MA2108 and MA3210 Mathematical Analysis I and II

Thang Pang Ern

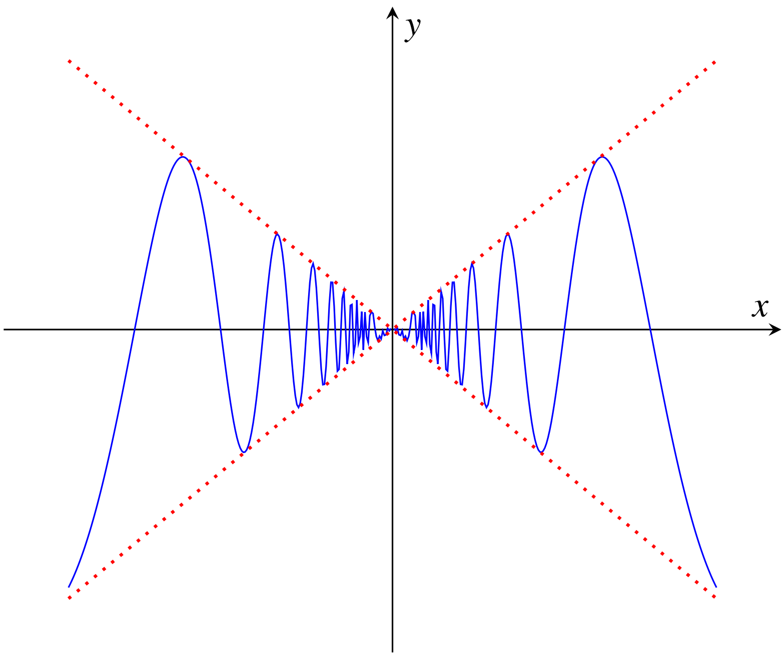
Contents

References

1 Rudin, W. (1953). *Principles of Mathematical Analysis*. McGraw Hill.

2 Johnsonbaugh, R., & Pfaffenberger, W. E. (2010). *Foundations of Mathematical Analysis*. Dover Publications.

3 Munkres, J. (2000). *Topology (2nd ed.)*. Pearson.



Sandwiched.

Chapter 1
The Real Numbers, $\mathbb{R}$

1.1
Fields

We say that $\mathbb{R}$ is a complete ordered field. There are three big ideas to be discussed — completeness, ordering, and fields! We will first discuss the property on fields, and we say that $\mathbb{R}$ satisfies the field axioms (Definition 1.1)[†]. There are many properties which might be deemed *trivial* but we will still discuss them. For example, the trichotomy property of $\mathbb{R}$[‡] states that

$$\text{if } a,b\in\mathbb{R} \quad \text{then} \quad \text{either } a<b, a>b \text{ or } a=b.$$

This is intuitive!

Definition 1.1 (field axioms)**.** A field consists of a set F satisfying the following properties:

(i) an additive map

$$+:F\times F\to F \quad \text{where} \quad (x,y)\mapsto x+y$$

(ii) the existence of an additive identity $0\in F$

(iii) a negation map

$$-:F\times F\to F \quad \text{where} \quad x\mapsto -x$$

(iv) a multiplication map

$$\cdot:F\times F\to F \quad \text{where} \quad (x,y)\mapsto xy$$

(v) the existence of a multiplicative identity $1\in F$

(vi) a reciprocal map

$$(-)^{-1}:F\backslash\{0\}\to F\backslash\{0\} \quad \text{where} \quad x\mapsto x^{-1}$$

such that the following properties are satisfied:

(i) $+$ is commutative, i.e. for all $x,y\in F$, we have $x+y=y+x$

(ii) $+$ is associative, i.e. for all $x,y,z\in F$, we have $(x+y)+z=x+(y+z)$

(iii) 0 is the identity for $+$, i.e. for all $x\in F$, we have $x+0=x=0+x$

(iv) $-$ is the additive inverse of addition, i.e. for all $x\in F$, we have $x+(-x)=0=(-x)+x$

(v) $\cdot$ is commutative, i.e. for all $x,y\in F$, we have $xy=yx$

(vi) $\cdot$ is associative, i.e. for all $x,y,z\in F$, we have $(xy)z=x(yz)$

(vii) 1 is the identity for $\cdot$, i.e. for all $x\in F$, we have $x1=x=1x$

(viii) $(-1)^{-1}$ is the inverse of $\cdot$, i.e. for all $x\in F$, we have $xx^{-1}=1=x^{-1}x$

(ix) $1\neq 0$, i.e. F is not the zero (trivial) field

(x) $\cdot$ is distributive over $+$, i.e. for all $x,y,z\in F$, we have

$$x(y+z)=xy+xz \quad \text{and} \quad (x+y)z=xz+yz$$

[†] An abrupt introduction.

[‡] In fact, we can regard the trichotomy property of $\mathbb{R}$ as a combination of the reflexivity and antisymmetry properties in Definition 1.2 and the comparability property in Definition 1.3. Alternatively, one can refer to **(iii)** of Proposition 1.5.

Remark 1.1. When we were discussing the properties of a field in Definition 1.1, recall that multiplication is denoted by $\cdot$, and we can *condense* $x \cdot y$ as xy. For example, refer to **(v)**, which can also be written as $x \cdot y = y \cdot x$.

Example 1.1. The best known fields are those of

$$\begin{aligned}\mathbb{Q} &= \text{field of rational numbers}\\ \mathbb{R} &= \text{field of real numbers}\\ \mathbb{C} &= \text{field of complex numbers}\end{aligned}$$

Example 1.2. In Number Theory or Abstract Algebra in general,

$$\begin{aligned}\mathbb{Q}_p &= \text{field of } p\text{-adic numbers}\\ \mathbb{F}_p &= \text{finite field of } p \text{ elements}\end{aligned}$$

Example 1.3. Let k be a field. Then, define

$$K(t) \quad \text{to be the field of rational functions over } K.$$

We then discuss the general properties of fields.

Proposition 1.1. The axioms for addition in Definition 1.1 imply the following statements: for all $x,y,z \in F$,

(i) Cancellation for $+$: if $x+y=x+z$, then $y=z$;
(ii) Uniqueness of 0: if $x+y=x$, then $y=0$;
(iii) Uniqueness of negative: if $x+y=0$, then $y=-x$;
(iv) Negative of negative: $-(-x)=x$

We will only prove **(i)** and **(iv)**.

Proof. First, we prove **(i)**. Suppose $x,y,z \in F$ such that $x+y=x+z$. Then, as $-x \in F$, we have

$$\begin{aligned}((-x))+x+y &= ((-x))+x+z\\ ((-x)+x)+y &= ((-x)+x)+z \quad \text{by associativity of } +\\ 0+y &= 0+z \quad \text{since 0 is the additive identity in } F\end{aligned}$$

and we conclude that $y=z$. □

We then prove **(iv)**.

Proof. Recall that $x+(-x)=0$. The trick now is to consider

$$-(-x)+(-x)=0 \quad \text{which again follows by the axiom for negation!}$$

As such,

$$\begin{aligned}x+(-x) &= -(-x)+(-x)\\ x &= -(-x) \quad \text{by the cancellation property in } \textbf{(i)}\end{aligned}$$

so **(iv)** holds. □

Proposition 1.2. The axioms for multiplication in Definition 1.1 imply the following statements: for all $x, y, z \in F$,

(i) Cancellation for $\cdot$: if $x \neq 0$ and $xy = xz$, then $y = z$;

(ii) Uniqueness of multiplicative identity: if $x \neq 0$ and $xy = x$, then $y = 1$;

(iii) Uniqueness of reciprocal: if $x \neq 0$ and $xy = 1$, then $y = 1/x$;

(iv) Reciprocal of reciprocal: if $x \neq 0$, then $1/(1/x) = x$

Proposition 1.3. The field axioms (Definition 1.1) imply the following statements: for all $x, y \in F$,

(i) $0x = 0$;

(ii) if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$

(iii) $(-x)y = -(xy) = x(-y)$

(iv) $(-x)(-y) = xy$

We now discuss what it means for a set to be ordered.

Definition 1.2 (partial order). Let S be a set. A partial ordering relation on S is a relation $\leq$ on S satisfying the following properties:

(i) Reflexivity: for all $x \in S$, we have $x \leq x$

(ii) Transitivity: for all $x, y, z \in S$, we have $x \leq y$ and $y \leq z$ imply $x \leq z$

(iii) Antisymmetry: for all $x, y \in S$, we have $x \leq y$ and $y \leq x$ implies $x = y$

Definition 1.3 (total order). A total ordering relation on S is partial ordering relation $\leq$ (Definition 1.2) on S which also satisies the following property that $\leq$ is comparable:

$$\text{for all } x, y \in S \quad \text{we have} \quad x \leq y \text{ or } y \leq x.$$

Example 1.4. Let S be a set. Then, the subset relation $\subseteq$ on $\mathcal{P}(S)$ is a partial ordering but not a total ordering when $|S| > 1$.

Definition 1.4 (ordered field). An ordered field consists of a field F and a total ordering $\leq$ on F saitsyfing the following properties:

(i) $\leq$ is compatible with $+$: for all $x, y, z \in F$, we have

$$x \leq y \quad \text{implies} \quad x + z \leq y + z$$

(ii) $\leq$ is compatible with $\cdot$: for all $x, y, z \in F$, we have

$$x \leq y \text{ and } z > 0 \quad \text{implies} \quad xz \leq yz$$

Definition 1.5. If

$$x > 0 \text{ we call } x \text{ positive} \quad \text{and if} \quad x < 0 \text{ we call } x \text{ negative} \quad \text{and if}$$
$$x \geq 0 \text{ we call } x \text{ non-negative} \quad \text{and if} \quad x \leq 0 \text{ we call } x \text{ non-positive}$$

Definition 1.6. We have

$$\begin{aligned} F_{>0} &= \{x \in F : x > 0\} \\ F_{<0} &= \{x \in F : x < 0\} \\ F_{\geq 0} &= \{x \in F : x \geq 0\} = F_{>0} \cup \{0\} \\ F_{\leq 0} &= \{x \in F : x \leq 0\} = F_{<0} \cup \{0\} \end{aligned}$$

Example 1.5. $\mathbb{Q}$ given with the usual ordering $\leq$ is an ordered field. We will eventually construct $\mathbb{R}$ as an ordered field.

Proposition 1.4. Let F be an ordered field. Then,

$$\text{for all } x, y \in F \quad \text{we have} \quad x \leq y \text{ if and only if } -x \geq -y.$$

In particular, $F_{<0} = -F_{>0}$ and $F_{\leq 0} = -F_{\geq 0}$.

Proof. We first prove the forward direction. If $x \leq y$, we take $z = (-x) + (-y)$ in F. As such, $x + z \leq y + z$, which implies $-y \leq -x$. For the reverse direction, we apply the same idea to $(x, y) = (-y, -x)$ to obtain $-(-x) \leq -(-y)$. As such, $x \leq y$. □

Proposition 1.5 (closure properties and trichotomy)**.** For any ordered field F,

(i) $F_{>0}$ **is closed under addition:** $F_{>0} + F_{>0} \subseteq F_{>0}$

(ii) $F_{>0}$ **is closed under multiplication:** $F_{>0} \cdot F_{>0} \subseteq F_{>0}$

(iii) **Trichotomy:** $F = F_{>0} \sqcup \{0\} \cup (-F_{>0})$

Proposition 1.6. For any ordered field F, the following hold:

(i) for all $x \in F$, we have $x^2 \geq 0$

(ii) for all $x, y \in F$ such that $0 < x < y$, we have $0 < 1/y < 1/x$

Proof. We first prove **(i)**. Suppose $x \geq 0$. Then, $x^2 = x \cdot x \geq 0 \cdot x = 0$ by the compatibility of $\leq$ with $\cdot$ (recall **(ii)** of Definition 1.4). If $x < 0$, then $-x > 0$, so $x^2 = (-x)(-x) > 0 \cdot (-x) = 0$ again by **(ii)** of Definition 1.4.

We then prove **(ii)**. Suppose $x > 0$. If $x^{-1} \leq 0$, then $0 = x \cdot 0 \geq x \cdot x^{-1} = 1$, which is a contradiction. As such, we must have $x^{-1} > 0$. If $0 < x < y$, then $xy > 0$ since $F_{>0}$ is closed under multiplication (**(ii)** of Proposition 1.5). As such, $(xy)^{-1} > 0$. Hence,

$$0 < y^{-1} = x \cdot (xy)^{-1} < y \cdot (xy)^{-1} = x^{-1}$$

by the compatibility of $\leq$ with $\cdot$ as mentioned in **(ii)** of Definition 1.4. □

Proposition 1.7 (field characteristic)**.** Let F be an ordered field. Then,

$$\text{for all } n \in \mathbb{N} \quad \text{we have} \quad n \cdot 1 = \underbrace{1_F + 1_F + \cdots + 1_F}_{n \text{ terms}} \quad \text{in } F.$$

In Abstract Algebra, we say that ordered fields have characteristic zero.

Proof. We shall induct on n. The base case $n=1$ is trivial as $1>0$ in F. Next, for any $n\in\mathbb{N}$, if $n\cdot 1>0$ in F, then $(n+1)\cdot 1=n\cdot 1+1>0$ because $n\cdot 1>0$ by the inductive hypothesis and $1>0$ trivially. As such, the proposition holds. □

For those who are interested in Abstract Algebra, Definition 1.7 would appeal to you.

Definition 1.7. Let F be an ordered field. Then, F is of characteristic zero. Also, there exists a unique *homomorphism* of fields

$$\iota:\mathbb{Q}\hookrightarrow F \quad \text{called} \quad \text{the canonical inclusion of } \mathbb{Q} \text{ into } F.$$

Moreover, ι is injective and order-preserving.

Via the canonical inclusion $\iota:\mathbb{Q}\hookrightarrow F$ of $\mathbb{Q}$ into F, we will identify $\mathbb{Q}$ with $\iota(\mathbb{Q})\subseteq F$ and regard $\mathbb{Q}$ as a subfield of F. All these will be covered in MA3201.

Remark 1.2. It follows that ordered fields must be infinite. Also, ordered fields cannot be algebraically closed. To see why, we note that $x^2+1=0$ has no solution in the ordered field F.

1.2
Supremum, Infimum and Completeness

Definition 1.8 (upper and lower bound)**.** Let S be an ordered set, i.e. a set given with a total ordering. We say that a subset $R\subseteq S$ is

$$\begin{aligned} &\text{bounded above} && \text{if and only if} && \text{there exists } B\in S \text{ such that for all } x\in E && \text{we have } x\leq B\\ &\text{bounded below} && \text{if and only if} && \text{there exists } A\in S \text{ such that for all } x\in E && \text{we have } A\leq x\\ &\text{bounded} && \text{if and only if} && \text{it is bounded above and bounded below} \end{aligned}$$

We say that

$$\begin{aligned} A\in S &\quad \text{is a lower bound of } E \text{ in } S\\ B\in S &\quad \text{is an upper bound of } E \text{ in } S \end{aligned}$$

Definition 1.9 (supremum and infimum)**.** Let S be an ordered set and $E\subseteq S$ be any subset. A real number α is the supremum (least upper bound or LUB) of E if

$$\alpha \text{ is an upper bound of } E \quad \text{and} \quad \alpha\leq u \text{ for every upper bound } u\in E, \text{ i.e. } \alpha=\sup(E).$$

A real number β is the infimum (greatest lower bound or GLB) of E if

$$\beta \text{ is a lower bound of } E \quad \text{and} \quad \beta\geq u \text{ for every lower bound } u\in E, \text{ i.e. } \beta=\inf(E).$$

Proposition 1.8. For an ordered set S, let $E \subseteq S$. Then, the set of upper bounds of E in S is always a subset of S. However, it may be empty. We remark that

$$\text{the set of upper bounds} = \emptyset \quad \text{if and only if} \quad E \text{ is not bounded above in } S.$$

Example 1.6. Take $S = \mathbb{Q}$ and $E = \mathbb{Z}$. Then, $E \subseteq S$, and we note that the set of upper bounds of $\mathbb{Z}$ in $\mathbb{Q}$ is $\emptyset$ as the $\sup(\mathbb{Z})$ does not exist.

Remark 1.3. The supremum and infimum of a set may or may not be elements of the set.

Example 1.7. Consider

$$E = \{x \in \mathbb{R} : 0 < x < 1\} \quad \text{where } \inf(E) = 0 \notin E \text{ and } \sup(E) = 1 \notin E.$$

Lemma 1.1 (supremum is unique)**.** Let S be an ordered set. Given $E \subseteq S$,

$$\text{if there exists a least upper bound of } E \text{ in } S \quad \text{then} \quad \sup(E) \text{ is unique.}$$

As mentioned, we write $\sup(E) \in S$ for the unique least upper bound of E in S if it exists.

Proof. The proof is very straightforward. Suppose both α and α' are least upper bounds of E in S. Then, one can show that $\alpha \leq \alpha'$ and $\alpha' \leq \alpha$ by using the two conditions mentioned in Definition 1.9. □

At this juncture, we note that a number of properties of the infimum, or greatest lower bound of a set, have not been discussed. These draw parallelisms with the definition of the supremum (both in Definition 1.9).

Definition 1.10 (least upper bound property)**.** An ordered set S has the least upper bound property if and only if for any non-empty subset $E \subseteq S$ which is bounded above, there exists a least upper bound $\sup(E) \in S$ of E in S.

Example 1.8. Let S_0 be an ordered set[†], and let $S \subseteq S_0$ be any finite subset. Then, S, which is regarded as an ordered set, has the least upper bound property (Definition 1.10). In fact, for any non-empty subset $E \subseteq S$ (which is necessarily finite since any subset of a finite set is also finite),

$$\sup(E) = \max(E) \quad \text{exists in } S \text{ (in fact in } E).$$

Lemma 1.2. $\mathbb{Z}$, as an ordered set, has the least upper bound property.

Proof. Suppose $E \subseteq \mathbb{Z}$ is any non-empty subset which is bounded above by $b_0 \in \mathbb{Z}$. Then, the set

$$b_0 \setminus E = \{b_0 - x \in \mathbb{Z} : x \in E\} = \{k \in \mathbb{Z} : b_0 - k \in E\} \quad \text{is a non-empty subset of } \mathbb{Z}_{\geq 0}.$$

Note that $b_0 \setminus E$ is indeed non-empty as $E \neq \emptyset$. By the well-ordering property of $\mathbb{Z}_{\geq 0}$, there exists a smallest element $k_0 \in b_0 \setminus E$. As such,

$$\sup(E) = \max(E) = b_0 - k_0 \quad \text{exists in } S.$$

□

[†]If you are unable to appreciate this example well, always make reference to sets, or number systems, that you already know which would be applicable here. For example, we can take $S_0 = \mathbb{Q}$. Consequently as we would see later, $S \subseteq S_0$ is a finite subset of the rationals. Suppose $S = \{-1/2, 3, 10/7\}$ and $E = \{-1/2, 10/7\}$. Then, $\sup(E)$ exists and it is equal to $\max(E) = 10/7$.

We then continue our discussion by showing that $\mathbb{Q}$ does not have the least upper bound property. There are some things to address first. One would know that the equation

$$p^2 = 2 \quad \text{is not satisfied by any } p \in \mathbb{Q}.$$

This shows that $\sqrt{2}$ is irrational, and consequently, $\mathbb{Q}$ does not have the least upper bound property. Anyway, the proof using the unique factorisation of $\mathbb{Z}$ is as follows:

$$p = \frac{a}{b} \quad \text{for some } a, b \in \mathbb{Z} \text{ and } b \neq 0.$$

Then, consider the prime factorisations of a and b to obtain

$$p = \frac{p_1^{\alpha_1} \ldots p_r^{\alpha_r}}{q_1^{\beta_1} \ldots q_s^{\beta_s}} \quad \text{so} \quad p_1^{2\alpha_1} \ldots p_r^{\alpha_r} = 2 \cdot q_1^{\beta_1} \ldots q_s^{\beta_s}.$$

The exponent of 2 on the LHS is even but it is odd on the RHS, resulting in a contradiction.

Now, let

$$A = \left\{ p \in \mathbb{Q}^+ : p^2 < 2 \right\}.$$

Note that A is non-empty and bounded above in $\mathbb{Q}$ since $1 \in A$ and for all $p \in A$, we have $p > 0$ and $p^2 < 2$, so we must have $p < 2$. We shall prove that A contains no largest number. More explicitly,

$$\text{for every } p \in A \quad \text{there exists } q \in A \text{ such that } p < q.$$

Now, for every $p \in A$, we construct q as follows:

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \in \mathbb{Q}^+.$$

Also,

$$q^2 - 2 = \left(\frac{2p + 2}{p + 2} \right)^2 - 2 = \frac{2\left(p^2 - 2\right)}{(p + 2)^2}.$$

Since $p \in A$, then $p^2 - 2 < 0$, so $q > p$ and $q^2 - 2 < 0$. Hence, $q \in A$ and is $> p$. As such, A contains no largest number. Anyway, here is a geometrical interpretation of the relationship between p and q (Figure 1). By constructing the line segment joining $\left(p, p^2 - 2\right)$ and $(2, 2)$ and defining $(q, 0)$ to be the point where this line intersects the x-axis, one can indeed deduce that

$$q = p - \frac{p^2 - 2}{p + 2}.$$

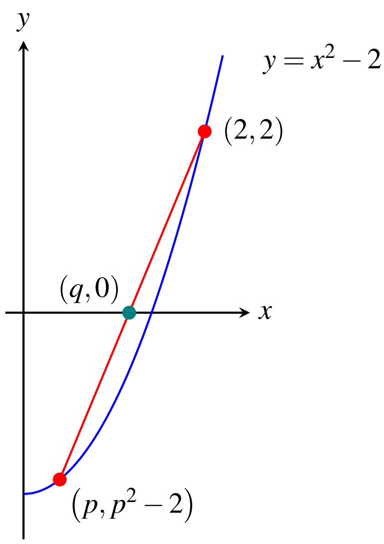


Figure 1: Graph of $y = x^2 - 2$

We are very close to showing that $\mathbb{Q}$ does not have the least upper bound property. More explicitly, for every p in the set of upper bounds of A in $\mathbb{Q}$, one can deduce that there exists q in this set such that $q < p$. As such, this set will not contain a smallest element.

Previously, we showed that A contains no largest number, so no element of A can be an upper bound of A. Similarly, for every p in the set of upper bounds of A in $\mathbb{Q}$, we construct q as follows (Figure 2):

$$q = p - \frac{p^2 - 2}{2p} = \frac{p^2 + 2}{2p} \in \mathbb{Q}^+$$

Also, as $p^2 - 2 > 0$, it follows that $q < p$, so

$$q^2 - 2 = \left(\frac{p^2 + 2}{2p}\right)^2 - 2 = \left(\frac{p^2 - 2}{2p}\right)^2 > 0$$

so $q \notin A$. As such, q is in the set of upper bounds of A in $\mathbb{Q}$.

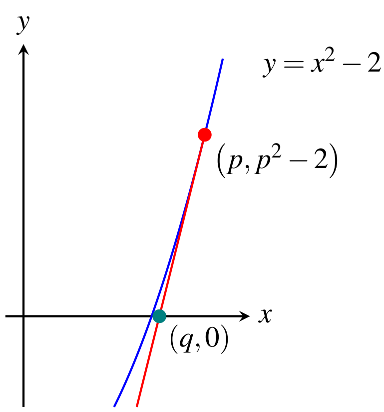


Figure 2: Graph of $y = x^2 - 2$

It follows that $\mathbb{Q}$ does not have the least upper bound property.

Proposition 1.9. Let S be an ordered set with the least upper bound property. Then, it also has the greatest lower bound property. That is to say, for any non-empty subset $B \subseteq S$ which is bounded below,

$$\text{there exists} \quad \text{a greatest lower bound } \inf(B) \in S \text{ of } B \text{ in } S.$$

Proof. Suppose $B \neq \emptyset$ is bounded below. Then, the set of lower bounds of B in S is non-empty and L is bounded above. By the least upper bound property of S, $\alpha = \sup(L)$ exists in S.

We claim that $\alpha = \inf(B)$ as well. We first prove that α is a lower bound of B. Note that for all $x \in B$, x is also in the set of upper bounds of L in S so $\alpha \leq x$. As such, α is also in the set of upper bounds of L in S. Next, we justify that α is the greatest among all lower bounds of B, which holds because $\alpha = \sup(L)$. □

Example 1.9 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 1). Let a, b, c, d be numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, and one where $bd < ac$.

Solution. For the first part, we can choose $a = 1$, $b = 2$, $c = -3$, $d = -1$ so that

$$ac = -3 \text{ and } bd = -2 \quad \text{so} \quad ac < bd.$$

For the second part, we can choose $a = 1$, $b = 2$, $c = -2$, $d = -2$ so that

$$bd = -4 \text{ and } ac = -2 \quad \text{so} \quad bd < ac.$$

□

Example 1.10 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 2). If $0 \leq a < b$, show that $a^2 \leq ab < b^2$. Show by example that it does *not* follow that $a^2 < ab < b^2$.

Solution. Suppose we are given that $0 \leq a < b$. We first prove that $a^2 \leq ab$, which is equivalent to showing that $ab - a^2 \geq 0$. As such, $a(b-a) \geq 0$. Since $a \geq 0$ and $b > a$ implies $b - a > 0$, then it follows that their product is non-positive, i.e. $a(b-a) \geq 0$.

We then prove that $ab < b^2$, which is equivalent to showing that $b^2 - ab > 0$. As such, $b(b-a) > 0$. Since $b > 0$ and $b > a$ implies $b - a > 0$, their product is positive, i.e. $b(b-a) > 0$.

Having said all these, we show by example that

$$0 \leq a < b \quad \text{does not imply} \quad a^2 < ab < b^2.$$

We choose $a = 0$ so $a^2 = 0$ and $ab = 0$, so the inequality $a^2 < ab$ does not hold. □

Example 1.11 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 3). Show the following: If $a \in \mathbb{R}$ is such that

$$0 \leq a \leq \varepsilon \text{ for every } \varepsilon > 0 \quad \text{then} \quad a = 0.$$

Solution. Since $\varepsilon > 0$ is arbitrary, we can choose $\varepsilon = a/2$, so $a \leq a/2$. As such, $a/2 \leq 0$, which implies $a \leq 0$. Combining with the fact that $a \geq 0$, we conclude that $a = 0$. □

Example 1.12 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 4). Let $a, b \in \mathbb{R}$, and suppose that for every $\varepsilon > 0$, we have $a \leq b + \varepsilon$. Show that $a \leq b$.

Solution. Suppose on the contrary that $a > b$. Then, $a - b > 0$. Choose $\varepsilon = (a-b)/2 > 0$, so

$$b + \varepsilon - a = b + \frac{a-b}{2} - a = \frac{2b + a - b - 2a}{2} = \frac{b-a}{2} < 0.$$

This implies $b + \varepsilon > a$ but this contradicts the fact that $a \leq b + \varepsilon$. To conclude, we must have $a \leq b$. □

Example 1.13 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 5)**.** Let

$$S_1 = \{x \in \mathbb{R} : x \geq 0\}.$$

Show in detail that the set S_1 has lower bounds but no upper bounds. Show that $\inf(S_1) = 0$.

Solution. We claim that the set

$$A = \{y \in \mathbb{R} : y \leq 0\} \quad \text{is} \quad \text{the set of lower bounds of } S_1.$$

Let $x \in S_1$ be an arbitrary element. Then, $x \geq 0$. Moreover, for any $y \in \mathbb{R}_{\leq 0}$, we have $x \geq 0 \geq y$, which implies that S_1 has lower bounds and they are all contained in A.

Next, we prove that S has no upper bound. Suppose on the contrary that it has one, say M. Then, $M \in \mathbb{R}_{\geq 0}$ is such that for every $x \in S_1$, we have $x \leq M$. Then, consider the inequality $x \leq M < M + 1$. As such, $M + 1$ is an upper bound of S_1. By definition of an upper bound, we must have $M + 1 \leq M$, which leads to a contradiction. We conclude that S_1 has no upper bound.

Lastly, we prove that $\inf(S_1) = 0$. Recall that A is the set of lower bounds of S_1. As the greatest value of A is 0, then by definition of infimum (greatest lower bound), we conclude that $\inf(S_1) = 0$. □

Example 1.14 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 6)**.** Let

$$S_2 = \{x \in \mathbb{R} : x > 0\}.$$

Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf(S_2)$ exist? Does $\sup(S_2)$ exist? Prove your statements.

Solution. Similar to Example 1.13, one can show that S_2 has lower bounds (take for example 0) but does not have any upper bound. Next, we claim that $\inf(S_2) = 0$. Consider the set

$$B = \{y \in \mathbb{R} : y \leq 0\} \quad \text{which is} \quad \text{the set of lower bounds of } S_2.$$

One can use the argument in Example 1.13 to justify this. Then, the greatest element of B is 0, so $\inf(S_2) = 0$.

Lastly, we claim that $\sup(S_2)$ does not exist. This follows from the fact that S_2 is not bounded above, so S_2 does not have a least upper bound. □

Example 1.15 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 7)**.** Let

$$S_3 = \{1/n : n \in \mathbb{N}\}.$$

Show that $\sup(S_3) = 1$ and $\inf(S_3) \geq 0$.

Solution. Let $\sup(S_3) = \alpha$. Then, for all $x \in S_3$, we must have $x \le \alpha$. That is to say, for any $n \in \mathbb{N}$, we must have $1/n \le \alpha$. We note that the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \text{ is strictly decreasing} \quad \text{as} \quad \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0.$$

As $\mathbb{N}$ satisfies the well-ordering property, it has a least element, which is 1. So, $1/1 = 1$ is the largest value of S_3, i.e. 1 is an upper bound for S_3. We then prove that 1 is indeed the least upper bound. Suppose on the contrary that there exists $\varepsilon > 0$ such that

$$1 - \varepsilon \quad \text{is} \quad \text{the least upper bound for } S_3 \quad \text{where} \quad n \in \mathbb{N}.$$

We claim that there exists $m \in \mathbb{N}$ such that

$$1 - \varepsilon < \frac{1}{m} \quad \text{or equivalently} \quad \varepsilon > 1 - \frac{1}{m} > 0.$$

This leads to a contradiction.

Next, we prove that $\inf(S_3) \ge 0$. Suppose $\inf(S_3) = \beta$. Then, as $1/n > 0$ for all $n \in \mathbb{N}$, then $\beta \ge 0$. In fact, we can further show that $\beta = 0$. Suppose there exists another lower bound $\beta' \ge 0$. Then, there exists $N \in \mathbb{N}$ such that $1/N < \beta'$, contradicting the fact that β' is a lower bound. We conclude that $\inf(S_3) = 0$. ☐

Example 1.16 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 8)**.** Let

$$S_4 = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$$

Find $\inf(S_4)$ and $\sup(S_4)$.

Solution. For any $x \in S_4$, we have

$$x = \begin{cases} 1 + 1/n & \text{if } n \text{ is odd;} \\ 1 - 1/n & \text{if } n \text{ is even.} \end{cases}$$

Clearly, $\inf(S_4) = 1/2$ and $\sup(S_4) = 2$. ☐

Example 1.17 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 9)**.** Find the infimum and supremum, if they exist, of each of the following sets:

(a) $A = \{x \in \mathbb{R} : 2x + 5 > 0\}$
(b) $B = \{x \in \mathbb{R} : x + 2 \ge x^2\}$
(c) $C = \{x \in \mathbb{R} : x < 1/x\}$
(d) $D = \{x \in \mathbb{R} : x^2 - 2x - 5 < 0\}$

Solution.

(a) The inequality is equivalent to $x > -5/2$, so $\sup(A)$ does not exist but $\inf(A) = -5/2$.
(b) The solution to the inequality is $-1 \le x \le 2$, so $\sup(B) = 2$ and $\inf(B) = -1$[†].
(c) We have

$$\frac{x^2 - 1}{x} < 0 \quad \text{so} \quad \frac{(x+1)(x-1)}{x} < 0.$$

Hence, $x \in (-\infty, -1) \cup (0, 1)$, so $\sup(C) = 1$ and $\inf(C) = -1$.

[†]Actually, to really argue this, we note that the solution set to the inequality is a compact set (just to jump the gun here, we can apply what is known as the Heine-Borel theorem. It states that for a subset of the Euclidean n-space $S \subseteq \mathbb{R}^n$, S is compact if and only if S is closed and bounded). In a compact set, we have $\sup(S) = \max(S)$ and $\inf(S) = \min(S)$.

(d) The solution to the inequality is $1-\sqrt{6}<x<1+\sqrt{6}$ so $\sup(D)=1+\sqrt{6}$ and $\inf(D)=1-\sqrt{6}$. $\square$

Example 1.18 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 18)**.** Show that

$$\sup\left\{1-\frac{1}{n}:n\in\mathbb{N}\right\}=1.$$

Solution. Let S be the mentioned set and suppose $\sup(S)=\alpha$. Then,

$$\text{for all } n\in\mathbb{N} \quad \text{we have} \quad 1-\frac{1}{n}\le\alpha.$$

Since the sequence $\{1-1/n\}_{n=1}^{\infty}$ is decreasing and bounded above by 1, then S is bounded above by 1. Proving that α is indeed 1 is trivial (we discussed this method multiple times). $\square$

Let us try to better understand the least upper bound property.

Example 1.19. Let $E\subseteq\mathbb{R}$ be any non-empty subset that is bounded above. For any $a\in\mathbb{R}$, consider the set

$$a+E=\{a+x\in\mathbb{R}:x\in E\} \quad \text{which is also non-empty and bounded above.}$$

Then, we have

$$\sup(a+E)=a+\sup(E) \quad \text{in } \mathbb{R}.$$

We will justify this result, i.e. show the equality of two real numbers. One common way to go about proving this is to show that the LHS $\le$ RHS and RHS $\le$ LHS directly. However, we see that proving the latter directly is difficult, so we will resort to using contradiction.

Proof. We first prove that $\sup(a+E)=a+\sup(E)$. Note that for any $y\in a+E$, there exists $x\in E$ such that $y=a+x$. So, $x\le\sup(E)$. Adding a to both sides of the inequality yields $y\le a+$E, so $a+\sup(E)$ is an upper bound of $a+E$. As such, $\sup(a+E)\le a+\sup(E)$.

We then prove that LHS $<$ RHS leads to a contradiction, which would assert that RHS $\le$ LHS. Suppose

$$\sup(a+E)<a+\sup(E) \quad \text{or equivalently} \quad \sup(E)>\sup(a+E)-a.$$

We claim that $\sup(a+E)-a$ is still an upper bound for E. To see why, for all $x\in E$, we have $a+x\in a+E$ so $a+x\le a+\sup(a+E)$. As such, $x\le\sup(a+E)-a$, contradicting the least upper bound property of $\sup(E)$. $\square$

Lemma 1.3. Let S be an ordered set, and suppose $E\subseteq S$. Let u be an upper bound of E. Then,

$$u=\sup(E) \quad \text{if and only if} \quad \text{for all } \varepsilon>0 \text{ there exists } x\in E \text{ such that } u-\varepsilon<x.$$

We refer to Figure 3 for an illustration of Lemma 1.3.

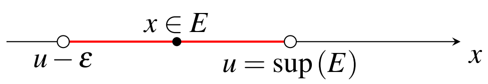


Figure 3: Illustration of the supremum condition in Lemma 1.3

Example 1.20 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 17)**.** Let S be a set that is bounded below. Prove that a lower bound w of S is the infimum of S if and only if for any $\varepsilon>0$, there exists $t\in S$ such that $t<w+\varepsilon$.

Solution. For the forward direction, suppose $w = \inf(S)$. So, for any $t \in S$, we have $w \le t$. Suppose on the contrary that

$$t \ge w+\varepsilon \quad \text{for every } t \in S.$$

So, $w+\varepsilon$ is also a lower bound for S. However, by definition of the infimum, w is the greatest lower bound for S, which implies $w \ge w+\varepsilon$. As such, $\varepsilon \le 0$, which is a contradiction.

For the reverse direction, suppose for any $\varepsilon > 0$, there exists $t \in S$ such that $t < w+\varepsilon$. We already know that w is a lower bound for S. Suppose v is another lower bound for w. We claim that $v \le w$. Suppose on the contrary that $v > w$. Choose $\varepsilon = w-v > 0$. Then, we have

$$t < w+\varepsilon \quad \text{which implies} \quad t < w+(v-w) = v.$$

As such, v cannot be a lower bound for S, which leads to a contradiction. □

Axiom 1.1. Every non-empty subset of $\mathbb{R}$ which is

$$\begin{matrix} \text{bounded above} \\ \text{bounded below} \end{matrix} \quad \text{has a} \quad \begin{matrix} \text{supremum;} \\ \text{infimum.} \end{matrix}$$

Example 1.21 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 2)**.** If $a < x < b$ and $a < y < b$, show that

$$|x-y| < b-a.$$

Solution. Suppose on the contrary that $|x-y| \ge b-a$. Then, either

$$x-y \ge b-a \quad \text{or} \quad x-y \le a-b.$$

Note that $a<y<b$ implies $-b<-y<-a$, so $x-y<b-a$, contradicting the claim that $x-y \ge b-a$. Similarly, $b-a < x-y$, but again this leads to a contradiction. Hence, we must have $|x-y| < b-a$. □

Example 1.22 (MA2108 AY19/20 Sem 1 Tutorial 1)**.** Let $a,b \in \mathbb{R}$. Show that

$$\max(a,b) = \frac{1}{2}(a+b+|a-b|) \quad \text{and} \quad \min(a,b) = \frac{1}{2}(a+b-|a-b|).$$

Solution. We consider two cases, namely $a \ge b$ and $a < b$. If $a \ge b$, then $a-b \ge 0$, then

$$\frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = a = \max(a,b).$$

Similarly,

$$\frac{1}{2}(a+b-|a-b| = \frac{1}{2}(a+b-(a-b) = b = \min(a,b).$$

The case where $a < b$ has similar working. □

Example 1.23 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 11)**.** If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.

Solution. Let $u \in \mathbb{S}$ be an upper bound for S. Suppose v is another upper bound for S such that $v < u$. Choosing $S \ni s = u$, there exists $s \in S$ such that $v < s$, which contradicts our claim that $v < u$. As such, we must have $u \le v$, i.e. if we have another upper bound v of S, then $u \le v$. We conclude that $u = \sup(S)$. □

Example 1.24 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 12). Let $S \subseteq \mathbb{R}$ be nonempty. Show that

$$u \in \mathbb{R} \text{ is an upper bound of } S \quad \text{if and only if} \quad \text{the conditions } t \in \mathbb{R} \text{ and } t > u \text{ imply } t \notin S.$$

Solution. We first prove the forward direction. Suppose $u \in \mathbb{R}$ is an upper bound of S. Then, for all $s \in S$, we have $s \leq u$. Say $t \in \mathbb{R}$ is such that $t > u$. Suppose on the contrary that $t \in S$. Then, because S is bounded above by u, we must have $t \leq u$, contradicting $t > u$. We conclude that $t \notin S$.

For the reverse direction, we argue by contradiction — say u is not an upper bound of S. Then, there exists $s_0 \in S$ such that $u < s_0$. Let $t = s_0$, then $t > u$, but this contradicts our hypothesis as any $t \in \mathbb{R}$ such that $t > u$ implies $t \notin S$. However, we have $t = s_0 \in S$, which is a contradiction. □

Proposition 1.10 (Archimedean property). For any $x, y \in \mathbb{R}$ such that $0 < x < y$,

$$\text{there exists } n \in \mathbb{N} \quad \text{such that} \quad nx > y \text{ in } \mathbb{R}.$$

Corollary 1.1. For any $\varepsilon \in \mathbb{R}_{>0}$,

$$\text{there exists } n \in \mathbb{N} \quad \text{such that} \quad n\varepsilon > 1 \text{ in } \mathbb{R}.$$

Theorem 1.1 (density theorem). The rational numbers are dense in $\mathbb{R}$, i.e.

$$\text{if } a, b \in \mathbb{R} \text{ such that } a < b \quad \text{then} \quad \text{there exists } r \in \mathbb{Q} \text{ such that } a < r < b.$$

In short, we are always able to find another rational number that lies between two real numbers.

Corollary 1.2. The irrationals are dense in $\mathbb{R}$, i.e.

$$\text{if } a, b \in \mathbb{R} \text{ such that } a < b \quad \text{then} \quad \text{there exists } x \in \mathbb{Q}' \text{ such that } a < x < b.$$

Example 1.25. $\mathbb{Q}$ satisfies the Archimedean property. Also, $\mathbb{Q}$ is dense in $\mathbb{Q}$ with respect to ordering — clearly, for any rational numbers a and b such that $a < b$, we can always find another rational number r strictly in between them. Take for example, $r = \frac{1}{2}(a+b)$.

Theorem 1.2 (existence and uniqueness of radicals). Let $x > 0$ and $n \in \mathbb{N}$. Then, exists a unique positive real number y such that $y^n = x$. The number y is known as the positive n^{th} root of x and thus,

$$y = \sqrt[n]{x} = x^{1/n}.$$

Proof. The uniqueness claim is quite obvious — suppose we have two positive real numbers $0 < y_1 < y_2$. Then, $0 < y_1^n < y_2^n$. As such, given that $y_1^n = y_2^n$, it implies $y_1 = y_2$.

We then prove the existence claim†. Let E denote the set consisting of all positive real numbers t such that $t^n < x$, i.e.

$$E = \{t \in \mathbb{R} : t > 0 \text{ and } t^n < x\}.$$

†As mentioned by Prof. Chin Chee Whye, when you encounter the proof in Rudin's book for the first time (referring to 'Principles of Mathematical Analysis'), you will feel so angry to the extent that you will throw the book away.

First, we claim that $y = \sup E$ exists in $\mathbb{R}$. By the least upper bound property of $\mathbb{R}$, it suffices to show that $E \neq \emptyset$ and E is bounded above. Consider $t = x/(1+x) \in \mathbb{R}$[‡]. Since $x > 0$, then we get $0 \leq t < 1$ and $t < x$. By induction on k, we note that for all $k \in \mathbb{N}$, we have $0 \leq t^k \leq t < 1$, so $t^n \leq t < x$. To see why, note that $t^k \geq 0$ is clear since $t \geq 0$. As such, it suffices to prove that $t^k \leq t$. Equivalently, $t\left(t^{k-1} - 1\right) \leq 0$, so $t^{k-1} - 1 \leq 0$, so $t^{k-1} \leq 1$ (in fact, this inequality is strict) which holds by the induction hypothesis.

The above shows that $t \in E$, so $E \neq \emptyset$.

We then claim that $1 + x \in \mathbb{R}$ is an upper bound of E, i.e. for all $t \in E$, one has $t \leq 1 + x$. Suppose on the contrary that there exists $t \in E$ such that $t > 1 + x$, i.e. $t > 1$ and $t > x$. Again, by induction on k, we note that for all $k \in \mathbb{N}$, we have $t^k \geq t + 1$. To see why, we consider

$$\begin{aligned} t^{k+1} &= t \cdot t^k \\ &\geq t(t+1) \quad \text{by induction hypothesis} \\ &> t+1 \quad \text{since } t > 1 \end{aligned}$$

Hence, $t^n \geq t > x$, contradicting the hypothesis that $t \in E$. So, $t \notin E$.

Lastly, we shall prove that $y = \sup E$, i.e. y is a positive real number satisfying $y^n = x$. This is clear for the case when $n = 1$. As such, we will prove the claim for $n \geq 2$. To do this, we will show that $y^n < x$ and $y^n > x$ both lead to a contradiction. First, note that for any $a, b \in \mathbb{R}$ such that $0 < a < b$, we have the inequality

$$b^n - a^n < (b-a)nb^{n-1}.$$

To see why this inequality holds, recall the geometric series formula

$$\left(\frac{b}{a}\right)^n - 1 = \left[1 + \frac{b}{a} + \left(\frac{b}{a}\right)^2 + \ldots + \left(\frac{b}{a}\right)^{n-1}\right]\left(\frac{b}{a} - 1\right)$$

so

$$\frac{b^n - a^n}{a^n} = \frac{1}{a^{n-1}}\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}\right)\left(\frac{b-a}{a}\right).$$

Since the expressions on each side contain $1/a^n$, it follows that

$$\begin{aligned} b^n - a^n &= (b-a)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1}\right) \\ &< (b-a)\left(b^{n-1} + b^{n-2} \cdot b + b^{n-3} \cdot b^2 + \ldots + b^{n-1}\right) \quad \text{since } a < b \\ &= (b-a)nb^{n-1} \end{aligned}$$

Assume that $y^n < x$. Choose h such that

$$0 < h < \min\left\{1, \frac{x - y^n}{n(y+1)^{n-1}}\right\}.$$

[‡]Actually, it is fairly intuitive to consider this function (though we will jump the gun). We can think of the problem as follows: construct a sequence of positive numbers x_n that is increasing, bounded between 0 and 1. Try to think of $x_n = 1 - 1/n$. However, as the sequence must be defined at index 0, we simply do a translation to obtain $x_n = 1 - 1/(n+1)$. One checks that $x_n = n/(n+1)$.

Setting $a = y$ and $b = y+h$, we have

$$\begin{aligned} b^n - a^n &= (y+h)^n - y^n \\ &< hn(y+h)^{n-1} \quad \text{since } b^n - a^n < (b-a)nb^{n-1} \text{ as deduced earlier} \\ &\le hn(y+1)^{n-1} \quad \text{since } h \le 1 \\ &= x - y^n \end{aligned}$$

So, $(y+h)^n < x$. Also, $y+h \in E$. Since $y+h > y$, this contradicts the fact that y is an upper bound of E.

Next, assume that $y^n > x$. Again, we will show that this leads to a contradiction. Choose

$$k \quad \text{such that} \quad 0 < k < \frac{y^n - x}{ny^{n-1}}.$$

Then, as $y^n > x$ implies $y^n - x > 0$, and $ny^{n-1} > 0$, it follows that $k > 0$. Next, by setting $b = y$ and $a = y-k$, we have

$$\begin{aligned} b^n - a^n &= y^n - (y-k)^n \\ &< kny^{n-1} \quad \text{since } b^n - a^n < (b-a)nb^{n-1} \text{ as deduced earlier} \end{aligned}$$

Since $y^n - (y-k)^n < y^n - x$ is equivalent to saying that $(y-k)^n > x$, our goal is to choose $k > 0$ such that $kny^{n-1} < y^n - x$. To be precise, the chosen value of k should be

$$k = \frac{y^n - x}{ny^{n-1}}.$$

If $t \ge y-k$, then

$$y^n - t^n \le y^n - (y-k)^n < kny^{n-1} = y^n - x.$$

As such, $t^n > x$ and $t \notin E$. It follows that $y-x$ is an upper bound of E. However, $y-k < y$, contradicting the fact that y is the least upper bound of E. To conclude, we must have $y^n = x$. □

We analyse the proof of Theorem 1.2. Recall that

$$y = \sup(E) \quad \text{where} \quad E = \{t \in \mathbb{R} : t > 0 \text{ and } t^n < x\}.$$

If $y^n < x$, then as shown in Figure 4, we define $\varepsilon = x - y^n$ in $\mathbb{R}_{>0}$. Try to choose a number of the form $y^n + \delta$, where $\delta \in \mathbb{R}_{>0}$ and $\delta < \varepsilon$ such that $y^n + \delta$ is of the form $(y+h)^n$, where $h \in \mathbb{R}_{>0}$. Consequently, this would contradict the fact that y is an upper bound for E.

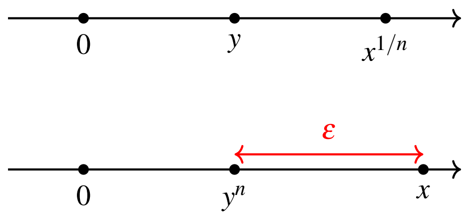


Figure 4

In Figure 5, we wish to choose $h \in \mathbb{R}_{>0}$ such that $0 < (y+h)^n - y^n < \varepsilon$, where $(y+h)^n - y^n = \delta$. In the proof of Theorem 1.2, we used the inequality $b^n - a^n = (b-a)nb^{n-1}$ (which follows by considering some finite

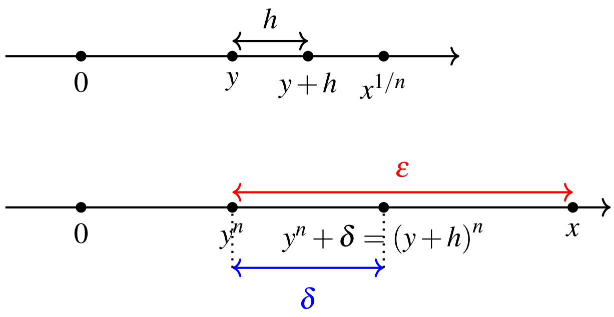


Figure 5

geometric series). As such, $(y+h)^n - y^n < hn(y+h)^{n-1}$, so it suffices to choose $h \in \mathbb{R}_{>0}$ so that $hn(y+h)^{n-1} < \varepsilon$.
The above is equivalent to choosing $h \in \mathbb{R}_{>0}$ so that

$$h < \frac{\varepsilon}{n(y+h)^{n-1}} \quad \text{which is impossible.}$$

On the other hand, if $y^n > x$, then we define $\varepsilon = y^n - x$, which is in $\mathbb{R}_{>0}$ (Figure 6).

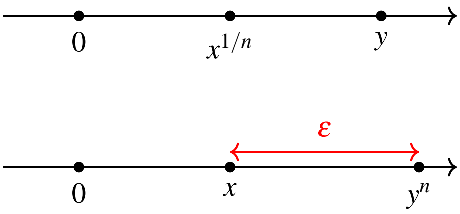


Figure 6

In Figure 7, we wish to choose a real number of the form $y^n - \delta$, where $\delta \in \mathbb{R}_{>0}$ and $\delta \le \varepsilon$ such that $y^n - \delta$ is of the form $(y-k)^n$, where $k \in \mathbb{R}_{>0}$.

Equivalently, we try to choose $k \in \mathbb{R}_{>0}$ such that $0 < y^n - (y-k)^n \le \varepsilon$. By the inequality $b^n - a^n = (b-a)nb^{n-1}$, we have $y^n - (y-k)^n < kny^{n-1}$ so it suffices to choose $k \in \mathbb{R}_{>0}$ such that $kny^{n-1} \le \varepsilon$. This suggests to choose

$$k = \frac{\varepsilon}{ny^{n-1}}.$$

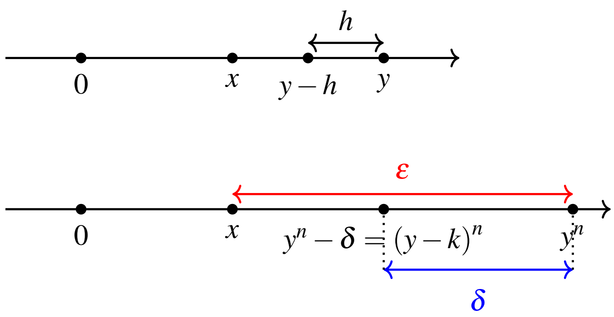


Figure 7

Corollary 1.3. If a and b are positive real numbers and $n \in \mathbb{N}$, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Proof. Let $\alpha = a^{1/n}$ and $\beta = b^{1/n}$, so $ab = \alpha^n \beta^n = (\alpha\beta)^n$. By the uniqueness of Theorem 1.2, we are done. □

We have a nice corollary on $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ (Corollary 1.4). This set is defined to be the multiplicative group of real numbers (will encounter in MA2202 and beyond).

Corollary 1.4. Let $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Then,

$$\mathbb{R}_{>0} = \left(\mathbb{R}^\times\right)^2 \quad \text{as subsets of } \mathbb{R}.$$

Proof. Let F be an ordered field. The inclusion $F_{>0} \supseteq (F^\times)^2$ is obvious. In particular, this holds for $F = \mathbb{R}$. For the other inclusion, if $x \in \mathbb{R}_{>0}$, then by Theorem 1.2, there exists $y \in \mathbb{R}_{>0} \subseteq \mathbb{R}^\times$ such that $y^2 = x$. Hence, the result follows. □

Example 1.26 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 10). Let S be a non-empty subset of $\mathbb{R}$ that is bounded below. Prove that

$$\inf(S) = -\sup\{-s : s \in S\}.$$

Solution. This problem aims to prove

$$\inf(S) = -\sup(-S).$$

Note that $\inf(S) \le s$ for all $s \in S$. So, $-\inf(S) \ge -s$, so $-\inf(S)$ is an upper bound for $-S$, which implies

$$-\inf(S) \ge \sup(-S) \quad \text{so} \quad \inf(S) \ge -\sup(-S).$$

For the other direction, since S is bounded below, then $-S$ is bounded above so $-s \le \sup(-S)$ for all $s \in S$. Hence,

$$s \le -\sup(-S) \quad \text{which implies} \inf(S) \le -\sup(-S).$$

Combining both inequalities yields the desired result. □

Example 1.27 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 20). Let S be a set of non-negative real numbers that is bounded above. Let $a > 0$, and let $aS = \{as : s \in S\}$. Prove that

$$\sup(aS) = a\sup(S).$$

Solution. Suppose $u = \sup(aS)$. Then, for all $as \in aS$, we have $as \le u$. Since $a > 0$, we have

$$s \le \frac{u}{a} = \frac{\sup(aS)}{a} \quad \text{which implies} \quad \sup(S) \le \frac{\sup(aS)}{a}.$$

We then prove the reverse direction. Suppose $\sup(S) = v$. Then, for all $s \in S$, we have $s \le v$. As such,

$$as \le av = a\sup(S) \quad \text{which implies} \quad \sup(aS) \le a\sup(S).$$

□

Example 1.28 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 21)**.** Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let

$$A+B=\{a+b:a\in A,b\in B\}.$$

This is known as the *Minkowski sum* of two sets. Prove that

$$\sup(A+B)=\sup(A)+\sup(B)\quad\text{and}\quad\inf(A+B)=\inf(A)+\inf(B).$$

Solution. We only prove the first result as the second result can be proven similarly. We first prove that

$$\sup(A)+\sup(B)\le\sup(A+B).$$

For all $a\in A$ and $b\in B$, we have

$$a+b\le\sup(A+B).$$

Subtracting b from both sides yields

$$a\le\sup(A+B)-b.$$

If we fix b, we see that $\sup(A+B)-b$ is an upper bound for A. By definition of supremum, we have

$$\sup(A)\le\sup(A+B)-b\quad\text{which implies}\quad b\le\sup(A+B)-\sup(A),$$

i.e. $\sup(A+B)-\sup(A)$ is an upper bound for any b. As such,

$$\sup(B)\le\sup(A+B)-\sup(A)\quad\text{or equivalently}\quad\sup(A)+\sup(B)\le\sup(A+B).$$

We then prove that $\sup(A)+\sup(B)\ge\sup(A+B)$. Since $\sup(A)$ is an upper bound for A, then $a\le\sup(A)$ for all $a\in A$. Similarly, $b\le\sup(B)$ for all $b\in B$. It follows that $a+b\le\sup(A)+\sup(B)$. So, $\sup(A)+\sup(B)\ge\sup(A+B)$. By considering both inequalities, the result follows. □

The real numbers satisfy the *completeness axiom*[†].

Definition 1.11 (completeness of $\mathbb{R}$)**.** There are no gaps or missing points in $\mathbb{R}$.

Corollary 1.5. $\mathbb{N}$ is not bounded above.

Proof. For any $\varepsilon>0$, there exists $n\in\mathbb{N}$ such that $1/n<\varepsilon$. This is justified by setting $x=\varepsilon$ and $y=1$. □

Example 1.29 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 19)**.** If

$$S=\left\{\frac{1}{n}-\frac{1}{m}:n,m\in\mathbb{N}\right\},$$

find $\inf(S)$ and $\sup(S)$.

Solution. Note that

$$\frac{1}{n}-\frac{1}{m}<\frac{1}{n}.$$

Since $n_{\min}=1$, then $1=1/1$ is an upper bound for S. To see why, we can fix $n=1$ and then consider the following sequence of numbers:

$$1-\frac{1}{1},1-\frac{1}{2},1-\frac{1}{3},\ldots,1-\frac{1}{m}$$

[†]Definition 1.11 is rather intuitive and simple. In fact, this was coined by Dedekind.

For large m, the sequence increases and tends to 1, S is bounded above by 1. Next, by the Archimedean property, for any $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \varepsilon \quad \text{so} \quad 1 - \frac{1}{m} > 1 - \varepsilon.$$

This shows that 1 is an upper bound for S but $1 - \varepsilon$ is not an upper bound for S for any $\varepsilon > 0$. As such, 1 is the least upper bound for S, so $\sup S = 1$.

Next, note that

$$\frac{1}{n} - \frac{1}{m} > \frac{1}{n} - 1 > -1,$$

which implies that -1 is a lower bound for S. Again, to see why, we fix $m = 1$ and then consider the following sequence of numbers:

$$\frac{1}{1} - 1, \frac{1}{2} - 1, \frac{1}{3} - 1, \ldots, \frac{1}{n} - 1$$

For large n, the sequence decreases and tends to -1, which implies that S is bounded below by -1. Next, by the Archimedean property, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon \quad \text{so} \quad \frac{1}{n} - 1 < \varepsilon - 1.$$

This shows that -1 is a lower bound for S but $-1 + \varepsilon$ is not a lower bound for S for any $\varepsilon > 0$. As such, -1 is the greatest lower bound for S, so $\inf S = -1$. $\square$

We have a very nice geometric interpretation of Example 1.29. Actually, we can also let

$$S = \left\{ \frac{1}{m} - \frac{1}{n} : m, n \in \mathbb{N} \right\} \quad \text{because} \quad m \text{ comes before } n \text{ in the English alphabet.}$$

Consider the following infinite matrix:

$$\begin{array}{cccccc}
\frac{1}{1} - \frac{1}{1} & \frac{1}{1} - \frac{1}{2} & \frac{1}{1} - \frac{1}{3} & \frac{1}{1} - \frac{1}{4} & \cdots & \sup(S) = 1 \\
\frac{1}{2} - \frac{1}{1} & \frac{1}{2} - \frac{1}{2} & \frac{1}{2} - \frac{1}{3} & \frac{1}{2} - \frac{1}{4} & \cdots & \vdots \\
\frac{1}{3} - \frac{1}{1} & \frac{1}{3} - \frac{1}{2} & \frac{1}{3} - \frac{1}{3} & \frac{1}{3} - \frac{1}{4} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\inf(S) = -1 & \cdots & \cdots & \cdots & \cdots & 0
\end{array}$$

The *matrix* is skew-symmetric since its transpose is equal to negative of itself. Next, for any element, as we travel rightwards, its value increases; as we travel downwards, its value decreases. By observation, the *maximum* and *minimum* values of this matrix (technically they should be the supremum and infimum respectively) occur on the boundary[†].

Example 1.30 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 13). Let $S \subseteq \mathbb{R}$ be non-empty. Show that if $u = \sup(S)$, then for every number $n \in \mathbb{N}$, the number $u - 1/n$ is not an upper bound of S, but the number $u + 1/n$ is an upper bound of S.

Solution. Let $u = \sup(S)$ for some $\emptyset \neq S \subseteq \mathbb{R}$. By definition, the supremum u is an upper bound of S, so $u + 1/n$ is also an upper bound of S.

[†]There is a nice result in Real Analysis which states that when a function is monotonic on a domain, extrema occur at the boundary. By the term 'monotonic', we mean that the sequence of numbers is either increasing or decreasing. For example, the sequence $1, 2, -1, 4, \ldots$ is not monotonic but the sequence of positive odd numbers $1, 3, 5, 7, 9, \ldots$ is monotonic.

We then prove that $u-1/n$ is not an upper bound of S. Suppose on the contrary that it is. Since $u=\sup(S)$, then $u-1/n<u$. As such, there exists $s_0\in S$ such that

$$u-\frac{1}{n}<s_0<u,$$

which is a contradiction as this shows that $u-1/n$ is not an upper bound of S. □

Example 1.31 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 14). Let $S\subseteq\mathbb{R}$ be non-empty. Prove that if a number u in $\mathbb{R}$ has the properties

(i) for every $n\in\mathbb{N}$, the number $u-1/n$ is not an upper bound of S;

(ii) For every $n\in\mathbb{N}$, the number $u+1/n$ is an upper bound of S,

then $u=\sup(S)$.

Solution. By **(ii)**, for any $s\in S$, we have

$$s\le u+\frac{1}{n}\quad\text{for all } n\in\mathbb{N}.$$

Since n can be made arbitrarily large, i.e. $n\to\infty$, then $s\le u$, which holds for all $s\in S$. As such, u is an upper bound of S.

Next, fix some $\varepsilon>0$. By the Archimedean property, there exists $n_0\in\mathbb{N}$ such that $1/n_0\le\varepsilon$. Hence,

$$u-\varepsilon\le u-\frac{1}{n_0}.$$

Since $u-1/n_0$ is not an upper bound of S by **(i)**, then there exists $s_0\in S$ such that

$$u-\varepsilon\le u-\frac{1}{n_0}\le s_0.$$

Hence, u is the least upper bound of S, which implies $u=\sup(S)$. □

Example 1.32 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 15). Show that if A and B are bounded subsets of $\mathbb{R}$, then $A\cup B$ is a bounded set. Show that

$$\sup(A\cup B)=\sup\{\sup A,\sup B\}.$$

Solution. Since A and B are bounded subsets of $\mathbb{R}$, then there exist $m_1,m_2,M_1,M_2\in\mathbb{R}$ such that for any $a\in A$ and $b\in B$, we have

$$m_1\le a\le M_1\quad\text{and}\quad m_2\le b\le M_2.$$

Let $x\in A\cup B$. Then $x\in A$ or $x\in B$, which implies

$$\min\{m_1,m_2\}\le c\le\max\{M_1,M_2\}.$$

This shows that $A\cup B$ is also a bounded subset of $\mathbb{R}$.

Next, we prove that

$$\sup(A\cup B)=\sup\{\sup A,\sup B\}.$$

It suffices to prove that $\sup(A\cup B)=\max\{\sup A,\sup B\}$. From the previous part, we already deduced that $\sup(A\cup B)\le\max\{\sup A,\sup B\}$. To prove the reverse inequality, note that $\sup(A)$ is an upper bound for A, so $\sup(A\cup B)\ge\sup(A)$. A similar argument shows that $\sup(A\cup B)\ge\sup(B)$. Hence,

$$\sup(A\cup B)\ge\max\{\sup A,\sup B\}.$$

The result follows. □

Example 1.33 (MA2108 AY24/25 Sem 2 Problem Set 1 Question 16). Let S be a bounded set in $\mathbb{R}$ and let S_0 be a non-empty subset of S. Show that

$$\inf S \le \inf S_0 \le \sup S_0 \le \sup S.$$

Solution. We first prove that $\sup(S_0) \le \sup(S)$. Note that one can deduce that $\inf(S) \le \inf(S_0)$. Suppose on the contrary that $\sup(S_0) > \sup(S)$. Then, $\sup(S)$ is not an upper bound for S_0, which is a contradiction because $S_0 \subseteq S$.

Lastly, we prove that $\inf(S_0) \le \sup(S_0)$. Let $\sup(S_0) = \alpha$ and $\inf(S_0) = \beta$. Let $x \in S_0$. Since β is a lower bound for x, then $x \ge \beta$. Similarly, since α is an upper bound of x, then $x \le \alpha$, which shows that $\beta \le x \le \alpha$. Hence, $\beta \le \alpha$. □

Let us take a look at Example 1.34. The result states that The result states that for a function of two variables $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ which is bounded above, the order of taking the supremum does not matter.

Example 1.34 (a Fubini-like identity). Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function which is bounded above. Prove that

$$\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} f(m,n) = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} f(m,n) = \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n).$$

Here is a nice geometric interpretation of the problem. We can view $f(m,n)$ as a surface. The supremum represents the highest *peak* or maximum value attained by this surface.

The process of taking $\sup_{m\in\mathbb{N}}$ first means that for each fixed n, we look at the highest value along the column $\{(m,n)\}_{m\in\mathbb{N}}$. Then, we take the supremum of these column-wise maxima over all n, which corresponds to finding the highest peak among these values. Conversely, taking $\sup_{n\in\mathbb{N}}$ first means scanning along the *row* $\{(m,n)\}_{n\in\mathbb{N}}$ for each fixed m, then finding the highest peak among those row-wise maxima.

On the other hand, directly taking the supremum over all ordered pairs $(m,n) \in \mathbb{N}\times\mathbb{N}$ means looking at all points at once and finding the highest value. Since taking the supremum column-first or row-first still results in scanning all points, they must all yield the same value.

Hence, this result shows that regardless of whether we take the maximum first across rows or columns, we always reach the same overall highest point in the grid. We now formally discuss the solution.

Solution. Since f is bounded above, we have

$$f(m,n) \le \sup_{m\in\mathbb{N}} f(m,n).$$

Taking the supremum over all $(m,n) \in \mathbb{N}\times\mathbb{N}$, we have

$$\sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n) \le \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} f(m,n).$$

Conversely, we have

$$\sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} f(m,n) \le \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n) \quad \text{so it follows that} \quad \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} f(m,n) = \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n).$$

In a similar fashion, one can deduce that

$$\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} f(m,n) = \sup_{(m,n)\in\mathbb{N}\times\mathbb{N}} f(m,n)$$

The result follows. □

Theorem 1.3. If n is non-square, then $\sqrt{n}$ is irrational.

Proof. Suppose on the contrary that $\sqrt{n}$ is rational, where n is non-square. Then,

$$\sqrt{n} = p/q \text{ implies } nq^2 = p^2 \quad \text{where } p,q \in \mathbb{N}, q \neq 0 \text{ and } \gcd(p,q) = 1.$$

We consider the prime factorisations of p^2 and q^2, each one of them having an even number of primes. Thus, n must also have an even number of primes. As n is non-square, there exists at least a prime with an odd multiplicity, which is a contradiction. □

Theorem 1.4. Every non-empty interval $I \subseteq \mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

1.3 Important Inequalities

Bernoulli's inequality (named after Jacob Bernoulli) is an inequality that approximates exponentiations of $1+x$. We discuss a widely-used version of this result.

Theorem 1.5 (Bernoulli's inequality)**.** For every $r \in \mathbb{Z}_{\geq 0}$ and $x \geq -1$, we have

$$(1+x)^n \geq 1+nx.$$

The inequality is strict if $x \neq 0$ and $r \geq 2$.

One can use induction to prove Theorem 1.5.

Example 1.35 (MA2108 AY19/20 Sem 1 Tutorial 1)**.** Use Bernoulli's inequality to deduce that for any integer $n > 1$, the following hold:

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{1}{n} \quad \text{and} \quad \left(1+\frac{1}{n-1}\right)^{n-1} < \left(1+\frac{1}{n}\right)^n$$

Solution. The first result is obvious by setting $x = -1/n^2$ in Theorem 1.5. For the second result, we wish to prove that

$$\frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n-1})^{n-1}} > 1.$$

Using some algebraic manipulation, we have

$$1+\frac{1}{n-1} = \frac{n}{n-1} = \frac{1}{1-\frac{1}{n}}.$$

Hence,

$$\text{LHS} = \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{n-1} = \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{-1} = \left(1-\frac{1}{n^2}\right)^n\left(1-\frac{1}{n}\right)^{-1}$$

which is > 1. Here, the inequality follows from the first result. □

Theorem 1.6 (QM-AM-GM-HM inequality). Let $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$. Let

$$
\begin{aligned}
Q(n) &= \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2} && \text{denote} \quad \text{the quadratic mean} \\
A(n) &= \frac{1}{n}\sum_{i=1}^{n} x_i && \text{denote} \quad \text{the arithmetic mean} \\
G(n) &= \sqrt[n]{\prod_{i=1}^{n} x_i} && \text{denote} \quad \text{the geometric mean} \\
H(n) &= n\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^{-1} && \text{denote} \quad \text{the harmonic mean}
\end{aligned}
$$

Then, $Q(n) \geq A(n) \geq G(n) \geq H(n)$. Equality is attained if and only if $x_1 = \ldots = x_n$.

Remark 1.4. The quadratic mean $Q(n)$ is also referred to as root mean square or RMS.

We first prove that $Q(n) \geq A(n)$.

Proof. By the Cauchy-Schwarz inequality,

$$
n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2 \quad \text{which implies} \quad \frac{n[Q(n)]^2}{n} \geq [nA(n)]^2.
$$

With some simple rearrangement, the result follows. □

Example 1.36 (MA2108S AY16/17 Sem 2 Homework 5). For each $n \in \mathbb{Z}^+$, let

$$
a_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n = \left(1 + \frac{1}{n}\right)^{n+1}.
$$

- **(a)** Show that a_n is strictly monotonically increasing.
- **(b)** Show that b_n is strictly monotonically decreasing.
  Hint: Use the GM-HM Inequality.
- **(c)** Show that for each $n \in \mathbb{Z}^+$, one has $a_n < b_n$.

Solution.

- **(a)** A special form of the AM-GM inequality states that

$$
\frac{x + ny}{n+1} \geq (xy^n)^{1/(n+1)}.
$$

  Setting $x = 1$ and $y = 1 + 1/n$, we have

$$
1 + \frac{1}{n+1} > \left(1 + \frac{1}{n}\right)^{n/(n+1)} \quad \text{so} \quad \left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n
$$

  which shows that $a_{n+1} > a_n$. Note that the inequality is strict since $x \neq y$.
- **(b)** Similar to **(a)**.
- **(c)** Let $(1 + 1/n) = u$. Then, $b_n - a_n = u^{n+1} - u^n = u^n/n$, which is positive. □

In fact, both sequences a_n and b_n in Example 1.36 converge in $\mathbb{R}$ and to the same limit. Justifying this requires the use of the monotone convergence theorem (Theorem 2.9) which will be covered in due course. The real number which is the common limit of the sequences a_n and b_n is called Euler's number[†] and it is denoted by e.

[†]Not to be confused with Euler's constant as this typically denotes the Euler-Mascheroni constant γ.

We now return to the proof of the QM-AM-GM-HM inequality (Theorem 1.6). There are numerous proofs of the AM-GM inequality like using backward-forward induction (Cauchy), considering e^x (Pólya), Lagrange Multipliers (MA2104) etc. This proof hinges on Jensen's inequality.

Theorem 1.7 (Jensen's inequality). For a concave function $f(x)$,

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \le f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Proof. Consider the logarithmic function $f(x) = \ln x$, where $x \in \mathbb{R}^+$. It can be easily verified that $f(x)$ is concave as $f''(x) = -1/x^2 < 0$ (this is a simple exercise using knowledge from MA2002). We wish to prove

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right).$$

Using Jensen's inequality (Theorem 1.7),

$$\frac{1}{n}\sum_{i=1}^{n} \ln(x_i) \le \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Note that

$$\sum_{i=1}^{n} \ln(x_i) = \ln(x_1) + \ln(x_2) + \ldots + \ln(x_n) = \ln\left(\prod_{i=1}^{n} x_i\right).$$

As such, the inequality becomes

$$\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{1}{n}\ln\left(\prod_{i=1}^{n} x_i\right).$$

With some simple rearrangement, the AM-GM inequality follows. □

Lastly, we will prove the GM-HM Inequality using the AM-GM Inequality.

Proof. Note that

$$\prod_{i=1}^{n} \frac{1}{x_i} = \left(\prod_{i=1}^{n} x_i\right)^{-1},$$

so we have

$$\frac{n/H(n)}{n} \ge \frac{1}{G(n)}.$$

Upon rearranging, we are done. □

Theorem 1.8 (triangle inequality). For $x, y \in \mathbb{R}$,

$$|x+y| \le |x| + |y|$$

and equality is attained if and only if $xy \ge 0$.

Corollary 1.6. The following hold:

(i) $|x-y| \le |x| + |y|$

(ii) **Reverse triangle inequality:** $||x| - |y|| \le |x-y|$

Proof. **(i)** can be easily proven by replacing $-y$ with y. We now prove **(ii)**. Write x as $x-y+y$ and y as $y-x+x$. Hence,

$$\begin{aligned}|x| &= |x-y+y| \leq |x-y|+|y| \\ |y| &= |y-x+x| \leq |y-x|+|x| = |x-y|+|x|\end{aligned}$$

As such, $|x|-|y| \leq |x-y|$ and $|x|-|y| \leq -|y-x|$, and taking the absolute value of $|x|-|y|$, the result follows. □

Example 1.37 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 1)**.** If $a,b \in \mathbb{R}$, show that

$$|a+b| = |a|+|b| \quad \text{if and only if} \quad ab \geq 0.$$

Solution. We first prove the forward direction. Suppose $|a+b| = |a|+|b|$. Squaring both sides yields

$$|a+b|^2 = |a|^2 + 2|ab| + |b|^2.$$

We note that $|a|^2 = a^2$ for any $a \in \mathbb{R}$. As such,

$$\begin{aligned}(a+b)^2 &= a^2 + 2|ab| + b^2 \\ a^2 + 2ab + b^2 &= a^2 + 2|ab| + b^2\end{aligned}$$

which implies $ab = |ab|$. Hence, $ab \geq 0$.

Conversely, suppose we know that $ab \geq 0$. Then, either

$$a \geq 0 \text{ and } b \geq 0 \quad \text{or} \quad a \leq 0 \text{ and } b \leq 0.$$

For the first case, $a+b$ is the sum of two non-negative numbers, which is also non-negative. Hence, $|a+b| = a+b$. Since $|a| = a$ and $|b| = b$, it follows that $|a+b| = |a|+|b|$. For the second case, $a+b$ is the sum of two non-positive numbers, which is also non-positive. As such, $|a+b| = -(a+b)$. Similarly, we also know that $|a| = -a$ and $|b| = -b$. The result follows. □

Corollary 1.7 (generalised triangle inequality)**.** For $x_1, \ldots, x_n \in \mathbb{R}$,

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|.$$

Proof. Repeatedly apply the triangle inequality (Theorem 1.8). □

Example 1.38 (MA2108 AY19/20 Sem 1 Tutorial 1)**.** Prove that if $x,y \in \mathbb{R}$, $y \neq 0$ and $|x| \leq \frac{|y|}{2}$, then

$$\frac{|x|}{|x-y|} \leq 1.$$

Solution. We wish to prove that $|x| \leq |x-y|$. Using the given inequality, we apply the triangle inequality, so

$$|x| \leq |y|/2 = \frac{|y+x-x|}{2} \leq \frac{|y-x|+|x|}{2}.$$

The result follows with some simple rearrangement and using the property that $|x-y| = |y-x|$. □

Example 1.39 (MA2108S AY16/17 Sem 2 Homework 5; Chebyshev's sum inequality)**.** Let $n \in \mathbb{N}$. Show that for any elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}$ with $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$, one has Chebyshev's inequality, i.e.

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} a_i b_i.$$

Solution. For any $1 \leq i, j \leq n$, we have

$$\begin{aligned}(a_i - a_j)(b_i - b_j) &\geq 0 \\ a_i b_i + a_j b_j &\geq a_i b_j + a_j b_i\end{aligned}$$

Taking the double sum over all i and j on both sides,

$$\begin{aligned}\sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_i + a_j b_j &\geq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j + a_j b_i \\ \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_i + \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_j &\geq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j + \sum_{j=1}^{n}\sum_{i=1}^{n} a_j b_i \\ n\sum_{i=1}^{n} a_i b_i + n\sum_{j=1}^{n} a_j b_j &\geq \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j + \sum_{i=1}^{n} b_i \sum_{j=1}^{n} a_j \\ 2n\sum_{i=1}^{n} a_i b_i &\geq 2\left(\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j\right) \\ \sum_{i=1}^{n} a_i b_i &\geq \frac{1}{n}\sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j\end{aligned}$$

Changing the right sum of b_j's to run from $i = 1$ to $i = n$, and dividing both sides by n, the result follows. □

Example 1.40 (MA2108S AY16/17 Sem 2 Homework 5; Hölder's inequality)**.** Let $n \in \mathbb{N}$. Show that for any $a_1, \ldots, a_n$ in $\mathbb{R}$ with $a_i \geq 0$ for each $1 \leq i \leq n$ and for any $p \in \mathbb{N}$, one has the inequality

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^p \leq \frac{1}{n}\sum_{i=1}^{n} a_i^p.$$

Solution. We use Hölder's inequality, which states that for $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}^+$ and $p, q > 1$ such that $1/p + 1/q = 1$,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

Set $q = p/(p-1)$ so

$$\begin{aligned}\sum_{i=1}^{n} a_i b_i &\leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^{p/(p-1)}\right)^{(p-1)/p} \\ \left(\sum_{i=1}^{n} a_i b_i\right)^p &\leq \left(\sum_{i=1}^{n} a_i^p\right)\left(\sum_{i=1}^{n} b_i^{p/(p-1)}\right)^{p-1}\end{aligned}$$

We can set $b_i = 1$ for all $1 \leq i \leq n$ so the inequality becomes

$$\left(\sum_{i=1}^{n} a_i\right)^p \leq \left(\sum_{i=1}^{n} a_i^p\right) n^{p-1}$$

and with some simple algebraic manipulation, the result follows. □

Chapter 2
Sequences

2.1
Limit of a Sequence

Definition 2.1 (sequence)**.** Let X be a set. A sequence in X is

$$\text{a function } x \text{ with domain } \mathbb{N} \quad \text{i.e. } x : \mathbb{N} \to X$$

which assigns to each natural number n an element $x_n \in X$. The notation x_n is commonly used to denote the image of n under x, meaning $x_n = x(n)$.

Some authors might also use $X(n)$ in Definition 2.1 but x_n is the more standard notation.

We give some examples of sequences.

Example 2.1 (constant sequence)**.** Given $p \in X$, for all $n \in \mathbb{N}$, define $x_n = p$ so we obtain the constant sequence of value p in X.

Example 2.2. We have the exponential sequence 2^n and the factorial sequence $n!$.

Example 2.3 (recursively defined sequences)**.** Also known as recurrence relations, we can apply the recursion theorem for $\mathbb{N}$ (*formally*) to construct a map $\mathbb{N} \to X$. For example, we have the Fibonacci sequence

$$\text{for all } n \in \mathbb{N} \text{ we have } x_{n+1} = x_n + x_{n-1} \quad \text{defined by the initial conditions} \quad x_0 = x_1 = 1.$$

Definition 2.2 (absolute value)**.** Let F be an ordered field. The absolute value on F is the map

$$|\cdot| : F \to F_{\geq 0} \quad \text{defined by} \quad x \mapsto |x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Proposition 2.1. For any $x, y \in F$, we have the following:

(i) Positive-definiteness: $|x| \geq 0$ in F and equality holds if and only if $x = 0$ in F

(ii) Multiplicativity: $|xy| = |x||y|$ in $\mathbb{F}_{\geq 0}$

(iii) Triangle inequality: $|x+y| \leq |x| + |y|$ (recall Theorem 1.8)

Definition 2.3 (neighbourhood)**.** Let F be an ordered field. For any $a \in F$ and $\varepsilon > 0$, define

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon) \text{ (Figure 8)} \quad \text{to be} \quad \text{the } \varepsilon\text{-neighbourhood of } a.$$

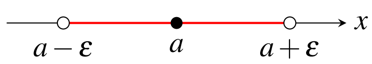


Figure 8: ε-neighbourhood of a

Definition 2.4 (formal definition of limit). Let F be an ordered field and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in F. We say that L is the limit of the sequence if

$$\text{for every } \varepsilon > 0 \text{ there exists } K \in \mathbb{N} \quad \text{such that} \quad \text{for all } n \geq K \text{ we have } |x_n - L| < \varepsilon.$$

Equivalently, $x_n \in V_\varepsilon(L)$. If L exists, then we say that $\{x_n\}_{n\in\mathbb{N}}$ converges to L in F (or simply $\{x_n\}_{n\in\mathbb{N}}$ is convergent); $\{x_n\}_{n\in\mathbb{N}}$ diverges otherwise.

Theorem 2.1 (uniqueness of limit). The limit of a sequence $\{x_n\}_{n\in\mathbb{N}}$, if it exists, is unique. That is to say, if $L, L' \in F$ for some ordered field F such that

$$\lim_{n\to\infty} x_n = L \text{ and } \lim_{n\to\infty} x_n = L' \quad \text{then} \quad L = L'.$$

Proof. Suppose on the contrary that L and L' are two distinct limits of $\{x_n\}_{n\in\mathbb{N}}$. By way of contradiction, say $L \neq L'$. Then, we can write $\varepsilon = |L - L'| \in F_{>0}$. The trick is to observe that $\varepsilon/2 \in F_{>0}$. Since $x_n \to L$, there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$, the inequality

$$|x_n - L| < \varepsilon' = \frac{\varepsilon}{2} \quad \text{holds.}$$

Similarly, as $x_n \to L'$, then there exists $K_2 \in \mathbb{N}$ such that for all $n \geq K_2$, the inequality

$$\left|x_n - L'\right| < \varepsilon' = \frac{\varepsilon}{2} \quad \text{holds.}$$

We define $K = \max\{K_1, K_2\}$, which is also $\in \mathbb{N}$. Then, for all $n \geq K$, we have

$$\left|L - L'\right| = \left|L - x_n + x_n - L'\right| \leq |x_n - L| + \left|x_n - L'\right| < 2\varepsilon' = \varepsilon.$$

Here, the first inequality follows from the triangle inequality. Since ε is arbitrary, we can set $|L - L'| = 0$, resulting in $L = L'$, contradicting the earlier assumption that L and L' are distinct. □

Remark 2.1. The triangle inequality is a helpful tool when finding limits. Note that changing a finite number of terms in a sequence does not affect its convergence or its limit.

Same as the formal definition of a limit in MA2002, to prove that a given sequence x_n converges to L, we first express $|x_n - L|$ in terms of n, and find a *simple* upper bound, L, for it. Then, let $\varepsilon > 0$ be arbitrary. We find $K \in \mathbb{N}$ such that

$$\text{for all } n \geq K \text{ we have } L < \varepsilon \quad \text{or equivalently} \quad |x_n - L| < \varepsilon.$$

Example 2.4. Prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Solution. Let $\varepsilon > 0$. By the Archimedean property (Proposition 1.10), there exists $K \in \mathbb{N}$ such that $K > 1/\varepsilon$. So, if $n \geq K$, then $n > 1/\varepsilon$. As such, $1/n < \varepsilon$. We conclude that for all $n \geq K$, $|1/n - 0| < \varepsilon$. □

Example 2.5. Prove that

$$\lim_{n\to\infty} \frac{2n^2+1}{n^2+3n} = 2.$$

Solution. We have
$$\left|\frac{2n^2+1}{n^2+3n}-2\right| = \left|\frac{1-6n}{n^2+3n}\right| \le \frac{1+6n}{n^2+3n} < \frac{1+6n}{n^2} < \frac{n+6n}{n^2} = \frac{7}{n}.$$
Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ such that $K > 7/\varepsilon$. Then, for all $n \ge K$, we have
$$\left|\frac{2n^2+1}{n^2+3n}-2\right| < \frac{7}{n} \le \frac{7}{K} < \varepsilon$$
and the result follows. □

Example 2.6 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 4)**.** Show that

(a) $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$ **(b)** $\lim_{n\to\infty} \frac{2n}{n+1} = 2$ **(c)** $\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}$ **(d)** $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$

Solution.

(a) Let $\varepsilon > 0$ be arbitrary. Choose $N = 1/\lceil \varepsilon \rceil$ in $\mathbb{N}$. Then, for all $n \ge N$, we have
$$\left|\frac{n}{n^2+1}-0\right| = \left|\frac{n}{n^2+1}\right| \le \left|\frac{n}{n^2}\right| = \frac{1}{|n|} \le \frac{1}{N} < \varepsilon.$$

(b) Let $\varepsilon > 0$ be arbitrary. Choose $N = 2/\lceil \varepsilon \rceil$ in $\mathbb{N}$. Then, for all $n \ge N$, we have
$$\left|\frac{2n}{n+1}-2\right| = \left|\frac{2n-2n-2}{n+1}\right| = \frac{2}{|n|} \le \frac{2}{N} < \varepsilon.$$

(c) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 13/4\varepsilon \rceil$. Then, for all $n \ge N$, we have
$$\left|\frac{3n+1}{2n+5}-\frac{3}{2}\right| = \left|\frac{13}{2(2n+5)}\right| \le \frac{13}{4|n|} \le \frac{13}{4N} < \varepsilon.$$

(d) Let $\varepsilon > 0$ be arbitrary. Choose $N = \left\lceil \sqrt{5/4\varepsilon} \right\rceil$. Then, for all $n \ge N$, we have
$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right| = \frac{5}{2|2n^2+3|} \le \frac{5}{4n^2} \le \frac{5}{4N^2} < \varepsilon.$$
□

Example 2.7 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 5)**.** Show that

(a) $\lim_{n\to\infty} \frac{1}{\sqrt{n+7}} = 0$ **(b)** $\lim_{n\to\infty} \frac{2n}{n+2} = 2$ **(c)** $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ **(d)** $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$

Solution.

(a) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon^2 \rceil$. Then, for all $n \ge N$, we have
$$\left|\frac{1}{\sqrt{n+7}}-0\right| \le \frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{N}} < \varepsilon.$$

(b) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 4/\varepsilon \rceil$. Then, for all $n \ge N$, we have
$$\left|\frac{2n}{n+2}-2\right| = \left|\frac{4}{n+2}\right| \le \frac{4}{|n|} \le \frac{4}{N} < \varepsilon.$$

(c) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon^2 \rceil$. Then, for all $n \ge N$, we have
$$\left|\frac{\sqrt{n}}{n+1}-0\right| = \frac{\sqrt{n}}{n+1} \le \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{N}} < \varepsilon.$$

(d) Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\sqrt{\varepsilon} \rceil$. Then, for all $n \geq N$, we have

$$\left| \frac{(-1)^n}{n^2+1} - 0 \right| = \frac{1}{n^2+1} \leq \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

$\square$

Example 2.8 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 13). For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = \{x_n\}_{n \in \mathbb{N}}$:

(a) $x_n = \dfrac{n}{n+1}$ **(b)** $x_n = \dfrac{(-1)^n n}{n+1}$ **(c)** $x_n = \dfrac{n^2}{n+1}$ **(d)** $x_n = \dfrac{2n^2+3}{n^2+1}$

Solution.

(a) The sequence converges to 1. We will formally prove this. Let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. Then, for all $n \geq N$, we have

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| \leq \frac{1}{|n|} \leq \frac{1}{N} < \varepsilon.$$

(b) We claim that the sequence diverges. Suppose on the contrary that the limit is L. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{(-1)^n n}{n+1} - L \right| < \varepsilon.$$

Let $\varepsilon = 1$. If n is even, then there exists $k \in \mathbb{Z}$ such that $n = 2k$ so

$$\left| \frac{(-1)^{2k} \cdot 2k}{2k+1} - L \right| < 1 \quad \text{so} \quad \left| \frac{2k}{2k+1} - L \right| < 1.$$

Upon expansion, we have

$$L - 1 < \frac{2k}{2k+1} < L + 1 \quad \text{and we see that} \quad \lim_{k \to \infty} \frac{2k}{2k+1} = 1.$$

On the other hand, if n is odd, then there exists $k \in \mathbb{Z}$ such that $n = 2k+1$ so

$$\left| \frac{(-1)^{2k+1} \cdot (2k+1)}{2k+2} - L \right| < 1 \quad \text{so} \quad \left| \frac{-2k-1}{2k+2} - L \right| < 1.$$

Upon expansion, we have

$$L - 1 < \frac{-2k-1}{2k+2} < L + 1 \quad \text{but however} \quad \lim_{k \to \infty} \frac{-2k-1}{2k+2} = -1.$$

Since both limits are different, this leads to a contradiction.

(c) We claim that the sequence diverges. To see why, we have the following inequality:

$$\left| \frac{n^2}{n+1} \right| \geq \left| \frac{n^2}{n^2} \right| = 1 \quad \text{so for sufficiently large } n \quad \left| \frac{n^2}{n+1} \right| \geq 1$$

so the sequence diverges.

(d) We claim that the sequence converges to 2. Let $\varepsilon > 0$ be arbitrary. Then, choose $N = \lceil 1/\sqrt{\varepsilon} \rceil$. As such,

$$\left| \frac{2n^2+3}{n^2+1} - 2 \right| = \left| \frac{1}{n^2+1} \right| \leq \left| \frac{1}{n^2} \right| < \frac{1}{N^2} < \varepsilon.$$

$\square$

Example 2.9 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 14). Find the limits of the following sequences:

(a) $\lim_{n\to\infty}\left(2+\frac{1}{n^2}\right)^2$ **(b)** $\lim_{n\to\infty}\frac{(-1)^n}{n+2}$ **(c)** $\lim_{n\to\infty}\frac{\sqrt{n}-1}{\sqrt{n}+1}$ **(d)** $\lim_{n\to\infty}\frac{n+1}{n\sqrt{n}}$

Solution.

(a) 4

(b) We claim that the limit is 0. To see why, let $\varepsilon>0$ be arbitrary. Choose $N=\lceil 1/\varepsilon\rceil$ in $\mathbb{N}$. As such,

$$\left|\frac{(-1)^n}{n+2}-0\right|=\frac{1}{|n+2|}\le\frac{1}{N}<\varepsilon.$$

(c) We have

$$\lim_{n\to\infty}\frac{\sqrt{n}-1}{\sqrt{n}+1}=\lim_{n\to\infty}\frac{1-1/\sqrt{n}}{1+1/\sqrt{n}}=1.$$

(d) The limit is

$$\lim_{n\to\infty}\frac{n+1}{n}\cdot\lim_{n\to\infty}\frac{1}{\sqrt{n}}=1\cdot 0=0.$$

□

Example 2.10 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 8). Show that

$$\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)=0.$$

Solution. We will use the formal definition of a limit to prove that the limit is 0. Before that, to see why one can make this deduction, we have

$$\sqrt{n^2+1}-n=\frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2+1}+n}$$

so as $n\to\infty$, the limit goes to zero. We now prove this formally. Let $\varepsilon>0$ be arbitrary. Then, choose $N=\lceil 1/2\varepsilon\rceil$ in $\mathbb{N}$. So, for all $n\ge N$, we have

$$\begin{aligned}\left|\sqrt{n^2+1}-n-0\right|&=\left|\sqrt{n^2+1}-n\right|\\&=\left|\frac{\left(\sqrt{n^2+1}-n\right)\left(\sqrt{n^2+1}+n\right)}{\sqrt{n^2+1}+n}\right|\\&=\left|\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}\right|\\&=\frac{1}{\sqrt{n^2+1}+n}\end{aligned}$$

Now, note that $\sqrt{n^2+1}\ge\sqrt{n^2}=n$ so $\sqrt{n^2+1}+n\ge 2n$. Hence,

$$\frac{1}{\sqrt{n^2+1}+n}\le\frac{1}{2n}\le\frac{1}{2N}<\varepsilon.$$

□

Example 2.11 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 17). Determine the limits of the following sequences:

(a) $\sqrt{4n^2+n}-2n$

(b) $\sqrt{n^2+5n}-n$

Solution.

(a) We have

$$\lim_{n\to\infty}\left(\sqrt{4n^2+n}-2n\right)=\lim_{n\to\infty}\frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n}=\lim_{n\to\infty}\frac{n}{\sqrt{4n^2+n}+2n}=\lim_{n\to\infty}\frac{1}{\sqrt{4+1/n}+2}=\frac{1}{4}.$$

(b) We have

$$\lim_{n\to\infty}\left(\sqrt{n^2+5n}-n\right)=\lim_{n\to\infty}\frac{n^2+5n-n^2}{\sqrt{n^2+5n}+n}=\lim_{n\to\infty}\frac{5}{\sqrt{1+5/n}+1}=\frac{5}{2}.$$

□

Example 2.12 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 19). If $a>0,b>0$, show that

$$\lim_{n\to\infty}\left(\sqrt{(n+a)(n+b)}-n\right)=\frac{a+b}{2}.$$

Solution. Let $\varepsilon>0$ be arbitrary. Then, choose $N=\left\lceil\frac{(a-b)^2}{4\varepsilon}\right\rceil$ in $\mathbb{N}$, and also let $k=\frac{1}{2}(a+b)$ for convenience. So, for all $n\geq N$, we have

$$\begin{aligned}\left|\sqrt{(n+a)(n+b)}-n-\frac{a+b}{2}\right|&=\left|\sqrt{(n+a)(n+b)}-n-k\right|\\&=\left|\frac{\left(\sqrt{(n+a)(n+b)}-(n+k)\right)\left(\sqrt{(n+a)(n+b)}+(n+k)\right)}{\sqrt{(n+a)(n+b)}+(n+k)}\right|\\&=\left|\frac{(n+a)(n+b)-(n+k)^2}{\sqrt{(n+a)(n+b)}+(n+k)}\right|\\&=\left|\frac{n^2+an+bn+ab-n^2-2kn-k^2}{\sqrt{(n+a)(n+b)}+(n+k)}\right|\\&=\left|\frac{an+bn+ab-(a+b)n-\left(\frac{a+b}{2}\right)^2}{\sqrt{(n+a)(n+b)}+(n+k)}\right|\end{aligned}$$

At this juncture, note that the numerator simplifies to

$$ab-\left(\frac{a+b}{2}\right)^2=ab-\frac{a^2+2ab+b^2}{4}=-\frac{(a-b)^2}{4}.$$

By considering the denominator, we have

$$\sqrt{(n+a)(n+b)}+(n+k)\geq\sqrt{(n+a)(n+b)}\geq\sqrt{n^2}=n$$

so

$$\left|\sqrt{(n+a)(n+b)}-n-\frac{a+b}{2}\right|\leq\frac{(a-b)^2}{4n}\leq\frac{(a-b)^2}{4N}<\varepsilon.$$

□

In Example 2.12, it was stated that $a,b>0$. This condition is not surprising because the expression $\sqrt{(n+a)(n+b)}$ must be well-defined for all relevant values of n. Specifically, the square root function requires that its argument be non-negative, meaning that $(n+a)(n+b)>0$.

For this inequality to hold for all sufficiently large n, both $n+a$ and $n+b$ must be either simultaneously non-negative or simultaneously non-positive. If either a or b were negative, there would exist some values of n for which $(n+a)(n+b)<0$, making the square root expression undefined in the real number system. Thus, ensuring that $a,b>0$ guarantees the validity of the expression for all sufficiently large n.

Example 2.13 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 8)**.** Show that if $x_n > 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{x_n} = \infty.$$

Solution. We first prove the forward direction. Suppose

$$\lim_{n\to\infty} x_n = 0.$$

Let $\varepsilon > 0$ be arbitrary and set $M = 1/\varepsilon$. Then, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n| < \varepsilon = 1/M$. Thus, for $n \geq N$, we have $1/x_n > M$, and the result follows.
the reverse direction, we note that there exists $N \in \mathbb{N}$ such that for $n \geq N$, $1/x_n > M$. Let $\varepsilon > 0$ be arbitrary and set $M = 1/\varepsilon$. Then, $1/x_n > 1/\varepsilon$, so $|x_n| < \varepsilon$. The result follows. □

Example 2.14 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 9)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0.$$

(a) Show that if $\lim x_n = \infty$, then $\lim y_n = \infty$.
(b) Show that if $\{y_n\}_{n\in\mathbb{N}}$ is bounded, then $\lim x_n = 0$.

Solution.

(a) Since

$$\lim_{n\to\infty} x_n = \infty \quad \text{then} \quad \text{there exists } K \in \mathbb{N} \text{ such that for all } n \geq K \text{ we have } x_n > 1.$$

Since

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 0 \quad \text{then} \quad \text{there exists } N \in \mathbb{N} \text{ such that for all } n \geq N \text{ we have } \left|\frac{x_n}{y_n}\right| < \varepsilon.$$

Thus,

$$\frac{1}{|y_n|} < \left|\frac{x_n}{y_n}\right| < \varepsilon \quad \text{for all } n \geq \max\{K, N\},$$

which shows that

$$\lim_{n\to\infty} \frac{1}{y_n} = 0.$$

By Example 2.13, the result follows.

(b) Since y_n is bounded, then there exists $M > 0$ such that $0 < |y_n| \leq M$. We wish to prove that there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, $|x_n| < \varepsilon$. We have $|x_n/y_n| < \varepsilon/M$ so $|x_n| < \varepsilon/M \cdot M = \varepsilon$. □

Definition 2.5 (eventually constant)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an ordered field F. We say that the sequence is eventually constant if and only if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n = x_N$.

Definition 2.6 (boundedness)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in an ordered field F. We say that the sequence is bounded in F if and only if

$$\text{the set } \{x_n \in F : n \in \mathbb{N}\} \quad \text{is} \quad \text{bounded in } F.$$

Theorem 2.2 (limit theorems)**.** Let F be an ordered field and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be convergent sequences in F. Then, the following properties hold:

(i) $\{x_n\}_{n\in\mathbb{N}}$ is convergent, i.e. if

$$\lim_{n\to\infty} x_n = L \quad \text{then} \quad |x_n| \leq M \text{ for some } M \in \mathbb{R}$$

(ii) Linearity: Just like how linear operators (i.e. derivatives and integrals) work, we have a similar result for limits. Suppose $\alpha, \beta \in F$ and

$$\lim_{n\to\infty} x_n = L_1 \quad \text{and} \quad \lim_{n\to\infty} y_n = L_2.$$

Then,

$$\{\alpha x_n \pm \beta y_n\}_{n\in\mathbb{N}} \text{ converges} \quad \text{i.e.} \quad \lim_{n\to\infty} (\alpha x_n \pm \beta y_n) = \alpha L_1 \pm \beta L_2.$$

(iii) Product and quotient: Considering the sequences x_n and y_n as mentioned in **(ii)**,

$$\lim_{n\to\infty} x_n y_n = L_1 L_2 \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{L_1}{L_2} \quad \text{provided that } y_n, y \neq 0 \text{ for all } n \in \mathbb{N}$$

(iv) If there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \leq y_n$ in F, then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \quad \text{in } F$$

The converse of Theorem 2.2 is not true as not all bounded sequences are convergent.

Example 2.15. As an example, the sequence $x_n = (-1)^n$ is bounded by -1 and 1 and it oscillates about only these two values. We claim that $\{x_n\}_{n\in\mathbb{N}}$ does not converge in F. By way of contradiction, say

$$\lim_{n\to\infty} x_n = L \quad \text{in } F.$$

Set $\varepsilon = 1$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < 1$. Hence, for odd $n \geq N$, we have $|-1-p| < 1$, which implies $-1 < -1-p < 1$, so $p < 0$. On the other hand, for even $n \geq N$, we have $|1-p| < 1$, which implies $-1 < 1-p < 1$, which implies $p > 0$. This leads to a contradiction!

We first prove **(i)** of Theorem 2.2.

Proof. We wish to prove that every convergent sequence is bounded. Suppose

$$\lim_{n\to\infty} x_n = L.$$

Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < \varepsilon$. Set $\varepsilon = 1$. Then, take $K \in \mathbb{N}$ such that $|x_n - L| < 1$ for all $n \geq K$. So, $L - 1 < x_n < L + 1$. Let $|x_n| = \max\{|L-1|, |L+1|\}$ for all $n \geq K$. Since $\{|x_1|, \ldots, |x_{K-1}|\}$ is a finite set of numbers in F, it is bounded, so it contains a maximum. As such, for $1 \leq n \leq K-1$, we have $|x_n| \leq A$ for some $A \in F$. Define

$$M = \max\{|L-1|, |L+1|, A\}$$

so $|x_n| \leq M$ and the result follows. □

Example 2.16 (sequences in $\mathbb{Q}$ diverge)**.** Let $x_n = n$. Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{Q}$ is not bounded in $\mathbb{Q}$ (simple application of the Archimedean property in $\mathbb{Q}$). By **(i)** of Theorem 2.2, $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$.

We then prove **(ii)** of Theorem 2.2.

Proof. We shall prove that

$$\lim_{n\to\infty} (x_n + y_n) = L_1 + L_2.$$

We know that there exist $K_1, K_2 \in \mathbb{N}$ such that

$$|x_n - L_1| < \frac{\varepsilon}{2} \text{ for all } n \geq K_1 \quad \text{and} \quad |y_n - L_2| < \frac{\varepsilon}{2} \text{ for all } n \geq K_2.$$

Set $K = \max\{K_1, K_2\}$. By the triangle inequality (Theorem 1.8),

$$|x_n - L_1 + y_n - L_2| < |x_n - L_1| + |y_n - L_2| < \varepsilon$$

and the result follows. □

For **(iii)** of Theorem 2.2, we only prove the result involving the product of two sequences.

Proof. Since $|x_n|$ is convergent, then it is bounded by **(i)** of Theorem 2.2, i.e. $|x_n| \leq M_1$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned}
|x_n y_n - L_1 L_2| &= |x_n y_n - x_n L_2 + x_n L_2 - L_1 L_2| \\
&\leq |x_n y_n - x_n L_2| + |x_n L_2 - L_1 L_2| \quad \text{by the triangle inequality (Theorem 1.8)} \\
&= |x_n||y_n - L_2| + |L_2||x_n - L_1| \\
&\leq M_1|y_n - L_2| + |L_2||x_n - L_1|
\end{aligned}$$

Set $M = \max\{M_1, |L_2|\} > 0$. So,

$$M_1|y_n - L_2| + |L_2||x_n - L_1| \leq M(|y_n - L_2| + |x_n - L_1|).$$

Let $\varepsilon > 0$ be arbitrary. Then, there exist $K_1, K_2 \in \mathbb{N}$ such that

$$\begin{aligned}
|x_n - L_1| &< \varepsilon/2M \text{ for all } n \geq K_1 \\
|y_n - L_2| &< \varepsilon/2M \text{ for all } n \geq K_2
\end{aligned}$$

Let $K = \max\{K_1, K_2\}$. Hence,

$$|x_n y_n - L_1 L_2| < M\left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}\right) < \varepsilon.$$

and we are done. □

Example 2.17 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 7). Prove that if

$$\lim_{n\to\infty} x_n = x > 0,$$

then there exists $M \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq M$.

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \varepsilon$. Hence,

$$x - \varepsilon < x_n < x + \varepsilon.$$

Since $\varepsilon > 0$ can be made sufficiently small, we can let $\varepsilon = x/2$ so $x_n > x/2 > 0$. Choosing $M = N$, the result follows. □

Example 2.18 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 12). If

$$\lim_{n\to\infty} x_n = x > 0,$$

show that there exists $K \in \mathbb{N}$ such that if $n \geq K$, then $x/2 < x_n < 2x$.

Solution. Let $\varepsilon > 0$ be arbitrary. Then, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, we have $|x_n - x| < \varepsilon$. So, $x - \varepsilon < x_n < x + \varepsilon$. Since $\varepsilon > 0$ can be made sufficiently small, then we can let $\varepsilon = x/2$ so $x/2 < x_n < 3x/2 < 2x$. □

Corollary 2.1. If x_n converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = \left(\lim_{n\to\infty} x_n\right)^k.$$

Theorem 2.3 (squeeze theorem)**.** Let x_n, y_n and z_n be sequences of numbers such that for all $n \in \mathbb{N}$, $x_n \le y_n \le z_n$. If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L \quad \text{then} \quad \lim_{n\to\infty} y_n = L.$$

Proof. Let $\varepsilon > 0$. Then, there exists $K \in \mathbb{N}$ such that for all $n \ge K$, we have

$$|x_n - a| < \varepsilon \quad \text{and} \quad |z_n - a| < \varepsilon.$$

Working with the modulus, we have

$$-\varepsilon < x_n - a < \varepsilon \quad \text{and} \quad -\varepsilon < z_n - a < \varepsilon.$$

Thus,

$$-\varepsilon < x_n - a \le y_n - a \le z_n - a < \varepsilon$$

which implies $|y_n - a| < \varepsilon$. □

Example 2.19. Evaluate the following limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

Even though one might think that the Riemann sum comes into play, it actually does not work in this case because

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+k/n^2}}$$

and setting

$$f\left(\frac{k}{n}\right) = \sqrt{1+\frac{k}{n^2}},$$

it is impossible to obtain an explicit expression for $f(x)$.

Solution. We use the squeeze theorem to help us. As

$$\frac{n}{\sqrt{n^2+n}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}},$$

then

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} &\le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le \frac{n}{\sqrt{n^2}} \\
\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} &\le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le 1 \\
1 &\le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le 1
\end{aligned}$$

By the squeeze theorem, the required limit is 1. □

Theorem 2.4 (limit theorems)**.** The following hold:

(i) For any $p, q \in \mathbb{N}$, we have
$$\lim_{n\to\infty} \frac{1}{n^{p/q}} = 0$$

(ii) For any $p > 0$, we have
$$\lim_{n\to\infty} \sqrt[n]{p} = 1$$

(iii) We have
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

(iv) For any $a > 1$ and $k \in \mathbb{Z}_{\geq 0}$ sufficiently large, we have
$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0$$

(v) For any $x \in \mathbb{R}$ with $|x| < 1$, one has
$$\lim_{n\to\infty} x^n = 0$$

We first prove **(i)** of Theorem 2.4.

Proof. Given any $\varepsilon > 0$, by Theorem 1.2, there exists a unique $\left(\varepsilon^{1/p}\right)^q > 0$ such that $\left(\varepsilon^{q/p}\right)^p = \varepsilon^q$. By the Archimedean property (Proposition 1.10), there exists $N \in \mathbb{N}$ such that $N \cdot \varepsilon^{q/p} > 1$. Thus, for all $n \geq N$,
$$n \cdot \varepsilon^{q/p} > 1 \quad \text{so} \quad n^p \cdot \varepsilon^q > 1.$$

As such,
$$0 < \frac{1}{n^p} < \varepsilon^q \quad \text{so} \quad 0 < \frac{1}{n^{p/q}} < \varepsilon^{1/q}.$$

Hence, the result follows. □

We then prove **(ii)** of Theorem 2.4.

Proof. There are three cases to consider. Firstly, if $p = 1$, then we obtain the constant sequence 1, so obviously the limit is 1 as well. Next, if $p > 1$, for every $n \in \mathbb{N}$, set $x_n = \sqrt[n]{p} - 1$. Then, $p = (1 + x_n)^n$. By the binomial theorem (one can also interpret it as Bernoulli's inequality in Theorem 1.5),
$$\text{for all } n \in \mathbb{N} \quad \text{we have} \quad p \geq 1 + nx_n \text{ so } 0 \leq x_n \leq \frac{p-1}{n}.$$

By the squeeze theorem 2.3, the result follows.

For the case where $p < 1$, then $1/p > 1$ so
$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1.$$

Hence,
$$\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{1/p}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{1/p}} = \frac{1}{1} = 1.$$

The result follows. □

Next, we prove **(iii)** of Theorem 2.4.

Proof. For each $n \in \mathbb{N}$, set $x_n = \sqrt[n]{n} - 1$, so $n = (1 + x_n)^n$. By Bernoulli's inequality (Theorem 1.5), for all $n \geq N$, we have

$$n \geq 1 + nx_n \quad \text{so} \quad 0 < x_n \leq \frac{n-1}{n} = 1 - \frac{1}{n}$$

but this is a *useless* statement because it just shows that $0 < x_n \leq 1$. Sadly, we are unable to apply the squeeze theorem here. As such, we use the binomial theorem. Observe that for $n \geq 2$, we have

$$n = (1 + x_n)^n = 1 + nx_n + \binom{n}{2} x_n^2 + \ldots + x_n^n \geq \frac{n(n-1)}{2} x_n^2.$$

As such, for $n \geq 2$, we have

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

By the squeeze theorem (Theorem 2.3), the limit of x_n is 0, so the limit of $\sqrt[n]{n}$ is 1. □

We then prove **(iv)** and **(v)** of Theorem 2.4.

Proof. Write $a = 1 + p$ with $p > 0$. Consider $n \in \mathbb{N}$ with $n > 2k$. Then, we have

$$a^n = (1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k} p^k$$

where we used the binomial theorem. Upon *expansion*, the above is equal to

$$\frac{n(n-1)\ldots(n-k+1)}{k!} \cdot p^k > \left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!}.$$

Hence,

$$0 < \frac{n^k}{a^n} < \frac{2^k k!}{p^k}.$$

However, the RHS is some constant (independent of n), so similar to our proof of **(iii)** of Theorem 2.4, we run into an error. So, we take a detour and consider

$$a^n = (1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k+1} p^{k+1} = \frac{n(n-1)\ldots(n-k+1)(n-k)}{k!(k+1)} \cdot p^{k+1} > \left(\frac{n}{2}\right)^{k+1} \cdot \frac{p^{k+1}}{(k+1)!}.$$

Hence,

$$0 < \frac{n^k}{a^n} < \frac{2^{k+1}(k+1)!}{p^{k+1}} \cdot \frac{1}{n}.$$

By the squeeze theorem, the original limit is equal to 0. **(v)** follows from **(iv)** by setting $k = 0$ and $a = 1/|x|$. □

Example 2.20 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 9)**.** Show that

$$\lim_{n\to\infty} (2n)^{1/n} = 1.$$

Solution. Recall **(iii)** of Theorem 2.4, where it was mentioned that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

Hence,

$$\lim_{n\to\infty} (2n)^{1/n} = \lim_{n\to\infty} 2^{1/n} \cdot \lim_{n\to\infty} n^{1/n} = 1 \cdot 1 = 1.$$

□

Example 2.21 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 10)**.** Show that

$$\lim_{n\to\infty}\frac{n^2}{n!}=0.$$

Solution. We have

$$\lim_{n\to\infty}\frac{n^2}{n!}=\lim_{n\to\infty}\frac{n^2}{n(n-1)}\cdot\lim_{n\to\infty}\frac{1}{(n-2)!}=1\cdot 0=0.$$

By the squeeze theorem (Theorem 2.3), the result follows. □

Example 2.22 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 11)**.** Show that

$$\lim_{n\to\infty}\frac{2^n}{n!}=0.$$

Hint: If $n\geq 3$, then $0<\frac{2^n}{n!}\leq 2\left(\frac{2}{3}\right)^{n-2}$

Solution. If $n\geq 3$, then $n!\geq 3^{n-2}$. As such,

$$0=\lim_{n\to\infty}0\leq\lim_{n\to\infty}\frac{2^n}{n!}\leq\lim_{n\to\infty}\frac{2^n}{3^{n-2}}=0.$$

By the squeeze theorem (Theorem 2.3), the result follows. □

Example 2.23 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 20)**.** Determine the limits of the following sequences:

(a) $\lim_{n\to\infty}n^{1/n^2}$

(b) $\lim_{n\to\infty}(n!)^{1/n^2}$

Solution.

(a) Let $x_n=n^{1/n^2}-1$. Then, $(x_n+1)^{n^2}=n$. By the binomial theorem, we have

$$\sum_{k=0}^{n^2}\binom{n^2}{k}x_n^k=n\quad\text{so}\quad n\geq 1+n^2x_n.$$

As such,

$$x_n\leq\frac{1}{n}-\frac{1}{n^2}\quad\text{which implies}\quad\lim_{n\to\infty}x_n\leq 0.$$

As for the lower bound, note that for $n\geq 2$, we have $x_n\geq 0$, so by **(iv)** of Theorem 2.2,

$$\lim_{n\to\infty}x_n\geq 0.$$

Combining both inequalities shows that the limit of x_n is 0, so the original limit is 1.

(b) We note that for $n\geq 4$, we have

$$n^2\leq n!\leq n^n$$

so

$$\lim_{n\to\infty}\left(n^2\right)^{1/n^2}\leq\lim_{n\to\infty}(n!)^{1/n^2}\leq\lim_{n\to\infty}(n^n)^{1/n^2}\quad\text{or equivalently}\quad\left(\lim_{n\to\infty}n^{1/n^2}\right)^2\leq\lim_{n\to\infty}(n!)^{1/n^2}\leq\lim_{n\to\infty}n^{1/n}.$$

By **(a)**, the lower bound is 1 and by **(iii)** of Theorem 2.4, the upper bound is 1. Hence, by the squeeze theorem (Theorem 2.3), the desired limit is 1. □

Theorem 2.5 (limit theorems). The following hold:

(i) If
$$\lim_{n\to\infty} |x_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} x_n = 0.$$
If $c = n$, the limit is still the same.

(ii) If
$$\lim_{n\to\infty} x_n = L \quad \text{then} \quad \lim_{n\to\infty} |x_n| = |L|.$$

(iii) Suppose $x_n \geq 0$ for all $n \in \mathbb{N}$. Then,
$$\lim_{n\to\infty} x_n = L \quad \text{implies} \quad \lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}.$$

(iv) If $x_n \geq 0$ for all $n \in \mathbb{N}$ and x_n converges, then
$$\lim_{n\to\infty} x_n \geq 0.$$

To see an application/proof of **(ii)** of Theorem 2.5 as well as its reverse direction, see Example 2.24.

Example 2.24 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 6). Prove that
$$\lim_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} |x_n| = 0.$$
Give an example to show that the convergence of $\{|x_n|\}_{n\in\mathbb{N}}$ need not imply the convergence of $\{x_n\}_{n\in\mathbb{N}}$.

Solution. For the first part, we first prove the forward direction. Suppose $x_n \to 0$. Then, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n| < \varepsilon$. Since applying the absolute value function twice is the same as applying it once, the forward direction holds.

For the proof of the reverse direction, suppose $|x_n| \to 0$. Then, for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, we have $||x_n| - 0| < \varepsilon$. Same as the reasoning provided earlier, the reverse direction holds.

For the second part, let $x_n = (-1)^n$. Then, $\{|x_n|\}_{n\in\mathbb{N}}$ converges because $|x_n| = 1$ which is the constant sequence 1 but by Example 2.15, $\{x_n\}_{n\in\mathbb{N}}$ is not convergent. □

Example 2.25 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 15). If $\{b_n\}_{n\in\mathbb{N}}$ is a bounded sequence and $\lim_{n\to\infty} a_n = 0$, show that
$$\lim_{n\to\infty} a_n b_n = 0.$$

Solution. Since b_n is bounded, then for all $n \in \mathbb{N}$, there exists $M \in \mathbb{R}$ such that $-M \leq b_n \leq M$. As such,
$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \quad \text{under the assumption that both limits exist}$$
Note that
$$0 = 0 \cdot (-M) \leq \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) \leq 0 \cdot M = 0$$
so by the squeeze theorem (Theorem 2.3), the result follows. □

Alternatively, we can prove the result in Example 2.25 more formally.

Solution. Since
$$\lim_{n\to\infty} a_n = 0,$$

then for every $\varepsilon, M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n| < \varepsilon/M$. Again, since b_n is bounded, then for all $n \in \mathbb{N}$, there exists $M \in \mathbb{R}^+$ such that $-M \leq b_n \leq M$. So, $(-\varepsilon/M) \cdot M \leq a_n b_n \leq (\varepsilon/M) \cdot M$. Since ε can be made sufficiently small, by the squeeze theorem (Theorem 2.3), the result follows. □

Example 2.26 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 22). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} x_n^{1/n} = L < 1.$$

Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that

$$\lim_{n\to\infty} x_n = 0.$$

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|x_n^{1/n} - L\right| < \varepsilon$. We choose $\varepsilon = (1-L)/2$. Then,

$$L - \frac{1-L}{2} < x_n^{1/n} < L + \frac{1-L}{2} \quad \text{so} \quad \frac{3L-1}{2} < x_n^{1/n} < \frac{L+1}{2}.$$

Raising each side to the power n yields the inequality

$$\left(\frac{3L-1}{2}\right)^n < x_n < \left(\frac{L+1}{2}\right)^n.$$

We can choose $r = (L+1)/2$. By the squeeze theorem, the result follows. □

Corollary 2.2. If $a, b \in \mathbb{R}$ and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ and x_n is convergent, then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

Example 2.27. Suppose we wish to evaluate the following limit:

$$\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}}$$

Solution. Recognise that for $0 \leq a < 1$, then $a^n \to 0$ as $n \to \infty$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n + 3^{n+1} + 5^{n+2}}{2^{n+2} + 3^n + 5^{n+1}} &= \lim_{n\to\infty} \frac{2^n + 3(3^n) + 25(5^n)}{4(2^n) + 3^n + 5(5^n)} \\
&= 5 - \lim_{n\to\infty} \frac{19(2^n) + 2(3^n)}{4(2^n) + 3^n + 5(5^n)} \\
&= 5 - \lim_{n\to\infty} \frac{19\left(\frac{2}{5}\right)^n + 2\left(\frac{3}{5}\right)^n}{4\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n + 5} \\
&= 5
\end{aligned}$$

□

Example 2.28 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 18). If $0 < a < b$, determine

$$\lim_{n\to\infty} \frac{a^{n+1} + b^{n+1}}{a^n + b^n}.$$

Solution. We have

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = b \cdot \frac{(a/b)^{n+1} + 1}{(a/b)^n + 1}$$

so the limit evaluates to b. □

Example 2.29 (MA2108S AY16/17 Sem 2 Homework 4). Let $X = x_n$ and $Y = y_n$ be given sequences, and let the "shuffled" sequence $Z = z_n$ be defined by

$$z_1 = x_1, z_2 = y_1, \ldots, z_{2n-1} = x_n, z_{2n} = y_n.$$

Show that

$$Z \text{ is convergent} \quad \text{if and only if} \quad X \text{ and } Y \text{ are convergent and } \lim_{n\to\infty} X = \lim_{n\to\infty} Y.^{\dagger}$$

Solution. We first prove the reverse direction. Suppose X and Y are convergent and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = L.$$

By the definition of a limit of a sequence, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \text{ whenever } n \geq N_1 \quad \text{and} \quad |y_n - L| < \varepsilon \text{ whenever } n \geq N_2.$$

Set $N = \max\{2N_1, 2N_2\}$. Then, whenever $n \geq N$, $|z_n - L| < \varepsilon$ and we are done.

Now, we prove the reverse direction. Suppose Z is convergent. That is

$$\lim_{n\to\infty} z_n = L.$$

By the definition of the limit of a sequence, there exists $N \in \mathbb{N}$ such that

$$|z_n - L| < \varepsilon \text{ whenever } n \geq N.$$

We need to show that

$$|x_n - L| < \varepsilon \text{ and } |y_n - L| < \varepsilon \text{ whenever } n \geq N,$$

which are

$$|z_{2n-1} - L| < \varepsilon \text{ and } |z_{2n} - L| < \varepsilon \quad \text{equivalently}.$$

Thus, we need $2n - 1 \geq N$ and $2n \geq N$, which are obviously true. Hence, the result follows. □

Example 2.30 (MA2108 AY19/20 Sem 1). Let $f : (a, \infty) \to \mathbb{R}$ be a function such that it is bounded in any interval (a, b) and

$$\lim_{x\to\infty} (f(x+1) - f(x)) = A.$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{x} = A.$$

Solution. Let $\varepsilon > 0$ be arbitrary. By the given limit, there exists $M > 0$ such that for all $x > M$,

$$|f(x+1) - f(x) - A| < \varepsilon.$$

So,

$$A - \varepsilon < f(x+1) - f(x) < A + \varepsilon.$$

Since f is locally bounded, then for $M < x \leq M+1$, $-B < f(x) < B$ for some $B \in \mathbb{R}$. Hence,

$$-B + (A+\varepsilon) \cdot \lfloor x - M \rfloor < f(x) < B + (A+\varepsilon) \cdot \lceil x - M \rceil.$$

Dividing by x on both sides, since ε is made arbitrarily small, by the squeeze theorem, $f(x)/x$ tends to A as $x \to \infty$. □

† Refer to this problem on StackExchange here.

Theorem 2.6 (L'Hôpital's Rule). If f and g are differentiable functions such that $g'(x) \neq 0$ on an open interval I containing a,

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \text{ exists} \quad \text{then} \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Theorem 2.7 (Stolz-Cesàro theorem). Let x_n and y_n be two sequences of real numbers. If y_n is strictly monotone and divergent and

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \text{ exists} \quad \text{then} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

We will give a proof of the Stolz-Cesàro theorem (Theorem 2.7) in Example 2.31, where without loss of generality, we assume that the sequence $\{b_n\}_{n\in\mathbb{N}}$ (in place of $\{y_n\}_{n\in\mathbb{N}}$ in Theorem 2.7) is monotonically increasing.

Example 2.31 (MA2108S AY24/25 Sem 2 Tutorial 3; Stolz-Cesàro). Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ be sequences, where $\{b_n\}_{n\in\mathbb{N}}$ is strictly increasing and divergent. Prove that

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = A \quad \text{implies} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = A.$$

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{a_{n+1} - a_n}{b_{n+1} - b_n} - A \right| < \varepsilon.$$

So,

$$(A - \varepsilon)(b_{n+1} - b_n) < a_{n+1} - a_n < (A + \varepsilon)(b_{n+1} - b_n)$$

By the method of difference,

$$\sum_{n=N}^{k-1} (A - \varepsilon)(b_{n+1} - b_n) < \sum_{n=N}^{k-1} a_{n+1} - a_n < \sum_{n=N}^{k-1} (A + \varepsilon)(b_{n+1} - b_n)$$

so

$$(A - \varepsilon)(b_k - b_N) < a_k - a_N < (A + \varepsilon)(b_k - b_N).$$

Adding a_N to each side yields

$$(A - \varepsilon)(b_k - b_N) + a_N < a_k < (A + \varepsilon)(b_k - b_N) + a_N.$$

For k sufficiently large, we have $b_k > 1$ so $1/b_k < 1$. As such,

$$\frac{(A - \varepsilon)(b_k - b_N) + a_N}{b_k} < \frac{a_k}{b_k} < \frac{(A + \varepsilon)(b_k - b_N) + a_N}{b_k}.$$

So,

$$(A - \varepsilon)\left(1 - \frac{b_N}{b_k}\right) + \frac{a_N}{b_k} < \frac{a_k}{b_k} < (A + \varepsilon)\left(1 - \frac{b_N}{b_k}\right) + \frac{a_N}{b_k}.$$

Letting $k \to \infty$, we see that a_k/b_k is sandwiched between A and A, so by the squeeze theorem (Theorem 2.3), the result follows. $\square$

Theorem 2.8 (Stolz-Cesàro theorem, alt.). If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$$

where y_n is strictly decreasing and

$$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L \quad \text{then} \quad \lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

Example 2.32 (MA2108 AY19/20 Sem 1). Let a_n be a sequence in $\mathbb{R}$.

(i) Prove that if

$$\lim_{n\to\infty} a_n = a \quad \text{then} \quad \lim_{n\to\infty} \frac{a_1+a_2+\ldots+a_n}{n} = a.$$

(ii) Suppose the sequence

$$\frac{a_1+a_2+\ldots+a_n}{n}$$

converges. Can we deduce that a_n converges? Justify your answer.

Solution.

(i) Let $\varepsilon > 0$ be arbitrary. There exists $K_1 \in \mathbb{N}$ such that $|a_j - a| \leq \varepsilon/2$ for all $j \geq K_1$. Then,

$$\left|\frac{a_1+a_2+\ldots+a_n}{n} - a\right| = \left|\frac{1}{n}\sum_{j=1}^{n}(a_j-a)\right|.$$

We can bound this sum accordingly. For $n \geq K_1$,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n}(a_j-a)\right| &\leq \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j-a)\right| + \frac{1}{n}\left|\sum_{j=K_1+1}^{n}(a_j-a)\right| \quad \text{by triangle inequality} \\
&\leq \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j-a)\right| + \frac{n-K_1}{n}\cdot\frac{\varepsilon}{2} \\
&< \frac{1}{n}\left|\sum_{j=1}^{K_1}(a_j-a)\right| + \frac{\varepsilon}{2} \\
&= \frac{C}{n} + \frac{\varepsilon}{2}
\end{aligned}$$

Here, we let C be the sum of $a_j - a$ from $j=1$ to $j=K_1$. Next, for $K \in \mathbb{N}$, where $K > \max\{K_1, 2C/\varepsilon\}$, it is now easy to see that

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n}(a_j-a)\right| &< \frac{C}{n} + \frac{\varepsilon}{2} \\
&\leq \frac{C}{K} + \frac{\varepsilon}{2} \quad \text{since } n \geq K \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{since } K > \frac{2C}{\varepsilon} \\
&= \varepsilon
\end{aligned}$$

(ii) No. Define $s_n = (a_1+a_2+\ldots+a_n)/n$. Setting $a_n = (-1)^n$,

$$s_n = \begin{cases} -1/n & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

By the squeeze theorem, as $n\to\infty$, $s_n \to 0$, so it converges. However, a_n diverges. □

2.2 Monotone Sequences

Definition 2.7 (monotone sequence). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that it is

(i) monotonically increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$;

(ii) monotonically decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$

Proposition 2.2. Let F be an ordered field. If F has the least upper bound property, then

$$\begin{aligned} \text{every monotone sequence in } F &\quad \text{is} \quad \text{bounded} \\ \text{every monotone increasing sequence in } F &\quad \text{is} \quad \text{bounded above} \\ \text{every monotone decreasing sequence in } F &\quad \text{is} \quad \text{bounded below} \end{aligned}$$

Theorem 2.9 (monotone convergence theorem). Let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone sequence. Then,

$$\{x_n\}_{n\in\mathbb{N}} \text{ converges} \quad \text{if and only if} \quad \text{it is bounded.}$$

In particular, if

$$x_n \text{ is increasing then } \lim_{n\to\infty} x_n = \sup x_n \quad \text{and if} \quad x_n \text{ is decreasing then } \lim_{n\to\infty} x_n = \inf x_n.$$

Proof. It suffices to show that if $\{x_n\}_{n\in\mathbb{N}}$ is a monotonically increasing sequence in F which is bounded above, then there exists $x \in F$ such that $x_n \to x$ in F. Let

$$S = \{x_n \in F : n \in \mathbb{N}\} = \{x \in F : \text{there exists } n \in \mathbb{N} \text{ such that } x = px_n\}$$

denote the image set of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Since $\mathbb{N} \neq \emptyset$, then $S \neq \emptyset$. As $\{x_n\}_{n\in\mathbb{N}}$ is bounded above, then $S \subseteq F$ is also bounded above.

By the least upper bound property of F (Definition 1.10), there exists $x = \sup S$ in F. We claim that $x_n \to x$ in F, i.e.

$$\text{for any } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } n \geq N \text{ we have } |x_n - x| < \varepsilon.$$

Since $\varepsilon > 0$, then $x - \varepsilon \in F$ is not an upper bound of S. So, there exists $x' \in S$ such that $x - \varepsilon < x'$, i.e. there exists $N \in \mathbb{N}$ such that $x - \varepsilon < x_N$. However, as $x = \sup S$ is an upper bound of S, then $x_n \leq x$. As $\{x_n\}_{n\in\mathbb{N}}$ is monotonically increasing, we know that

$$\text{for all } n \geq N \quad \text{we have} \quad x_N \leq x_n.$$

Equivalently, we have $x - \varepsilon < x_n \leq x$, so $|x - x_n| < \varepsilon$. We conclude that x_n tends to $\sup x_n$. □

Example 2.33 (MA2108S AY16/17 Sem 2 Homework 4). Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that x_n converges and find the limit.

Solution. It is clear that x_n is bounded above by 2. Given that $x_1 = 1$, we show that x_n is strictly increasing. That is, for $n \in \mathbb{N}$, $x_{n+1} > x_n$.

$$x_{n+1} - x_n = \sqrt{2 + x_n} - x_n = \frac{2 + x_n - x_n^2}{\sqrt{2 + x_n} + x_n} = \frac{(x_n - 2)(x_n + 1)}{\sqrt{2 + x_n} + x_n}.$$

It is clear that x_n is a sequence of positive terms so we consider the numerator of $x_{n+1} - x_n$, which is $(2 - x_n)(x_n + 1)$. For $1 \le x_n \le 2$, this product is always positive, and hence $x_{n+1} - x_n \ge 0$. By the monotone convergence theorem (Theorem 2.9), x_n converges.

Suppose

$$\lim_{n\to\infty} x_n = L.$$

Then, $L = \sqrt{2+L}$, but since $L > 0$, then $L = 2$. $\square$

Example 2.34. Consider the recurrence relation

$$a_{n+1} = \frac{3+a_n}{1+a_n} \quad \text{with the initial condition} \quad a_1 = 3.$$

Prove that a_n is a convergent sequence and find its limit.

Solution. We first prove that $\sqrt{3} \le a_n \le 3$. To show that $a_n \le 3$, we have

$$\begin{aligned} a_{k+1} = \frac{3+a_k}{1+a_k} &\le \frac{3+3}{1+\sqrt{3}} \quad \text{by the induction hypothesis} \\ &\le 3 \end{aligned}$$

Similarly, we have

$$a_{k+1} = \frac{3+a_k}{1+a_k} \ge \frac{3+\sqrt{3}}{1+3} \ge \sqrt{3}$$

where again, the first inequality follows by the induction hypothesis. This shows that a_n is bounded.

We then prove that a_n is decreasing using strong induction. We have

$$a_{k+1} - a_k = \frac{3+a_k}{1+a_k} - a_k = \frac{3-a_k^2}{1+a_k}.$$

It suffices to prove that $3 - a_k^2 \le 0$ since the denominator $1 + a_k > 0$. Since $\sqrt{3} \le a_k \le 3$, then $3 \le a_k^2 \le 9$ so $3 - a_k^2 \le 0$. So, a_n is decreasing. By the monotone convergence theorem, a_n converges to some limit L. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L \quad \text{then} \quad L = \frac{3+L}{1+L}.$$

So, either $L = \sqrt{3}$ or $L = -\sqrt{3}$. We reject the latter as we earlier established that a_n is a sequence of positive numbers (from $\sqrt{3} \le a_n \le 3$) so $L = 3$. $\square$

Example 2.35 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 16)**.** Let

$$y_n = \sqrt{n+1} - \sqrt{n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\{\sqrt{n}y_n\}_{n\in\mathbb{N}}$ converges. Find the limit.

Solution. Let $x_n = \sqrt{n}y_n$. Then,

$$x_n = \sqrt{n(n+1)} - n.$$

We first prove that $\{x_n\}_{n\in\mathbb{N}}$ is bounded, i.e. $0 \le x_n \le 1/2$. Proving the lower bound is obvious because it is equivalent to showing that

$$n(n+1) \ge n^2 \quad \text{or equivalently} \quad n \ge 0.$$

The aforementioned statement holds trivially. We then justify the upper bound, i.e.

$$\sqrt{n(n+1)} - n - \frac{1}{2} \le 0 \quad \text{or equivalently} \quad n(n+1) \le \left(n + \frac{1}{2}\right)^2.$$

We have

$$n^2 + n \le n^2 + n + \frac{1}{4} \quad \text{or equivalently} \quad \frac{1}{4} \ge 0.$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

Next, we prove that x_n is increasing by induction. We have

$$\begin{aligned} x_{k+1} - x_k &= \sqrt{(k+1)(k+2)} - (k+1) - \sqrt{k(k+1)} + k \\ &= \sqrt{k+1}\left(\sqrt{k+2} - \sqrt{k}\right) - 1 \\ &= \frac{2\sqrt{k+1} - \sqrt{k+2} - \sqrt{k}}{\sqrt{k+2} + \sqrt{k}} \end{aligned}$$

As such, it suffices to prove that $2\sqrt{k+1} - \sqrt{k+2} - \sqrt{k} \ge 0$. To see why this holds, define $z_k = \sqrt{k+1} - \sqrt{k}$. Then, the mentioned inequality is equivalent to $z_k - z_{k+1} \ge 0$, or $z_{k+1} \le z_k$. As it is known that z_k is a decreasing sequence, then $x_{k+1} \ge x_k$, i.e. x_n is increasing. By the monotone convergence theorem (Theorem 2.9), $\{x_n\}_{n\in\mathbb{N}}$ converges.

Hence,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n(n+1) - n^2}{\sqrt{n(n+1)} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + 1/n} + 1} = \frac{1}{2}.$$

□

Example 2.36 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 24). Let $x_1 > 1$ and

$$x_{n+1} = 2 - \frac{1}{x_n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded and monotone. Find the limit.

Solution. We first prove that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. We claim that $x_n > 1$ for all $n \in \mathbb{N}$. The base case holds trivially. Assume that $x_k > 1$ for some $k \in \mathbb{N}$. Then, $-1/x_k > -1$ so $x_{k+1} > 1$. As such, $\{x_n\}_{n\in\mathbb{N}}$ is bounded below by induction.

We then prove that $\{x_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. Assume that $x_{k+1} - x_k \le 0$ for all $k \le n-1$. Then,

$$x_{n+1} - x_n = 2 - \frac{1}{x_n} - x_n = \frac{2x_n - x_n^2 - 1}{x_n} = -\frac{(x_n-1)^2}{x_n}.$$

Since x_n is a sequence of positive numbers (we deduce that x_n is bounded below by 1 earlier), then $x_{n+1} - x_n < 0$, so $x_{n+1} < x_n$. By induction, $\{x_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. Hence, $\{x_n\}_{n\in\mathbb{N}}$ converges. Suppose the limit is L. Then,

$$L = 2 - \frac{1}{L}.$$

Hence, $L = 1$. □

Example 2.37 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 25). Let $x_1 \ge 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below by 2. Find the limit.

Solution. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded below by 2. We have

$$x_{n+1} = 1+\sqrt{x_n-1} \geq 1+\sqrt{2-1} = 2$$

so by induction, $\{x_n\}_{n\in\mathbb{N}}$ is bounded below. We then prove that $\{x_n\}_{n\in\mathbb{N}}$ is decreasing. We have

$$x_{n+1}-x_n = 1+\sqrt{x_n-1}-x_n = \sqrt{x_n-1}\left(1-\sqrt{x_n-1}\right).$$

Since $\sqrt{x_n-1}\geq 1$, then it follows that $x_{n+1}-x_n<0$, i.e. $\{x_n\}_{n\in\mathbb{N}}$ is decreasing. By the monotone convergence theorem, $\{x_n\}_{n\in\mathbb{N}}$ converges. Suppose the limit is L. Then, $L=1+\sqrt{L-1}$. As such, $L=2$. □

Example 2.38 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 28)**.** Let $a>0$ and let $z_1>0$. Define

$$z_{n+1}=\sqrt{a+z_n} \quad \text{for } n\in\mathbb{N}.$$

Show that $\{z_n\}_{n\in\mathbb{N}}$ converges and find the limit.

Solution. We first observe that $\{z_n\}_{n\in\mathbb{N}}$ is a positive sequence of numbers. Consider the equation $L=\sqrt{a+L}$, which yields $L^2-L-a=0$. The positive root of this quadratic equation is

$$r=\frac{1+\sqrt{1+4a}}{2}.$$

As such, we shall consider three cases as follows:

(i) $z_1<r$

(ii) $z_1=r$

(iii) $z_1>r$

For **(i)**, if $z_1<r$, then we claim that $\{z_n\}_{n\in\mathbb{N}}$ is increasing and bounded above by r. We first prove the latter by induction. The base case holds trivially as we earlier mentioned that $z_1<r$. Suppose $z_k<r$ for some $k\in\mathbb{N}$. Then,

$$z_{k+1}=\sqrt{a+z_k}\leq\sqrt{a+r}=r.$$

The last equality holds because it is equivalent to $r^2-r-a=0$. As mentioned, r is a root of the quadratic equation $L^2-L-a=0$, so indeed $\sqrt{a+r}=r$. As such, $\{z_n\}_{n\in\mathbb{N}}$ is bounded above.

We then prove that $\{z_n\}_{n\in\mathbb{N}}$ is increasing. We have

$$z_{n+1}-z_n=\sqrt{a+z_n}-z_n=\frac{\left(\sqrt{a+z_n}-z_n\right)\left(\sqrt{a+z_n}+z_n\right)}{\sqrt{a+z_n}+z_n}=\frac{a+z_n-z_n^2}{\sqrt{a+z_n}+z_n}=-\frac{z_n^2-z_n-a}{\sqrt{a+z_n}+z_n}.$$

By considering the denominator, as z_n is a positive sequence of numbers, then $\sqrt{a+z_n}, z_n>0$ so it suffices to prove that $z_n^2-z_n-a<0$. Let

$$r'=\frac{-1-\sqrt{1+4a}}{2} \quad \text{be} \quad \text{the negative root of the quadratic equation } L^2-L-a=0.$$

Then, the solution to the quadratic inequality $z_n^2-z_n-a<0$ is $r'<z_n<r$. As we earlier deduced that $0<z_n<r$, then $z_n<r$ holds so $z_{n+1}-z_n>0$, i.e. $\{z_n\}_{n\in\mathbb{N}}$ is increasing. By the monotone convergence theorem, $\{z_n\}_{n\in\mathbb{N}}$ converges.

For **(ii)**, we have the constant sequence $z_n=\sqrt{a+r}$ so $\{z_n\}_{n\in\mathbb{N}}$ converges.

Lastly, for **(iii)**, if $z_1>r$, we claim that $\{z_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below by r. We first prove the latter by induction. Again, the base case holds trivially as we earlier mentioned that $z_1 > r$. Suppose $z_k > r$ for some $k \in \mathbb{N}$. Then,

$$z_{k+1} = \sqrt{a+z_k} \geq \sqrt{a+r} = r.$$

Again, the last equality holds due to the same argument made previously, i.e. r is the positive root of the quadratic equation $L^2 - L - a = 0$. As such, $\{z_n\}_{n\in\mathbb{N}}$ is bounded below.

We then prove that $\{z_n\}_{n\in\mathbb{N}}$ is decreasing. We have

$$z_{n+1} - z_n = -\frac{z_n^2 - z_n - a}{\sqrt{a+z_n}+z_n}.$$

Again, it suffices to consider the numerator $z_n^2 - z_n - a$. We wish to prove that it is positive. The solution to the quadratic inequality $z_n^2 - z_n - a > 0$ is $z_n > r$ or $z_n < -r'$. As we earlier deduced that $z_n > r$, then it follows that $\{z_n\}_{n\in\mathbb{N}}$ is decreasing. By the monotone convergence theorem, $\{z_n\}_{n\in\mathbb{N}}$ converges.

As mentioned, the limit is r, which is equal to

$$\frac{1+\sqrt{1+4a}}{2}.$$

We chose the positive root here because $\{z_n\}_{n\in\mathbb{N}}$ is a positive sequence of numbers. □

Example 2.39 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 29). Let $x_1 = a > 0$ and

$$x_{n+1} = x_n + \frac{1}{x_n} \quad \text{for } n \in \mathbb{N}.$$

Determine whether $\{x_n\}_{n\in\mathbb{N}}$ converges or diverges.

Solution. We have

$$x_{n+1}^2 = x_n^2 + \frac{1}{x_n^2} + 2 \quad \text{so} \quad x_{n+1}^2 - x_n^2 = 2 + \frac{1}{x_n^2} > 2.$$

By the method of difference, $x_N^2 - x_1^2 > 2(N-1)$ so

$$x_N > \sqrt{a+2N-2}.$$

Hence, $x_N \to \infty$ as $N \to \infty$, which implies $\{x_n\}_{n\in\mathbb{N}}$ diverges. □

Example 2.40 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 30). Establish the convergence or the divergence of the sequence (y_n), where

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \quad \text{for } n \in \mathbb{N}.$$

Solution. We first prove that $\{y_n\}_{n\in\mathbb{N}}$ is bounded. We have

$$\frac{1}{2} \leq \underbrace{\frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n}}_{n \text{ copies}} \leq \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \leq \underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{n \text{ copies}} = 1$$

so $1/2 \leq y_n \leq 1$ for all $n \in \mathbb{N}$. This shows that $\{y_n\}_{n\in\mathbb{N}}$ is bounded.

We then prove that $\{y_n\}_{n\in\mathbb{N}}$ is increasing. We have

$$y_{n+1} - y_n = \left(\frac{1}{n+1} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \ldots + \frac{1}{2n}\right) = \frac{1}{2n+1} + \frac{1}{2n+2} > 0$$

so $\{y_n\}_{n\in\mathbb{N}}$ is increasing. By the monotone convergence theorem, $\{y_n\}_{n\in\mathbb{N}}$ converges. □

Example 2.41 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 31)**.** Let

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \quad \text{for each } n \in \mathbb{N}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ converges.

Solution. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is increasing. We have

$$x_{n+1} - x_n = \frac{1}{(n+1)^2} > 0$$

so $x_{n+1} > x_n$, so $\{x_n\}_{n\in\mathbb{N}}$ is increasing.

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$. We use the fact $k^2 \geq k(k-1)$ for $k \geq 2$ so

$$\frac{1}{k^2} \leq \frac{1}{k(k-1)}.$$

As such,

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \leq 2$$

where the last inequality uses the method of difference. By the monotone convergence theorem (Theorem 2.9), $\{x_n\}_{n\in\mathbb{N}}$ converges. □

Example 2.42 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 21)**.** Let $X = \{x_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1.$$

Show that X is not a bounded sequence and hence is not convergent.

Solution. By the formal definition of a limit, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon.$$

So,

$$L - \varepsilon < \frac{x_{n+1}}{x_n} < L + \varepsilon.$$

Hence,

$$x_{n+1} > (L-\varepsilon)x_n > (L-\varepsilon)^2 x_{n-1} > \ldots > (L-\varepsilon)^{n-N+1} x_N.$$

As $n \to \infty$, then it shows that x_n is not bounded above. As such X is not a bounded sequence. By the monotone convergence theorem (Theorem **??**), X is not a convergent sequence. □

Example 2.43 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 23)**.** Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a convergent sequence and $\{y_n\}_{n\in\mathbb{N}}$ is such that for any $\varepsilon > 0$, there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that $\{y_n\}_{n\in\mathbb{N}}$ is convergent?

Solution. Yes. Let $\varepsilon' = 2\varepsilon$ (which is > 0) be arbitrary. Since $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - L| < \varepsilon$, where L is the limit of x_n. As such, choose

$K = \max\{N, M\}$. Then, for all $n \geq K$, we have

$$\begin{aligned} |y_n - L| &= |y_n - x_n + x_n - L| \\ &\leq |y_n - x_n| + |x_n - L| \quad \text{by the triangle inequality} \\ &< \varepsilon + \varepsilon \\ &= \varepsilon' \end{aligned}$$

So, $\{y_n\}_{n\in\mathbb{N}}$ is also a convergent sequence. □

Example 2.44 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 34)**.** Suppose

$$x_n \geq 0 \text{ for all } n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} (-1)^n x_n \text{ exists.}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ converges.

Solution. Since the aforementioned limit, say L, exists, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|(-1)^n x_n - L| < \varepsilon.$$

So,

$$L - \varepsilon < (-1)^n x_n < L + \varepsilon.$$

Since this inequality holds for all $n \in \mathbb{N}$, suppose n is even. Then, there exists $k \in \mathbb{N}$ such that $n = 2k$. Moreover, as $\varepsilon > 0$ can be made arbitrarily small, we can choose $\varepsilon = 1$, so

$$L - 1 < x_{2k} < L + 1.$$

On the other hand, suppose n is odd. Then, there exists $k \in \mathbb{N}$ such that $n = 2k + 1$. So,

$$L - 1 < -x_{2k+1} < L + 1.$$

As $k \to \infty$, by the squeeze theorem, we see that

$$\lim_{k\to\infty} x_{2k} = L.$$

Similarly, by the squeeze theorem,

$$\lim_{k\to\infty} x_{2k+1} = -L.$$

Since $x_n \geq 0$ for all $n \in \mathbb{N}$, then $L \geq 0$ and $-L \geq 0$, which forces $L = 0$. So, $x_n \to 0$, i.e. $\{x_n\}_{n\in\mathbb{N}}$ converges. □

Methods of computing square roots are numerical analysis algorithms for approximating the principal, or non-negative, square root of a real number, say S.

Theorem 2.10 (Babylonian method)**.** We start with an initial value somewhere near $\sqrt{S}$. That is $x_0 \approx \sqrt{S}$. We then use the following recurrence relation to find a better estimate for $\sqrt{S}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{S}{x_n}\right) \quad \text{where} \quad \lim_{n\to\infty} x_n = \sqrt{S}$$

Proof. Suppose

$$\lim_{n\to\infty} x_n = L.$$

Substituting this into the recurrence relation yields

$$L = \frac{1}{2}\left(L + \frac{S}{L}\right).$$

Rearranging and the result follows. □

Theorem 2.11 (nested interval theorem). Let $I_n = [a_n, b_n]$, where $n \in \mathbb{N}$, be a nested sequence of closed and bounded sequences. That is, $I_n \supseteq I_{n+1}$. Then, the intersection

$$\bigcap_{n=1}^{\infty} I_n = \{x : x \in I_n \text{ for all } n \in \mathbb{N}\}$$

is non-empty. In addition, if $b_n - a_n \to 0$ (i.e. length of I_n tends to 0), then the intersection contains exactly one point.

Definition 2.8 (harmonic numbers). The harmonic numbers, H_n, are defined to be

$$\sum_{k=1}^{n} \frac{1}{k}.$$

Definition 2.9 (harmonic series). The harmonic series is defined to be the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n\to\infty} H_n$$

Note that the harmonic numbers are increasing (since $H_{n+1} - H_n > 0$) and

$$\lim_{n\to\infty} H_n = 0.$$

However, the harmonic series is divergent! Another interesting property is that other than H_1, the harmonic numbers are never integers, whose proof hinges on some elementary Number Theory.

2.3 Euler's Number, e

Definition 2.10 (Euler's number). Euler's number, $e \approx 2.71828$, is defined to be

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Theorem 2.12. The sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n \quad \text{is strictly increasing.}$$

That is, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

Proof. It is easier to prove $x_n > x_{n-1}$, so we wish to prove

$$\left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

First, we write $1 + 1/n$ as

$$1 + \frac{1}{n-1} = \frac{n}{n-1} = \frac{1}{1 - 1/n}.$$

Hence,

$$\begin{aligned}\frac{(1+1/n)^n}{(1+1/(n-1))^{n-1}} &= \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{n-1}\\ &= \left(1+\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^n\left(1-\frac{1}{n}\right)^{-1}\\ &= \left(1-\frac{1}{n^2}\right)^n\left(1-\frac{1}{n}\right)^{-1}\end{aligned}$$

By Bernoulli's inequality (Theorem 1.5), this is greater than 1, and so $x_n > x_{n-1}$. □

Theorem 2.13. $2 \le e \le 3$

Proof. We use the series expansion of x_n.

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\ldots+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3\\ &= 1+1+\frac{n-1}{(2!)n}+\frac{(n-1)(n-2)}{3!(n^2)}+\ldots\end{aligned}$$

It is clear that $e \ge 2$. To prove that $e \le 3$, we consider the infinite series, but starting from the third term of the expansion of x_n. It suffices to show that

$$\frac{n-1}{2n}+\frac{(n-1)(n-2)}{6n^2}+\frac{(n-1)(n-2)(n-3)}{24n^3}+\ldots \le 1.$$

Observe that the r^{th} term can be written as

$$\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r} = \frac{1}{(r+1)!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\ldots\left(1-\frac{r}{n}\right) \le \frac{1}{(r+1)!}.$$

It is clear that

$$\frac{1}{(r+1)!} \le \frac{1}{2^r},$$

since the factorial grows much faster than the geometric series, and so taking the reciprocal, the result follows. To conclude,

$$\sum_{r=1}^{\infty}\frac{(n-1)(n-2)(n-3)\ldots(n-r)}{(r+1)!n^r} \le \sum_{r=1}^{\infty}\frac{1}{2^r} = 1,$$

and we are done. □

Though the incredible constant is named after the Swiss mathematician Leonhard Euler, its discovery is actually accredited to another Swiss mathematician, Jacob Bernoulli. Just like π, e is also irrational (Theorem 2.14), which can be proven by contradiction.

Theorem 2.14. e is irrational

Proof. Suppose on the contrary that e is rational. Then, there exist $p, q \in \mathbb{Z}$ with $q \neq 0$ such that $e = p/q$. As e can be expressed as the following infinite series

$$\sum_{k=0}^{\infty}\frac{1}{k!},$$

we have

$$e = \frac{p}{q} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{m!} + \frac{1}{(m+1)!} + \ldots$$
$$m!e = \frac{m!}{0!} + \frac{m!}{1!} + \frac{m!}{2!} + \frac{m!}{3!} + \frac{m!}{4!} + \ldots + \frac{m!}{m!} + \frac{m!}{(m+1)!} + \ldots$$

By setting $q = m!$, we see that $m!e \in \mathbb{Z}$. Next, we take a look at the RHS. Observe that

$$\frac{m!}{0!} + \frac{m!}{1!} + \frac{m!}{2!} + \frac{m!}{3!} + \frac{m!}{4!} + \ldots + \frac{m!}{m!}$$

is an integer but

$$\frac{m!}{(m+1)!} + \frac{m!}{(m+2)!} + \frac{m!}{(m+3)!} + \ldots = \frac{1}{m+1} + \frac{1}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} + \ldots$$

is not an integer, which is a contradiction. □

2.4 The Euler-Mascheroni Constant, γ

Definition 2.11 (Euler-Mascheroni constant)**.** The Euler-Mascheroni constant, $\gamma \approx 0.5772$, is the limiting difference between the harmonic series and the natural logarithm. That is,

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

γ is an epic constant. From Figure 9, the Euler-Mascheroni constant can be regarded as the sum of areas of the yellow rectangles minus the area under the curve $y = 1/x$ for $x \geq 1$.

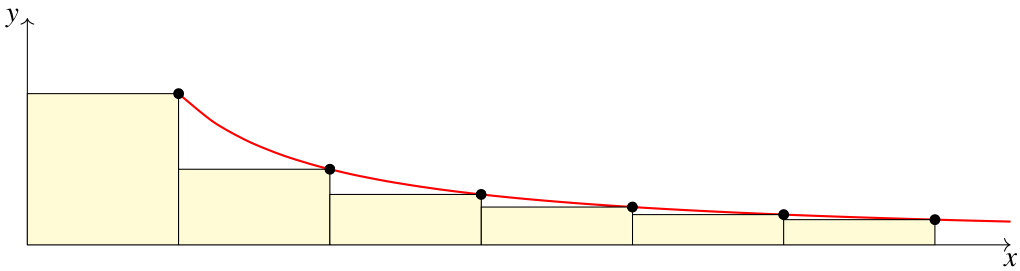


Figure 9: The graph of $y = 1/x$ and an approximation for the area under the curve

It is interesting to note that the Euler-Mascheroni constant converges even though the harmonic series diverges and $\ln n$ tends to infinity as n tends to infinity. Let us prove this result using the monotone convergence theorem.

Lemma 2.1. Let x_n be the following sequence:

$$x_n = \sum_{k=1}^{n}\frac{1}{k} - \ln n$$

Then, the following properties hold:

(i) x_n is a decreasing sequence.

(ii) $0 < x_n \leq 1$, i.e. x_n is bounded.

Proof. We first prove **(i)**, i.e. $x_n > x_{n+1}$. Consider

$$x_n - x_{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n+1) = \ln\left(\frac{n+1}{n}\right) - \frac{1}{n+1}.$$

Consider the graph of $f(x) = 1/x$, for $n \le x \le n+1$. We can regard

$$\begin{aligned} \ln((n+1)/n) \text{ as } & \text{ the area under the curve from } x = n \text{ to } x = n+1 \quad \text{and} \\ 1/(n+1) \text{ as } & \text{ the area of a rectangle bounded by } x = n, x = n+1 \text{ and } y = 1/n \end{aligned}$$

Since f is strictly decreasing and concave up, then the area under the curve is less than the area of the rectangle. Hence, $x_n - x_{n+1} > 0$ and the result follows.

We then prove that $0 < x_n \le 1$, i.e. x_n is bounded. Note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then

$$1 = x_1 > x_2 > x_3 > \ldots$$

and so x_n is bounded above by 1.

Write x_n as

$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx.$$

Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up. □

With the two facts established in Lemma 2.1, by the monotone convergence theorem (Theorem 2.9), x_n converges, and it converges to γ. It is still unknown whether γ is rational or irrational. This remains an open problem.

Example 2.45 (MA2108 AY21/22 Sem 1 Midterm).

(i) Let $n \in \mathbb{N}$. Prove that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

(ii) Use the above inequalities to prove that

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$$

has a limit as $n \to \infty$.

Solution.

(i) Let

$$f(n) = \frac{1}{n+1}, \quad g(n) = \ln\left(1 + \frac{1}{n}\right) \quad \text{and} \quad h(n) = \frac{1}{n}.$$

Note that f, g and h are concave up on $(0, \infty)$. If we establish that $f'(n) > g'(n) > h'(n)$ for all $n \in (0, \infty)$, then we are done.

Consider

$$g'(n) - f'(n) = \frac{1}{(n+1)^2} - \frac{1}{n^2+n} = -\frac{1}{n(n+1)^2}$$

and since $n > 0$, then $g'(n) < f'(n)$.

Next, consider

$$g'(n) - h'(n) = -\frac{1}{n^2+n} + \frac{1}{n^2} = \frac{1}{n^2(n+1)}$$

and in a similar fashion, $g'(n) > h'(n)$. We are done.

(ii) It suffices to show that x_n is decreasing and bounded.

To show x_n is decreasing, consider

$$x_n - x_{n+1} = -\frac{1}{n+1} + \ln\left(1+\frac{1}{n}\right) > 0$$

by **(i)**.

To show x_n is bounded, note that $x_1 = 1$. Since x_n is a strictly decreasing sequence, then

$$1 = x_1 > x_2 > x_3 > \dots$$

and so x_n is bounded above by 1.

Write x_n as

$$\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx.$$

Construct a rectangle of width 1 and height $1/n$ (taking the left endpoint) and note that the sum of areas of the rectangles is strictly greater than the area under the curve, so $x_n > 0$ since the graph of f is strictly decreasing and concave up.

Since x_n is decreasing and between 0 and 1, its limit exists. □

2.5 Subsequences

Definition 2.12 (subsequence)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, i.e. a map

$$x : \mathbb{N} \to X \quad \text{where} \quad n \mapsto x_n.$$

A subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where the indices n_k form a strictly increasing sequence of natural numbers, i.e. $n_1 < n_2 < \dots$. Formally, it can be seen as the composition of the following two maps:

$$\mathbb{N} \xrightarrow{k} \mathbb{N} \xrightarrow{x} \mathbb{R} \quad \text{where} \quad k \mapsto n_k \mapsto x_{n_k}$$

Lemma 2.2. We have

$$\{x_n\}_{n\in\mathbb{N}} \text{ converges in } \mathbb{R} \quad \text{if and only if} \quad \text{every subsequence of } \{x_n\}_{n\in\mathbb{N}} \text{ converges in } \mathbb{R}.$$

Proof. We first prove the reverse direction. Then, $\{x_n\}_{n\in\mathbb{N}}$ as a subsequence of itself must converge in $\mathbb{R}$, i.e. consider $\mathbb{N} \to \mathbb{N}$ where $i \mapsto i$ is strictly increasing.

For the forward direction, suppose $x_n \to x$ in $\mathbb{R}$ and we have $\{x_{n_k}\}_{k\in\mathbb{N}}$ as a subsequence. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - x| < \varepsilon$. Since $k \mapsto n_k$ is strictly increasing, then for all $k \geq N$, we have $n_k \geq N$, so $|x_{n_k} - x| < \varepsilon$. □

Corollary 2.3. If $\{x_n\}_{n\in\mathbb{N}}$ has two convergent subsequences with distinct limits, then $\{x_n\}_{n\in\mathbb{N}}$ is divergent.

Theorem 2.15 (existence of monotone subsequences)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which is monotone.

Proof. For any $k \in \mathbb{N}$, we say that

$$\{x_n\}_{n\in\mathbb{N}} \text{ has a peak at } k \quad \text{if and only if} \quad \text{for all } n \geq k \text{ we have } x_k \geq x_n.$$

Let

$$S = \left\{k \in \mathbb{N} : \{x_n\}_{n\in\mathbb{N}} \text{ has a peak at } k\right\} = \{k \in \mathbb{N} : x_k \geq x_n \text{ for all } n \geq k\} \subseteq \mathbb{N}.$$

Then, either

$$\begin{aligned} \{x_n\}_{n\in\mathbb{N}} &\text{ has infinitely many peaks} \quad \text{i.e. } S \text{ is infinite} \quad \text{or} \\ \{x_n\}_{n\in\mathbb{N}} &\text{ has finitely many peaks} \quad \text{i.e. } S \text{ is finite} \end{aligned}$$

For the first case, suppose $\{x_n\}_{n\in\mathbb{N}}$ has infinitely many peaks. Define the map $h : \mathbb{N} \to \mathbb{N}$ recursively as follows. Set $n_1 = 1$. Given $i \in \mathbb{N}$ such that n_i has been defined, set

$$n \quad \text{to be} \quad \text{the smallest element of the set } \left\{k \in \mathbb{N} : k > n_i \text{ and } \{x_n\}_{n\in\mathbb{N}} \text{ has a peak at } k\right\}$$

This set is $S \setminus \{1, \ldots, n_i\}$. As S is an infinite set and $\{1, \ldots, n_i\}$ is a finite set, then $S \setminus \{1, \ldots, n_i\}$ is an infinite set, which is non-empty.

By induction, for all $i \in \mathbb{N}$, we have $n_{i+1} > n_i$ in $\mathbb{N}$. So, for all $i, j \in \mathbb{N}$ such that $i < j$, we have $n_i < n_j$ in $\mathbb{N}$ and $x_{n_i} \geq x_{n_j}$ in $\mathbb{R}$. We conclude that $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a monotonically decreasing subsequence of $\{x_n\}_{n\in\mathbb{N}}$.

For the second case, suppose $\{x_n\}_{n\in\mathbb{N}}$ has finitely many peaks. Then, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $\{x_n\}_{n\in\mathbb{N}}$ has a peak at k, i.e. $S \subseteq \{1, \ldots, N\}$. Define a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}\}$ recursively as follows: set $n_1 = N + 1$. Given n_i, choose n_{i+1} to be the smallest index $m > n_k$ such that $x_m > x_{n_k}$. Such an m always exists by the non-peak property of indices greater than N. This ensures that $\{x_{n_k}\}$ is strictly increasing. □

Theorem 2.16 (Bolzano-Weierstrass theorem)**.** Every bounded sequence has a convergent subsequence.

Example 2.46 (MA2108S AY16/17 Sem 2 Homework 4)**.** Suppose that every subsequence of x_n has a subsequence that converges to 0. Show that

$$\lim_{n\to\infty} x_n = 0.^{\dagger}$$

† This also appears in MA2108 AY24/25 Sem 2 Problem Set 2 Question 32.

Solution. We first show that x_n is bounded[‡]. Suppose on the contrary that it is not. Then, for every $M, N > 0$, there exists $n \geq N$ such that $|x_n| \geq M$. Then, we can find a subsequence x_{n_k} such that

$$\lim_{k\to\infty} x_{n_k} = \infty.$$

However, this subsequence does not have a subsequence that converges to 0, which is a contradiction. Hence, x_n is bounded.

Next, x_n must have only one limit point. Let L be a limit point of x_n. Then, x_{n_k} converges to L. Also, any subsequence of x_{n_k} converges to L. By the uniqueness of the limit, as every subsequence of x_{n_k} converges to 0, then x_{n_k} converges to 0, we establish the required result. □

Here is an alternative solution.

Solution. Suppose on the contrary that $\{x_n\}_{n\in\mathbb{N}}$ does not converge to 0. Then, there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there exists $n_k \geq k$ such that

$$|x_{n_k}| \geq \varepsilon.$$

Let $\left\{x_{n_{k_\ell}}\right\}_{\ell\in\mathbb{N}}$ be a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$. Then, for every $k \in \mathbb{N}$, we have $\left|x_{n_{k_\ell}}\right| \geq \varepsilon$, i.e. there exists a subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$ that does not converge to 0, contradicting the hypothesis that $\{x_n\}_{n\in\mathbb{N}}$ does not converge to 0. □

Example 2.47 (MA2108S AY16/17 Sem 2 Homework 4)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence and for each $n \in \mathbb{N}$, let

$$s_n = \sup\{x_k : k \geq n\} \quad \text{and} \quad S = \inf\{s_n\}.$$

Show that there exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to S[†].

Solution. In Example 1.33, we proved that if $A \subseteq B$, where $A \neq \emptyset$, then $\sup A \leq \sup B$. From here, we claim that $\{s_n\}_{n\in\mathbb{N}}$ is monotonically decreasing. We have

$$\begin{aligned} s_{n+1} - s_n &= \sup\{x_k : k \geq n+1\} - \sup\{x_k : k \geq n\} \\ &= \sup\{x_{n+1}, x_{n+2}, \ldots\} - \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \end{aligned}$$

If $x_n = \sup\{x_{n+1}, x_{n+2}, \ldots\}$, then $s_{n+1} - s_n = 0$. On the other hand, if $x_n \geq \sup\{x_{n+1}, x_{n+2}, \ldots\}$, then $s_{n+1} - s_n \leq 0$, i.e. $\{s_n\}_{n\in\mathbb{N}}$ is decreasing. As $\{x_n\}_{n\in\mathbb{N}}$ is bounded, then $\{s_n\}_{n\in\mathbb{N}}$ is also bounded. By the monotone convergence theorem (Theorem 2.9), $\{s_n\}_{n\in\mathbb{N}}$ converges. In particular, as $\{s_n\}_{n\in\mathbb{N}}$ is bounded below and decreasing, then $\{s_n\}_{n\in\mathbb{N}}$ converges to the infimum. □

Example 2.48 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 35)**.** Show that if $\{x_n\}_{n\in\mathbb{N}}$ is unbounded, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$$

Solution. Without loss of generality, suppose $\{x_n\}_{n\in\mathbb{N}}$ is not bounded above. Then, for all $M \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n > M$. Take some subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, so for each $k \in \mathbb{N}$, choose n_k such that

$$x_{n_k} > k \quad \text{and} \quad n_1 < n_2 < \ldots < n_k.$$

[‡]See here for a reference.
[†]Also appears in MA2108 AY24/25 Sem 2 Problem Set 2 Question 33.

Hence,

$$0 < \frac{1}{x_{n_k}} < \frac{1}{k}.$$

As $k \to \infty$, by the squeeze theorem, the result follows. □

Example 2.49 (MA2108 AY24/25 Sem 2 Problem Set 2 Question 36)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence and let $s = \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to s.

Solution. By definition of supremum, we know that for every $\varepsilon > 0$, there exists x_n such that

$$s - \varepsilon < x_n < s \quad \text{where } s = \sup\{x_n : n \in \mathbb{N}\}.$$

Choose $\varepsilon = 1$ so

$$\text{there exists } x_{n_1} \quad \text{such that} \quad s - 1 < x_n < s.$$

Similarly, choose $\varepsilon = 1/2$ so

$$\text{there exists } x_{n_2} \quad \text{such that} \quad s - \frac{1}{2} < x_{n_2} < s.$$

As such, we obtain an increasing subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that

$$s - \frac{1}{k} < x_{n_k} < s.$$

Letting k tend to infinity, we have

$$\lim_{k\to\infty}\left(s - \frac{1}{k}\right) < \lim_{k\to\infty} x_{n_k} < \lim_{k\to\infty} s.$$

By the squeeze theorem, the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converges to s. □

2.6 Cauchy Sequences

Definition 2.13 (Cauchy sequence)**.** Let F be an ordered field A sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if

$$\text{for every } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that} \quad \text{for all } m, n \geq N \text{ we have } |x_m - x_n| < \varepsilon.$$

Intuitively, what Definition 2.13 means is that for large n, the x_n's are very close to each other. For instance, the sequence $\{1/n\}_{n\in\mathbb{N}}$ is obviously Cauchy. To see why, we consider Figure 10, whereby for some $N \in \mathbb{N}$, the distance between x_m and x_n is *sufficiently small* (to be precise, this distance is at most ε, but not including it). Here, we have constructed an *open ball*[†] which contains x_n and x_m, where the distance between x_m and x_n is strictly contained in this ball!

Theorem 2.17 (convergent implies Cauchy)**.** Let F be an ordered field. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent in F, then it is Cauchy.

[†]Do not need to care too much what 'open ball' means for now. Loosely speaking, you can regard it as an open interval, but this notion applies to arbitrary topological spaces.

Figure 10: If $x_n = 1/n$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy

Proof. Suppose $\{x_n\}_{in\mathbb{N}}$ is convergent in F, i.e. $x_n \to x$ in F. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|x_n - x| < \frac{\varepsilon}{2}.$$

So, for all $m, n \geq N$, we have

$$\begin{aligned}|p_n - p_m| &= |(p_n - p) - (p_m - p)| \\ &\leq |p_n - p| + |p_m - p| \quad \text{by the triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\end{aligned}$$

which is bounded above by ε. This shows that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. □

We take a look Examples 2.50 and 2.51 for an application of Theorem 2.17.

Example 2.50. Let $x_n = n$. Then, $\{x_n\}_{n\in\mathbb{N}}$ is not convergent, so it is not Cauchy.

Example 2.51. Let

$$y_n = \frac{1}{2^n} \quad \text{and} \quad z_n = \frac{1}{n^2}.$$

Note that $\{y_n\}_{n\in\mathbb{N}}$ is a geometric sequence with a common ratio of $1/2$ so it is convergent. As such, it is Cauchy. In fact, we can prove that $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy by directly applying Definition 2.13. Let $\varepsilon > 0$ be arbitrary. Choose $N = \left\lceil 1 - \frac{\ln \varepsilon}{\ln 2} \right\rceil$ in $\mathbb{N}$. Then, for all $m \geq n \geq N$ we have

$$|y_m - y_n| = \left| \frac{1}{2^m} - \frac{1}{2^n} \right| = \frac{|2^n - 2^m|}{2^{m+n}} \leq \frac{2^{m+1}}{2^{m+n}} = 2^{1-n} \leq 2^{1-N} < \varepsilon.$$

Moreover, $\{z_n\}_{n\in\mathbb{N}}$ is convergent, so it is Cauchy.

Remark 2.2. The converse of Theorem 2.17 is not true, i.e. if we are given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in an ordered field F, then it may not converge to some element in F. Take for example $F = \mathbb{Q}$ and a sequence of positive rational numbers that converge to $\sqrt{2}$.

Example 2.52 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 1). Show directly from the definition that the following are Cauchy sequences:

(a) $\frac{n+1}{n}$

(b) $1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$

Solution.

(a) Let

$$x_n = \frac{n+1}{n}.$$

Then, let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. Then, for $m, n \in \mathbb{N}$ sufficiently large such that $m \geq n \geq N$, we have

$$|x_m - x_n| = \left| \frac{m+1}{m} - \frac{n+1}{n} \right| = \left| \frac{m-n}{mn} \right| \leq \left| \frac{m}{mn} \right| = \frac{1}{|n|} \leq \frac{1}{N} < \varepsilon.$$

(b) Let
$$x_n = 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$
Then, let $\varepsilon > 0$ be arbitrary. Choose $N = \lceil 1/\varepsilon \rceil$ in $\mathbb{N}$. Then, for $m, n \in \mathbb{N}$ sufficiently large such that $m, n \geq N$, we have
$$\begin{aligned}
|x_m - x_n| &= \left| \left(1 + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!} + \ldots + \frac{1}{m!}\right) - \left(1 + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) \right| \\
&= \left| \frac{1}{(n+1)!} + \ldots + \frac{1}{m!} \right| \\
&\leq \underbrace{\frac{1}{n!} + \ldots + \frac{1}{n!}}_{m-n \text{ times}} \quad \text{since } (n+1)!, \ldots, m! \geq n! \\
&= \frac{m-n}{n!} \\
&\leq \frac{m-n}{mn} \\
&\leq \frac{m}{mn} \\
&= \frac{1}{n} < \varepsilon
\end{aligned}$$
so $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. □

Example 2.53 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 2)**.** Show directly from the definition that the following are not Cauchy sequences:

(a) $(-1)^n$

(b) $n + \frac{(-1)^n}{n}$

(c) $\ln n$

Solution.

(a) Let $x_n = (-1)^n$. Choose $\varepsilon = 1$. Then, consider
$$|x_{n+1} - x_n| = \left|(-1)^{n+1} - (-1)^n\right| = |(-1)^n||-1-1| = 2 > 1 = \varepsilon,$$
so $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence.

(b) Let
$$x_n = n + \frac{(-1)^n}{n}.$$
Choose $\varepsilon = 1$. Then, consider
$$|x_{n+1} - x_n| = \left| n + 1 + \frac{(-1)^{n+1}}{n+1} - n - \frac{(-1)^n}{n} \right| = \left| 1 + \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} \right|.$$
By the reverse triangle inequality,
$$|x_{n+1} - x_n| \geq 1 - \left| \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} \right|$$
so $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence.

(c) Let $x_n = \ln n$. Choose $\varepsilon = \frac{1}{2}$. Then, consider
$$|x_{2n} - x_n| = \ln 2 > 0.5 = \varepsilon,$$
so $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. □

Example 2.54 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 3). If $x_n = \sqrt{n}$, show that $\{x_n\}_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0 \quad \text{but that} \quad \text{it is not a Cauchy sequence.}$$

Solution. We have

$$\lim_{n\to\infty} |x_{n+1} - x_n| = \lim_{n\to\infty} \left|\sqrt{n+1} - \sqrt{n}\right| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

However, we claim that $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Consider

$$|x_{2n} - x_n| = \sqrt{2n} - \sqrt{n} = \frac{n}{\sqrt{2n}+\sqrt{n}} = \frac{\sqrt{n}}{1+\sqrt{2}}$$

which grows without bound. The result follows. □

Theorem 2.18 (Cauchy implies bounded). Let F be an ordered field. If $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in F, then it is bounded.

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,n \geq N$, we have $|x_m - x_n| < \varepsilon$. In particular, we can choose $\varepsilon = 1$. Set $M = \max\{|x_1|, \ldots, |x_N|\} + 1$.

Consider any $n \in \mathbb{N}$. If $n \leq N$, then we have $|x_n| \leq M$. On the other hand, if $n \geq N$, then

$$\begin{aligned}
|x_n| &= |x_n - x_N + x_N| \\
&\leq |x_n - x_N| + |x_N| \quad \text{by the triangle inequality} \\
&< 1 + |x_N| \leq M
\end{aligned}$$

This shows that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. □

Proposition 2.3. Let F be an ordered field and $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in F. If

$$\text{there exists a subsequence of } \{x_n\}_{n\in\mathbb{N}} \text{ that converges in } F \quad \text{then} \quad \{x_n\}_{n\in\mathbb{N}} \text{ also converges in } F.$$

In fact, both limits would be the same.

Proof. Suppose $\{x_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and $x_{n_k} \to x$ in F. Given $\varepsilon > 0$, there exists $N \in N$ such that for all $k \geq N$, we have

$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists $M \in \mathbb{N}$ such that for all $m,n \in \mathbb{N}$, we have

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

Consider the map

$$\mathbb{N} \to \mathbb{N} \quad \text{where} \quad i \mapsto n_i,$$

which is strictly increasing. As such, we can choose $k_0 \geq N$ such that $n_{k_0} \geq M$, i.e. choose $k_0 = \max\{N, M\}$. As such, for all $n \geq M$, we have

$$\begin{aligned}
|x_n - x| &= \left|x_n - x_{n_{k_0}} + x_{n_{k_0}} - x\right| \\
&\leq \left|x_n - x_{n_{k_0}}\right| + \left|x_{n_{k_0}} - x\right| \quad \text{by the triangle inequality} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\end{aligned}$$

which is bounded above by ε. Then, the result follows. □

Proposition 2.4 (properties of Cauchy sequences)**.** Let F be an ordered field. Suppose $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in F. Then, the following hold:

(i) $\{x_n+y_n\}_{n\in\mathbb{N}}$ is also Cauchy in F

(ii) $\{-x_n\}_{n\in\mathbb{N}}$ is also Cauchy in F

(iii) $\{x_ny_n\}_{n\in\mathbb{N}}$ is also Cauchy in F

We will only prove **(iii)** of Proposition 2.4 (actually, **(i)** and **(ii)** can be deduced in the midst of our proof).

Proof. Since $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences, by Theorem 2.18, they are bounded, i.e. there exists $M>0$ such that for all $n\in\mathbb{N}$, we have $|x_n|\le M$ and $|y_n|\le M$.

Also, there exists $N\in\mathbb{N}$ such that for all $m,n\ge N$, we have

$$|x_m-x_n|<\frac{\varepsilon}{2M}\quad\text{and}\quad|y_m-y_n|<\frac{\varepsilon}{2M}.$$

As such, for every $m,n\ge N$, we have

$$\begin{aligned}|s_nt_n-s_mt_m| &= |(s_n-s_m)t_n+s_mt_n-s_mt_m|\\ &= |(s_n-s_m)t_n+(t_n-t_m)s_m|\\ &\le |s_n-s_m|\,|t_n|+|t_n-t_m|\,|s_m| \quad \text{by the triangle inequality}\\ &< \frac{\varepsilon}{2M}\cdot M+\frac{\varepsilon}{2M}\cdot M\end{aligned}$$

which is bounded above by ε. The result follows. □

Definition 2.14 (Cauchy complete)**.** An ordered field F is Cauchy complete if and only if every Cauchy sequence in F is convergent in F.

In general, given a sequence $\{x_n\}_{n\in\mathbb{N}}$ in an ordered field F, it is difficult to decide whether $\{x_n\}_{n\in\mathbb{N}}$ converges in F or ont. In principle, one needs to test every $x\in F$ as a possible limit. However, if F is a Cauchy complete field, then

$$\{x_n\}_{n\in\mathbb{N}}\text{ converges in }F\quad\text{if and only if}\quad\text{it is Cauchy in }F.$$

So, it suffices to just check whether $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence — a much simpler task!

Definition 2.15 (Cauchy criterion for convergence)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges if and only if it is Cauchy.

Example 2.55 ($\mathbb{Q}$ is not Cauchy complete)**.** Consider the Fibonacci sequence defined as follows:

$$\text{for all } n\in\mathbb{Z}_{\ge 0}\text{ we have } F_{n+1}=F_n+F_{n-1}\quad\text{with initial condition}\quad F_0=F_1=1.$$

Let $\{x_n\}_{n\in\mathbb{Z}_{\ge 0}}$ be the sequence in $\mathbb{Q}$ defined as follows:

$$x_n=\frac{F_n}{F_{n+1}}.$$

Then, $x_0=1$ and

$$\text{for all } n\in\mathbb{N}\quad\text{we have}\quad x_{n+1}=\frac{1}{1+x_n}\text{ in }\mathbb{Q}.$$

This is because

$$x_{n+1} = \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+1}}{F_{n+1}+F_n} = \frac{1}{1+p_n}.$$

For all $n \in \mathbb{N}$, we claim that $1/2 \le x_n \le 2/3$. To see why, first note that we have $x_1 = 1/2$. By induction, if we have

$$\frac{1}{2} \le x_n \le \frac{2}{3} \quad \text{then} \quad \frac{3}{2} \le 1+x_n \le \frac{5}{3} < 2.$$

This shows that

$$\frac{1}{2} \le x_{n+1} = \frac{1}{1+p_n} \le \frac{2}{3}.$$

We claim that $\{x_n\}_{n\in\mathbb{N}}$ is a contractive sequence (just to jump the gun, this appears in Definition 2.16). To see why, note that for all $n \in \mathbb{N}$, we have

$$|x_{n+1}-x_n| = \left|\frac{1}{1+x_n} + \frac{1}{1+x_{n-1}}\right| \le \frac{|x_n - x_{n-1}|}{(1+x_n)(1+x_{n-1})} \le \frac{4}{9}|x_n - x_{n-1}|.$$

By applying the definition of x_n recursively (this is just induction), for all $n \in \mathbb{N}$, we have

$$|x_{n+1}-x_n| \le \frac{4}{9}|x_n - x_{n-1}| \le \ldots \le \left(\frac{4}{9}\right)^n |x_1 - x_0| < \left(\frac{4}{9}\right)^n.$$

Writing this compactly, we have

$$|x_{n+1}-x_n| \le \left(\frac{4}{9}\right)^n.$$

Moreover, for all $n, r \in \mathbb{N}$, we have

$$|x_{n+r}-x_{n+r-1}| \le \frac{4}{9}|x_{n+r-1}-x_{n+r-2}| \le \ldots \le \left(\frac{4}{9}\right)^{r-1}|x_{n+1}-x_n|.$$

So,

$$\begin{aligned}
|x_{n+r}-x_n| &= |x_{n+r}-x_{n+r-1}+\ldots+x_{n+1}-x_n| \\
&= |x_{n+r}-x_{n+r-1}|+\ldots+|x_{n+1}-x_n| \quad \text{by the triangle inequality} \\
&\le \left[\left(\frac{4}{9}\right)^{r-1} + \left(\frac{4}{9}\right)^{r-2} + \ldots + 1\right]|x_{n+1}-x_n| \\
&< \frac{1}{1-\frac{4}{9}}|x_{n+1}-x_n| \\
&= \frac{9}{5}|x_{n+1}-x_n|
\end{aligned}$$

Again, writing this compactly, we have

$$|x_{n+r}-x_n| < \frac{9}{5}|x_{n+1}-x_n|.$$

Hence,

$$|x_{n+r}-x_n| < \frac{9}{5}\left(\frac{4}{9}\right)^n.$$

First, we claim that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. Let $\varepsilon > 0$ be arbitrary. We shall choose $N \in \mathbb{N}$ such that $\frac{9}{5}\left(\frac{4}{9}\right)^N < \varepsilon$. Then, for all $m, n \ge N$ with $m = n+r$, we have

$$|x_m - x_n| = |x_{n+r}-x_n| < \frac{9}{5}\left(\frac{4}{9}\right)^n < \varepsilon.$$

However, $\{x_n\}_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$. To see why, suppose on the contrary that $x_n \to x$ in $\mathbb{Q}$. Recall that for all $n \in \mathbb{N}$, we have

$$\frac{1}{2} \leq x_n \leq \frac{2}{3} \quad \text{which implies} \quad \frac{1}{2} \leq x \leq \frac{2}{3}.$$

Then, $1+x_n \to 1+x$ in $\mathbb{Q}$ and $3/2 \leq 1+x \leq 5/3$. So,

$$x_{n+1} = \frac{1}{1+x_n} \to \frac{1}{1+x} \quad \text{in } \mathbb{Q}.$$

Since $\{x_{n+1}\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ have the same limit, then

$$\frac{1}{1+x} = x \quad \text{in } \mathbb{Q}.$$

This means that $x \in \mathbb{Q}$ satisfies the equation $x^2+x=1$, i.e. $(2x+1)^2 = 5$ in $\mathbb{Q}$. However, there does not exist $x \in \mathbb{Q}$ such that $(2x+1)^2 = 5$.

Example 2.56 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 4)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant.

Solution. Since the sequence is Cauchy, for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $m,n \geq N$, we have $|x_m - x_n| < \varepsilon$. In particular, choose $\varepsilon = \frac{1}{2}$. Since $x_n \in \mathbb{Z}$, then $x_m - x_n \in \mathbb{Z}$. The only integer with absolute value less than $\frac{1}{2}$ is 0. Therefore, for all $m,n \geq N$, $|x_m - x_n| = 0$ implies $x_m = x_n$. □

Example 2.57 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 5)**.** If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solution. Let $\varepsilon > 0$ be arbitrary. Let N in $\mathbb{N}$ be chosen such that

$$\frac{r^N}{1-r} < \varepsilon.$$

Then, for $m \geq n \geq N$ sufficiently large, we have

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \ldots + (x_{n+1} - x_n)| \\
&\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \\
&\leq r^{m-1} + r^{m-2} + \ldots + r^n \\
&= \frac{r^n\left(1 - r^{m-n}\right)}{1-r} \\
&\leq \frac{r^n}{1-r} < \varepsilon
\end{aligned}$$

□

Definition 2.16 (contractive sequence)**.** Let F be an ordered field. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be contractive if

$$\text{there exists } 0 < C < 1 \quad \text{such that} \quad |x_{n+2} - x_{n+1}| \leq C\,|x_{n+1} - x_n| \text{ for all } n \in \mathbb{N}.$$

Lemma 2.3 (contractive implies convergent)**.** Every contractive sequence is convergent, and hence Cauchy.

Lemma 2.4. A sequence x_n is contractive if

$$\text{there exists } 0 < C < 1 \quad \text{such that} \quad |x_{n+2} - x_{n+1}| \leq C^{n-1} |x_2 - x_1| \text{ for all } n \in \mathbb{N}.$$

Proof. Repeatedly apply the inequality in Definition 2.16. □

Example 2.58 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 7)**.** If $y_1 < y_2$ are arbitrary real numbers and

$$y_n = \frac{1}{3} y_{n-1} + \frac{2}{3} y_{n-2} \quad \text{for } n > 2,$$

show that $\{y_n\}_{n \in \mathbb{N}}$ is convergent. What is its limit?

Solution. We have

$$\begin{aligned}
|y_n - y_{n-1}| = \left| \frac{1}{3} y_{n-1} + \frac{2}{3} y_{n-2} - y_{n-1} \right| &= \frac{2}{3} |y_{n-1} - y_{n-2}| \\
&= \left(\frac{2}{3}\right)^2 |y_{n-2} - y_{n-3}| \\
&= \ldots \quad \text{by applying the formula recursively} \\
&= \left(\frac{2}{3}\right)^{n-2} |y_2 - y_1|
\end{aligned}$$

Hence, for $m, n \in \mathbb{N}$ sufficiently large enough, where $m \geq n$, we have

$$\begin{aligned}
|y_m - y_n| &= |(y_m - y_{m-1}) + (y_{m-1} - y_{m-2}) + \ldots + (y_{n+1} - y_n)| \\
&\leq \left[\left(\frac{2}{3}\right)^{m-2} + \left(\frac{2}{3}\right)^{m-3} + \ldots + \left(\frac{2}{3}\right)^{n-1} \right] |y_2 - y_1| \\
&\leq \frac{\left(\frac{2}{3}\right)^{n-1} \left[1 - \left(\frac{2}{3}\right)^{m-n} \right]}{\frac{1}{3}} |y_2 - y_1| \\
&= 3 \left(\frac{2}{3}\right)^{n-1} \left[1 - \left(\frac{2}{3}\right)^{m-n} \right] |y_2 - y_1| \\
&\leq 3 \left(\frac{2}{3}\right)^{n} |y_2 - y_1|
\end{aligned}$$

This shows that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, so it converges.

We then compute the limit of the sequence. We have

$$\begin{aligned}
y_n - y_{n-1} &= -\frac{2}{3} (y_{n-1} - y_{n-2}) \\
&= \left(-\frac{2}{3}\right)^2 (y_{n-2} - y_{n-3}) \\
&= \ldots \quad \text{by applying the formula recursively} \\
&= \left(-\frac{2}{3}\right)^{n-2} (y_2 - y_1)
\end{aligned}$$

so

$$y_{n-1} - y_{n-2} = \left(-\frac{2}{3}\right)^{n-3} (y_2 - y_1)$$

and so on. Hence,

$$(y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \ldots + (y_3 - y_2) = \left[\left(-\frac{2}{3}\right)^{n-2} + \left(-\frac{2}{3}\right)^{n-3} + \ldots + \left(-\frac{2}{3}\right)\right](y_2 - y_1)$$

$$y_n - y_2 = \frac{\left(-\frac{2}{3}\right)\left[1 - \left(-\frac{2}{3}\right)^{n-2}\right]}{\frac{5}{3}}(y_2 - y_1)$$

Taking the limit as n goes to infinity,

$$\lim_{n\to\infty} y_n = -\frac{2}{5}(y_2 - y_1) + y_2 = \frac{2}{5}y_1 + \frac{3}{5}y_2$$

which is the desired limit of the sequence. □

Example 2.59 (MA2108 AY19/20 Sem 1)**.** Let $a_1 \geq 0$ and for $n \geq 1$, define

$$a_{n+1} = \frac{3(1+a_n)}{3+a_n}$$

(a) Prove that a_n converges.

(b) Find the limit.

Solution.

(a) We have

$$|a_{n+2} - a_{n+1}| = \left|\frac{3(1+a_{n+1}) - 3a_{n+1} - a_{n+1}^2}{3+a_{n+1}}\right| = \left|\frac{3-a_{n+1}^2}{3+a_{n+1}}\right|,$$

which simplifies to

$$\left|\frac{3-a_n^2}{(3+a_n)(2+a_n)}\right| = \left|\frac{3-a_n^2}{3+a_n}\right| \cdot \frac{1}{|2+a_n|} < \left|\frac{3-a_n^2}{3+a_n}\right| = |a_{n+1} - a_n|$$

so a_n is a contractive sequence. By Lemma 2.3, a_n converges.

(b) Suppose

$$\lim_{n\to\infty} a_n = L.$$

Thus,

$$L = \frac{3(1+L)}{3+L}.$$

Since $L > 0$, then $L = \sqrt{3}$. □

Theorem 2.19 (equivalent characteristics of $\mathbb{R}$)**.** Let F be an ordered field. Then, the following are equivalent:

(i) F has the least upper bound property

(ii) Every monotonically increasing sequence in F which is bounded above converges in F (precisely the monotone convergence theorem)

(iii) F is Archimedean and Cauchy complete

Proof. **(i)** implies **(ii)** follows from Proposition 2.2. We then prove **(ii)** implies **(iii)**. Suppose on the contrary that the ordered field F does not satisfy the Archimedean property. Then, there exist $a, b \in F_{>0}$ such that for all $n \in \mathbb{N}$, we have $na \leq b$, so $\{na\}_{n\in\mathbb{N}}$ is a monotonically increasing sequence in F (as $a > 0$) which is bounded above by b. As such, there exists $x \in F$ such that $na \to x$.

So, with $\varepsilon = a \in F_{>0}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|na - x| < \varepsilon = a$. We can rewrite this as $x - a < Na < x + a$, but then $x + a < (N+1)\,a$, which leads to a contradiction. This forces F to be Archimedean.

We then prove that F is Cauchy complete. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in F. Then, choose a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ which is monotone. Without loss of generality, assume that it is monotonically increasing. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded above, then so is the subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. By the hypothesis, as the subsequence converges in F, then the original sequence $\{x_n\}_{n\in\mathbb{N}}$ also converges in F. Note that similar claims can be made for the case when $\{x_{n_k}\}_{k\in\mathbb{N}}$ is monotonically decreasing.

Lastly, we prove **(iii)** implies **(i)**. Suppose F is Archimedean and Cauchy complete. Let $S \subseteq F$ to be a non-empty subset which is bounded above. We wish to prove that there exists a least upper bound of S in F. Since $S \neq \emptyset$, then there exists $a_0 \in F$ such that a_0 is not the upper bound of S. To achieve this, for instance, one can choose $s_0 \in S$ and set $a_0 = s_0 - 1$. Also, since S is bounded above, then there exists $b_0 \in F$ such that b_0 is an upper bound of S. It is clear that

$$a_0 < b_0 \quad \text{in } F \quad \text{or equivalently} \quad b_0 - a_0 > 0 \quad \text{in } F.$$

Define the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in F recursively as follows. If a_n and b_n have been defined, consider $(a_n + b_n)/2 \in F$, which would either be an upper bound of S or not.

Thereafter, set

$$a_{n+1} = \begin{cases} a_n & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound of } S; \\ \frac{a_n+b_n}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad b_{n+1} = \begin{cases} \frac{a_n+b_n}{2} & \text{if } \frac{a_n+b_n}{2} \text{ is an upper bound of } S; \\ b_n & \text{otherwise} \end{cases}$$

By induction, one can deduce the following. First, a_n is not an upper bound of S but b_n is an upper bound of S. Moreover,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

Lastly,

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}.$$

It is a simple exercise (one can use the formal definition of limits) to deduce that

$$\lim_{n\to\infty} (b_n - a_n) = 0.$$

We claim that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Cauchy sequences in F. To see why, given any $\varepsilon \in F_{>0}$, since $b_n - a_n \to 0$ in F, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has $0 < b_n - a_n < \varepsilon$. In particular, $b_N - a_N < \varepsilon$. By using the fact that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are Cauchy sequences, for all $m, n \in \mathbb{N}$ where $n \geq m \geq N$, we have

$$a_N \leq a_m \leq a_n \leq b_n \leq b_m \leq b_N.$$

So,

$$|a_n - a_m| \leq b_N - a_N < \varepsilon \quad \text{and} \quad |b_n - b_m| \leq b_N - a_N < \varepsilon.$$

By the Cauchy completeness of F, $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ converge in F. Let

$$a = \lim_{n\to\infty} a_n \quad \text{and} \quad b = \lim_{n\to\infty} b_n.$$

First, we claim that $a = b$ in F. We know that for all $n \in \mathbb{N}$, $a_n \leq b_n$ so $a \leq b$ in F. By way of contradiction, if $a < b$ in F, we shall define $\varepsilon = b - a > 0$ in F. By convergence (the last expression $|b_n - a_n|$ involves considering a Cauchy sequence in a Cauchy complete field, which converges), there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - a|, |b_n - b|, |b_n - a_n| \quad \text{are all} \; < \frac{\varepsilon}{3}.$$

Hence,

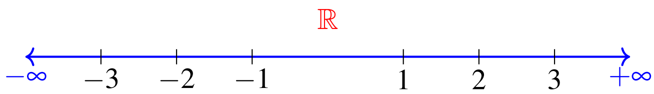
$$\varepsilon = |b - a| \leq |b - b_n| + |b_n - a_n| + |a_n - a|$$

by the triangle inequality, but the sum of terms on the right is bounded by three copies of $\varepsilon/3$, which adds to ε. As such, $\varepsilon < \varepsilon$, which is a contradiction. One can then prove that b is the least upper bound of S in F, justifying that F has the least upper bound property. $\square$

2.7
The Extended Real Number System

We begin our discussion with the extended real numbers. Let

$$[-\infty, +\infty] = \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\} \quad \text{where } \sqcup \text{ denotes disjoint union.}$$



For all $x \in \mathbb{R}$, define $-\infty < x < +\infty$ which preserves the original order in $\mathbb{R}$. This is a total ordering on $[-\infty, +\infty]$. Note that for any subset E of $[-\infty, +\infty]$, we say that

$$+\infty \text{ is an upper bound of } E \quad \text{and} \quad -\infty \text{ is a lower bound of } E.$$

Thus, $\sup(E), \inf(E)$ always exist in $[-\infty, +\infty]$.

Example 2.60. We have $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = +\infty$.

Example 2.61. Let

$$A \subseteq \mathbb{R} \text{ be a set which is not bounded above in } \mathbb{R} \quad \text{and} \quad B \subseteq \mathbb{R} \text{ be a set which is not bounded below in } \mathbb{R}.$$

Then, $\sup(A) = +\infty$ and $\inf(B) = -\infty$.

Remark 2.3. The extended real number system does not form a field.

Here are some conventions. If x is real, then

$$x + \infty = \infty \quad \text{and} \quad x - \infty = -\infty \quad \text{and} \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

If $x > 0$, then

$$x \cdot (+\infty) = +\infty \quad \text{and} \quad x \cdot (-\infty) = -\infty.$$

Lastly, if $x < 0$, then

$$x \cdot (+\infty) = -\infty \quad \text{and} \quad x \cdot (-\infty) = +\infty.$$

In contrast, in Measure Theory, the convention is that $0 \cdot \{\pm\infty\} = 0$. On the other hand, in Complex Analysis, $+\infty = -\infty$ in $\mathbb{C}$ but $0 \cdot \infty$ is undefined.

Definition 2.17. A sequence $\{s_n\}_{n\in\mathbb{N}}$ in $[-\infty,\infty]$ converges to ∞ if and only if

$$\text{for all } A\in[-\infty,\infty] \quad \text{there exists } N\in\mathbb{N} \text{ such that for all } n\geq N \quad \text{one has } s_n>A \text{ in } [-\infty,\infty],$$

i.e. s_n is closer to ∞ than A is.

Similarly, a sequence $\{s_n\}_{n\in\mathbb{N}}$ in $[-\infty,\infty]$ converges to $-\infty$ if and only if

$$\text{for all } B\in[-\infty,\infty] \quad \text{there exists } N\in\mathbb{N} \text{ such that for all } n\geq N \quad \text{one has } s_n<B \text{ in } [-\infty,\infty],$$

i.e. s_n is closer to $-\infty$ than B is.

We write

$$\lim_{n\to\infty} s_n=\pm\infty \quad \text{or} \quad \{s_n\}_{n\in\mathbb{N}}\to\pm\infty \quad \text{in } [-\infty,\infty].$$

Proposition 2.5. Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence in $[-\infty,\infty]$ and $x\in[-\infty,\infty]$. Then, $\{s_n\}_{n\in\mathbb{N}}\to x$ in $[-\infty,\infty]$ if and only if the following properties hold:

(a) for all $A\in[-\infty,\infty]$ with $A<x$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, we have $A<s_n$

(b) for all $B\in[-\infty,\infty]$ with $x<B$, there exists $M\in\mathbb{N}$ such that for all $n\geq M$, we have $s_n<B$

Lemma 2.5. For any sequence $\{s_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, there exists a subsequence $\{s_{n_i}\}_{i\in\mathbb{N}}$ which converges in $[-\infty,\infty]$.

Proof. Recall Theorem 2.15 on the existence of monotone subsequences. Given a monotone subsequence $\{s_{n_i}\}_{i\in\mathbb{N}}$. If

the sequence is increasing and bounded above	then	the subsequence converges to $\mathbb{R}$
the sequence is increasing and not bounded above	then	the subsequence converges to $+\infty$
the sequence is decreasing and bounded below	then	the subsequence converges to $\mathbb{R}$
the sequence is decreasing and not bounded below	then	the subsequence converges to $-\infty$

□

2.8
Cluster Point, Limit Superior and Limit Inferior

Definition 2.18 (cluster point)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. A cluster point of a sequence is a number that is the limit of some convergent subsequence. Equivalently, a point L is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ if every neighbourhood around L contains infinitely many terms of the sequence.

Let

$$E=\left\{\text{the limits in } [-\infty,\infty] \text{ of all convergent subsequences of } \{x_n\}_{n\in\mathbb{N}}\right\}.$$

By Lemma 2.5, E is non-empty. In fact, we call it the set of cluster points of x_n.

Definition 2.19 (limit superior and limit inferior). We define the limit superior and limit inferior of x_n to be the following:

$$\limsup_{n\to\infty} x_n = \sup(E) \quad \text{and} \quad \liminf_{n\to\infty} x_n = \inf(E)$$

where E is the set of cluster points of x_n as mentioned in Definition 2.18.

Proposition 2.6. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then,

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \quad \text{in } [-\infty,\infty].$$

Equality holds if and only if $\{s_n\}_{n\in\mathbb{N}}$ converges in $[-\infty,\infty]$ in which case

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n \quad \text{in } [-\infty,\infty].$$

Proof. The set

$$E = \left\{\text{the limits in } [-\infty,\infty] \text{ of all convergent subsequences of } \{x_n\}_{n\in\mathbb{N}}\right\} \neq \emptyset$$

as mentioned earlier. Hence,

$$\liminf_{n\to\infty} x_n = \inf(E) \quad \text{is} \quad \leq \sup(E) = \limsup_{n\to\infty} x_n$$

One has

$$\lim_{n\to\infty} \inf\, x_n = \limsup_{n\to\infty} \quad \text{in } [-\infty,\infty]$$

if and only if either of the following hold:

(i) $E = \{x^*\}$ is a singleton subset of $[-\infty,\infty]$

(ii) there exists $x^* \in [-\infty,\infty]$ such that every convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* in $[-\infty,\infty]$

(iii) there exists $x^* \in [-\infty,\infty]$ such that every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* in $[-\infty,\infty]$

(iv) there exists $x^* \in [-\infty,\infty]$ such that $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* in $[-\infty,\infty]$

□

Proposition 2.7 (equivalent characteristics of limit supremum). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let $x^* \in [-\infty,\infty]$. Then, the following are equivalent:

(i) We have

$$x^* = \limsup_{n\to\infty} x_n$$

(ii) For all $A \in [-\infty,\infty]$ with $A < x^*$,

$$\text{there are infinitely many } n \in \mathbb{N} \quad \text{such that } A < x_n$$

or equivalently, for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $A < x_n$.

Moreover, for all $B \in [-\infty,\infty]$ with x^*,

$$\text{there are only finitely many } n \in \mathbb{N} \quad \text{such that } B \leq x_n$$

or equivalently, there exists $N \in \mathbb{N}$ usch that for all $n \geq N$, we have $x_n < B$.

(iii) We have

$$x^* = \inf\{x \in [-\infty,\infty] : \text{there are only finitely many } n \in \mathbb{N} \text{ such that } x < x_n\}$$

Example 2.62 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 12). Alternate the terms of the sequences $\left\{1+\frac{1}{n}\right\}_{n\in\mathbb{N}}$ and $\left\{-\frac{1}{n}\right\}_{n\in\mathbb{N}}$ to obtain the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by

$$2,-1,\frac{3}{2},-\frac{1}{2},\frac{4}{3},-\frac{1}{3},\frac{5}{4},-\frac{1}{4},\ldots.$$

Determine the values of $\limsup x_n$ and $\liminf x_n$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.

Solution. We begin by writing the two sequences $a_n = 1+\frac{1}{n}$ and $b_n = -\frac{1}{n}$ which are interlaced to form the sequence

$$a_1,b_1,a_2,b_2,a_3,b_3,\ldots.$$

The odd-indexed subsequence is

$$x_{2k-1} = a_k = 1+\frac{1}{k}.$$

As $k\to\infty$, we have

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right) = 1.$$

The even-indexed subsequence is

$$x_{2k} = b_k = -\frac{1}{k}.$$

As $k\to\infty$, we have

$$\lim_{k\to\infty}\left(-\frac{1}{k}\right) = 0.$$

Hence, $\limsup x_n = 1$ and $\liminf x_n = 0$.

Then, we find the supremum and infimum. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is strictly decreasing and its largest term is $a_1 = 2$. Also, the even-indexed terms are all negative. Hence, $\sup\{x_n\} = 2$. The sequence $\{b_n\}_{n\in\mathbb{N}}$ is increasing (becoming less negative) with the smallest term $b_1 = -1$. The odd-indexed terms are all greater than 1. Therefore, $\inf\{x_n\} = -1$. □

Example 2.63 (MA2108 AY18/19 Sem 1 Midterm). For each $n\in\mathbb{N}$, let

$$y_n = \frac{2n-\sqrt{n}+1}{n+2\sqrt{n}+1}\cos\left(\frac{(n-1)\pi}{4}\right).$$

(i) Find $\limsup y_n$ and $\liminf y_n$.
(ii) Is the sequence y_n convergent? Justify your answer.

Solution.

(i) We first find $\sup y_n$. Since cosine is bounded above by 1, then

$$y_n \le \frac{2n-\sqrt{n}+1}{n+2\sqrt{n}+1} = 2-\frac{5\sqrt{n}+1}{n+2\sqrt{n}+1}.$$

On the right side of the inequality, the denominator grows much faster than the numerator, so $\sup y_n = 2$. Now, we show that $\limsup y_n = 2$. Define

$$a_n = \cos\left(\frac{(n-1)\pi}{4}\right).$$

so that $a_{8n+1} = 1$ for all $n\in\mathbb{N}$. The result follows. Use the same method to find $\liminf y_n$.

(ii) No, since $\liminf y_n \neq \limsup y_n$. □

Example 2.64 (MA2108 AY18/19 Sem 1 Midterm)**.** Let a_n and b_n be bounded sequences, and let

$$c_n = \max\{a_n, b_n\} \quad \text{for all } n \in \mathbb{N}.$$

Prove that

$$\limsup c_n = \max\{\limsup a_n, \limsup b_n\}.$$

Solution. Note that $a_n, b_n \leq c_n$. Define $M_1 = \limsup a_n$, $M_2 = \limsup b_n$ and $M = \max\{M_1, M_2\}$. So, $M_1 \leq \limsup c_n$ and $M_2 \leq \limsup c_n$. Thus, $M \leq \limsup c_n$. Now, we prove that $M = \limsup c_n$.

Let c be a cluster point of c_{n_k} and $c_{n_k} \to c$. For any arbitrary $\varepsilon > 0$, there exists $K_1, K_2 \in \mathbb{N}$ such that for all $n > K_1$ and $n > K_2$, we have

$$|a_n - M_1| < \varepsilon \quad \text{and} \quad |b_n - M_2| < \varepsilon \text{ respectively.}$$

The expansion of these two inequalities yields $a_n < M_1 + \varepsilon$ and $b_n < M_2 + \varepsilon$. We'll now relate this to $c_n = \max\{a_n, b_n\}$. Let $K = \max\{K_1, K_2\}$. Then, for all $n > K$,

$$a_n < M_1 + \varepsilon < M + \varepsilon \quad \text{and} \quad b_n < M_2 + \varepsilon < M + \varepsilon.$$

Hence, $c_n < M + \varepsilon$. As mentioned, c is a cluster point of c_{n_k}, so $c_{n_k} < M + \varepsilon$. As $k \to \infty$, it is clear that $c < M + \varepsilon$. Hence, M is an upper bound for the cluster points of c_n, and so $\limsup c_n \leq M$. Combining the purple inequalities yields the result. □

Example 2.65 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 13)**.** Show that if $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are bounded sequences, then

$$\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

Give an example in which the two sides are not equal.

Solution. Let u be a subsequential limit of $x_n + y_n$. Then, there exists a subsequence $\{x_{n_k} + y_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n + y_n\}_{n\in\mathbb{N}}$ which converges to u. Let $\varepsilon > 0$. Then, there exist $K_1, K_2 \in \mathbb{N}$ such that

$$x_n \geq K_1 \text{ implies } x_n > \liminf x_n + \frac{\varepsilon}{2} \quad \text{and} \quad y_n \geq K_2 \text{ implies } y_n > \liminf y_n + \frac{\varepsilon}{2}.$$

Define $K = \max\{K_1, K_2\}$. Since $n_k \geq k$, then for all $k \geq K$, we have

$$u = \lim_{k\to\infty} (x_{n_k} + y_{n_k}) \geq \liminf x_n + \liminf y_n + \varepsilon.$$

Since ε is some arbitrary small positive number, it follows that $\liminf x_n + \liminf y_n$ is a lower bound for $x_n + y_n$. As there exists a subsequence $x_{n_k} + y_{n_k}$ converging to $\inf(x_n + y_n)$, then the result follows.

For the second part, let $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then,

$$\limsup (x_n + y_n) = 0 \quad \text{but} \quad \limsup x_n = \limsup y_n = 1 \text{ so } \limsup x_n + \limsup y_n = 2.$$

□

Example 2.66 (MA2108 AY19/20 Sem 1)**.** Let x_n and y_n be two bounded sequences in $\mathbb{R}$. Suppose there exists an $N \in \mathbb{N}$ such that when $n > N$, one has $x_n \leq y_n$. Prove that

$$\liminf x_n \leq \liminf y_n.$$

Solution. Define

$$a_n = \inf\{x_k : k \geq n\} \text{ and } b_n = \inf\{y_k : k \geq n\}.$$

As each $x_n \leq y_n$, then $\inf x_k \leq \inf y_k$ for all $1 \leq k \leq n$. That is, $a_n \leq b_n$. To conclude,

$$\liminf x_n = \inf a_n \leq \inf b_n = \liminf y_n.$$

□

Chapter 3
Infinite Series

3.1
Series

Let V be $\mathbb{R}$ or $\mathbb{C}$[†] We let $\{a_k\}_{k\in\mathbb{N}}$ be any sequence in V. The map

$$\sum_{k=1}^{*} a_k : \mathbb{N}\cup\{0\} \to V \quad \text{where} \quad n \mapsto \sum_{k=1}^{n} a_k \quad \text{can be defined recursively.}$$

For the case when $n=0$, we have

$$\sum_{k=1}^{0} a_k = 0_V \quad \text{where} \quad 0_V \text{ is the additive identity of } V$$

and for all $n\in\mathbb{N}\cup\{0\}$, we have

$$\sum_{k=1}^{n+1} a_k = \left(\sum_{k=1}^{n} a_k\right) + a_{n+1}.$$

This means that

$$\sum_{k=1}^{n} a_k = (\ldots((a_1+a_2)+a_3)+\ldots)+a_n.$$

From the associativity and commutativity of addition $+$ in V, one can prove that associativity and commutativity holds for n terms by induction. Hence, for all $n\in\mathbb{N}$ and permutation $\sigma\in$ set of permutations on $\{1,\ldots,n\}$, we have

$$\sum_{k=1}^{n} a_{\sigma(k)} = \sum_{k=1}^{n} a_k \quad \text{in } V.$$

Hence, given any finite set I and any map $a : I\to V$, where $i\mapsto a_i$, i.e. any finite family $\{a_i\}_{i\in I}$ of elements of V indexed by I, one can define the sum of the given series, denoted by

$$\sum_{i\in I} a_i \in V$$

as follows. First, set $n=|I|$, where $n\in\mathbb{N}\cup\{0\}$. We then choose any bijective map $\tau : \{1,\ldots,n\}\to I$. Define

$$\sum_{i\in I} a_i = \sum_{k=1}^{n} a_{\tau(k)} = a_{\tau(1)} + a_{\tau(2)} + \ldots + a_{\tau(n)} \in V.$$

This is a well-defined map which is independent of the choice of the bijection τ. With this definition, one can prove easily the following two properties in Proposition 3.1.

Proposition 3.1 (rearrangement and repartitioning). We have the following:

(i) Rearrangement: for every permutation σ on the set $\{1,\ldots,n\}$, we have

$$\sum_{i\in I} a_{\sigma(i)} = \sum_{i\in I} a_i \quad \text{in } V$$

[†]For those who are interested in MA2202, more generally, the set V can be regarded as an Abelian group, i.e. a group where the group operation is commutative.

(ii) Repartitioning: for every finite partition $\{I_j\}_{j\in J}$ of I, we have

$$\sum_{j\in J}\left(\sum_{i\in I_j} a_i\right) = \sum_{i\in I} a_i \quad \text{in } V.$$

By a finite partition, we mean that J is a finite set and for all $j \in J$, there exists $I_j \subseteq I$ such that

$$\text{for all distinct } j, j' \in J \quad \text{we have} \quad I_j \cap I_{j'} = \emptyset \text{ and } \bigcup_{j\in J} I_j = I.$$

Definition 3.1 (norm). Let V be a vector space over $\mathbb{R}$. A norm on V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ which satisfies the following properties:

(i) Positive-definite: for all $v \in V$, we have $\|v\| = 0$ if and only if $v = 0$

(ii) Homogeneity: for all $v \in V$ and $a \in \mathbb{R}$, we have $\|av\| = |a|\,\|v\|$

(iii) Triangle inequality: for all $v, w \in V$, we have $\|v + w\| \leq \|v\| + \|w\|$

A normed vector space consists of an $\mathbb{R}$-vector space V which is equipped with a norm $\|\cdot\|$ on V.

In Definition 3.1, we gave the definition of the norm of a vector. We mentioned that V is a vector space over $\mathbb{R}$, which means that the entries of V are the real numbers! Alternatively, we say that V is an $\mathbb{R}$-vector space. Note that

$$\mathbb{R} \text{ is a one-dimensional vector space over } \mathbb{R} \quad \text{but} \quad \mathbb{R} \text{ is an infinite-dimensional vector space over } \mathbb{Q}.$$

In particular, one can easily deduce that the dimension of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable.

Example 3.1. $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$.

Example 3.2. $\mathbb{R}^k$ and $\mathbb{C}^k$ are finite-dimensional vector spaces over $\mathbb{R}$. In $\mathbb{R}^k$, the norm function is given by the usual Euclidean k-norm, i.e.

$$\text{for any } v = (v_1, \ldots, v_k) \in \mathbb{R}^k \quad \text{we have} \quad \|v\| = \sqrt{v_1^2 + \ldots + v_k^2} \in \mathbb{R}_{\geq 0}.$$

Definition 3.2. Let V be a normed vector space and $\{a_k\}_{k\in\mathbb{N}}$ be any sequence in V. The notation

$$\sum_{k=1}^{\infty} a_k$$

is called the series in V defined by the sequence $\{a_k\}_{k\in\mathbb{N}}$. For each $n \in \mathbb{N}$, the element

$$\sum_{k=1}^{n} a_k \in V \quad \text{is} \quad \text{the } n^{\text{th}} \text{ partial sum of the series.}$$

Definition 3.3 (geometric sequence). A geometric sequence, u_n, has first term a and common ratio r. The first few terms are

$$u_1 = a,\ u_2 = ar,\ u_3 = ar^2,\ u_4 = ar^3.$$

The general term, u_n, is $u_n = ar^{n-1}$, where $n \in \mathbb{N}$.

Proposition 3.2. The sum to n terms of a geometric sequence is denoted by S_n. We establish the formula

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

For the sum to infinity, S_∞, we impose a restriction on r for the sum to exist. That is, $|r|<1$. Hence, S_∞ is

$$S_\infty = \frac{a}{1-r}.$$

Remark 3.1. If $r=-1$, we obtain the famous Grandi's series $1-1+1-1+\ldots$.

Definition 3.4 (telescoping series). A telescoping series is a series whose general term can be written in the form $a_n - a_{n-1}$.

Let $b_n = a_n - a_{n-1}$. Then,

$$\sum_{k=1}^{n} b_k = a_n - a_0.$$

This process is known as the method of differences. There are times when the partial fraction decomposition method has to be used on b_n (Example 3.3).

Example 3.3 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 14). By using partial fractions, show that

(a)

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$$

(b)

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0 \text{ if } \alpha > 0$$

(c)

$$\sum_{n=0}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

Solution. These are trivial — one should recall from H2 Mathematics on how to evaluate telescoping sums using the method of difference. □

3.2 Properties of Convergence and Divergence

Theorem 3.1. The series

$$\sum_{k=1}^{\infty} a_k \text{ converges in } V \text{ to the sum } s \in V \quad \text{if and only if} \quad \lim_{n\to\infty} \sum_{k=1}^{n} a_k = s \quad \text{in } V.$$

Example 3.4. Suppose the series

$$\sum_{k=1}^{\infty} a_k \quad \text{has} \quad \text{only finitely many non-zero terms in } V.$$

This means that the sequence $\{a_k\}_{k\in\mathbb{N}}$ in V is eventually zero. By the formal definition of a limit, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we have $a_k = 0_V$ in V. Then,

$$\sum_{k=1}^{\infty} a_k \quad \text{converges sequentially in } V \text{ to } \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N} a_k \quad \text{in } V.$$

Proposition 3.3 (linearity properties of convergent series)**.** Let

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=1}^{\infty} b_k \quad \text{be} \quad \text{two convergent series in } V.$$

Then, the following hold:

(i)

$$\sum_{k=1}^{\infty} (a_k + b_k) \text{ is convergent} \quad \text{and} \quad \sum_{k=1}^{\infty} (a_k + b_k) = \left(\sum_{k=1}^{\infty} a_k\right) + \left(\sum_{k=1}^{\infty} b_k\right) \quad \text{in } V$$

(ii) For every $c \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} ca_k \text{ is also convergent} \quad \text{and} \quad \sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k \quad \text{in } V$$

Here is a prelude into Functional Analysis (MA4211), where we define Banach spaces (Definition 3.5).

Definition 3.5 (Banach space)**.** A Banach space is a normed vector space V where every Cauchy sequence converges with respect to the metric induced by its norm $\|\cdot\|$.

Example 3.5. Every finite-dimensional Euclidean space is a Banach space. For example,

$$\mathbb{R}^k \text{ equipped with the norm } \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} \quad \text{is a Banach space.}$$

In particular, $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces.

Theorem 3.2 (Cauchy criterion for series)**.** Let V be a Banach space and $\{a_k\}_{k\in\mathbb{N}}$ be a sequence in V. The series

$$\sum_{k=1}^{\infty} a_k \text{ converges sequentially in } V \quad \text{if and only if} \quad \text{the Cauchy criterion holds.}$$

That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ with $m \geq n$, one has

$$\left\| \sum_{k=n+1}^{m} a_k \right\| < \varepsilon.$$

Theorem 3.3. If

$$\sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

The converse of Theorem 3.3 does not hold in general. That is to say, the condition $a_n \to 0$ is not sufficient to ensure the convergence of the sum of a_n. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges in } \mathbb{R}.$$

Example 3.6 (MA2108 AY18/19 Sem 1)**.** Let

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

be two series with the property that there exists $K \in \mathbb{N}$ such that

$$a_n = b_n \text{ for all } n \geq K.$$

Prove that

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \quad \text{if and only if} \quad \sum_{n=1}^{\infty} b_n \text{ is convergent.}$$

Solution. Just use Cauchy criterion. □

Example 3.7 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 21)**.** Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of strictly positive numbers. If

$$\sum a_n \text{ converges} \quad \text{show that } \lim_{n\to\infty} na_n = 0.$$

Give an example of a divergent series

$$\sum a_n \text{ with } \{a_n\}_{n\in\mathbb{N}} \text{ decreasing} \quad \text{for which } \lim_{n\to\infty} na_n = 0.$$

Solution. For the first part, since $\{a_n\}_{n\in\mathbb{N}}$ is decreasing and positive, for any $m \geq n$, we have

$$a_m \leq a_n \quad \text{whenever } m \geq n.$$

For $n \in \mathbb{N}$ sufficiently large, we have $a_{n+1} + a_{n+2} + \ldots + a_{2n} \geq na_{2n}$. If the sum of a_k converges, then its tail sums must go to 0. In particular,

$$\lim_{n\to\infty} (a_{n+1} + \cdots + a_{2n}) = 0.$$

Hence,

$$0 \leq \lim_{n\to\infty} na_{2n} \leq \lim_{n\to\infty} (a_{n+1} + \cdots + a_{2n}) = 0.$$

So,

$$\lim_{n\to\infty} na_{2n} = 0.$$

Finally, by monotonicity $a_n \geq a_{2n}$, and the result follows. For the second part, let

$$a_n = \frac{1}{n\ln n} \quad \text{where } n \geq 2.$$

□

Example 3.8 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 22)**.** Give an example of a divergent series

$$\sum a_n \text{ with } \{a_n\}_{n\in\mathbb{N}} \text{ decreasing} \quad \text{and} \quad \text{such that } \lim_{n\to\infty} na_n = 0$$

Solution. Let $a_n = \frac{1}{n\ln n}$. Then, the result follows. □

Example 3.9 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 24)**.** Let $a > 0$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \quad \text{is divergent if } 0 < a \leq 1 \text{ and is convergent if } a > 1.$$

Solution. We note that if $a > 1$, then $a^n > 1$, so

$$a^n < 1 + a^n < 2a^n \quad \text{which implies} \quad \frac{1}{2a^n} < \frac{1}{1+a^n} < \frac{1}{a^n}.$$

Note that

$$\sum_{n=1}^{\infty} \frac{1}{a^n} = \frac{1/a}{1-1/a} = \frac{1}{a-1} \quad \text{so} \quad \frac{1}{2(a-1)} < \sum_{n=1}^{\infty} \frac{1}{1+a^n} < \frac{1}{a-1}.$$

So, we conclude that if $a > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \text{ converges.}$$

We then claim that

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \text{ diverges if } a = 1.$$

This is clear because

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} = \sum_{n=1}^{\infty} \frac{1}{2}$$

which is a divergent series. Lastly, we prove that the mentioned sum diverges for $0 < a < 1$. Note that

$$\lim_{n\to\infty} \frac{1}{1+a^n} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} (1+a^n)} = \frac{1}{1+0} = 1.$$

Therefore, the terms $\frac{1}{1+\alpha^n}$ do not go to 0 as $n \to \infty$. In fact, they approach 1. A necessary condition for the convergence of an infinite series (sum of a_n) is its terms $a_n \to 0$. Since $\frac{1}{1+\alpha^n}$ does not tend to 0, the series must diverge. □

Example 3.10 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 29). Let $a_n > 0$ and suppose that

$$\sum a_n \text{ converges.}$$

Construct a convergent series

$$\sum b_n \text{ with } b_n > 0 \quad \text{such that} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Hence,

$$\sum b_n \text{ converges less rapidly than } \sum a_n.$$

Hint: Let A_n be the partial sums of

$$\sum a_n \quad \text{and} \quad \text{let } A \text{ denote its limit.}$$

Define $b_1 = \sqrt{A} - \sqrt{A - A_1}$ and $b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n}$ for $n \geq 1$.

Solution. We have

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \left(\sqrt{A - A_{n-1}} - \sqrt{A - A_n}\right) = \lim_{N\to\infty} \left(\sqrt{A - A_1} - \sqrt{A - A_N}\right)$$

so

$$\begin{aligned}\sum_{n=1}^{\infty} b_n &= \sqrt{A} - \sqrt{A - A_1} + \sqrt{A - A_1} - \lim_{N\to\infty} \sqrt{A - A_N} \\ &= \sqrt{A} - \lim_{N\to\infty} \sqrt{A - A_N} \\ &= \sqrt{A}\end{aligned}$$

so the sum of b_n converges. We then prove that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

To see why this holds, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{\sqrt{A - A_{n-1}} - \sqrt{A - A_n}} &= \lim_{n\to\infty} \frac{a_n\left(\sqrt{A - A_{n-1}} + \sqrt{A - A_n}\right)}{A_n - A_{n-1}} \\ &= \lim_{n\to\infty} \left(\sqrt{A - A_{n-1}} + \sqrt{A - A_n}\right)\end{aligned}$$

which tends to 0 because $A_n \to A$ and $A_{n-1} \to A$. □

Example 3.11 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 30)**.** Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers converging to 0 and suppose that

$$\sum a_n \text{ diverges.}$$

Construct a divergent series

$$\sum b_n \text{ with } b_n > 0 \quad \text{such that} \quad \lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Hence,

$$\sum b_n \text{ diverges less rapidly than } \sum a_n.$$

Hint: Let

$$b_n = \frac{a_n}{\sqrt{A_n}} \quad \text{where } A_n \text{ is the } n^{\text{th}} \text{ partial sum of } \sum a_n.$$

Solution. Since $\{a_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers converging to 0, then $a_n > 0$ for all $n \in \mathbb{N}$. So, the n^{th} partial sum of the sum of a_n is also > 0, which implies $b_n > 0$ for all $n \in \mathbb{N}$. We then note that

$$\sqrt{A_{n+1}} - \sqrt{A_n} = \frac{A_{n+1} - A_n}{\sqrt{A_{n+1}} + \sqrt{A_n}} = \frac{a_{n+1}}{\sqrt{A_{n+1}} + \sqrt{A_n}} \leq \frac{a_{n+1}}{2\sqrt{A_n}} \leq \frac{a_n}{2\sqrt{A_n}} = \frac{b_n}{2}.$$

By the method of difference, it follows that the sum of b_n diverges. We then prove that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

To see why this holds, we have

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt{A_n}} = 0$$

and the result follows. □

3.3 Tests for Convergence

Theorem 3.4. A series of non-negative terms converges if and only if its partial sums form a bounded sequence.

Proof. We note that for all $k \in \mathbb{N}$, the sequence of partial sums is monotonically increasing on $\mathbb{R}$. So,

the series converges in $\mathbb{R}$	if and only if	the sequence of partial sums converges in $\mathbb{R}$
	if and only if	the sequence of partial sums is bounded above in $\mathbb{R}$

The result follows. □

Definition 3.6 (p-series)**.** The p-series is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Theorem 3.5 (p-series test)**.** If $p > 1$, the p-series converges. If $0 < p \leq 1$, the p-series diverges.

Example 3.12 (MA2108S AY16/17 Sem 2 Homework 6). Show that

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent.

Solution. As $|\cos n| \leq 1$, the above sum is bounded above by

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{which is convergent}$$

since it is the p-series when $p = 2$, or rather the Basel problem, which has a value of $\pi^2/6$. □

Example 3.13 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 15).

(a) Show that the series

$$\sum_{n=1}^{\infty} \cos n \quad \text{is not convergent.}$$

(b) Show that the series

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} \quad \text{is convergent.}$$

Solution.

(a) Let $N \in \mathbb{N}$. Then,

$$\begin{aligned}\sum_{n=1}^{N} \cos n \sin 1 &= \frac{1}{2} \sum_{n=1}^{N} [\sin(n+1) - \sin(n-1)] \\ &= \frac{1}{2} \sin(N+1) \\ \sum_{n=1}^{N} \cos n &= \frac{\sin(N+1)}{2 \sin 1}\end{aligned}$$

Now, it suffices to show that the limit

$$\lim_{N \to \infty} \sin N \quad \text{does not exist.}$$

Consider

$$\sin(k\pi) = 0 \quad \text{but} \quad \sin\left(\frac{\pi}{2} + 2k\pi\right) = 1$$

which shows that the aforementioned limit does not exist. Hence, the sum of $\cos n$ is not convergent.

(b) Use the fact that $-1 \leq \cos n \leq 1$, then consider the 2-series (Definition 3.6). □

Example 3.14 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 25).

(a) Does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} \quad \text{converge?}$$

(b) Does the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \quad \text{converge?}$$

Solution.

(a) We have

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n+1} + \sqrt{n}\right)} \geq \sum_{n=1}^{\infty} \frac{1}{2(n+1)}$$

which diverges.

(b) We have

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} \leq \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$

which converges. □

Theorem 3.6 (comparison test). Suppose there exists $K \in \mathbb{N}$ such that $0 \leq a_n \leq b_n$ for all $n \geq K$. Then,

$$\sum_{n=1}^{\infty} b_n \text{ converges implies } \sum_{n=1}^{\infty} a_n \text{ converges} \quad \text{and} \quad \sum_{n=1}^{\infty} a_n \text{ diverges implies } \sum_{n=1}^{\infty} b_n \text{ diverges}$$

Example 3.15 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 16). If

$$\sum a_n \text{ with } a_n > 0 \text{ is convergent} \quad \text{is } \sum a_n^2 \text{ always convergent?}$$

Either prove it or give a counterexample.

Solution. Observe that

$$\left(\sum_{n=1}^{N} a_n\right)^2 = \sum_{n=1}^{N} a_n^2 + 2\sum_{i<j} a_i a_j$$

so

$$\left(\sum_{n=1}^{N} a_n\right)^2 \geq \sum_{n=1}^{N} a_n^2.$$

Since $\sum a_n$ converges, then $\sum a_n^2$ converges too. □

Example 3.16 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 17). If

$$\sum a_n \text{ with } a_n > 0 \text{ is convergent} \quad \text{is } \sum \sqrt{a_n} \text{ always convergent?}$$

Either prove it or give a counterexample.

Solution. Not always convergent. Consider $a_n = 1/n^2$. □

Example 3.17 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 18). If

$$\sum a_n \text{ with } a_n > 0 \text{ is convergent} \quad \text{and if} \quad b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \text{ for } n \in \mathbb{N},$$

then show that

$$\sum b_n \text{ is not always convergent.}$$

Solution. Let $a_n = \frac{1}{2^n}$. Then, the sum of a_n converges, but

$$b_n = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{n}\left(1 - \frac{1}{2^n}\right)$$

which diverges due to the presenceof the harmonic series. □

Theorem 3.7 (limit comparison test). Let

$$\sum_{i=n}^{\infty} a_n \text{ and } \sum_{i=n}^{\infty} b_n \quad \text{be} \quad \text{series of positive terms.}$$

Define

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

(i) If $L>0$, then the series are either both convergent or both divergent.

(ii) If $L=0$ and

$$\sum_{i=1}^{\infty} b_i \text{ converges} \quad \text{then} \quad \sum_{i=1}^{\infty} a_i \text{ converges.}$$

Definition 3.7 (alternating series). An alternating series is a series of the form

$$\sum_{n=1}^{\infty} a_n(-1)^n = a_1 - a_2 + a_3 - a_4 + \ldots \quad \text{where all } a_n > 0 \text{ or all } a_n < 0.$$

Definition 3.8 (absolute convergence). Let V be a normed vector space and $\{a_n\}_{n\in\mathbb{N}}$ be any sequence in V. Then,

$$\sum_{n=1}^{\infty} a_n \text{ is absolutely convergent in } V \quad \text{if and only if} \quad \text{the series } \sum_{n=1}^{\infty} \|a_n\| \text{ converges in } \mathbb{R}_{\geq 0}.$$

We give a classic result on the convergence of a geometric series (Theorem 3.8).

Theorem 3.8 (geometric series). If $0 \leq x < 1$, then

$$\sum_{n=0}^{\infty} x^n \quad \text{converges absolutely in } \mathbb{R} \text{ to } \frac{1}{1-x} \text{ in } \mathbb{R}.$$

If $x \geq 1$, the series diverges.

There is a more general result for Theorem 3.8, for which we extend it to $x\in\mathbb{C}$. More generally,

$$\text{if } |x|<1 \quad \text{then} \quad \sum_{n=0}^{\infty} x^n \text{ converges absolutely to } \frac{1}{1-x} \text{ in } \mathbb{C} \quad \text{and}$$

$$\text{if } |x|\geq 1 \quad \text{then} \quad \sum_{n=0}^{\infty} x^n \text{ does not converge in } \mathbb{C}$$

We now prove Theorem 3.8.

Proof. If $x=1$, then for all $n\in\mathbb{N}$, the n^{th} partial sum is unbounded so the series does not converge in $\mathbb{R}$. Hence, we assume that $x\neq 1$. Recall from H2 Mathematics that for any $n\in\mathbb{N}$, the n^{th} partial sum is

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

which implies

$$\sum_{n=0}^{\infty} x^n \text{ converges in } \mathbb{R} \quad \text{if and only if} \quad x^{n+1} \text{ converges in } \mathbb{R}.$$

If $|x|<1$, then x^{n+1} tends to 0 for large n, and

$$\sum_{n=0}^{\infty} x^n \text{ converges to } \frac{1}{1-x}.$$

The convergence is absolute because

$$\sum_{n=0}^{\infty} |x^n| = \sum_{n=0}^{\infty} |x|^n \quad \text{converges to } \frac{1}{1-|x|}.$$

On the other hand, if $|x| > 1$, then the sequence $\{x^{n+1}\}_{n\in\mathbb{N}}$ is unbounded, so it cannot converge in $\mathbb{R}$. Lastly, for the case where $|x| = 1$ but $x \neq 1$ (this can be applied to arbitrary $x \in \mathbb{C}$), we leave it as an exercise. □

Example 3.18 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 23). If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence and if

$$\lim_{n\to\infty} n^2 a_n \quad \text{exists in } \mathbb{R},$$

show that

$$\sum a_n \quad \text{is} \quad \text{absolutely convergent.}$$

Solution. Since the aforementioned limit exists, then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|n^2 a_n - L\right| < \varepsilon \quad \text{where} \quad \lim_{n\to\infty} n^2 a_n = L.$$

So,

$$\frac{L-\varepsilon}{n^2} < a_n < \frac{L+\varepsilon}{n^2}.$$

Taking absolute value, we have

$$|a_n| < \max\left\{\frac{L-\varepsilon}{n^2}, \frac{L+\varepsilon}{n^2}\right\} \leq \frac{|L|+\varepsilon}{n^2}.$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \\
&\leq (N-1) \max_{i\leq 1\leq N-1} |a_i| + (|L|+\varepsilon) \sum_{n=N}^{\infty} \frac{1}{n^2} \\
&\leq (N-1) \max_{i\leq 1\leq N-1} |a_i| + (|L|+\varepsilon) \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{since } \frac{1}{n^2} \geq 0 \text{ for all } 1 \leq n \leq N-1
\end{aligned}$$

It is a well-known fact that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges — in fact to $\pi^2/6$. As such,

$$\sum_{n=1}^{\infty} |a_n|$$

is bounded above by some constant, implying that the sum of a_n is absolutely convergent. □

Example 3.19 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 19). Find an explicit expression for the n^{th} partial sum of

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$$

to show that this series converges to $-\ln 2$. Is this convergence absolute?

Solution. Let

$$s_N = \sum_{n=2}^{N} \ln\left(1 - \frac{1}{n^2}\right).$$

Then,

$$s_N = \sum_{n=2}^{N} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{N} \ln(n+1) + \ln(n-1) - 2\ln n = \ln\left(\frac{N+1}{2N}\right)$$

where the last equality uses the method of difference. Letting $N \to \infty$, we have

$$\lim_{N\to\infty} s_N = -\ln 2 + \lim_{N\to\infty} \ln\left(1+\frac{1}{N}\right) = -\ln 2.$$

Yes, the convergence is absolute. Let

$$a_n = \ln\left(1-\frac{1}{n^2}\right).$$

Then, for all $n \geq 2$, $a_n < 0$, so it follows that the sum of the absolute values is $\ln 2$. □

Theorem 3.9 (D'Alembert's ratio test). Let

$$\sum_{i=1}^{\infty} a_n \quad \text{be} \quad \text{a series of positive terms.}$$

Define

$$L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If $L < 1$ the series converges;
(ii) if $L > 1$ the series diverges;
(iii) if $L = 1$, the test is inconclusive

Proof. We first prove **(i)**. Suppose $L < 1$. Then, one may choose $\beta \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1.$$

Then, by property of limit supremum, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has

$$\frac{|a_{n+1}|}{|a_n|} < \beta.$$

By induction, we see that

$$\text{for all } p \in \mathbb{N} \quad \text{we have} \quad |a_{N+p}| < \beta^p |a_N|.$$

That is to say, for all $n \geq N$, one has $|a_n| \leq \beta^{-N} |a_N| \beta^n$. Since the sum of $|a_n|$ is termwise bounded, i.e. eventually by

$$\beta^{-N} |a_N| \sum_{n=1}^{\infty} \beta^n,$$

then by the comparison test (Theorem 3.6), it follows that the sum of a_n converges absolutely.

(ii) is obvious because for all $n \geq n_0$, we have $|a_n| \geq |a_{n_0}|$. Hence, $|a_n|$ does not tend to 0, which implies that the sum of a_n cannot converge. □

We note that for **(iii)** of the ratio test (Theorem 3.9), it is possible for the sum to diverge or converge if $L = 1$. As such, it makes sense to say that when $L = 1$, the ratio test is inconclusive. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges absolutely.}$$

However, in both series, $L = 1$.

Theorem 3.10 (Cauchy's root test)**.** We wish to determine if the series

$$\sum_{i=1}^{\infty} a_n \text{ of positive terms} \quad \text{is absolutely convergent.}$$

Define

$$L = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

(i) If $L < 1$, the series is absolutely convergent;
(ii) if $L > 1$, the series diverges;
(iii) if $L = 1$, the test is inconclusive

Proof. If $L < 1$, then one can choose

$$\beta \in \mathbb{R} \quad \text{such that} \quad \limsup_{n\to\infty} \sqrt[n]{|a_n|} < \beta < 1.$$

Then, by property of limit supremum, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, one has $\sqrt[n]{|a_n|} < \beta$. As $\beta < 1$, then

$$\sum_{n=1}^{\infty} |a_n| \quad \text{is} \quad \text{termwise dominated by } \sum_{n=1}^{\infty} \beta^n.$$

By the comparison test (Theorem 3.6), it follows that the sum of a_n converges absolutely.

On the other hand, if $L > 1$, then one can choose

$$\beta \in \mathbb{R} \quad \text{such that} \quad \limsup_{n\to\infty} \sqrt[n]{|a_n|} > \beta > 1.$$

Again, by property of limit supremum, there exist infinitely many $n \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} > \beta$. Hence, $|a_n| > 1$. As such, $|a_n|$ does not tend to 0 for large n, which implies that the sum of a_n diverges. ☐

We note that for **(iii)** of the root test (Theorem 3.10), it is possible for the sum of a_n to converge or diverge if $L = 1$. Hence, it makes sense to say that when $L = 1$, the root test is inconclusive. For example, we note that

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad \text{but} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges absolutely.}$$

However, in both series, $L = 1$.

Example 3.20 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 28)**.** Show that the series

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \ldots \quad \text{is convergent}$$

but that both the ratio and the root tests fail to apply.

Solution. The sum can be written as

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \ldots \leq \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which converges by the p-series test.

We then claim that the ratio test fails here. Let x_n denote the terms of the sequence. Then, $x_n = \frac{1}{n^2}$ if n is odd; $x_n = \frac{1}{n^3}$ if n is even. Note that if n is even, then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{n^2}{(n+1)^3} < 1.$$

On the other hand, if n is odd, then

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)^3}{n^2} > 1.$$

Hence, the ratio test does not apply here. Next, we claim that the root test also fails here. We have

$$|x_{2n}|^{1/n} = \frac{1}{(2n)^{2/n}} \quad \text{and} \quad |x_{2n+1}|^{1/n} = \frac{1}{(2n+1)^{3/n}}$$

so

$$\lim_{n\to\infty} |x_{2n}|^{1/n} = \lim_{n\to\infty} |x_{2n+1}|^{1/n} = 1 \quad \text{which implies} \quad \lim_{n\to\infty} |x_n|^{1/n} = 1.$$

Hence, the root test is inconclusive. □

At this juncture, we emphasise that the ratio test is frequently easier to apply than the root test. However, the root test has wider scope — whenever the ratio test shows convergence, the root test does too, and whenever the root test is inconclusive, the ratio test is too. That is to say, for any positive sequence of numbers $\{a_n\}_{n\in\mathbb{N}}$, we have

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \liminf_{n\to\infty} \sqrt[n]{a_n} \text{ in } [-\infty,\infty) \quad \text{and} \quad \limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \text{ in } (-\infty,\infty].$$

Essentially, we can combine both inequalities as well. To see why this chain of inequalities holds in the first place, it suffices to prove the one involving lim sup.

Proof. Define

$$\alpha = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

We wish to show that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \alpha.$$

If $\alpha = +\infty$, then we are done. As such, we assume that $\alpha \in (-\infty,\infty)$. Let $\beta \in \mathbb{R}$ be arbitrary such that $\alpha < \beta$. Then, by property of limit supremum, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, we have

$$\frac{a_{n+1}}{a_n} \le \beta.$$

For any $p \in \mathbb{N}$, we have

$$\frac{a_{N+k+1}}{a_{N+k}} \le \beta \text{ for all } 0 \le k \le p-1 \quad \text{so} \quad \frac{a_{N+p}}{a_N} \le \beta^p.$$

Here, we have used the cancellation property of a telescoping product. That is to say, for all $n \ge N$, one has $a_n \le \beta^{-N} a_N \beta^n$, where we set $n = N+p$, i.e. $\sqrt[n]{a_n} \le \sqrt[n]{a_N \beta^{-N}} \cdot \beta$. Hence,

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \beta$$

and the result follows. □

We take a look at Example 3.21 which discusses the superiority of the root test in comparison to the ratio test.

Example 3.21 (root test stronger than ratio test). Let

$$a_1 = \frac{1}{2} \quad a_2 = \frac{1}{3} \quad a_3 = \frac{1}{2^2} \quad a_4 = \frac{1}{3^2},$$

and I believe that you get the idea from here. Then,

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots.$$

Here, we see that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}} < 1 \quad \text{but} \quad \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = +\infty.$$

This shows that the root test indicates convergence but the ratio test fails!

Example 3.22 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 20).

(a) If

$$\sum a_n \text{ is absolutely convergent} \quad \text{and} \quad \{b_n\}_{n\in\mathbb{N}} \text{ is a bounded sequence,}$$

show that

$$\sum a_n b_n \text{ is absolutely convergent}$$

(b) Give an example to show that if the convergence of

$$\sum a_n \text{ is conditional and } \{b_n\}_{n\in\mathbb{N}} \text{ is a bounded sequence} \quad \text{then} \quad \sum a_n b_n \text{ may diverge}$$

Solution.

(a) Since

$$\sum a_n \text{ is absolutely convergent,}$$

then suppose the limit of the sum of absolute values is L_1, i.e. for every $\varepsilon_1 > 0$, there exists $N \in \mathbb{N}$ such that

$$||a_1| + \ldots + |a_n| - L| < \varepsilon.$$

Since $\{b_n\}_{n\in\mathbb{N}}$ is bounded, then there exists $M \in \mathbb{R}$ such that $-M \le b_n \le M$ for all $n \in \mathbb{N}$. Hence, for n sufficiently large, we have

$$-M(|a_1| + \ldots + |a_n|) \le |a_1 b_1| + |a_2 b_2| + \ldots + |a_n b_n| \le M(|a_1| + \ldots + |a_n|)$$

Since

$$L - \varepsilon < |a_1| + \ldots + |a_n| < L + \varepsilon,$$

then

$$-ML + \varepsilon M < |a_1 b_1| + \ldots + |a_n b_n| < ML + \varepsilon M$$

so

$$||a_1 b_1| + |a_n b_n| - \varepsilon M| < ML$$

so the sum $a_n b_n$ is absolutely convergent.

(b) Let $a_n = \frac{(-1)^n}{n}$ and $b_n = (-1)^n$, then the sum of $a_n b_n$ is the harmonic series, which diverges! □

Theorem 3.11 (Cauchy's condensation test). For a non-increasing sequence of non-negative real numbers $f(n)$,

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \quad \text{if and only if} \quad \text{the condensed series } \sum_{n=0}^{\infty} 2^n f(2^n) \text{ converges.}$$

Observe the difference in the lower indices — one of them is 1 and another is 0.

Proof. Suppose the original series converges. We wish to prove that the condensed series converges. Consider twice the original series.

$$\begin{aligned} 2\sum_{n=1}^{\infty} f(n) &= (f(1)+f(1)) + (f(2)+f(2)+f(3)+f(3)) + \dots \\ &\geq (f(1)+f(2)) + (f(2)+f(4)+f(4)+f(4)) + \dots \\ &= f(1) + (f(2)+f(2)) + (f(4)+f(4)+f(4)+f(4)) + \dots \\ &= \sum_{n=0}^{\infty} 2^n f(2^n) \end{aligned}$$

Dividing both sides by 2, the condensed series converges.

Now, suppose the condensed series converges. We wish to prove the original series converges.

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n f(2^n) &= f(1)+f(2)+f(2)+f(4)+f(4)+f(4)+f(4)+\dots \\ &\geq f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)+f(8)+\dots \\ &= \sum_{n=1}^{\infty} f(n) \end{aligned}$$

This concludes the proof. □

Corollary 3.1. If both series converge, the sum of the condensed series is no more than twice as large as the sum of the original. We have the inequality

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2\sum_{n=1}^{\infty} f(n).$$

Corollary 3.2. Consider a variant of the p-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

If $p > 1$, the series converges. If $p \leq 1$, the series diverges.

Proof. We use Cauchy's condensation test (Theorem 3.11). Note that

$$f(n) = \frac{1}{n(\ln n)^p},$$

so

$$2^n f(2^n) = \frac{2^n}{2^n(\ln(2^n))^p} = \frac{1}{n^p(\ln 2)^p}.$$

We have

$$\frac{1}{(\ln 2)^p}\sum_{n=2}^{\infty}\frac{1}{n^p}$$

so the result follows by the conventional p-series test. □

Theorem 3.12 (partial summation formula). Given two sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ indxed by the non-negative integers $\mathbb{Z}_{\geq 0}$, set $A_{-1}=0$ and for any $n\geq 0$, put

$$A_n=\sum_{k=0}^{n}a_k.$$

Then, for any $0\leq p\leq q$, we have

$$\sum_{n=p}^{q}a_nb_n=\left(\sum_{n=p}^{q-1}A_n\left(b_n-b_{n+1}\right)\right)+A_qb_q-A_{p-1}b_p.$$

Proof. This is easy to see because

$$\begin{aligned}\sum_{n=p}^{q}a_nb_n&=\sum_{n=p}^{q}\left(A_n-A_{n-1}\right)b_n\\&=\sum_{n=p}^{q}A_nb_n-\sum_{n=p}^{q}A_{n-1}b_n\\&=\sum_{n=p}^{q}A_nb_n-\sum_{n=p-1}^{q-1}A_nb_{n+1}\end{aligned}$$

and the result follows from here. □

We note that the partial summation formula (Theorem 3.12) is useful in the investigation of series of the form

$$\sum_{n=1}^{\infty}a_nb_n.$$

Theorem 3.13 (Dirichlet's test). Suppose the partial sums A_n form a bounded sequence in $\mathbb{R}$, $\{b_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of numbers such that

$$\lim_{n\to\infty}b_n=0.$$

Then,

$$\sum_{n=0}^{\infty}a_nb_n\quad\text{converges in }\mathbb{R}.$$

Proof. Since $\{A_n\}_{n\in\mathbb{N}}$ forms a bounded sequence, then there exists $M\geq 0$ such that for all $n\in\mathbb{Z}_{\geq 0}$, we have $|A_n|\leq M$. Let $\varepsilon>0$ be arbitrary. As

$$\lim_{n\to\infty}b_n=0,$$

then there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq N$, we have $0 \leq b_n \leq \frac{\varepsilon}{2M}$. As such, for all $p, q \geq N$ with $p \leq q$, we apply the partial summation formula (Theorem 3.12) to obtain

$$\begin{aligned}
\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\
&\leq \sum_{n=p}^{q-1} |A_n| |b_n - b_{n+1}| + |A_q| |b_q| + |A_{p-1}| |b_p| \quad \text{by the triangle inequality} \\
&\leq M \left(\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right) \quad \text{since } \{b_n\}_{n \in \mathbb{N}} \text{ is a decreasing sequence} \\
&\leq 2M b_p
\end{aligned}$$

Since $2Mb_p \leq \varepsilon$, then it follows that the partial sums of the sum of $a_n b_n$ form a Cauchy sequence in $\mathbb{R}$. As such, the aforementioned sum converges in $\mathbb{R}$. $\square$

Example 3.23 (Fourier series). For example,

$$\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}} \quad \text{converge.}$$

More generally, for any sequence of numbers $\{b_n\}_{n \in \mathbb{N}}$ which decreases to 0 in $\mathbb{R}$, and for any $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$,

$$\sum_{n=1}^{\infty} b_n \cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \sin nx \quad \text{converge in } \mathbb{C}.$$

These are known as a Fourier cosine series and a Fourier sine series respectively. In many contexts, a function can be represented as a sum of sine and cosine series, which together form a full Fourier series. Fourier cosine series are typically used for even extensions of functions, while Fourier sine series are used for odd extensions.

By Dirichlet's test (Theorem 3.13), it suffices to show that the partial sums of the Fourier cosine and Fourier sine series are bounded, i.e. we should show something like

$$\sum_{n=1}^{N} \cos nx \text{ and } \sum_{n=1}^{N} \sin nx \quad \text{are} \quad \text{bounded.}$$

Indeed, by Lagrange's trigonometric identities (can be easily proved using techniques taught in H2 Mathematics), we have

$$\left|\sum_{k=1}^{n} \cos kx\right| = \left|\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\right| \leq \frac{1}{\left|\sin\frac{1}{2}x\right|} \quad \text{and} \quad \left|\sum_{k=1}^{n} \sin kx\right| = \left|\frac{\cos\frac{1}{2}x - \cos\left(\left(n+\frac{1}{2}x\right)\right)}{2\sin\frac{1}{2}x}\right| \leq \frac{1}{\left|\sin\frac{1}{2}x\right|}$$

which hold for all $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

We then discuss the alternating series test (Theorem 3.14), which can be seen as a special case of Dirichlet's test (Theorem 3.13).

Theorem 3.14 (alternating series test). If a_n is an alternating series with

$$\left|\frac{a_{n+1}}{a_n}\right| \leq 1 \quad \text{for } n \geq 1 \text{, i.e. } a_n \text{ decreases monotonically} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0,$$

then the sum of a_n converges.

Example 3.24 (alternating harmonic series). The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \quad \text{converges} \quad \text{by the alternating series test.}$$

Example 3.25 (MA2108 AY21/22 Sem 1 Midterm). Consider the following alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2/3}}.$$

Is it convergent? Prove your conclusion.

Solution. Let

$$a_n = \frac{1}{n^{2/3}} = n^{-2/3}.$$

We verify if

$$\lim_{n\to\infty} a_n = 0$$

and a_n is monotonically decreasing. The limit property is obviously true.

Consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{-2/3}}{n^{-2/3}}\right| = \left|\left(1+\frac{1}{n}\right)^{-2/3}\right| < 1$$

and so $a_{n+1} < a_n$. By the alternating series test (Theorem 3.14), the series is convergent. □

3.4 Grouping and Rearrangement of Series

Theorem 3.15 (convergence is stable under grouping). Let V be a Banach space. If

$$\sum_{n=1}^{\infty} a_n \quad \text{converges absolutely in } V,$$

then any series obtained by

$$\text{grouping the terms of } \sum_{n=1}^{\infty} a_n \text{ is also absolutely convergent in } V \quad \text{and} \quad \text{has the same value as } \sum_{n=1}^{\infty} a_n.$$

Definition 3.9 (rearrangement). A series

$$\sum_{n=1}^{\infty} b_n \quad \text{is} \quad \text{a rearrangement of the series } \sum_{n=1}^{\infty} a_n$$

if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$.

Theorem 3.16 (absolute convergence is stable under rearrangement). Let V be a Banach space. Suppose the series

$$\sum_{k=1}^{\infty} a_k \quad \text{converges absolutely in } V.$$

Then, for all $\sigma \in$ set of permutations of $\mathbb{N}$, the series

$$\sum_{k=1}^{\infty} a_{\sigma(k)} \text{ also converges absolutely in } V \quad \text{and} \quad \sum_{k=1}^{\infty} a_{\sigma(k)} = \sum_{k=1}^{\infty} a_k \text{ in } V.$$

We can generalise to the following result. Given a Banach space V, for any countably infinite set I and any map $a : I \to V$, where $k \mapsto a_k$, the series

$$\sum_{i \in I} a_i \quad \text{is} \quad \text{absolutely convergent in } V$$

if and only if for every bijection $\tau : \mathbb{N} \to I$, the series

$$\sum_{k=1}^{\infty} a_{\tau(k)} \quad \text{converges absolutely in } V \quad \text{or equivalently} \quad \sum_{k=1}^{\infty} \left\| a_{\tau(k)} \right\| < \infty.$$

When this happens, we define

$$\sum_{i \in I} a_i = \sum_{k=1}^{\infty} a_{\tau(k)} \in V \quad \text{which is} \quad \text{called the sum of the given series.}$$

In fact, this is well-defined and independent of the choice of the bijection τ. Hence, for every $\varepsilon > 0$, there exists a finite set $I_0 \subseteq I$ such that for every finite set $I' \subseteq I$ with $I_0 \subseteq I'$, we have

$$\left\| \sum_{i \in I'} a_i - \sum_{i \in I} a_i \right\| < \varepsilon.$$

It follows that we have the triangle inequality for absolutely convergent series, which states that

$$\left\| \sum_{i \in I} a_i \right\| \leq \sum_{i \in I} \|a_i\| \quad \text{in } \mathbb{R}_{\geq 0}.$$

Corollary 3.3 (rearrangement)**.** If

$$\sum_{i \in I} a_i \quad \text{is} \quad \text{absolutely convergent in } V,$$

then for every permutation $\sigma \in$ the set of permutations of I,

$$\sum_{i \in I} a_{\sigma(i)} \text{ is absolutely convergent in } V \quad \text{and} \quad \sum_{i \in I} a_{\sigma(i)} = \sum_{i \in I} a_i \text{ in } V.$$

The proof of Corollary 3.3 is immediate from the well-defined property of the sum of a_i, where $i \in I$.

Corollary 3.4 (repartitioning)**.** Let V be a Banach space. If

$$\sum_{i \in I} a_i \quad \text{is} \quad \text{absolutely convergent in } V,$$

then for every partition $\{I_j\}_{j \in J}$ of I, we have the following:

(i) for all $j \in J$, the series

$$\sum_{i \in I_j} a_i \quad \text{converges absolutely in } V$$

(ii) the series

$$\sum_{j \in J} \left(\sum_{i \in I_j} a_i \right) \quad \text{converges absolutely in } V$$

(iii) we have

$$\sum_{j \in J} \left(\sum_{i \in I_j} a_i \right) = \sum_{i \in I} a_i \quad \text{in } V$$

Example 3.26 (paradoxical?). The series

$$1 + 2 + 3 + 4 + \dots$$

is an interesting one. Although it is a divergent series, by certain methods such as rearrangement of the original series or by Ramanujan summation, we obtain the formula

$$1 + 2 + 3 + 4 + \dots = -\frac{1}{12}.$$

Example 3.27 (MA2108S AY16/17 Sem 2 Homework 4). For x_n given by the following formulae, establish either the convergence or the divergence of the series

$$\sum_{n=1}^{\infty} x_n.$$

(a) $x_n = \dfrac{n}{n+1}$ **(b)** $x_n = \dfrac{(-1)^n n}{n+1}$ **(c)** $x_n = \dfrac{n^2}{n+1}$ **(d)** $x_n = \dfrac{2n^2+3}{n^2+1}$

Solution.

(a) Note that

$$x_n = 1 - \frac{1}{n+1}$$

so

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n+1} \right)$$

which diverges.

(b) Pairing the terms,

$$\begin{aligned}
\sum_{n=1}^{\infty} x_n &= (x_1 + x_2) + (x_3 + x_4) + (x_5 + x_6) + \dots \\
&= \frac{1}{6} + \frac{1}{20} + \frac{1}{42} + \frac{1}{72} + \dots \\
&= \sum_{n=1}^{\infty} \frac{1}{(2n)(2n+1)} \\
&= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \dots
\end{aligned}$$

The above alternating sum motivates us to use the infinite series representation of $\ln 2$, so the sum of x_n is $1 - \ln 2$, implying that x_n converges.

(c) x_n can be written as

$$x_n = n - 1 + \frac{1}{n+1}$$

so the sum is divergent.

(d) x_n can be written as

$$x_n = 2 + \frac{1}{n^2+1}$$

so the sum is divergent. □

Example 3.28 (MA2108 AY18/19 Sem 1). Determine whether each of the following sequences is convergent. Justify your answers.

(i)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}+\sqrt{n^2+1}}$$

(ii)
$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{2n^2-\cos n}$$

(iii)
$$\sum_{n=1}^{\infty} x_n,$$

where x_n is defined to be the following:

$$x_n = \frac{3^{n+1}}{(n+1)!} \text{ if } n \text{ is odd}, \; x_n = -\frac{3^{n-1}}{(n-1)!} \text{ if } n \text{ is even.}$$

Solution.

(i) We use the alternating series test as $(-1)^{n+1}$ is present here. Define

$$a_n = \frac{1}{\sqrt{n}+\sqrt{n^2+1}}.$$

We prove that a_n is monotonically decreasing. Note that

$$a_{n+1} = \frac{1}{\sqrt{n+1}+\sqrt{(n+1)^2+1}}.$$

It is clear that $a_n > a_{n+1}$ because $\sqrt{n} < \sqrt{n+1}$ and $\sqrt{n^2+1} < \sqrt{(n+1)^2+1}$, thus the sequence is decreasing. Lastly,

$$\lim_{n\to\infty} a_n = 0 \quad \text{so} \quad \text{the series converges by the alternating series test.}$$

(ii) We use the limit comparison test. Let

$$a_n = \frac{\sqrt{n}+1}{2n^2-\cos n} \text{ and } b_n = \frac{1}{n^{3/2}}.$$

Note that b_n is the p-series, where $p = 3/2$, so b_n converges. Consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{2n^2-\cos n} = \lim_{n\to\infty} \frac{1+n^{-1/2}}{2-\cos n/n^2} = \frac{1}{2}.$$

As this limit is finite, the series converges by the limit comparison test.

(iii) The sum of x_n is a rearrangement of the following alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!}$$

Use the ratio test to prove that this alternating series converges. □

Example 3.29 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 26). Establish the convergence or the divergence of the series whose n^{th} term is

(a) $\dfrac{1}{(n+1)(n+2)}$ **(b)** $\dfrac{n}{(n+1)(n+2)}$ **(c)** $2^{-1/n}$ **(d)** $\dfrac{n}{2^n}$

Solution.

(a) Converges — use method of difference.

(b) Diverges — use method of difference.

(c) Diverges. Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{2^{1/n}} \geq \sum_{n=1}^{\infty} \frac{1}{2}.$$

(d) Converges[†]. The trick is to first let the sum be S. Then,

$$\begin{aligned} S &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \frac{5}{2^5} + \ldots \\ &= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots\right) + \left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{4}{2^5} + \ldots\right) \\ &= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots\right) + \frac{1}{2}S \\ &= 1 + \frac{1}{2}S \end{aligned}$$

So, $S = 2$.

Here is an interesting perspective to the arithmetic-geometric series in **(d)** of Example 3.29. Let X be a random variable denoting the number of occurrences up to and including the first occurrence of a heads. Then,

$$P(X = k) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{k-1} = \frac{1}{2^k}$$

as this is equivalent to saying that the first $k-1$ trials are failures (i.e. tails) and the k^{th} trial is a success (i.e. a head). This essentially models a geometric distribution with probability of success $1/2$. So, we write $X \sim \text{Geo}(1/2)$. One notes that the expectation can be computed as follows:

$$\begin{aligned} E(X) &= 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) \\ &= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + \ldots \end{aligned}$$

It is a well-known fact that the expectation can be computed easily — if $X \sim \text{Geo}(p)$, then $E(X) = 1/p$. Since $p = 1/2$, then the expectation is 2.

Example 3.30 (MA2108 AY24/25 Sem 2 Problem Set 3 Question 27)**.** Establish the convergence or divergence of the series whose n^{th} term is:

(a) $\dfrac{1}{\sqrt{n(n+1)}}$ **(b)** $\dfrac{1}{\sqrt{n^2(n+1)}}$ **(c)** $\dfrac{n!}{n^n}$ **(d)** $\dfrac{(-1)^n n}{n+1}$

Solution.

(a) Diverges. Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \cdot n}} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

(b) Converges. Use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2(n+1)}} \geq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}.$$

[†]This is known as an arithmetic-geometric series since it is the product of an arithmetic sequence and a geometric sequence. In fact, it is known as Gabriel's staircase.

(c) Converges. By the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right| = \lim_{n\to\infty}\left|(n+1)\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}\right| = \frac{1}{e} < 1.$$

(d) Converges by the alternating series test. □

Theorem 3.17 (Riemann rearrangement theorem)**.** If a series

$$\sum_{n=1}^{\infty} a_n$$

of real numbers converges conditionally (i.e. it converges, but the series of absolute values diverges), then for any real number L there exists a rearrangement of the terms of the series such that the rearranged series converges to L. Moreover, it is also possible to rearrange the series so that it diverges to $+\infty$ or $-\infty$, or even fails to have a limit in the extended real sense.

Example 3.31 (alternating harmonic series)**.** Recall that the alternating harmonic series is given by

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots,$$

and it is well-known that this series converges to $\ln 2$, but it does not converge absolutely. The Riemann rearrangement theorem (Theorem 3.17) tells us that for any real number L, there exists a rearrangement of the terms of a conditionally convergent series (like the alternating harmonic series) that converges to L. The idea behind the rearrangement is as follows.

We first accumulate the positive terms. The positive terms are

$$P = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right\}.$$

Add these terms until the partial sum exceeds the target L. We then consider the negative terms

$$N = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \ldots\right\}.$$

Add enough negative terms to bring the partial sum below L. Continue alternating between adding positive terms until the sum exceeds L and then negative terms until it drops below L. This process creates a sequence of partial sums that oscillate around L with the oscillations diminishing in size, ensuring convergence to L in the limit.

Chapter 4
Limits of Functions

4.1
Limit Theorems

We will not discuss the formal definition of limits that have already been discussed in MA2002 and the early 2 and the front section of this set of notes.

Theorem 4.1 (sequential criterion)**.** We have

$$\lim_{x\to a} f(x) = L$$

if and only if x_n is any sequence in the domain of f such that $x_n \neq a$ for all n and

$$\lim_{n\to\infty} x_n = a \quad \text{implies} \quad \lim_{n\to\infty} f(x_n) =$$

Theorem 4.2 (divergent criteria)**.** Say we wish to prove that

$$\lim_{x\to a} f(x) \quad \text{does not exist.}$$

(i) Method 1: find a sequence x_n such that $x_n \neq a$ for all $n \in \mathbb{N}$ with

$$\lim_{n\to\infty} x_n = a \quad \text{but} \quad \lim_{n\to\infty} f(x_n) \text{ diverges}$$

(ii) Method 2: Find two sequences x_n and y_n such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a \quad \text{but} \quad \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$$

Lemma 4.1. Let $c \in \mathbb{R}$. Then, there exist sequences $x_n \in \mathbb{Q}, y_n \in \mathbb{Q}'$ such that

$$x_n, y_n \neq c \quad \text{and} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c.$$

Example 4.1. It is a well-known fact that

$$\lim_{x\to 0} \cos\left(\frac{1}{x^2}\right)$$

does not exist, but how do we prove it?

Solution. We make use of the fact that $\cos(n\pi) = (-1)^n$ for all $n \in \mathbb{N}$ to establish the result. We set $f(x) = \cos(1/x^2)$ and $x_n = 1/\sqrt{n\pi}$. Note that $x_n \neq 0$. Hence, $f(x_n) = \cos(n\pi) = (-1)^n$ but $f(x_n)$ is divergent. By Method 1 of the divergent criteria (Theorem 4.2), the limit as $x \to 0$ does not exist. $\square$

4.2
One-Sided Limits

Proposition 4.1. We have

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

Example 4.2 (signum function)**.** The signum function, or $\operatorname{sgn}(x)$, is defined by the following piece-wise function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0. \end{cases}$$

Note that

$$\lim_{x\to 0^+} \operatorname{sgn}(x) = 1 \text{ but } \lim_{x\to 0^-} \operatorname{sgn}(x) = -1 \quad \text{so} \quad \lim_{x\to 0} \operatorname{sgn}(x) \text{ does not exist.}$$

Definition 4.1 (floor function)**.** The floor function of a number x, which is denoted by $\lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lfloor x \rfloor = n \quad \text{if } x \in [n, n+1).$$

Example 4.3. $\lfloor \pi \rfloor = 3$ and $\lfloor -4.8 \rfloor = -5$

Definition 4.2 (ceiling function)**.** The ceiling function of a number x, which is denoted by $\lceil x \rceil$, is defined to be the least integer greater than or equal to x. Hence, for $n \in \mathbb{Z}$,

$$\lceil x \rceil = n \quad \text{if } x \in (n, n+1].$$

Example 4.4. $\lceil 6.1 \rceil = 7$ and $\lceil -7.8 \rceil = -7$

Two important inequalities in relation to the floor and ceiling function respectively are

$$n \le \lfloor x \rfloor < n+1 \text{ and } n < \lceil x \rceil \le n+1 \quad \text{for } n \in \mathbb{Z},$$

which can be used to solve equations, inequalities and limits involving them.

Definition 4.3 (fractional part)**.** For any number x, the fractional part of it is defined by $\{x\}$. So, for any $x > 0$, we have

$$\{x\} = x - \lfloor x \rfloor.$$

Chapter 5
Continuous Functions

5.1
Types of Discontinuity

These are covered in MA2002 so we shall not emphasise much here.

Definition 5.1 (continuity). A function $f(x)$ is continuous at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

Definition 5.2 (removable discontinuity). A removable discontinuity is a point on the graph that is undefined or does not fit the rest of the graph.

Example 5.1. The graph of $f(x) = x^2/x$ is discontinuous at $x = 0$ even though the right side can be simplified to $f(x) = x$. However, based on the original domain of the function, if $x = 0$, then the denominator will be 0 as well, which is impossible!

Example 5.2 (infinite discontinuity). Consider the graph of $g(x) = 1/x$, where

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = \infty.$$

g is said to have infinite discontinuity at $x = 0$.

Example 5.3 (jump discontinuity). In relation to the signum function discussed in Example 4.2, there is a jump discontinuity at $x = 0$.

Definition 5.3 (oscillating discontinuity). An oscillating discontinuity exists when the values of the function appear to be approaching two or more values simultaneously.

Example 5.4. Consider the graph of $h(x) = \sin(1/x)$, where $x = 0$ is regarded as a point of oscillating discontinuity.

Here is the formal definition for continuity at a point.

Definition 5.4 (formal definition of continuity). A function f is continuous at $x = a$ if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that} \quad |f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

5.2
Special Functions

Definition 5.5 (Dirichlet function). Named after mathematician Peter Gustav Lejeune Dirichlet, the Dirichlet function, $f(x)$, is defined to be the following:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

It is an example of a function that is nowhere continuous.

Theorem 5.1. The Dirichlet function is nowhere continuous.

Proof. Suppose $x \in \mathbb{Q}$, so $f(x) = 1$. We show that f is discontinuous at x. Let $\delta > 0$ be arbitrary and $y \in \mathbb{Q}$ such that $|x-y| < \delta$. Choose $\varepsilon = 1/2$. Without a loss of generality, assume $x < y$. Since there exists $z \in \mathbb{Q}'$ such that $x < z < y$ (due to the density of the irrationals in the reals), then

$$|f(x) - f(z)| = |1-0| = 1 > \frac{1}{2} = \varepsilon.$$

In a similar fashion, we now consider the case where $x > y$. There exists $z' \in \mathbb{Q}'$ such that $y < z' < x$, so

$$|f(x) - f(z')| = |1-0| = 1 > 1/2 = \varepsilon.$$

Therefore, if $x \in \mathbb{Q}$, f is discontinuous at x. For the case where $x \in \mathbb{Q}'$, the proof is very similar. $\square$

Lemma 5.1. The Dirichlet function can be constructed as the double limit of a sequence of continuous function. That is,

$$f(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Definition 5.6 (Thomae's function). Thomae's function maps all real numbers to the unit interval $[0,1]$. The function can be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}; \\ 1/q & \text{if } x = p/q,\ p,q \in \mathbb{N} \text{ and } \gcd(p,q) = 1 \end{cases}$$

It is named after Carl Johannes Thomae, and the function is also known as the popcorn function due to its nature.

It is a well-known fact that Thomae's function is not continuous at all rational points but continuous at all irrational points.

5.3 Properties of Continuous Functions

Proposition 5.1. The following hold:

(i) Combinations: Suppose f and g are continuous at $x = a$. Then,

$$\text{for any } \alpha \in \mathbb{R} \quad f \pm g, fg, \alpha f \text{ are also continuous at } x = a.$$

If $g(a) \neq 0$, then f/g is also continuous at $x = a$.

(ii) Composite functions: Suppose f and g are such that $g \circ f$ is defined. If

$$f \text{ is continuous at } a \text{ and } g \text{ is continuous at } f(a) \quad \text{then} \quad g \circ f \text{ is continuous at } a.$$

Moreover, suppose $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subseteq B$ so that $g \circ f$ is defined. If f is continuous on A and g is continuous on B, then $g \circ f$ is continuous on A.

Theorem 5.2 (extreme value theorem). If f is continuous on $[a,b]$, then there exist $x_1, x_2 \in [a,b]$ such that

$$f(x_1) \le f(x) \le f(x_2) \text{ for all } x \in [a,b].$$

In short, if f is continuous on a closed and bounded interval, then its range is also bounded.

Theorem 5.3 (intermediate value theorem). If f is continuous on $[a,b]$, and $f(a) < k < f(b)$, then there exists a point $c \in (a,b)$ such that $f(c) = k$.

Corollary 5.1. If f is continuous on $[a,b]$, then

$$f([a,b]) = [m,M] \quad \text{where } m = \inf f([a,b]) \text{ and } M = \sup f([a,b]).$$

Corollary 5.2 (location of roots). If f is continuous on $[a,b]$, and $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

Example 5.5 (MA2108 AY19/20 Sem 1). Let f be continuous on $[0,1]$ and $f(0) = f(1)$. Prove that for any positive integer n, there exists a $\zeta \in [0,1]$ such that

$$f\left(\zeta + \frac{1}{n}\right) = f(\zeta).$$

Solution. Define

$$g(x) = f\left(x + \frac{1}{n}\right) - f(x).$$

By the intermediate value theorem, g does not experience a change in its polarity for all $x \in [0,1]$. Suppose on the contrary that this claim is false. Then, by the method of differences,

$$\begin{aligned} \sum_{i=1}^{n} g\left(1 - \frac{i}{n}\right) &= f\left(\frac{1}{n}\right) - f(0) \\ g(0) &= f\left(\frac{1}{n}\right) - f(0) \end{aligned}$$

Without a loss of generality, assume that $g(x) > 0$ for all $x \in [0,1]$. Then, setting $n = 1$, it implies that $f(1) - f(0) > 0$, which is a contradiction! □

5.4
Monotone and Inverse Functions

Definition 5.7. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $x_1, x_2 \in A$. Then, the following hold:

(i) f is increasing on A if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$

(ii) f is strictly increasing on A if $x_1 < x_2$ implies $f(x_1) < f(x_2)$

(iii) f is decreasing on A if $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$

(iv) f is strictly decreasing on A if $x_1 < x_2$ implies $f(x_1) > f(x_2)$

(v) f is monotone if it is either increasing or decreasing

(vi) f is strictly monotone if it either strictly increasing or strictly decreasing

Proposition 5.2. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be an increasing function. If $c \in I$ is not an endpoint of I, then

$$\lim_{x \to c^-} f(x) = \sup\{f(x) : x \in I, x < c\} \quad \text{and} \quad \lim_{x \to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}.$$

Theorem 5.4 (continuous inverse theorem)**.** Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a strictly monotone function. If f is continuous on I and $J = f(I)$, then its inverse function $f^{-1} : J \to \mathbb{R}$ is strictly monotone and continuous on J.

5.5
Uniform Continuity

Definition 5.8 (uniform continuity)**.** Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. f is uniformly continuous on I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{for any } x, y \in I \quad |x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.$$

Corollary 5.3. If a function is uniformly continuous on I, then it is continuous on I.

Example 5.6. We claim that $f(x) = x^2$ is uniformly continuous on $[0, 1]$. To see why, let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/2$. For $x, y \in [0, 1]$, suppose $|x - y| < \delta$. Then,

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 2 \cdot \delta = \varepsilon$$

and we are done.

Theorem 5.5. A function f is uniformly continuous on I if and only if f' is bounded.

It is worth noting that $f(x) = x^2$ is uniformly continuous on $[a, b]$ in general, where $a, b \in \mathbb{R}$, but it is not uniformly continuous on $\mathbb{R}$!

Example 5.7 (MA2108 AY19/20 Sem 1)**.** Prove that the function $f(x) = \sqrt{x^2 - x + 1}$ is uniformly continuous on $[1, \infty)$[†].

[†]The original question had an error. It used $f(x) = \sqrt{x(x-1)}$, which is undefined at $x = 1$. Confirmed the change with one of the students.

Solution. Since

$$f'(x) = \frac{1}{2}\left(x^2 - x + 1\right)^{-1/2} \cdot (2x-1) = \frac{2x-1}{2\sqrt{x^2-x+1}},$$

and noting that $x^2 - x + 1 > 0$ for all $x \in [1, \infty)$, as well as $|f'(x)| < 1$, by Theorem 5.5, the result follows. □

Theorem 5.6 (sequential criterion for uniform continuity)**.** $f : I \to \mathbb{R}$ is uniformly continuous on I if and only if for any two sequences $x_n, y_n \in I$ such that

$$\text{if } \lim_{n\to\infty}(x_n - y_n) = 0 \quad \text{then} \quad \lim_{n\to\infty}[f(x_n) - f(y_n)] = 0.$$

Definition 5.9 (Lipschitz continuity)**.** Let I be an interval and $f : I \to \mathbb{R}$ satisfies the Lipschitz condition on I. Then, there is $K > 0$ such that

$$|f(x) - f(y)| \le K|x - y|, \quad \text{for all } x, y \in I.$$

Theorem 5.7. If a function is Lipschitz continuous on I, then it is uniformly continuous on I.

Example 5.8. We verify that $f(x) = x^2$, in the interval $[0, 1]$, satisfies the Lipschitz condition.

Solution. Since $f(x) - f(y) = x^2 - y^2$, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \left|\frac{x^2 - y^2}{x - y}\right| = |x + y| \le 2,$$

and since $2 > 0$, $f(x) = x^2$, in $[0, 1]$, is said to satisfy the Lipschitz condition. In other words, f is Lipschitz continuous. □

Theorem 5.8. If $f : I \to \mathbb{R}$ is uniformly continuous on I and x_n is Cauchy, then $f(x_n)$ is Cauchy.

If the function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b), then $f(a)$ and $f(b)$ can be defined so that the extended function is continuous on $[a, b]$.

Chapter 6
The Topology of the Real Numbers

6.1
Open and Closed Sets in $\mathbb{R}$

Recall that a neighbourhood of a point $x \in \mathbb{R}$ is any set V that

$$\text{contains an } \varepsilon\text{-neighbourhood } V_{\varepsilon}(x) = (x-\varepsilon, x+\varepsilon) \text{ of } x \quad \text{for some } \varepsilon > 0.$$

Definition 6.1 (open set). A subset G of $\mathbb{R}$ is open in $\mathbb{R}$ if for each $x \in G$, there exists a neighbourhood V of x such that $V \subseteq G$.

To show that $G \subseteq \mathbb{R}$ is open, it suffices to show that each point in G has an ε-neighbourhood contained in G. In fact, G is open if and only if for each $x \in G$, there exists $\varepsilon_x > 0$ such that $(x-\varepsilon_x, x+\varepsilon_x)$ is contained in G.

Definition 6.2 (closed set). A subset F of $\mathbb{R}$ is closed in $\mathbb{R}$ if the complement $\mathbb{R} \backslash F$ is open in $\mathbb{R}$.

To show that F is closed, it suffices to show that each point $y \notin F$ has an ε-neighbourhood disjoint from F. In fact, F is closed if and only if for each $y \notin F$, there exists $\varepsilon_y > 0$ such that $F \cap (y-\varepsilon_y, y+\varepsilon_y) = \emptyset$.

Proposition 6.1. Any open interval $I = (a,b)$ is an open set.

Proof. For $x \in I$, take $\varepsilon_x = \min\{x-a, b-x\}$. Then, $x+\varepsilon_x = b$ and $x-\varepsilon_x = a$, so $(x-\varepsilon_x, x+\varepsilon_x) \subseteq I$. □

Example 6.1. We can show that the set $[0,1]$ is not open but closed. Note that in English, the terms 'open' and 'closed' are antonyms, but in Topology, these are not the opposite of each other.

We first show that $[0,1]$ is not open. This is clear since every ε-neighbourhood of $0 \in I$ contains points not in I. In a similar fashion, instead of 0, we can take the other endpoint, which is 1. Note that $(0-\varepsilon_x, 0+\varepsilon_x)$ is the same as $(-\varepsilon_x, \varepsilon_x)$, which clearly shows that this interval contains points not in I.

To show that $[0,1]$ is closed, let $y \notin I$. Then, $y < 0$ or $y > 1$. If $y < 0$, take $\varepsilon_y = |y|$, and if $y > 1$, take $\varepsilon_y = y-1$. For the former, $[0,1] \cap (y-|y|, y+|y|)$ is equivalent to $(2y, 0) \cap [0,1]$, which is clearly $\emptyset$. For the latter, $[0,1] \cap (1, 2y-1)$ is equivalent to $\emptyset$ too.

Chapter 7
Differentiable Functions

7.1
First Principles

Definition 7.1 (differentiable function)**.** A function f is differentiable at a point a if f is defined in some open interval containing a and the limit

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \quad \text{exists.}$$

In this case, $f'(a)$ is the derivative of f at $x=a$.

Geometrically, $f'(a)$ is the slope of the tangent to the curve $y=f(x)$ at $x=a$. The formula in Definition 7.1 is similar to one that students have learnt in H2 Mathematics, which is the derivative of $f(x)$, denoted by $f'(x)$, can be expressed as

$$\lim_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x}.$$

Example 7.1 (MA2108 AY22/23 Sem 1 Warmup)**.** Let $y=f(x)$ be continuous everywhere for $x\in(-\infty,\infty)$ and satisfy

$$f(0)=1, f(1)=e \text{ and } f(x+y)=f(x)f(y).$$

Prove that $f(x)=e^x$ for $x\in(-\infty,\infty)$.

Solution. It is clear that

$$f\left(\sum_{i=1}^n x_i\right) = \prod_{i=1}^n f(x_i).$$

By first principles,

$$f'(x) = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = \lim_{\delta x\to 0}\frac{f(x)f(\delta x)-f(x)}{\delta x} = f(x)\lim_{\delta x\to 0}\frac{f(\delta x)-1}{\delta x} = f(x)f'(0).$$

We set $c=f'(0)$ so $f'(x)=cf(x)$. Integrating, we have $\ln|f(x)|=cx+d$. Since $f(0)=1$, then $d=0$. Since $f(1)=e$, then $c=1$. Thus, $\ln|f(x)|=x$. As such, we conclude that $f(x)=e^x$. $\square$

Definition 7.2. If f is differentiable at every point in (a,b), then f is differentiable on (a,b).

Proposition 7.1. If the function $f:[a,b]\to\mathbb{R}$ is such that f is differentiable on (a,b) and the one sided limits

$$L_1 = \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a} \quad \text{and} \quad L_2 = \lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$$

exist, then f is differentiable on $[a,b]$. In this case, $f'(a)=L_1$ and $f'(b)=L_2$.

7.2
Continuity and Differentiability

Definition 7.3. f is continuously differentiable on I if f is differentiable on I and f' is continuous on I.

Definition 7.4. The collection of all functions which are continuously differentiable on I is denoted by $\mathcal{C}^1(I)$.

Proposition 7.2. If f is differentiable at a, then it is continuous at a.

Proof. We have

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right)+f(a)\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a} x-a\right)+f(a)\\ &= f'(a)\cdot 0+f(a)\end{aligned}$$

which is just $f(a)$. □

The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere. It is an example of a fractal curve named after its discoverer German mathematician Karl Weierstrass[†].

Definition 7.5 (Weierstrass function)**.** In Weierstrass's original paper, the function was defined as the following Fourier series:

$$f(x)=\sum_{n=0}^{\infty} a^n\cos(b^n\pi x),$$

where $0<a<1$, b is a positive odd integer and $ab>1+3\pi/2$.

7.3
Derivative Rules and Theorems

Theorem 7.1 (Carathéodory's theorem)**.** Let I be an interval, $f:I\to\mathbb{R}$ and $c\in I$. Then $f'(c)$ exists if and only if there exists a function ϕ on I such that ϕ is continuous at c and

$$f(x)-f(c)=\phi(x)(x-c)\quad \text{for all } x\in I.$$

Proposition 7.3 (chain rule)**.** Let I and J be intervals, and let $g:I\to\mathbb{R}$ and $f:J\to\mathbb{R}$ be such that $f(J)\subseteq I$. If $a\in J$, f is differentiable at a and g is differentiable at $f(a)$, then $h=g\circ f$ is differentiable at

[†] This link provides an analysis of the Weierstrass function involving its uniform convergence (this term will be studied in due course) and it being nowhere differentiable. This involves the Weierstrass M-test.

a, and

$$h'(a) = g'(f(a))f'(a).$$

Theorem 7.2 (inverse function theorem). If f is a continuously differentiable function with non-zero derivative at a; then f is invertible in a neighbourhood of a, the inverse is continuously differentiable, and the derivative of the inverse function at $b = f(a)$ is the reciprocal of the derivative of f at a. As an equation, we have

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

7.4 Mean Value Theorem and Applications

Definition 7.6. Let I be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$.

(i) If $f(x_0) \geq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute maximum of f on I

(ii) If $f(x_0) \leq f(x)$ for all $x \in I$, then $f(x_0)$ is the absolute minimum of f on I

(iii) If there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f

(iv) If there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative maximum of f

(v) If there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq I$, then $f(x_0)$ is a relative minimum of f

(vi) If $f(x_0)$ is either a relative minimum or relative maximum of f, then $f(x_0)$ is a relative extremum of f

Remark 7.1. A relative extremum can only occur at an interior point, but an absolute extremum may occur at one of the end points of the interval. So if a function has an absolute maximum at a point x_0, it may not have a relative maximum at x_0. If f has an absolute maximum at an interior point x_0 of I, then $f(x_0)$ is also a relative maximum of f.

Lemma 7.1. Let $f : (a,b) \to \mathbb{R}$ and $f'(c)$ exists for some $c \in (a,b)$.

(i) If $f'(c) > 0$, then there exists $\delta > 0$ such that

$$f(x) < f(c) \text{ for every } x \in (c - \delta, c) \text{ and } f(x) > f(c) \text{ for every } x \in (c, c + \delta).$$

(ii) If $f'(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) > f(c) \text{ for every } x \in (c - \delta, c) \text{ and } f(x) < f(c) \text{ for every } x \in (c, c + \delta).$$

Theorem 7.3 (Fermat's extremum theorem). Suppose c is an interior point of an interval I and $f : I \to \mathbb{R}$ is differentiable at c. If f has a relative extremum at c, then $f'(c) = 0$.

Proof. Without a loss of generality, assume that f has a relative maximum at c (the proof if f has a relative minimum is similar). Suppose on the contrary that either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then by the lemma above, there exists $\delta > 0$ such that $f(x) < f(c)$ for every $x \in (c-\delta, c)$ and $f(x) > f(c)$ for every $x \in (c, c+\delta)$. This contradicts the assumption that f has a relative maximum at c. The proofs for other cases are similar. □

Remark 7.2. A function f may have a relative extremum at x_0, but $f'(x_0)$ does not exist.

Example 7.2. Consider $f(x) = |x|$. There is a relative (absolute) minimum at $x = 0$, but $f'(0)$ does not exist.

The converse of Fermat's theorem is false. For example, consider $f(x) = x^3$, where $f'(0) = 0$ but $x = 0$ is not a relative extremum point of f. It is merely a point of inflection.

Theorem 7.4 (Rolle's theorem). If f is continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

Proof. The proof where $f(x)$ is a constant will not be discussed since it is trivial. For the more meaningful cases, we have $f(x) > f(a)$ or $f(x) < f(a)$ for some $x \in (a,b)$. Without a loss of generality, we shall prove the former case since the proof for the latter is similar.

By the extreme value theorem (Theorem 5.2), we know that $f(x)$ has a maximum, M in the closed interval $[a,b]$. As $f(a) = f(b)$, the maximum value is attained at $x = c$. That is, $f(c) = M$. So, f has a local maximum at c. Since f is differentiable, the result follows. □

Theorem 7.5 (mean value theorem). If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We wish to construct a function $g : [a,b] \to \mathbb{R}$ such that $g(a) = g(b) = 0$, with a point $c \in (a,b)$ such that $g'(c) = 0$. Suppose

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} \cdot (x - a).$$

It is clear that g is continuous on $[a,b]$ and differentiable on (a,b), and $g(a) = g(b) = 0$. By Rolle's theroem, there exists $c \in (a,b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

Rearranging the equation, we are done. □

Corollary 7.1. If f is continuous on $[a,b]$, differentiable on (a,b) and $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Example 7.3 (Berkeley Problems in Mathematics P1.1.25). Let the function f from $[0,1]$ to $[0,1]$ have the following properties:

(i) f is C^1 (i.e. f is differentiable and its derivative is continuous)
(ii) $f(0) = f(1) = 0$
(iii) f' is non-increasing (i.e. f is concave down)

Prove that the arc length of the graph of f does not exceed 3.

Solution. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, by Rolle's theorem, there exists $c \in (0,1)$ such that $f'(c) = 0$. Since f is concave down, then f is increasing on $(0,c)$ and decreasing on $(c,1)$. On $[0,c]$, the arc length of f is given by

$$\int_0^c \sqrt{1+[f'(x)]^2}\,dx.$$

We shall partition $(0,c)$ into n equally sized subintervals. Hence, each interval has width c/n. So,

$$\int_0^c \sqrt{1+[f'(x)]^2}\,dx = \lim_{n\to\infty} \frac{c}{n}\sum_{k=1}^{n}\sqrt{1+[f'(\zeta_k)]^2} \quad \text{where } \zeta_k = \left(\frac{c\,(k-1)}{n}, \frac{ck}{n}\right).$$

By the mean value theorem, each ζ_k satisfies

$$f'(\zeta_k) = \frac{f(ck/n) - f(c(k-1)/n)}{c/n}$$

so

$$\begin{aligned}
\int_0^c \sqrt{1+[f'(x)]^2}\,dx &\le \lim_{n\to\infty} \frac{c}{n}\sum_{k=1}^{n}\sqrt{1+\left[\frac{f(ck/n)-f(c(k-1)/n)}{c/n}\right]^2}\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\left(\frac{c}{n}\right)^2+\left[f\left(\frac{ck}{n}\right)-f\left(\frac{c\,(k-1)}{n}\right)\right]^2}\\
&\le \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{c}{n}+f\left(\frac{ck}{n}\right)-f\left(\frac{c\,(k-1)}{n}\right)\right] \quad \text{since } \sqrt{a^2+b^2}\le a+b\\
&= c+f(c) \quad \text{by method of difference.}
\end{aligned}$$

In a similar fashion, one can prove that

$$\int_c^1 \sqrt{1+[f'(x)]^2}\,dx \le 1-c+f(c).$$

so

$$\int_0^1 \sqrt{1+[f'(x)]^2}\,dx \le c+f(c)+1-c+f(c) = 1+2f(c) \le 1+2=3$$

where we used the fact that the range of f is $[0,1]$. □

Proposition 7.4 (increasing and decreasing functions). Let f be differentiable on (a,b).

- **(i)** If $f'(x)\ge 0$ for all $x\in(a,b)$, then f is increasing on (a,b)
- **(ii)** If $f'(x)\le 0$ for all $x\in(a,b)$, then f is decreasing on (a,b)

Theorem 7.6 (first derivative test). Let f be a continuous function on $[a,b]$ and $c\in(a,b)$. Suppose f is differentiable on (a,b) except possibly at c.

- **(i)** If there is a neighbourhood $(c-\delta,c+\delta)\subseteq I$ of c such that $f'(x)\ge 0$ for $x\in(c-\delta,c)$ and $f'(x)\le 0$ for $x\in(c,c+\delta)$, then $f(c)\ge f(x)$ $\forall x\in(c-\delta,c+\delta)$. Hence, f has a relative maximum at c
- **(ii)** If there is a neighbourhood $(c-\delta,c+\delta)\subseteq I$ of c such that $f'(x)\le 0$ for $x\in(c-\delta,c)$ and $f'(x)\ge 0$ for $x\in(c,c+\delta)$, then $f(c)\le f(x)$ $\forall x\in(c-\delta,c+\delta)$. Hence, f has a relative minimum at c

Consider a function $f(x)$. Its first derivative is denoted by $f'(x)$, second derivative is denoted by $f''(x) = f^{(2)}(x)$, and so on. In general, for $n\in\mathbb{N}$, the n^{th} derivative of f at c is defined as

$$f^{(n)}(c) = (f^{n-1})'(c).$$

Let I be an interval. Then, for $n \in \mathbb{N}$, $\mathcal{C}^n(I)$ is defined to be the set of functions f such that $f^{(n)}$ exists and is continuous on I. Note that

$$\mathcal{C}^\infty(I) = \bigcap_{n=1}^{\infty} \mathcal{C}^n(I).$$

If $\in \mathcal{C}^\infty(I)$, then f is infinitely differentiable on I.

Proposition 7.5. For $m > n \geq 1$, where $m, n \in \mathbb{Z}$,

$$\mathcal{C}^\infty(I) \subseteq \mathcal{C}^m(I) \subseteq \mathcal{C}^n(I) \subseteq \mathcal{C}(I).$$

Theorem 7.7 (second derivative test)**.** Let f be defined on an interval I and let its derivative f' exist on I. Suppose c is an interior point of f such that $f'(c) = 0$ and $f''(c)$ exists.

(i) If $f''(c) > 0$, then f has a relative minimum at c

(ii) If $f''(c) < 0$, then f has a relative maximum at c

(iii) If $f''(c) = 0$, then the test is inconclusive. Hence, we have to use the first derivative test to prove whether c is a relative minimum, relative maximum, or a point of inflection

Theorem 7.8 (Cauchy's mean value theorem)**.** Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. We first claim that $g(a) \neq g(b)$. Suppose otherwise, then $g(a) = g(b)$, so by Rolle's theorem, there exists $x_0 \in (a,b)$ such that $g'(x_0) = 0$, contradicting the assumption that $g'(x) \neq 0$ for all $x \in (a,b)$. Next, define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)} \cdot ((g(x)-g(a)) - (f(x)-f(a)),$$

where $x \in [a,b]$. Since h is continuous on $[a,b]$, differentiable on (a,b) and $h(a) = h(b) = 0$, by Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. The result follows. □

Theorem 7.9 (Taylor's theorem)**.** Let f be a function such that $f \in \mathcal{C}^n([a,b])$ and $f^{(n+1)}$ exists on (a,b). If $x_0 \in [a,b]$, then for any $x \in [a,b]$, there exists a point c between x and x_0 such that

$$f(x) = \sum_{k=0}^{n+1} \frac{f^k(c)}{k!}(x-x_0)^k.$$

Corollary 7.2. If $n = 0$, then $f(x) = f(x_0) + f'(c)(x-x_0)$, which is the mean value theorem.

The polynomial $P_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x-x_0)^k$$

is the n^{th} Taylor polynomial for f at x_0.

By Taylor's theorem, as $f(x) = P_n(x) + R_n(x)$, then

$$R_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1}$$

for some point c_n between x and x_0. This formula for R_n is the Lagrange form of the remainder.

Let f be infinitely differentiable on $I = (x_0 - r, x_0 + r)$ and $x \in I$. Then, recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

if and only if

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1} = 0,$$

where each c_n is between x and x_0.

Chapter 8
The Riemann–Stieltjes Integral

8.1
Definition and Existence

Let $I = [a,b]$. A finite set $P = \{x_0, x_1, x_2, \dots, x_n\}$ where

$$a < x_0 < x_1 < x_2 < \dots < x_n < b$$

is a partition of I. It divides I into the subintervals as

$$I = [x_0,x_1] \cup [x_1,x_2] \cup [x_2,x_3] \cup \dots \cup [x_{n-1},x_n] = \bigcup_{i=1}^{n} [x_{i-1},x_i].$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, x_2, \dots, x_n\}$ be a partition of $[a,b]$. For each $1 \le i \le n$, let

$$M_i = M_i(f,P) = \sup\{f(x) : x \in [x_{i-1},x_i]\},$$

$$m_i = m_i(f,P) = \inf\{f(x) : x \in [x_{i-1},x_i]\} \text{ and}$$

$\Delta x_i = x_i - x_{i-1}$. Define the upper sum and lower sum of f with respect to P to be

$$U(f,p) = \sum_{i=1}^{n} M_i \Delta x_i \text{ and } L(f,p) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Note that each partition may not be of uniform length.

By setting $m = \inf\{f(x) : x \in [a,b]\}$ and $M = \sup\{f(x) : x \in [a,b]\}$, then

$$m(b-a) \le L(f,p) \le U(f,p) \le M(b-a).$$

Furthermore,

$$m(b-a) \le \int_a^b f \le M(b-a)$$

and if $f(x) \ge 0$ for all $x \in [a,b]$, then

$$\int_a^b f \ge 0.$$

Definition 8.1 (Darboux integral). The upper Darboux integral of f on $[a,b]$ is defined to be

$$U(f) = \overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

and the lower Darboux integral of f on $[a,b]$ is defined to be

$$L(f) = \underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

Lemma 8.1. $L(f) \le U(f)$

Proof. We prove by contradiction. Suppose $U(f) < L(f)$. Then, there exists a partition P_1 of $[a,b]$ such that

$$U(f) \le U(f,P_1) < L(f).$$

Also, there exists a partition P_2 of $[a,b]$ such that

$$U(f) \le U(f,P_1) < L(f,P_2) \le L(f).$$

However, $L(f,P_2) \le U(f,P_1)$, which is a contradiction. □

From Lemma 8.1, it is clear that for partitions P and Q of $[a,b]$,

$$L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,Q),$$

and consequently,

$$L(f) = \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f).$$

Definition 8.2. If P and Q are partitions of $[a,b]$, then Q is a refinement of P if $P \subseteq Q$.

Proposition 8.1. If P and Q are partitions of $[a,b]$ with Q a refinement of P, then

$$L(f,P) \le L(f,Q) \quad \text{and} \quad U(f,Q) \le U(f,P).$$

Definition 8.3. A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

The Riemann integral is only defined for bounded functions (i.e. if f is unbounded on $[a,b]$, it is not integrable on $[a,b]$) .

Example 8.1.

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

is not integrable since $\lim_{x\to 0} 1/x^2 = \infty$, implying that the function is unbounded on $[-1,1]$.

Consider the Dirichlet function and denote it by $f(x)$. Since the rational and irrational numbers both form dense subsets of $\mathbb{R}$, then f takes on the value of 0 and 1 on every sub-interval of any partition. Thus for any partition P, $U(f,P) = 1$ and $L(f,P) = 0$. By noting that the upper and lower Darboux integrals are unequal, we conclude that f is not Riemann integrable on $[0,1]$[†].

8.2 Riemann Integrability Criterion and Consequences

Theorem 8.1 (Riemann integrability criterion)**.** For a bounded function $f : [a,b] \to \mathbb{R}$, then f is integrable on $[a,b]$ if and only if for any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

[†] A fun fact is that the Dirichlet function is actually Lebesgue integrable (covered in MA4262).

Proof. We first prove that if $U(f,P)-L(f,P)<\varepsilon$, then f is integrable on $[a,b]$. Note that $\varepsilon>0$ is arbitrary. Recall that

$$L(f,P)\le L(f)\le U(f)\le U(f,P).$$

Hence,

$$U(f)-L(f)\le U(f,P)-L(f,P)<\varepsilon,$$

and we are done.

Now, suppose f is integrable on $[a,b]$. We wish to prove that $U(f,P)-L(f,P)<\varepsilon$. Note that there exists a partition P_1 on $[a,b]$ such that $U(f,P_1)<U(f)$ so

$$U(f,P_1)<U(f)+\frac{\varepsilon}{2}.$$

In a similar fashion, there exists a partition P_2 such that

$$L(f,P_2)>L(f)-\frac{\varepsilon}{2}.$$

Let $P=P_1\cup P_2$ be the common refinement of the previous two partitions. Since

$$0\le U(f,P)-L(f,P),$$

then

$$0\le U(f,P)-L(f,P)<U(f)-L(f)+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon.$$

$\square$

Corollary 8.1. If $f:[a,b]\to\mathbb{R}$ is monotone on $[a,b]$, then f is integrable on $[a,b]$.

Corollary 8.2. If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Corollary 8.3. Let $f,g:[a,b]\to\mathbb{R}$ be bounded functions, P be a partition of $[a,b]$ and $c\in\mathbb{R}$. Then,

(i)
$$L(cf,P)=\begin{cases} cL(f,P) & \text{if } c>0\\ cU(f,P) & \text{if } c<0\end{cases}$$

(ii)
$$U(cf,P)=\begin{cases} cU(f,P) & \text{if } c>0\\ cL(f,P) & \text{if } c<0\end{cases}$$

(iii)
$$L(f,P)+L(g,P)\le L(f+g,P)\le U(f+g,P)\le U(f,P)+U(g,P)$$

Proposition 8.2. Let $f,g:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ and $c\in\mathbb{R}$. Then,

(i) Just like linear transformations, the function $cf+g$ is integrable on $[a,b]$ and

$$\int_a^b (cf+g)=c\int_a^b f+\int_a^b g.$$

(ii) If $f(x)\le g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f\le\int_a^b g.$$

(iii) $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

(iv) fg is integrable on $[a,b]$.

Proposition 8.3. If f is integrable on $[a,b]$, then for any $c \in (a,b)$, f is integrable on $[a,c]$ and $[c,b]$. The converse is true and we have the following result:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Let f be a continuous function on $[a,b]$. If $P = \{x_0, x_1, x_2, \dots, x_n\}$ is a partition of $[a,b]$, define

$$L(f,P) = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2},$$

where the supremum is taken over all possible partitions $a = x_0 < x_1 < x_2, \dots < x_n = b$. This definition as the supremum of the all possible partition sums is also valid if f is merely continuous, not differentiable.

8.3
Fundamental Theorems of Calculus

Theorem 8.2 (First Fundamental Theorem of Calculus)**.** Let f be integrable on $[a,b]$ and for $x \in [a,b]$, let

$$F(x) = \int_a^x f.$$

If f is continuous at a point $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Remark 8.1. Not all functions have an elementary antiderivative. That is, for example, there do not exist elementary functions $F(x)$ and $G(x)$ such that

$$F(x) = \int e^{-x^2}\, dx \text{ and } G(x) = \int \frac{1}{\ln x}.$$

Theorem 8.3 (Second Fundamental Theorem of Calculus)**.** Let g be a differentiable function on $[a,b]$ and assume that g' is continuous on $[a,b]$. Then,

$$\int_a^b g' = g(b) - g(a).$$

Theorem 8.4 (Cauchy's Fundamental Theorem of Calculus)**.** Let g be a differentiable function on $[a,b]$ and assume that g' is integrable on $[a,b]$. Then,

$$\int_a^b g' = g(b) - g(a).$$

Example 8.2. It is possible for the derivative of a function to not be integrable. Consider the following function:

$$f(x) = \begin{cases} x^2 \sin\left(1/x^2\right) & x \neq 0; \\ 0 & x = 0. \end{cases}$$

For $x \neq 0$,

$$f'(x) = -\frac{2}{x}\cos\left(\frac{1}{x^2}\right) + 2x\sin\left(\frac{1}{x^2}\right)$$

but $f'(x)$ is not integrable on $[-1,1]$ as this is a region of oscillating discontinuity!

8.4
Riemann Sum

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$ and let $\Delta x = x_i - x_{i-1}$ for $1 \leq i \leq n$. Then, the norm of P, denoted by $\|P\|$, is defined by

$$\|P\| = \max\{\Delta x_i : 1 \leq i \leq n\}.$$

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$, $\|P\| < \delta$ implies that

$$U(f,P) < \overline{\int_a^b} f + \varepsilon \text{ and } L(f,P) > \underline{\int_a^b} f - \varepsilon.$$

We are now ready to define the Riemann sum of f with respect to P.

Definition 8.4 (Riemann sum). Let ξ_i be a point in the i^{th} sub-interval $[x_{i-1}, x_i]$ for $1 \leq i \leq n$. The sum

$$S(f,P)(\xi) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

is the Riemann sum of f with respect to P and $\xi = (\xi_1, \ldots, \xi_n)$.

If there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a,b]$ and any choice of $\xi = (\xi_1, \ldots, \xi_n)$,

$$\|P\| < \delta \quad \text{implies} \quad |S(f,P)(\xi) - A| < \varepsilon,$$

then

$$\lim_{\|P\| \to 0} S(f,P)(\xi) = A.$$

Note that

$$L(f,P) \leq S(f,P)(\xi) \leq U(f,P).$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then,

$$\lim_{\|P\| \to 0} U(f,P) = \overline{\int_a^b} f \quad \text{and} \quad \lim_{\|P\| \to 0} L(f,P) = \underline{\int_a^b} f.$$

Hence, f is integrable on $[a,b]$ and $\int_a^b f = A$ if and only if

$$\lim_{\|P\| \to 0} S(f,P)(\xi) = A.$$

Corollary 8.4. Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. For each $n \in \mathbb{N}$, let $P_n = \left\{x_0^{(n)}, x_1^{(n)}, \ldots, x_{m_n}^{(n)}\right\}$ be a partition of $[a,b]$ and let $\xi^{(n)} = \left(\xi_1^{(n)}, \ldots, \xi_{m_n}^{(n)}\right)$ be such that $\xi_i^{(n)} \in \left[x_{i-1}^{(n)}, x_i^{(n)}\right]$ for all $1 \leq i \leq m_n$. Define the sequence y_n as follows:

$$y_n = S(f,p)(\xi^{(n)})$$

If $\lim_{n\to\infty} \|P_n\| = 0$, then

$$\lim_{n\to\infty} y_n = \int_a^b f.$$

8.5
Improper Integrals

Definition 8.5 (improper integral). An improper integral is one such that either the integrand, f, is unbounded on (a,b) or the interval of integration is unbounded.

Proposition 8.4. Suppose f is defined on $[a,b)$ and f is integrable on $[a,c]$ for every $c \in (a,b)$. If the limit

$$L = \lim_{c\to b^-} \int_a^c f(x)\, dx$$

exists, then

$$\text{the improper integral } \int_a^b f(x)\, dx \text{ converges} \quad \text{and} \quad \int_a^b f(x)\, dx = L.$$

If the limit does not exist, then the improper integral diverges.

Similarly, if f is defined on $(a,b]$ and f is integrable on $[c,b]$ for every $c \in (a,b)$, then

$$\int_a^b f(x)\, dx = \lim_{c\to a^+} f(x)\, dx \quad \text{provided that the limit exists.}$$

Chapter 9
Sequences and Series of Functions

9.1
Pointwise and Uniform Convergence

An example of a sequence of functions $f_n(x)$, where $n \in \mathbb{N}$ is

$$f_n(x) = \frac{x+x^n}{2+x^n}$$

for $x \in [0,1]$. Then, consider the integral

$$\int_0^{\frac{1}{2}} f_n(x).$$

As $n \to \infty$, what can be deduced?

This section deals with questions like these. To start off, we need to introduce the ideas of pointwise convergence and uniform convergence.

Definition 9.1 (pointwise convergence). Let E be a non-empty subset of $\mathbb{R}$. Suppose for each $n \in \mathbb{N}$, we have a function $f_n : E \to \mathbb{R}$. Then, f_n is a sequence of functions on E. For each $x \in E$, the sequence $f_n(x)$ of real numbers converges. Define the function $f : E \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \text{for all } x \in E.$$

Then, f_n converges to f pointwise on E, and so $f_n \to f$ pointwise on E.

Definition 9.2 (pointwise convergence). $f_n \to f$ pointwise on E if and only if for every $x \in E$ and for every $\varepsilon > 0$, there exists $K = K(\varepsilon, x) \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon.$$

Remark 9.1. If $f_n \to f$ pointwise on I and each f_n is continuous on I, then f is not necessarily continuous on I.

Example 9.1. Consider $f_n(x) = x^n$ for $x \in [0,1]$. Note that each f_n is continuous on $[0,1]$. However, f is not continuous at $x = 1$ since for $x \in [0,1)$, then

$$\lim_{n\to\infty} f_n(x) = 0$$

but for $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = 1.$$

Remark 9.2. If $f_n \to f$ pointwise on $[a,b]$ and each f_n is integrable on $[a,b]$, then

(1). f is not necessarily integrable on $[a,b]$

(2). the pointwise convergence

$$\int_a^b g_n \to \int_a^b g \quad \text{is not necessarily true.}$$

Remark 9.3. If $f_n \to f$ pointwise on $[a,b]$ and each f_n and f are differentiable on $[a,b]$, then $f_n' \to f'$ not necessarily pointwise on $[a,b]$.

Example 9.2. Consider $f_n(x) = \sin(nx)/\sqrt{n}, x \in \mathbb{R}$. $f(x) = 0$ for all $x \in \mathbb{R}$, and thus $f_n \to f$ pointwise on $\mathbb{R}$. As $f'(x) = \sqrt{n}\cos(nx)$, for each $n \in \mathbb{N}$, $f' = 0$, but $f_n' \to f'$ pointwise on $\mathbb{R}$. Then, $f_n'(0) = \sqrt{n} \to \infty$ as $n \to \infty$, but $f'(0) = 0$.

Definition 9.3 (uniform convergence)**.** A sequence of functions f_n converges uniformly to f on E if for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$n \geq K \implies |f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$. In this case, $f_n \to f$ uniformly on E. We say that the sequence f_n of functions converges uniformly on E if there exists a function f such that f_n converges to f uniformly on E.

Definition 9.4 (uniform norm)**.** Let $E \subseteq \mathbb{R}$ and let $\phi : E \to \mathbb{R}$ be a bounded function. The uniform norm of ϕ on E is defined as

$$\|\phi\|_E = \sup\{|\phi(x)| : x \in E\}.$$

Then, $|\phi(x)| \leq \|\phi\|_E$ for all $x \in E$.

Lemma 9.1. A sequence of functions f_n converges to f uniformly on E if and only if $\|f_n - f\|_E \to 0$.

Proposition 9.1 (Cauchy criterion)**.** A sequence of functions f_n converges uniformly on E if and only if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_E < \varepsilon \quad \text{for all } m, n \geq K.$$

Proposition 9.2. The following hold:

(i) If f_n converges uniformly on E, then f_n converges pointwise on E

(ii) If f_n converges uniformly on E and $F \subseteq E$, then f_n converges uniformly on F

Proposition 9.3. If f_n converges uniformly to f on an interval I and each f_n is continuous at $x_0 \in I$, then f is continuous at x_0.

Corollary 9.1. If f_n converges uniformly to f on I and each f_n is continuous on I, then f is continuous on I. Hence,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x),$$

implying that we can interchange the order of the two limit operations.

Proposition 9.4. Suppose $f_n \to f$ uniformly on $[a,b]$ and each f_n is integrable on $[a,b]$. Then,
(i): f is integrable on $[a,b]$ and

(ii): for each $x_0 \in [a,b]$, the sequence of functions

$$F_n(x) = \int_{x_0}^{x} f_n(t)\, dt$$

converges uniformly to the function

$$F(x) = \int_{x_0}^{x} f(t)\, dt$$

on $[a,b]$. Hence,

$$\lim_{n\to\infty}\int_{x_0}^{x} f_n(t)\, dt = \int_{x_0}^{x}\lim_{n\to\infty} f_n(t)\, dt$$

and in particular,

$$\lim_{n\to\infty}\int_{a}^{b} f_n(t)\, dt = \int_{a}^{b} f(t)\, dt.$$

Proposition 9.5. Suppose f_n is a sequence of differentiable functions on $[a,b]$ such that

$$f_n(x_0) \text{ converges for some } x_0 \in [a,b] \quad \text{and} f_n' \text{ converges uniformly on } [a,b].$$

Then, f_n converges uniformly on $[a,b]$ to a differentiable function f and for $a \le x \le b$,

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

9.2 Infinite Series of Functions

If f_n is a sequence of functions on E, then

$$S = \sum_{n=1}^{\infty} f_n \quad \text{is} \quad \text{an infinite series of functions.}$$

For each $n \in \mathbb{N}$ and $x \in E$, the n^{th} partial sum of S is the function

$$S_n(x) = \sum_{i=1}^{n} f_i(x).$$

Proposition 9.6. The following hold:

(i) S converges pointwise to a function S on E if the sequence S_n of functions converges pointwise to S on E

(ii) S converges uniformly to a function S on E if the sequence S_n of functions converges uniformly to S on E

(iii) S converges absolutely on E if the series

$$\sum_{n=1}^{\infty} |f_n| \quad \text{converges pointwise on } E$$

Proposition 9.7 (Cauchy criterion). Let f_n be a sequence of functions on E. Then,

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } E$$

if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\text{for all } n > m \geq K \quad \text{we have} \quad \left\| \sum_{i=m+1}^{n} f_i \right\|_E < E.$$

Corollary 9.2. If

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } E \quad \text{then} \quad f_n \to 0 \text{ uniformly on } E.$$

Theorem 9.1 (Weierstrass M-test). Let f_n be a sequence of functions on E and M_n be a sequence of positive real numbers such that $\|f_n\|_E \leq M_n$ for all $n \in \mathbb{N}$. If

$$\sum_{n=1}^{\infty} M_n \text{ converges} \quad \text{then} \quad \sum_{n=1}^{\infty} f_n \text{ converges uniformly and absolutely on } E.$$

Example 9.3. We can prove that the series expansion of the exponential function can be uniformly convergent on any bounded subset $S \subseteq \mathbb{C}$.

Solution. Let $z \in \mathbb{C}$. Note that the series expansion of the complex exponential function is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Any bounded subset is a subset of some disc D_R of radius R centred on the origin on the complex plane. The Weierstrass M-test requires us to find an upper bound, M_n, on the terms of the series, with M_n independent of the position in the disc. Observe that

$$\left| \frac{z^n}{n!} \right| \leq \frac{|z|^n}{n!} \leq \frac{R^n}{n!}$$

so by setting $M = R^n/n!$, we are done. □

Proposition 9.8. If

$$\sum_{n=1}^{\infty} f_n \to f \quad \text{uniformly on } I$$

and each f_n is continuous on each $x_0 \in I$, then f is continuous at x_0.

We now state some properties related to differentiability and integrability.

Proposition 9.9. If

$$\sum_{n=1}^{\infty} f_n \to f \text{ uniformly on } [a,b] \quad \text{and} \quad \text{each } f_n \text{ is integrable on } [a,b],$$

(1). f is integrable on $[a,b]$

(2). for every $x \in [a,b]$,

$$\sum_{n=1}^{\infty} \int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt = \int_a^x \sum_{n=1}^{\infty} f_n(t)\, dt$$

where the convergence is uniform on $[a,b]$

Proposition 9.10. Suppose f_n is a sequence of differentiable functions on $[a,b]$ such that

$$\sum_{n=1}^{\infty} f_n(x_0) \text{ converges for some } x_0 \in [a,b] \quad \text{and} \quad \sum_{n=1}^{\infty} f_n' \text{ converges uniformly on } [a,b].$$

Then,

$$\sum_{n=1}^{\infty} f_n \text{ converges uniformly on } [a,b] \text{ to a differentiable function } f \quad \text{and}$$

$$\sum_{n=1}^{\infty} f_n'(x) = f'(x) \text{ for } a \leq x \leq b$$

Chapter 10
Power Series

10.1
Introduction

Definition 10.1 (power series). A series of functions of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots$$

where $x_0, a_1, a_2, \ldots$ are constants, is a power series in $x - x_0$. So,

$$\sum_{n=0}^{\infty} (x-x_0)^n = \sum_{n=0}^{\infty} f_n(x)$$

where for each n, $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = a_n(x-x_0)^n$.

If $x_0 = 0$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Proposition 10.1. Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{be} \quad \text{a power series.}$$

(i) If it converges at $x = x_1$, then it is absolutely convergent for all values of x for which $|x - x_0| < |x_1 - x_0|$

(ii) If it diverges for $x = x_2$, then it diverges for all values of x such that $|x - x_0| > |x_2 - x_0|$

10.2
Radius of Convergence

Definition 10.2 (radius of convergence). Given a power series, let

$$S = \left\{ |x - x_0| : x \in \mathbb{R} \quad \text{and} \quad \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges} \right\}$$

The radius of convergence of the series, R, is defined as follows:

(i) $R = 0$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges only for } x = x_0$$

(ii) $R = \infty$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for all } x \in \mathbb{R}$$

(iii) $R = \sup S$ if

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges for some } x \text{ and diverges for others}$$

Example 10.1. The series

$$\sum_{n=0}^{\infty} n!x^n$$

converges only at $x = 0$, implying that $R = 0$.

Example 10.2. The exponential function e^x converges at every point of $\mathbb{R}$, and so $R = \infty$.

Example 10.3. Consider the geometric series $1+x+x^2+x^3+\ldots$, which converges for all $x \in \mathbb{R}$ and diverges for all oher x's. Hence, $R = 1$.

Definition 10.3 (absolute convergence).

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges absolutely}$$

for all $x \in (x_0 - R, x_0 + R)$ and diverges for all x with $|x - x_0| > R$.

Theorem 10.1 (ratio test). Suppose a_n is non-zero for all n. Let

$$\rho = \lim_{n=0} \left| \frac{a_{n+1}}{a_n} \right|.$$

(i) If the limit ρ exists, then the radius of convergence, R, of the power series is

$$R = \begin{cases} 1/\rho & \text{if } \rho > 0; \\ \infty & \text{if } \rho = 0 \end{cases}$$

(ii) If $\rho = \infty$, then $R = 0$

The ratio test is frequently easier to apply than the root test since it is usually easier to evaluate ratios than n^{th} roots. However, the root test is a *stronger* test for convergence. This means that whenever the ratio test shows convergence, the root test does too and whenever the root test is inconclusive, the ratio test is too (merely the contrapositive statement).

For any sequence x_n of positive numbers,

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \liminf_{n\to\infty} \sqrt[n]{x_n} \text{ and } \limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Theorem 10.2 (Cauchy-Hadamard formula). Let

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ be a power series} \quad \text{and} \quad \rho = \limsup |a_n|^{1/n}.$$

The radius of convergence, R, is

$$R = \begin{cases} 0 & \text{if } \rho = \infty; \\ 1/\rho & \text{if } 0 < \rho < \infty; \\ \infty & \text{if } \rho = 0 \end{cases}$$

10.3
Properties of Power Series

Proposition 10.2. Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a radius of convergence $R > 0$. Then, f is infinitely differentiable on $(x_0 - R, x_0 + R)$, i.e.

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \quad \text{where } x \in (x_0 - R, x_0 + R).$$

Proposition 10.3. For every $k \in \mathbb{N}$, we have the following result:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1) a_n (x - x_0)^{n-k} \quad \text{where } x \in (x_0 - R, x_0 + R)$$

and the radius of convergence of each of these derived series is also R.

Although a power series and its derived series have the same values of R, they may converge on different sets.

Example 10.4. Consider the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

By the ratio test, $R = 1$, so the series converges in $(-1, 1)$. The series also converges at $x = \pm 1$. In fact, when $x = 1$, we obtain the famous p-series for which $p = 2$, and it is also known as the Basel problem. When $x = -1$, we obtain a variant of the Basel problem which can be evaluated as well. Hence, the series converges in $[-1, 1]$.

Differentiating both sides of the power series gives

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n},$$

where $x \in (-1, 1)$. $f'(x)$ converges at $x = -1$ but diverges at $x = 1$, which is the harmonic series. Hence, $f'(x)$ converges on $[-1, 1)$.

Corollary 10.1. If there exists $r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{for all } x \in (x_0 - r, x_0 + r),$$

then

$$a_k = \frac{f^{(k)}(x_0)}{k!} \quad \text{for all } k \in \mathbb{Z}_{\geq 0}.$$

Corollary 10.2 (uniqueness of power series)**.** If

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

for all $x \in (x_0 - r, x_0 + r)$ for some $r > 0$, then $a_n = b_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Corollary 10.3. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{have} \quad \text{a non-zero radius of convergence } R.$$

Then, for any a and b for which $x_0 - R < a < b < x_0 + R$,

$$\int_a^b f(x)\,dx = \int_a^b \sum_{n=0}^{\infty} a_n (x - x_0)^n\,dx = \sum_{n=0}^{\infty} \int_a^b a_n (x - x_0)^n\,dx.$$

In other words, a power series can be integrated term-by-term over any closed interval $[a, b]$ contained in the interval of convergence.

Theorem 10.3 (Abel summation formula)**.** Let b_n and c_n be sequences of real numbers, and for each pair of integers $n \geq m \geq 1$, set

$$B_{n,m} = \sum_{k=m}^{n} b_k.$$

Then,

$$\sum_{k=m}^{n} b_k c_k = B_{n,m} c_n - \sum_{k=m}^{n-1} B_{k,m}(c_{k+1} - c_k).$$

for all $n > m \geq 1$, $n, m \in \mathbb{N}$.

Theorem 10.4 (Abel's theorem)**.** Suppose

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{has} \quad \text{a finite non-zero radius of convergence } R.$$

(i) If the series converges at $x = x_0 + R$, then it converges uniformly on $[x_0, x_0 + R]$

(ii) If the series converges at $x = x_0 - R$, then it converges uniformly on $[x_0 - R, x_0]$

10.4 Taylor Series

A function f is infinitely differentiable on (a, b) if $f^{(n)}(x)$ exists for all $x \in (a, b)$ and for all $n \in \mathbb{N}$. This class of functions is denoted by $\mathcal{C}^{\infty}$.

Definition 10.4 (Taylor series)**.** Let f be infinitely differentiable on $(x_0 - r, x_0 + r)$ for some $r > 0$. The power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad \text{is} \quad \text{the Taylor series of } f \text{ about } x_0.$$

Definition 10.5 (Taylor series)**.** Considering the Taylor series, set $x_0 = 0$. We then obtain the Maclaurin Series of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Definition 10.6 (analytic function). A function f is analytic on (a,b) if f is infinitely differentiable on (a,b) and for any $x_0 \in (a,b)$, the Taylor Series of f about x_0 converges to f in a neighbourhood of x_0.

Example 10.5. The functions e^x, $\sin x$ and $\cos x$ are analytic on $\mathbb{R}$ and the infinite geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

is analytic on $(-1,1)$.

10.5 Arithmetic Operations with Power Series

Definition 10.7 (Cauchy product). The Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where for each $n \in \mathbb{N}$,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0.$$

Proposition 10.4. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n,\ |x-x_0| < R_1 \text{ and } g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n,\ |x-x_0| < R_2.$$

For $\alpha, \beta \in \mathbb{R}$, we have the following:

(i)

$$\alpha f(x) + \beta g(x) = \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n \text{ for } |x-x_0| < \min(R_1, R_2)$$

(ii)

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n,\ |x-x_0| < \min(R_1,R_2) \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Theorem 10.5 (Merten's theorem). If

$$\sum_{n=0}^{\infty} a_n \text{ converges absolutely and } \sum_{n=0}^{\infty} b_n \text{ converges} \quad \text{then} \quad \text{the Cauchy product } \sum_{n=0}^{\infty} c_n \text{ converges.}$$

Also,

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right).$$

Recall that for Merten's theorem, we just need at least one of the series to converge absolutely.

Definition 10.8 (conditional convergence). A series is conditionally convergent if it converges but does not converge absolutely.

Remark 10.1. If

$$\sum_{n=0}^{\infty} a_n \text{ and } \sum_{n=0}^{\infty} b_n \quad \text{converge conditionally,}$$

then their Cauchy product may not converge.

Example 10.6. Set

$$a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n+1}},$$

where $n \geq 0$. It is clear that both series are conditionally convergent (but not absolutely convergent) by the alternating series test. The Cauchy product of these two series is

$$c_n = \sum_{k=0}^{\infty} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}} \quad \text{for all } n \in \mathbb{N}.$$

Note that $n \geq k$ so $n+1 \geq k+1$ and $n+1 \geq n-k+1$ so we are able to obtain a lower bound for $|c_n|$. Hence,

$$|c_n| \geq \sum_{k=0}^{n} \frac{1}{n+1} = 1 \quad \text{which implies} \quad \sum_{n=0}^{\infty} c_n \text{ diverges.}$$

Theorem 10.6 (Riemann rearrangement theorem)**.** Suppose a_n is a sequence of real numbers, and that

$$\sum_{n=1}^{\infty} a_n \quad \text{is} \quad \text{conditionally convergent.}$$

Let $M \in \mathbb{R}$. Then, there exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

There also exists a permutation σ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \infty.$$

The sum can also be rearranged to diverge to $-\infty$ or to fail to approach any limit, finite or infinite.

10.6 Some Special Functions

Definition 10.9 (exponential function)**.** The function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

for all $x \in \mathbb{R}$, is the exponential function.

Proposition 10.5. $E : \mathbb{R} \to \mathbb{R}$ has the following properties:

(i) $E(0) = 1$ and $E'(x) = E(x)$ for all $x \in \mathbb{R}$

(ii) $E(x+y) = E(x)E(y)$ for all $x, y \in \mathbb{R}$

(iii) $E(x) > 0$ for all $x \in \mathbb{R}$

(iv) E is strictly increasing (i.e. $E'(x) > 0$ for all $x \in \mathbb{R}$)

(v)

$$\lim_{x \to \infty} E(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} E(x) = 0$$

For **(i)**, any function $f(x)$ that has this property is invariant under successive levels of differentiation. Actually, one can verify that the exponential function is indeed the only function that is invariant under the differential operator by treating the differential equation $f'(x) = f(x)$ as a separable one.

Proposition 10.6. The functional equation

$$f(x+y) = f(x)f(y)$$

holds true only for the exponential function.

Euler's number, e (≈ 2.71828459045) is defined as the following limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Proposition 10.7. By considering the Maclaurin series of e^x, setting $x = 1$ gives the expansion

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

Proposition 10.8. In relation to sequences, for $x \in \mathbb{R}$, e^x is defined as

$$e^x = \lim_{n\to\infty} e^{r_n},$$

where r_n is an increasing rational sequence which converges to x.

Proposition 10.9. For $x \in \mathbb{R}$, e^x is continuous on $\mathbb{R}$.

Since the exponential function E is strictly increasing on $\mathbb{R}$ and $E(\mathbb{R}) = (0,\infty)$, then it implies that E is injective and thus has an inverse function $L : (0,\infty) \to \mathbb{R}$, which is also strictly increasing.

We have the following composition of functions

$$L(E(x)) = x \;\forall x \in \mathbb{R}$$

and

$$E(L(y)) = y \;\forall y > 0.$$

Definition 10.10 (natural logarithm)**.** By the Fundamental Theorem of Calculus, we define $L(y)$ to be the following integral:

$$L(y) = \int_1^y \frac{1}{t}\,dt$$

The function $L : (0,\infty) \to \mathbb{R}$ is the natural logarithm, $\ln(x)$.

Proposition 10.10. The natural logarithm $\ln : (0,\infty) \to \mathbb{R}$ has the following properties:

(i)

$$\frac{d}{dy}\ln y = \frac{1}{y} \quad \text{for all } y > 0$$

(ii)

$$\ln y = \int_1^y \frac{1}{t}\,dt \quad \text{for all } y > 0$$

(iii) $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y, > 0$

(iv) $\ln(1) = 0$ and $\ln(e) = 1$

(v) For $x > 0$ and $\alpha \in \mathbb{R}$, $x^\alpha = e^{\alpha \ln x}$

Proposition 10.11. The functional equation

$$f(xy) = f(x) + f(y)$$

holds true only for the logarithmic function.

Corollary 10.4. Let $\alpha \in \mathbb{R}$. Then, the function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$f(x) = x^\alpha$$

for all $x > 0$ is differentiable on $(0,\infty)$ and

$$f'(x) = \alpha x^{\alpha-1}$$

for all $x > 0$ as well.

Definition 10.11 (cosine)**.** The function

$$C(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

for all $x \in \mathbb{R}$, is the cosine function.

Definition 10.12 (sine)**.** The function

$$S(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all $x \in \mathbb{R}$, is the sine function.

These two trigonometric functions have the following relationship, that for all $x \in \mathbb{R}$,

$$C'(x) = -S(x) \text{ and } S'(x) = C(x).$$

Differentiating both sides of each equation will yield

$$C''(x) = -C(x) \text{ and } S''(x) = -S(x),$$

which are second order linear homogeneous differential equations which constant coefficients.

Thus, we make the claim that if $g : \mathbb{R} \to \mathbb{R}$ has the property that $g''(x) = -g(x)$ for all $x \in \mathbb{R}$, then

$$g(x) = \alpha C(x) + \beta S(x)$$

for all $x \in \mathbb{R}$ too, where $\alpha = g(0)$ and $\beta = g'(0)$. The two functions satisfy the identity $(C(x))^2 + (S(x))^2 = 1$ for all $x \in \mathbb{R}$, which is also known as the Pythagorean identity.

The cosine function is even. That is, $C(-x) = C(x)$ (i.e. the graph is symmetrical about the y-axis). It satisfies the following addition formula:

$$C(x+y) = C(x)C(y) - S(x)S(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The sine function is odd. That is, $S(-x) = -S(x)$ (i.e. the graph is symmetrical about the origin). It satisfies the following addition formula:

$$S(x+y) = S(x)C(y) + C(x)S(y) \quad \text{for all } x, y \in \mathbb{R}.$$

The four other trigonometric functions, as well as all the inverse trigonometric functions, will not be discussed. Moreover, respective small angle approximations will not be discussed too.

Definition 10.13 (gamma function)**.** The gamma function is one commonly used extension of the factorial function to complex numbers. Denoted by $\Gamma(z)$, it is defined to be the following convergent improper integral:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \quad \text{where } \Re(z) > 0.$$

Theorem 10.7. For $z \geq 0$, we have the following relationship:

$$\Gamma(z+1) = z\Gamma(z),$$

which has some semblance to the functional equation $f(x+1) = xf(x)$. Hence,

$$\Gamma(n) = (n-1)! \quad \text{where } n \in \mathbb{N}.$$

Proof. Using integration by parts,

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt = -t^z e^{-t} \Big|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} \, dt = z\Gamma(z)$$

and we are done.

Next, to prove the closed form for $\Gamma(n)$, as $\Gamma(1) = 1$, so

$$\begin{aligned} \prod_{i=1}^{n-1} \frac{\Gamma(i+1)}{\Gamma(i)} &= \prod_{i=1}^{n-1} i \\ \frac{\Gamma(n)}{\Gamma(1)} &= (n-1)! \end{aligned}$$

and the result follows by the telescoping product. □

Theorem 10.8. $\ln \Gamma(z)$ is convex on $(0, \infty)$

Theorem 10.9 (Bohr-Mollerup theorem)**.** The gamma function is the only function satisfying $f(1) = 1$, $f(x+1) = xf(x)$ and f is logarithmically convex.

The Bohr-Mullerup theorem characterises the gamma function.

There are two types of Euler integral. The gamma function is also known as the Euler integral of the first kind and the beta function (discussed in the next section) is also known as the Euler integral of the second kind.

Euler's reflection formula and Legendre's duplication formula are examples of functional equations closely related to the gamma function.

Theorem 10.10 (Euler's reflection formula)**.** For $z \notin \mathbb{Z}$,

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z).$$

Theorem 10.11 (Legendre duplication formula)**.**

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

Definition 10.14 (beta function)**.** For $x, y \in \mathbb{C}$, where $\Re(x) > 0$ and $\Re(y) > 0$, the Beta Function $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

Proposition 10.12. $B(x,y)$ is symmetric.

Proof. Use the substitution $x = y$ so $B(x,y) = B(y,x)$, □

Theorem 10.12. The beta function is closely related to the gamma function and the binomial coefficients by the following equation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

The proof of Theorem 10.12 hinges on writing $\Gamma(x)\Gamma(y)$ as a double integral and using the technique of change of variables.

Chapter 11
Functions of Several Variables