MA4211 Functional Analysis

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Reference books:

(1) Kreyszig, E. (1989). '*Introductory Functional Analysis with Applications*'. Wiley.

Contents

1. Metric Spaces

1.1. *Introduction*

Functional Analysis is essentially the study of infinite-dimensional Linear Algebra.

Example 1.1 (Euclidean metric/distance). Recall the familiar metric in Euclidean space $\mathbb{R}$

$$d(x,y) = |x-y|.$$

We call this the Euclidean metric or Euclidean distance. Naturally, we can extend this to the Euclidean 2-space $\mathbb{R}^2$. Consider $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Then,

$$d(\mathbf{x},\mathbf{y}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We give the definition of a metric space (Definition 1.1).

Definition 1.1 (metric space). Let X be a set. A metric space is an ordered pair (X,d) equipped with a distance function $d : X \times X \to \mathbb{R}$ such that the following properties are satisifed:

- **(i) Non-negativity:** $d(x,y) \geq 0$
- **(ii) Positive-definiteness:** $x = y$ if and only if $d(x,y) = 0$
- **(iii) Symmetry:** for all $x, y \in SX$, we have $d(x,y) = d(y,x)$
- **(iv) Triangle inequality:** for all x, y, z, we have $d(x,z) \leq d(x,y) + d(y,z)$

Definition 1.2 ($\mathbb{R}^\infty$). Define $\mathbb{R}^\infty$ to be the space of all infinite sequences of real numbers, i.e. $(x_1, x_2, \ldots)$ where $x_1, x_2, \ldots \in \mathbb{R}$.

Example 1.2 ($\mathbb{R}^\infty$). We have the infinite sequences $(0,0,\ldots,)$ and $(1,2,3,\ldots,100,\ldots)$ in $\mathbb{R}^\infty$.

Example 1.3. For $X = \mathbb{R}$, we can define

$$d(x,y) = \min\{|x-y|, 1\} \quad \text{such that it is a metric.}$$

Example 1.4. For $X = \mathbb{R}^\infty$, let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, where each element is in $\mathbb{R}$. Then, one can check that

$$d(\mathbf{x},\mathbf{y}) = \sup d(x_i, y_i) \quad \text{is a metric.}$$

Example 1.5. We give an introduction to the sequence space l^∞. This example gives one an impression of how surprisingly general the concept of a metric space is. We can define

$$X = \{\text{bounded sequences of complex numbers}\}.$$

So, every element of X is a complex sequence ξ_j such that for all $j = 1, 2, \ldots$, we have

$$|\xi_j| \leq c_x \quad \text{where} \quad c_x \text{ is a real number which may depend on } x.$$

Then, the following is a metric:

$$d(\mathbf{x},\mathbf{y}) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \quad \text{where} \quad y = (\eta_1, \eta_2, \ldots) \in X$$

Definition 1.3 (function space)**.** Let

$$\mathcal{C}[a,b] \quad \text{denote} \quad \text{the set of continuous functions on } [a,b].$$

Example 1.6 (function space)**.** Let $X = \mathcal{C}[a,b]$, for which we recall that this refers to the set of continuous functions on $[a,b]$. Let $f,g \in \mathcal{C}[a,b]$. Then,

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)| \quad \text{and} \quad d(f,g) = \sqrt{\int_0^L |f(x) - g(x)|^2 \, dx} \quad \text{are metrics.}$$

Example 1.7 (Hamming distance)**.** Consider the two English words 'word' and 'wind' of the same length for which the second and third letters differ. Since two letters differ, we say that their Hamming distance is 2. We write

$$d(\text{wind}, \text{word}) = 2.$$

In this case, d is a metric. The reader can read Kreyszig p. 9 Question 10 to prove that the Hamming distance is indeed a metric.

Definition 1.4 (l^p-space)**.** Let $p \geq 1$ be a fixed real number. Each element in the space l^p is a sequence $(x_1, \ldots)$ such that $|x_1|^p + \ldots$ converges. So,

$$l^p = \left\{ \mathbf{x} \in \mathbb{R}^\infty : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

Definition 1.5 (p-norm)**.** Every element in l^p-space is equipped with a norm, known as the p-norm. We define it as follows (will not be strict with the use of either x or $\mathbf{x}$):

$$\text{if } x \in l^p \quad \text{then} \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

Theorem 1.1 (Young's inequality)**.** Suppose $\alpha, \beta > 0$. Then,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Example 1.8. Let $t = 1/p$. Then,

$$\begin{aligned}
\ln\left(t\alpha^p + (1-t)\beta^q\right) &\geq t\ln(\alpha^p) + (1-t)\ln(\beta^q) \quad \text{since } \ln \text{ is concave down} \\
&= \frac{1}{p}\ln(\alpha^p) + \frac{1}{q}\ln(\beta^q) \\
&= \ln\alpha + \ln\beta \\
&= \ln\alpha\beta
\end{aligned}$$

Taking exponentials on both sides yields the desired result.

Theorem 1.2 (Hölder's inequality)**.** We have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q .$$

Theorem 1.3 (Minkowski's inequality)**.** We have

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

Please refer to my MA4262 notes for proofs of Theorems 1.2 and 1.3.