MA4211 Functional Analysis

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Reference books:

(1) Kreyszig, E. (1989). '*Introductory Functional Analysis with Applications*'. Wiley.

(2) Sasane, A. (2017). '*A Friendly Approach to Functional Analysis*'. World Scientific.

Contents

1. Metric Spaces

1.1. *Metric Spaces*

Functional Analysis is essentially the study of infinite-dimensional Linear Algebra.

Example 1.1 (Euclidean metric/distance). Recall the familiar metric in Euclidean space $\mathbb{R}$

$$d(x,y) = |x-y|.$$

We call this the Euclidean metric or Euclidean distance. Naturally, we can extend this to the Euclidean 2-space $\mathbb{R}^2$. Consider $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in $\mathbb{R}^2$. Then,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We give the definition of a metric space (Definition 1.1).

Definition 1.1 (metric space). Let X be a set. A metric space is an ordered pair (X,d) equipped with a distance function $d : X \times X \to \mathbb{R}$ such that the following properties are satisifed:

(i) Non-negativity: $d(x,y) \geq 0$

(ii) Positive-definiteness: $x = y$ if and only if $d(x,y) = 0$

(iii) Symmetry: for all $x, y \in X$, we have $d(x,y) = d(y,x)$

(iv) Triangle inequality: for all x,y,z, we have $d(x,z) \leq d(x,y) + d(y,z)$

Example 1.2 (MA4211 AY24/25 Sem 2 Tutorial 1). Given two non-empty subsets A, B of a metric space (X,d), their distance is defined as

$$D(A,B) = \inf_{a \in A, b \in B} d(a,b).$$

Consider the power set of X and the function D. Which of the axioms of a metric space does this pair satisfy?

Solution. We first claim that non-negativity is satisfied. For $A, B \subseteq X$, we have

$$d(a,b) \geq 0 \quad \text{which implies} \quad \inf_{a \in A, b \in B} d(a,b) \geq 0.$$

Next, symmetry is satisfied since

$$D(B,A) = \inf_{a \in A, b \in B} d(b,a) = \inf_{a \in A, b \in B} d(a,b)$$

which follows from the fact that d satisfies symmetry. Next, the triangle inequality is satisfied. To see why, let $A, B, C \subseteq X$. Then,

$$\begin{aligned} D(A,C) &= \inf_{a \in A, c \in C} d(a,c) \\ &\leq \inf_{a \in A, b \in B} d(a,b) + \inf_{b \in B, c \in C} d(b,c) \\ &= D(A,B) + D(B,C) \end{aligned}$$

We claim that the property $D(A,B) = 0$ if and only if $A = B$ is not satisfied. Recall from Definition 1.1 that this is called positive-definiteness. Anyway, to see why, let $A = \{x\}$ and $B = \{x_n\}_{n=1}^{\infty}$, where $x_n \to x$. Then, $D(A,B) = 0$ since x_n can be arbitrarily close to x for large n. However, $A \neq B$. $\square$

Definition 1.2 ($\mathbb{R}^\infty$)**.** Define $\mathbb{R}^\infty$ to be the space of all infinite sequences of real numbers, i.e. $(x_1, x_2, \ldots)$ where $x_1, x_2, \ldots \in \mathbb{R}$.

Example 1.3 ($\mathbb{R}^\infty$)**.** We have the infinite sequences $(0, 0, \ldots,)$ and $(1, 2, 3, \ldots, 100, \ldots)$ in $\mathbb{R}^\infty$.

Example 1.4. For $X = \mathbb{R}$, we can define

$$d(x,y) = \min\{|x-y|, 1\} \quad \text{such that it is a metric.}$$

Example 1.5. For $X = \mathbb{R}^\infty$, let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, where each element is in $\mathbb{R}$. Then, one can check that

$$d(\mathbf{x}, \mathbf{y}) = \sup d(x_i, y_i) \quad \text{is a metric.}$$

Example 1.6 (ℓ^∞)**.** We give an introduction to the sequence space ℓ^∞. This example gives one an impression of how surprisingly general the concept of a metric space is. We can define

$$X = \{\text{bounded sequences of complex numbers}\}.$$

So, every element of X is a complex sequence ξ_j such that for all $j = 1, 2, \ldots$, we have

$$|\xi_j| \leq c_x \quad \text{where} \quad c_x \text{ is a real number which may depend on } x.$$

Then, the following is a metric:

$$d(\mathbf{x}, \mathbf{y}) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \quad \text{where} \quad y = (\eta_1, \eta_2, \ldots) \in X$$

Definition 1.3 (function space)**.** A function space is a set of functions holding the properties of a vector space structure, norm, or inner product. In particular, it has either of the following properties:

- **Vector space structure:**
  - **Closure under addition:** for any $f, g \in \mathcal{C}[a,b]$, we have $f + g \in \mathcal{C}[a,b]$
  - **Closure under scalar multiplication:** for any $k \in \mathbb{R}$, $kf \in \mathcal{C}[a,b]$ for $f \in \mathcal{C}[a,b]$
- **Norm:** A norm $\|\cdot\|$ is a function $\|\cdot\| : \mathcal{C}[a,b] \to \mathbb{R}$ that satisfies the following:
  - **Non-negativity:** $\|f\| \geq 0$ for all $f \in \mathcal{C}[a,b]$, and $\|f\| = 0$ if and only if $f = 0$
  - **Scalar multiplication:** $\|kf\| = |k|\|f\|$ for any $k \in \mathbb{R}$ and $f \in \mathcal{C}[a,b]$
  - **Triangle inequality:** $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in \mathcal{C}[a,b]$
- **Inner product:** An inner product $\langle \cdot, \cdot \rangle$ is a function $\langle \cdot, \cdot \rangle : \mathcal{C}[a,b] \times \mathcal{C}[a,b] \to \mathbb{R}$ that satisfies the following:
  - **Conjugate symmetry:** $\langle f, g \rangle = \langle g, f \rangle$
  - **Linearity in the first argument:** $\langle kf + g, h \rangle = k\langle f, h \rangle + \langle g, h \rangle$ for any $k \in \mathbb{R}$
  - **Positive-definiteness:** $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{C}[a,b]$, and $\langle f, f \rangle = 0$ if and only if $f = 0$

Example 1.7. Let

$$\mathcal{C}[a,b] \quad \text{denote} \quad \text{the set of continuous functions on } [a,b].$$

Example 1.8 (function space)**.** Let $X = \mathcal{C}[a,b]$, for which we recall that this refers to the set of continuous functions on $[a,b]$. Let $f, g \in \mathcal{C}[a,b]$. Then,

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)| \quad \text{and} \quad d(f,g) = \sqrt{\int_0^L |f(x) - g(x)|^2 \, dx} \quad \text{are metrics.}$$

Example 1.9 (Hamming distance). Consider the two English words 'word' and 'wind' of the same length for which the second and third letters differ. Since two letters differ, we say that their Hamming distance is 2. We write

$$d\left(\text{wind}, \text{word}\right) = 2.$$

In this case, d is a metric. The reader can read Kreyszig p. 9 Question 10 to prove that the Hamming distance is indeed a metric.

Example 1.10 (Hamming metric; Kreyszig p. 9 Question 10). Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X defined by

$$d\left(x, y\right) = \text{number of places where } x \text{ and } y \text{ have different entries.}$$

Solution. First, non-negativity clearly holds as it is impossible for two words of the same length to differ by a negative number of letters. So, $d\left(x, y\right) \geq 0$. Symmetry is also obvious.

For positive-definiteness, suppose $x = y$. Then, x and y are the same word. Hence, they are two words of the same length which differ by 0 letters. By definition of d, we have $d\left(x, y\right) = 0$. Similarly, if $d\left(x, y\right) = 0$, then x and y are two words of the same length that are of Hamming distance 0, which implies that they differ by 0 letters. As such, $x = y$.

Lastly, we prove that d satisfies the triangle inequality. Let x, y and z be three words of length n. We can explicitly define d as follows:

$$d\left(x, y\right) = \sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq y_i\}} \quad \text{where} \quad \mathbf{1}_{\{x_i \neq y_i\}} = \begin{cases} 1 & \text{if } x_i \neq y_i; \\ 0 & \text{if } x_i = y_i \end{cases}$$

We note that for each position $1 \leq i \leq n$, the inequality

$$\mathbf{1}_{\{x_i \neq z_i\}} \leq \mathbf{1}_{\{x_i \neq y_i\}} + \mathbf{1}_{\{y_i \neq z_i\}}.$$

To see why, if $x_i = z_i$, then $\mathbf{1}_{\{x_i \neq z_i\}} = 0$ and the inequality holds trivially since $\mathbf{1}_{\{x_i \neq y_i\}}$ and $\mathbf{1}_{\{y_i \neq z_i\}}$ are non-negative. If $x_i \neq z_i$, then either $x_i \neq y_i$, or $y_i \neq z_i$, or both. Hence, at least one of $\mathbf{1}_{\{x_i \neq y_i\}}$ or $\mathbf{1}_{\{y_i \neq z_i\}}$ is 1, and the inequality holds. Hence,

$$\sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq z_i\}} \leq \sum_{i=1}^{n} \mathbf{1}_{\{x_i \neq y_i\}} + \sum_{i=1}^{n} \mathbf{1}_{\{y_i \neq z_i\}} \quad \text{or equivalently} \quad d\left(x, z\right) \leq d\left(x, y\right) + d\left(y, z\right).$$

so the triangle inequality is satisfied. □

Definition 1.4 (ℓ^p-space). Let $p \geq 1$ be a fixed real number. Each element in the space ℓ^p is a sequence $(x_1, \ldots)$ such that $|x_1|^p + \ldots$ converges. So,

$$\ell^p = \left\{ \mathbf{x} \in \mathbb{R}^\infty : \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

Definition 1.5 (p-norm). Every element in ℓ^p-space is equipped with a norm, known as the p-norm. We define it as follows (will not be strict with the use of either x or $\mathbf{x}$):

$$\text{if } x \in \ell^p \quad \text{then} \quad \|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

Example 1.11 (MA4211 AY24/25 Sem 2 Tutorial 1). Give an example of a sequence x_n such that

$$x_n \to 0 \quad \text{but} \quad x_n \notin \ell^p \text{ for any } 1 \le p < \infty.$$

Solution. Consider

$$x_n = \frac{1}{\log(n+1)} \quad \text{for which} \quad \lim_{n\to\infty} x_n = 0.$$

Then, we have

$$\|x_n\|_p = \frac{1}{[\log(n+1)]^p} \quad \text{so} \quad \sum_{n=1}^{\infty} \|x_n\|_p = \sum_{n=1}^{\infty} \frac{1}{[\log(n+1)]^p} \text{ diverges.}$$

□

Definition 1.6 (L^p). Define the L^p space as follows:

$$L^p[a,b] = \left\{ f : [a,b] \to \mathbb{R} \quad \text{such that} \quad \int_a^b |f(x)|^p \, dx < \infty \right\}$$

Theorem 1.1 (Young's inequality). Suppose $\alpha, \beta > 0$. Then,

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Example 1.12. Let $t = 1/p$. Then,

$$\begin{aligned} \ln(t\alpha^p + (1-t)\beta^q) &\ge t\ln(\alpha^p) + (1-t)\ln(\beta^q) \quad \text{since } \ln \text{ is concave down} \\ &= \frac{1}{p}\ln(\alpha^p) + \frac{1}{q}\ln(\beta^q) \\ &= \ln\alpha + \ln\beta \\ &= \ln\alpha\beta \end{aligned}$$

Taking exponentials on both sides yields the desired result.

Theorem 1.2 (Hölder's inequality). We have

$$\sum_{i=1}^{\infty} |x_i y_i| \le \|x\|_p \|y\|_q.$$

Theorem 1.3 (Minkowski's inequality). We have

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Please refer to my MA4262 notes for proofs of Theorems 1.2 and 1.3.

Example 1.13 (MA4211 AY24/25 Sem 2 Tutorial 1). Prove the reverse triangle inequality

$$|d(x,z)-d(z,y)| \leq d(x,y) \quad \text{for all } x,y,z \text{ in a metric space } (X,d).$$

Solution. Recall the triangle inequality which states that

$$d(x,z) \leq d(x,y)+d(y,z) \quad \text{for } x,y,z \text{ in a metric space } (X,d).$$

So,

$$d(x,z)-d(z,y) \leq d(x,y) \quad \text{where we used the fact that the metric } d \text{ is symmetric.}$$

If $d(x,z)-d(z,y) \geq 0$, then the result follows by taking absolute value; otherwise, we now consider the case where $d(x,z) < d(z,y)$. So,

$$d(x,y) \geq d(y,z)-d(x,z) > 0.$$

Taking absolute value again yields the desired result. □

1.2. *Convergence, Completeness, and Topology*

Definition 1.7 (convergence of sequence). Let x_n be a sequence in $\mathbb{R}$. We say that

$$x_n \to x \quad \text{if} \quad \text{there exists } x \in \mathbb{R} \text{ such that } \lim_{n\to\infty} d(x_n,x) = 0.$$

One should recall from MA2108 that this is equivalent to saying that

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } n \geq N \text{ we have } d(x_n,x) < \varepsilon.$$

Definition 1.8 (continuous function). Let (X,d_X) and (Y,d_Y) be metric spaces. We say that f is continuous at $x_0 \in X$ if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that } d_X(x,x_0) < \delta \text{ implies } d_Y(f(x),f(x_0)) < \varepsilon.$$

Theorem 1.4. Let (X,d_X) and (Y,d_Y) be metric spaces and $T : X \to Y$ be a map. Then,

$$T \text{ is continuous} \quad \text{if and only if} \quad x_n \to x \text{ implies } T(x_n) \to T(x).$$

Definition 1.9 (isometry). Let (X,d_X) and (Y,d_Y) be metric spaces and $T : X \to Y$. We say that

$$T \text{ is an isometry} \quad \text{if} \quad d_X(x_1,x_2) = d_Y(T(x_1),T(x_2)).$$

Definition 1.10 (isometric spaces). If there exists a bijective isometry T between two metric spaces X and Y, then we say that X and Y are isometric.

Definition 1.11 (open and closed intervals). Let

$$(a,b) \text{ and } [a,b] \quad \text{denote} \quad \text{the open interval and closed interval in } \mathbb{R} \text{ respectively.}$$

Definition 1.12 (open and closed balls). Define

$$B(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\} \quad \text{denote} \quad \text{the open ball in } \mathbb{R}^n$$
$$\overline{B}(x,r) = \{y \in \mathbb{R}^n : d(x,y) \leq r\} \quad \text{denote} \quad \text{the closed ball in } \mathbb{R}^n$$

Here, each ball is centred at x and is of radius r.

Definition 1.13 (open set). A subset $A \subseteq X$ is open if

$$\text{for all } x \in A \text{ there exists } r > 0 \quad \text{such that} \quad B(x,r) \subseteq A.$$

Definition 1.14 (closed set). A subset of a metric space $S \subseteq X$ is closed if

$$\text{its complement } S^c \quad \text{is open.}$$

Definition 1.15 (topology). Given a set X, a topology $\mathcal{T}$ on X is a collection of subsets of X satisfying the following properties:

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under arbitrary unions

(iii) $\mathcal{T}$ is closed under finite intersection

Definition 1.16 (limit point). In a topological space, a point x is a limit point of a sequence x_n if for every neighbourhood of x, there exists $N \in \mathbb{N}$ such that for all $n \geq n$, x_n belongs to that neighbourhood.

Definition 1.17 (continuous function). If $f : X \to Y$ is a function between two topological spaces X and Y, then f is continuous if the pre-image of every open set in Y is open in X.

Definition 1.18 (closure). Let X be a topological space. For $A \subseteq X$, the closure of A, denoted by $\mathrm{cl}(A)$ or $\overline{A}$, is defined as follows:

$$\overline{A} = A \cup \{\text{limit points of } A\}$$

Definition 1.19 (dense set). Let X be a topological space. If $D \subseteq X$ such that

$$\overline{D} = X \quad \text{then} \quad D \text{ is dense in } X.$$

Definition 1.20 (separable space). A topological space is separable if it has a countable dense subset.

Theorem 1.5. Let $1 \leq p < \infty$. Then, ℓ^p is separable.

Proof. Define $X \subseteq \ell^p$ to be the collection of sequences of the form

$$(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \quad \text{where } x_i \in \mathbb{Q}.$$

As X is a countable union of countable sets, X is countable. Let $y \in \ell^p$ and $\varepsilon > 0$ be arbitrary. That is,

$$\sum_{i=1}^{\infty} |y_i|^p < \infty.$$

Also, there exists $n \in \mathbb{N}$ such that

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon^p}{2}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $x \in X$ such that

$$\sum_{i=1}^{n} |y_i - x_i|^p < \frac{\varepsilon^p}{2}.$$

Then,

$$\|y - x\|_p^p = \sum_{i=1}^{n} |y_i - x_i|^p + \sum_{i=n+1}^{\infty} |y_i|^p < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p.$$

Taking the p^{th} root, we obtain $d(x,y) = \|x - y\|_p < \varepsilon$. □

Theorem 1.6. ℓ^∞ is not separable.

Proof. Let $y \in \mathbb{R}^\infty$ be a sequence of 0s and 1s. Define $z \in \mathbb{R}$ to be as follows:

$$z = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \ldots \quad \text{so} \quad \text{we infer that } 0 \le z \le 1.$$

Because there are uncountably many real numbers in $[0,1]$, it follows there are uncountably many distinct sequences $y \in \{0,1\}^{\mathbb{N}}$. Denote this uncountable family by $\mathcal{Y} \subset \ell^\infty$. Note that for any two distinct sequences

$$y = (y_1, y_2, y_3, \ldots) \quad y' = (y'_1, y'_2, y'_3, \ldots) \quad \in \mathcal{Y},$$

there is at least one index i such that $y_i \neq y'_i$. Because each coordinate is either 0 or 1, at index i, we have$|y_i - y'_i| = 1$. Hence,

$$\|y - y'\|_\infty = \sup_{n \in \mathbb{N}} |y_n - y'_n| \ge 1.$$

In fact, it is exactly 1 if the two sequences differ in at least one place (and cannot exceed 1 because each coordinate difference is 0 or 1). Around each $y \in \mathcal{Y}$, consider the open ball $B(y, 1/3)$ of radius $1/3$. Since any two distinct y, y' are at distance $\|y - y'\|_\infty = 1$, their balls $B(y, 1/3)$ and $B(y', 1/3)$ cannot overlap. In other words, these balls are pairwise disjoint.

Suppose on the contrary that $D \subseteq \ell^\infty$ is a countable dense set. Then, for each $y \in \mathcal{Y}$, $B(y, 1/3)$ must contain at least one point of D. However, there are uncountably many such disjoint balls $B(y, 1/3)$ since $\mathcal{Y}$ is uncountable. A single countable set D cannot meet each of these uncountably many disjoint balls in a distinct point. This leads to a contradiction. □

Definition 1.21 (Cauchy sequence). A sequence x_n in a metric space X is Cauchy if

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } m, n \ge N \text{ we have } d(x_m, x_n) < \varepsilon.$$

Proposition 1.1. Every convergent sequence is Cauchy.

Remark 1.1. The converse of Proposition 1.1 is not true, i.e. not every Cauchy sequence is convergent.

Definition 1.22 (complete metric space). A metric space is called complete if every Cauchy sequence in the space converges.

Example 1.14. ℓ^p and ℓ^∞ are complete.

Example 1.15. The space of continuous functions on $[a,b]$

$$\mathcal{C}[a,b] \quad \text{equipped with the norm} \quad d_\infty(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| \quad \text{is complete.}$$

However, for any $1 \le p < \infty$,

$$\mathcal{C}[a,b] \quad \text{equipped with the norm} \quad d_p(f,g) = \left(\int_a^b |f(x)-g(x)|^p \, dx\right)^{1/p} \quad \text{is not complete.}$$

Having said that, the completion of this is denoted by $L^p[a,b]$.

Example 1.16 (MA4211 AY24/25 Sem 2 Tutorial 1). Show that the limits of Cauchy sequences in a complete metric space are unique.

Solution. Let x_n be a Cauchy sequence in a complete metric space. Suppose x_n has two limits x and y. Then, for all $\varepsilon > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\text{for all } n \ge N_1 \text{ we have } |x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad \text{for all } n \ge N_2 \text{ we have } |x_n - y| < \frac{\varepsilon}{2}.$$

By the triangle inequality,

$$|x-y| \le |x_n - x| + |x_n - y| < \varepsilon.$$

Since ε can be made arbitrarily small, then $x = y$, i.e. the limits of the Cauchy sequence are the same. $\square$

Theorem 1.7 (completion of metric space). Every metric space can be completed, and the completion is unique up to isometry.

We first state a rough proof sketch of Theorem 1.7 before formally proving it. Given a metric space (X,d), we can construct a new metric space $\left(\widetilde{X}, \widetilde{d}\right)$. Then, there exists $W \subseteq \widetilde{X}$ such that X is isometric to W and $\overline{W} = \widetilde{X}$. So, $\widetilde{X}$ is complete and $\widetilde{X}$ is unique up to isometry.

Proof. Let X be a metric space. Suppose x_n and y_n be Cauchy sequences of X. We say that

$$x_n \sim y_n \quad \text{if and only if} \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Define $\widetilde{X}$ to be the set of equivalence clases determined by $\sim$. We write $x_n \in \widetilde{X}$. Then,

$$\widetilde{d}(\widetilde{x}, \widetilde{y}) = \lim_{n\to\infty} d(x_n, y_n) \quad \text{where} \quad x_n, y_n \in X.$$

By applying the triangle inequality twice, we obtain

$$\begin{aligned} d(x_n, y_n) &\le d\left(x_n, x_n'\right) + d\left(x_n', y_n\right) \\ &\le d\left(x_n, x_n'\right) + d\left(x_n', y_n'\right) + d\left(y_n', y_n\right) \end{aligned}$$

Also, by applying the triangle inequality, we have

$$d\left(x_n', y_n'\right) \le d\left(x_n', x_n\right) + d\left(x_n, y_n\right) + d\left(y_n, y_n'\right).$$

We claim that $d\left(x_n, y_n\right)$ is a Cauchy sequence. To see why, let $\varepsilon > 0$ be arbitrary. Recall that x_n and y_n are both Cauchy sequences. So, there exists $N \in \mathbb{N}$ such that for all $m, n \ge N$, we have

$$d\left(x_n, x_m\right) < \varepsilon \quad \text{and} \quad d\left(y_n, y_m\right) < \varepsilon.$$

Then,

$$\begin{aligned} d\left(x_n, y_n\right) &\le d\left(x_n, x_m\right) + d\left(x_m, y_n\right) \\ &\le d\left(x_n, x_m\right) + d\left(x_m, y_m\right) + d\left(y_m, y_n\right) \end{aligned}$$

Let $x \in X$ and consider the infinite sequence $(x, x, \ldots)$. This sequence belongs to some element of $\widetilde{X}$, so we call it $\widetilde{x}$. Define $\varphi : X \to \widetilde{X}$ via $\varphi(x) = \widetilde{x}$. We also define $W = \varphi(X)$. Then,

$$\widetilde{d}\left(\varphi(x), \varphi(y)\right) = \widetilde{d}\left(\widetilde{x}, \widetilde{y}\right) = \lim_{n\to\infty} d\left(x_n, y_n\right) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Let $\widetilde{x} \in \widetilde{X}$ and consider $x_n \in \widetilde{x}$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d\left(x_n, x_N\right) < \frac{\varepsilon}{2} \quad \text{for all } n \ge N.$$

Now, consider the sequence $(x_N, x_N, \ldots)$, so $\widetilde{x}_N \in W$. As such,

$$\widetilde{d}\left(\widetilde{x}, \widetilde{x}_N\right) = \lim_{n\to\infty} d\left(x_n, x_N\right) \le \frac{\varepsilon}{2} < \varepsilon.$$

This shows that W is dense in $\widetilde{X}$.

Next, let $\widetilde{x}_n$ be a Cauchy sequence of elements in $\widetilde{X}$. Since W is dense in $\widetilde{X}$, for each $n \in \mathbb{N}$, there exists $\widetilde{w}_n \in W$ such that $\widetilde{d}\left(\widetilde{x}_n, \widetilde{w}_n\right) < 1/n$. Hence,

$$\begin{aligned} \widetilde{d}\left(\widetilde{w}_m, \widetilde{w}_n\right) &\le \widetilde{d}\left(\widetilde{w}_m, \widetilde{x}_m\right) + \widetilde{d}\left(\widetilde{x}_m, \widetilde{x}_n\right) + \widetilde{d}\left(\widetilde{x}_n, \widetilde{w}_n\right) \\ &< \frac{1}{m} + \widetilde{d}\left(\widetilde{x}_m, \widetilde{x}_n\right) + \frac{1}{n} \end{aligned}$$

Hence, $\widetilde{w}_n$ is Cauchy. For each $\widetilde{w}_n$, define $w_n \in X$ by $w_n = \varphi^{-1}\left(\widetilde{w}_n\right)$. Since φ is an isometry, then w_n is Cauchy. Let $w_n \in \widetilde{X}$. Then,

$$\begin{aligned} \widetilde{d}\left(\widetilde{x}_n, \widetilde{x}\right) &\le \widetilde{d}\left(\widetilde{x}_n, \widetilde{w}_m\right) + d\left(\widetilde{w}_m, \widetilde{x}\right) \\ &< \frac{1}{n} + d\left(\widetilde{w}_m, \widetilde{x}\right) \end{aligned}$$

So,

$$\widetilde{d}\left(\widetilde{w}_m, \widetilde{x}\right) = \lim_{n\to\infty} d\left(w_m, w_n\right).$$

As such, the sequence w_n is Cauchy, which implies

$$\lim_{n\to\infty} \widetilde{d}\left(\widetilde{x}_n, \widetilde{x}\right) = 0 \quad \text{or equivalently} \quad \widetilde{x}_n \to \widetilde{x}.$$

Finally, we need to argue that Y is another complete metric space and there is an isometric embedding $f : X \to Y$ with $f(X)$ dense in Y, then Y is isometric to $\widetilde{X}$. Concretely, each Cauchy sequence x_n in X gives a Cauchy sequence $f\left(x_n\right)$ in Y. Since Y is complete, then $f\left(x_n\right)$ converges to some point in Y. Define a map $\psi : \widetilde{X} \to Y$ by sending $\widetilde{x}$ to the limit of $f\left(x_n\right)$ in Y. ψ is well-defined, isometric and surjective by the density of $f(X)$ in Y. We conclude that $\widetilde{X}$ is unique up to isometry. $\square$

2. Normed Spaces and Banach Spaces

2.1. *Vector Spaces*

Definition 2.1 (vector space). A vector space is a set V together with a field $\mathbb{F}$ equipped with two operations (addition $+$ and multiplication $\cdot$)

$$+ : V \times V \to V \quad \text{and} \quad \cdot : \mathbb{F} \times V \to V$$

satisfying the following properties

- **(i)** $+$ is commutative
- **(ii)** $+$ is associative
- **(iii)** There exists an additive identity
- **(iv)** There exists an additive inverse
- **(v)** $\cdot$ is associative
- **(vi)** There exists a multiplicative identity
- **(vii)** The distributivity properties hold

In this course, we are interested in infinite-dimensional vector spaces. Next, we are interested in continuous functions over $\mathbb{R}$. Recall that the set of continuous functions on $[a,b]$ over $\mathbb{R}$ is denoted by $\mathcal{C}[a,b]$. We define the addition and scalar multiplication functions by the obvious way — that is for any $f,g \in \mathcal{C}[a,b]$, we have

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha \cdot f(x),$$

where $\alpha \in \mathbb{R}$.

Example 2.1. ℓ^p for $1 \le p \le \infty$ is a vector space.

Example 2.2. $\mathbb{R}^\infty$ is a vector space.

Definition 2.2 (linear combination and span). A linear combination is a vector of the form

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n \quad \text{for all } \alpha_i \in \mathbb{F} \text{ and } \mathbf{v}_i \in V.$$

Given a set of vectors S, its span, denoted by $\mathrm{span}(S)$, is the set of all linear combinations of its elements.

Definition 2.3 (linearly independent). A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent if for any $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, we have

$$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \mathbf{0} \quad \text{implies} \quad \alpha_1 = \ldots = \alpha_n = 0.$$

Definition 2.4 (dimension). Let V be a vector space. If there exists $S \subseteq V$ such that $|S| < \infty$ and $\mathrm{span}(S) = V$, then V is finite-dimensional. Otherwise, V is infinite-dimensional.

Definition 2.5 (basis). If $\mathcal{B}$ is an independent spanning set for a finite-dimensional vector space V, then we say that $\dim(V) = |\mathcal{B}|$. Such a set $\mathcal{B}$ is a basis. More formally, we call it a Hamel basis.

In Definition 1.3 on the definition of a function space, we briefly discussed the definition of a norm. We also talked about the p-norm in Definition 1.5. Now, we will formally introduce p-norms.

Definition 2.6 (norm). Given a vector space V, a norm on V is a function $\|\cdot\| : V \to R$ satisfying the following properties for any $\mathbf{u}, \mathbf{v} \in V$:

- **(i) Non-negativity:** $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in V$
- **(ii) Positive-definiteness:** $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = 0$
- **(iii) Homogeneity:** For any $\alpha \in \mathbb{F}$, we have $\|\alpha\mathbf{v}\| = |\alpha| \|\mathbf{v}\|$
- **(iv) Triangle inequality:** $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

It is important to note that a metric space is a space where we wish to measure distance, but a normed space is a space where we wish to measure the Euclidean distance between two vectors. We then discuss some properties of normed spaces.

Definition 2.7. Let $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ be vectors in a normed space and α be a real-valued scalar. Then, we have the following:

- **(i)** $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$, which defines the metric induced by the norm, also known as the norm-induced metric
- **(ii)** We have the familiar p-norm

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

- **(iii)** $d(\mathbf{x} + \mathbf{z}, \mathbf{y} + \mathbf{z}) = d(\mathbf{x}, \mathbf{y})$, which denotes translation invariance
- **(iv)** $d(\alpha\mathbf{x}, \alpha\mathbf{y}) = |\alpha| d(\mathbf{x}, \mathbf{y})$, which denotes the homogeneity of the metric

Definition 2.8 (Banach space). A normed space where the space is complete with respect to the induced metric is called a Banach space.

Theorem 2.1. A subspace Y of a Banach space X is complete if and only if it is closed in X.

Theorem 2.1 can be reformulated as saying that a subspace of a Banach space X is a Banach space if and only if X is closed.

Proof. We first prove the forward direction. Suppose Y is complete. Let $y \in \overline{Y}$. Then, there exists a subsequence $y_n \subseteq Y$ such that

$$\lim_{n \to \infty} y_n = y.$$

So, y_n is Cauchy and in fact, y_n converges in Y. So, Y is closed in X.

For the reverse direction, suppose Y is closed in X. Let y_n be Cauchy in Y. Since $Y \subseteq X$ and X is complete, then

$$\lim_{n \to \infty} y_n = y.$$

So, $y \in \overline{Y} = Y$. □

Definition 2.9 (convergence). If X is a normed space and x_n is a sequence of elements in X, then we can define

$$s_n = \sum_{k=1}^{n} x_k.$$

If there exists $s \in X$ such that $\lim_{n\to\infty} \|s_n - s\| = 0$, then we write

$$s = \sum_{k=1}^{\infty} x_k.$$

Note that if $\sum_{k=1}^{\infty} \|x_k\| < \infty$, the series is said to be absolutely convergent.

Definition 2.10 (Schauder basis). If X is a normed space and e_k is a sequence of elements of x such that for all $x \in X$, there exists a sequence of scalars a_n such that

$$\left\| \sum_{k=1}^{n} a_k e_k - x \right\| = 0 \quad \text{then} \quad \{e_1, \ldots, e_n\} \text{ is a Schauder basis.}$$

In a *slightly more general notion*, a Schauder basis is also called a countable basis.

Definition 2.11 (partial order). A partial order $\leq$ on a set S is a binary relation that satisfies the following properties:

(i) Reflexivity: $a \leq a$

(ii) Transitivity: If $a \leq b$ and $b \leq b$, then $a \leq c$

(iii) Antisymmetry: If $a \leq b$ and $b \leq a$, then $a = b$

Definition 2.12 (total order). A total order relation is a partial order in which every element of the set is comparable with every other element of the set, i.e. if $\leq$ is a partial order on a set S, then

$$\text{for any } x, y \in S \text{ such that either } x \leq y \text{ or } y \leq x \quad \text{then} \quad \leq \text{ is a total order on } S.$$

Note that every total order is a partial order, but the converse does not hold.

Theorem 2.2 (Zorn's lemma). If S is a partially ordered set such that every totally ordered subset of S has an upper bound in S, then

$$\text{there exists an element } m \in S \quad \text{such that} \quad \text{for all } a \in S \text{ we have } m \geq a.$$

Theorem 2.3. Every vector space has a Hamel basis.

Proof. If $V = \{\mathbf{0}\}$, then $\emptyset$ is its basis. Suppose $V \neq \emptyset$. Let P be the set of linearly independent subsets of V ordered by inclusion.

Let S_α be a totally ordered subset of P and define

$$M = \bigcup_{\alpha} S_\alpha.$$

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n \in M$ such that $\alpha_1 \mathbf{v}_1 + \alpha_n \mathbf{v}_n = \mathbf{0}$ with not all $\alpha_i = 0$. Then, there exists an α such that S_α contains all $\mathbf{v}_1, \ldots, \mathbf{v}_n$. By Zorn's lemma (Theorem 2.2), S_α has a maximal element, say $\mathcal{B}$.

Suppose on the contrary that $\mathcal{B}$ does not span V. Then, there exists $\mathbf{v} \in V$ such that $\mathbf{v} \notin \operatorname{span}(\mathcal{B})$. So, $\mathcal{B} \cup \{\mathbf{v}\}$ is linearly independent but $\mathcal{B} \subseteq \mathcal{B} \cup \{\mathbf{v}\}$. This is a contradiction. Hence, $\mathcal{B}$ spans V and is a Hamel basis. Since V was an arbitrary vector space, we conclude that every vector space has a Hamel basis. $\square$