MA4271 Differential Geometry of Curves and Surfaces

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These notes are based off **Prof. Loke Hung Yean's** MA3215 Three-Dimensional Differential Geometry materials. Additional references are cited in the bibliography. Note that the old course code is MA3215 and it has been upgraded to the current MA4271 Differential Geometry of Curves and Surfaces.

This set of notes was last updated on February 11, 2026. If you would like to contribute a nice discussion to the notes or point out a typo, please send me an email at thangpangern@u.nus.edu.

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Chapter 1

Smooth Functions

1.1 Open Subsets of $\mathbb{R}$

Recall from MA2108 Mathematical Analysis I that an open interval in $\mathbb{R}$ means any set of the form (a,b) or (a,∞) or $(-\infty,b)$ or $\mathbb{R}$. A subset $U \subseteq \mathbb{R}$ is open if it can be written as a (possibly infinite) union of open intervals. In practice, one can treat 'U open' as the hypothesis that every point of U sits inside some small interval still contained in U. This is the natural domain condition for differentiation.

Definition 1.1 (smoothness on an open set). Let $U \subseteq \mathbb{R}$ be open and let $f: U \to \mathbb{R}$. We say that f is smooth (or $\mathcal{C}^\infty$) on U if for every $n \in \mathbb{Z}_{\geq 0}$ and every $t \in U$, the n^{th} derivative

$$f^{(n)}(t) = \frac{d^n}{dt^n} f(t) \quad \text{exists.}$$

Proposition 1.1. Let $U \subseteq \mathbb{R}$ be open and let $f: U \to \mathbb{R}$. If f is smooth on U, then f is continuous on U.

Proof. If f is smooth, then in particular the first derivative exists for all $t \in U$. Differentiability implies continuity at each point of U, hence f is continuous on U. □

Recall that a smooth function need not equal to its Taylor series. That is to say, even if all derivatives exist, the Taylor series built from these derivatives may fail to represent the original function near the expansion point. Equivalently, $\mathcal{C}^\infty$ does not imply analytic. Take for example

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Then, f is smooth on $\mathbb{R}$ and $f^{(n)}(0) = 0$ for all $n \geq 0$. Hence, the Taylor series of f at 0 is identically 0, but $f(x) > 0$ for $x \neq 0$. So f does not have a Taylor series expansion at 0 (in the sense of being equal to its Taylor series in a neighbourhood of 0). Well, to make it more rigorous, for $x \neq 0$, repeated differentiation yields

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-1/x^2},$$

where P_n is a polynomial (depending on n). As $x \to 0$, the exponential decay of e^{-1/x^2} dominates any polynomial growth in $1/x$, so

$$\lim_{x \to 0} f^{(n)}(x) = 0.$$

Thus, defining $f^{(n)}(0) = 0$ makes each derivative continuous at 0, and inductively shows $f \in C^\infty(\mathbb{R})$ with all derivatives at 0 equal to 0.

In this course, as we would see eventually, *charts* and *transition maps* are required to be smooth. It is crucial to distinguish *smooth* from *analytic*: many geometric constructions live naturally in the C^∞ category without being representable by convergent power series.

1.2 Open Subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$

When we differentiate $f : U \to \mathbb{R}$ at a point $p \in U$, we take small perturbations of the input and compare values of f. Thus, we want the inputs to be able to wander freely in a small vicinity of p while staying inside U. This is precisely what 'U is open' formalises.

Recall from MA2108 Mathematical Analysis I or even MA3209 Metric and Topological Spaces the notion of open balls and open subsets of $\mathbb{R}^n$. We shall briefly state these definitions. Let $p \in \mathbb{R}^n$ and $\varepsilon > 0$. The open ball of radius ε centred at p is

$$B(p,\varepsilon) = \{q \in \mathbb{R}^n : \|q - p\| < \varepsilon\}.$$

Next, a subset $U \subseteq \mathbb{R}^n$ is open if for every $p \in U$ there exists $\varepsilon > 0$ such that $B(p,\varepsilon) \subseteq U$. By definition, every open ball $B(p,\varepsilon)$ is an open subset of $\mathbb{R}^n$. This notion is the Euclidean special case of the general definition of openness in metric/topological spaces.

Let $p \in \mathbb{R}^n$. A subset $N \subseteq \mathbb{R}^n$ is an open neighbourhood of p if $p \in N$ and N is open in $\mathbb{R}^n$. In particular, $B(p,\varepsilon)$ is an open neighbourhood of p for every $\varepsilon > 0$.

Example 1.1. We state some basic examples and non-examples.

(i) A unit square in $\mathbb{R}^2$ without its boundary is open; with its boundary it is not open

(ii) A unit cube in $\mathbb{R}^3$ without its boundary is open; with its boundary it is not open

(iii) The plane $\mathbb{R}^2$ is an open subset of itself, but the xy-plane $\{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ is not open in $\mathbb{R}^3$

(iv) The upper half-space $\{(x,y,z) \in \mathbb{R}^3 : z > 0\}$ is open in $\mathbb{R}^3$, while $\{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$ is not open in $\mathbb{R}^3$

Lastly, a subset $C \subseteq \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus C$ is open in $\mathbb{R}^n$. Intuitively, a closed set contains its boundary points. For instance, an open ball is not closed because it excludes its boundary sphere.

1.3 Smooth Multivariable Functions

For $f:(a,b)\to\mathbb{R}$, the existence of derivatives of all orders forces those derivatives to be continuous. In several variables, one uses partial derivatives, but existence of higher partial derivatives alone does not guarantee continuity of those derivatives, so continuity must be built into the definition.

Definition 1.2 (higher-order partial derivatives). Let $V \subseteq \mathbb{R}^n$ be open, and let $f : V \to \mathbb{R}$. Write $f(x_1,\dots,x_n)$. Let $\alpha_1,\dots,\alpha_n \in \mathbb{Z}_{\geq 0}$ and let

$$N = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

If the partial derivative

$$\frac{\partial^N f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x_1,\dots,x_n)$$

exists for all $(x_1,\dots,x_n) \in V$, then it is called a partial derivative of order N.

Definition 1.3 ($\mathcal{C}^N(V)$ and $\mathcal{C}^\infty(V)$). Let $V \subseteq \mathbb{R}^n$ be open. For $N \in \mathbb{N}$, denote by $\mathcal{C}^N(V)$ the set of continuous functions $f : V \to \mathbb{R}$ such that all partial derivatives of order not greater than N exist and are continuous on V. A function in $\mathcal{C}^N(V)$ is called differentiable of order N on V.

Define

$$\mathcal{C}^\infty(V) = \bigcap_{N=0}^{\infty} \mathcal{C}^N(V).$$

A function $f \in \mathcal{C}^\infty(V)$ is called a smooth function (in the sense of Calculus) on V.

Definition 1.4 (equivalent smoothness criterion). A function $f : V \to \mathbb{R}$ is smooth (in the sense of Calculus) if the following hold:

(i) f is continuous

(ii) all partial derivatives of any order exist

(iii) all higher derivatives are continuous functions on V

The continuity requirements in Definition 1.4 cannot be removed, otherwise one admits undesirable functions with many existing partial derivatives but poor regularity. For example, let $V = \mathbb{R}^2$ and define

$$f : \mathbb{R}^2 \to \mathbb{R} \quad \text{where} \quad f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then, f is not continuous[1] at $(0,0)$ even though certain partial derivatives at $(0,0)$ exist.

Definition 1.5 (smoothness at a point). Let $V \subseteq \mathbb{R}^n$ be open and $p \in V$. We say that $f : V \to \mathbb{R}$ is smooth at p if there exists an open subset $V' \subseteq V$ with $p \in V'$ such that the restriction

$$f|_{V'} : V' \to \mathbb{R}$$

is smooth. We call such a V' an open neighbourhood of p.

The pointwise notion 'smooth at p' is typically less useful than smoothness on an open domain; in Differential Geometry we usually specify a domain and ask for smoothness throughout it.

Lemma 1.1 (algebra of smooth functions). Let $V \subseteq \mathbb{R}^n$ be open and let $f, g : V \to \mathbb{R}$ be smooth. Then, the following functions are smooth on V:

$$f + g \quad \text{and} \quad f - g \quad \text{and} \quad fg \quad \text{and} \quad f^n \text{ for } n \in \mathbb{N}.$$

Moreover, the following hold:

(i) If $g(v) \neq 0$ for all $v \in V$, then $\frac{1}{g}$ is smooth on V.

(ii) If $g(v) > 0$ for all $v \in V$, then $\sqrt{g}$ is smooth on V.

We give a rough sketch of the proof.

Proof. Use the product rule to show closure under fg, and the quotient rule to show smoothness of $1/g$ when g is nowhere zero. The remaining statements follow by repeated applications of the chain rule. □

Example 1.2 (smoothness away from the singular point). Let $V' = \mathbb{R}^2 \setminus \{(0,0)\}$. Then, V' is open in $\mathbb{R}^2$. Define $g(x,y) = x^2 + y^2$. On V', we have $g(x,y) > 0$, hence $\frac{1}{g}$ is smooth on V' by Lemma 1.1. As such,

$$(x,y) \mapsto \frac{xy}{x^2+y^2} = xy \cdot \frac{1}{g(x,y)}$$

is a smooth function on V'. In particular, if $p \in \mathbb{R}^2$ and $p \neq (0,0)$, then the original f is smooth at p.

[1] One can use the two path test taught in MA2104 Multivariable Calculus to show that f is not continuous at $(0,0)$ by approaching $(0,0)$ along two different paths.

Definition 1.6 (smooth maps into $\mathbb{R}^m$). Let $V \subseteq \mathbb{R}^n$ be open and let $f : V \to \mathbb{R}^m$ be given by

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

We say that f is smooth (in the sense of Calculus) if each component function $f_i : V \to \mathbb{R}$ is smooth for $i = 1, \ldots, m$.

Here is a course convention — unless otherwise stated, functions of the form $f(x)$, $g(x,y)$, $h(x,y,z)$, etc. are assumed to be smooth. Also in this course, parametrisations, coordinate charts, and transition maps are required to be smooth maps between open subsets of Euclidean spaces. The definitions above are the Euclidean baseline that will be transplanted to smooth functions on a surface via charts.

CHAPTER 2

Curves

2.1 Curves in $\mathbb{R}^3$

In this chapter, we begin the study of curvature of a curve; much of the basic material overlaps with MA2104 Multivariable Calculus.

Definition 2.1 (parametrised smooth curve). A parametrised smooth curve in $\mathbb{R}^3$ is a map

$$\alpha : (a,b) \to \mathbb{R}^3 \quad \text{where} \quad \alpha(t) = (x(t), y(t), z(t)),$$

where x, y, z are smooth functions on (a,b). That is to say, they are differentiable as many times as we want.

Definition 2.2 (tangent/velocity vector and speed). The tangent vector (also called the velocity vector) of α at t is $\alpha'(t)$. The speed is $|\alpha'(t)|$.

Definition 2.3 (tangent line and acceleration). If $\alpha'(t) \neq (0,0,0)$, then the line through $\alpha(t)$ and parallel to $\alpha'(t)$ is the tangent line at t. The second derivative $\alpha''(t)$ is the acceleration.

Example 2.1 (helix). Let $a, b > 0$ and define

$$\alpha : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \alpha(t) = (a\cos t, a\sin t, bt).$$

Then,

$$\alpha'(t) = (-a\sin t, a\cos t, b) \quad \text{and} \quad \alpha''(t) = (-a\cos t, -a\sin t, 0),$$

and the speed is constant because $|\alpha'(t)| = \sqrt{a^2 + b^2}$.

Example 2.2 (a smooth curve with zero velocity at an instant). Consider

$$\alpha : \mathbb{R} \to \mathbb{R}^2 \quad \text{where} \quad \alpha(t) = \left(t^3,\ t^2\right).$$

Then, $\alpha'(t) = \left(3t^2,\ 2t\right)$, $\alpha''(t) = (6t,\ 2)$, and $\alpha'(t) = |t|\sqrt{9t^2+4}$. In particular, $\alpha'(0) = (0,0)$, so the velocity vanishes at $t = 0$ even though α is smooth.

Example 2.3 (MA3215 AY14/15 Sem 2 Tutorial 1). Find the velocity, speed and acceleration for the curve $\alpha(t) = (\cosh t, \sinh t, t)$.

Solution. We have $\alpha'(t) = (\sinh t, \cosh t, 1)$ and $\alpha''(t) = (\cosh t, \sinh t, 0)$. Then, use Definition 2.2 to compute the velocity and speed and Definition 2.3 to compute the acceleration. □

Definition 2.4 (non-singular curve). A smooth curve α is non-singular if $\alpha'(t) \neq \mathbf{0}$.

Example 2.4 (MA3215 AY14/15 Sem 2 Tutorial 1). Let $\alpha(t) = (x(t), y(t), z(t))$ be a non-singular smooth curve, where x, y, z are smooth functions. Let

$$s(t) = \int_0^t |\alpha'(\tau)| \, d\tau.$$

(i) Compute $s'(t)$.

(ii) Compute $s''(t)$ in terms of x, y, z and its higher derivatives.

(iii) Show that $s(t)$ is a smooth function of t. That is to say (by Definition 2.1), all its higher derivatives $s^{(n)}(t)$ exist.

Solution.

(i) By the Fundamental Theorem of Calculus, we have $s'(t) = |\alpha'(t)|$. Equivalently,

$$s'(t) = \sqrt{(x')^2 + (y')^2 + (z')^2}.$$

(ii) By the chain rule,

$$s''(t) = \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}}.$$

(iii) Since x, y, z are smooth functions, then so are their derivatives. Let $g(t)$ denote the sum of squares of the derivatives. Since α is non-singular, then by Definition 2.4, $g(t) > 0$. From **(i)** and **(ii)**, we have seen that the derivatives s' and s'' exist. It follows that all higher derivatives $s^{(n)}(t)$ exist. □

Example 2.5 (self-intersection). Let

$$\alpha : \mathbb{R} \to \mathbb{R}^2 \quad \text{where} \quad \alpha(t) = \left(t^3 - 4t, \, t^2 - 4\right).$$

Then, $\alpha(2) = \alpha(-2) = (0, 0)$ so the curve may intersect itself.

In this course, we are mainly concerned with the shape of the curve rather than how fast the point moves along it; hence the same geometric curve may admit many different parametrisations. Take for example the parametrisations $\alpha(t) = (\cos t, \sin t)$, $\beta(t) = (\cos(2t), \sin(2t))$, and $\gamma(t) = (\cos(t-2), \sin(t-2))$. All three trace the unit circle, but with different *timings* — β moves with twice the speed of α, while γ is a time-shift of α.

Having said that, a natural choice is to reparametrise so that the point moves with constant unit speed, i.e. $|\alpha'(t)| = 1$ for all t; in particular, such a parametrisation is automatically non-singular. We will return to this in Chapter 2.2. The standing assumption from now on is that we will only consider non-singular smooth curves.

2.2 Arc Length

Definition 2.5 (arc length). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a (non-singular) smooth curve. The arc length of α from $t = a$ to $t = b$ is

$$\int_a^b |\alpha'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

Fix a reference time (often 0). Define

$$s(t_0) = \int_0^{t_0} |\alpha'(t)|\, dt.$$

Then, the arc length from $t = a$ to $t = b$ is

$$\int_a^b |\alpha'(t)|\ dt = s(b) - s(a).$$

Example 2.6 (semicircle of radius r). Let $r > 0$ and

$$\gamma : (0,\pi) \to \mathbb{R}^2 \quad \text{where} \quad \gamma(t) = (r\cos t,\ r\sin t).$$

Then, $\gamma'(t) = (-r\sin t,\ r\cos t)$ so $|\gamma'| = r$, so the arc length from $t = 0$ to $t = \pi$ is

$$\int_0^\pi |\gamma'(t)|\ dt = \int_0^\pi r\, dt = r\pi.$$

Example 2.7 (MA3215 AY14/15 Sem 2 Tutorial 1). Parametrise the curve in Example 2.3 by arc length so that when $s = 0$, we have

$$\alpha(0) = \left(\frac{e+e^{-1}}{2}, \frac{e-e^{-1}}{2}, 1\right).$$

Note that there is no typo in the question.

Solution. Note that the arc length is

$$s(t) = \int_1^t \sqrt{\sinh^2 u + \cosh^2 u + 1}\, du = \sqrt{2}(\sinh t - \sinh 1).$$

So,

$$t(s) = \operatorname{arsinh}\left(\frac{s}{\sqrt{2}} + \sinh 1\right)$$

and the result follows using this parametrisation. $\square$

Example 2.8 (MA3215 AY14/15 Sem 2 Tutorial 1). Consider the helix

$$\alpha : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \alpha(t) = (3\cos t, 3\sin t, 4t).$$

Parametrise α by arc length so that when $s = 0$, the curve α passes through the point $(3,0,0)$. If you are interested in how a helix looks like, refer to Figure 2.1.

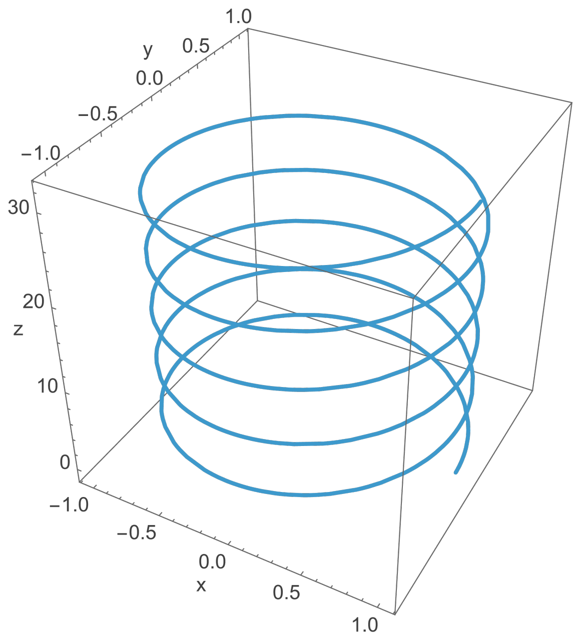

Figure 2.1: Helix

Solution. The arc length is

$$\int_0^t \sqrt{9\sin^2\tau + 9\cos^2\tau + 16}\, d\tau = 5t.$$

So, $s(t) = 5t$, which implies that $t(s) = \frac{s}{5}$. We conclude that the parametrisation is $\alpha(s) = \left(3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4s}{5}\right)$. $\square$

> Definition 2.6. Let $\alpha : (a,b) \to \mathbb{R}^3$ (or $\mathbb{R}^2$) be a curve. If $|\alpha'(t)| = 1$ for all t, then we say that α is parametrised by arc length (i.e. unit speed). In this case, the arc length from $t = a$ to $t = b$ is simply $b - a$. Moreover, we often switch notation and write the parameter as s instead of t.

Example 2.9. For the semicircle in Example 2.6, we have

$$s(t) = \int_0^t |\gamma'(u)|\, du = \int_0^t r\, du = rt,$$

so $t = \frac{s}{r}$. As such, define the new parametrisation

$$\alpha(s) = \gamma\left(\frac{s}{r}\right) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\right).$$

Then, $|\alpha'(s)| = 1$, so α is unit speed and traces the same semicircle.

Say we are given a non-singular smooth curve. Can we change variable from t to s so that the curve becomes parametrised by arc length? The answer is yes. Here is the general strategy.

(i) Define the distance-travelled function

$$s(t) = \int_{t_0}^{t} \left|\alpha'(u)\right| du.$$

(ii) Find the inverse function $t = t(s)$.

(iii) Define the reparametrised curve

$$\beta(s) = \alpha(t(s))$$

which will be parametrised by arc length.

Recall by the Fundamental Theorem of Calculus from MA2002 Calculus that

$$\frac{ds}{dt} = \left|\alpha'(t)\right|.$$

Since α is non-singular, then $|\alpha'(t)| > 0$, hence $s(t)$ is strictly increasing, and so it admits an inverse $t(s)$ (obtained by flipping the t- and s-axes).

Theorem 2.1 (inverse function theorem). Suppose s is continuous, smooth, and $s'(t) > 0$ for all t in an interval. Then, s is strictly increasing, the inverse $t = t(s)$ exists, and the inverse is smooth.

Theorem 2.2 (existence of arc-length parametrisation). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a non-singular smooth curve. Define

$$s(t) = \int_{a}^{t} \left|\alpha'(u)\right| du \quad \text{where } c = s(a) \text{ and } d = s(b).$$

Then, $s(t)$ is strictly increasing and has a smooth inverse $t = t(s)$ on (c,d). If we set $\beta(s) = \alpha(t(s))$, then β is parametrised by arc length, i.e. $|\beta'(s)| = 1$ for all $s \in (c,d)$.

Proof. Using the chain rule, we have $\beta'(s) = \alpha'(t(s)) \cdot t'(s)$. Since $\frac{ds}{dt} = |\alpha'(t)|$, we have

$$t'(s) = \frac{1}{|\alpha'(t(s))|}.$$

As such,

$$\left|\beta'(s)\right| = \left|\alpha'(t(s))\right| \cdot \left|t'(s)\right| = \left|\alpha'(t(s))\right| \cdot \frac{1}{|\alpha'(t(s))|} = 1.$$

□

Example 2.10 (MA3215 AY14/15 Sem 2 Tutorial 2). Let $\alpha(s)$ be a curve parametrised by arc length. Suppose $s = s(t)$ is a function of time t. Prove the chain rule for curves. That is,

$$\frac{d}{dt}\alpha(s(t)) = \left(\frac{d}{ds}\alpha(s)\right)\frac{ds}{dt}.$$

Proof. Let I and J be intervals. Suppose $\alpha : I \to \mathbb{R}^n$ be a curve parametrised by arc length. Thus, α is differentiable as a map of the real variable s by Definition 2.5, and $\alpha'(s)$ exists. Let $s : J \to I$ be a differentiable function of time t, and define the composed curve $\beta(t) = \alpha(s(t))$, where $t \in J$. It suffices to prove that

$$\beta'(t) = \alpha'(s(t))\, s'(t).$$

Take note of the following commutative diagram:

$$\begin{tikzcd} J \subseteq \mathbb{R} \arrow[r, "s"] \arrow[dr, "\beta"'] & I \subseteq \mathbb{R} \arrow[d, "\alpha"] \\ & \mathbb{R}^n \end{tikzcd} \qquad \text{where } \beta = \alpha \circ s$$

Fix $t \in J$. For $h \neq 0$ sufficiently small so that $t + h \in J$, we have

$$\frac{\beta(t+h) - \beta(t)}{h} = \frac{\alpha(s(t+h)) - \alpha(s(t))}{h}.$$

We then insert and remove the factor $s(t+h) - s(t)$ so

$$\frac{\alpha(s(t+h)) - \alpha(s(t))}{h} = \frac{\alpha(s(t+h)) - \alpha(s(t))}{s(t+h) - s(t)} \cdot \frac{s(t+h) - s(t))}{h}.$$

This holds whenever $s(t+h) \neq s(t)$. If equality holds for some h, then the left-hand difference quotient is 0 for that h, and the following argument still goes through by taking limits along $h \to 0$.

We take the limit $h \to 0$. Since s is differentiable at t, then $s(t+h) \to s(t)$. Since α is differentiable at $s(t)$, then

$$\lim_{u \to s(t)} \frac{\alpha(u) - \alpha(s(t))}{u - s(t)} = \alpha'(s(t)).$$

Apply this with $u = s(t+h)$ and the result follows. □

Example 2.11 (MA3215 AY14/15 Sem 2 Tutorial 2). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a curve. Show that

$$\int_a^b \alpha'(t)\, dt = \alpha(b) - \alpha(a).$$

Solution. Apply the Fundamental Theorem of Calculus to each component and the result follows. □

Example 2.12 (MA3215 AY14/15 Sem 2 Tutorial 2). The goal of this exercise is to show that the curve of shortest length from one point to another is the straight line joining these points Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, and let $\alpha : [a,b] \to \mathbb{R}^3$ be a smooth curve such that $\alpha(a) = \mathbf{p}$ and $\alpha(b) = \mathbf{q}$.

(i) Show that for any constant vector $\mathbf{v}$ such that $|\mathbf{v}| = 1$ satisfies

$$(\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} = \int_a^b \alpha'(t) \cdot \mathbf{v}\, dt \leq \int_a^b |\alpha'(t)|\, dt.$$

(ii) Set

$$\mathbf{v} = \frac{\mathbf{q}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|}$$

and show that

$$|\alpha(b) - \alpha(a)| = \left| \int_a^b \alpha'(t)\ dt \right| \leq \int_a^b |\alpha'(t)|\ dt.$$

That is to say, the curve of shortest length from $\mathbf{p}$ to $\mathbf{q}$ is the straight line joining these points.

Solution.

(i) Suppose $\mathbf{v}$ is a constant vector such that $|\mathbf{v}| = 1$. Then,

$$\int_a^b \alpha'(t) \cdot \mathbf{v}\, dt = \int_a^b \alpha'(t)\ dt \cdot \mathbf{v} = (\mathbf{q}-\mathbf{p}) \cdot \mathbf{v},$$

thus proving the equality statement. To prove that the inequality holds, use the fact that $\alpha'(t) \cdot \mathbf{v} = |\alpha'(t)| \cos\theta$, where θ is the subtended angle between $\alpha'(t)$ and $\mathbf{v}$. Then, use the fact that $\cos\theta \leq 1$ and $|\alpha'(t)| \geq 0$.

(ii) We have

$$(\mathbf{q}-\mathbf{p}) \cdot \frac{\mathbf{q}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|} = |\mathbf{q}-\mathbf{p}| = |\alpha(b) - \alpha(a)|$$

and the result follows by using **(i)**. □

2.3 Orientation

In $\mathbb{R}^3$, we write the vector product (cross product) as $\mathbf{v} \wedge \mathbf{w} = \mathbf{v} \times \mathbf{w}$.

Theorem 2.3 (product rules in $\mathbb{R}^3$). Let $\mathbf{w}(t), \mathbf{w}(t)$ be differentiable vector-valued functions in $\mathbb{R}^3$. Then, the following hold

$$\frac{d}{dt}(\mathbf{v}(t) \cdot \mathbf{w}(t)) = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t)$$

and

$$\frac{d}{dt}(\mathbf{v}(t) \wedge \mathbf{w}(t)) = \mathbf{v}'(t) \wedge \mathbf{w}(t) + \mathbf{v}(t) \wedge \mathbf{w}'(t).$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis for $\mathbb{R}^3$.

(i) We say $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has positive orientation if $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (right-hand rule)

(ii) We say $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ has negative orientation if $\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_3$ (left-hand rule)

A coordinate system in $\mathbb{R}^3$ is a choice of mutually perpendicular x-axis, y-axis and z-axis. There are two types: positively oriented and negatively oriented.

Theorem 2.4. We state the effects of rotations, translations, and reflections on orientation.

(i) **Rotations preserve orientation:** if we rotate the axes, the axes change but the orientation remains positive (respectively negative)

(ii) **Translations preserve orientation:** translating the axes by a vector does not change orientation

(iii) **Reflections reverse orientation:** reflection sends a positively oriented coordinate system to a negatively oriented one (right hand becomes left hand)

2.4 Curvature

How do we tell that one curve is more *curvy* than the other? Let us discuss curvature. Let $\alpha(s)$ be a curve parametrised by arc length, and write the unit tangent vector as $\mathbf{t}(s) = \alpha'(s)$. To measure how much the curve bends, we compare $\mathbf{t}(s)$ and $\mathbf{t}(s+\Delta s)$ and consider the rate of change

$$\frac{\mathbf{t}(s+\Delta s) - \mathbf{t}(s)}{\Delta s}.$$

Taking $\Delta s \to 0$ gives

$$\frac{d\mathbf{t}(s)}{ds} = \frac{d}{ds}\left(\frac{d\alpha(s)}{ds}\right) = \alpha''(s).$$

Hence, curvature is governed by the acceleration $\alpha''(s)$. We give some physical intuition. If you travel at constant speed and the train goes around a bend, you feel a centripetal force; a sharper bend gives larger acceleration, hence larger curvature. Note that if the curve is not travelling at constant speed, i.e. not parametrised by arc length, then using acceleration to gauge the bend can be misleading.

Definition 2.7 (curvature and radius of curvature). Let $\alpha : (a,b) \to \mathbb{R}^3$ be a curve parametrised by arc length. The curvature at s and the radius of curvature at s are

$$k(s) = |\alpha''(s)| \text{ and } \rho(s) = \frac{1}{k(s)} \quad \text{respectively.}$$

Example 2.13 (straight line has zero curvature). Let

$$\alpha(s) = (a_1, a_2, a_3) + s(t_1, t_2, t_3)$$

be a straight line parametrised by arc length. Then $\alpha'(s) = (t_1, t_2, t_3)$ is a unit vector and $\alpha''(s) = (0,0,0)$ so $k(s) = 0$. Thus, a straight line has zero curvature.

Example 2.14 (circle of radius r has curvature $1/r$). Let $\alpha(s)$ be a circle of radius r parametrised by arc length (for instance $\alpha(s) = (r\cos(s/r), r\sin(s/r))$). Then, one computes

$$k(s) = |\alpha''(s)| = \frac{1}{r} \quad \text{so} \quad \rho(s) = r.$$

Hence, the radius of curvature of a circle of radius r is r.

Lemma 2.1 (orthogonality lemma). Let $\alpha(s)$ be parametrised by arc length. Then,

$$\alpha''(s) \cdot \alpha'(s) = 0,$$

i.e. the acceleration is perpendicular to the unit tangent vector.

Proof. Since $\alpha(s)$ is parametrised by arc length, we have $|\alpha'(s)| = 1$, so

$$1 = |\alpha'(s)|^2 = \alpha'(s) \cdot \alpha'(s).$$

Differentiating both sides with respect to s, we have

$$0 = \frac{d}{ds}\left(\alpha'(s) \cdot \alpha'(s)\right) = \alpha''(s) \cdot \alpha'(s) + \alpha'(s) \cdot \alpha''(s) = 2\alpha'(s) \cdot \alpha''(s).$$

Hence, $\alpha''(s) \cdot \alpha'(s) = 0$. □

More generally, if $\mathbf{v}(t)$ is a unit vector for all t, then $\mathbf{v}'(t)$ is perpendicular to $\mathbf{v}(t)$.

Definition 2.8 (singular point of order 1). Suppose for some s_0, we have $\alpha''(s_0) = (0,0,0)$. Then, s_0 is called a singular point of order 1.

We will only consider curves such that $\alpha''(s) \neq (0,0,0)$ for all s.

2.5 The Frenet Trihedron

For the rest of this chapter, we only consider curves $\alpha(s)$ parametrised by arc length such that $\alpha''(s) \neq (0,0,0)$ for all s.

Definition 2.9 (unit tangent vector). If $\alpha(s)$ is parametrised by arc length, we define the unit tangent vector $\mathbf{t}(s) = \alpha'(s)$.

Definition 2.10 (normal vector). Since $\alpha''(s) \neq 0$, the unit vector in the direction of $\alpha''(s)$ is well-defined. We call it the normal vector. That is,

$$\mathbf{n}(s) = \frac{\alpha''(s)}{|\alpha''(s)|}.$$

Equivalently, $\alpha''(s) = k(s)\,\mathbf{n}(s)$ and $k(s) = |\alpha''(s)|$.

Definition 2.11 (osculating plane). The plane spanned by $\mathbf{t}(s)$ and $\mathbf{n}(s)$ is called the osculating plane at s (Figure 2.2).

Definition 2.12 (binormal vector). Define the binormal vector by

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s).$$

We see that $\mathbf{b}(s)$ is a unit vector, and $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ are mutually perpendicular.

Definition 2.13 (Frenet trihedron). The three unit vectors $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ are called the Frenet trihedron. They form a moving frame along the curve.

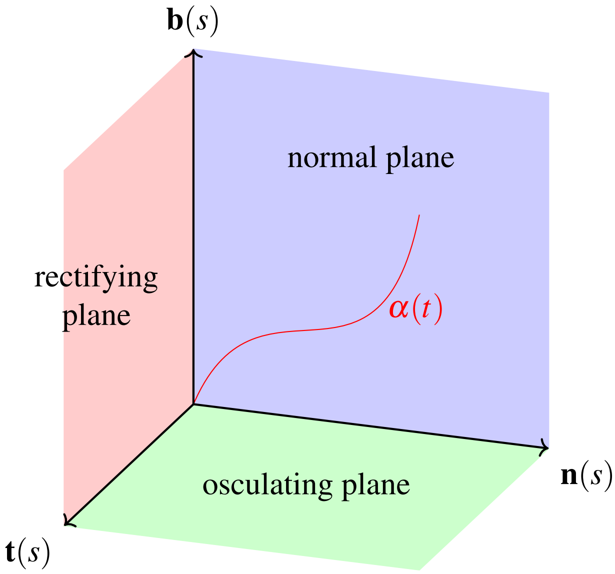

Figure 2.2: The Frenet trihedron

Example 2.15 (planar curves have constant binormal). Suppose $\alpha(s)$ lies in the xy-plane. Then, $\mathbf{t}(s)$ and $\mathbf{n}(s)$ lie in the xy-plane, so $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$ is perpendicular to the xy-plane, i.e. $\mathbf{b}(s) = (0,0,1)$ or $\mathbf{b}(s) = (0,0,-1)$.

Proposition 2.1. Let $\alpha(s)$ be parametrised by arc length in $\mathbb{R}^3$. Assume that for all $s \in \mathbb{R}$, we have

$$\alpha(0) = (0,0,0) \quad \text{and} \quad \mathbf{b}(s) = (0,0,1).$$

Then, $\alpha(s)$ lies in the xy-plane.

Proof. Write $\alpha(s) = (x(s), y(s), z(s))$ and consider $\mathbf{z}(s) = \alpha(s) \cdot \mathbf{b}(s)$. Differentiating both sides yields

$$\mathbf{z}'(s) = \alpha'(s) \cdot \mathbf{b}(s) + \alpha(s) \cdot \mathbf{b}'(s).$$

Since $\mathbf{b}(s) \perp \mathbf{t}(s)$, where $\mathbf{t}(s) = \alpha'(s)$, and as we will see, $\mathbf{b}'(s)$ is parallel to $\mathbf{n}(s)$, we obtain $\mathbf{z}'(s) = \mathbf{0}$. As such, $\mathbf{z}(s)$ is constant. Since $\mathbf{z}(0) = \mathbf{0}$, then $\mathbf{z}(s) = \mathbf{0}$ for all s. $\square$

Lemma 2.2. Suppose $\alpha''(s) \neq 0$. Then, $\mathbf{b}'(s)$ is parallel to $\mathbf{n}(s)$.

Proof. Since $\mathbf{b}(s)$ is a unit vector, we have $\mathbf{b}(s) \cdot \mathbf{b}(s) = 1$, hence

$$0 = \frac{d}{ds}(\mathbf{b}(s) \cdot \mathbf{b}(s)) = 2\mathbf{b}'(s) \cdot \mathbf{b}(s),$$

so $\mathbf{b}'(s) \perp \mathbf{b}(s)$. Also, $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$, and differentiating shows that $\mathbf{b}'(s) \perp \mathbf{t}(s)$ as well. Since $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ is an orthonormal basis, a vector perpendicular to both $\mathbf{t}$ and $\mathbf{b}$ must be parallel to $\mathbf{n}$. $\square$

Definition 2.14 (torsion). By Lemma 2.2, we may write

$$\mathbf{b}'(s) = \tau(s)\,\mathbf{n}(s),$$

where the scalar $\tau(s)$ is called the torsion of α at s.

Theorem 2.5 (Frenet-Serret formulae for arc length parametrisation). Let $\alpha(s)$ be parametrised by arc length with $\alpha''(s) \neq 0$. Then,

$$\mathbf{t}'(s) = k(s)\,\mathbf{n}(s) \quad \text{and} \quad \mathbf{n}'(s) = -k(s)\,\mathbf{t}(s) - \tau(s)\,\mathbf{b}(s) \quad \text{and} \quad \mathbf{b}'(s) = \tau(s)\,\mathbf{n}(s).$$

Equivalently,

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix},$$

where the coefficient matrix is skew-symmetric.

We now discuss the Frenet-formulae for curves not parametrised by arc length. In theory, a non-singular curve $\alpha(t)$ can be reparametrised by arc length $s = s(t)$, but in computations this can be tedious or impossible. Hence, we want formulae for $\mathbf{t}, \mathbf{n}, \mathbf{b}, k, \tau$ directly in terms of t-derivatives. In this section, let $', '', ''', \ldots$ denote differentiation with respect to t (not s).

Theorem 2.6 (Frenet data for a general time parametrisation). Let $\alpha(t)$ be a non-singular curve in $\mathbb{R}^3$. Define

$$\mathbf{t}(t) = \frac{\alpha'(t)}{|\alpha'(t)|} \quad \text{and} \quad \mathbf{b}(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|} \quad \text{and} \quad \mathbf{n}(t) = \mathbf{b}(t) \wedge \mathbf{t}(t). \tag{2.1}$$

Then,

$$k(t) = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3} \quad \text{and} \quad \tau(t) = -\frac{((\alpha'(t) \wedge \alpha''(t)) \cdot \alpha'''(t))}{|\alpha'(t) \wedge \alpha''(t)|^2}. \tag{2.2}$$

Moreover,

$$\alpha'(t) = \left|\alpha'(t)\right| \mathbf{t}(t),$$

and

$$\alpha''(t) = \frac{d}{dt}\left(\left|\alpha'(t)\right|\right)\mathbf{t}(t) + k(t)\left|\alpha'(t)\right|^2 \mathbf{n}(t). \tag{2.3}$$

Finally, the Frenet-Serret equations become

$$\mathbf{t}'(t) = k(t)\left|\alpha'(t)\right|\mathbf{n}(t) \tag{2.4}$$
$$\mathbf{n}'(t) = -k(t)\left|\alpha'(t)\right|\mathbf{t}(t) - \tau(t)\left|\alpha'(t)\right|\mathbf{b}(t) \tag{2.5}$$
$$\mathbf{b}'(t) = \tau(t)\left|\alpha'(t)\right|\mathbf{n}(t) \tag{2.6}$$

One might ask what changed compared to the s-version? When differentiating with respect to t, the extra factor $|\alpha'(t)| = \frac{ds}{dt}$ appears everywhere. In particular, the rates of turning measured per unit time are the arc length rates multiplied by speed.

Given a non-singular curve $\alpha(t)$, in theory we can convert it into an arc length parametrisation $\alpha(s)$ via $s = s(t)$. In practice, this conversion can be tedious (or impossible to do explicitly), so we want formulas for $\mathbf{t}, \mathbf{n}, \mathbf{b}, k, \tau$ directly in terms of $\mathbf{t}$-derivatives. Now, we let $', '', ''', \ldots$ denote differentiation with respect to $\mathbf{t}$.

We now prove Theorem 2.6.

Proof. Let $\alpha(t) = \alpha(s(t))$, where s denotes arc length. Then,

$$\frac{ds}{dt} = \left|\alpha'(t)\right|.$$

Write $t(t) = t(s(t))$ etc. Using the chain rule, we have

$$\mathbf{t}'(t) = \frac{dt}{ds}\bigg|_{s=s(t)} \cdot \frac{ds}{dt} = t'(s)\bigg|_{s=s(t)} \cdot \left|\alpha'(t)\right| = k(t)\left|\alpha'(t)\right|\mathbf{n}(t),$$

so (2.4) holds. We have similar results for $\mathbf{n}'$ and $\mathbf{b}'$, giving (2.5) and (2.6) respectively.

For the curvature formula, start from

$$\alpha'(t) = \left|\alpha'(t)\right|\mathbf{t}(t).$$

Then, differentiate both sides to obtain

$$\alpha''(t) = \frac{d}{dt}\left(\left|\alpha'(t)\right|\right)\mathbf{t}(t) + \left|\alpha'(t)\right|\mathbf{t}'(t) = \frac{d}{dt}\left(\left|\alpha'(t)\right|\right)\mathbf{t}(t) + k(t)\left|\alpha'(t)\right|^2\mathbf{n}(t),$$

which is the displayed decomposition in (2.3). Now, wedge with $\alpha'(t)$ to obtain

$$\begin{aligned}\alpha'(t) \wedge \alpha''(t) &= \left(\left|\alpha'(t)\right|\mathbf{t}(t)\right) \wedge \left(\frac{d}{dt}\left(\left|\alpha'(t)\right|\right)t(t) + k(t)\left|\alpha'(t)\right|^2\mathbf{n}(t)\right) \\ &= k(t)\left|\alpha'(t)\right|^3(\mathbf{t}(t) \wedge \mathbf{n}(t))\end{aligned}$$

Since $\mathbf{t} \wedge \mathbf{n} = \mathbf{b}$, taking norms, we have

$$\left|\alpha'(t) \wedge \alpha''(t)\right| = k(t)\left|\alpha'(t)\right|^3,$$

hence,

$$k(t) = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}.$$

We have proven that (2.2) holds. This also shows that

$$\mathbf{b}(t) = \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t) \wedge \alpha''(t)|}.$$

For (2.2) on torsion, one differentiates the decomposition of $\alpha''(t)$ once more, dots with $\mathbf{b}(t)$, and uses that among the resulting terms, the only component not perpendicular to $\mathbf{b}$ arises from the $\mathbf{n}'$-term, which contains $\tau(t)|\alpha'(t)|\mathbf{b}(t)$. This yields

$$\tau(t) = -\frac{((\alpha'(t) \wedge \alpha''(t)) \cdot \alpha'''(t))}{|\alpha'(t) \wedge \alpha''(t)|^2}.$$

□

Example 2.16 (MA3215 AY14/15 Sem 2 Tutorial 1). Continuing from Example 2.8, consider the helix

$$\alpha : \mathbb{R} \to \mathbb{R}^3 \quad \text{where} \quad \alpha(t) = (3\cos t, 3\sin t, 4t).$$

(i) Compute $\mathbf{t}(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$.

(ii) Compute $k(s)$, $\mathbf{b}'(s)$, and $\tau(s)$.

(iii) Compute $\mathbf{n}'(s)$. Verify that $\mathbf{n}'(s) = -k(s)\mathbf{t}(s) - \tau(s)\mathbf{b}(s)$.

Solution.

(i) In Example 2.8, we saw that the arc length is $s(t) = 5t$, so $\alpha(s) = \left(3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4s}{5}\right)$. By (2.1), we have

$$\mathbf{t}(s) = \frac{\alpha'(s)}{|\alpha'(s)|} = \left(-\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}\right).$$

By Definition 2.10, we have

$$\mathbf{n}(s) = \frac{\alpha''(s)}{|\alpha''(s)|} = \left(-\cos\frac{s}{5}, -\sin\frac{s}{5}, 0\right).$$

By Definition 2.12, $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$.

(ii) From Definition 2.7, $k(s) = |\alpha''(s)| = \frac{3}{25}$. It is easy to compute $\mathbf{b}'(s)$ from **(i)**. By Definition 2.14, we have $\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$, so $\tau(s) = -\frac{4}{25}$.

(iii) This is merely verifying one of the Frenet-Serret formulae as in Theorem 2.5. □

Continuing our discussion from Example 2.8 and 2.16, note that translating a curve does not alter its curvature $k(s)$ and torsion $\tau(s)$. For example, let β denote the curve obtained by translating α by $(7,8,9)$. That is, $\beta(t) = \alpha(t) + (7,8,9)$ for all $t \in \mathbb{R}$. Then, $\beta(s) = \alpha(s) + (7,8,9)$ trivially (one can verify this). Consequently, by Proposition 2.2 and the proof of Theorem 2.8 as we would see in due course, α and β are the same curve because they differ by rigid motion, which implies that they have the same curvature k and torsion τ.

Example 2.17. Let $\alpha(s)$ and $\beta(s)$ denote two smooth curves in $\mathbb{R}^3$ parametrised by arc length. Let $\mathbf{t}_1(s), \mathbf{n}_1(s), \mathbf{b}_1(s), k_1(s), \tau_1(s)$ denote the Frenet trihedron, curvature, and torsion of $\alpha(s)$. Similarly, let $\mathbf{t}_2(s), \mathbf{n}_2(s), \mathbf{b}_2(s), k_2(s), \tau_2(s)$ denote those of $\beta(s)$. Suppose $k_1(s) = k_2(s)$ and $\tau_1(s) = \tau_2(s)$.

(i) Show that

$$\frac{d}{ds}\left(\left|\mathbf{t}_1(s)-\mathbf{t}_2(s)\right|^2+\left|\mathbf{n}_1(s)-\mathbf{n}_2(s)\right|^2+\left|\mathbf{b}_1(s)-\mathbf{b}_2(s)\right|^2\right)=0.$$

(ii) Suppose further that $\mathbf{t}_1(0)=\mathbf{t}_2(0)$, $\mathbf{n}_1(0)=\mathbf{n}_2(0)$, and $\mathbf{b}_1(0)=\mathbf{b}_2(0)$. Show that

$$\mathbf{t}_1(s)=\mathbf{t}_2(s) \quad \text{and} \quad \mathbf{n}_1(s)=\mathbf{n}_2(s) \quad \text{and} \quad \mathbf{b}_1(s)=\mathbf{b}_2(s).$$

Solution.

(i) Let $\Delta\mathbf{t}=\mathbf{t}_1-\mathbf{t}_2$, and let $\Delta\mathbf{n}$ and $\Delta\mathbf{b}$ be defined similarly. Since we see the square of the length of a vector, it is natural to consider dot products. Let $F(s)$ denote the expression on the left side of the equation. We wish to prove that its derivative is equal to 0. We have

$$F(s)=\|\Delta\mathbf{t}\|^2+\|\Delta\mathbf{n}\|^2+\|\Delta\mathbf{b}\|^2=\Delta\mathbf{t}\cdot\Delta\mathbf{t}+\Delta\mathbf{n}\cdot\Delta\mathbf{n}+\Delta\mathbf{b}\cdot\Delta\mathbf{b}.$$

As such,

$$F'(s)=2\Delta\mathbf{t}\cdot(\Delta\mathbf{t})'+2\Delta\mathbf{n}\cdot(\Delta\mathbf{n})'+2\Delta\mathbf{b}\cdot(\Delta\mathbf{b})'. \tag{2.7}$$

Since α and β are parametrised by arc length, by Theorem 2.2, $|\alpha'(s)|=|\beta'(s)|=1$. By the Frenet-Serret equations in Theorem 2.6, we see that

$$\mathbf{t}'(t)=k(t)\,\mathbf{n}(t) \quad \text{and} \quad \mathbf{n}'(t)=-k(t)\,\mathbf{t}(t)-\tau(t)\,\mathbf{b}(t) \quad \text{and} \quad \mathbf{b}'(t)=\tau(t)\,\mathbf{n}(t).$$

Substitute the Frenet-Serre equations into (2.7), and use the fact that the curves have the same curvature and torsion to conclude that $F'(s)=0$.

(ii) From **(i)**, we infer that $F(s)$ is constant in s. Now, we know that $\Delta\mathbf{t}(0)=0$, $\Delta\mathbf{n}(0)=0$, and $\Delta\mathbf{b}(0)=0$, so $F(0)=0$. Since F is a constant, then $F(s)=0$ for all s. The result follows. □

Example 2.18 (do Carmo p. 27 Question 17). In general, a curve α is called a helix (see Figure 2.1) if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s)\neq 0$, where $s\in I$, and prove the following:

(a) α is a helix if and only if $\frac{k}{\tau}$ is a constant

(b) The curve

$$\alpha(s)=\left(\frac{a}{c}\int\sin\theta(s)\,ds,\ \frac{a}{c}\int\cos\theta(s)\,ds,\ \frac{b}{c}s\right),$$

where $c^2=a^2+b^2$, is a helix, and that $k/\tau=a/b$.

Solution.

(a) A tangent line is $\mathbf{t}(s)=\alpha'(s)$. Fix a non-zero vector $\mathbf{v}\in\mathbb{R}^3$. Then, we have $\langle\mathbf{t},\mathbf{v}\rangle=\|\mathbf{v}\|\cos\theta$, where θ is constant. Differentiating both sides yields $\langle\mathbf{t}',\mathbf{v}\rangle=0$

so $\langle \mathbf{n}, \mathbf{v} \rangle = 0$. Differentiating again yields $\langle \mathbf{n}', \mathbf{v} \rangle = 0$. By the Frenet-Serret equations (Theorem 2.5),

$$\langle k(s)\mathbf{t}(s), \mathbf{v} \rangle = -\langle \tau(s)\mathbf{b}(s), \mathbf{v} \rangle .$$

As such,

$$\frac{k}{\tau} = \frac{\langle \mathbf{b}, \mathbf{v} \rangle}{\langle \mathbf{t}, \mathbf{v} \rangle}.$$

We wish to prove that this quantity is a constant. Upon differentiating both sides using the quotient rule and using the fact that $\langle \mathbf{n}, \mathbf{v} \rangle = 0$, the forward direction follows.

We then prove the reverse direction. The idea is to let $k/\tau = c$ where c is a constant. Then, define $\mathbf{u} = \mathbf{t} + c\mathbf{b}$, and show that $\mathbf{u}' = \mathbf{0}$. One can then compute the angle between $\mathbf{t}$ and $\mathbf{v}$ to prove that the angle θ satisfies $\cos\theta = \frac{1}{\sqrt{1+c^2}}$.

(b) We have

$$\mathbf{t}(s) = \alpha'(s) = \left(\frac{a}{c}\sin\theta(s), \frac{a}{c}\cos\theta(s), \frac{b}{c} \right) \quad \text{so} \quad |\mathbf{t}(s)| = 1.$$

Observe that $\langle \mathbf{t}, \mathbf{e}_3 \rangle = \frac{b}{c}\mathbf{e}_3$ so α is a helix. Next,

$$\alpha''(s) = \left(\frac{a}{c}\theta'(s)\cos\theta(s), -\frac{a}{c}\theta'(s)\sin\theta(s), 0 \right) \quad \text{so} \quad k(s) = |\alpha''(s)| = \frac{a}{c}|\theta'(s)|.$$

By (2.2), $\tau(s) = \frac{b}{c}|\theta'(s)|$. Hence, $k/\tau = a/b$. □

Example 2.19 (MA3215 AY14/15 Sem 2 Tutorial 2). Let $\alpha(s)$ be a curve parametrised by arc length. Suppose $\alpha(0) = (0,0,0)$ and that the curve has constant curvature $k(s) = C > 0$ for all s, and $\tau(s) = 0$ for all s.

(i) Suppose $\mathbf{b}(0) = (0,0,1)$. Show that $\mathbf{b}(s) = (0,0,1)$ for all s.

(ii) Show that $\alpha(s)$ lies on the xy-plane.

(iii) Show that $\alpha(s)$ is an arc on a circle of radius $1/C$.

Solution.

(i) Since the torsion τ is 0 for all s. by Definition 2.14, $\mathbf{b}'(s) = \mathbf{0}$, so $\mathbf{b}$ is a constant vector. It follows that $\mathbf{b}(s) = (0,0,1)$ for all s.

(ii) Since α is parametrised by arc length, then $\alpha'(s) = \mathbf{t}(s)$ by Definition 2.9. Also, $\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s)$ is orthogonal to $\mathbf{t}(s)$, so $\mathbf{t}(s) \cdot \mathbf{b}(s) = 0$ for all s. Since $\mathbf{b}(s) = (0,0,1)$, then $\mathbf{t}(s) \cdot (0,0,1) = 0$. As such, the z-component of $\alpha'(s)$ is 0. Since $\alpha(0) = (0,0,0)$, it follows that $\alpha(s)$ lies in the xy-plane.

(iii) By the Frenet-Serret equations (Theorem 2.6), we have

$$\mathbf{t}'(t) = C\mathbf{n}(t) \quad \text{and} \quad \mathbf{n}'(t) = -C\mathbf{t}(t) \quad \text{and} \quad \mathbf{b}'(t) = \mathbf{0}.$$

Use the fact that $\mathbf{n}$ has norm 1, and the result follows. □

2.6 The Fundamental Theorem of the Local Theory of Curves

In this chapter, we emphasise that we only care about the shape of a curve. Hence, two curves which differ by a rigid motion (translations and rotations) are regarded as the same curve. On the other hand, reflections are not considered rigid motions here.

Definition 2.15 (rigid motion). A map $\mathbf{R} : \mathbb{R}^3 \to \mathbb{R}^3$ is called a rigid motion if the following hold:

(i) it preserves distances, i.e. for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $|\mathbf{R}(\mathbf{v}) - \mathbf{R}(\mathbf{w})| = |\mathbf{v} - \mathbf{w}|$

(ii) it preserves orientation (it sends a positively oriented (x, y, z)-axes to a positively oriented axes)

Lemma 2.3. A rigid motion satisfies the following properties:

(i) sends straight lines to straight lines

(ii) preserves angles

(iii) preserves inner products in the sense that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, we have

$$(\mathbf{R}(\mathbf{v}) - \mathbf{R}(\mathbf{u})) \cdot (\mathbf{R}(\mathbf{w}) - \mathbf{R}(\mathbf{u})) = (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{w} - \mathbf{u}).$$

Theorem 2.7 (structure of rigid motions). Every rigid motion can be written as a rotation (about an axis through the origin) followed by a translation. Equivalently, every rigid motion has the form

$$\mathbf{R}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{v},$$

where $\mathbf{A}$ is an orthogonal matrix with $\det(\mathbf{A}) = 1$ and $\mathbf{v} \in \mathbb{R}^3$.

Proposition 2.2 (invariance of $k(s)$ and $\tau(s)$). Let $\alpha(s)$ be a curve and let $\mathbf{R}$ be a rigid motion. Then, the curvature $k(s)$ and torsion $\tau(s)$ of α are unchanged under the rigid motion of the curve.

Under a rigid motion, the Frenet trihedron $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ and their derivatives transform the same way, so their lengths are preserved. Since $k(s)$ and $\tau(s)$ are defined via ratios of these lengths (Frenet-Serret formulas), they remain unchanged.

By a translation, we may assume that $\alpha(0) = (0,0,0)$. By rotations about the origin, we may further assume that the initial Frenet trihedron aligns with the standard basis $\mathbf{t}(0) = (1,0,0)$, $\mathbf{n}(0) = (0,1,0)$, and $\mathbf{b}(0) = (0,0,1)$.

Theorem 2.8 (uniqueness up to rigid motion). Suppose $\alpha(s)$ and $\beta(s)$ are two smooth curves parametrised by arc length, and they have the same curvature and torsion for all $s \in \mathbb{R}$. That is,

$$k_\alpha(s) = k_\beta(s) \quad \text{and} \quad \tau_\alpha(s) = \tau_\beta(s).$$

Then, α and β differ by a rigid motion.

Proof. Apply a rigid motion to each curve so that both satisfy the normalisation

$$\alpha(0) = \beta(0) = (0,0,0) \quad \text{and} \quad (t(0), n(0), b(0)) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3).$$

Let $(\mathbf{t}_1, \mathbf{n}_1, \mathbf{b}_1)$ be the Frenet trihedron of α and $(\mathbf{t}_2, \mathbf{n}_2, \mathbf{b}_2)$ that of β. Since $k(s)$ and $\tau(s)$ agree, the Frenet-Serret equations (Theorem 2.5) give the same system of differential equations for both frames with the same initial condition at $s = 0$. By the uniqueness theorem for ordinary differential equations, we obtain for all s,

$$\mathbf{t}_1(s) = \mathbf{t}_2(s) \quad \text{and} \quad \mathbf{n}_1(s) = \mathbf{n}_2(s) \quad \text{and} \quad \mathbf{b}_1(s) = \mathbf{b}_2(s).$$

In particular,

$$\alpha'(s) = \mathbf{t}_1(s) = \mathbf{t}_2(s) = \beta'(s).$$

Integrating both sides, we have $\alpha(s) = \beta(s) + \mathbf{C}$ for some constant vector $\mathbf{C}$, and $\mathbf{C} = \mathbf{0}$ from $\alpha(0) = \beta(0)$. Undoing the normalisation gives that the original curves differ by a rigid motion. □

Theorem 2.9 (existence). Given any two smooth functions $k(s)$ and $\tau(s)$ with $k(s) > 0$, there exists a smooth curve $\alpha(s)$ whose curvature is $k(s)$ and whose torsion is $\tau(s)$.

Example 2.20 (do Carmo p. 24 Question 7). Let $\alpha : I \to \mathbb{R}^2$ be a regular parametrised plane curve with arbitrary parameter, and define $\mathbf{n} = \mathbf{n}(t)$ and $k = k(t)$. Assume that $k(t) \neq 0$ for all $t \in I$. In this situation, the curve

$$\beta(t) = \alpha(t) + \frac{1}{k(t)}\mathbf{n}(t)$$

is called the evolute of α.

(a) Show that the tangent at t of the evolute of α is the normal to α at t.

(b) Consider the normal lines of α at two neighbouring points t_1 and t_2, where $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of α.

Solution.

(a) Use Theorem 2.6.

(b) The idea is to let $t_2 = t$ and $t_1 = t + h$, and then let $h \to 0$. □

Example 2.21 (do Carmo p. 25 Question 8). The trace of the parametrised curve $\alpha(t) = (t, \cosh t)$, where $t \in \mathbb{R}$, is called the catenary. One can show that the signed curvature of the catenary is

$$k(t) = \frac{1}{\cosh^2 t}$$

and by considering Example 2.20, the evolute of the catenary is

$$\beta(t) = (t - \sinh t \cosh t, 2\cosh t).$$

Chapter 3

Surfaces

3.1 Surfaces

Intuitively, a surface in $\mathbb{R}^3$ is something on which an ant can move with two degrees of freedom in a small vicinity of any point. Locally, the ant's neighbourhood should look more or less like a flat plane. In this course, *nice* surfaces are expected to be smooth and non-self-intersecting (the surface should not cross itself). By smooth, intuitively, we cannot have sharp bends/edges so we exclude corners like a box, or a cone with a sharp tip. Some prototypical examples include a plane, a sphere, a disc, and a torus.

Figure 3.1: Where you are

On a plane, we represent a point by coordinates (u, v) after choosing axes. On a sphere, longitude/latitude also give coordinates, but there are built-in defects. Some of which are as follows:

(i) crossing an international date line causes a sudden jump in the coordinate value

(ii) at the poles, longitude becomes ill-defined

These defects are fatal for Calculus, because differentiation depends on making arbitrarily small movements without sudden coordinate jumps.

Definition 3.1 (local coordinate region). A local coordinate region on a surface is a small region of the surface equipped with a coordinate system (u,v) so that each point in the region is described by a pair of real numbers. Such a region is also called a local coordinate chart.

Example 3.1. Let $S = \mathbb{R}^2 \subseteq \mathbb{R}^3$ as the $z = 0$ plane. A chart is just $\mathbf{X}(u,v) = (u,v,0)$ with $(u,v) \in U \subseteq \mathbb{R}^2$ open.

We insist that neighbouring regions overlap along borders, so that in the overlap, a point has (at least) two coordinate descriptions. To perform Calculus consistently, we must specify and control how coordinates change from one chart to another on overlaps.

Definition 3.2 (coordinate change/transition map). Let (U,φ) and (V,ψ) be two charts on a surface, where

$$\varphi : U \to \mathbb{R}^2 \quad \text{and} \quad \psi : V \to \mathbb{R}^2.$$

On the overlap $U \cap V$, the transition map from φ-coordinates to ψ-coordinates is

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

$$\begin{tikzcd}
 & U \cap V \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
\varphi(U \cap V) \arrow[rr, "\psi\circ\varphi^{-1}"] & & \psi(U \cap V)
\end{tikzcd}$$

Definition 3.3 (smoothly compatible charts). Two charts (U,φ) and (V,ψ) are smoothly compatible if the transition maps

$$\psi \circ \varphi^{-1} \quad \text{and} \quad \varphi \circ \psi^{-1}$$

are smooth maps between open subsets of $\mathbb{R}^2$.

Note that if you differentiate a function using (u,v)-coordinates, and I differentiate the same function using (s,t)-coordinates, then the chain rule compares our derivatives using the transition map. Smooth transition maps guarantee that derivatives computed in different charts are consistent.

Definition 3.4 (atlas). An atlas on a set S (Figure 3.2) is a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that the following hold:

(i) the domains cover S, i.e. $S = \bigcup_{\alpha \in A} U_\alpha$

(ii) charts are pairwise smoothly compatible on overlaps

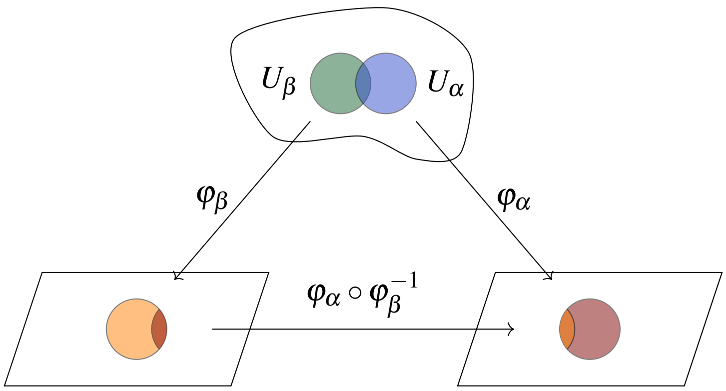

Figure 3.2: An atlas on a set S

Note that one can reverse engineer a surface as follows. We start with many planar pieces (regions) with coordinates, then prescribe transition maps on overlaps, and lastly glue the regions along overlaps so that the coordinate identifications match the transition maps. This produces a surface intrinsically without first embedding it into $\mathbb{R}^3$.

What is the difference between intrinsic geometry and embedding? If we only care about what happens on the surface (distances, angles, curvature, differentiation along the surface), it is not always necessary to know how the surface sits in $\mathbb{R}^3$. This viewpoint allows exotic examples that may not embed nicely in $\mathbb{R}^3$. Take for example the Klein bottle (Figure 3.3) — the Klein bottle cannot be embedded in $\mathbb{R}^3$ without any self-intersection and any drawing in $\mathbb{R}^3$ is an *immersion*, not an embedding.

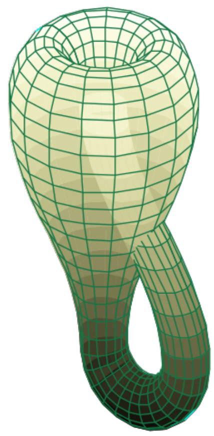

Figure 3.3: A two-dimensional representation of a Klein bottle embedded in $\mathbb{R}^3$

We now introduce the concept of a manifold (Definition 3.5).

Definition 3.5 (manifold). A set M is an n-dimensional manifold if it can be covered by charts mapping into $\mathbb{R}^n$ with smooth transition maps on overlaps.

From Definition 3.5, we see that a surface is a 2-dimensional manifold. In this course, we treat surfaces as 2-manifolds (often embedded in $\mathbb{R}^3$), and we build Calculus on them by transporting Euclidean Calculus through charts. A surface $S \subseteq \mathbb{R}^3$ is a set which is locally 2-dimensional. That is to say, for each $p \in S$, if we take a sufficiently small open ball $V \subseteq \mathbb{R}^3$ centred at p, then the piece $V \cap S$ should look like an open disk in $\mathbb{R}^2$.

Definition 3.6 (open subsets of a surface). Let $S \subseteq \mathbb{R}^3$ be a surface and let $U \subseteq S$. We say that U is open in S if for every $p \in U$, there exists $\varepsilon > 0$ such that

$$B(p,\varepsilon) \cap S \subseteq U \quad \text{where} \quad B(p,\varepsilon) = \left\{q \in \mathbb{R}^3 : |q-p| < \varepsilon\right\}.$$

Equivalently, U is open in S if there exists an open set $V \subseteq \mathbb{R}^3$ such that $U = S \cap V$. If $p \in S$, an open neighbourhood of p (in S) is a set $U \subseteq S$ such that $p \in U$ and U is open in S.

3.2 Regular Surfaces

Definition 3.7 (regular surface). A subset $S \subseteq \mathbb{R}^3$ is called a regular surface if for each $p \in S$, there exists an open neighbourhood $V \subseteq \mathbb{R}^3$ of p, an open set $U \subseteq \mathbb{R}^2$, and a map

$$\mathbf{x} : U \to V \cap S \quad \text{where} \quad \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

such that the following hold:

- **(i)** The component functions $x(u,v)$, $y(u,v)$, $z(u,v)$ are smooth on U
- **(ii)** $\mathbf{x}$ is a bijective homeomorphism onto its image $V \cap S$, so the inverse $\mathbf{x}^{-1} : V \cap S \to U$ exists and is continuous
- **(iii)** The differential

$$d\mathbf{x} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

has rank 2 at every $\mathbf{q} = (u_0, v_0) \in U$, i.e. $d\mathbf{x}_{\mathbf{q}}$ has rank 2 as a real 3×2 matrix

Such a map $\mathbf{x}$ is called a coordinate function (or parametrisation), and the pair $(U, \mathbf{x})$ is called a coordinate chart.

$$\begin{tikzcd} U \arrow[r, "\mathbf{x}"] \arrow[dr, "\iota"'] & V \cap S \arrow[d, hook, "\subseteq"] \\ & V \subseteq \mathbb{R}^3 \end{tikzcd}$$

A standard way to prove that a subset $S \subset \mathbb{R}^3$ is a regular surface is to explicitly exhibit coordinate functions (charts) $\mathbf{x} : U \to V \cap S$, where $U \subseteq \mathbb{R}^2$ and $V \subseteq \mathbb{R}^3$ are open, $\mathbf{x}$ is smooth, bijective onto $V \cap S$ with smooth inverse, and $d\mathbf{x}$ has rank 2 everywhere.

In practice, one often checks the smoothness and rank condition (**(i)** and **(iii)** respectively in Definition 3.7). In fact, for a regular surface, conditions **(i)** and **(iii)** already force the required topological behaviour in **(ii)**. Also, the regularity condition is frequently stated as $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ being an injective linear transformation — this is equivalent to $\text{rank}\left(d\mathbf{x}_q\right) = 2$.

One can think of $\mathbf{x} : U \to S \cap V$ as assigning coordinates (u, v) to points on the surface patch $S \cap V$. Firstly, injective means a point on $S \cap V$ is not assigned two different coordinate pairs; surjective means every point of the patch $S \cap V$ has some coordinate pair. For **(i)**, we need the map to be smooth since we want to differentiate the parametrisation, i.e. perform some calculus on S. Lastly, with regards to **(iii)** on the rank 2 condition, fix $q = (u_0, v_0) \in U$ and put $p = \mathbf{x}(u_0, v_0) \in S$. If we fix $v = v_0$ and vary u, we obtain a curve

$$\alpha(u) = \mathbf{x}(u, v_0) \quad \text{with} \quad \alpha'(u_0) = \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0).$$

If we fix $u = u_0$ and vary v, we obtain a curve

$$\beta(v) = \mathbf{x}(u_0, v) \quad \text{with} \quad \beta'(v_0) = \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0).$$

These two vectors $\alpha'(u_0)$ and $\beta'(v_0)$ are precisely the two columns of $d\mathbf{x}_q$. Thus, $\text{rank}\left(d\mathbf{x}_q\right) = 2$ means they are linearly independent, and hence they span a 2-dimensional plane in $\mathbb{R}^3$, which we interpret as the tangent plane to the surface at p.

Example 3.2 (MA3215 AY14/15 Sem 2 Tutorial 2). Consider the plane

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot (3, 2, 0) = -1\right\}.$$

Verify that this is a regular surface by checking the three conditions in Definition 3.7.

Solution. Take an arbitrary point $p = (x_0, y_0, z_0) \in S$. Then, $3x_0 + 2y_0 = -1$. Define a parametrisation of the plane by

$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{and} \quad \mathbf{x}(u, v) = \left(u, \frac{-1 - 3u}{2}, v\right).$$

Clearly, $\mathbf{x}(u, v) \in S$ since $\mathbf{x}(u, v) \cdot (3, 2, 0) = -1$. Now, let $V \subseteq \mathbb{R}^3$ be an open neighbourhood of p. Set

$$U = \mathbf{x}^{-1}(V) = \left\{(u, v) \in \mathbb{R}^2 : \mathbf{x}(u, v) \in V\right\}.$$

Since $\mathbf{x}$ is continuous and V is open, then U is open in $\mathbb{R}^2$. We claim that $\mathbf{x} : U \to V \cap S$ satisfies the three conditions in Definition 3.7.

For **(i)**, the component functions $x(u, v) = u$, $y(u, v) = \frac{-1-3u}{2}$, and $z(u, v) = v$ are polynomials, so they are smooth on U. For **(ii)**, we claim that $\mathbf{x}$ is a bijective homeomorphism onto $V \cap S$. The fact that $\mathbf{x}$ is a bijection is clear. Next, we shall prove that $\mathbf{x}^{-1}$ is continuous. Observe that

$$\mathbf{x}^{-1}\left(u, \frac{-1 - 3u}{2}, v\right) = (u, v).$$

So, $\mathbf{x}^{-1}$ exists and it is equal to (x,z).

Lastly for **(iii)**, we have $\mathbf{x}_u(u,v) = \left(1, -\frac{3}{2}, 0\right)$ and $\mathbf{x}_v(u,v) = (0,0,1)$ which are linearly independent in $\mathbb{R}^3$. As such, the 3×2 Jacobian matrix has rank 2 at every $(u,v) \in U$. $\square$

Example 3.3. We now give a non-example illustrating condition **(iii)** on rank in the definition of a regular surface (Definition 3.7). Let

$$\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{where} \quad \mathbf{x}(u,v) = \left(u^2, u^3, v\right).$$

This is known as a cuspidal cubic (Figure 3.4)[1]. Then,

$$d\mathbf{x} = \begin{pmatrix} 2u & 0 \\ 3u^2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{so} \quad d\mathbf{x}_q(0, v_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which has rank 1. Geometrically, for a fixed $v = v_0$, the curve $u \mapsto \left(u^2, u^3, v_0\right)$ is the standard planar cusp $(x,y) = \left(u^2, u^3\right)$ sitting at height $z = v_0$. Stacking these cusps over all v gives a cusp-edge[2] along the z-axis, which is not a smooth surface.

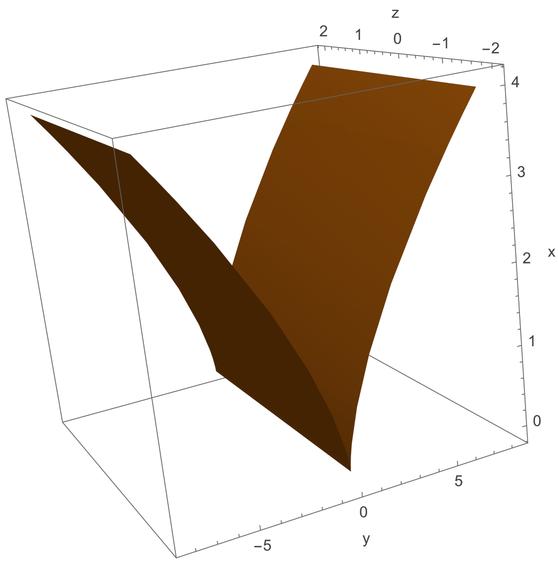

Figure 3.4: Cuspidal cubic

Lemma 3.1. Suppose $\mathbf{x} : U \to V \cap S$ is a coordinate function. If $U' \subseteq U$ is open, then the restriction $U' \to \mathbf{x}(U')$ is also a coordinate function for an appropriate open set $V' \subseteq \mathbb{R}^3$ with $\mathbf{x}(U') = V' \cap S$.

[1] Historically, the semicubical parabola was discovered in 1657 by William Neile who computed its arc length [3].

[2] Geometrically, a regular surface in $\mathbb{R}^3$ is a surface that, when you zoom in enough around any point, looks like a smooth (non-creasing, non-folding) patch of the plane.

$$\begin{array}{ccc} U' & \xrightarrow{\mathbf{x}|_{U'}} & \mathbf{x}(U') = S \cap V' \\ \Big\downarrow & & \Big\downarrow \\ U & \xrightarrow{\mathbf{x}} & \mathbf{x}(U) = S \cap V \end{array}$$

Example 3.4 (the unit sphere S^2). Define the unit sphere

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\}.$$

Let

$$U = \left\{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\right\} \quad \text{and} \quad V = \left\{(x,y,z) \in \mathbb{R}^3 : z > 0\right\}.$$

Then, $V \cap S^2$ is the open northern hemisphere, and we define

$$\mathbf{x}_N : U \to V \cap S^2 \quad \text{where} \quad \mathbf{x}_N(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right). \tag{3.1}$$

This is a bijective map with inverse $\mathbf{x}_N^{-1}(x,y,z) = (x,y)$. Smoothness is clear for the first two coordinates; for the third coordinate, note that $1-u^2-v^2 > 0$ on U, so the square root is smooth on U.

Lastly, we verify that $\operatorname{rank}(d\mathbf{x}_N) = 2$. We shall compute

$$\frac{\partial \mathbf{x}_N}{\partial u}(u,v) = \left(1, 0, -\frac{u}{\sqrt{1-u^2-v^2}}\right) \quad \text{and} \quad \frac{\partial \mathbf{x}_N}{\partial v}(u,v) = \left(0, 1, -\frac{v}{\sqrt{1-u^2-v^2}}\right),$$

which are clearly linearly independent in $\mathbb{R}^3$. Hence, $\operatorname{rank}(d\mathbf{x}_N) = 2$ for all $(u,v) \in U$.

Using the same U and $V_- = \left\{(x,y,z) \in \mathbb{R}^3 : z < 0\right\}$, define

$$\mathbf{x}_S : U \to V_- \cap S^2 \quad \text{where} \quad \mathbf{x}_S(u,v) = \left(u, v, -\sqrt{1-u^2-v^2}\right), \tag{3.2}$$

with inverse $\mathbf{x}_S^{-1}(x,y,z) = (x,y)$. The rank check is identical. The two charts in (3.1) and (3.2) do not cover the equator though. One can add four further charts using open sets $\{y > 0\}$, $\{y < 0\}$, $\{x > 0\}$, and $\{x < 0\}$ to obtain a cover of S^2 by six coordinate functions.

From this example, we see that coordinate charts are not unique as there are many valid ways to cover the same surface. Next, redundant charts are allowed: extra charts do not invalidate anything; we simply ignore redundancy.

Example 3.5 (MA3215 AY14/15 Sem 2 Tutorial 3). Let

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+1 = z^2\right\}.$$

Recall that this quadric surface is a hyperboloid of one sheet.

(i) If $(x,y,z) \in S$, show that either $z \geq 1$ or $z \leq -1$. This shows that S is the union of the regions

$$S_+ = \{(x,y,z) \in S : z \geq 1\} \quad \text{and} \quad S_- = \{(x,y,z) \in S : z \leq -1\}.$$

(ii) Find parametrisations of S_+ and S_-.

(iii) Use the parametrisations in **(ii)** to show that S_+ and S_- are regular surfaces.

Solution.

(i) Since $x^2+y^2 \geq 0$, then $z^2 \geq 1$, so $z \geq 1$ or $z \leq -1$. It follows that S is the union of the regions S_+ and S_-.

(ii) We can parametrise S_+ and S_- using

$$\Phi_+ = \left(x, y, \sqrt{1+x^2+y^2}\right) \quad \text{and} \quad \Phi_- = \left(x, y, -\sqrt{1+x^2+y^2}\right)$$

respectively.

(iii) We will only show that S_+ is a regular surface. Note that

$$(\Phi_+)_x = \left(1, 0, \frac{1}{\sqrt{1+x^2+y^2}}\right) \quad \text{and} \quad (\Phi_+)_y = \left(0, 1, \frac{1}{\sqrt{1+x^2+y^2}}\right).$$

So, it is easy to see that the Jacobian matrix is of rank 2. □

Example 3.6 (union of two planes that is not regular at the origin). Consider

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : \; yz = 0\right\}.$$

Equivalently, S is the union of the xy-plane $z=0$ and the xz-plane $y=0$. Then, S is not a regular surface at $p=(0,0,0)$ since near the origin, it looks like two smooth sheets crossing, so it fails to be locally parametrised by a single smooth coordinate patch with rank 2 everywhere. Intuitively, the tangent plane is not well-defined as a single plane there.

To make it more explicit, write $F(x,y,z) = yz$ and define $S = F^{-1}(0)$. So, $\nabla F(x,y,z) = (0,z,y)$. At $p=(0,0,0)$, we have $\nabla F(p) = \mathbf{0}$, so the rank condition fails here.

Proposition 3.1 (graphs are regular surfaces). Let $U \subseteq \mathbb{R}^2$ be open and let $f : U \to \mathbb{R}$ be smooth. Consider the graph

$$S := \left\{(x,y,z) \in \mathbb{R}^3 : (x,y) \in U, z = f(x,y)\right\}.$$

Then, S is a regular surface.

Proof. Define

$$\mathbf{x} : U \to S \quad \text{where} \quad \mathbf{x}(u,v) = (u, v, f(u,v)).$$

Then, $\mathbf{x}$ is smooth, bijective, and the inverse is the projection $\mathbf{x}^{-1}(x,y,f(x,y)) = (x,y)$ which is smooth. Finally,

$$\frac{\partial x}{\partial u}(u,v) = \left(1, 0, \frac{\partial f}{\partial u}(u,v)\right) \quad \text{and} \quad \frac{\partial x}{\partial v}(u,v) = \left(0, 1, \frac{\partial f}{\partial v}(u,v)\right),$$

which are linearly independent. Hence, $\operatorname{rank}(d\mathbf{x}) = 2$ everywhere. □

Example 3.7 (surface of revolution). Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and assume $f(z) > 0$ for all $z \in \mathbb{R}$. Rotate the curve $x = f(z)$ in the xz-plane about the z-axis. The resulting surface is

$$S = \{(f(z)\cos u, f(z)\sin u, z) : u \in \mathbb{R}, z \in \mathbb{R}\}.$$

See Figure 3.5 for an example of a surface of revolution.

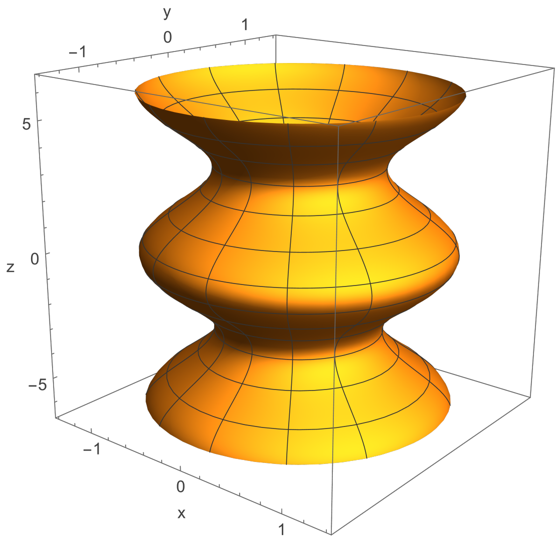

Figure 3.5: An example of a surface of revolution

Let

$$U = (0, 2\pi) \times \mathbb{R} \quad \text{and} \quad \mathbf{x} : U \to S \text{ where } \mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, v).$$

Then, $\mathbf{x}$ is smooth. For the rank condition, we compute

$$\frac{\partial x}{\partial u}(u, v) = (-f(v)\sin u, f(v)\cos u, 0) \quad \text{and} \quad \frac{\partial x}{\partial v}(u, v) = \left(f'(v)\cos u, f'(v)\sin u, 1\right).$$

The first vector has length $|f(v)|$, which is non-zero by assumption, so the two vectors are linearly independent. Hence, $\operatorname{rank}(d\mathbf{x}) = 2$.

Next, given $(x, y, z) \in S$, we can recover $v = z$. To recover u, we need an angle function $\theta(x, y)$ with values in $(0, 2\pi)$, but a single continuous choice forces us to remove a ray (a branch cut), e.g.

$$W = \mathbb{R}^2 \setminus \left\{(x, 0) \in \mathbb{R}^2 : x \geq 0\right\} \quad \text{where} \quad \theta : W \to (0, 2\pi).$$

Then, on the corresponding subset of S, one may write $\mathbf{x}^{-1}(x, y, z) = (\theta(x, y), z)$. To cover the missing ray, one introduces an additional chart.

Example 3.8 (MA3215 AY14/15 Sem 2 Tutorial 2). One way to define a system of coordinates for the sphere S^2, given by $x^2 + y^2 + (z-1)^2 = 1$ is to consider the so-called stereographic projection $\pi : S^2 \setminus \{N\} \to \mathbb{R}^2$ which carries a point $p = (x, y, z)$ of S^2 minus the north pole $N = (0, 0, 2)$ onto the intersection of the xy-plane with the straight line that connects N to p. Figure 3.6 depicts stereographic projection for the sphere $x^2+y^2+z^2=1$.

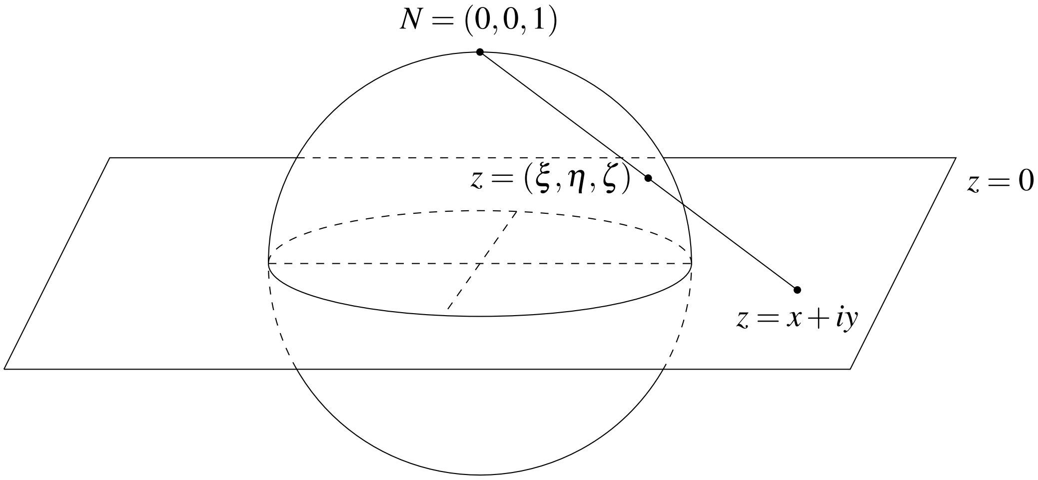

Figure 3.6: Stereographic projection

Let $(u,v)=\pi(x,y,z)$, where $(x,y,z)\in S^2\setminus\{N\}$ and (u,v) is contained in the xy-plane.

(i) Show that $\pi^{-1}:\mathbb{R}^2\to S^2\setminus\{N\}$ is given by

$$\pi^{-1}(u,v)=\frac{1}{u^2+v^2+4}\left(4u,4v,2\left(u^2+v^2\right)\right).$$

(ii) Let U denote the uv-plane. Using stereographic projection, define a coordinate function $\mathbf{x}:U\to S^2\setminus\{N\}$. There is no need to verify the 3 conditions (Definition 3.7) on $\mathbf{x}$.

(iii) Briefly explain how one can find a coordinate function $\mathbf{y}:U\to S^2\setminus\{S\}$, where S denotes the south pole. There is no need to give exact formulae[3].

Solution.

(i) Fix $(u,v)\in\mathbb{R}^2$. Consider the line passing through N and $(u,v,0)$, which is parametrised by $(tu,tv,2-2t)$. We want the point on this line that lies on S^2, so

$$(tu)^2+(tv)^2+(1-2t)^2=1.$$

The solution $t=0$ corresponds to the north pole, so we reject. The other solution is

$$t=\frac{4}{u^2+v^2+4}.$$

By using the fact that $(x,y,z)=(tu,tv,2-2t)$, the result follows.

(ii) We can use $\mathbf{x}(u,v)=\pi^{-1}(u,v)$.

[3]Having said that, see 3.26 for an answer.

(iii) Let $S = (0,0,0)$ denote the south pole. We perform the analogous construction but project from the south pole onto a plane that does not pass through the plane. Take for example $z = 2$, the plane tangent at N. For $p \in S^2 \setminus \{S\}$, draw the line through S and p and let it intersect the plane $z = 2$. The intersection point has coordinates $(u, v, 2)$. Then, identify the plane $z = 2$ with $\mathbb{R}^2$ by $(u, v, 2) \mapsto (u, v)$. This yields a stereographic projection $\overline{\pi} : S^2 \setminus \{S\} \to \mathbb{R}^2$. Then, $\mathbf{y}$ can be defined as $\mathbf{y} = \overline{\pi}^{-1}$. □

Example 3.9 (MA3215 AY14/15 Sem 2 Tutorial 3). Let S be the torus which has equation

$$\left(\sqrt{x^2+y^2}-2\right)^2 + z^2 = 1.$$

(i) Find an open subset V of $\mathbb{R}^3$ such that $\mathbf{x} : U \to S \cap V$ is a bijection and

$$\mathbf{x}(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u).$$

Here, $U = \left\{(u,v) \in \mathbb{R}^2 : 0 < u < 2\pi, 0 < v < 2\pi\right\}$ is a coordinate function for the surface. Show that $\mathbf{x}(u,v) \in S$.

(ii) Verify the three conditions in Definition 3.7 that $\mathbf{x}$ is a coordinate function.

Solution.

(i) It is clear that $\mathbf{x}(u,v) \in S$. We then find an open subset $V \subseteq \mathbb{R}^3$ such that $\mathbf{x} : U \to S \cap V$ is a bijection. The idea is to take V to be the image of $\mathbf{x}$, so V corresponds to the portion of the torus where $0 < u, v < 2\pi$.

(ii) We first verify **(i)** of Definition 3.7. This is clear because $\mathbf{x}$ is composed of polynomials and trigonometric functions.

To verify **(ii)**, note that $\mathbf{x}$ is continuous and injective on U since each $(u,v) \in (0,2\pi)^2$ gives a unique point on the torus without overlap, and its inverse (recovering (u,v) from (x,y,z)) is continuous on $S \cap V$. Hence, **(ii)** is satisfied. Verifying **(iii)** is trivial. □

3.3 The Inverse Function Theorem

As we would see in Theorem 3.1 eventually, the inverse function theorem is the precise statement that $\det(dF_p) \neq 0$ implies F is locally invertible near p with a smooth local inverse. In Differential Geometry, this is one of the key tools behind the local description of surfaces by coordinate charts.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, where

$$F(u_1, \ldots, u_n) = (F_1(u_1, \ldots, u_n), \ldots, F_m(u_1, \ldots, u_n)).$$

We define its differential dF to be the $m \times n$ matrix of first partial derivatives. That is,

$$dF = \begin{pmatrix} \frac{\partial F_1}{\partial u_1} & \cdots & \frac{\partial F_m}{\partial u_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial u_n} & \cdots & \frac{\partial F_m}{\partial u_n} \end{pmatrix}.$$

When $n = m$, dF is an $n \times n$ matrix and we define the Jacobian determinant

$$\det(dF) = \frac{\partial(F_1, \ldots, F_n)}{\partial(u_1, \ldots, u_n)}.$$

Definition 3.8 (critical point and regular point). Let $F : U \to \mathbb{R}^n$ be a smooth map, where $U \subseteq \mathbb{R}^n$ is open, and let $p \in U$.

(i) p is a critical point of F if dF_p is singular (i.e. not invertible)

(ii) p is a regular point of F if dF_p is invertible

Lemma 3.2. Let $F : U \to \mathbb{R}^n$ be a smooth map and $p \in U$. Then, the following are equivalent:

(i) p is a regular point of F

(ii) $\det(dF_p) \neq 0$

(iii) The $n \times n$ matrix dF_p is invertible

Proof. This is standard Linear Algebra from MA2001 — an $n \times n$ matrix is invertible if and only if its determinant is non-zero. □

Example 3.10 (the case $n = 1$). We perform a sanity check for the 1-dimensional case. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and write $y = f(u)$. Then,

$$df = \frac{\partial f}{\partial u} = \frac{df}{du} = f'(u).$$

So, a point $p \in \mathbb{R}$ is critical if $f'(p) = 0$, which is exactly the situation where f fails to be locally injective (e.g. local maxima/minima or other flattening behaviour). If instead p was a regular point, i.e. $f'(p) \neq 0$, then f is locally invertible near p. As such, one can solve $y = f(x)$ for x as a function of y in a neighbourhood).

From Example 3.10, we give a remark from MA2002 Calculus that even if $f'(p) \neq 0$ at some point p, this only guarantees local invertibility near p. It does not mean f is invertible on all of $\mathbb{R}$.

Example 3.11. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined $F(x,y) = (2x + y, x + 3y)$. Then,

$$dF_{(x,y)} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

which has non-zero determinant. By Lemma 3.2, any point is a regular point of F.

We now state the inverse function theorem (Theorem 3.1).

Theorem 3.1 (inverse function theorem). Let $U \subseteq \mathbb{R}^n$ be open and let $F : U \to \mathbb{R}^n$ be a smooth function. Suppose $p \in U$ is a regular point, i.e. $\det(dF_p) \neq 0$. Then, the following hold:

(i) There exists an open neighbourhood V of p in U and an open set $W \subseteq \mathbb{R}^n$ containing $F(p)$ such that $F : V \to W$ is a bijection

(ii) The inverse map $F^{-1} : W \to V$ exists and is smooth

The inverse function theorem (Theorem 3.1) is often summarised as follows. If $\det(dF_p) \neq 0$, then F is a local diffeomorphism[4] near p. In particular, near a regular point, F has a well-behaved coordinate change. We shall give some applications of Theorem 3.1 in due course, but first, we shall give a nice interpretation of regular surfaces. That is, a very efficient way to prove that many subsets $S \subseteq \mathbb{R}^3$ are regular surfaces is to realise them as level sets

$$S = f^{-1}(a) = \{p \in V : \ f(p) = a\}$$

of a smooth function $f : V \to \mathbb{R}$. If the gradient never vanishes on the level set, then S is a regular surface.

We now discuss regular points of a scalar field. Let $V \subseteq \mathbb{R}^3$ be open and let $f : V \to \mathbb{R}$ be smooth.

Definition 3.9. Let $p \in V$. Define

$$df_p = \left(\frac{\partial f}{\partial x} |_p, \frac{\partial f}{\partial y} |_p, \frac{\partial f}{\partial z} |_p \right) \in \mathbb{R}^3.$$

We say that p is a regular point of f if $df_p \neq 0$. In MA2104 Multivariable Calculus, df_p is the gradient vector and is denoted by $\operatorname{grad} f(p)$ or $\nabla f(p)$.

Proposition 3.2 (level sets as regular surfaces). Let $V \subseteq \mathbb{R}^3$ be open and let $f : V \to \mathbb{R}$ be smooth. Fix $a \in \mathbb{R}$ and consider the level set

$$S = f^{-1}(a) = \{p \in V : f(p) = a\}.$$

If every point of S is a regular point of f, i.e. $df_p \neq 0$ for all $p \in S$, then S is a regular surface in $\mathbb{R}^3$.

Here is a geometric remark with regards to Proposition 3.2 and MA2104 Multivariable Calculus. For a regular level set $S = f^{-1}(a)$, the gradient $\operatorname{grad} f(p)$ is always perpendicular to the surface at p.

Proof. Fix $p = (x_0, y_0, z_0) \in S$. Since $df_p \neq 0$, at least one partial derivative is non-zero. Without a loss of generality, assume that $\frac{\partial f}{\partial z} |_p \neq 0$. Then, define

$$F : V \to \mathbb{R}^3 \quad \text{where} \quad F(x, y, z) = (x, y, f(x, y, z)).$$

[4]Just to jump the gun, we will formally what a diffeomorphism is in Definition 5.1. In this context, a map $F : V \to W$ is a diffeomorphism if F is smooth, F is bijective so F^{-1} exists, and F^{-1} is smooth.

Then,

$$dF = \begin{pmatrix} 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \\ 0 & 0 & \frac{\partial f}{\partial z} \end{pmatrix}$$

so it implies that $\det(dF)$ evaluated at p is $\frac{\partial f}{\partial z}\mid_p \neq 0$. By the inverse function theorem (Theorem 3.1), there exist open sets $V_0 \subseteq V$ containing p and $W \subseteq \mathbb{R}^3$ containing $F(p) = (x_0, y_0, a)$ such that $F : V_0 \to W$ is a bijection with smooth inverse F^{-1}.

Write

$$F^{-1}(u,v,w) = (g_1(u,v,w), g_2(u,v,w), g_3(u,v,w)).$$

Since $F(x,y,z) = (x,y,f(x,y,z))$, one obtains

$$g_1(u,v,w) = u \quad \text{and} \quad g_2(u,v,w) = v \quad \text{and} \quad f(u,v,g_3(u,v,w)) = w.$$

Now, restrict to the slice $w = a$. Let

$$U = \{(u,v) : (u,v,a) \in W\} \subseteq \mathbb{R}^2,$$

and define the local parametrisation

$$\mathbf{x} : U \longrightarrow S \quad \text{where} \quad \mathbf{x}(u,v) = F^{-1}(u,v,a) = (u,v,g_3(u,v,a)).$$

Then, $\mathbf{x}(U) \subseteq S$ and $\mathbf{x}$ is smooth. Moreover, the rank condition holds (it is a genuine chart), so S is a regular surface near p. □

Example 3.12 (MA3215 AY14/15 Sem 2 Tutorial 4). Let $f : \mathbb{R}^3 \to \mathbb{R}$. Suppose $f(x,y,z) = 0$ defines a regular surface S. Let $\mathbf{x} : U \to S \cap V$ be a parametrisation and let $\mathbf{p} = \mathbf{x}(u_0, v_0)$ be a point on the surface. Show that at $\mathbf{p}$, ∇f is perpendicular to the tangent space.

Solution. We wish to show that ∇f is perpendicular to the surface spanning $\mathbf{x}_u$ and $\mathbf{x}_v$, which is the tangent space. That is, $\nabla f \cdot \mathbf{x}_u = 0$ and $\nabla f \cdot \mathbf{x}_v = 0$. For example, expanding the first dot product yields

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} = 0. \tag{3.3}$$

Since $\mathbf{x}(U) \subseteq S$, then $(f \circ \mathbf{x})(u,v) = 0$ for all $(u,v) \in U$. Differentiating with respect to u and evaluating at (u_0, v_0) yields

$$\frac{\partial}{\partial u}(f \circ \mathbf{x})(u_0, v_0) = 0.$$

By the chain rule,

$$f_x(\mathbf{x}(u_0,v_0))\, x_u(u_0,v_0) + f_y(\mathbf{x}(u_0,v_0))\, y_u(u_0,v_0) + f_z(\mathbf{x}(u_0,v_0))\, z_u(u_0,v_0) = 0.$$

This is equivalent to (3.3), so the result follows. □

Corollary 3.1 (local graph form). If $p \in S = f^{-1}(a)$ and $\frac{\partial f}{\partial z}|_p \neq 0$, then S can be locally parametrised near p by (x,y)-coordinates. That is, there exists a smooth function h such that, near p,

$$S = \{(x,y,h(x,y))\}.$$

Equivalently, a local coordinate function is

$$\mathbf{x}(u,v) = (u,v,h(u,v)).$$

Example 3.13 (the unit sphere S^2 via a level set). Let $f(x,y,z) = x^2+y^2+z^2$, $V = \mathbb{R}^3$, and $a = 1$. Then,

$$f^{-1}(1) = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\} = S^2.$$

Next, $df = (2x,2y,2z)$. On S^2, we cannot have $(x,y,z) = (0,0,0)$, so $df \neq 0$ everywhere on S^2. Hence, S^2 is a regular surface.

Example 3.14 (A hyperboloid level set). Let $f(x,y,z) = -x^2-y^2+z^2-1$, $V = \mathbb{R}^3$, and $a = 0$. Then,

$$f^{-1}(0) = \left\{(x,y,z) : z^2 = x^2+y^2+1\right\}.$$

Next, $df = (-2x,-2y,2z)$. If $df = (0,0,0)$, then $x = y = z = 0$, but $(0,0,0) \notin f^{-1}(0)$ since $f(0,0,0) = -1 \neq 0$. Hence, $df \neq 0$ on $f^{-1}(0)$, so $f^{-1}(0)$ is a regular surface.

Example 3.15 (MA3215 AY06/07 Sem 2). Prove that

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : x^5 + \sin y + z^3 = 2\right\}$$

is a regular surface.

Solution. Let $f(x,y,z) = x^5 + \sin y + z^3 - 2$. Then,

$$f^{-1}(0) = \left\{(x,y,z) : x^5 + \sin y + z^3 - 2 = 0\right\}.$$

Then, $df = \left(5x^4, \cos y, 3z^2\right)$. If $df = (0,0,0)$, then $x = z = 0$ and $\sin y = \pm 1$. However, we see that $f\left(0, \sin^{-1}(\pm 1), 0\right) \neq 2$, which shows that $df \neq 0$ on $f^{-1}(0)$. As such, $f^{-1}(0)$ is a regular surface. $\square$

3.4 Tangent Spaces

One of the central ideas in Differential Geometry is that, although a surface $S \subseteq \mathbb{R}^3$ may be curved globally, it can be well approximated near each point by a flat plane. This plane captures the first-order behaviour of the surface and serves as the geometric setting in which notions such as velocity, direction, and differentiation of curves on the surface are defined. The object that formalises this idea is the tangent space (Definition 3.11). To construct the tangent space rigorously, we make use of local parametrisations of the surface. A regular surface admits coordinate functions whose differentials have maximal rank, ensuring that locally the surface behaves like a smooth deformation of an open subset of $\mathbb{R}^2$. The differential of a parametrisation (Definition 3.10) encodes the velocity vectors of coordinate curves, and these vectors span a two-dimensional subspace of $\mathbb{R}^3$. This subspace is precisely the tangent space at the point.

Definition 3.10 (differential of a parametrisation). Let $S \subseteq \mathbb{R}^3$ be a regular surface, and let

$$\mathbf{x} : U \to S \cap V \quad \text{where} \quad \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$$

be a coordinate function. The differential at $p = x(u_0, v_0)$ is the 3×2 matrix

$$(d\mathbf{x})_{\mathbf{p}} = \begin{pmatrix} \frac{\partial}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}_{\mathbf{p}} .$$

Its columns are the velocity vectors $(\mathbf{x}_u)_{\mathbf{p}}$ and $(\mathbf{x}_v)_{\mathbf{p}}$.

Definition 3.11 (tangent space). The tangent space of S at $\mathbf{p} \in S$ is

$$T_{\mathbf{p}}S = \operatorname{span}\left\{(\mathbf{x}_u)_{\mathbf{p}}, (\mathbf{x}_v)_{\mathbf{p}}\right\}.$$

Since S is regular, $(d\mathbf{x})_{\mathbf{p}}$ has rank 2 by Definition 3.7, so $(\mathbf{x}_u)_{\mathbf{p}}$ and $(\mathbf{x}_v)_{\mathbf{p}}$ are linearly independent and $T_{\mathbf{p}}S$ is a 2-dimensional subspace of $\mathbb{R}^3$. Note that the cross product $(\mathbf{x}_u)_{\mathbf{p}} \wedge (\mathbf{x}_v)_{\mathbf{p}} \neq (0,0,0)$ is perpendicular to T_pS, hence gives a normal direction to the tangent plane. Equivalently, one may view it in terms of Jacobians. That is,

$$\mathbf{x}_u \wedge \mathbf{x}_v = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right) \quad \text{and} \quad \mathbf{x}_u \wedge \mathbf{x}_v \neq (0,0,0).$$

If we translate coordinates so that p is the origin, the tangent plane can be written as

$$\left\{(X,Y,Z) \in \mathbb{R}^3 : \left((\mathbf{x}_u)_p \times (\mathbf{x}_v)_p\right) \cdot (X,Y,Z) = 0\right\}.$$

The tangent plane is said to be a first-order approximation to the surface near p. If we choose a different coordinate function, the definition via column space gives the same plane, i.e. T_pS is intrinsic.

Proposition 3.3 (local graph criterion). Let S be a regular surface and $x(u,v)$ a coordinate function. Suppose at $p = \mathbf{x}(u_0, v_0)$ we have

$$\frac{\partial(x,y)}{\partial(u,v)}(u_0, v_0) \neq 0.$$

Then, there exists a neighbourhood W of p in S such that W is the graph of a smooth function $z = f(x,y)$.

The condition

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

in Proposition 3.3 geometrically means that the tangent plane at p (and nearby) is not perpendicular to the xy-plane.

Proof. Let $\pi : S \to \mathbb{R}^2$ denote the projection of S onto the xy-plane. Next, define

$$h = \pi \circ x \quad \text{where} \quad h(u,v) = (x(u,v), y(u,v)).$$

The hypothesis says that $\det(dh)_{(u_0,v_0)} \neq 0$, so by the inverse function theorem (Theorem 3.1), h has a local inverse h^{-1} near $(x(u_0,v_0), y(u_0,v_0))$. Write

$$h^{-1}(x,y) = (u(x,y), v(x,y)).$$

Then, define

$$\gamma(x,y) = x \circ h^{-1}(x,y) = (x, y, z(u(x,y), v(x,y))).$$

Hence, locally S is parametrised as $(x, y, f(x,y))$ with $f(x,y) = z(u(x,y), v(x,y))$, so the surface is a graph. □

Example 3.16 (sphere and the equator). On the unit sphere, at the equator it is not possible to write the surface locally as $z = f(x,y)$. The relevant Jacobian condition fails precisely on the equator so the local graph criterion (Proposition 3.3) does not apply there.

Away from the equator, we may write the upper and lower hemispheres as the following graphs:

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}$$

Example 3.17 (a surface that is globally a graph). Consider the parametrised surface

$$x(u,v) = (3u + 4v, 4u + 5v, \cos(uv)).$$

Compute

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} = -1 \neq 0,$$

so the graph criterion applies. Moreover, solving

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad \text{so} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5x + 4y \\ 4x - 3y \end{pmatrix},$$

we get

$$z = \cos(uv) = \cos((-5x + 4y)(4x - 3y)).$$

Hence the surface is the graph $z = \cos((-5x + 4y)(4x - 3y))$.

Example 3.18 (MA3215 AY14/15 Sem 2 Tutorial 4). Let S denote the surface of revolution by rotating the curve $x = z^2 + 1$ about the z-axis. Let

$$U = \left\{(u,v) \in \mathbb{R}^2 : 0 < u < 2\pi, v \in \mathbb{R}\right\}$$
$$V = \mathbb{R}^3 \setminus \left\{(x,0,z) \in \mathbb{R}^3 : x \geq 0\right\}$$

Note that

$$\mathbf{x} : U \to S \cap V \quad \text{where} \quad \mathbf{x}(u,v) = \left(\left(v^2+1\right)\cos u, \left(v^2+1\right)\sin u, v\right)$$

is a coordinate function. Let $\mathbf{x}_u$ and $\mathbf{x}_v$ denote the first and second columns of $d\mathbf{x}$.

(i) Calculate $\mathbf{x}_u$ and $\mathbf{x}_v$ at $\mathbf{p} = \mathbf{x}\left(\frac{\pi}{2}, 1\right) = (0,2,1)$ on S.

(ii) Show that the vector $(2,4,2)$ is a tangent vector at $\mathbf{p}$.

(iii) Find a curve $\alpha(t)$ on the surface which passes through $\mathbf{p}$ at $t = 0$ and $\alpha'(0) = (2,4,2)$.

(iv) Find a vector perpendicular to the tangent space at a point $\mathbf{q} = \mathbf{x}(u_0, v_0)$.

(v) Find the equation of the tangent plane.

(vi) Determine the points $\mathbf{x}(u,v)$ on S where $\frac{\partial(x,y)}{\partial(u,v)}$ is non-zero.

Solution.

(i) We have

$$\mathbf{x}_u = \left(-\left(v^2+1\right)\sin u, \left(v^2+1\right)\cos u, 0\right) \quad \text{and} \quad \mathbf{x}_v = (2v\cos u, 2v\sin u, 1)$$

so $\mathbf{x}_u\left(\frac{\pi}{2}, 1\right) = (-2,0,0)$ and $\mathbf{x}_v\left(\frac{\pi}{2}, 1\right) = (0,2,1)$.

(ii) Observe that $(2,4,2)$ is contained in the span of $(-2,0,0)$ an $(0,2,1)$.

(iii) $\alpha(t) = (2t, 4t, 2t) + (0,2,1)$

(iv) We have

$$\mathbf{x}_u \wedge \mathbf{x}_v = \left(v^2+1\right)(\cos u, \sin u, -2v)$$

so $\mathbf{n}(u_0, v_0) = (\cos u_0, \sin u_0, -2v_0)$, which is a vector perpendicular to the tangent space at $\mathbf{q}$.

(v) Note that the tangent plane passes through

$$\mathbf{q} = \left(\left(v_0^2+1\right)\cos u_0, \left(v_0^2+1\right)\sin u_0, \cos v_0\right).$$

Then, one can use the formula $\mathbf{r} \cdot \mathbf{n} = \mathbf{q} \cdot \mathbf{n}$ from high school to deduce the equation of the tangent plane.

(vi) The Jacobian determinant is

$$\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} -\left(v^2+1\right)\sin u & 2v\cos u \\ -\left(v^2+1\right)\cos u & 2v\sin u \end{pmatrix} = -2v\left(v^2+1\right)$$

For the determinant to be non-zero, we must have $v \neq 0$. $\square$

3.5 Change of Parameters

If $\mathbf{x} : U \to S$ is a coordinate function (chart) of a regular surface $S \subseteq \mathbb{R}^3$, and $y : Y \to \mathbb{R}^3$ is a smooth parametrised patch whose image lies inside $\mathbf{x}(U)$, then we can change parameters by passing from y-coordinates to x-coordinates via $\mathbf{x}^{-1} \circ y : Y \to U$. The key point is that this map is smooth.

Proposition 3.4 (change of parameters). Let S be a regular surface and let $\mathbf{x} : U \to S$ be a coordinate function. Let $Y \subseteq \mathbb{R}^n$ be open and let $y : Y \to \mathbb{R}^3$ be a smooth map, i.e.

$$y(\xi_1, \ldots, \xi_n) = (x(\xi_1, \ldots, \xi_n), y(\xi_1, \ldots, \xi_n), z(\xi_1, \ldots, \xi_n))$$

with smooth component functions. Assume that $y(Y) \subseteq \mathbf{x}(U)$. Then, the map $x^{-1} \circ y : Y \to U$ is smooth.

Proof. This is an application of the inverse function theorem (Theorem 3.1) to show that the local inverse x^{-1} behaves smoothly on the image[5]. □

Let $\mathbf{x} : U \to S$ and $\mathbf{y} : V \to S$ be two coordinate functions on the same regular surface S. Assume their images overlap. That is,

$$W = \mathbf{x}(U) \cap \mathbf{y}(V) \neq \emptyset \quad \text{where} \quad U' = \mathbf{x}^{-1}(W) \subseteq U \text{ and } V' = \mathbf{y}^{-1}(W) \subseteq V.$$

Then, we have a bijection (the change of coordinates)

$$h = \mathbf{x}^{-1} \circ \mathbf{y} : V' \to U'. \tag{3.4}$$

Proposition 3.5 (transition maps are diffeomorphisms). The bijection

$$h = \mathbf{x}^{-1} \circ \mathbf{y} : V' \to U'$$

in (3.4) is a diffeomorphism. Equivalently, both h and $h^{-1} = y^{-1} \circ x$ are smooth.

Proof. Smoothness of h follows from Proposition 3.4. Interchanging the roles of x and y yields the smoothness of h^{-1}. □

Example 3.19. Let $S^2 \subseteq \mathbb{R}^3$ denote the unit sphere and let

$$U = V = \left\{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\right\}.$$

We define two charts, namely the northern hemisphere

$$x(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$$

and the hemisphere pointing in the direction of the x-axis, which is

$$y(\xi, \eta) = \left(\sqrt{1 - \xi^2 - \eta^2}, \xi, \eta\right).$$

[5] We treat this as a technical tool; the detailed proof is omitted.

On the overlap $W = x(U) \cap y(V)$, the transition map is

$$h := x^{-1} \circ y : V' \to U' \quad \text{where} \quad h(\xi, \eta) = \left(\sqrt{1-\xi^2-\eta^2}, \xi\right),$$

and its inverse is

$$h^{-1}(u,v) = \left(v, \sqrt{1-u^2-v^2}\right).$$

Both are smooth on the appropriate restricted domains U', V'.

One can interpret h and h^{-1} as the precise gluing data that tells us how two flat parameter domains U' and V' fit together to form the surface. This viewpoint generalises to the abstract construction of manifolds by gluing open sets in $\mathbb{R}^n$.

We can define the tangent space T_pS intrinsically using smooth curves on the surface, and then show this definition agrees with the span of $\mathbf{x}_u$ and $\mathbf{x}_v$ from a coordinate chart. The punchline is that T_pS does not depend on which chart you use. Let S be a regular surface and let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a (short) smooth curve on S. We do not assume arc length parametrisation; we even allow $\alpha'(t) = 0$ for some t. Fix $p = \alpha(0) \in S$, and choose a chart $\mathbf{x} : U \to S$ such that the image $\mathbf{x}(U)$ contains the whole curve. Then by change of parameters (Proposition 3.4), the map

$$h(t) = (h_1(t), h_2(t)) = \left(\mathbf{x}^{-1} \circ \alpha\right)(t)$$

is smooth, and

$$\alpha(t) = \mathbf{x} \circ h(t) = \mathbf{x}(h_1(t), h_2(t)).$$

By the chain rule,

$$\alpha'(0) = \left.\frac{\partial x}{\partial u}\right|_{(0,0)} h_1'(0) + \left.\frac{\partial x}{\partial v}\right|_{(0,0)} h_2'(0).$$

Writing $a_1 = h_1'(0)$ and $a_2 := h_2'(0)$, we obtain the key formula

$$\alpha'(0) = a_1 \mathbf{x}_u(0,0) + a_2 \mathbf{x}_v(0,0),$$

so $\alpha'(0)$ lies in the tangent plane (column space of $d\mathbf{x}_p$). Conversely, every tangent plane vector comes from a curve. We formally describe this in Proposition 3.6.

Proposition 3.6. Let v be any vector in the tangent plane at $p = x(0,0)$, say

$$v = a_1 \mathbf{x}_u(0,0) + a_2 \mathbf{x}_v(0,0).$$

Then, there exists a smooth curve α on S with $\alpha(0) = p$ and $\alpha'(0) = v$.

Proof. Define $\alpha(t) = \mathbf{x}(a_1 t, a_2 t)$ be a parametrisation of a curve. Then, $\alpha(0) = \mathbf{x}(0,0) = p$, and differentiating yields $\alpha'(0) = v$. □

Theorem 3.2 (intrinsic description of the tangent plane). The tangent plane at $p \in S$ is equal to the union of all tangent vectors $\alpha'(0)$ where α ranges over smooth curves on S with $\alpha(0) = p$.

Corollary 3.2 (independence of coordinates). The tangent plane T_pS (equivalently, the column space of $d\mathbf{x}_p$) is independent of the choice of coordinate function $\mathbf{x}$.

Example 3.20 (prescribing a tangent vector on S^2). Pick a chart $\mathbf{x}(u,v)$ on S^2 and a point $p = \mathbf{x}(u_0, v_0)$. Given a vector $v = a_1\mathbf{x}_u(u_0,v_0) + a_2\mathbf{x}_v(u_0,v_0)$, a curve with $\alpha(0) = p$ and $\alpha'(0) = v$ is

$$\alpha(t) = \mathbf{x}(u_0 + a_1 t, v_0 + a_2 t).$$

3.6 Smooth Functions on Surfaces

We begin with a motivation on how to differentiate a function on a curved surface. Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $f : S \to \mathbb{R}$. Since S is curved, we do not differentiate f by moving in straight lines in $\mathbb{R}^3$. Instead, we differentiate along a parametrised patch $\mathbf{x}(u,v)$, i.e. we pull back f to a function on an open set in $\mathbb{R}^2$.

Definition 3.12 (smoothness via coordinate functions). Let $f : S \to \mathbb{R}$ and let $\mathbf{p} \in S$. Let $\mathbf{x} : U \to S$ be a coordinate function with $\mathbf{p} = \mathbf{x}(u_0, v_0) \in \mathbf{x}(U)$.

- **(i)** We say that f is smooth on $\mathbf{x}(U)$ if $f \circ \mathbf{x} : U \to \mathbb{R}$ is smooth in the usual MA2104 Multivariable Calculus sense
- **(ii)** We say that f is smooth at $\mathbf{p}$ if f is smooth on some open subset of S containing $\mathbf{p}$

With $p = \mathbf{x}(u_0, v_0)$, we define

$$\left.\frac{\partial f}{\partial u}\right|_p = \left.\frac{\partial (f \circ x)}{\partial u}\right|_{(u_0,v_0)} \quad \text{and} \quad \left.\frac{\partial^2 f}{\partial u \partial v}\right|_p = \left.\frac{\partial^2 (f \circ x)}{\partial u \partial v}\right|_{(u_0,v_0)}$$

and similarly for other mixed/second derivatives.

Example 3.21. Let $S = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$ and define $f(x,y,z) = xz^2$. We then parametrise the northern hemisphere by

$$\mathbf{x}(u,v) = \left(u,\ v,\ \sqrt{1 - u^2 - v^2}\right),$$

so that the north pole is $p = \mathbf{x}(0,0)$. Then,

$$(f \circ \mathbf{x})(u,v) = u\left(1 - u^2 - v^2\right),$$

so

$$\frac{\partial (f \circ \mathbf{x})}{\partial u}(u,v) = 1 - 3u^2 - v^2 \quad \text{and} \quad \frac{\partial^2 (f \circ \mathbf{x})}{\partial u^2}(u,v) = -6u.$$

In particular, at $(u,v) = (0,0)$, we have $\left.\frac{\partial f}{\partial u}\right|_p = 1$ and $\left.\frac{\partial^2 f}{\partial u^2}\right|_p = 0$. Since $(f \circ \mathbf{x})(u,v) = u\left(1 - u^2 - v^2\right)$ is a polynomial (hence smooth) on the open unit disk in the uv-plane, it follows from the definition that f is smooth on the northern hemisphere (i.e. on $\mathbf{x}(U)$).

Example 3.22 (MA3215 AY14/15 Sem 2 Tutorial 4). Let $f(x,y,z) = xz^2$ be a function on the unit sphere S centered at the origin. Let $\mathbf{x} : \mathbb{R}^2 \to S \setminus \{N\}$ be the coordinate function coming from the stereographic projection in Example 3.8. That is,

$$\mathbf{x}(u,v) = \frac{1}{u^2+v^2+4}\left(4u, 4v, u^2+v^2-4\right).$$

(i) Compute $(f \circ \mathbf{x})(u,v)$. Does this function have higher derivatives with respect to u and v?

(ii) Compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

(iii) Using **(i)** or otherwise, show that f is smooth at every point on $S \setminus \{N\}$.

Solution.

(i) We have

$$\begin{aligned}(f \circ \mathbf{x})(u,v) &= f\left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4}\right) \\ &= \frac{4u}{u^2+v^2+4}\left(\frac{u^2+v^2-4}{u^2+v^2+4}\right)^2 \\ &= \frac{4u\left(u^2+v^2-4\right)^2}{\left(u^2+v^2+4\right)^3}\end{aligned}$$

This is a rational function of u and v, whose denominator $\left(u^2+v^2+4\right)^3$ never vanishes on $\mathbb{R}^2$. As such, $f \circ \mathbf{x}$ is infinitely differentiable on $\mathbb{R}^2$, and it hence has derivatives of all orders with respect to u and v.

(ii) This is trivial. For example,

$$\frac{\partial f}{\partial u}(u,v) = \frac{\partial}{\partial u}(f \circ \mathbf{x})(u,v).$$

(iii) We see that $f \circ \mathbf{x}$ is smooth on $S \setminus \{N\}$, where $N = (0,0,1)$ denotes the north pole. By **(i)** of Definition 3.12, it follows that f is smooth at every point on $S \setminus \{N\}$. □

Note that partial derivatives depend on the coordinate function. That is to say, if $y : V \to S$ is another coordinate function around p, then in general

$$\left.\frac{\partial(f \circ x)}{\partial u}\right|_{(u_0,v_0)} \neq \left.\frac{\partial(f \circ y)}{\partial \xi}\right|_{(\xi_0,\eta_0)}.$$

So the numerical value of $\partial f / \partial u$ depends on the chosen parameters.

Also, smoothness at a point is chart-independent. Although derivatives depend on the chart, the property of being smooth at p does not. That is, if $x : U \to S$ and $y : V \to S$ are coordinate functions with $p = x(u_0,v_0) = y(\xi_0,\eta_0)$, let $h = x^{-1} \circ y$ be defined near (ξ_0,η_0) (a change of parameters map). Then,

$$(f \circ y)(\xi,\eta) = (f \circ x \circ h)(\xi,\eta).$$

Since h is smooth (change of parameters) and $f \circ x$ is smooth, the composition $f \circ x \circ h$ is smooth. Hence, $f \circ y$ is smooth. Therefore, the definition of f being smooth at p is independent of the coordinate function.

Example 3.23. Let S^2 denote the unit sphere and let $\mathbf{x}$ and $\mathbf{y}$ as in Example 3.26. Let $f(x,y,z) = xz^2$ be a function on S^2. One can show that

$$(f \circ \mathbf{x})(u,v) = \frac{4u\left(u^2+v^2-4\right)^2}{\left(u^2+v^2+4\right)^3} \quad \text{and} \quad (f \circ \mathbf{x})(\xi,\eta) = \frac{4\xi\left(4-\xi^2-\eta^2\right)^2}{\left(\xi^2+\eta^2+4\right)^3}. \tag{3.5}$$

In a similar fashion to Example 3.4, one can prove that f is a smooth function on the sphere by composing it with the six coordinate functions. Using the previous discussion, give an alternative proof that f is smooth.

Solution. We use **(i)** of Definition 3.12, so it suffices to prove that $f \circ \mathbf{x}$ is smooth. Observe that each of the quotients in (3.5) has a non-vanishing denominator on $\mathbb{R}^2$, so $f \circ \mathbf{x}$ is a smooth function. Hence, f is smooth. $\square$

Example 3.24 (MA3215 AY14/15 Sem 2 Tutorial 5). Let S be the surface in $\mathbb{R}^3$ given by $z^2 = 1 + x^2 + y^2$. It is a well-known fact that this is a regular surface.

(i) Show that S does not intersect the xy-plane.

(ii) We define a function $f(x,y,z) = \frac{x^2+y^2+z^2}{z^3}$ for (x,y,z) on the surface S. Show that f is a smooth function on S.

Solution.

(i) Suppose $z = 0$. Then, noting that $1 + x^2 + y^2 \geq 1$ for all $(x,y) \in \mathbb{R}^2$, S does not intersect the xy-plane.

(ii) We parametrise using

$$\mathbf{x}(r,\theta) = \left(r\cos\theta, r\sin\theta, \pm\sqrt{1+r^2}\right) \quad \text{where } r \geq 0 \text{ and } 0 \leq \theta \leq 2\pi.$$

Then,

$$f\left(r\cos\theta, r\sin\theta, \pm\sqrt{1-r^2}\right) = \pm\frac{1+2r^2}{\left(1+r^2\right)^{3/2}}.$$

This expression depends smoothly on r and is independent of θ. So, it is infinitely differentiable on the parameter domain. By Definition 3.12, it follows that f is a smooth function on S. $\square$

Example 3.25 (checking smoothness on the whole sphere). For $f(x,y,z) = xz^2$ on the unit sphere, we can cover the sphere by several coordinate patches (northern hemisphere, southern hemisphere, and additional hemispheres to cover the equator). On each patch, $f \circ x$ is a smooth function of (u,v), hence f is smooth on the entire sphere.

Example 3.26 (MA3215 AY14/15 Sem 2 Tutorial 5). We recall the stereographic projection of a unit sphere S^2 as in Example 3.8. We will shift everything downwards by a unit length so that the centre of the ball is now at the origin. Let N and S denote the north pole and the south pole respectively. Then, we have seen in Example 3.8 that the map

$$\mathbf{x}(u,v) = \pi^{-1}(u,v) = \frac{1}{u^2+v^2+4}\left(4u, 4v, u^2+v^2-4\right)$$

is a parametrisation of $\mathbf{x} : \mathbb{R}^2 \to S^2 \setminus \{N\}$. We also have a parametrisation $\mathbf{y} : \mathbb{R}^2 \to S^2 \setminus \{S\}$ given by

$$\mathbf{y}(\xi,\eta) = \frac{1}{\xi^2+\eta^2+4}\left(4\xi, 4\eta, 4-\xi^2-\eta^2\right).$$

Note that $\mathbf{x}^{-1}(x,y,z) = \frac{2}{1-z}(x,y)$ and $\mathbf{y}^{-1}(x,y,z) = \frac{2}{1+z}(x,y)$.

(i) Describe the overlap W of the two parametrisations on the sphere S^2.

(ii) Describe $\mathbf{x}^{-1}(W)$ and $\mathbf{y}^{-1}(W)$.

(iii) Compute $h : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$, where $(u,v) = h(\xi,\eta) = \left(\mathbf{x}^{-1} \circ \mathbf{y}\right)(\xi,\eta)$.

(iv) Explain using **(iii)** that $h(\xi,\eta)$ is a smooth function on $\mathbf{y}^{-1}(W)$. Note that $(0,0) \notin \mathbf{y}^{-1}(W)$.

Solution.

(i) The overlap W is $S^2 \setminus \{N,S\}$.

(ii) For $\mathbf{x}^{-1}(W)$, let

$$x = \frac{4u}{u^2+v^2+4} \quad \text{and} \quad y = \frac{4v}{u^2+v^2+4} \quad \text{and} \quad z = \frac{u^2+v^2-4}{u^2+v^2+4}.$$

Consider $(x,y,z) = (0,0,-1)$, so $u = 0$ and $v = 0$. Indeed, $z = -1$, so we exclude the south pole. One can check that $z = 1$ is impossible, so we exclude the north pole as well. As such, $\mathbf{x}^{-1}(W) = \mathbb{R}^2 \setminus \{(0,0)\}$.

Next, for $\mathbf{y}^{-1}(W)$, let

$$x = \frac{4\xi}{\xi^2+\eta^2+4} \quad \text{and} \quad y = \frac{4\eta}{\xi^2+\eta^2+4} \quad \text{and} \quad z = \frac{4-\xi^2-\eta^2}{\xi^2+\eta^2+4}.$$

Consider $(x,y,z) = (0,0,-1)$, so $\xi = \eta = 0$. One can deduce that $\mathbf{y}^{-1}(W) = \mathbb{R}^2 \setminus \{(0,0)\}$.

(iii) We have

$$\begin{aligned}
h(\xi,\eta) &= \mathbf{x}^{-1}(\mathbf{y}(\xi,\eta)) \\
&= \mathbf{x}^{-1}\left(\frac{4\xi}{\xi^2+\eta^2+4}, \frac{4\eta}{\xi^2+\eta^2+4}, \frac{4-\xi^2-\eta^2}{\xi^2+\eta^2+4}\right) \\
&= \left(\frac{4\xi}{\xi^2+\eta^2+4}, \frac{4\eta}{\xi^2+\eta^2+4}\right)
\end{aligned}$$

(iv) Each component of h is a quotient of polynomials, for which the denominator never vanishes on $\mathbb{R}^2 \setminus \{(0,0)\}$. $\square$

Proposition 3.7 (restriction of a smooth $\mathbb{R}^3$ function is smooth on S). Let $V \subseteq \mathbb{R}^3$ be open and let $S \subseteq V$ be a regular surface. If $F : V \to \mathbb{R}$ is smooth, then its restriction $f := F|_S : S \to \mathbb{R}$ is a smooth function on S.

Proof. For any coordinate function $x : U \to S$, we have

$$f \circ x = (F|_S) \circ x = F \circ x$$

which is smooth since F and x are smooth. □

In many situations, one proves $f : S \to \mathbb{R}$ is smooth by writing $f = F|_S$ where F is a smooth function on an open set $V \subseteq \mathbb{R}^3$ containing S. If F is only defined/smooth away from a *bad set* like an axis, one chooses V to avoid that set; then f is smooth by Proposition 3.7.

Example 3.27. Let

$$F(x,y,z) = \left(x^2+y^2+z^2\right)\sin x \quad \text{on } \mathbb{R}^3.$$

On the unit sphere, we have $x^2+y^2+z^2 = 1$, hence the restriction of F to S is $f(x,y,z) = \sin x$. This illustrates that restriction means having the same formula but a smaller domain.

Theorem 3.3 (extension theorem). Conversely, every smooth function on S is the restriction of some smooth function defined on an open set in $\mathbb{R}^3$ containing S.

The proof of Theorem 3.3 is complicated and it uses a technique known as partition of unity.

3.7 Smooth Functions between Surfaces

A map $\phi : S_1 \to S_2$ between regular surfaces is declared smooth if, after choosing local coordinates $\mathbf{x}_1 : U_1 \to S_1$ and $\mathbf{x}_2 : U_2 \to S_2$, the coordinate expression

$$\mathbf{x}_2^{-1} \circ \phi \circ \mathbf{x}_1 : U_1 \to U_2 \tag{3.6}$$

is a smooth map in the usual sense. Here, $U_i \subseteq \mathbb{R}^2$ is an open set in $\mathbb{R}^2$.

$$\begin{tikzcd}
U_1 \arrow[r, "\mathbf{x}_2^{-1}\circ\phi\circ\mathbf{x}_1"] \arrow[d, "\mathbf{x}_1"'] & U_2 \arrow[d, "\mathbf{x}_2"] \\
S_1 \arrow[r, "\phi"'] & S_2
\end{tikzcd}$$

Next, let S_1, S_2 be regular surfaces and let $\phi : S_1 \to S_2$ be continuous. Fix $p_1 \in S_1$ and set $\mathbf{p}_2 = \phi(\mathbf{p}_1) \in S_2$. Choose coordinate functions

$$x_1 : U_1 \to S_1 \text{ with } \mathbf{p}_1 \in \mathbf{x}_1(U_1) \quad \text{and} \quad \mathbf{x}_2 : U_2 \to S_2 \text{ with } \mathbf{p}_2 \in \mathbf{x}_2(U_2).$$

By shrinking U_1 if necessary, we may assume $\phi(\mathbf{x}_1(U_1)) \subseteq \mathbf{x}_2(U_2)$ so the coordinate expression (3.6) is well-defined.

Definition 3.13 (smoothness). Let $\phi : S_1 \to S_2$ be as in our above discussion.

(i) We say that ϕ is smooth on $x_1(U_1)$ if

$$x_2^{-1} \circ \phi \circ x_1 : U_1 \to U_2$$

is smooth in the sense of Calculus

(ii) We say that ϕ is smooth at p_1 if ϕ is smooth on some open subset of S_1 containing p_1

(iii) We say that ϕ is a smooth map if it is smooth at every point of S_1

Example 3.28 (antipodal map is smooth). Let $S_1 = S_2 = S^2$ and define

$$\phi(x, y, z) = (-x, -y, -z).$$

Pick p_1 on the northern hemisphere so that $\phi(p_1)$ lies on the southern hemisphere. Use the standard charts

$$x_1(u_1, v_1) = \left(u_1, v_1, \sqrt{1 - u_1^2 - v_1^2}\right) \quad \text{and} \quad x_2(u_2, v_2) = \left(u_2, v_2, -\sqrt{1 - u_2^2 - v_2^2}\right).$$

Then the coordinate expression becomes

$$\left(x_2^{-1} \circ \phi \circ x_1\right)(u_1, v_1) = (-u_1, -v_1),$$

which is clearly smooth on the open unit disk. Hence, ϕ is smooth on the northern hemisphere. To finish the global statement, one must also check the southern hemisphere and the equator using additional charts.

The definition of smoothness in Definition 3.13 looks like it depends on the chosen coordinate functions x_1 and x_2, but it actually does not: replacing charts does not change whether ϕ is smooth.

Lemma 3.3 (smoothness is independent of charts). If ϕ is smooth at p_1 with respect to one choice of coordinate functions x_1 at p_1 and x_2 at p_2, then it is smooth at p_1 for any other choice of coordinate functions around these points.

Proof. Let y_1 be another coordinate function around p_1. Set the transition map $h = x_1^{-1} \circ y_1$ so that $y_1 = x_1 \circ h$. Then locally,

$$x_2^{-1} \circ \phi \circ y_1 = x_2^{-1} \circ \phi \circ x_1 \circ h.$$

The right side is a composition of smooth maps (by hypothesis $x_2^{-1} \circ \phi \circ x_1$ is smooth, and h is smooth by change of parameters), hence $x_2^{-1} \circ \phi \circ y_1$ is smooth. This shows independence of x_1 and similarly for x_2). □

Example 3.29. Let S^2 denote the unit sphere and $\mathbf{x}$ and $\mathbf{x},\mathbf{y},W$ as in Example 3.26.

(i) Compute $d\mathbf{x}$ and $d\mathbf{y}$.

(ii) Let $\mathbf{p}=(x,y,z)=\mathbf{x}(1,1)=\mathbf{y}(2,2)$ be a point on the overlap W. Show that the column spaces of $d\mathbf{x}$ and $d\mathbf{y}$ at this point are both equal to the span of $(1,0,2)$ and $(0,1,2)$.

(iii) Show that the tangent space at $\mathbf{p}$ is independent of the coordinate functions $\mathbf{x}$ and $\mathbf{y}$ used.

Solution.

(i) Recall that

$$\mathbf{x}(u,v)=\frac{1}{u^2+v^2+4}\left(4u,4v,u^2+v^2-4\right).$$

One can look up Example 3.26 to find the expression for $\mathbf{y}$. Let $R=u^2+v^2+4$ and $T=\xi^2+\eta^2+4$. Then,

$$d\mathbf{x}=4R^{-2}\begin{pmatrix} v^2-u^2+4 & -2uv \\ -2uv & u^2-v^2+4 \\ 4u & 4v \end{pmatrix}$$

$$d\mathbf{y}=4T^{-2}\begin{pmatrix} \eta^2-\xi^2+4 & -2\xi\eta \\ -2\xi\eta & \xi^2-\eta^2+4 \\ -4\xi & -4\eta \end{pmatrix}$$

(ii) Let $u=v=1$ and $\xi=\eta=2$. Then, one can show that the column spaces of $d\mathbf{x}$ and $d\mathbf{y}$ at this point are both equal to the span of $(1,0,2)$ and $(0,1,2)$.

(iii) If $\mathbf{x}:U\to S^2$ is a coordinate map with $\mathbf{x}(u_0,v_0)=\mathbf{p}$, then the tangent plane at $\mathbf{p}$ is the span of $\mathbf{x}_u$ and $\mathbf{x}_v$ at (u_0,v_0). Repeat the same argument for $\mathbf{y}$. From **(ii)**, we know that the column spaces of the differentials $d\mathbf{x}$ and $d\mathbf{y}$ at $\mathbf{p}$ are equal to the span of $(1,0,2)$ and $(0,1,2)$. So, both coordinate maps produce the same tangent space at $\mathbf{p}$, and we conclude that $T_{\mathbf{p}}\left(S^2\right)$ is independent of the choice of coordinate function. □

Proposition 3.8 (restriction of a smooth map in $\mathbb{R}^3$). Let $V_1,V_2\subseteq\mathbb{R}^3$ be open and let $\Phi:V_1\to V_2$ be smooth. Let $S_1\subseteq V_1$ and $S_2\subseteq V_2$ be regular surfaces such that $\Phi(S_1)\subseteq S_2$. Define

$$\phi:S_1\to S_2 \quad \text{where} \quad \phi=\Phi|_{S_1}.$$

Then, ϕ is smooth.

$$\begin{tikzcd} S_1 \arrow[r, "\phi=\Phi|_{S_1}"] \arrow[d, hook, "i_1"'] & S_2 \arrow[d, hook, "i_2"] \\ V_1 \arrow[r, "\Phi"'] & V_2 \end{tikzcd}$$

Proof. Fix $p_1 \in S_1$ and $p_2 = \phi(p_1) \in S_2$. Near p_2, we may assume S_2 is locally a graph and choose coordinates so that

$$x_2^{-1}(x,y,z) = (x,y).$$

Write $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. Then for a chart x_1 on S_1, we have

$$\left(x_2^{-1} \circ \phi \circ x_1\right)(u_1, v_1) = \left(\Phi_1 \circ x_1(u_1, v_1),\ \Phi_2 \circ x_1(u_1, v_1)\right),$$

which is smooth because Φ_1, Φ_2 and x_1 are smooth. $\square$

Example 3.30. We are now in position to give a short proof that the antipodal map in Example 3.28 is smooth. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be $\Phi(x,y,z) = (-x,-y,-z)$. Then, Φ is smooth, and $\Phi\left(S^2\right) \subseteq S^2$. Hence $\phi = \Phi\,|_{S^2}$ is smooth by Proposition 3.8.

Example 3.31 (rotation on a tube). Let S be the tube obtained by rotating the line $x = 1$ in the xz-plane about the z-axis. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the z-axis by 30°. So,

$$\Phi(x,y,z) = \left(x\cos 30^\circ + y\sin 30^\circ,\ -x\sin 30^\circ + y\cos 30^\circ,\ z\right).$$

Then, Φ is smooth and $\Phi(S) \subseteq S$, so $\phi = \Phi\,|_S : S \to S$ is smooth by Example 3.8.

Lemma 3.4 (composition of smooth maps is smooth). If $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ are smooth maps between regular surfaces, then $\psi \circ \phi : S_1 \to S_3$ is smooth.

3.8 Maps between Tangent Planes

A smooth map between surfaces induces a linear map between tangent planes. This is the main bridge from Differential Geometry to Linear Algebra. Let $\phi : S_1 \to S_2$ be a smooth map between regular surfaces. Fix $p_1 \in S_1$ and set $p_2 = \phi(p_1) \in S_2$. Let $p_2 = \phi(p_1) \in S_2$. Say $v_1 \in T_{p_1}S_1$ is a tangent vector, so there exists a curve

$$\alpha : (-\varepsilon, \varepsilon) \to S_1 \quad \text{such that} \quad \alpha(0) = p_1 \text{ and } \alpha'(0) = v_1.$$

Define a curve on S_2, say $\beta(t) := \phi(\alpha(t))$. Then, $\beta(0) = p_2$ and $\beta'(0) \in T_{p_2}S_2$.

Definition 3.14 (differential). We define the map

$$d\phi_{p_1} : T_{p_1}S_1 \to T_{p_2}S_2 \quad \text{where} \quad d\phi_{p_1}(v_1) = \beta'(0) = (\phi \circ \alpha)'(0).$$

We must check that this is well-defined, i.e. independent of the curve α used to represent v_1.

Definition 3.15 (coordinate functions and the induced map Φ). Let $x_1 : U_1 \to S_1$ be a coordinate function covering p_1 and let $x_2 : U_2 \to S_2$ be a coordinate function covering p_2. Shrink U_1 so that $\phi(x_1(U_1)) \subseteq x_2(U_2)$. Define the induced map between parameter domains

$$\Phi : U_1 \to U_2 \quad \text{where} \quad \Phi = x_2^{-1} \circ \phi \circ x_1.$$

Equivalently, $\phi \circ x_1 = x_2 \circ \Phi$. Write

$$\Phi(u_1, v_1) = (\Phi_1(u_1, v_1), \Phi_2(u_1, v_1)).$$

We give a formula for $d\phi_{p_1}$ in coordinates. Let $p_1 = x_1(u_1, v_1)$. Any tangent vector $v_1 \in T_{p_1}S_1$ can be written as

$$v_1 = a(x_{1,u})_{p_1} + b(x_{1,v})_{p_1} = (dx_1)_{p_1}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Then,

$$d\phi_{p_1}(v_1) = (dx_2)_{p_2}(d\Phi)_{(u_1,v_1)}\begin{pmatrix} a \\ b \end{pmatrix}.$$

In particular, $d\phi_{p_1}$ is a linear transformation. Note that this map is well-defined because the right side depends only on the coefficients (a, b) of v_1 in the basis $\left\{(x_{1,u})_{p_1}, (x_{1,v})_{p_1}\right\}$, and on the Jacobian matrix $(d\Phi)_{(u_1,v_1)}$. Hence, different curves α representing the same v_1 produce the same $d\phi_{p_1}(v_1)$.

We give a matrix representation of the above. Let

$$(d\Phi)_{(u_1,v_1)} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Let

$$\mathcal{B}_1 = \left\{(x_{1,u})_{p_1}, (x_{1,v})_{p_1}\right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{(x_{2,u})_{p_2}, (x_{2,v})_{p_2}\right\}.$$

Then,

$$\begin{aligned} d\phi_{p_1}\left((x_{1,u})_{p_1}\right) &= a_{11}(x_{2,u})_{p_2} + a_{21}(x_{2,v})_{p_2} \\ d\phi_{p_1}\left((x_{1,v})_{p_1}\right) &= a_{12}(x_{2,u})_{p_2} + a_{22}(x_{2,v})_{p_2} \end{aligned}$$

Equivalently,

$$[d\phi_{p_1}]_{\mathcal{B}_2,\mathcal{B}_1} = (d\Phi)_{(u_1,v_1)}.$$

Example 3.32. Let $S_1 = S_2 = S^2$ be the unit sphere, and define $\phi(x, y, z) = (-y, x, z)$ to be a 90° anticlockwise rotation about the z-axis. Use the northern hemisphere coordinate functions

$$x_1(u_1, v_1) = \left(u_1, v_1, \sqrt{1 - u_1^2 - v_1^2}\right) \quad \text{and} \quad x_2(u_2, v_2) = \left(u_2, v_2, \sqrt{1 - u_2^2 - v_2^2}\right).$$

Then the induced map is

$$\Phi(u_1, v_1) = (-v_1, u_1) \quad \text{where} \quad d\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence for $v_1 = a\left(x_{1,u}\right)_{p_1} + b\left(x_{1,v}\right)_{p_1}$, we obtain

$$d\phi_{p_1}\left(v_1\right) = \left(dx_2\right)_{p_2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -b\left(x_{2,u}\right)_{p_2} + a\left(x_{2,v}\right)_{p_2}.$$

3.9 The First Fundamental Form

We first give the big picture. The first fundamental form is the object that assigns lengths (and hence angles and distances) on a surface by restricting the Euclidean inner product of $\mathbb{R}^3$ to tangent vectors on the surface. Recall that $\mathbb{R}^3$ has the standard inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2 + u_3v_3$. This allows us to measure lengths by considering $|\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

Let S be a regular surface and let $p \in S$. Choose a coordinate function $\mathbf{x} : U \to S$ covering p, with $p = \mathbf{x}\left(u_0, v_0\right)$. Then, every tangent vector $\mathbf{v} \in T_pS$ can be written as

$$\mathbf{v} = a\mathbf{x}_u\left(p\right) + b\mathbf{x}_v\left(p\right) \quad \text{for some } a, b \in \mathbb{R}.$$

Definition 3.16 (first fundamental form). For $\mathbf{v} \in T_pS$, define $I_p\left(v\right) = \langle \mathbf{v}, \mathbf{v} \rangle$, known as the first fundamental form of S at $\mathbf{p}$.

We then define the functions (evaluated at $\left(u_0, v_0\right)$), known as the E, F, G coefficients. These are namely

$$\begin{aligned}
E\left(u_0, v_0\right) &= \langle \mathbf{x}_u\left(p\right), \mathbf{x}_u\left(p\right) \rangle \\
F\left(u_0, v_0\right) &= \langle \mathbf{x}_u\left(p\right), \mathbf{x}_v\left(p\right) \rangle \\
G\left(u_0, v_0\right) &= \langle \mathbf{x}_v\left(p\right), \mathbf{x}_v\left(p\right) \rangle
\end{aligned}$$

Then, for $v = a\mathbf{x}_u\left(p\right) + b\mathbf{x}_v\left(p\right)$, we have

$$I_p\left(\mathbf{v}\right) = \langle \mathbf{v}, \mathbf{v} \rangle = a^2E\left(u_0, v_0\right) + 2abF\left(u_0, v_0\right) + b^2G\left(u_0, v_0\right).$$

E, F, G record how the coordinate directions $\mathbf{x}_u$ and $\mathbf{x}_v$ sit in $\mathbb{R}^3$. That is, $E = |\mathbf{x}_u|^2$ and $G = |\mathbf{x}_v|^2$, and $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$ measures the failure of orthogonality.

Example 3.33 (plane with orthonormal coordinates). Let S be a plane and choose an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ of the plane. Take the parametrisation

$$\mathbf{x}\left(u, v\right) = p_0 + u\mathbf{w}_1 + v\mathbf{w}_2.$$

Then, $\mathbf{x}_u = \mathbf{w}_1$ and $\mathbf{x}_v = \mathbf{w}_2$, so $E = 1$, $F = 0$, and $G = 1$. As such, $I_p\left(a\mathbf{x}_u + b\mathbf{x}_v\right) = a^2 + b^2$.

Example 3.34 (tube). Let

$$\mathbf{x}\left(u, v\right) = \left(\cos u, \sin u, v\right) \quad \text{where } 0 < u < 2\pi, v \in \mathbb{R}.$$

Then, $\mathbf{x}_u = \left(-\sin u, \cos u, 0\right)$ and $\mathbf{x}_v = \left(0, 0, 1\right)$. As such, $E = 1$, $F = 0$ and $G = 1$. As such, $I_p\left(a\mathbf{x}_u + b\mathbf{x}_v\right) = a^2 + b^2$.

Example 3.35 (do Carmo p. 233 Question 16). Let $\mathbf{x} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, where

$$U = \left\{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < \pi, 0 < \varphi < 2\pi\right\},$$

and

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

be a parametrisation of the unit sphere S^2. Let $\log\tan\frac{1}{2}\theta = u$ and $\varphi = v$ and show that a new parametrisation of the coordinate neighbourhood $\mathbf{x}(U) = V$ can be given by

$$\mathbf{y}(u, v) = (\operatorname{sech} u\cos v, \operatorname{sech} u\sin v, \tanh u).$$

Prove that in the parametrisation $\mathbf{y}$, the coefficients of the first fundamental form are $E = G = \operatorname{sech}^2 u$ and $F = 0$. Thus, $\mathbf{y}^{-1} : V \subseteq S^2 \to \mathbb{R}^2$ is a conformal map which takes the meridians and parallels of S^2 into straight lines of the plane. This is called Mercator's projection.

Solution. The parametrisation is easy to obtain. Starting from $\log\tan\frac{1}{2}\theta$, one can obtain

$$\tan\theta = \frac{2e^u}{1-u^2}.$$

We omit the remaining details. Next, we compute the first fundamental form. We have

$$\mathbf{y}_u(u, v) = \left(-\operatorname{sech} u\tanh u\cos v, -\operatorname{sech} u\tanh u\sin v, \operatorname{sech}^2 u\right)$$
$$\mathbf{y}_v(u, v) = (-\operatorname{sech} u\sin v, \operatorname{sech} u\cos v, 0)$$

and $\mathbf{N}(u, v) = -\mathbf{y}(u, v)$. Recall that $E = \langle\mathbf{y}_u, \mathbf{y}_u\rangle$, $F = \langle\mathbf{y}_u, \mathbf{y}_v\rangle$ and $E = \langle\mathbf{y}_v, \mathbf{y}_v\rangle$, and the result follows. □

We now discuss the arc length on a surface using the first fundamental form.

Proposition 3.9 (arc length using first fundamental form). Let $\alpha(t)$ be a smooth curve on S. Since $\alpha'(t) \in T_{\alpha(t)}S$, its speed is

$$|\alpha'(t)| = \sqrt{I_{\alpha(t)}(\alpha'(t))}.$$

Hence, the arc length from 0 to t is

$$s(t) = \int_0^t |\alpha'(\tau)|\, d\tau = \int_0^t \sqrt{I_{\alpha(\tau)}(\alpha'(\tau))}\, d\tau.$$

Now, suppose $\alpha(t)$ is written in local coordinates as $\alpha(t) = \mathbf{x}(u(t), v(t))$. Then,

$$\alpha'(t) = \mathbf{x}_u u'(t) + \mathbf{x}_v v'(t),$$

so the first fundamental form gives

$$I(\alpha'(t)) = E(u, v)\left(u'(t)\right)^2 + 2F(u, v)u'(t)v'(t) + G(u, v)\left(v'(t)\right)^2.$$

Therefore,

$$s(t)=\int_0^t\sqrt{E(u,v)\left(\frac{du}{d\tau}\right)^2+2F(u,v)\left(\frac{du}{d\tau}\right)\left(\frac{dv}{d\tau}\right)+G(u,v)\left(\frac{dv}{d\tau}\right)^2}\,d\tau.$$

Formally, we write the metric expression

$$ds^2=Edu^2+2Fdudv+Gdv^2.$$

At this point, du and dv are treated as symbols encoding the quadratic form determined by I. They become genuinely meaningful once we develop the differential-geometric interpretation of the metric.

Let θ be the angle between $\mathbf{x}_u$ and $\mathbf{x}_v$. Then, we can write

$$\cos\theta=\frac{\langle\mathbf{x}_u,\mathbf{x}_v\rangle}{|\mathbf{x}_u|\,|\mathbf{x}_v|}=\frac{F}{\sqrt{E}\sqrt{G}}.$$

We say that x is an orthogonal parametrisation if $\mathbf{x}_u\perp\mathbf{x}_v$ at all points, i.e. $\cos\theta=0$ if and only if $F=0$ on U.

We then explain how the first fundamental form computes the area of a surface patch. Let $S\subseteq\mathbb{R}^3$ be a regular surface and let $\mathbf{x}:U\to S$ be a coordinate function. Consider the parametrised patch $\mathbf{x}(U)\subseteq S$. Suppose $(u,v)\in U$ and take a small rectangle

$$[u,u+\Delta u]\times[v,v+\Delta v]\subseteq U.$$

Its image on the surface is approximately a parallelogram spanned by the vectors $\mathbf{x}_u(u,v)\Delta u$ and $\mathbf{x}_v(u,v)\Delta v$. Hence, the side lengths satisfy

$$\Delta a=\|\mathbf{x}(u+\Delta u,v)-x(u,v)\|\approx\|\mathbf{x}_u(u,v)\|\,\Delta u\quad\text{and similarly}\quad\Delta b\approx\|\mathbf{x}_v(u,v)\|\,\Delta v,$$

and the (approximate) area is

$$\Delta A=\Delta a\Delta b\,|\sin\theta|\approx\|\mathbf{x}_u(u,v)\wedge\mathbf{x}_v(u,v)\|\,\Delta u\Delta v.$$

Then, the area of the parametrised patch $\mathbf{x}(U)$ is

$$\lim_{\Delta u,\Delta v\to 0}\sum_U\|\mathbf{x}_u\wedge\mathbf{x}_v\|\,\Delta u\Delta v=\iint_U\|\mathbf{x}_u\wedge\mathbf{x}_v\|\;dudv. \tag{3.7}$$

Proposition 3.10 (area formula). Let $\mathbf{x}:U\to S$ be a coordinate function, and set $E=\langle\mathbf{x}_u,\mathbf{x}_u\rangle$, $F=\langle\mathbf{x}_u,\mathbf{x}_v\rangle$, and $G=\langle\mathbf{x}_v,\mathbf{x}_v\rangle$. Then,

$$\operatorname{Area}(\mathbf{x}(U))=\iint_U\|\mathbf{x}_u\wedge\mathbf{x}_v\|\;dudv=\iint_U\sqrt{EG-F^2}\,dudv.$$

Proof. Recall the vector identity $\|\mathbf{a}\wedge\mathbf{b}\|^2=\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2-\langle\mathbf{a},\mathbf{b}\rangle^2$. Applying this with $\mathbf{a}=\mathbf{x}_u$ and $\mathbf{b}=\mathbf{x}_v$ gives

$$\|\mathbf{x}_u\wedge\mathbf{x}_v\|^2=\|\mathbf{x}_u\|^2\,\|\mathbf{x}_v\|^2-\langle\mathbf{x}_u,\mathbf{x}_v\rangle^2=EG-F^2.$$

Since $\|\mathbf{x}_u\wedge\mathbf{x}_v\|\geq 0$, we have $\|\mathbf{x}_u\wedge\mathbf{x}_v\|=\sqrt{EG-F^2}$ and substituting into (3.7) yields the result. $\square$

The key point is that the area formula[6] in Proposition 3.10 depends only on E, F, G, hence depends only on the first fundamental form. So, we do not need to know the lengths of all vectors in $\mathbb{R}^3$ to compute area on a surface; knowledge of lengths of tangent vectors is sufficient. This viewpoint generalizes to Riemannian geometry.

Example 3.36 (area of torus). Let $a > r$ and define

$$\mathbf{x}(u,v) = ((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u) \quad \text{where } 0 < u, v < 2\pi.$$

This parametrises a torus. We then compute the E, F, G coefficients. First,

$$\mathbf{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$
$$\mathbf{x}_v = (-(a + r\cos u)\sin v, (a + r\cos u)\cos v, 0)$$

so $E = r^2$, $F = 0$, and $G = (a + r\cos u)^2$. Hence,

$$\sqrt{EG - F^2} = r(a + r\cos u) = r(r\cos u + a)$$

Therefore, by Proposition 3.10, the area is

$$A = \int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\, dudv = \int_0^{2\pi}\int_0^{2\pi} r(r\cos u + a)\ dudv = 4\pi^2 ra.$$

3.10 The Chain Rule

Whenever we compose smooth maps

$$U \xrightarrow{F} V \xrightarrow{G} W,$$

the differential of the composite $H = G \circ F$ is given by matrix multiplication. That is, $dH = dGdF$. This is the multivariable chain rule written in the language of Jacobian matrices.

Now, let $U \subseteq \mathbb{R}^k$, $V \subseteq \mathbb{R}^m$, $W \subseteq \mathbb{R}^n$ be open sets. Let $F : U \to V$ be smooth, and write

$$F(x_1, \ldots, x_k) = (F_1(\mathbf{x}), \ldots, F_m(\mathbf{x})) \quad \text{where } \mathbf{x} = (x_1, \ldots, x_k).$$

Definition 3.17 (differential/Jacobian matrix). The differential of F is the $m \times k$ matrix

$$dF = (F_{ij})_{1 \le i \le m, 1 \le j \le k} \quad \text{where} \quad F_{ij} = \frac{\partial F_i}{\partial x_j}.$$

It is also called the Jacobian matrix of F.

If $k = m$, then dF is a square matrix and we define the Jacobian determinant

$$\det(dF) = \frac{\partial(F_1, \ldots, F_m)}{\partial(x_1, \ldots, x_m)}.$$

[6]In Do Carmo's text [1], the integration is sometimes carried out by approximating an open parameter set by closed sets with boundary because Riemann integration is typically presented for such domains. Using Lebesgue integration, one can integrate over open subsets directly.

Similarly, let $G : V \to W$ be smooth, written as

$$G(y_1, \ldots, y_m) = (G_1(\mathbf{y}), \ldots, G_n(\mathbf{y})) \quad \text{where } \mathbf{y} = (y_1, \ldots, y_m),$$

with differential $dG = (G_{i\ell})$, where $G_{i\ell} = \frac{\partial G_i}{\partial y_\ell}$.

Proposition 3.11 (chain rule). Define the composite map

$$H = G \circ F : U \to W \quad \text{where } H(\mathbf{x}) = (H_1(\mathbf{x}), \ldots, H_n(\mathbf{x})).$$

We have $dH = dGdF$, where the right side is the product of the $n \times m$ matrix dG with the $m \times k$ matrix dF.

Proof. It suffices to check entries. Fix $1 \le i \le n$ and $1 \le j \le k$. Since $H_i = G_i \circ F$, the usual MA2104 Multivariable Calculus chain rule gives

$$\frac{\partial H_i}{\partial x_j} = \sum_{\ell=1}^{m} \frac{\partial G_i}{\partial y_\ell}\bigg|_{y=F(x)} \cdot \frac{\partial F_\ell}{\partial x_j}.$$

Using matrices, the (i, j)-entry of dH equals to the (i, j)-entry of $dGdF$. $\square$

Note that if $k = m = n$, then dF, dG, dH are all $m \times m$ matrices. Hence,

$$\det(dH) = \det(dGdF) = \det(dG)\det(dF).$$

In particular, if $G = F^{-1}$, then $dG = (dF)^{-1}$.

Example 3.37. Let $F(t)$ be a smooth curve on the uv-plane, say

$$F(t) = (F_1(t), F_2(t)).$$

Let $X : U \to S$ be a coordinate function of a surface S, written as

$$X(u, v) = (X(u, v), Y(u, v), Z(u, v)).$$

Define

$$H = X \circ F \quad \text{where} \quad H(t) = (H_1(t), H_2(t), H_3(t)).$$

Then, $H(t)$ is a curve on the surface S, and by the chain rule (Proposition 3.11), we have

$$H'(t) = dX_{F(t)} F'(t).$$

Writing this out, we have

$$H'(t) = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix}_{(u,v)=F(t)} \begin{pmatrix} F_1'(t) \\ F_2'(t) \end{pmatrix}.$$

To interpret this example, the tangent vector $H'(t)$ of the curve on the surface is obtained by pushing forward the tangent vector $F'(t)$ in parameter space via the Jacobian matrix dX.

Example 3.38 (MA3215 AY14/15 Sem 2 Tutorial 4). Let $V_1 = V_2 = V_3 = \mathbb{R}^2$, and let $F : V_1 \to V_2$ and $G : V_2 \to V_3$ be two functions. Let $H = G \circ F : V_1 \to V_3$. In order to avoid confusion, we will denote V_1 and V_2 as the x_1x_2-plane and the y_1y_2-plane respectively.

Write

$$\begin{aligned} F(x_1,x_2) &= (F_1(x_1,x_2), F_2(x_1,x_2)) \\ G(y_1,y_2) &= (G_1(y_1,y_2), G_2(y_1,y_2)) \\ H(x_1,x_2) &= (H_1(x_1,x_2), H_2(x_1,x_2)) \\ &= (G_1(F_1(x_1,x_2), F_2(x_1,x_2)), G_2(F_1(x_1,x_2), F_2(x_1,x_2))) \end{aligned}$$

Let

$$dF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} \quad dG = \begin{pmatrix} \frac{\partial G_1}{\partial y_1} & \frac{\partial G_1}{\partial y_2} \\ \frac{\partial G_2}{\partial y_1} & \frac{\partial G_2}{\partial y_2} \end{pmatrix} \quad dH = \begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{pmatrix}.$$

We will denote their determinants (call the Jacobians) by

$$\frac{\partial(F_1,F_2)}{\partial(x_1,x_2)} \quad \frac{\partial(G_1,G_2)}{\partial(y_1,y_2)} \quad \frac{\partial(H_1,H_2)}{\partial(x_1,x_2)}.$$

(i) Show that $dH = d(G \circ F)$ is equal to the product of the two matrices $dG \cdot dF$.

(ii) Show that

$$\frac{\partial(H_1,H_2)}{\partial(x_1,x_2)} = \frac{\partial(G_1,G_2)}{\partial(y_1,y_2)} \frac{\partial(F_1,F_2)}{\partial(x_1,x_2)}.$$

Solution.

(i) One checks that

$$dG \cdot dF = \begin{pmatrix} \frac{\partial G_1}{\partial y_1}\frac{\partial F_1}{\partial x_1} + \frac{\partial G_1}{\partial y_2}\frac{\partial F_2}{\partial x_1} & \frac{\partial G_1}{\partial y_1}\frac{\partial F_1}{\partial x_2} + \frac{\partial G_1}{\partial y_2}\frac{\partial F_2}{\partial x_2} \\ \frac{\partial G_2}{\partial y_1}\frac{\partial F_1}{\partial x_1} + \frac{\partial G_2}{\partial y_2}\frac{\partial F_2}{\partial x_1} & \frac{\partial G_2}{\partial y_1}\frac{\partial F_1}{\partial x_2} + \frac{\partial G_2}{\partial y_2}\frac{\partial F_2}{\partial x_2} \end{pmatrix}.$$

One just needs to verify that

$$\frac{\partial H_1}{\partial x_1} = \frac{\partial G_1}{\partial y_1}\frac{\partial F_1}{\partial x_1} + \frac{\partial G_1}{\partial y_2}\frac{\partial F_2}{\partial x_1}$$

which is true by the chain rule.

(ii) Some crazy computations. □

CHAPTER 4

Curvature of Surfaces

4.1 The Gauss Map

In Definition 2.7, we defined the curvature of a curve. For a surface, curvature depends on a direction. Fix a point p on a regular surface S and choose a unit normal vector $\mathbf{N}(p)$. Given a unit tangent direction $\mathbf{v} \in T_pS$, slice the surface by the plane spanned by $\mathbf{v}$ and $\mathbf{N}(p)$. This produces a plane curve on S called a normal section. The curvature of this normal section at p is the normal curvature[1] of S at p in the direction $\mathbf{v}$.

As we vary the tangent direction $\mathbf{v}$, the normal curvature changes. Gauss's key observations [4] were the following:

(i) There exists a direction where the normal curvature is maximal, and a perpendicular direction where it is minimal

(ii) These extremal values are the principal curvatures $k_1(p)$ and $k_2(p)$ The Gauss curvature at p is the product

$$K(p) = k_1(p)k_2(p).$$

Heuristically, $K(p) > 0$ means sphere-like, $K(p) = 0$ means cylinder-like, and $K(p) < 0$ means saddle-like.

Definition 4.1 (Gauss map). Let S be a regular surface. A Gauss map is a smooth map $N : S \to S^2$ such that for each $q \in S$, the vector $N(q)$ is a unit vector perpendicular to the tangent plane T_qS.

Definition 4.2 (orientability). Locally, if $\mathbf{x} : U \to S$ is a parametrisation, one can

[1] Here is a sign convention. The normal curvature is positive if the normal section bends to the same side of the chosen normal vector $N(p)$. Naturally, the normal curvature is negative if the normal section bends to the opposite side of the chosen normal vector $N(p)$.

define a unit normal by

$$\mathbf{N}(\mathbf{x}(u,v)) = \pm \frac{\mathbf{x}_u(u,v) \wedge \mathbf{x}_v(u,v)}{|\mathbf{x}_u(u,v) \wedge \mathbf{x}_v(u,v)|}. \tag{4.1}$$

However, globally we may fail to choose the sign continuously. If a global continuous choice exists, we say S is orientable. If not, S is non-orientable.

Recall from MA2104 Multivariable Calculus that an example of a non-orientable surface is the Möbius strip.

Example 4.1 (plane). If S is the plane $ax+by+cz=0$, a unit normal vector is

$$\mathbf{n} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}.$$

Then, $N(q) = \mathbf{n}$ for all $q \in S$, so the image of N is a single point on S^2.

Example 4.2 (unit sphere). For $S = S^2$, at $p = (x,y,z) \in S^2$, the outward unit normal equals p itself, so the Gauss map is

$$N : S^2 \to S^2 \quad \text{where } N(p) = p,$$

i.e. the identity map.

Example 4.3 (hyperbolic paraboloid). Let S be given by $f(x,y,z) = z - y^2 + x^2 = 0$ and parametrise by $\mathbf{x}(u,v) = \left(u,v,v^2-u^2\right)$. Since $\nabla f = (2x,-2y,1)$ is perpendicular to S, we obtain the Gauss map (unit normal)

$$N(\mathbf{x}(u,v)) = \frac{(2u,-2v,1)}{\sqrt{4u^2+4v^2+1}}.$$

Example 4.4 (MA3215 AY14/15 Sem 2 Tutorial 7). Describe the region of the unit sphere S^2 covered by the image of the Gauss map of the following surfaces:

(a) paraboloid of revolution $z = x^2+y^2$

(b) hyperboloid of revolution $x^2+y^2-z^2=1$

(c) catenoid $x^2+y^2 = \cosh^2 z$

Solution.

(a) Let a parametrisation of the surface S be $\mathbf{x}(u,v) = \left(u,v,u^2+v^2\right)$. Then, $\mathbf{x}_u = (1,0,2u)$ and $\mathbf{x}_v = (0,1,2v)$. By (4.1),

$$\mathbf{N}(u,v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(-2u,-2v,1)}{\sqrt{1+4u^2+4v^2}}.$$

The z-component of $\mathbf{N}$ is

$$\frac{1}{\sqrt{1+4u^2+4v^2}}.$$

As u and v tend to 0, the z-component tends to 1; as u and v tend to infinity, the z-component tends to 0. Since the x- and y-components are unaffected, we see that the image of the Gauss map is the northern hemisphere of the unit sphere (excluding the equator).

(b) Let a parametrisation of the top half of the hyperboloid's surface S be

$$\mathbf{x}(u,v) = \left(u, v, \sqrt{u^2+v^2-1}\right).$$

Then,

$$\mathbf{x}_u = \left(1, 0, \frac{u}{\sqrt{u^2+v^2-1}}\right) \quad \text{and} \quad \mathbf{x}_v = \left(0, 1, \frac{v}{\sqrt{u^2+v^2-1}}\right).$$

As such,

$$\mathbf{N}(u,v) = \sqrt{\frac{u^2+v^2-1}{2u^2+2v^2-1}}\left(-\frac{u}{\sqrt{u^2+v^2-1}}, -\frac{v}{\sqrt{u^2+v^2-1}}, 1\right).$$

Let $r = u^2+v^2$. Then, the z-component of $\mathbf{N}$ is

$$\sqrt{\frac{r^2-1}{2r^2-1}}.$$

Note that $r > 1$, and as $r \to \infty$, the z-component tends to $\frac{1}{\sqrt{2}}$. Hence, for this parametrisation, the image of the Gauss map is the band $0 < z < \frac{1}{\sqrt{2}}$. One can then consider a parametrisation of the bottom half of the hyperboloid's surface S and by combining both cases, we see that the overall image of the Gauss map is the band $|z| < \frac{1}{\sqrt{2}}$.

(c) Let a parametrisation of the top half of the surface S be

$$\mathbf{x}(u,v) = \left(u, v, \cosh^{-1}\left(\sqrt{u^2+v^2}\right)\right).$$

Then,

$$\mathbf{N}(u,v) = \left(-\frac{u}{u^2+v^2}, -\frac{v}{u^2+v^2}, \frac{\sqrt{u^2+v^2-1}}{\sqrt{u^2+v^2}}\right).$$

Let $r^2 = u^2+v^2$. Then, the z-component of $\mathbf{N}$ is $\frac{\sqrt{r^2-1}}{r}$. As r tends to infinity, the z-component of $\mathbf{N}$ tends to 1, and this means that the image of the Gauss map excludes the north pole. Then, by considering a parametrisation of the bottom half of the surface S, we see that the image of the overall Gauss map is the unit sphere excluding the north and south poles. □

4.2 The Second Fundamental Form

Fix $q \in S$. The differential $dN_q : T_qS \to T_{N(q)}S^2$ lands in the tangent plane of the sphere at $N(q)$. However, both planes T_qS and $T_{N(q)}S^2$ pass through the origin and are perpendicular to $N(q)$, hence they coincide as the same 2-plane in $\mathbb{R}^3$. Therefore, we may view

$$dN_q : T_qS \to T_qS$$

as a linear transformation of a 2-dimensional inner product space.

Proposition 4.1. For each $q \in S$, the linear map $dN_q : T_qS \to T_qS$ is self-adjoint with respect to the induced inner product on T_qS.

We now introduce the second fundamental form (Definition 4.3). Previously in Definition 3.16, we encountered the first fundamental form, which measures lengths and angles on the surface. The second fundamental form is the basic tool that measures how a surface bends in $\mathbb{R}^3$, i.e. extrinsic curvature.

Definition 4.3 (second fundamental form). For $\mathbf{v} \in T_qS$, define the quadratic form

$$\mathrm{II}_q(\mathbf{v}) = -\left\langle dN_q(\mathbf{v}), \mathbf{v} \right\rangle.$$

This is called the second fundamental form of S at q.

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthonormal basis of T_qS and write

$$-\left[dN_q\right]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}.$$

Then, the coefficients are determined by the quadratic form II_q via

$$a = \mathrm{II}_q(\mathbf{v}_1) \quad \text{and} \quad d = \mathrm{II}_q(\mathbf{v}_2) \quad \text{and} \quad b = \frac{1}{2}\left(\mathrm{II}_q(\mathbf{v}_1 + \mathbf{v}_2) - \mathrm{II}_q(\mathbf{v}_1) - \mathrm{II}_q(\mathbf{v}_2)\right).$$

Hence, knowing $\mathrm{II}_q(\cdot)$ for all tangent vectors is equivalent to knowing the linear map dN_q.

4.3 Normal Curvature

Definition 4.4 (normal curvature). Let $\alpha(s)$ be a curve on a regular surface S, parametrised by arc length, and suppose $p = \alpha(0) \in S$. Recall from Definition 2.7 that

$$\alpha''(0) = k(0)\,\mathbf{n}(0),$$

where $k(0)$ is the curvature of the space curve α and $\mathbf{n}(0)$ is its principal normal. Fix a unit normal $\mathbf{N}$ along S. Define

$$k_n(0) = \left\langle \alpha''(0), \mathbf{N}(p) \right\rangle = k(0)\left\langle \mathbf{n}(0), \mathbf{N}(p) \right\rangle = k(0)\cos\theta,$$

where θ is the angle between $n(0)$ and $N(p)$. The number $k_{\mathbf{n}}(0)$ is called the normal curvature of α on S at p.

Proposition 4.2. Let $\alpha(s)$ be a curve on S parametrised by arc length with $p = \alpha(0)$. Then,

$$\mathrm{II}_p\left(\alpha'(0)\right) = k_{\mathbf{n}}(0).$$

In particular, the normal curvature depends only on the tangent vector $\mathbf{v} = \alpha'(0) \in$

T_pS, and not on the choice of curve with that tangent direction.

Proof. Since $\alpha'(s) \in T_{\alpha(s)}S$, we have

$$\langle \mathbf{N}(\alpha(s)), \alpha'(s) \rangle = 0.$$

Differentiate with respect to s and use $\alpha''(s) = k(s)n(s)$ to obtain

$$0 = \langle d\mathbf{N}(\alpha'(s)), \alpha'(s) \rangle + \langle \mathbf{N}(\alpha(s)), \alpha''(s) \rangle = -\mathrm{II}_{\alpha(s)}(\alpha'(s)) + k(s)\langle \mathbf{N}, \mathbf{n} \rangle.$$

Evaluating at $s = 0$ gives $\mathrm{II}_p(\alpha'(0)) = k(0)\langle \mathbf{N}, \mathbf{n} \rangle = k_{\mathbf{n}}(0)$. □

Given a unit tangent vector $\mathbf{v} \in T_pS$, consider the plane spanned by $\mathbf{v}$ and $N(p)$. Its intersection with the surface (for s small) is a curve $\alpha(s)$ called the normal section of S at p along v. For this curve, the principal normal $\mathbf{n}(0)$ is parallel to $N(p)$, so $\theta = 0$ or π and hence $|k_{\mathbf{n}}(0)| = k(0)$. So, among all curves through p with tangent $\mathbf{v}$, the normal section is the one whose curvature equals $|k_n|$.

Example 4.5 (unit sphere). On S^2, the normal sections are great circles (radius 1), hence curvature 1. With the outward normal, one gets constant normal curvature (sign depending on convention) for every unit tangent vector $\mathbf{v}$.

We shall justify this. Note that a parametrisation of S^2 is

$$\mathbf{x}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) \quad \text{where } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \pi.$$

One can then compute

$$\mathbf{N}(\mathbf{x}(\theta, \phi)) = \frac{\mathbf{x}_\theta \wedge \mathbf{x}_\phi}{|\mathbf{x}_\theta \wedge \mathbf{x}_\phi|} = -\sin\phi\, \mathbf{x}(\theta, \phi).$$

Let $\alpha(t)$ be a curve on the surface of the sphere. Note that $k_{\mathbf{n}}(t) = k(t)\langle \mathbf{n}, \mathbf{N} \rangle$ and $\alpha''(t) = k(t)\mathbf{n}(t)$. So, $k_{\mathbf{n}}(t) = \langle \alpha''(t), k(t)\mathbf{N}(t) \rangle$. Suppose α is of unit speed and $\mathbf{N} = \alpha$. Then, $k_{\mathbf{n}}(t) = \langle \alpha'', \alpha \rangle = -1$.

Example 4.6 (MA3215 AY14/15 Sem 2 Tutorial 7). Let S be the surface with a parametrisation given by

$$\mathbf{x}(u, v) = \left(u, v, u^2 + v^2\right)$$

and let $\alpha(t)$ be a curve on S defined by $\alpha(t) = \mathbf{x}\left(t^2, t\right) = \left(t^2, t, t^4 + t^2\right)$.

(i) Compute the Frenet trihedron and the curvature $k(t)$ of $\alpha(t)$. *This part is tedious though.*

(ii) Compute

$$\mathbf{N}(\mathbf{x}(u, v)) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}.$$

(iii) Find the normal curvature $k_{\mathbf{n}}(t) = k(t)\langle \mathbf{n}, \mathbf{N} \rangle$ of $\alpha(t)$.

Solution.

(i) We leave it to the reader to check the following for $\mathbf{t},\mathbf{n},\mathbf{b}$:

$$\mathbf{t} = \frac{\left(2t,1,4t^3+2t\right)}{\sqrt{16t^6+16t^4+8t^2+1}}$$
$$\mathbf{n} = \frac{\left(-32t^6-16t^4+1,-24t^5-16t^3-4t,16t^4+6t^2+1\right)}{\sqrt{64t^6+36t^4+12t^2+2}\sqrt{16t^6+16t^4+8t^2+1}}$$
$$\mathbf{b} = \frac{\left(6t^2+1,-8t^3,-1\right)}{\sqrt{64t^6+36t^4+12t^2+2}}$$

By Definition 2.7, the curvature is $k(t) = |\alpha''(t)|$. Note that

(ii) We have $\mathbf{x}_u = (1,0,2u)$ and $\mathbf{x}_v = (0,1,2v)$. Then,

$$\mathbf{N}(\mathbf{x}(u,v)) = \frac{(-2u,-2v,1)}{\sqrt{1+4u^2+4v^2}}.$$

(iii) By (2.2),

$$k(t) = \frac{|\alpha'(t)\times\alpha''(t)|}{|\alpha'(t)|^3} = \frac{2\sqrt{64t^6+36t^4+12t^2+2}}{\left(16t^6+16t^4+8t^2+1\right)^{3/2}}.$$

By **(i)** and **(ii)**, we have

$$\langle \mathbf{n},\mathbf{N}\rangle = \frac{2t\left(32t^6+16t^4-1\right)+2t\left(24t^5+16t^3+4t\right)+\left(16t^4+6t^2+1\right)}{\sqrt{64t^6+36t^4+12t^2+2}\sqrt{16t^6+16t^4+8t^2+1}\sqrt{1+8t^2}}.$$

Then, use the formula $k_{\mathbf{n}}(t) = k(t)\langle \mathbf{n},\mathbf{N}\rangle$ that was mentioned in the question to conclude the proof. □

Definition 4.5 (principal directions and principal curvatures). At each $p\in S$, the map $dN_p : T_pS \to T_pS$ is self-adjoint, hence $-dN_p$ is self-adjoint as well. By the spectral theorem, there exists an orthonormal basis $\mathcal{B} = \{v_1,v_2\}$ of T_pS such that

$$(-dN_p)(v_1) = k_1v_1 \quad \text{and} \quad (-dN_p)(v_2) = k_2v_2.$$

In this basis,

$$-[dN_p]_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

The directions v_1 and v_2 are the principal directions at p, and the eigenvalues $k_1(p), k_2(p)$ are the principal curvatures.

Example 4.7 (MA3215 AY14/15 Sem 2 Tutorial 8). Let S be the surface $z = 1+x^2+y^2$. It is a well-known fact that S is a regular surface with coordinate function $\mathbf{x}(u,v) = \left(u,v,1+u^2+v^2\right)$. The Gauss map $N : S\to S^2$ is given by

$$(N\circ\mathbf{x})(u,v) = \frac{(-2u,-2v,1)}{R} \quad \text{and} \quad [-dN]_{\mathcal{B},\mathcal{B}} = \frac{2}{R^3}\begin{pmatrix} 4v^2+1 & -4uv \\ -4uv & 4u^2+1 \end{pmatrix},$$

where $R = \sqrt{1+4u^2+4v^2}$ and $\mathcal{B} = \{\mathbf{x}_u,\mathbf{x}_v\}$.

(i) Find the principal curvatures k_1 and k_2, and the principal directions when $u = 0$ or $v = 0$.

(ii) Now suppose u and v are non-zero. Diagonalise the symmetric matrix $[-dN]$. Find the principal curvatures k_1 and k_2, and the principal directions.

Solution.

(i) When $u = 0$, we obtain

$$[-dN]_{\mathcal{B},\mathcal{B}} = \frac{2}{R^3}\begin{pmatrix} 4v^2+1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, the principal curvature is $2R^{-3}\left(4v^2+1\right)$ with principal directions are given by $\mathbf{x}_u$ and $\mathbf{x}_v$. We have a similar claim for the $v = 0$ case.

(ii) The eigenvalues of the matrix $[-dN]_{\mathcal{B},\mathcal{B}}$ are $2R^{-3}$ and $2R^{-3}\left(1+4u^2+4v^2\right)$. So, the principal curvatures are $k_1 = 2R^{-3}$ and $k_2 = 2R^{-1}$. Obtaining the principal directions is trivial. □

Proposition 4.3 (extremal normal curvatures occur in principal directions). Let $v \in T_pS$ be a unit vector. Write $v = av_1 + bv_2$ with $a^2 + b^2 = 1$. Then,

$$\mathrm{II}_p(v) = k_1 a^2 + k_2 b^2.$$

Hence, the maximal and minimal values of $\mathrm{II}_p(v)$ (equivalently k_n) occur at $v = v_1$ and $v = v_2$ respectively, and these directions are perpendicular.

Definition 4.6 (Gauss curvature and mean curvature). Let k_1, k_2 be the eigenvalues of $-dN_p$. Then, define the Gauss curvature $K(p)$ and the mean curvature $H(p)$ as follows:

$$K(p) = k_1(p)\,k_2(p) \quad \text{and} \quad H(p) = \frac{1}{2}\left(k_1(p) + k_2(p)\right)$$

Equivalently,

$$K(p) = \det\left(dN_p\right) \quad \text{and} \quad H(p) = \frac{1}{2}\operatorname{tr}\left(-dN_p\right),$$

which are independent of basis.

Recall from Definition 4.6 that K denotes Gauss curvature. We give a geometric meaning of K. If $K(p) > 0$, then the surface is locally elliptic at p. That is to say, in a neighbourhood of p, it lies on one side of the tangent plane, and the normal curvature has the same sign in every tangent direction, so it is sphere-like. If $K(p) < 0$, then the surface is locally hyperbolic at p. That is, it crosses its tangent plane and the normal curvature takes both signs depending on direction, so it is saddle-like. If $K(p) = 0$, then the surface is locally parabolic (or flat in at least one direction). That is, at least one principal curvature vanishes, so the surface is cylinder-like (or planar, if both principal curvatures vanish).

4.4 The Gauss Map in Local Coordinates

Let $x : U \to S$ be a local parametrisation, and define

$$\mathbf{M}(u,v) = (\mathbf{N} \circ \mathbf{x})(u,v) = \mathbf{N}(\mathbf{x}(u,v)).$$

We want formulae for the matrix of $d\mathbf{N}$ (equivalently $d\mathbf{M}$) in terms of the first and second fundamental forms, so that we can compute K, H, and k_1, k_2 in coordinates. Write

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \quad \text{and} \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \quad \text{and} \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle. \tag{4.2}$$

Also, let $D = EG - F^2$. Let N be the chosen unit normal and define

$$e = \langle \mathbf{N}, \mathbf{x}_{uu} \rangle \quad \text{and} \quad f = \langle \mathbf{N}, \mathbf{x}_{uv} \rangle \quad \text{and} \quad g = \langle \mathbf{N}, \mathbf{x}_{vv} \rangle. \tag{4.3}$$

Then, for a tangent vector $v = a\mathbf{x}_u + b\mathbf{x}_v$ at $p = \mathbf{x}(u,v)$, we have

$$\mathrm{II}_p(v) = ea^2 + 2fab + gb^2. \tag{4.4}$$

Proposition 4.4 (Weingarten equations and curvature formulas). Since $M(u,v)$ is orthogonal to both $\mathbf{x}_u$ and $\mathbf{x}_v$, one can write

$$M_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \quad \text{and} \quad M_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v.$$

Let $\mathbf{A} = (a_{ij})$ be the corresponding 2×2 matrix of dN in the basis $\{x_u, x_v\}$. Then,

$$a_{11} = \frac{fF - eG}{D} \quad a_{21} = \frac{eF - fE}{D} \quad a_{12} = \frac{gF - fG}{D} \quad a_{22} = \frac{fF - gE}{D}.$$

Moreover,

$$K = \det(\mathbf{A}) = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = -\frac{1}{2}(a_{11} + a_{22}) = \frac{eG - 2fF + gE}{2(EG - F^2)}. \tag{4.5}$$

Finally, the principal curvatures are the eigenvalues of $-dN$, i.e.

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

Example 4.8. For the torus parametrisation

$$\mathbf{x}(u,v) = ((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u),$$

one can compute E, F, G and e, f, g explicitly and obtain

$$K(u,v) = \frac{\cos u}{r(a + r\cos u)}.$$

Let S be the graph of $z = h(x,y)$ and parametrise using $\mathbf{x}(u,v) = (u, v, h(u,v))$. Then, $\mathbf{x}_u = (1, 0, h_u)$ and $\mathbf{x}_v = (0, 1, h_v)$. Hence,

$$E = 1 + h_u^2 \quad \text{and} \quad F = h_u h_v \quad \text{and} \quad G = 1 + h_v^2.$$

Also,

$$\mathbf{x}_{uu} = (0,0,h_{uu}) \quad \text{and} \quad \mathbf{x}_{uv} = (0,0,h_{uv}) \quad \text{and} \quad \mathbf{x}_{vv} = (0,0,h_{vv}).$$

A unit normal vector is

$$\mathbf{N}(\mathbf{x}(u,v)) = \frac{(-h_u,-h_v,1)}{\sqrt{1+h_u^2+h_v^2}}.$$

Thus,

$$e = \frac{h_{uu}}{\sqrt{1+h_u^2+h_v^2}} \quad \text{and} \quad f = \frac{h_{uv}}{\sqrt{1+h_u^2+h_v^2}} \quad \text{and} \quad g = \frac{h_{vv}}{\sqrt{1+h_u^2+h_v^2}}.$$

Using

$$K = \frac{eg-f^2}{EG-F^2} \quad \text{and} \quad EG-F^2 = 1+h_u^2+h_v^2, \tag{4.6}$$

we get

$$K = \frac{h_{uu}h_{vv}-h_{uv}^2}{\left(1+h_u^2+h_v^2\right)^2}.$$

In particular, at a critical point where $h_u(u,v) = h_v(u,v) = 0$, we have

$$K = \left(h_{uu}h_{vv}-h_{uv}^2\right)\big|_{(u,v)}.$$

So, the sign of K at a critical point is exactly the sign of the Hessian determinant. We then give a geometric interpretation of the second derivative test. Let (u_0,v_0) be a critical point, i.e. $h_u(u_0,v_0) = h_v(u_0,v_0) = 0$ (so the tangent plane is horizontal and $N = (0,0,1)$). Then, we have the following:

(i) If $K(u_0,v_0) > 0$ (equivalently $h_{uu}h_{vv}-h_{uv}^2 > 0$), then the point is locally elliptic: it is a local maximum or local minimum

(ii) If $K(u_0,v_0) < 0$ (equivalently $h_{uu}h_{vv}-h_{uv}^2 < 0$), then the point is saddle-like

(iii) If $K(u_0,v_0) = 0$, the test is inconclusive

Example 4.9. Let us investigate the plane $x+y+z=1$. We can parametrise it using

$$\mathbf{x}(u,v) = (u,v,1-u-v).$$

Then,

$$\mathbf{x}_u = (1,0,-1) \quad \text{and} \quad \mathbf{x}_v = (0,1,-1)$$

so

$$\mathbf{N} = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{1}{\sqrt{3}}(1,1,1).$$

Clearly, $\mathbf{N}_u = \mathbf{N}_v = \mathbf{0}$ so for any point p, the differential $d\mathbf{N}_p$ is the 2×2 zero matrix. More precisely, the map $d\mathbf{N}_p : T_pS \to T_pS^2$ is the zero linear map, where S denotes the surface of the plane and S^2 denotes the unit sphere in $\mathbb{R}^3$. Geometrically, both principal curvatures of the plane are zero, so the Gauss curvature is equal to zero.

Example 4.10 (MA3215 AY14/15 Sem 2 Tutorial 8).

(i) Compute the principal curvatures k_1, k_2, and the Gaussian curvature K at a point $\mathbf{p}$ on a sphere S_1 of radius 10. Briefly explain your answers.

(ii) Compute the principal curvatures k_1, k_2 and the Gaussian curvature K at a point $\mathbf{p}$ on a circular tube S_2 of radius 10. Briefly explain your answers.

Solution.

(i) We have $x^2+y^2+z^2=10^2$. We parametrise using spherical coordinates, so

$$\mathbf{x}(\theta,\phi)=(10\cos\theta\sin\phi,10\sin\theta\sin\phi,10\cos\phi),$$

where $0\le\theta\le 2\pi$ and $0\le\phi\le\pi$. Then, one can compute

$$\mathbf{N}=\frac{\mathbf{x}_\theta\wedge\mathbf{x}_\phi}{|\mathbf{x}_\theta\wedge\mathbf{x}_\phi|}=(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi).$$

Since $\mathbf{x}=10\mathbf{N}$, then $-d\mathbf{N}_{\mathbf{p}}=-\frac{1}{10}d\mathbf{x}_{\mathbf{p}}$. Alternatively, to see why, we have

$$\mathbf{N}_\theta=(-\sin\theta\sin\phi,\cos\theta\sin\phi,0)\quad\text{and}\quad\mathbf{N}_\phi=(\cos\theta\cos\phi,\sin\theta\cos\phi,-\sin\phi).$$

Observe that $-\frac{1}{10}\mathbf{x}_\theta=\mathbf{N}_\theta$ and $-\frac{1}{10}\mathbf{x}_\phi=\mathbf{N}_\phi$. As such, the principal curvatures are $k_1=k_2=-\frac{1}{10}$, so $K=\frac{1}{100}$.

(ii) We have $x^2+y^2=10^2$. We parametrise using cylindrical coordinates, so

$$\mathbf{x}(\theta,z)=(10\cos\theta,10\sin\theta,z)\quad\text{where } 0\le\theta\le 2\pi.$$

We have

$$\mathbf{x}_\theta=(-10\sin\theta,10\cos\theta,0)\quad\text{and}\quad\mathbf{x}_z=(0,0,1).$$

Next,

$$\mathbf{N}=\frac{\mathbf{x}_\theta\wedge\mathbf{x}_z}{|\mathbf{x}_\theta\wedge\mathbf{x}_z|}=(\cos\theta,\sin\theta,0).$$

One can then show that $K=\frac{eg-f^2}{EG-F^2}=0$. Alternatively, we have

$$\mathbf{N}_\theta=(-\sin\theta,\cos\theta,0)=\frac{1}{10}\mathbf{x}_\theta\quad\text{and}\quad\mathbf{N}_z=(0,0,0)=0\mathbf{x}_z.$$

So,

$$d\mathbf{N}_p=\begin{pmatrix}\frac{1}{10}&0\\0&0\end{pmatrix}$$

which implies that the principal curvatures are $k_1=\frac{1}{10}$ and $k_2=0$. So, the Gaussian curvature is $K=k_1k_2=0$. □

Example 4.11 (MA3215 AY14/15 Sem 2 Tutorial 9). Let $\gamma(v)=(a(v),b(v))$ be a non-singular smooth curve on the xz-plane (Definition 2.4). That is, $\gamma'(v)\ne(0,0)$. We assume that $a(v)>0$ so that the curve does not intersect the z-axis. Let S denote the surface of revolution about the z-axis. Let

$$\mathbf{x}(u,v)=(a(v)\cos u,a(v)\sin u,b(v))$$

be a parametrisation of S. Express the following in terms of $a,a',a'',b,b',b'',\sin u,\cos u$:

(i) $\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uu}, \mathbf{x}_{uv}, \mathbf{x}_{vv}$ and $\mathbf{N} = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$

(ii) E, F, G, e, f, g

(iii) $\mathrm{II}(\mathbf{v})$ where $\mathbf{v} = a_1 \mathbf{x}_u + a_2 \mathbf{x}_v$

Solution.

(i) Trivial computation. We have

$$\begin{aligned} \mathbf{x}_u &= (a(v)\sin u, a(v)\cos u, 0) \\ \mathbf{x}_v &= \left(a'(v)\cos u, a'(v)\sin u, b'(v)\right) \\ \mathbf{x}_{uu} &= (a(v)\cos u, a(v)\sin u, 0) \\ \mathbf{x}_{uv} &= \left(a'(v)\sin u, a'(v)\cos u, 0\right) \\ \mathbf{x}_{vv} &= \left(a''(v)\cos u, a''(v)\sin u, b''(v)\right) \end{aligned}$$

Next,

$$\mathbf{N} = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = \frac{1}{\sqrt{a'(v)^2 + b'(v)^2}} \left(b'(v)\cos u, b'(v)\sin u, a'(v)\right)$$

(ii) We then compute the first and second fundamental form coefficients. Using (4.2) and (4.3), we would obtain the following: $E = a^2$, $F = 0$, $G = (a')^2 + (b')^2$. Also, $e = -s^{-1}ab'$, $f = 0$ and $g = s^{-1}(a''b' - a'b'')$.

(iii) By (4.4),

$$\mathrm{II}(\mathbf{v}) = ea_1^2 + 2fa_1a_2 + ga_2^2.$$

From **(ii)**, we know that $e = -s^{-1}ab'$, $f = 0$ and $g = s^{-1}(a''b' - a'b'')$. Substituting this into the second fundamental form yields the answer. □

Example 4.12 (MA3215 AY14/15 Sem 2 Tutorial 9). We continue from Example 4.11. Let $\gamma(v) = (a(v), b(v))$ be a non-singular smooth curve on the xz-plane. That is, $\gamma'(v) \neq (0,0)$. We assume that $a(v) > 0$ so that the curve does not intersect the z-axis. Let S denote the surface of revolution about the z-axis. Let

$$\mathbf{x}(u,v) = (a(v)\cos u, a(v)\sin u, b(v))$$

be a parametrisation of S.

(i) Compute the Gaussian curvature K and the mean curvature H in terms of a, a', a'' and b, b', b''.

(ii) Suppose $\gamma(v)$ is parametrised by arc length. That is, $|\gamma'|^2 = (a')^2 + (b')^2 = 1$. Show that $a'a'' + b'b'' = 0$ and express K in terms of a, a', a''.

Solution.

(i) We consider Definition 4.6. From **(ii)** of Example 4.11, we found e, f, g, E, F, G. By (4.6),

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-s^{-1}ab' \cdot s^{-1}\left(a''b' - a'b''\right)}{a^2 s^2} = -\frac{b'\left(a''b' - a'b''\right)}{as^4},$$

where $s^2 = (a')^2 + (b')^2$.

Next, by (4.5), we can compute the mean curvature as follows:

$$H = \frac{eG - 2fF + gE}{2\left(EG - F^2\right)} = \frac{aa''b' - aa'b'' - b'\left(a'\right)^2 - \left(b'\right)^3}{2as^3}$$

(ii) The first result follows by differentiation. Next, by using **(i)**, $K = -\frac{a''}{a}$. □

Example 4.13 (MA3215 AY14/15 Sem 2 Tutorial 9). We continue from Example 4.11. Let $\gamma(v) = (a(v), b(v))$ be a non-singular smooth curve on the xz-plane. That is, $\gamma'(v) \neq (0,0)$. We assume that $a(v) > 0$ so that the curve does not intersect the z-axis. Let S denote the surface of revolution about the z-axis. Let

$$\mathbf{x}(u, v) = (a(v)\cos u, a(v)\sin u, b(v))$$

be a parametrisation of S. Suppose $K = 0$ at all points on S.

(i) Show that $a'(v) = C_1$ and $a(v) = C_1 v + C_2$ for some constants C_1 and C_2.

(ii) Furthermore, show that $b'(v) = C_3$ and $b(v) = C_3 v + C_4$, where $C_3 = \pm\sqrt{1 - C_1}$ and C_4 are constants.

(iii) Conclude that $\gamma(v)$ is a straight line.

(iv) Conclude that S is either (part of) a vertical cylinder, a cone, or a plane perpendicular to the z-axis.

Solution.

(i) We previously showed that

$$K = -\frac{b'\left(a''b' - a'b''\right)}{as^4}.$$

Suppose $K = 0$ at all points on S. Then, either $b' = 0$ or $a''b' = a'b''$. If $b' = 0$, then b is constant. This corresponds to a circular cross-section and generates a cylinder, which answers **(iv)**.

On the other hand, if $a''b' = a'b''$, then $a' = C_1 b'$. The result follows.

(ii) As we have seen earlier, the curve is parametrised by arc length, so $(a')^2 + (b')^2 = 1$. The result then follows.

(iii) By **(i)** and **(ii)**, we see that

$$\gamma(v) = (a, b) = (C_1 v + C_2, C_3 v + C_4)$$

which is a linear function in terms of v. So, γ is a straight line.

(iv) The surface of revolution is

$$\mathbf{x}(u,v) = ((C_1 v + C_2)\cos u, (C_1 v + C_2)\sin u, C_3 v + C_4).$$

Suppose $a' = 0$. Then, $C_1 = 0$ so $\mathbf{x}(u,v) = (C_2 \cos u, C_2 \sin u, C_3 v + C_4)$. This produces a cylinder since the sum of squares of the x- and y-components is a constant.

If $b' = 0$, then $C_3 = 0$ so $\mathbf{x}(u,v) = ((C_1 v + C_2)\cos u, (C_1 v + C_2)\sin u, C_4)$. So, S is a plane perpendicular to the z-axis.

Lastly, if C_1 and C_3 are both non-zero, then consider the following: $x^2 + y^2 = (C_1 v + C_2)^2$ and $z = C_3 v + C_4$. As such,

$$\frac{\sqrt{x^2 + y^2} - C_2}{C_1} = \frac{z - C_4}{C_3}.$$

We obtain a cone. $\square$

CHAPTER 5

Isometries

5.1 Isometries

Definition 5.1 (diffeomorphism). A map $\phi : S_1 \to S_2$ is a diffeomorphism if ϕ is smooth, ϕ is bijective (so ϕ^{-1} exists), and ϕ^{-1} is smooth. Two surfaces are diffeomorphic if there exists a diffeomorphism between them.

In Differential Geometry, two diffeomorphic surfaces are regarded as the same object. That is, one cannot tell them apart by properties that are invariant under diffeomorphisms.

Definition 5.2 (isometry). A diffeomorphism $\phi : S_1 \to S_2$ is an isometry if arc lengths of curves are preserved. Equivalently, for every $p \in S_1$ and every $\mathbf{u}, \mathbf{v} \in T_p(S_1)$, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle_p = \left\langle d\phi_p(\mathbf{u}), d\phi_p(\mathbf{v}) \right\rangle_{\phi(p)}. \tag{5.1}$$

In particular, taking $\mathbf{u} = \mathbf{v}$ gives

$$\|\mathbf{u}\|_p = \left\| d\phi_p(\mathbf{u}) \right\|_{\phi(p)}.$$

$$\begin{array}{ccc} S_1 & \xrightarrow{\phi} & S_2 \\ \uparrow & & \uparrow \\ T_pS_1 & \xrightarrow{d\phi_p} & T_{\phi(p)}S_2 \end{array}$$

Proposition 5.1 (same coefficients in first fundamental form). Let S_1 and S_2 be regular surfaces, and $\phi : S_1 \to S_2$ be an isometry. Fix a parametrisation $\mathbf{x} : U \to S_1$ and define

$$\widetilde{\mathbf{x}} : \phi \circ \mathbf{x} : U \to S_2.$$

Let E, F, G denote the coefficients of the first fundamental form of $\mathbf{x}$, and $\widetilde{E}, \widetilde{F}, \widetilde{G}$ denote the coefficients of the first fundamental form of $\widetilde{\mathbf{x}}$. Then, $E = \widetilde{E}$, $F = \widetilde{F}$, and

$G = \widetilde{G}$, so the first fundamental form coefficients are the same.

Example 5.1 (MA3215 AY14/15 Sem 2 Tutorial 9). Let $\phi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ be two isometries.

(i) Show that $\phi^{-1} : S_2 \to S_1$ is an isometry.

(ii) Show that $\psi \circ \phi : S_1 \to S_3$ is an isometry.

(iii) Assume that $\mathbf{y} = \phi \circ \mathbf{x} : U \to S_2$ is a parametrisation. Show that ϕ is area-preserving, that is, if $\mathbf{x} : U \to S_1$ is a parametrisation, then the area of $V = \mathbf{x}(U)$ is the same as that of $\overline{V} = \phi \circ \mathbf{x}(U)$.

Solution.

(i) By Definition 5.2, for every $p \in S_1$ and $\mathbf{u}_1, \mathbf{u}_2$ contained in the tangent plane of S_1 at p, denoted by T_pS_1, we have

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle_p = \langle d\phi_p(\mathbf{u}_1), d\phi_p(\mathbf{u}_2) \rangle_{\phi(p)} \tag{5.2}$$

Let $q = \phi(p)$. Take arbitrary $\mathbf{v}_1, \mathbf{v}_2 \in T_qS_2$. Since $d\phi_p : T_pS_1 \to T_qS_2$ is a diffeomorphism (Definition 5.2), then it is a linear isomorphism (Definition 5.1). So, there exist unique $\mathbf{u}_1, \mathbf{u}_2 \in T_pS_1$ such that

$$d\phi_p(\mathbf{u}_1) = \mathbf{v}_1 \quad \text{and} \quad d\phi_p(\mathbf{u}_2) = \mathbf{v}_2.$$

Then,

$$\begin{aligned} \left\langle d\left(\phi^{-1}\right)_q(\mathbf{v}_1), d\left(\phi^{-1}\right)_q(\mathbf{v}_2) \right\rangle_{\phi^{-1}(q)} &= \langle \mathbf{u}_1, \mathbf{u}_2 \rangle_p \\ &= \langle d\phi_p(\mathbf{u}_1), d\phi_p(\mathbf{u}_2) \rangle_{\phi(p)} \quad \text{by (5.2)} \\ &= \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_q \end{aligned}$$

So, ϕ^{-1} is an isometry.

(ii) Again by Definition 5.2, for every $p \in S_1$ and $\mathbf{u}_1, \mathbf{u}_2 \in T_pS_1$, we have

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle_p = \langle d\phi_p(\mathbf{u}_1), d\phi_p(\mathbf{u}_2) \rangle_{\phi(p)}.$$

Since this is the same as (5.2), we will make reference to the earlier equation. Let $q = \phi(p) \in S_2$. Then because $d\phi_p : T_pS_1 \to T_qS_2$ is a diffeomorphism (Definition 5.1), then it is a linear isomorphism (Definition 5.2). So, there exist unique $\mathbf{u}_1, \mathbf{u}_2 \in T_pS_1$ such that

$$d\phi_p(\mathbf{u}_1) = \mathbf{v}_1 \quad \text{and} \quad d\phi_p(\mathbf{u}_2) = \mathbf{v}_2.$$

Similarly, we can write

$$d\psi_q(\mathbf{v}_1) = \mathbf{w}_1 \quad \text{and} \quad d\psi_q(\mathbf{v}_2) = \mathbf{w}_2.$$

Next, we also have

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle_q = \langle d\psi_q(\mathbf{v}_1), d\psi_q(\mathbf{v}_2) \rangle_{\psi(q)}. \tag{5.3}$$

Then use (5.2) and (5.3).

(iii) We recall Proposition 3.10, which states the following. Let $\mathbf{x} : U \to S$ be a coordinate function, and set $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle$, $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle$, and $G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$. Then,

$$\text{Area}\left(\mathbf{x}\left(U\right)\right) = \iint_U \|\mathbf{x}_u \wedge \mathbf{x}_v\| \, dudv = \iint_U \sqrt{EG - F^2} \, dudv.$$

Recall that this is related to the first fundamental form (Definition 3.16). As such,

$$\text{Area}\left(V\right) = \iint_U \|\mathbf{x}_u \wedge \mathbf{x}_v\| \, dudv.$$

Next,

$$\text{Area}\left(\overline{V}\right) = \text{Area}\left(\phi \circ \mathbf{x}\left(U\right)\right) = \iint_U \|(\phi \circ \mathbf{x})_u \wedge (\phi \circ \mathbf{x})_v\| \, dudv.$$

It is a known fact that if ϕ is an isometry, then the E, F, G coefficients in the first fundamental form are the same (Proposition 5.1). The result follows. □

Definition 5.3 (local isometry at a point). A smooth map $\phi : S_1 \to S_2$ is a local isometry at $p \in S_1$ if there exist open sets $V_1 \subseteq S_1$ containing p and $V_2 \subseteq S_2$ containing $\phi\left(p\right)$ such that

$$\phi : V_1 \to V_2$$

is an isometry.

From Definition 5.3, we see that ϕ being a local isometry at p is a stronger condition than just requiring the metric preserving condition at the single point p; it must hold on an open neighbourhood around p.

Example 5.2 (MA3215 AY14/15 Sem 2 Tutorial 9). Let $F : V \to W$ be a linear transformation between two inner product spaces V and W of dimension 2. It is a well-known fact that the following statements are equivalent:

(i) $\|F\left(\mathbf{v}\right)\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in V$

(ii) $\langle F\left(\mathbf{v}_1\right), F\left(\mathbf{v}_2\right) \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

(iii) If $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis for V, then $\{F\left(\mathbf{e}_1\right), F\left(\mathbf{e}_2\right)\}$ is an orthonormal basis for W

(iv) There is an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for V such that $\{F\left(\mathbf{e}_1\right), F\left(\mathbf{e}_2\right)\}$ is an orthonormal basis of W

If any of the mentioned statements are satisfied, then F is a bijection. Now, let $\phi : S \to \overline{S}$ be a diffeomorphism between two surfaces and let $\mathbf{p} \in S$ and $\overline{\mathbf{p}} = \phi\left(\mathbf{p}\right) \in \overline{S}$. By setting $F = d\phi$, $V = T_{\mathbf{p}}\left(S\right)$ and $W = T_{\overline{\mathbf{p}}}\left(\overline{S}\right)$ in the last problem, give four equivalent definitions of 'ϕ is an isometry'.

Solution. We can now formulate the original statement as follows. Let $d\phi : T_{\mathbf{p}}\left(S\right) \to T_{\overline{\mathbf{p}}}\left(\overline{S}\right)$ be a diffeomorphism between two surfaces. We will only provide the analogous statement for **(i)** and leave **(ii)**, **(iii)** and **(iv)** as an exercise to the reader. For **(i)**, we can write the statement as follows: for every $\mathbf{p} \in S$ and every $\mathbf{v} \in T_{\mathbf{p}}\left(S\right)$, we have $\left\|d\phi_{\mathbf{p}}\left(\mathbf{v}\right)\right\|_{\overline{\mathbf{p}}} = \|\mathbf{v}\|_{\mathbf{p}}$. □

Proposition 5.2. Using the setup in Definition 5.3, $\phi : S_1 \to S_2$ is a diffeomorphism between the open sets $V_1 \subseteq S_1$ and $V_2 \subseteq S_2$.

Example 5.3 (plane to cylinder is locally isometric but not bijective). Let

$$S_1 = \{(x,y,0) \in \mathbb{R}^3\} \cong \mathbb{R}^2 \quad \text{and} \quad S_2 = \{(X,Y,Z) \in \mathbb{R}^3 : \, Y^2 + Z^2 = 1\}$$

define the xy-plane and the unit cylinder respectively. Define

$$\phi(x,y,0) = (x, \cos y, \sin y).$$

Equivalently, in coordinates we consider the map

$$F : \mathbb{R}^2 \to S_2 \quad \text{where} \quad F(x,y) = (x, \cos y, \sin y).$$

The map is periodic in the y-variable because $F(x, y+2\pi k) = F(x,y)$ for all $k \in \mathbb{Z}$, so it is not injective on all of $\mathbb{R}^2$. It is also surjective onto S_2 (every point $(x, \cos\theta, \sin\theta)$ is hit by (x,θ)), hence globally it is a covering map rather than a bijection.

We then compute the coordinate derivatives $F_x = (1,0,0)$ and $F_y = (0, -\sin y, \cos y)$. These are linearly independent for every (x,y), so $\operatorname{rank}(dF_{(x,y)}) = 2$ everywhere. By the inverse function theorem (Theorem 3.1), for each (x_0, y_0), there exists a neighbourhood U of (x_0, y_0) such that $F\,|_U$ is a diffeomorphism onto its image. Concretely, if we take

$$U = \{(x,y) : |y - y_0| < \pi\},$$

then $y \mapsto (\cos y, \sin y)$ is injective on $(y_0 - \pi, y_0 + \pi)$, so F is injective on U.

On the plane with coordinates (x,y), the induced metric is the standard one. That is, $I_{S_1} = dx^2 + dy^2$. On the cylinder parametrised by F, the first fundamental form is

$$I_{S_2} = \langle F_x, F_x \rangle \, dx^2 + 2\langle F_x, F_y \rangle \, dxdy + \langle F_y, F_y \rangle \, dy^2.$$

Using the derivatives above, we have

$$\langle F_x, F_x \rangle = 1 \quad \text{and} \quad \langle F_x, F_y \rangle = 0 \quad \text{and} \quad \langle F_y, F_y \rangle = \sin^2 y + \cos^2 y = 1,$$

hence $I_{S_2} = dx^2 + dy^2 = I_{S_1}$. Therefore, F preserves lengths of tangent vectors (and hence lengths of sufficiently short curves) on each neighbourhood where it is one-to-one. In this precise sense, ϕ is a local isometry.

This is the standard 'rolling without stretching' identification: the cylinder can be developed (flattened) onto the plane without distortion, but one full turn around the cylinder corresponds to shifting the plane by 2π in the y-direction, which is why the global map cannot be injective.

Proposition 5.3 (matching E,F,G gives an isometry on coordinate patches). Let $x : U \to S_1$ and $y : U \to S_2$ be coordinate functions. Let (E_1, F_1, G_1) and (E_2, F_2, G_2) be the coefficients of the first fundamental forms on S_1 and S_2 induced by x and y. If

$$E_1(u,v) = E_2(u,v) \quad \text{and} \quad F_1(u,v) = F_2(u,v) \quad \text{and} \quad G_1(u,v) = G_2(u,v)$$

as functions on U, then

$$\Phi = y \circ x^{-1} : x(U) \to y(U)$$

is an isometry.

Definition 5.4 (intrinsic property). A property of a regular surface is intrinsic if it is invariant under (local) isometries.

Heuristically, if a quantity can be computed purely from E,F,G and their higher partial derivatives, then it is intrinsic.

5.2 Gauss' Theorema Egregium

Theorem 5.1 (Theorema Egregium). Let $\phi : S_1 \to S_2$ be a local isometry. Fix $p_1 \in S_1$ and set $p_2 = \phi(p_1) \in S_2$. Let $K(p_1)$ and $\overline{K}(p_2)$ be the Gaussian curvatures at p_1 and p_2. Then,

$$K(p_1) = \overline{K}(p_2).$$

Equivalently, Gaussian curvature is a local intrinsic quantity.

Theorem 5.1 is remarkable. Recall from Definition 4.6 that $K = k_1 k_2$ is defined using the shape operator (or second fundamental form), i.e. apparently extrinsic data in $\mathbb{R}^3$. Theorem 5.1 says that despite the definition, K is completely determined by the metric on the surface, so it cannot change under local distance-preserving deformations.

Example 5.4. For the plane, $k_1 = k_2 = 0$ so $K = 0$. For the cylinder, $k_1 = 1$ and $k_2 = 0$ so $K = 0$. By Theorem 5.1, there is no curvature obstruction to locally rolling a plane into a cylinder.

Example 5.5 (plane vs sphere). The plane has $K = 0$ while the unit sphere has $K = 1$. Hence, there cannot exist a local isometry from the plane to the sphere. Geometrically, one cannot wrap a flat sheet smoothly onto a sphere without crumpling.

To prove Theorem 5.1, it suffices to show that the Gaussian curvature K is determined by the first fundamental form. Concretely, we aim to show K is a function of E,F,G and their higher partial derivatives. Then, under a local isometry, the coefficients (E,F,G) (hence all derivatives) agree in corresponding coordinates, so K must agree as well.

Definition 5.5 (Christoffel symbols from the metric). Let $\mathbf{x}: U \to S$ be a coordinate function, and write $p = \mathbf{x}(u, v)$. Then, $\{\mathbf{x}_u, \mathbf{x}_v, \mathbf{N}\}$ is a basis of $\mathbb{R}^3$ at p, so we may expand to obtain

$$\mathbf{x}_{uu} = \Gamma_{11}^{1}\mathbf{x}_u + \Gamma_{11}^{2}\mathbf{x}_v + e\mathbf{N}$$
$$\mathbf{x}_{uv} = \Gamma_{12}^{1}\mathbf{x}_u + \Gamma_{12}^{2}\mathbf{x}_v + f\mathbf{N}$$
$$\mathbf{x}_{vv} = \Gamma_{22}^{1}\mathbf{x}_u + \Gamma_{22}^{2}\mathbf{x}_v + g\mathbf{N}$$

The coefficients Γ_{ij}^{k} are the Christoffel symbols.

A key point is that each Γ_{ij}^{k} can be written purely using E, F, G and their first derivatives $E_u, F_u, G_u, E_v, F_v, G_v$ by taking inner products of the above expansions with $\mathbf{x}_u$ and $\mathbf{x}_v$ and solving the resulting linear system. Hence,

$$\Gamma_{ij}^{k} = \Gamma_{ij}^{k}(E, F, G, E_u, F_u, G_u, E_v, F_v, G_v).$$

Geometrically, the Christoffel symbols are the coefficients that tell one how the tangent basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ changes as we move on the surface after discarding the normal component. More concretely, we decompose

$$\mathbf{x}_{ij} = \underbrace{\Gamma_{ij}^{1}\mathbf{x}_u + \Gamma_{ij}^{2}\mathbf{x}_v}_{\text{tangential part}} + \underbrace{(\mathrm{II})ij\mathbf{N}}_{\text{normal part}} \quad \text{where } i, j \in \{u, v\}.$$

Proposition 5.4 (Christoffel symbols are invariant under local isometries). If $\phi: S_1 \to S_2$ is a local isometry and $x: U \to S_1$ is a coordinate function, then $\mathbf{y} = \phi \circ \mathbf{x}: U \to S_2$ is also a coordinate function. Moreover, the first fundamental form coefficients agree. That is, $\overline{E} = E$, $\overline{F} = F$, and $\overline{G} = G$ as functions on U, and hence their partial derivatives agree. Therefore the Christoffel symbols computed from $(\overline{E}, \overline{F}, \overline{G})$ coincide with those from (E, F, G), so

$$\overline{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k}.$$

One derives identities by differentiating $\mathbf{x}_{uv} = \mathbf{x}_{vu}$ and expanding everything in the basis $\{\mathbf{x}_u, \mathbf{x}_v, N\}$, and also using the Weingarten equations for N_u, N_v. After simplifying, one obtains formulas that express EK and FK (hence K) in terms of E, F, G, Γ_{ij}^{k} and the first derivatives of Γ_{ij}^{k}. Since each Γ_{ij}^{k} is itself a function of E, F, G and their first derivatives, it follows that

$$K = K(E, F, G, \text{ partial derivatives of } E, F, G \text{ up to some finite order}).$$

This is the intrinsicness statement needed for Theorem 5.1.

A refinement of the computation also yields the Peterson-Mainardi-Codazzi equations (relations between E, F, G and e, f, g and derivatives). Together with the Gauss equation, they form the compatibility conditions in the fundamental theorem of surfaces: given smooth functions E, F, G, e, f, g satisfying positivity and these equations, one can locally realize them as the first and second fundamental forms of some surface (unique up to rigid motions).

Chapter 6

Tangent Vector Fields

6.1 Tangent Vector Fields

A useful way to think about a regular surface $S \subseteq \mathbb{R}^3$ is a two-dimensional *stage* sitting inside $\mathbb{R}^3$. Even though $\mathbb{R}^3$ has three linearly independent directions, motion constrained to the surface has only two degrees of freedom at each point. This local notion of the allowed directions is captured by the tangent plane T_pS: it is the best linear approximation to S near p and it contains precisely the velocity vectors of smooth curves on S passing through p.

As we would see in Definition 6.1, a tangent vector field is a consistent assignment of such allowed directions along the whole surface: to each point $p \in S$, we associate a vector $\mathbf{w}(p)$ that lies in T_pS. This notion is the surface analogue of an ordinary vector field in $\mathbb{R}^3$ with the crucial additional constraint that the vectors must remain tangent to S. Equivalently, if $\mathbf{N}(p)$ is a chosen unit normal to S at p, tangency is characterised by the orthogonality condition $\mathbf{w}(p) \cdot \mathbf{N}(p)$. This encodes the geometric idea that $\mathbf{w}(p)$ has no component pointing *off* the surface. In a coordinate patch ,this constraint becomes concrete since $\{\mathbf{x}_u, \mathbf{x}_v\}$ spans T_pS, so every tangent field admits a local decomposition as in (6.1). That is to say, studying tangent vector fields reduces to studying the coefficient functions a and b. Requiring these coefficients to be smooth (Definition 6.2) is precisely what ensures that $\mathbf{w}$ varies smoothly along the surface, making tangent vector fields the natural objects for describing surface flows and later, intrinsic differential operators such as the covariant derivative (Chapter 6.2).

Definition 6.1 (tangent vector fields in coordinates). Let $S \subseteq \mathbb{R}^3$ be a regular surface with a coordinate map $\mathbf{x}(u,v) : U \to S$. At a point $p = \mathbf{x}(u,v)$, the tangent space is

$$T_pS = \operatorname{span}\{\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)\}.$$

A tangent vector field on S is a map $\mathbf{w} : S \to \mathbb{R}^3$ such that

$$\mathbf{w}(p) \in T_pS \quad \text{for all } p \in S.$$

In a coordinate patch, any tangent vector field can be written as

$$\mathbf{w}(\mathbf{x}(u,v)) = a(u,v)\,\mathbf{x}_u(u,v) + b(u,v)\,\mathbf{x}_v(u,v). \tag{6.1}$$

Definition 6.2 (smooth vector field). Let $S \subseteq \mathbb{R}^3$ be a regular surface with a coordinate map $\mathbf{x}(u,v) : U \to S$. We say that tangent vector field $\mathbf{w}$ is smooth if in every coordinate patch, the coefficient functions $a(u,v)$ and $b(u,v)$ are smooth.

Example 6.1. Let S be a sphere and let $\mathbf{w}(x,y,z) = (-y,x,0)$ be a vector field in it. Prove that $\mathbf{w}$ is a smooth vector field.

Solution. We consider a standard local parametrisation on the sphere, which is

$$\mathbf{x}(u,v) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, \cos\phi) \quad \text{where } 0 < \theta < 2\pi \text{ and } 0 < \phi < \pi.$$

Here, R denotes the radius of the sphere. Then,

$$\mathbf{x}_\theta = (-R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0) \quad \text{and} \quad \mathbf{x}_\phi = (R\cos\theta\cos\phi, R\sin\theta\cos\phi, -\sin\phi).$$

We wish to prove that there exist coefficient functions a and b such that the decomposition in (6.1) holds. That is,

$$(-R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0)$$

is equal to

$$a(-R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0) + b(R\cos\theta\cos\phi, R\sin\theta\cos\phi, -\sin\phi).$$

Clearly, choose $a = 1$ and $b = 0$. □

Example 6.2 (MA3215 AY14/15 Sem 2 Tutorial 10). Let S be a sphere and let $\mathbf{w}(x,y,z) = (-y,x,0)$ be a vector field in it. By Example 6.2 is a smooth vector field on the sphere. Now, use the stereographic projection

$$\mathbf{x}(u,v) = \pi^{-1}(u,v) = R^{-1}\left(4u, 4v, u^2+v^2-4\right) \tag{6.2}$$

where $R = u^2+v^2+4$ and show that $\mathbf{w}$ is a smooth vector field on the sphere excluding the north pole.

Solution. We have

$$\mathbf{x}_u = R^{-1}(4,0,2u) \quad \text{and} \quad \mathbf{x}_v = R^{-1}(0,4,2v).$$

Again, we wish to prove that there exist coefficient functions a and b such that the decomposition in (6.1) holds. That is, $\left(-4R^{-1}v, 4R^{-1}u, 0\right)$ is equal to

$$aR^{-1}(4,0,2u) + bR^{-1}(0,4,2v).$$

This is equivalent to

$$(-4v, 4u, 0) = (4a, 4b, 2au + 2bv).$$

We choose $a = -v$ and $b = u$ and it follows that $\mathbf{w}$ is a smooth vector field on the sphere. However, we need to show that smoothness does not hold on the north pole. Recall (6.4), and we see that the z-component is $\frac{R-8}{R}$. As R tends to infinity, the z-component tends to 1. □

6.2 Differentiating Vector Fields

In Chapter 6.1, we studied tangent vector fields $\mathbf{w} : S \to \mathbb{R}^3$ on a regular surface $S \subseteq \mathbb{R}^3$. A natural next question is how to differentiate such fields. In Euclidean space, differentiation is straightforward: if $\mathbf{w}$ is a vector-valued function of parameters (u, v) or of a curve parameter t, we simply take partial derivatives or ordinary derivatives. On a surface, however, there is an immediate obstruction: even if $\mathbf{w}(p) \in T_pS$ for every $p \in S$, the ambient derivative $\frac{\partial \mathbf{w}}{\partial u}$ (or $\frac{d\mathbf{w}}{dt}$ along a curve) generally develops a component in the normal direction and hence need not lie in the tangent plane. Thus, ordinary differentiation does not preserve tangency.

The fundamental idea is to keep the usual derivative in $\mathbb{R}^3$ but then discard the extraneous normal component by projecting back onto the tangent plane. This leads to the covariant derivative on a surface: a differentiation operator that measures the intrinsic rate of change of a tangent field while remaining tangent. We first define covariant derivatives in coordinate directions (Definition 6.3), which are the basic building blocks for computations in a parametrised patch. We then extend the definition to covariant differentiation along a curve $\alpha : I \to S$ (Definition 6.4), which is the appropriate notion when tracking how a moving tangent vector changes as one travels on the surface.

Definition 6.3 (covariant derivatives in coordinate directions). Let $\mathbf{w}$ be a smooth tangent vector field on S, written in a coordinate patch as

$$\mathbf{w}(u, v) = \mathbf{w}(\mathbf{x}(u, v)) \in \mathbb{R}^3.$$

The ordinary partial derivatives $\frac{\partial \mathbf{w}}{\partial u}$ and $\frac{\partial \mathbf{w}}{\partial v}$ need not be tangent to S. We define the covariant derivatives by projecting back to the tangent plane:

$$\frac{D\mathbf{w}}{du} = \frac{\partial \mathbf{w}}{\partial u} - \left\langle \frac{\partial \mathbf{w}}{\partial u}, \mathbf{N} \right\rangle \mathbf{N} \quad \text{and} \quad \frac{D\mathbf{w}}{dv} = \frac{\partial \mathbf{w}}{\partial v} - \left\langle \frac{\partial \mathbf{w}}{\partial v}, \mathbf{N} \right\rangle \mathbf{N}, \tag{6.3}$$

where $\mathbf{N}(u, v)$ is the unit normal along the patch.

By construction, $\frac{D\mathbf{w}}{du}$ and $\frac{D\mathbf{w}}{dv}$ are tangent vector fields.

Example 6.3 (MA3215 AY14/15 Sem 2 Tutorial 10). This is a continuation of Example 6.2. Let S be a sphere and let $\mathbf{w}(x, y, z) = (-y, x, 0)$ be a vector field in it. We use the stereographic projection

$$\mathbf{x}(u,v) = \pi^{-1}(u,v) = R^{-1}\left(4u, 4v, u^2+v^2-4\right) \tag{6.4}$$

where $R = u^2+v^2+4$.

(i) Compute $\frac{\partial \mathbf{w}}{\partial u}$ and $\frac{\partial \mathbf{w}}{\partial v}$.

(ii) Compute $\frac{D\mathbf{W}}{\partial u}$ and $\frac{D\mathbf{W}}{\partial v}$.

Solution.

(i) We have

$$\frac{\partial \mathbf{w}}{\partial u} = 4R^{-2}\left(2uv, -u^2+v^2+4, 0\right) \quad \text{and} \quad \frac{\partial \mathbf{w}}{\partial v} = -4R^{-2}\left(u^2-v^2+4, 2uv, 0\right).$$

(ii) These are known as the covariant derivatives as seen in Definition 6.3. One can find

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$$

then use the formulae in (6.3). □

Definition 6.4 (covariant derivative along a curve). Let $\alpha : I \to S$ be a smooth curve and let $\mathbf{w}$ be a tangent vector field along α, i.e. for each $t \in I$ we have $\mathbf{w}(t) \in T_{\alpha(t)}S$. Differentiate $\mathbf{w}(t)$ in $\mathbb{R}^3$ and project to the tangent plane:

$$\frac{D\mathbf{w}}{dt} = \frac{d\mathbf{w}}{dt} - \left\langle \frac{d\mathbf{w}}{dt}, \mathbf{N}(\alpha(t)) \right\rangle \mathbf{N}(\alpha(t)).$$

Example 6.4 (MA3215 AY14/15 Sem 2 Tutorial 10). Let S be a regular surface, and let $\mathbf{v}(t)$ and $\mathbf{w}(t)$ be vector fields along a curve $\alpha(t)$ on S. Let $\frac{D\mathbf{v}}{dt}$ denote the orthogonal projection of $\mathbf{v}'(t)$ onto the tangent plane. Prove the following:

(i)

$$\langle \mathbf{v}'(t), \mathbf{w}(t) \rangle = \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{w}(t) \right\rangle$$

(ii)

$$\frac{d}{dt}\langle \mathbf{v}(t), \mathbf{w}(t) \rangle = \left\langle \frac{D\mathbf{v}}{dt}, \mathbf{w}(t) \right\rangle + \left\langle \mathbf{v}(t), \frac{D\mathbf{w}}{dt} \right\rangle \tag{6.5}$$

Solution.

(i) By Definition 6.4,

$$\frac{D\mathbf{v}}{dt} = \mathbf{v}' - \langle \mathbf{v}', \mathbf{N} \rangle \mathbf{N}.$$

Hence,

$$\left\langle \frac{D\mathbf{v}}{dt}, \mathbf{w}(t) \right\rangle = \langle \mathbf{v}' - \langle \mathbf{v}', \mathbf{N} \rangle \mathbf{N}, \mathbf{w} \rangle = \langle \mathbf{v}', \mathbf{w} \rangle - \langle \mathbf{v}', \mathbf{N} \rangle \langle \mathbf{N}, \mathbf{w} \rangle.$$

Since $\mathbf{w} \in T_\alpha S$ and $T_\alpha S \perp \mathbf{N}$, then $\langle \mathbf{N}, \mathbf{w} \rangle = 0$ and the result follows.

(ii) By the product rule for inner products,

$$\frac{d}{dt}\langle \mathbf{v},\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{w}'\rangle + \langle \mathbf{v}',\mathbf{w}\rangle .$$

By considering the right side of (6.5) and Definition 6.4,

$$\left\langle \frac{D\mathbf{v}}{dt},\mathbf{w}(t)\right\rangle + \left\langle \mathbf{v}(t),\frac{D\mathbf{w}}{dt}\right\rangle = \langle \mathbf{v}' - \langle \mathbf{v}',\mathbf{N}\rangle\mathbf{N},\mathbf{w}\rangle + \langle \mathbf{v},\mathbf{w}' - \langle \mathbf{w}',\mathbf{N}\rangle\mathbf{N}\rangle .$$

To conclude the proof, use the fact that $\langle \mathbf{v}',\mathbf{N}\rangle = 0$ and $\langle \mathbf{w}',\mathbf{N}\rangle = 0$. □

Definition 6.5. A vector field $\mathbf{w}$ along α is called parallel along α if

$$\frac{D\mathbf{w}}{dt} = (0,0,0) \quad \text{for all } t \in I.$$

We now introduce the Christoffel symbols (Definition 6.6), which package the tangent components of the derivatives of the coordinate basis vectors $\{\mathbf{x}_u,\mathbf{x}_v\}$. In Proposition 6.1, we will see that they provide a coordinate expression for covariant differentiation and lead to a practical formula for $\frac{D\mathbf{w}}{du}$ and $\frac{D\mathbf{w}}{dv}$ when $\mathbf{w}$ is written in the basis $\mathbf{x}_u,\mathbf{x}_v$.

Definition 6.6 (Christoffel symbols via tangent projections). Let $\mathbf{x}(u,v)$ be a coordinate map with tangent basis $\{\mathbf{x}_u,\mathbf{x}_v\}$. The tangent components of the second derivatives are encoded by the Christoffel symbols Γ^k_{ij}:

$$\begin{aligned}
\frac{D\mathbf{x}_u}{du} &= \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v \\
\frac{D\mathbf{x}_u}{dv} &= \Gamma^1_{12}\mathbf{x}_u + \Gamma^2_{12}\mathbf{x}_v \\
\frac{D\mathbf{x}_v}{du} &= \Gamma^1_{21}\mathbf{x}_u + \Gamma^2_{21}\mathbf{x}_v \\
\frac{D\mathbf{x}_v}{dv} &= \Gamma^1_{22}\mathbf{x}_u + \Gamma^2_{22}\mathbf{x}_v
\end{aligned}$$

For regular surfaces, one has $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, hence typically $\Gamma^k_{12} = \Gamma^k_{21}$.

Proposition 6.1. If $\mathbf{w} = a(u,v)\mathbf{x}_u + b(u,v)\mathbf{x}_v$, then

$$\begin{aligned}
\frac{D\mathbf{w}}{du} &= \left(a_u + a\Gamma^1_{11} + b\Gamma^1_{21}\right)\mathbf{x}_u + \left(b_u + a\Gamma^2_{11} + b\Gamma^2_{21}\right)\mathbf{x}_v \\
\frac{D\mathbf{w}}{dv} &= \left(a_v + a\Gamma^1_{12} + b\Gamma^1_{22}\right)\mathbf{x}_u + \left(b_v + a\Gamma^2_{12} + b\Gamma^2_{22}\right)\mathbf{x}_v
\end{aligned}$$

6.3 Geodesic Curvature

Definition 6.7 (geodesic curvature and normal curvature). Let $\alpha(s)$ be a curve on S parametrised by arc length. Fix $p = \alpha(s_0)$. Let $t = \alpha'(s_0)$, $N = N(p)$, and

$s = N \wedge t$. Then, $\{t, s, N\}$ is an orthonormal basis of $\mathbb{R}^3$ at p, and we decompose

$$\alpha''(s_0) = \langle N, \alpha''(s_0) \rangle N + \langle s, \alpha''(s_0) \rangle s.$$

Define

$$k_n = \langle N, \alpha''(s_0) \rangle \quad \text{and} \quad k_g = \langle s, \alpha''(s_0) \rangle.$$

Here k_n is the normal curvature and k_g is the geodesic curvature. Since $N \perp s$, Pythagoras' theorem gives

$$k^2 = k_n^2 + k_g^2, \tag{6.6}$$

where $k = |\alpha''(s_0)|$ is the usual curvature of α in $\mathbb{R}^3$.

(6.6) shows that the ambient curvature splits orthogonally into tangential and normal contributions. This decomposition singles out the distinguished curves on S that are as straight as possible: those for which the tangential component of the acceleration vanishes. Equivalently, as we would see in Proposition 6.2, $\frac{D\alpha'}{ds} = 0$.

We say that a curve $\alpha(s)$ on S parametrised by arc length is called a geodesic if

$$\frac{D\alpha'}{ds} = (0,0,0) \quad \text{for all } s.$$

See Proposition 6.2 for equivalent characterisations of geodesics.

Proposition 6.2 (equivalent characterizations of geodesics). Let $\alpha(s)$ be a curve on S parametrised by arc length. The following are equivalent:

(i) $\frac{D\alpha'}{ds} = (0,0,0)$ for all s

(ii) $\alpha''(s)$ is perpendicular to $T_{\alpha(s)}S$ for all s

(iii) $\alpha''(s)$ is parallel to the unit normal $N(\alpha(s))$ for all s

(iv) The geodesic curvature satisfies $k_g(s) = 0$ for all s

Proof. We first prove **(i)** implies **(ii)**. By Definition 6.4, □

Example 6.5 (MA3215 AY14/15 Sem 2 Tutorial 10). As we have seen from Proposition 6.2, the curve $\alpha(s)$ is called a geodesic if its geodesic curvature $k_g(s) = 0$ at every point of the curve, or equivalently, $\alpha''(s)$ is perpendicular to the surface S for all s.

(i) Suppose S is the unit sphere and $\alpha(s)$ is a latitude on a sphere. Show that a latitude is a geodesic if and only if it is the equator.

(ii) Suppose S is the xy-plane and $\alpha(s)$ is a geodesic on it. Show that $\alpha''(s) = (0,0,0)$. Hence conclude that $\alpha(s)$ is a geodesic if and only if it is a straight line.

Solution.

(i) From Definition 6.7, $k_g = \langle s, \alpha'' \rangle$. A non-degenerate latitude is the intersection of S^2 with a horizontal plane $z = h$, where $|h| < 1$. It is a circle of radius $\rho = \sqrt{1-h^2}$. We parametrise the curve using

$$\alpha(s) = \left(\rho \cos\left(\frac{s}{\rho}\right), \rho \sin\left(\frac{s}{\rho}\right), h\right) \quad \text{where } s \in \mathbb{R}.$$

It is easy to see that

$$\alpha''(s) = \left(-\frac{1}{\rho}\cos\left(\frac{s}{\rho}\right), -\frac{1}{\rho}\sin\left(\frac{s}{\rho}\right), 0\right).$$

Using the fact that $\mathbf{s} = \mathbf{N} \wedge \mathbf{t}$, one can prove the if and only if statement immediately.

(ii) Then the tangent plane at every point is the xy-plane itself, and a unit normal field is $(0,0,1)$. By Proposition 6.2, α'' must be parallel to the unit normal $\mathbf{N}$. Since α lies on the xy-plane, its z-component is zero. As such, the z-component of α'' is also 0, which implies $\alpha''(s) = (0,0,0)$.

We then prove that $\alpha(s)$ is a geodesic if and only if it is a straight line. For the forward direction, suppose α is a geodesic. Then, $\alpha(s) = \mathbf{v}s + \mathbf{p}$ for some constant vectors $\mathbf{v}, \mathbf{p} \in \mathbb{R}^3$. Since $\alpha(s) \in S$, then α is a straight line on the plane. For the reverse direction, suppose $\alpha(s) = \mathbf{v}s + \mathbf{p}$. Then, one can prove that $k_g = \langle s, \alpha'' \rangle = 0$ by using the fact that $\alpha'' = (0,0,0)$. $\square$

Proposition 6.3 (isometries preserve geodesics). Let $\varphi : S_1 \to S_2$ be an isometry. If α is a geodesic on S_1, then $\beta = \varphi \circ \alpha$ is a geodesic on S_2.

Proposition 6.4 (existence and uniqueness). Let $p \in S$ and let $v \in T_pS$ be a unit tangent vector. Then there exists a unique geodesic $\alpha(s)$ parametrised by arc length such that

$$\alpha(0) = p \quad \text{and} \quad \alpha'(0) = v.$$

Example 6.6. On the unit sphere, the geodesics are exactly the arcs of great circles.

Proposition 6.5 (geodesics are locally shortest paths). Given two points $p, q \in S$ that are sufficiently near, there exists a geodesic joining p to q, and among all nearby curves joining p to q, this geodesic has minimal arc length.

Let $x : U \to S$ be a coordinate map and write a curve as

$$\alpha(t) = x(a(t), b(t)).$$

Using $\alpha''(t) \perp \operatorname{span}\{x_u, x_v\}$ and expressing the tangent components via the Christoffel symbols, one obtains the geodesic equations

$$a''(t) + \Gamma^1_{11}(a(t),b(t))\left(a'(t)\right)^2 + 2\Gamma^1_{12}(a(t),b(t))\,a'(t)\,b'(t) + \Gamma^1_{22}(a(t),b(t))\left(b'(t)\right)^2 = 0$$
$$b''(t) + \Gamma^2_{11}(a(t),b(t))\left(a'(t)\right)^2 + 2\Gamma^2_{12}(a(t),b(t))\,a'(t)\,b'(t) + \Gamma^2_{22}(a(t),b(t))\left(b'(t)\right)^2 = 0$$

Given initial conditions $\alpha(0) = p$ and $\alpha'(0) = v$, the theory of ordinary differential equations yields a unique local solution, hence a unique geodesic with the prescribed initial data.

Chapter 7

The Gauss-Bonnet Theorem

7.1 The Local Gauss-Bonnet Theorem

Let S be an orientable regular surface, with a smooth unit normal field N (equivalently, a Gauss map exists). Then, the Gaussian curvature $K(p)$ is defined for each $p \in S$ and gives a function $K : S \to \mathbb{R}$. Let $R \subseteq S$ be a simple region bounded by three smooth curves C_1, C_2, C_3, each parametrised by arc length, and oriented so that the boundary direction and N obey the right hand rule. Along each boundary curve C_i, we have the geodesic curvature k_g, hence a function $k_g : C_i \to \mathbb{R}$. At the three vertices, record the interior angles $\lambda_1, \lambda_2, \lambda_3$.

Theorem 7.1 (local Gauss-Bonnet theorem). With the above setup,

$$\sum_{i=1}^{3} \int_{C_i} k_g \, ds + \iint_R K \, dS = 2\pi - \sum_{i=1}^{3} (\pi - \lambda_i) . \tag{7.1}$$

The same identity (7.1) holds when ∂R is a union of n smooth curves surrounding the region. Also, requiring arc length parametrisation is not essential since $\int k_g \, ds$ is a line integral and is independent of parametrisation.

Example 7.1 (Euclidean plane). If $S = \mathbb{R}^2$ (the xy-plane), then $K = 0$. If the boundary curves are geodesics (straight lines), then $k_g = 0$. By the local Gauss-Bonnet theorem (Theorem 7.1), we have

$$0 = 2\pi - \sum_{i=1}^{3} (\pi - \lambda_i) \quad \text{so} \quad \lambda_1 + \lambda_2 + \lambda_3 = \pi.$$

Example 7.2 (sphere). Let S be a sphere of radius r, so $K = r^{-2}$. If the boundary curves are arcs of great circles, then $k_g = 0$. Thus,

$$\iint_R K \, dS = 2\pi - \sum_{i=1}^{3} (\pi - \lambda_i) \quad \text{so} \quad \lambda_1 + \lambda_2 + \lambda_3 = \pi + \frac{\text{Area}(R)}{r^2}.$$

7.2 The Global Gauss-Bonnet Theorem

A triangulation of a nice region $R \subset S$ divides R into finitely many triangles. Let F denote the number of faces (triangles), E denote the number of edges, and V denote the number of vertices. The Euler-Poincaré characteristic of the triangulation is

$$\chi(R) = F - E + V.$$

It is a fact that $\chi(R)$ does not depend on the chosen triangulation. We now state the global Gauss-Bonnet theorem with boundary (Theorem 7.2).

Theorem 7.2 (global Gauss-Bonnet Theorem with boundary). Let R be a nice region in an oriented surface S. Suppose the boundary ∂R is a union of nice curves $C_1, \dots, C_n$, each positively oriented (with respect to the chosen normal field). Let $\lambda_1, \dots, \lambda_n$ be the internal angles at the corners of ∂R. Then,

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)\, ds + \iint_R K\, dS + \sum_{i=1}^{n} (\pi - \lambda_i) = 2\pi\chi(R).$$

We give a rough proof sketch of Theorem 7.2. First, triangulate R into sufficiently small triangles so that each triangle lies in a coordinate patch. Then, apply the local Gauss-Bonnet theorem (Theorem 7.1) to each triangle and sum over all triangles. The boundary terms telescope on interior edges, leaving only the contribution from ∂R, and the angle bookkeeping produces the Euler characteristic term.

Definition 7.1 (genus). For a bounded surface without boundary (i.e. a compact surface without boundary), the topology is determined by the number of holes/handles, called the genus, denoted by $\operatorname{genus}(S)$. For such surfaces,

$$\chi(S) = 2 - 2\operatorname{genus}(S).$$

We now state the Gauss-Bonnet theorem for compact surfaces without boundary.

Theorem 7.3 (Gauss-Bonnet theorem for compact surfaces without boundary). Let S be a compact regular surface (without boundary). Then,

$$\iint_S K\, dS = 2\pi\chi(S).$$

Theorem 7.3 is a prototypical bridge between a local differential-geometric quantity (K) and a global topological invariant (χ or genus).

Example 7.3 (ellipsoid). An ellipsoid has genus 0, hence $\chi(S) = 2$ by Definition 7.1. By Theorem 7.3, we have

$$\iint_S K\, dS = 2\pi\chi(S) = 4\pi.$$

In particular, the value 4π depends only on the topology, not on the specific shape (round sphere vs. rugby ball).

Example 7.4 (MA3215 AY14/15 Sem 2 Tutorial 10). Let

$$U = \left\{(u,v) \in \mathbb{R}^2 : 0 < u < 2\pi, 0 < v < 2\pi\right\}.$$

Also, let

$$\mathbf{x}(u,v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v, r\sin u)$$

for $(u,v) \in U$. This yields a parametrisation of the torus S. In Example 3.36 on our discussion of the first fundamental form, we showed that

$$\sqrt{EG - F^2} = r(a + r\cos u).$$

In Example 4.8, we computed the Gaussian curvature

$$K = \frac{\cos u}{r(a + r\cos u)}.$$

The torus is a compact regular surface without boundary. Then,

$$\iint_S K\, dS = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{r(a + r\cos u)} \cdot \sqrt{EG - F^2}\, du dv = \int_0^{2\pi}\int_0^{2\pi} \cos u\, du dv = 0.$$

By Theorem 7.3, $\chi(S) = 0$. By Definition 7.1, we know that $\chi(S) = 2 - 2\,\text{genus}(S)$, and because the genus of a torus is 1, the Gauss-Bonnet theorem indeed holds.

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