several coordinate patches (northern hemisphere, southern hemisphere, and additional hemispheres to cover the equator). On each patch, $f\circ x$ is a smooth function of (u,v), hence f is smooth on the entire sphere.

Proposition 3.7 (restriction of a smooth $\mathbb{R}^3$ function is smooth on S). Let $V \subseteq \mathbb{R}^3$ be open and let $S \subseteq V$ be a regular surface. If $F: V \to \mathbb{R}$ is smooth, then its restriction $f := F|_S : S \to \mathbb{R}$ is a smooth function on S.

Proof. For any coordinate function $x: U \to S$, we have

$$f\circ x = (F|_S)\circ x = F\circ x$$

which is smooth since F and x are smooth. □

In many situations, one proves $f: S \to \mathbb{R}$ is smooth by writing $f = F|_S$ where F is a smooth function on an open set $V \subseteq \mathbb{R}^3$ containing S. If F is only defined/smooth away from a *bad set* like an axis, one chooses V to avoid that set; then f is smooth by Proposition 3.7.

Example 3.15. Let

$$F(x,y,z) = \left(x^2+y^2+z^2\right)\sin x \quad \text{on } \mathbb{R}^3.$$

On the unit sphere, we have $x^2+y^2+z^2=1$, hence the restriction of F to S is $f(x,y,z) = \sin x$. This illustrates that restriction means having the same formula but a smaller domain.

Theorem 3.3 (extension theorem). Conversely, every smooth function on S is the restriction of some smooth function defined on an open set in $\mathbb{R}^3$ containing S.

The proof of Theorem 3.3 is complicated and it uses a technique known as partition of unity.

3.7 Smooth Functions between Surfaces

A map $\phi : S_1 \to S_2$ between regular surfaces is declared smooth if, after choosing local coordinates

$$x_1 : U_1 \to S_1 \quad \text{and} \quad x_2 : U_2 \to S_2,$$

the coordinate expression

$$x_2^{-1} \circ \phi \circ x_1 : U_1 \to U_2$$

is a smooth map in the usual sense. Next, let S_1, S_2 be regular surfaces and let $\phi : S_1 \to S_2$ be continuous. Fix $p_1 \in S_1$ and set $p_2 = \phi(p_1) \in S_2$. Choose coordinate functions

$$x_1 : U_1 \to S_1 \text{ with } p_1 \in x_1(U_1) \quad \text{and} \quad x_2 : U_2 \to S_2 \text{ with } p_2 \in x_2(U_2).$$

By shrinking U_1 if necessary, we may assume $\phi(x_1(U_1)) \subseteq x_2(U_2)$ so the coordinate expression

$$x_2^{-1} \circ \phi \circ x_1 : U_1 \to U_2$$

is well-defined.

Definition 3.14 (smoothness). Let $\phi : S_1 \to S_2$ be as in our above discussion.

(i) We say that ϕ is smooth on $x_1(U_1)$ if

$$x_2^{-1} \circ \phi \circ x_1 : U_1 \to U_2$$

is smooth in the sense of Calculus

(ii) We say that ϕ is smooth at p_1 if ϕ is smooth on some open subset of S_1 containing p_1

(iii) We say that ϕ is a smooth map if it is smooth at every point of S_1

Example 3.16 (antipodal map is smooth). Let $S_1 = S_2 = S^2$ and define

$$\phi(x,y,z) = (-x,-y,-z).$$

Pick p_1 on the northern hemisphere so that $\phi(p_1)$ lies on the southern hemisphere. Use the standard charts

$$x_1(u_1,v_1) = \left(u_1, v_1, \sqrt{1-u_1^2-v_1^2}\right) \quad \text{and} \quad x_2(u_2,v_2) = \left(u_2, v_2, -\sqrt{1-u_2^2-v_2^2}\right).$$

Then the coordinate expression becomes

$$\left(x_2^{-1} \circ \phi \circ x_1\right)(u_1,v_1) = (-u_1,-v_1),$$

which is clearly smooth on the open unit disk. Hence, ϕ is smooth on the northern hemisphere. To finish the global statement, one must also check the southern hemisphere and the equator using additional charts.

The definition of smoothness in Definition 3.14 looks like it depends on the chosen coordinate functions x_1 and x_2, but it actually does not: replacing charts does not change whether ϕ is smooth.

Lemma 3.3 (smoothness is independent of charts). If ϕ is smooth at p_1 with respect to one choice of coordinate functions x_1 at p_1 and x_2 at p_2, then it is smooth at p_1 for any other choice of coordinate functions around these points.

Proof. Let y_1 be another coordinate function around p_1. Set the transition map $h = x_1^{-1} \circ y_1$ so that $y_1 = x_1 \circ h$. Then locally,

$$x_2^{-1} \circ \phi \circ y_1 = x_2^{-1} \circ \phi \circ x_1 \circ h.$$

The right side is a composition of smooth maps (by hypothesis $x_2^{-1} \circ \phi \circ x_1$ is smooth, and h is smooth by change of parameters), hence $x_2^{-1} \circ \phi \circ y_1$ is smooth. This shows independence of x_1 and similarly for x_2). □

Proposition 3.8 (restriction of a smooth map in $\mathbb{R}^3$). Let $V_1, V_2 \subseteq \mathbb{R}^3$ be open and let $\Phi : V_1 \to V_2$ be smooth. Let $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ be regular surfaces such that $\Phi(S_1) \subseteq S_2$. Define

$$\phi : S_1 \to S_2 \quad \text{where} \quad \phi = \Phi|_{S_1}.$$

Then, ϕ is smooth.

Proof. Fix $p_1 \in S_1$ and $p_2 = \phi(p_1) \in S_2$. Near p_2, we may assume S_2 is locally a graph and choose coordinates so that

$$x_2^{-1}(x,y,z) = (x,y).$$

Write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. Then for a chart x_1 on S_1, we have

$$\left(x_2^{-1} \circ \phi \circ x_1\right)(u_1,v_1) = \left(\Phi_1 \circ x_1(u_1,v_1),\ \Phi_2 \circ x_1(u_1,v_1)\right),$$

which is smooth because Φ_1, Φ_2 and x_1 are smooth. □

Example 3.17. We are now in position to give a short proof that the antipodal map in Example 3.16 is smooth. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be $\Phi(x,y,z) = (-x,-y,-z)$. Then, Φ is smooth, and $\Phi\left(S^2\right) \subseteq S^2$. Hence $\phi = \Phi\,|_{S^2}$ is smooth by Proposition 3.8.

Example 3.18 (rotation on a tube). Let S be the tube obtained by rotating the line $x = 1$ in the xz-plane about the z-axis. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the z-axis by $30°$:

$$\Phi(x,y,z) = \left(x\cos 30° + y\sin 30°,\ -x\sin 30° + y\cos 30°,\ z\right).$$

Then, Φ is smooth and $\Phi(S) \subseteq S$, so $\phi = \Phi\,|_S \colon S \to S$ is smooth by Example 3.8

Lemma 3.4 (composition of smooth maps is smooth). If $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are smooth maps between regular surfaces, then $\psi \circ \phi : S_1 \to S_3$ is smooth.

3.8 Maps between Tangent Planes

A smooth map between surfaces induces a linear map between tangent planes. This is the main bridge from Differential Geometry to Linear Algebra. Let $\phi : S_1 \to S_2$ be a smooth map between regular surfaces. Fix $p_1 \in S_1$ and set $p_2 = \phi(p_1) \in S_2$. Let $p_2 = \phi(p_1) \in S_2$. Say $v_1 \in T_{p_1}S_1$ is a tangent vector, so there exists a curve

$$\alpha : (-\varepsilon, \varepsilon) \to S_1 \quad \text{such that} \quad \alpha(0) = p_1 \text{ and } \alpha'(0) = v_1.$$

Define a curve on S_2, say $\beta(t) := \phi(\alpha(t))$. Then, $\beta(0) = p_2$ and $\beta'(0) \in T_{p_2}S_2$.

Definition 3.15 (differential). We define the map

$$d\phi_{p_1} : T_{p_1}S_1 \to T_{p_2}S_2 \quad \text{where} \quad d\phi_{p_1}(v_1) = \beta'(0) = (\phi \circ \alpha)'(0).$$

We must check that this is well-defined, i.e. independent of the curve α used to represent v_1.

Definition 3.16 (coordinate functions and the induced map Φ). Let $x_1 : U_1 \to S_1$ be a coordinate function covering p_1 and let $x_2 : U_2 \to S_2$ be a coordinate function covering p_2. Shrink U_1 so that $\phi(x_1(U_1)) \subseteq x_2(U_2)$. Define the induced map between parameter domains

$$\Phi : U_1 \to U_2 \quad \text{where} \quad \Phi = x_2^{-1} \circ \phi \circ x_1.$$

Equivalently, $\phi \circ x_1 = x_2 \circ \Phi$. Write

$$\Phi(u_1, v_1) = (\Phi_1(u_1, v_1), \Phi_2(u_1, v_1)).$$

We give a formula for $d\phi_{p_1}$ in coordinates. Let $p_1 = x_1(u_1, v_1)$. Any tangent vector $v_1 \in T_{p_1}S_1$ can be written as

$$v_1 = a\,(x_{1,u})_{p_1} + b\,(x_{1,v})_{p_1} = (dx_1)_{p_1}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Then,

$$d\phi_{p_1}(v_1) = (dx_2)_{p_2}(d\Phi)_{(u_1,v_1)}\begin{pmatrix} a \\ b \end{pmatrix}.$$

In particular, $d\phi_{p_1}$ is a linear transformation. Note that this map is well-defined because the right side depends only on the coefficients (a,b) of v_1 in the basis $\left\{(x_{1,u})_{p_1},(x_{1,v})_{p_1}\right\}$, and on the Jacobian matrix $(d\Phi)_{(u_1,v_1)}$. Hence, different curves α representing the same v_1 produce the same $d\phi_{p_1}(v_1)$.

We give a matrix representation of the above. Let

$$(d\Phi)_{(u_1,v_1)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Let

$$\mathcal{B}_1 = \left\{(x_{1,u})_{p_1},(x_{1,v})_{p_1}\right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{(x_{2,u})_{p_2},(x_{2,v})_{p_2}\right\}.$$

Then,

$$\begin{aligned} d\phi_{p_1}\left((x_{1,u})_{p_1}\right) &= a_{11}(x_{2,u})_{p_2} + a_{21}(x_{2,v})_{p_2} \\ d\phi_{p_1}\left((x_{1,v})_{p_1}\right) &= a_{12}(x_{2,u})_{p_2} + a_{22}(x_{2,v})_{p_2} \end{aligned}$$

Equivalently,

$$[d\phi_{p_1}]_{\mathcal{B}_2,\mathcal{B}_1} = (d\Phi)_{(u_1,v_1)}.$$

Example 3.19. Let $S_1 = S_2 = S^2$ be the unit sphere, and define $\phi(x,y,z) = (-y,x,z)$ to be a 90° anticlockwise rotation about the z-axis. Use the northern hemisphere coordinate functions

$$x_1(u_1,v_1) = \left(u_1,v_1,\sqrt{1-u_1^2-v_1^2}\right) \quad \text{and} \quad x_2(u_2,v_2) = \left(u_2,v_2,\sqrt{1-u_2^2-v_2^2}\right).$$

Then the induced map is

$$\Phi(u_1,v_1) = (-v_1,u_1) \quad \text{where} \quad d\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence for $v_1 = a(x_{1,u})_{p_1} + b(x_{1,v})_{p_1}$, we obtain

$$d\phi_{p_1}(v_1) = (dx_2)_{p_2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -b(x_{2,u})_{p_2} + a(x_{2,v})_{p_2}.$$

3.9 The First Fundamental Form

We first give the big picture. The first fundamental form is the object that assigns lengths (and hence angles and distances) on a surface by restricting the Euclidean inner product of $\mathbb{R}^3$ to tangent vectors on the surface. Recall that $\mathbb{R}^3$ has the standard inner product

$$\langle u,v\rangle = u_1v_1 + u_2v_2 + u_3v_3.$$

This allows us to measure lengths by considering $|v| = \sqrt{\langle v, v\rangle}$.

Let S be a regular surface and let $p \in S$. Choose a coordinate function $x : U \to S$ covering p, with $p = x(u_0, v_0)$. Then, every tangent vector $v \in T_pS$ can be written as

$$v = ax_u(p) + bx_v(p) \quad \text{for some } a, b \in \mathbb{R}.$$

Definition 3.17 (first fundamental form). For $v \in T_pS$, define $I_p(v) = \langle v, v\rangle$, known as the first fundamental form of S at p.

We then define the functions (evaluated at (u_0, v_0)), known as the E, F, G coefficients. These are namely

$$\begin{aligned} E(u_0, v_0) &= \langle x_u(p), x_u(p)\rangle \\ F(u_0, v_0) &= \langle x_u(p), x_v(p)\rangle \\ G(u_0, v_0) &= \langle x_v(p), x_v(p)\rangle \end{aligned}$$

Then, for $v = ax_u(p) + bx_v(p)$, we have

$$I_p(v) = \langle v, v\rangle = a^2E(u_0, v_0) + 2abF(u_0, v_0) + b^2G(u_0, v_0).$$

E, F, G record how the coordinate directions x_u and x_v sit in $\mathbb{R}^3$. That is, $E = |x_u|^2$ and $G = |x_v|^2$, and $F = \langle x_u, x_v\rangle$ measures the failure of orthogonality.

Example 3.20 (plane with orthonormal coordinates). Let S be a plane and choose an orthonormal basis $\{w_1, w_2\}$ of the plane. Take the parametrisation

$$x(u, v) = p_0 + uw_1 + vw_2.$$

Then, $x_u = w_1$ and $x_v = w_2$, so $E = 1$, $F = 0$, and $G = 1$. As such, $I_p(ax_u + bx_v) = a^2 + b^2$.

Example 3.21 (tube). Let

$$x(u, v) = (\cos u, \sin u, v) \quad \text{where } 0 < u < 2\pi, v \in \mathbb{R}.$$

Then, $x_u = (-\sin u, \cos u, 0)$ and $x_v = (0, 0, 1)$. As such, $E = 1$, $F = 0$ and $G = 1$. As such, $I_p(ax_u + bx_v) = a^2 + b^2$.

We now discuss the arc length on a surface using the first fundamental form. Let $\alpha(t)$ be a smooth curve on S. Since $\alpha'(t) \in T_{\alpha(t)}S$, its speed is

$$|\alpha'(t)| = \sqrt{I_{\alpha(t)}(\alpha'(t))}.$$

Hence, the arc length from 0 to t is

$$s(t) = \int_0^t |\alpha'(\tau)|\, d\tau = \int_0^t \sqrt{I_{\alpha(\tau)}(\alpha'(\tau))}\, d\tau.$$

Now, suppose $\alpha(t)$ is written in local coordinates as $\alpha(t) = x(u(t), v(t))$. Then,

$$\alpha'(t) = x_u u'(t) + x_v v'(t),$$

so the first fundamental form gives

$$I\left(\alpha'(t)\right) = E(u,v)\left(u'(t)\right)^2 + 2F(u,v)\,u'(t)\,v'(t) + G(u,v)\left(v'(t)\right)^2.$$

Therefore,

$$s(t) = \int_0^t \sqrt{E(u,v)\left(\frac{du}{d\tau}\right)^2 + 2F(u,v)\left(\frac{du}{d\tau}\right)\left(\frac{dv}{d\tau}\right) + G(u,v)\left(\frac{dv}{d\tau}\right)^2}\, d\tau.$$

Formally, we write the metric expression

$$ds^2 = Edu^2 + 2F\,dudv + Gdv^2.$$

At this point, du and dv are treated as symbols encoding the quadratic form determined by I. They become genuinely meaningful once we develop the differential-geometric interpretation of the metric.

Let θ be the angle between x_u and x_v. Then, we can write

$$\cos\theta = \frac{\langle x_u, x_v\rangle}{|x_u|\,|x_v|} = \frac{F}{\sqrt{E}\sqrt{G}}.$$

We say that x is an orthogonal parametrisation if $x_u \perp x_v$ at all points, i.e. $\cos\theta = 0$ if and only if $F = 0$ on U.

We then explain how the first fundamental form computes the area of a surface patch. Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $x : U \to S$ be a coordinate function. Consider the parametrised patch $x(U) \subseteq S$. ix $(u,v) \in U$ and take a small rectangle

$$[u, u+\Delta u] \times [v, v+\Delta v] \subseteq U.$$

Its image on the surface is approximately a parallelogram spanned by the vectors $x_u(u,v)\,\Delta u$ and $x_v(u,v)\,\Delta v$. Hence, the side lengths satisfy

$$\Delta a = \|x(u+\Delta u, v) - x(u,v)\| \approx \|x_u(u,v)\|\,\Delta u \quad \text{and similarly} \quad \Delta b \simeq \|x_v(u,v)\|\,\Delta v,$$

and the (approximate) area is

$$\Delta A = \Delta a \Delta b\,|\sin\theta| \approx \|x_u(u,v) \wedge x_v(u,v)\|\,\Delta u\,\Delta v.$$

Then, the area of the parametrised patch $x(U)$ is

$$\lim_{\Delta u, \Delta v \to 0} \sum_U \|x_u \wedge x_v\|\,\Delta u \Delta v = \iint_U \|x_u \wedge x_v\|\; dudv. \tag{3.4}$$

Proposition 3.9 (area formula). Let $x : U \to S$ be a coordinate function, and set $E = \langle x_u, x_u\rangle$, $F = \langle x_u, x_v\rangle$, and $G = \langle x_v, x_v\rangle$. Then,

$$\operatorname{Area}(x(U)) = \iint_U \|x_u \wedge x_v\|\; dudv = \iint_U \sqrt{EG - F^2}\, dudv.$$

Proof. Recall the vector identity

$$\|a \wedge b\|^2 = \|a\|^2 \|b\|^2 - \langle a, b \rangle^2 .$$

Applying this with $a = x_u$ and $b = x_v$ gives

$$\|x_u \wedge x_v\|^2 = \|x_u\|^2 \|x_v\|^2 - \langle x_u, x_v \rangle^2 = EG - F^2.$$

Since $\|x_u \wedge x_v\| \geq 0$, we have

$$\|x_u \wedge x_v\| = \sqrt{EG - F^2},$$

and substituting into (3.4) yields the result. □

The key point is that the area formula in (3.9) depends only on E, F, G, hence depends only on the first fundamental form. So, we do not need to know the lengths of all vectors in $\mathbb{R}^3$ to compute area on a surface; knowledge of lengths of tangent vectors is sufficient. This viewpoint generalizes to Riemannian geometry.

Example 3.22 (area of torus). Let $a > r$ and define

$$x(u, v) = ((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u) \quad \text{where } 0 < u, v < 2\pi.$$

This parametrises a torus. We then compute the E, F, G coefficients. First,

$$x_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u) \quad \text{and} \quad x_v = (-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0)$$

so $E = r^2$, $F = 0$, and $G = (a + r\cos u)^2$. Hence,

$$\sqrt{EG - F^2} = r(a + r\cos u) = r(r\cos u + a)$$

Therefore, by Proposition 3.9, the area is

$$A = \int_0^{2\pi} \int_0^{2\pi} \sqrt{EG - F^2}\, dudv = \int_0^{2\pi} \int_0^{2\pi} r(r\cos u + a)\ dudv = 4\pi^2 ra.$$

In Do Carmo's text [1], the integration is sometimes carried out by approximating an open parameter set by closed sets with boundary because Riemann integration is typically presented for such domains. Using Lebesgue integration, one can integrate over open subsets directly.

3.10 The Chain Rule

Whenever we compose smooth maps

$$U \xrightarrow{F} V \xrightarrow{G} W,$$

the differential of the composite $H = G \circ F$ is given by matrix multiplication. That is, $dH = dGdF$. This is the multivariable chain rule written in the language of Jacobian matrices.

Now, let $U \subseteq \mathbb{R}^k$, $V \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^n$ be open sets. Let $F : U \to V$ be smooth, and write

$$F(x_1, \ldots, x_k) = (F_1(x), \ldots, F_m(x)) \quad \text{where } x = (x_1, \ldots, x_k).$$

Definition 3.18 (differential/Jacobian matrix). The differential of F is the $m \times k$ matrix

$$dF = \left(F_{ij}\right)_{1 \le i \le m, 1 \le j \le k} \quad \text{where} \quad F_{ij} = \frac{\partial F_i}{\partial x_j}.$$

It is also called the Jacobian matrix of F.

If $k = m$, then dF is a square matrix and we define the Jacobian determinant

$$\det(dF) = \frac{\partial(F_1, \ldots, F_m)}{\partial(x_1, \ldots, x_m)}.$$

Similarly, let $G : V \to W$ be smooth, written as

$$G(y_1, \ldots, y_m) = (G_1(y), \ldots, G_n(y)) \quad \text{where } y = (y_1, \ldots, y_m),$$

with differential $dG = (G_{i\ell})$, where $G_{i\ell} = \frac{\partial G_i}{\partial y_\ell}$.

Proposition 3.10 (chain rule). Define the composite map

$$H = G \circ F : U \to W \quad \text{where } H(x) = (H_1(x), \ldots, H_n(x)).$$

We have $dH = dGdF$, where the right side is the product of the $n \times m$ matrix dG with the $m \times k$ matrix dF.

Proof. It suffices to check entries. Fix $1 \le i \le n$ and $1 \le j \le k$. Since $H_i = G_i \circ F$, the usual MA2104 Multivariable Calculus chain rule gives

$$\frac{\partial H_i}{\partial x_j} = \sum_{\ell=1}^{m} \left.\frac{\partial G_i}{\partial y_\ell}\right|_{y=F(x)} \cdot \frac{\partial F_\ell}{\partial x_j}.$$

Using matrices, the (i, j)-entry of dH equals to the (i, j)-entry of $dGdF$. □

Note that if $k = m = n$, then dF, dG, dH are all $m \times m$ matrices. Hence,

$$\det(dH) = \det(dGdF) = \det(dG)\det(dF).$$

In particular, if $G = F^{-1}$, then $dG = (dF)^{-1}$.

Example 3.23. Let $F(t)$ be a smooth curve on the uv-plane, say

$$F(t) = (F_1(t), F_2(t)).$$

Let $X : U \to S$ be a coordinate function of a surface S, written as

$$X(u,v) = (X(u,v), Y(u,v), Z(u,v)).$$

Define

$$H = X \circ F \quad \text{where} \quad H(t) = (H_1(t), H_2(t), H_3(t)).$$

Then, $H(t)$ is a curve on the surface S, and by the chain rule (Proposition 3.10), we have

$$H'(t) = dX_{F(t)}F'(t).$$

Writing this out, we have

$$H'(t) = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix}_{(u,v)=F(t)} \begin{pmatrix} F_1'(t) \\ F_2'(t) \end{pmatrix}.$$

To interpret this example, the tangent vector $H'(t)$ of the curve on the surface is obtained by pushing forward the tangent vector $F'(t)$ in parameter space via the Jacobian matrix dX.

Chapter 4

Curvature of Surfaces

4.1 The Gauss Map

In Definition 2.9, we defined the curvature of a curve. For a surface, curvature depends on a direction. Fix a point p on a regular surface S and choose a unit normal vector $N(p)$. Given a unit tangent direction $v \in T_pS$, slice the surface by the plane spanned by v and $N(p)$. This produces a plane curve on S called a normal section. The curvature of this normal section at p is the normal curvature[1] of S at p in the direction v.

As we vary the tangent direction v, the normal curvature changes. Gauss's key observations [3] were the following:

(i) There exists a direction where the normal curvature is maximal, and a perpendicular direction where it is minimal

(ii) These extremal values are the principal curvatures $k_1(p)$ and $k_2(p)$ The Gauss curvature at p is the product

$$K(p) = k_1(p)\, k_2(p).$$

Heuristically, $K(p) > 0$ means sphere-like, $K(p) = 0$ means cylinder-like, and $K(p) < 0$ means saddle-like.

> Definition 4.1 (Gauss map). Let S be a regular surface. A Gauss map is a smooth map $N : S \to S^2$ such that for each $q \in S$, the vector $N(q)$ is a unit vector perpendicular to the tangent plane T_qS.

> orientability
>
> Definition 4.2. Locally, if $x : U \to S$ is a parametrisation, one can define a unit

[1] Here is a sign convention. The normal curvature is positive if the normal section bends to the same side of the chosen normal vector $N(p)$. Naturally, the normal curvature is negative if the normal section bends to the opposite side of the chosen normal vector $N(p)$.

normal by

$$N(x(u,v)) = \pm \frac{x_u(u,v) \wedge x_v(u,v)}{|x_u(u,v) \wedge x_v(u,v)|}.$$

However, globally we may fail to choose the sign continuously. If a global continuous choice exists, we say S is orientable. If not, S is non-orientable.

Recall from MA2104 Multivariable Calculus that an example of a non-orientable surface is the Möbius strip.

Example 4.1 (plane). If S is the plane $ax+by+cz=0$, a unit normal vector is

$$n = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}.$$

Then, $N(q) = n$ for all $q \in S$, so the image of N is a single point on S^2.

Example 4.2 (unit sphere). For $S = S^2$, at $p = (x,y,z) \in S^2$, the outward unit normal equals p itself, so the Gauss map is

$$N : S^2 \to S^2 \quad \text{where } N(p) = p,$$

i.e. the identity map.

Example 4.3 (hyperbolic paraboloid). Let S be given by $f(x,y,z) = z - y^2 + x^2 = 0$ and parametrise by

$$x(u,v) = \left(u, v, v^2 - u^2\right).$$

Since $\nabla f = (2x, -2y, 1)$ is perpendicular to S, we obtain the Gauss map (unit normal)

$$N(x(u,v)) = \frac{(2u, -2v, 1)}{\sqrt{4u^2+4v^2+1}}.$$

4.2 The Second Fundamental Form

Fix $q \in S$. The differential $dN_q : T_qS \to T_{N(q)}S^2$ lands in the tangent plane of the sphere at $N(q)$. However, both planes T_qS and $T_{N(q)}S^2$ pass through the origin and are perpendicular to $N(q)$, hence they coincide as the same 2-plane in $\mathbb{R}^3$. Therefore, we may view

$$dN_q : T_qS \to T_qS$$

as a linear transformation of a 2-dimensional inner product space.

Proposition 4.1. For each $q \in S$, the linear map $dN_q : T_qS \to T_qS$ is self-adjoint with respect to the induced inner product on T_qS.

Definition 4.3 (second fundamental form). For $v \in T_qS$, define the quadratic form

$$\mathrm{II}_q(v) = -\left\langle dN_q(v), v \right\rangle.$$

This is called the second fundamental form of S at q.

Let $\mathcal{B} = \{v_1, v_2\}$ be an orthonormal basis of T_qS and write

$$-\left[dN_q\right]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Then, the coefficients are determined by the quadratic form II_q via

$$a = \mathrm{II}_q(v_1) \quad \text{and} \quad d = \mathrm{II}_q(v_2) \quad \text{and} \quad b = \frac{1}{2}\left(\mathrm{II}_q(v_1+v_2) - \mathrm{II}_q(v_1) - \mathrm{II}_q(v_2)\right).$$

Hence, knowing $\mathrm{II}_q(\cdot)$ for all tangent vectors is equivalent to knowing the linear map dN_q.

4.3 Normal Curvature

Definition 4.4 (normal curvature). Let $\alpha(s)$ be a curve on a regular surface S, parametrised by arc length, and suppose $p = \alpha(0) \in S$. Recall that

$$\alpha''(0) = k(0)\, n(0),$$

where $k(0)$ is the curvature of the space curve α and $n(0)$ is its principal normal. Fix a unit normal N along S. Define

$$k_n(0) = \left\langle \alpha''(0), N(p)\right\rangle = k(0)\langle n(0), N(p)\rangle = k(0)\cos\theta,$$

where θ is the angle between $n(0)$ and $N(p)$. The number $k_n(0)$ is called the normal curvature of α on S at p.

Proposition 4.2. Let $\alpha(s)$ be a curve on S parametrised by arc length with $p = \alpha(0)$. Then,

$$\mathrm{II}_p\left(\alpha'(0)\right) = k_n(0).$$

In particular, the normal curvature depends only on the tangent vector $v = \alpha'(0) \in T_pS$, and not on the choice of curve with that tangent direction.

Proof. Since $\alpha'(s) \in T_{\alpha(s)}S$, we have

$$\left\langle N(\alpha(s)), \alpha'(s)\right\rangle = 0.$$

Differentiate with respect to s and use $\alpha''(s) = k(s)\, n(s)$ to obtain

$$0 = \left\langle dN\left(\alpha'(s)\right), \alpha'(s)\right\rangle + \left\langle N(\alpha(s)), \alpha''(s)\right\rangle = -\mathrm{II}_{\alpha(s)}\left(\alpha'(s)\right) + k(s)\langle N, n\rangle.$$

Evaluating at $s = 0$ gives $\mathrm{II}_p(\alpha'(0)) = k(0)\langle N, n\rangle = k_n(0)$. ☐

Given a unit tangent vector $v \in T_pS$, consider the plane spanned by v and $N(p)$. Its intersection with the surface (for s small) is a curve $\alpha(s)$ called the normal section of S

at p along v. For this curve, the principal normal $n(0)$ is parallel to $N(p)$, so $\theta = 0$ or π and hence

$$|k_n(0)| = k(0).$$

So, among all curves through p with tangent v, the normal section is the one whose curvature equals $|k_n|$.

Example 4.4 (unit sphere). On S^2, the normal sections are great circles (radius 1), hence curvature 1. With the outward normal, one gets constant normal curvature (sign depending on convention) for every unit tangent vector v.

Definition 4.5 (principal directions and principal curvatures). At each $p \in S$, the map $dN_p : T_pS \to T_pS$ is self-adjoint, hence $-dN_p$ is self-adjoint as well. By the spectral theorem, there exists an orthonormal basis $\mathcal{B} = \{v_1, v_2\}$ of T_pS such that

$$(-dN_p)(v_1) = k_1 v_1 \quad \text{and} \quad (-dN_p)(v_2) = k_2 v_2.$$

In this basis,

$$-[dN_p]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

The directions v_1 and v_2 are the principal directions at p, and the eigenvalues $k_1(p), k_2(p)$ are the principal curvatures.

Proposition 4.3 (extremal normal curvatures occur in principal directions). Let $v \in T_pS$ be a unit vector. Write $v = av_1 + bv_2$ with $a^2 + b^2 = 1$. Then,

$$\mathrm{II}_p(v) = k_1 a^2 + k_2 b^2.$$

Hence, the maximal and minimal values of $\mathrm{II}_p(v)$ (equivalently k_n) occur at $v = v_1$ and $v = v_2$ respectively, and these directions are perpendicular.

Definition 4.6 (Gauss curvature and mean curvature). Let k_1, k_2 be the eigenvalues of $-dN_p$. Then, define the Gauss curvature $K(p)$ and the mean curvature $H(p)$ as follows:

$$K(p) = k_1(p)k_2(p) \quad \text{and} \quad H(p) = \frac{1}{2}(k_1(p) + k_2(p))$$

Equivalently,

$$K(p) = \det(dN_p) \quad \text{and} \quad H(p) = \frac{1}{2}\operatorname{tr}(-dN_p),$$

which are independent of basis.

Recall from Definition 4.6 that K denotes Gauss curvature. We give a geometric meaning of K. If $K(p) > 0$, then the surface is locally elliptic at p. That is to say, in a neighbourhood of p, it lies on one side of the tangent plane, and the normal curvature has the same sign in every tangent direction, so it is sphere-like. If $K(p) < 0$, then the surface is locally hyperbolic at p. That is, it crosses its tangent plane and the normal curvature takes both signs depending on direction, so it is saddle-like. If $K(p) = 0$, then the surface is locally parabolic (or flat in at least one direction). That is, at least one principal curvature vanishes, so the surface is cylinder-like (or planar, if both principal curvatures vanish).

4.4 The Gauss Map in Local Coordinates

Let $x : U \to S$ be a local parametrisation, and define

$$M(u,v) = (N \circ x)(u,v) = N(x(u,v)).$$

We want formulae for the matrix of dN (equivalently dM) in terms of the first and second fundamental forms, so that we can compute K, H, and k_1, k_2 in coordinates. Write

$$E = \langle x_u, x_u \rangle \quad \text{and} \quad F = \langle x_u, x_v \rangle \quad \text{and} \quad G = \langle x_v, x_v \rangle .$$

Also, let $D = EG - F^2$. Let N be the chosen unit normal and define

$$e = \langle N, x_{uu} \rangle \quad \text{and} \quad f = \langle N, x_{uv} \rangle \quad \text{and} \quad g = \langle N, x_{vv} \rangle .$$

Then, for a tangent vector $v = ax_u + bx_v$ at $p = x(u,v)$, we have

$$\mathrm{II}_p(v) = ea^2 + 2fab + gb^2.$$

Proposition 4.4 (Weingarten equations and curvature formulas). Since $M(u,v)$ is orthogonal to both x_u and x_v, one can write

$$M_u = a_{11}x_u + a_{21}x_v \quad \text{and} \quad M_v = a_{12}x_u + a_{22}x_v.$$

Let $\mathbf{A} = (a_{ij})$ be the corresponding 2×2 matrix of dN in the basis $\{x_u, x_v\}$. Then,

$$a_{11} = \frac{fF - eG}{D} \qquad a_{21} = \frac{eF - fE}{D} \qquad a_{12} = \frac{gF - fG}{D} \qquad a_{22} = \frac{fF - gE}{D}.$$

Moreover,

$$K = \det(\mathbf{A}) = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = -\frac{1}{2}(a_{11} + a_{22}) = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

Finally, the principal curvatures are the eigenvalues of $-dN$, i.e.

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

Example 4.5. For the torus parametrisation

$$x(u,v) = ((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u),$$

one can compute E, F, G and e, f, g explicitly and obtain

$$K(u,v) = \frac{\cos u}{r(a + r\cos u)}.$$

Let S be the graph of $z = h(x,y)$ and parametrise using

$$x(u,v) = (u,v,h(u,v)).$$

Then,

$$x_u = (1,0,h_u) \quad \text{and} \quad x_v = (0,1,h_v).$$

Hence,

$$E = 1 + h_u^2 \quad \text{and} \quad F = h_u h_v \quad \text{and} \quad G = 1 + h_v^2.$$

Also,

$$x_{uu} = (0,0,h_{uu}) \quad \text{and} \quad x_{uv} = (0,0,h_{uv}) \quad \text{and} \quad x_{vv} = (0,0,h_{vv}).$$

A unit normal vector is

$$N(x(u,v)) = \frac{(-h_u,-h_v,1)}{\sqrt{1+h_u^2+h_v^2}}.$$

Thus,

$$e = \frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}} \quad \text{and} \quad f = \frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}} \quad \text{and} \quad g = \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}}.$$

Using

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad EG - F^2 = 1 + h_u^2 + h_v^2,$$

we get

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{\left(1+h_u^2+h_v^2\right)^2}.$$

In particular, at a critical point where $h_u(u,v) = h_v(u,v) = 0$, we have

$$K = \left(h_{uu}h_{vv} - h_{uv}^2\right)\Big|_{(u,v)}.$$

So, the sign of K at a critical point is exactly the sign of the Hessian determinant. We then give a geometric interpretation of the second derivative test. Let (u_0,v_0) be a critical point, i.e. $h_u(u_0,v_0) = h_v(u_0,v_0) = 0$ (so the tangent plane is horizontal and $N = (0,0,1)$). Then, we have the following:

(i) If $K(u_0,v_0) > 0$ (equivalently $h_{uu}h_{vv} - h_{uv}^2 > 0$), then the point is locally elliptic: it is a local maximum or local minimum

(ii) If $K(u_0,v_0) < 0$ (equivalently $h_{uu}h_{vv} - h_{uv}^2 < 0$), then the point is saddle-like

(iii) If $K(u_0,v_0) = 0$, the test is inconclusive

Chapter 5

Isometries

5.1 Isometries

Definition 5.1 (diffeomorphism). A map $\phi : S_1 \to S_2$ is a diffeomorphism if ϕ is smooth, ϕ is bijective (so ϕ^{-1} exists), and ϕ^{-1} is smooth. Two surfaces are diffeomorphic if there exists a diffeomorphism between them.

In Differential Geometry, two diffeomorphic surfaces are regarded as the same object. That is, one cannot tell them apart by properties that are invariant under diffeomorphisms.

Definition 5.2 (isometry). A diffeomorphism $\phi : S_1 \to S_2$ is an isometry if arc lengths of curves are preserved. Equivalently, for every $p \in S_1$ and every $u, v \in T_pS_1$, we have

$$\langle u, v\rangle_p = \left\langle d\phi_p(u), d\phi_p(v)\right\rangle_{\phi(p)}.$$

In particular, taking $u = v$ gives

$$\|u\|_p = \left\|d\phi_p(u)\right\|_{\phi(p)}.$$

Definition 5.3 (local isometry at a point). A smooth map $\phi : S_1 \to S_2$ is a local isometry at $p \in S_1$ if there exist open sets $V_1 \subseteq S_1$ containing p and $V_2 \subseteq S_2$ containing $\phi(p)$ such that

$$\phi : V_1 \to V_2$$

is an isometry.

From Definition 5.3, we see that ϕ being a local isometry at p is a stronger condition than just requiring the metric preserving condition at the single point p; it mustold on an open neighbourhood around p.

Proposition 5.1. Using the setup in Definition 5.3, $\phi : S_1 \to S_2$ is a diffeomorphism between the open sets $V_1 \subseteq S_1$ and $V_2 \subseteq S_2$.

Example 5.1 (plane to cylinder is locally isometric but not bijective). Let

$$S_1 = \{(x,y,0) \in \mathbb{R}^3\} \cong \mathbb{R}^2 \quad \text{and} \quad S_2 = \{(X,Y,Z) \in \mathbb{R}^3 : \ Y^2+Z^2 = 1\}$$

define the xy-plane and the unit cylinder respectively. Define

$$\phi(x,y,0) = (x, \cos y, \sin y).$$

Equivalently, in coordinates we consider the map

$$F : \mathbb{R}^2 \to S_2 \quad \text{where} \quad F(x,y) = (x, \cos y, \sin y).$$

The map is periodic in the y-variable because $F(x,y+2\pi k) = F(x,y)$ for all $k \in \mathbb{Z}$, so it is not injective on all of $\mathbb{R}^2$. It is also surjective onto S_2 (every point $(x, \cos\theta, \sin\theta)$ is hit by (x,θ)), hence globally it is a covering map rather than a bijection.

We then compute the coordinate derivatives $F_x = (1,0,0)$ and $F_y = (0,-\sin y, \cos y)$. These are linearly independent for every (x,y), so $\operatorname{rank}(dF_{(x,y)}) = 2$ everywhere. By the inverse function theorem (Theorem 3.1), for each (x_0,y_0), there exists a neighbourhood U of (x_0,y_0) such that $F|_U$ is a diffeomorphism onto its image. Concretely, if we take

$$U = \{(x,y) : |y-y_0| < \pi\},$$

then $y \mapsto (\cos y, \sin y)$ is injective on $(y_0-\pi, y_0+\pi)$, so F is injective on U.

On the plane with coordinates (x,y), the induced metric is the standard one. That is, $I_{S_1} = dx^2+dy^2$. On the cylinder parametrised by F, the first fundamental form is

$$I_{S_2} = \langle F_x, F_x\rangle\, dx^2 + 2\langle F_x, F_y\rangle\, dxdy + \langle F_y, F_y\rangle\, dy^2.$$

Using the derivatives above, we have

$$\langle F_x, F_x\rangle = 1 \quad \text{and} \quad \langle F_x, F_y\rangle = 0 \quad \text{and} \quad \langle F_y, F_y\rangle = \sin^2 y + \cos^2 y = 1,$$

hence $I_{S_2} = dx^2+dy^2 = I_{S_1}$. Therefore, F preserves lengths of tangent vectors (and hence lengths of sufficiently short curves) on each neighbourhood where it is one-to-one. In this precise sense, ϕ is a local isometry.

This is the standard 'rolling without stretching' identification: the cylinder can be developed (flattened) onto the plane without distortion, but one full turn around the cylinder corresponds to shifting the plane by 2π in the y-direction, which is why the global map cannot be injective.

Proposition 5.2 (matching E,F,G gives an isometry on coordinate patches). Let $x : U \to S_1$ and $y : U \to S_2$ be coordinate functions. Let (E_1,F_1,G_1) and (E_2,F_2,G_2) be the coefficients of the first fundamental forms on S_1 and S_2 induced by x and y.

If

$$E_1(u,v) = E_2(u,v) \quad \text{and} \quad F_1(u,v) = F_2(u,v) \quad \text{and} \quad G_1(u,v) = G_2(u,v)$$

as functions on U, then

$$\Phi = y \circ x^{-1} : x(U) \to y(U)$$

is an isometry.

Definition 5.4 (intrinsic property). A property of a regular surface is intrinsic if it is invariant under (local) isometries.

Heuristically, if a quantity can be computed purely from E, F, G and their higher partial derivatives, then it is intrinsic.

5.2 Gauss' Theorema Egregium

Theorem 5.1 (Theorema Egregium). Let $\phi : S_1 \to S_2$ be a local isometry. Fix $p_1 \in S_1$ and set $p_2 = \phi(p_1) \in S_2$. Let $K(p_1)$ and $\overline{K}(p_2)$ be the Gaussian curvatures at p_1 and p_2. Then,

$$K(p_1) = \overline{K}(p_2).$$

Equivalently, Gaussian curvature is a local intrinsic quantity.

Theorem 5.1 is remarkable. Recall from Definition 4.6 that $K = k_1 k_2$ is defined using the shape operator (or second fundamental form), i.e. apparently extrinsic data in $\mathbb{R}^3$. Theorem 5.1 says that despite the definition, K is completely determined by the metric on the surface, so it cannot change under local distance-preserving deformations.

Example 5.2. For the plane, $k_1 = k_2 = 0$ so $K = 0$. For the cylinder, $k_1 = 1$ and $k_2 = 0$ so $K = 0$. By Theorem 5.1, there is no curvature obstruction to locally rolling a plane into a cylinder.

Example 5.3 (plane vs sphere). The plane has $K = 0$ while the unit sphere has $K = 1$. Hence, there cannot exist a local isometry from the plane to the sphere. Geometrically, one cannot wrap a flat sheet smoothly onto a sphere without crumpling.

To prove Theorem 5.1, it suffices to show that the Gaussian curvature K is determined by the first fundamental form. Concretely, we aim to show K is a function of E, F, G and their higher partial derivatives. Then, under a local isometry, the coefficients (E, F, G) (hence all derivatives) agree in corresponding coordinates, so K must agree as well.

Definition 5.5 (Christoffel symbols from the metric). Let $x : U \to S$ be a coordinate function, and write $p = x(u,v)$. Then, $\{x_u, x_v, N\}$ is a basis of $\mathbb{R}^3$ at p, so we may expand to obtain

$$x_{uu} = \Gamma^1_{11} x_u + \Gamma^2_{11} x_v + eN$$
$$x_{uv} = \Gamma^1_{12} x_u + \Gamma^2_{12} x_v + fN$$
$$x_{vv} = \Gamma^1_{22} x_u + \Gamma^2_{22} x_v + gN$$

The coefficients Γ^k_{ij} are the Christoffel symbols. A key point is that each Γ^k_{ij} can be written purely using E, F, G and their first derivatives

$$E_u, F_u, G_u, E_v, F_v, G_v,$$

by taking inner products of the above expansions with x_u and x_v and solving the resulting linear system. Hence,

$$\Gamma^k_{ij} = \Gamma^k_{ij}(E, F, G, E_u, F_u, G_u, E_v, F_v, G_v).$$

Proposition 5.3 (Christoffel symbols are invariant under local isometries). If $\phi : S_1 \to S_2$ is a local isometry and $x : U \to S_1$ is a coordinate function, then $y = \phi \circ x : U \to S_2$ is also a coordinate function. Moreover, the first fundamental form coefficients agree. That is, $\overline{E} = E$, $\overline{F} = F$, and $\overline{G} = G$ as functions on U, and hence their partial derivatives agree. Therefore the Christoffel symbols computed from $(\overline{E}, \overline{F}, \overline{G})$ coincide with those from (E, F, G), so

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij}.$$

One derives identities by differentiating $x_{uv} = x_{vu}$ and expanding everything in the basis $\{x_u, x_v, N\}$, and also using the Weingarten equations for N_u, N_v. After simplifying, one obtains formulas that express EK and FK (hence K) in terms of E, F, G, Γ^k_{ij} and the first derivatives of Γ^k_{ij}. Since each Γ^k_{ij} is itself a function of E, F, G and their first derivatives, it follows that

$$K = K(E, F, G, \text{ partial derivatives of } E, F, G \text{ up to some finite order}).$$

This is the intrinsicness statement needed for Theorem 5.1.

A refinement of the computation also yields the Peterson-Mainardi-Codazzi equations (relations between E, F, G and e, f, g and derivatives). Together with the Gauss equation, they form the compatibility conditions in the fundamental theorem of surfaces: given smooth functions E, F, G, e, f, g satisfying positivity and these equations, one can locally realize them as the first and second fundamental forms of some surface (unique up to rigid motions).

CHAPTER 6

Tangent Vector Fields

6.1 Tangent Vector Fields

Definition 6.1 (tangent vector fields in coordinates). Let $S \subseteq \mathbb{R}^3$ be a regular surface with a coordinate map

$$x(u,v) : U \to S.$$

At a point $p = x(u,v)$, the tangent space is

$$T_pS = \operatorname{span}\{x_u(u,v), x_v(u,v)\}.$$

A tangent vector field on S is a map $w : S \to \mathbb{R}^3$ such that

$$w(p) \in T_pS \quad \text{for all } p \in S.$$

In a coordinate patch, any tangent vector field can be written as

$$w(x(u,v)) = a(u,v)\, x_u(u,v) + b(u,v)\, x_v(u,v).$$

We call w smooth if in every coordinate patch, the coefficient functions $a(u,v)$ and $b(u,v)$ are smooth.

It is crucial that w is tangent, i.e. $w(p) \perp N(p)$ where $N(p)$ is a unit normal to S at p.

6.2 Differentiating Vector Fields

Definition 6.2 (covariant derivatives in coordinate directions). Let w be a smooth tangent vector field on S, written in a coordinate patch as

$$w(u,v) = w(x(u,v)) \in \mathbb{R}^3.$$

The ordinary partial derivatives $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ need not be tangent to S. We define the covariant derivatives by projecting back to the tangent plane:

$$\frac{Dw}{du} = \frac{\partial w}{\partial u} - \left\langle \frac{\partial w}{\partial u}, N \right\rangle N \quad \text{and} \quad \frac{Dw}{dv} = \frac{\partial w}{\partial v} - \left\langle \frac{\partial w}{\partial v}, N \right\rangle N,$$

where $N = N(u,v)$ is the unit normal along the patch.

By construction, $\frac{Dw}{du}$ and $\frac{Dw}{dv}$ are tangent vector fields.

Definition 6.3 (covariant derivative along a curve). Let $\alpha : I \to S$ be a smooth curve and let w be a tangent vector field along α, i.e. for each $t \in I$ we have $w(t) \in T_{\alpha(t)}S$. Differentiate $w(t)$ in $\mathbb{R}^3$ and project to the tangent plane:

$$\frac{Dw}{dt} = \frac{dw}{dt} - \left\langle \frac{dw}{dt}, N(\alpha(t)) \right\rangle N(\alpha(t)).$$

Definition 6.4. A vector field w along α is called parallel along α if

$$\frac{Dw}{dt} = (0,0,0) \quad \text{for all } t \in I.$$

Definition 6.5 (Christoffel symbols via tangent projections). Let $x(u,v)$ be a coordinate map with tangent basis $\{x_u, x_v\}$. The tangent components of the second derivatives are encoded by the Christoffel symbols Γ^k_{ij}:

$$\begin{aligned}
\frac{Dx_u}{du} &= \Gamma^1_{11} x_u + \Gamma^2_{11} x_v, \\
\frac{Dx_u}{dv} &= \Gamma^1_{12} x_u + \Gamma^2_{12} x_v, \\
\frac{Dx_v}{du} &= \Gamma^1_{21} x_u + \Gamma^2_{21} x_v, \\
\frac{Dx_v}{dv} &= \Gamma^1_{22} x_u + \Gamma^2_{22} x_v.
\end{aligned}$$

For regular surfaces one has $x_{uv} = x_{vu}$, hence typically $\Gamma^k_{12} = \Gamma^k_{21}$.

Proposition 6.1. If $w = a(u,v)\,x_u + b(u,v)\,x_v$, then

$$\begin{aligned}
\frac{Dw}{du} &= \left(a_u + a\Gamma^1_{11} + b\Gamma^1_{21}\right) x_u + \left(b_u + a\Gamma^2_{11} + b\Gamma^2_{21}\right) x_v \\
\frac{Dw}{dv} &= \left(a_v + a\Gamma^1_{12} + b\Gamma^1_{22}\right) x_u + \left(b_v + a\Gamma^2_{12} + b\Gamma^2_{22}\right) x_v
\end{aligned}$$

6.3 Geodesic Curvature

Definition 6.6 (geodesic curvature and normal curvature). Let $\alpha(s)$ be a curve on S parametrised by arc length. Fix $p = \alpha(s_0)$. Let $t = \alpha'(s_0)$, $N = N(p)$, and $s = N \wedge t$. Then, $\{t, s, N\}$ is an orthonormal basis of $\mathbb{R}^3$ at p, and we decompose

$$\alpha''(s_0) = \langle N, \alpha''(s_0)\rangle N + \langle s, \alpha''(s_0)\rangle s.$$

Define

$$k_n = \langle N, \alpha''(s_0)\rangle \quad \text{and} \quad k_g = \langle s, \alpha''(s_0)\rangle.$$

Here k_n is the normal curvature and k_g is the geodesic curvature. Since $N \perp s$, Pythagoras' theorem gives

$$k^2 = k_n^2 + k_g^2,$$

where $k = |\alpha''(s_0)|$ is the usual curvature of α in $\mathbb{R}^3$.

We say that a curve $\alpha(s)$ on S parametrised by arc length is called a geodesic if

$$\frac{D\alpha'}{ds} = (0,0,0) \quad \text{for all } s.$$

Proposition 6.2 (equivalent characterizations of geodesics). Let $\alpha(s)$ be a curve on S parametrised by arc length. The following are equivalent:

- **(i)** $\frac{D\alpha'}{ds} = (0,0,0)$ for all s
- **(ii)** $\alpha''(s)$ is perpendicular to $T_{\alpha(s)}S$ for all s
- **(iii)** $\alpha''(s)$ is parallel to the unit normal $N(\alpha(s))$ for all s
- **(iv)** The geodesic curvature satisfies $k_g(s) = 0$ for all s

If $\alpha(s)$ is a geodesic and $c \neq 0$, then $\beta(t) = \alpha(ct)$ has constant speed and is also treated as a geodesic in some texts.

Proposition 6.3 (isometries preserve geodesics). Let $\varphi : S_1 \to S_2$ be an isometry. If α is a geodesic on S_1, then $\beta = \varphi \circ \alpha$ is a geodesic on S_2.

Proposition 6.4 (existence and uniqueness). Let $p \in S$ and let $v \in T_pS$ be a unit tangent vector. Then there exists a unique geodesic $\alpha(s)$ parametrised by arc length such that

$$\alpha(0) = p \quad \text{and} \quad \alpha'(0) = v.$$

Example 6.1. On the unit sphere, the geodesics are exactly the arcs of great circles.

Proposition 6.5 (geodesics are locally shortest paths). Given two points $p, q \in S$ that are sufficiently near, there exists a geodesic joining p to q, and among all nearby curves joining p to q, this geodesic has minimal arc length.

Let $x : U \to S$ be a coordinate map and write a curve as

$$\alpha(t) = x(a(t), b(t)).$$

Using $\alpha''(t) \perp \operatorname{span}\{x_u, x_v\}$ and expressing the tangent components via the Christoffel symbols, one obtains the geodesic equations

$$a''(t) + \Gamma^1_{11}(a(t), b(t))\left(a'(t)\right)^2 + 2\Gamma^1_{12}(a(t), b(t))\, a'(t)\, b'(t) + \Gamma^1_{22}(a(t), b(t))\left(b'(t)\right)^2 = 0$$
$$b''(t) + \Gamma^2_{11}(a(t), b(t))\left(a'(t)\right)^2 + 2\Gamma^2_{12}(a(t), b(t))\, a'(t)\, b'(t) + \Gamma^2_{22}(a(t), b(t))\left(b'(t)\right)^2 = 0$$

Given initial conditions $\alpha(0) = p$ and $\alpha'(0) = v$, ODE theory yields a unique local solution, hence a unique geodesic with the prescribed initial data.

Chapter 7

The Gauss-Bonnet Theorem

7.1 The Local Gauss-Bonnet Theorem

Let S be an orientable regular surface, with a smooth unit normal field N (equivalently, a Gauss map exists). Then, the Gaussian curvature $K(p)$ is defined for each $p \in S$ and gives a function $K : S \to \mathbb{R}$. Let $R \subseteq S$ be a simple region bounded by three smooth curves C_1, C_2, C_3, each parametrised by arc length, and oriented so that the boundary direction and N obey the right hand rule. Along each boundary curve C_i, we have the geodesic curvature k_g, hence a function $k_g : C_i \to \mathbb{R}$. At the three vertices, record the interior angles $\lambda_1, \lambda_2, \lambda_3$.

Theorem 7.1 (local Gauss-Bonnet theorem). With the above setup,

$$\sum_{i=1}^{3} \int_{C_i} k_g \, ds + \iint_R K \, dS = 2\pi - \sum_{i=1}^{3} (\pi - \lambda_i) \,. \tag{7.1}$$

The same identity (7.1) holds when ∂R is a union of n smooth curves surrounding the region. Also, requiring arc length parametrisation is not essential since $\int k_g \, ds$ is a line integral and is independent of parametrisation.

Example 7.1 (Euclidean plane). If $S = \mathbb{R}^2$ (the xy-plane), then $K = 0$. If the boundary curves are geodesics (straight lines), then $k_g = 0$. By the local Gauss-Bonnet theorem (Theorem 7.1), we have

$$0 = 2\pi - \sum_{i=1}^{3} (\pi - \lambda_i) \quad \text{so} \quad \lambda_1 + \lambda_2 + \lambda_3 = \pi.$$

Example 7.2 (sphere). Let S be a sphere of radius r, so $K = r^{-2}$. If the boundary curves are arcs of great circles, then $k_g = 0$. Thus,

$$\iint_R K \, dS = 2\pi - \sum_{i=1}^{3} (\pi - \lambda_i) \quad \text{so} \quad \lambda_1 + \lambda_2 + \lambda_3 = \pi + \frac{\text{Area}\,(R)}{r^2}.$$

7.2 The Global Gauss-Bonnet Theorem

A triangulation of a nice region $R \subset S$ divides R into finitely many triangles. Let F denote the number of faces (triangles), E denote the number of edges, and V denote the number of vertices. The Euler-Poincaré characteristic of the triangulation is

$$\chi(R) = F - E + V.$$

It is a fact that $\chi(R)$ does not depend on the chosen triangulation. We now state the global Gauss-Bonnet theorem with boundary (Theorem 7.2).

Theorem 7.2 (global Gauss-Bonnet Theorem with boundary). Let R be a nice region in an oriented surface S. Suppose the boundary ∂R is a union of nice curves $C_1, \ldots, C_n$, each positively oriented (with respect to the chosen normal field). Let $\lambda_1, \ldots, \lambda_n$ be the internal angles at the corners of ∂R. Then,

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \iint_R K\, dS + \sum_{i=1}^{n} (\pi - \lambda_i) = 2\pi\chi(R).$$

We give a rough proof sketch of Theorem 7.2. First, triangulate R into sufficiently small triangles so that each triangle lies in a coordinate patch. Then, apply the local Gauss-Bonnet theorem (Theorem 7.1) to each triangle and sum over all triangles. The boundary terms telescope on interior edges, leaving only the contribution from ∂R, and the angle bookkeeping produces the Euler characteristic term.

Definition 7.1 (genus). For a bounded surface without boundary (i.e. a compact surface without boundary), the topology is determined by the number of holes/handles, called the genus, denoted by $\operatorname{genus}(S)$. For such surfaces,

$$\chi(S) = 2 - 2\operatorname{genus}(S).$$

We now state the Gauss-Bonnet theorem for compact surfaces without boundary.

Theorem 7.3 (Gauss-Bonnet theorem for compact surfaces without boundary). Let S be a compact regular surface (without boundary). Then,

$$\iint_S K\, dS = 2\pi\chi(S).$$

Theorem 7.3 is a prototypical bridge between a local differential-geometric quantity (K) and a global topological invariant (χ or genus).

Example 7.3 (ellipsoid). An ellipsoid has genus 0, hence $\chi(S) = 2$ by Definition 7.1. By Theorem 7.3, we have

$$\iint_S K\, dS = 2\pi\chi(S) = 4\pi.$$

In particular, the value 4π depends only on the topology, not on the specific shape (round sphere vs. rugby ball).

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