MA3264 Mathematical Modelling

Thang Pang Ern

Contents

Chapter 1

Models using First-Order Differential Equations

1.1 Modelling with Separable Differential Equations

Mathematical models are rarely realistic. So, what is their purpose? The value lies in their ability to evolve. When a model predicts something nonsensical, it highlights where the simplifications fall short. You refine the model, adding complexity to better align it with reality. This iterative process mirrors life itself: start simple, and work towards something more realistic.

Take $\sin\theta = \theta$ as an example. Is it accurate? No, but it's a reasonable approximation for small θ. To improve it, we can use $\sin\theta = \theta - \theta^3/6$. Is that entirely true? Not quite, but it is closer to the truth. And with further refinements, we can get even better approximations.

Definition 1.1 (separable DE)**.** A first-order differential equation is separable if it can be written as

$$M(x)\ dx = N(y)\ dy.$$

Notice that in this form, we say that we have *separated the variables* as everything involving x is on one side and everything involving y is on the other.

One should know how to solve such differential equations as shown in Definition 1.1 from one's high school days (or even MA2002 Calculus) — simply integrate both sides, i.e.

$$\int M(x)\ dx = \int N(y)\ dy + c.$$

Example 1.1 (radioactive decay)**.** Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with a sample containing 2 mg of this substance at certain time, say $t = 0$, what can be said about the amount available at a later time?

Solution. There could be a variety of radioactive materials present, and some of them might contribute to generating the substance we are analysing. As such, we shall deliberately disregard all other materials. So,

$$\frac{dm}{dt} = -km$$

where $m(t)$ is the amount of substance at time t, so $m(0) = 2$. Also, k is some arbitrary constant. With some algebraic manipulation, we have

$$\frac{1}{m}\, dm = -k\, dt.$$

Note that this admits the form in Definition 1.1, so one can integrate both sides to obtain

$$\ln\left(\frac{m}{c}\right) = -kt \quad \text{which implies} \quad m = ce^{-kt}.$$

Here, c is also an arbitrary constant. Setting $m(0) = 2$, we have $c = 2$, so $m = 2e^{-kt}$ — anyway, this means that the radioactive substance will decay at an exponential rate. This process is commonly known as an example of exponential decay. □

Example 1.2 (black holes)**.** Stephen Hawking discovered that black holes lose mass over time, in addition to gaining it through the process of accreting matter. He developed a model to describe this complex phenomenon by simplifying the situation: he disregarded the details of matter falling into the black hole and concentrated solely on the radiation emitted, known as Hawking radiation. The rate of mass loss is described by the following differential equation:

$$\frac{dM}{dt} = -\frac{\hbar c^4}{15360\pi G^2 M^2}$$

where t is time, M is the mass of the black hole, $\hbar$ is the reduced Planck's constant, c is the speed of light, and G is the universal gravitational constant.

One can easily compute the time T it takes for a black hole to disappear completely, i.e. the lifetime of a black hole with initial mass M_0. We have

$$T = \frac{5120\pi G^2 M_0^3}{\hbar c^4}.$$

Example 1.3 (planetary orbit)**.** The orbit of a planet represents the path it follows as it moves around the Sun. In reality, this trajectory is highly complex due to gravitational influences from other planets, which pull on it from various directions. Isaac Newton, however, devised a simplified model of this situation by focusing exclusively on the interaction between the Sun and a single planet. He ignored the effects of other planets, asteroids, and miscellaneous items, as well as the fact that the Sun is not a perfect sphere, among other complexities. This approach allowed him to derive foundational insights into planetary motion.

In order to understand Newton's model of planetary orbits, one needs to recall polar coordinates (recall from MA2104)! Using his laws of motion, Newton discovered that a planet in his model has an orbit which satisfies the differential equation

$$\left(\frac{du}{d\theta}\right)^2 + (u - A)^2 = B^2,$$

where $u(\theta) = 1/r(\theta)$ and $A, B > 0$ are constants with $B/A < 1$. Note that $r(\theta)$ is the equation of our graph in polar coordinates.

This differential equation is separable, i.e. one can show that

$$d\theta = \frac{du/B}{\sqrt{1 - \left(\frac{u-A}{B}\right)^2}}.$$

Integrating both sides yields

$$\theta + c = \arcsin\left(\frac{u - A}{B}\right).$$

Since $u = 1/r$, it follows that

$$r = \frac{1/A}{1 + \frac{B}{A}\sin(\theta + c)}.$$

Since $B/A < 1$, we would see that this curve looks like an ellipse[†]! As such, in this simplified model of the solar system, all the planets have elliptical orbits (also known as Kepler's first law of planetary motion).

Example 1.4 (MA3264 AY25/26 Sem 1 Tutorial 1)**.** Solve the equation $y' = y$, $y(0) = 1$, in the following way: assume that y has an expansion of the form

$$y = a_0 + a_1 x + a_2 x^2 + \dots \tag{1.1}$$

[†]Given the range of values of the ratio B/A, we can obtain various conic sections.

and use the equation and the initial conditions to find the numbers a_n for all n. Next, consider the equation

$$y' = 2\sqrt{y} \quad \text{where} \quad y\,(x) \geq 0 \text{ and } y(0) = 0.$$

The previous method doesn't work. So find the solution in some other way.

Solution. This is a simple exercise that is also covered in MA3220 Ordinary Differential Equations. Suppose y admits the power series solution as in (1.1). Then,

$$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots.$$

Comparing the coefficients, we have $a_0 = a_1$, $2a_2 = a_1$, $3a_3 = a_2$ and so on. Suppose $a_0 = c$, where c is an arbitrary constant. Then, $a_1 = c$, $a_2 = c/2$, $a_3 = c/3!$ and so on. In general,

$$a_n = \frac{c}{n!} \quad \text{so} \quad y = \sum_{n=0}^{\infty} a_nx^n = c\sum_{n=0}^{\infty}\frac{x^n}{n!}. \tag{1.2}$$

Substituting the initial condition $y\,(0) = 1$ yields $c = 1$, so $y = e^x$ (recall that the infinite series obtained on the right side of (1.2) represents e^x).

On the other hand, for the differential equation $y' = 2\sqrt{y}$, a brief modification we can make is to let

$$\sqrt{y} = b_0 + b_1x + b_2x^2 + \ldots. \tag{1.3}$$

Squaring both sides and realising that the product of two infinite series can be interpreted as a convolution (search *Cauchy product*), we have

$$y = \left(\sum_{m=0}^{\infty} b_mx^m\right)\left(\sum_{n=0}^{\infty} b_nx^n\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} b_ib_{k-i}\right)x^k.$$

Upon expansion, we have

$$y = b_0^2 + 2b_0b_1x + (2b_0b_2 + b_1^2)x^2 + (2b_0b_3 + 2b_1b_2)x^3 + \ldots$$

so

$$y' = 2b_0b_1 + 2\left(2b_0b_2 + b_1^2\right)x + 3\left(2b_0b_3 + 2b_1b_2\right)x^2 + \ldots. \tag{1.4}$$

Since $y' = 2\sqrt{y}$, by comparing the coefficients in (1.3) and (1.4), we have $2b_0b_1 = 2b_0$, $4b_0b_2 + 2b_1^2 = 2b_1$, $6b_0b_3 + 6b_1b_2 = 2b_2$ and so on. Since $y(0) = 0$, then $b_0^2 = 0$, so $b_0 = 0$. From $2b_0b_1 = 2b_0$, we have $b_1 = 1$ or $b_0 = 0$. Hence, $b_1 = 1$. From $6b_0b_3 + 6b_1b_2 = 2b_2$, we have $6b_2 = 2b_2$, so $b_2 = 0$. One can then deduce that $b_n = 0$ for all $n \geq 3$. We conclude that $y = x^2$ and $y = 0$ are solutions to the second differential equation. Of course, we could have proceeded with the usual technique of separating the variables and integrating both sides. □

Example 1.5 (MA3264 AY25/26 Sem 1 Tutorial 1). One theory about the behaviour of moths states that they navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon. A certain moth flies near to a candle and mistakes it for the Moon. What will happen to the moth?

Note that in polar coordinates (r, θ), the formula for the angle ψ between the radius vector and the velocity vector is given by

$$\tan\psi = r\frac{d\theta}{dr}.$$

If you wish to derive this formula[†], recall that the tangential component of a small displacement in polar coordinates $(r, \theta) \mapsto (r + dr, \theta + d\theta)$ is $rd\theta$ and the radial component is just dr. Use the formula to solve for r as a function of θ.

Solution. We have

$$\frac{1}{r}\,dr = \cot\psi\;d\theta.$$

Then, integrate both sides. Solving this differential equation gives a logarithmic spiral of equation

$$r(\theta) = r_0 e^{\theta\cot\psi}.$$

A logarithmic spiral winds inexorably toward (or away from) the origin depending on the sign of $\cot\psi$. Since the moth is attempting to approach the candle, it traces the part of the logarithmic spiral that converges towards the candle. □

[†] As the point moves by an infinitesimal amount $(dr, d\theta)$, the displacement vector in Cartesian space is composed of a radial component of length dr and a tangential component of $rd\theta$. Hence, the tangent of the angle ψ between the radius vector and the velocity vector is given by the quotient of the opposite length with the adjacent length, which is $rd\theta/dr$.

1.2 Modelling with Linear Differential Equations

Example 1.6 (melting ice). The Arctic Ocean (Figure 1) plays a crucial role in the global climate system, as it is the region most impacted by global warming. It is warming at approximately three times the rate of the rest of the planet, with the pace continuing to accelerate.

Figure 1: The Arctic Ocean

The surface of the Arctic Ocean consists of both ice-covered areas and open water. Let $I(t)$ represent the area covered by ice and $W(t)$ the area of open water, both as functions of time. The temperature $T(t)$ is also time-dependent. The rate of change of the ice-covered area, $I(t)$, is negatively influenced by the temperature $T(t)$, while the rate of change of the temperature is positively affected by $W(t)$. As such, we obtain the following pair of differential equations:

$$\frac{dI}{dt} = -aT \quad \text{and} \quad \frac{dT}{dt} = bW \quad \text{where } a, b > 0 \text{ are constants}$$

This relationship arises because ice, being highly reflective (a property known as having high albedo), reflects most sunlight, preventing it from absorbing significant heat. In contrast, open water, which appears dark blue or nearly black, absorbs heat efficiently. Consequently, when $W(t)$ is large, more heat is absorbed, causing the temperature to rise.

The equations are called *linear* because of the absence of terms like T^3 or $\cos W$ — we only see T and W. The trick to solving such a pair of simultaneous equations is to differentiate one of the equations. In particular, we differentiate the first equation and substitute into the second one to obtain

$$\frac{d^2 I}{dt^2} = -abW.$$

We note that the total area of the Arctic Ocean is a constant, which is equal to $I + W$. Differentiating a constant twice yields 0, so

$$\frac{d^2 I}{dt^2} + \frac{d^2 W}{dt^2} = 0 \quad \text{which implies} \quad \frac{d^2 W}{dt^2} = abW.$$

We will learn how to solve such differential equations in Chapter 2. Anyway, one checks that

$$W(t) = Ae^{\sqrt{ab}t} + Be^{-\sqrt{ab}t} \quad \text{satisfies the differential equation.}$$

Here, A and B are some constants. Unless $A = 0$, the expression will blow up exponentially fast, with W increasing rapidly till it reaches the total area, and there will not be any ice at all. In fact it is feared that exactly this will happen some time this century!

Of course, this hasn't happened (yet), which suggests that there is a potential flaw in our model. However, nothing is actually wrong since it is just a model after all! The Arctic Ocean is an extraordinarily complex system governed by hundreds, if not thousands, of parameters and interrelated processes. Nevertheless, we need to begin somewhere. Having said that, see Example 1.8 for a modified setup.

Example 1.7 (cosmology; MA3264 AY25/26 Sem 1 Tutorial 1)**.** In Cosmology, the ratio of the sizes of the Universe at two different times is measured by a function of time called the *scale function*, denoted by $a(t)$. What are the units of $a(t)$?

The Friedmann equation relates this function to the energy density of the Universe and to its spatial curvature. In a particular cosmological model, the Friedmann equation takes the form

$$L^2\dot{a}^2 = a^2 - \frac{2}{a^2} + 1 \quad \text{where} \quad \dot{a} = \frac{da}{dt} \text{ denotes the time derivative.} \tag{1.5}$$

Also, L denotes a positive constant, and the initial condition is $a(0) = 1$. What are the units of L? Show, without solving this equation, that the universe described by this model is never smaller than a certain minimum size. Now solve the equation and describe the history of this universe.

Solution. First, note that $a(t)$ is dimensionless. We then solve the Friedmann differential equation (1.5). We have

$$L^2\left(\frac{da}{dt}\right)^2 = \frac{(a^2+2)(a^2-1)}{a^2} \quad \text{so} \quad \frac{da}{dt} = \frac{1}{L}\cdot\frac{\sqrt{(a^2+2)(a^2-1)}}{a}.$$

One can deduce that the minimum value of a is 1. Then, one can use the substitution $u = \sqrt{a^2+2}$ to integrate, which yields

$$\frac{t}{L} = \ln\left(\sqrt{a^2+2} + \sqrt{a^2-1}\right) + c.$$

When $t = 0$, $a = 1$, so $c = \frac{1}{2}\ln 3$. Making a the subject of the equation, we have

$$a(t) = \frac{1}{\sqrt{2}}\cdot\sqrt{3\cosh\left(\frac{2t}{L}\right) - 1}.$$

At $t = 0$, we have $a = 1$ and $\dot{a} = 0$. For $t < 0$, we have $\dot{a} < 0$ (contraction) down to the minimum $a_{\min} = 1$. Lastly, for $t > 0$, $\dot{a} > 0$ which denotes expansion. □

Example 1.8 (modified melting ice; MA3264 AY25/26 Sem 1 Tutorial 1)**.** In Example 1.6, we constructed a model of Arctic sea ice using the equations (all parameter values have been chosen just for convenience)

$$\frac{dI}{dt} = -T \quad \text{and} \quad \frac{dT}{dt} = \sin(5t)\, W, \tag{1.6}$$

where I denotes the area of the ice and W denotes the area of open water. The $\sin 5t$ represents seasonal variations. Notice that $\sin 5t$ is sometimes negative. This is because when the atmospheric weather gets hot enough, the water (which is always pretty cold) actually helps to lower temperatures. Argue that, if the total area of the Arctic Ocean is 10 in these units, then this is a model of the fluctuating area of the open sea in pre-industrial times. Notice that there is a long-term variation as well as the expected seasonal one.

Now the Industrial Revolution happens and carbon dioxide is emitted into the atmosphere, causing a slow global rise in temperatures. Let us model this with the equations

$$\frac{dI}{dt} = -T \quad \text{and} \quad \frac{dT}{dt} = \sin(5t)\,W + 0.01t. \tag{1.7}$$

What does your model predict now?

Solution. We begin by discussing the first pair of differential equations (1.6). Recall that I denotes the area of the ice and W denotes the area of the open water. Suppose $I + W = 10$. Clearly,

$$\frac{dI}{dt} + \frac{dW}{dt} = 0 \quad \text{so} \quad \frac{dW}{dt} = -\frac{dI}{dt} = T.$$

Hence,

$$\frac{d^2W}{dt^2} = \frac{dT}{dt} = W \sin 5t.$$

That is, the model describes seasonal oscillations superimposed on a possible slow (long-term) drift set by the initial heat content. As for the second model (1.7), the added $0.01t$ term imposes an accelerating warming, driving a superlinear increase in open water and eventual ice-free conditions, with the seasonal cycle riding on an ever-rising trend. □

First-order linear ODEs are very useful. However, they are not always separable. Having said that, there is a trick that allows us to solve them.

Proposition 1.1 (integrating factor)**.** Consider linear differential equations of the form

$$\frac{dy}{dx} + yP(x) = Q(x),$$

where P and Q are functions of x. One can solve such differential equations by multiplying both sides of the equation by an integrating factor $\mu(x)$, then use the product rule, where

$$\mu(x) = \exp\left(\int P(t)\ dt\right).$$

Proposition 1.1 has already been covered in MA2002 so we will not discuss further. Now, what happens if the differential equation is neither separable nor linear? One nice instance is when we come across a Bernoulli equation (Definition 1.2).

Definition 1.2 (Bernoulli equation)**.** The differential equation

$$\frac{dy}{dx} + yP(x) = Q(x)\,y^n,$$

where $n \in \mathbb{R}$, is a Bernoulli equation.

Again, Definition 1.2 has already been covered in MA2002 — the trick to solving such equations is to introduce the substitution $z = y^{1-n}$.

Example 1.9 (Bernoulli equation; MA3264 AY25/26 Sem 1 Tutorial 2)**.** Solve the differential equation

$$2xy\frac{dy}{dx} + (x-1)\,y^2 = x^2e^x.$$

Solution. We have

$$\frac{dy}{dx} + \left(\frac{x-1}{2x}\right)y = \frac{1}{2}xe^xy^{-1}.$$

This is a Bernoulli equation (Definition 1.2). We use the substitution $z = y^2$ and omit the remaining details. □

Example 1.10 (mixing problem)**.** At time $t = 0$, a tank contains 2 kg of salt dissolved in 100 ℓ of water. Assuming that the water containing 0.25 kg of salt per litre is entering the tank at a rate of 3 ℓ/min and the well-stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t. Again, such questions have already been discussed in MA2002 so we will skip.

Example 1.11 (mixing problem)**.** Imagine an experiment where a planet with a pristine atmosphere begins receiving 50 billion tons of CO_2 annually. The CO_2 mixes uniformly with the air, while biological and geological processes remove it, keeping the total atmospheric volume nearly constant. Based on what we have discussed thus far, the concentration of CO_2 would rise exponentially toward a limiting value.

Warned by their scientists, the planet's inhabitants immediately reduce the CO_2 concentration in their emissions at a rate inversely proportional to time.

Now, consider the following analogous problem. A tank contains 100 m^3 of pure air (negligible CO_2) at $t = 1$ second. At that moment, polluted air with a CO_2 concentration of $10/t\ \ell/\text{m}^3$ starts flowing in at 10 m^3/s. The mixture in the tank is pumped out at the same rate. Plot the quantity of CO_2 in the tank as a function of time.

Solution. We have

$$\frac{dQ}{dt} = \frac{100}{t} - \frac{Q}{10} \quad \text{with initial condition } Q(1) = 0.$$

The integrating factor is $e^{t/10}$ so we obtain

$$Q(t) = 100e^{-t/10}\left[\text{Ei}\left(\frac{t}{10}\right) - \text{Ei}\left(\frac{1}{10}\right)\right].$$

Here, $\text{Ei}(x)$ denotes the exponential integral (Definition 1.3).

Definition 1.3 (exponential integral)**.** For real non-zero values of x, define $\text{Ei}(x)$ to be

$$\text{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt.$$

One can plot the graph of $Q(t)$. We see that the amount of CO_2 in the tank increases for some time even though the concentration in the gas entering the tank is decreasing. It reaches a rather high maximum, before decreasing rather slowly. This is known as the dreaded *momentum effect*, i.e. even if we start drastic reductions of CO_2 release now, the amount of it in the atmosphere will increase for a long time and will only be reduced to safe levels in the distant future[†]. □

Example 1.12 (radioactive decay)**.** Sometimes, the product of radioactive decay is itself a radioactive substance that undergoes decay at a different rate. An example is uranium-thorium dating, a method used by paleontologists to estimate the age of fossils, particularly ancient corals.

Corals filter seawater (Figure 2), which contains trace amounts of uranium-234, a radioactive isotope. These corals absorb uranium-234 into their skeletons while alive. Over time, uranium-234 decays (to be precise, the type of radioactive

[†]One can look up 'representative concentration pathway' for more notes on this

decay that uranium-234 goes through is alpha decay) into thorium-230, another radioactive element via the equation

$$^{234}_{92}\mathrm{U} \rightarrow ^{230}_{90}\mathrm{Th} + ^{4}_{2}\alpha.$$

Uranium-234 has a half-life of 245,000 years, while thorium-230 has a shorter half-life of 75,000 years.

Thorium-230 is not naturally present in seawater, so when a coral dies, its skeleton contains uranium-234 but no thorium-230. This is because the coral's lifespan is negligible compared to uranium-234's half-life. However, over time, as uranium-234 decays, thorium-230 begins to accumulate in the coral skeleton. By measuring the ratio of uranium-234 to thorium-230 in a coral sample, we can estimate the time elapsed since the coral's death — its age.

This information is crucial for understanding events like mass coral die-offs. If corals have historically died off regularly over long periods, it might suggest that current coral deaths are part of a natural cycle rather than solely caused by global warming.

Figure 2: Corals in the Great Barrier Reef, Australia

To model this process, we make certain simplifying assumptions. Although other radioactive materials may be present, we ignore them because the decay products of thorium-230 typically decay much faster than uranium-234 or thorium-230 itself. Therefore, their contributions are negligible for our purposes.

Let $U(t)$ represent the number of uranium-234 atoms in the coral sample at time t, and $T(t)$ represent the number of thorium-230 atoms. Since each uranium-234 atom decay produces one thorium-230 atom, the rate at which thorium-230 is produced equals the rate at which uranium-234 decays. Consequently, we have the following relationships for the decay rates:

$$\frac{dU}{dt} = -k_U U \quad \text{and} \quad \frac{dT}{dt} = k_U U - k_T T, \tag{1.8}$$

where k_U and k_T are constants with $k_U \neq k_T$, and $U(0) = U_0$ and $T(0) = 0$. We wish to find t given that we know the ratio of $T(t)$ to $U(t)$ at the present time. Solving with the given data (first equation) yields

$$U = U_0 e^{-k_U t}.$$

One can attempt to solve for k_U and k_T, which are

$$k_U = \frac{\ln 2}{245,000} \quad \text{and} \quad k_T = \frac{\ln 2}{75,000}. \tag{1.9}$$

The second differential equation yields

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}.$$

Solving with $T(0) = 0$ yields

$$T(t) = \frac{k_U}{k_T - k_U} U_0 \left(e^{-k_U t} - e^{-k_T t} \right).$$

Although we do not know the value of U_0, we can consider the ratio T/U, which is

$$\frac{T}{U} = \frac{k_U}{k_T - k_U} \left[1 - e^{(k_U - k_T)t} \right] \tag{1.10}$$

So, if we compute the ratio T/U at the present time, we can solve for t and obtain our answer!

Example 1.13 (radioactive decay; MA3264 AY25/26 Sem 1 Tutorial 2)**.** The half-life of thorium-230 is about 75000 years, while that of uranium-234 is about 245000 years. A certain sample of ancient coral has a thorium/uranium ratio of 10 percent. How old is the coral?

Solution. Recall the radioactive differential equations (1.8) discussed in Example 1.12. We also deduced the values of k_U and k_T in (1.9), where 245,000 and 75,000 in the denominators are known as the respective decay constants. In this question, we are given that $T/U = 0.1$. One can substitute the known quantities into (1.10) to obtain the value of t, which is approximately 40083 years. $\square$

Example 1.14 (catenary; MA3264 AY25/26 Sem 1 Tutorial 2)**.** If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a *catenary*. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and Australia. For example, the Gateway Arch in Missouri, United States of America, is in the shape of an inverted catenary.

Figure 3: The Gateway Arch in Missouri, United States of America

It can be shown using simple physics that if the shape is given by a function $y(x)$, then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T} \int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt, \tag{1.11}$$

where $x = 0$ at the lowest point of the catenary and $y(0) = 0$, μ is the weight per unit length of the cable, and T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. One can use the Fundamental Theorem of Calculus, and think of the resulting equation as a first-order ordinary differential equation.

Solution. Differentiating both sides of the catenary differential equation (1.11) yields

$$\frac{d^2y}{dx^2} = \frac{\mu}{T}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \tag{1.12}$$

For compactness, let $c = \mu/T$. Then, squaring both sides of (1.12) yields

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^2 + 1.$$

Using the substitution

$$u = \frac{dy}{dx} \quad \text{we have} \quad \frac{du}{dx} = \sqrt{u^2+1}.$$

One can use the substitution $u = \tan\theta$ to solve the new differential equation. We omit the details. □

Example 1.15 (potato blight; MA3264 AY25/26 Sem 1 Tutorial 2)**.** The spread of potato blight[†] in a crop (Figure 4) can be modelled by the differential equation

$$\frac{dx}{dt} = Kf(t)(1-x),$$

where $x(t)$ denotes the fraction of plants that are infected at time t, $K > 0$ is a constant describing the infectivity of the disease, and $p > 0$ and $q > 0$ represent physiological delay parameters. We assume that no plant is infected until $t = 0$, and that at $t = 0$ a fraction $\alpha \in (0, 1)$ of susceptible plants suddenly become infected. Also,

$$f(t) = x(t-p) - x(t-p-q).$$

Note that this is a function in terms of t!

[†]A blight is a plant disease, typically one caused by fungi.

Figure 4: A potato blight

Show that the solution to the differential equation can be written in the form

$$x(t) = 1 - (1-\alpha)\exp\left(-K\int_0^t f(\tau)\,d\tau\right).$$

This expression still does not determine $x(t)$ explicitly since $x(t)$ also appears on the right side of the equation. However, it can be used iteratively as follows:

(i) Show that for any $a > 0$,

$$\int_0^t x(\tau - a)\,d\tau = \int_0^{t-a} x(\tau)\,d\tau,$$

interpreting $x(s) = 0$ for all $s < 0$.

(ii) Use **(i)** to prove the identity

$$\int_0^t f(\tau)\,d\tau = \int_{(t-p)-q}^{t-p} x(\tau)\,d\tau.$$

(iii) Deduce that

$$\lim_{t\to\infty}\int_0^t f(\tau)\,d\tau = q\,\beta \quad \text{where } \beta = \lim_{t\to\infty} x(t)$$

is the final fraction of plants which become infected. A hint is that $x(t)$ is monotonically increasing and bounded above by 1, and $f(\tau)$ consists of delayed versions of x.

(iv) Show that β satisfies the nonlinear equation

$$\beta = 1 - (1-\alpha)\exp(-Kq\beta).$$

Note that $\beta \neq 1$, except in the trivial case $\alpha = 1$: regardless of how long the epidemic continues, a positive fraction of plants remain uninfected.

Solution. We first show that the solution to the differential equation

$$\frac{dx}{dt} = Kf(t)(1-x) \quad \text{is} \quad x(t) = 1 - (1-\alpha)\exp\left(-K\int_0^t f(\tau)\ d\tau\right). \tag{1.13}$$

We have

$$\frac{1}{1-x}\ dx = Kf(t)\ dt.$$

We integrate both sides — the right side from $\tau = 0$ to $\tau = t$. As for the left side, the limits are from α to x. So,

$$-\ln|1-x| + \ln|1-\alpha| = K\int_0^t f(\tau)\ d\tau.$$

The result follows with some algebraic manipulation.

(i) Use the substitution $u = \tau - a$ on the left integral.

(ii) By the definition of f, we have

$$\begin{aligned} \int_0^t f(\tau)\ d\tau &= \int_0^t x(t-p)\ d\tau - \int_0^t x(t-p-q)\ d\tau \\ &= \int_0^{t-p} x(\tau)\ d\tau - \int_0^{t-p-q} x(\tau)\ d\tau \\ &= \int_{t-p-q}^{t-p} x(\tau)\ d\tau \end{aligned}$$

Here, the second equality uses **(i)**.

(iii) From **(ii)**, we have

$$\lim_{t\to\infty}\int_0^t f(\tau)\ d\tau = \lim_{t\to\infty}\int_{t-p-q}^{t-p} x(\tau)\ d\tau. \tag{1.14}$$

We are now integrating x over an interval of length q. As hinted from the blue part of the problem, x denotes the fraction of plants that are infected at time t, so x is monotonically increasing and bounded by 1. By the monotone convergence theorem, x converges to 1. As such, we can approximate the integral in (1.14) by considering the area of a rectangle of width q and height β, where β was defined earlier.

(iv) This is trivial from (1.13). $\square$

Chapter 2

Models using Second-Order Differential Equations

2.1 Introduction

We will need to study ordinary differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x),$$

where a,b,c are real constants and $f(x)$ is some given function. There is a systematic way of solving such ODEs.

Observe that since there are two derivatives present, we would need to integrate twice. Integrating once yields a constant, so the general solution of a second-order ODE must involve exactly two constants. To find them, we generally work with the initial conditions $y(0)$ and $y'(0)$ (which are usually known quantities). We then end up with two equations for two unknowns, which determine the two constants.

Definition 2.1 (characteristic equation)**.** Consider the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Its characteristic equation is

$$a\lambda^2+b\lambda+c=0.$$

Note that in Definition 2.1, the characteristic equation is also known as the auxiliary equation. Note that here, we have taken $f(x)=0$. To find two solutions S_1 and S_2 to the equation, we consider the corresponding characteristic equation $a\lambda^2+b\lambda+c=0$. The idea here is that if λ is real, then $e^{\lambda x}$ is a solution. Usually, we obtain two solutions[†], i.e. two numbers λ_1 and λ_2. As such, we obtain two

[†]Here is a fun exercise that is related to ST2131. Given the quadratic equation $Ax^2+Bx+C=0$ with $A,B,C\sim U(0,1)$, i.e. A,B,C are uniformly distributed on the interval $(0,1)$, what is the probability that the roots of the quadratic equation are real?

solutions $S_1 = e^{\lambda_1 x}$ and $S_2 = e^{\lambda_2 x}$.

However, two things can potentially go wrong.

- **Case 1:** The quadratic equation might have only one root λ (which must be real since a, b, c are real). Then, we will take $S_1 = e^{\lambda x}$ and $S_2 = xe^{\lambda x}$. One should verify by direct substitution that this indeed works.
- **Case 2:** We might obtain two solutions, which are complex. In fact, by the conjugate root theorem, the roots of the quadratic equation form conjugate pairs. We focus on one of them, and write it as

  $$\lambda = \alpha + \beta i \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

  Then, we take

  $$S_1 = e^{\alpha x} \cos \beta x \quad \text{and} \quad S_2 = e^{\alpha x} \sin \beta x.$$

  Again, one should substitute these into the differential equation to be convinced that S_1 and S_2 are indeed solutions. Actually, these do not look so strange if we recall Euler's formula, which states that

  $$e^{i\theta} = \cos \theta + i \sin \theta.$$

In all cases, we will be able to deduce the solutions S_1 and S_2, provided that a, b, c are constants.

For the case where $f(x) \neq 0$, suppose we have some *miraculous* way of deducing one solution to the differential equation. Call this solution $y = P(x)$, known as the particular solution. Then, we can find the general solution to the differential equation as follows. Consider

$$\begin{aligned}
a\frac{d^2 S_1}{dx^2} + b\frac{dS_1}{dx} + cS_1 &= 0 \\
a\frac{d^2 S_2}{dx^2} + b\frac{dS_2}{dx} + cS_2 &= 0 \\
a\frac{d^2 P}{dx^2} + b\frac{dP}{dx} + cP &= 0
\end{aligned}$$

We multiply the first equation by an arbitrary real number A and multiply the second by an arbitrary real number B. Thereafter, we add the three equations to obtain

$$a\left(AS_1 + BS_2 + P\right)'' + b\left(AS_1 + BS_2 + P\right)' + c\left(AS_1 + BS_2 + P\right) = f\left(x\right),$$

so we infer that $AS_1 + BS_2 + P$ is the general solution to the differential equation! This works due to the linearity of the derivative operator, and because the equation is also linear (inherently used the principle of superposition here). This method is not applicable to non-linear ODEs though.

Example 2.1. Solve the differential equation

$$\frac{d^2y}{dx^2} - y = e^{2x}.$$

Solution. Note that the differential equation can be written as

$$\frac{d^2y}{dx^2} - y = 0 + e^{2x}.$$

We first find the complementary solution. That is, the set of all y such that

$$\frac{d^2y}{dx^2} - y = 0.$$

The characteristic equation is $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$. Hence, $S_1 = e^x$ and $S_2 = e^{-x}$.

Now, we find the particular solution. The only way to obtain e^{2x} on the RHS is if it is already there on the LHS. As such, we try $P\left(x\right) = ce^{2x}$, where c has to be found. Since P satisfies the differential equation, we have

$$4ce^{2x} - ce^{2x} = e^{2x} \quad \text{which implies} \quad c = \frac{1}{3}.$$

Hence, $P = e^{2x}/3$. Combining the complementary solution and the particular solution yields the general solution, which is

$$y = Ae^x + Be^{-x} + \frac{1}{3}e^{2x}.$$

□

Example 2.2 (MA3264 AY25/26 Sem 1 Tutorial 3)**.** Consider the differential equation $\ddot{y} + y = 0$, with $y\left(0\right) = 0$ and $\dot{y}\left(0\right) = 1$. Solve it using the power series method as in Example 1.4. Infer that one needs two initial conditions to obtain a specific solution to a second-order ordinary differential equation.

Solution. Let

$$y = a_0 + a_1 t + a_2 t^2 + \dots.$$

Then,

$$\dot{y} = a_1 + 2a_2 t + 3a_3 t^2 + \dots \quad \text{so} \quad \ddot{y} = 2a_2 + 6a_3 t + \dots$$

Actually, we can write the differential equation as

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^n = 0.$$

We need to match the exponents of t, so

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=2}^{\infty} a_{n-2} t^{n-2} = 0 \quad \text{so} \quad \sum_{n=2}^{\infty} \left(n(n-1) a_n + a_{n-2}\right) t^{n-2} = 0.$$

We must have

$$n(n-1) a_n + a_{n-2} = 0 \quad \text{so} \quad a_n = -\frac{a_{n-2}}{n(n-1)}.$$

We have

$$a_2 = -\frac{a_0}{2 \cdot 1} \text{ and } a_4 = -\frac{a_2}{4 \cdot 3} \quad \text{so} \quad a_4 = \frac{a_0}{4!}.$$

One can deduce for n even,

$$a_n = (-1)^{\frac{n}{2}} \cdot \frac{a_0}{n!}$$

and for n odd,

$$a_n = (-1)^{\frac{n-1}{2}} \cdot \frac{a_1}{n!}.$$

So,

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + \frac{a_1}{1!} t + \frac{a_0}{2!} t^2 - \frac{a_1}{3!} t^3 + \frac{a_0}{4!} t^4 + \dots = a_0 \cos t + a_1 \sin t.$$

When $t = 0$, $y = 0$ so $a_0 = 0$. When $t = 0$, $\dot{y} = 1$. As $y = a_1 \cos t$, we have $a_1 = 1$. Hence, $y = \cos t$. □

However, there are times where the method to finding the particular solution in Example 2.1 does not work. Let us take a look at Example 2.3.

Example 2.3. Solve the differential equation

$$\frac{d^2 y}{dx^2} - y = e^x.$$

Finding the complementary solution is precisely the same as Example 2.1 since both differential equations share the same characteristic equation. Now, if we try

$P(x) = ce^x$ as our particular solution, we will see that $0 = e^x$, which is an obvious error. As such, we need to amend the particular solution (the method to finding the particular solution is somewhat systematic) — try $P(x) = cxe^x$. We will see that

$$\frac{d^2P}{dx^2} - P = 2ce^x \quad \text{which implies} \quad 2c = 1.$$

Hence, $c = 1/2$ and the desired general solution is

$$y = Ae^x + Be^{-x} + \frac{1}{2}xe^x.$$

Examples 2.1 and 2.3 are great examples of finding particular solutions. The given method works because if we have an exponential function on the RHS, taking derivatives of exponential functions would give exponential functions. Similarly, it always works for polynomials and for products of exponential functions with polynomials. However, this method does not work if we have functions like $\tan x$ on the RHS!

Example 2.4. Now, what if we wish to solve a differential equation like

$$\frac{d^2y}{dx^2} + y = \cos x?$$

The easy way to handle this is to remember that $\cos x$ and $\sin x$ are really just names of the real and imaginary parts of e^{ix} respectively. As such, consider $z(x)$ to be a complex function such that $\mathrm{Re}(z) = y$. Then, say we have the equation

$$\frac{d^2z}{dx^2} + z = e^{ix}.$$

This is easy to solve because we know what to do when we have an exponential function on the RHS! As such, we solve for z. As we are interested in y, upon finding z, we just take the real part of that.

Again, we first find the complementary solution. We first solve

$$\frac{d^2}{dx^2} + y = 0.$$

The characteristic equation is $\lambda^2 + 1 = 0$, so $\lambda = \pm i$. Hence,

$$S_1 = \cos x \quad \text{and} \quad S_2 = \sin x.$$

Next, try $P(x) = ce^{ix}$, which does not work. As such, we try $P(x) = cxe^{ix}$, for which we obtain $2ice^{ix} = e^{ix}$. So,

$$c = \frac{1}{2i} = -\frac{1}{2}i.$$

Hence,

$$P(x) = -\frac{1}{2}ixe^{ix} = -\frac{x}{2}(-\sin x + i\cos x).$$

The real part of P is $x\sin x/2$, so the general solution to the differential equation is

$$y = A\cos x + B\sin x + \frac{1}{2}x\sin x.$$

Example 2.5 (MA3264 AY25/26 Sem 1 Tutorial 3)**.** Find particular solutions to the differential equation

$$\frac{d^2y}{dx^2} - y = 2x\sin x.$$

Solution. The particular solution should be of the form

$$y_p = (Ax + B)\sin x + (Cx + D)\cos x.$$

Hence,

$$\frac{d^2y_p}{dx^2} = -(Ax + 2C + B)\sin x - (Cx + D - 2A)\cos x.$$

Substituting these into the differential equation, one can deduce that

$$y_p = -x\sin x - \cos x$$

is a particular solution. □

Example 2.6 (MA3264 AY25/26 Sem 1 Tutorial 1)**.** Find particular solutions to the differential equation

$$\frac{d^2y}{dx^2} + 4y = \sin^2 x.$$

Solution. The trick is to use the identity $\cos 2x = 1 - 2\sin^2 x$, then make $\sin^2 x$ the subject. One can check that

$$y_p = \frac{1}{8} - \frac{1}{8}x\sin 2x$$

is a particular solution. □

2.2 Stability

Definition 2.2 (pendulum equation)**.** Consider a pendulum. Let θ be the angle with the vertical and let L be the length of the pendulum. Then, using Physics (briefly see Figure 5), one can deduce that a differential equation governing θ is as follows:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

Sometimes, $d^2\theta/dt^2$ is written as $\ddot{\theta}$, which also denotes the second derivative of the angle with respect to time.

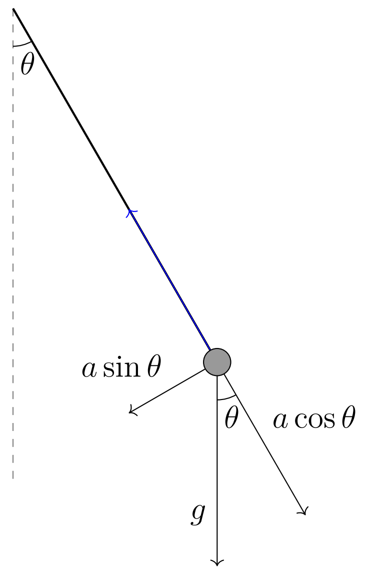


Figure 5: A free-body diagram of a pendulum bob

It is possible to solve the pendulum differential equation (Definition 2.2), but this involves complicated concepts in Mathematics like elliptic integrals.

Let us simplify the setup. An obvious solution is $\theta = 0$, which is known as an equilibrium solution, meaning that θ is a constant function. This means that if we set $\theta = 0$ initially, then θ will remain at 0 and the pendulum will not move — which of course know is correct. There is another equilibrium solution which is $\theta = \pi$. Again, in theory, if we set the pendulum exactly at $\theta = \pi$, then it will remain in that position forever. In reality, it will not due to gravity! As such, the equilibrium at $\theta = \pi$ is very much different from the one at $\theta = 0$ (an important distinction).

Definition 2.3 (equilibrium). The equilibrium of an object is said to be stable if a small push away from equilibrium remains small. If the small push tends to grow large, then the equilibrium is unstable.

The concept of equilibrium is particularly important for engineers as they want vibrations of structures, engines, etc. to remain small.

We shall analyse the case where $\theta = \pi$. By Taylor's theorem, near $\theta = \pi$, we have

$$f(\theta) = f(\pi) + f'(\pi)(\theta - \pi) + \frac{1}{2}f''(\pi)(\theta - \pi)^2 + \ldots,$$

and upon letting $f(\theta) = \sin\theta$, we obtain the following series expansion:

$$\sin\theta = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 + \ldots$$

For small deviations away from π, note that $\theta - \pi$ is small, and $(\theta - \pi)^3$ is much smaller. So, we have the following approximation:

$$\sin\theta \approx -(\theta - \pi)$$

As such, the pendulum differential equation in Definition 2.2 can be approximated as follows:

$$\frac{d^2\theta}{dt^2} \approx \frac{g}{L}(\theta - \pi).$$

Using the substitution $\phi = \theta - \pi$, the differential equation can be written as

$$\frac{d^2\phi}{dt^2} = \frac{g}{L}\phi.$$

This equation has the general solution

$$\phi = Ae^{\sqrt{g/L}t} + Be^{-\sqrt{g/L}t}$$

so

$$\theta = Ae^{\sqrt{g/L}t} + Be^{-\sqrt{g/L}t} + \pi.$$

Since the exponential function grows very quickly, even if θ is close to π initially, it will not stay near it very long. Very soon, θ will arrive at either $\theta = 0$ or $\theta = 2\pi$, which is far away from $\theta = \pi$. This equilibrium is unstable! So, we ask how long would it take for things to get out of control? This is determined by the quantity in the exponent of the exponential term which is $\sqrt{g/L}$. Note that it takes longer for the pendulum to fall if L is large.

2.3 Damped Oscillations

When an object moves fairly slowly through air, the resistance due to friction is approximately proportional to its speed, and of course in the opposite direction. One would recall Hooke's law from H2 Physics. In fact, we can extend it to the following differential equation (Definition 2.4):

Definition 2.4 (simple harmonic oscillator)**.**

$$m\frac{d^2x}{dt^2} + kx = 0$$

This equation describes the oscillation of a block of mass m on one of a spring and a nail on the other end. Here, x measures how much the spring is stretched and k is a positive constant that measures the stiffness of the spring (known as the spring constant).

If we include friction which is proportional to the speed, we obtain

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where b is a positive constant known as the damping coefficient. It quantifies the resistance to motion provided by the medium (such as air or fluid), often associated with dissipative forces like friction or drag. As such, the corresponding characteristic equation is

$$m\lambda^2 + b\lambda + k = 0.$$

Note that $m, b, k > 0$. Let us discuss the solutions to this differential equation. We consider three cases on the nature of the roots.

- **Case 1:** λ_1 and λ_2 are real, which results in overdamping

  Example 2.7. Consider the differential equation

  $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0.$$

Its characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$, which yields the roots $\lambda = -1$ and $\lambda = -2$. The general solution is

$$x = B_1 e^{-t} + B_2 e^{-2t}.$$

We see that the motion rapidly dies away to zero, which implies that there is much friction.

- **Case 2:** λ_1 and λ_2 are complex, which results in underdamping

  Example 2.8. Consider the differential equation

  $$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 0.$$

  Its characteristic equation is $\lambda^2 + 4\lambda + 13 = 0$, which yields the roots $\lambda = -2 \pm 3i$. The general solution is

  $$x = B_1 e^{-2t} \cos 3t + B_2 e^{-2t} \sin 3t,$$

  which by the R-formula (from O-Level Additional Mathematics), can be also written as

  $$x = \sqrt{B_1^2 + B_2^2} e^{-2t} \cos\left(3t - \frac{\pi}{4}\right).$$

  This acts like a simple harmonic oscillator, where the amplitude $\sqrt{B_1^2 + B_2^2} e^{-2t}$ is a function of time. Note that in this problem, there are two independent time scales. First, the factor e^{-2t} determines how quickly the oscillations decay over time. This decay is governed by the real part of the roots. Next, the angular frequency of oscillation is determined by the imaginary part of the roots. The oscillation period is

  $$T = \frac{2\pi}{\omega} = \frac{2\pi}{3} \quad \text{where } \omega = 3 \text{ is the angular frequency.}$$

  This represents the rapidity of oscillations within the decaying envelope.

2.4 Forced Oscillations

Now, consider the case where an external motor is attached to the block of mass m. This motor exerts a force of $F_0 \cos \alpha t$, where F_0 is the amplitude of the external force and α is the frequency. If $F_0 = 0$, then by Newton's second law, we have

$$m\frac{d^2x}{dt^2} + kx = 0$$

so we obtain the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}. \tag{2.1}$$

Here, ω is the frequency that the system has if we leave it alone, i.e. its natural frequency. It is not related to α.

If $F_0 \neq 0$, then we have

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \alpha t.$$

Let z be a complex function that satisfies the differential equation

$$m\frac{d^2z}{dt^2} + kz = F_0 e^{i\alpha t}.$$

The real part, $\operatorname{Re} z$, satisfies this differential equation, so we can solve for z and then take the real part. Try $z = ce^{i\alpha t}$ to be a solution. One can deduce that

$$c = \frac{F_0/m}{\omega^2 - \alpha^2} \quad \text{which implies} \quad \operatorname{Re} z = \frac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t.$$

We conclude that the general solution is

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \cos \alpha t,$$

where δ is some constant (we will explain in just a bit). Note that upon differentiation, we obtain

$$\frac{dx}{dt} = -A\omega \sin(\omega t - \delta) - \frac{aF_0/m}{\omega^2 - \alpha^2} \sin \alpha t.$$

The constants A and δ are fixed. One can deduce these values from $x(0)$ and $\dot{x}(0)$ as usual, where we recall that $\dot{x}(0)$ is dx/dt evaluated at $t = 0$.

Example 2.9. As an example, if $x(0) = \dot{x}(0) = 0$, then we have

$$A\cos\delta + \frac{F_0/m}{\omega^2 - \alpha^2} = 0 \quad \text{and} \quad A\omega \sin \delta = 0 \quad \text{respectively.}$$

Assuming that $F_0 \neq 0$, we cannot have $A = 0$, which forces $\delta = 0$. Hence,

$$A = -\frac{F_0/m}{\omega^2 - \alpha^2} \quad \text{which implies} \quad x = \frac{F_0/m}{\omega^2 - \alpha^2} (\cos \alpha t - \cos \omega t).$$

Notice that the amplitude function

$$A(t) = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left(\frac{\alpha - \omega}{2} t\right)$$

has a maximum value

$$\left| \frac{2F_0/m}{\alpha^2 - \omega^2} \right|$$

which becomes very large when α is very close to ω. What happens if we let $\alpha \to \omega$? We have

$$A(t) = \frac{2F_0/m}{\alpha + \omega} \cdot \frac{\sin\left(\frac{\alpha-\omega}{2} t\right)}{\alpha - \omega} \longrightarrow \frac{F_0}{m\omega} \cdot \frac{t}{2} = \frac{F_0 t}{2m\omega}$$

by L'Hopital's rule. So in this limit, we have

$$x = \frac{F_0 t}{2m\omega} \sin(\omega t)$$

and we see that the oscillations go completely out of control. This situation is called resonance. We see that if a system is forced in a way that agrees with its own natural frequency, it can oscillate uncontrollably. A vast number of things can be modelled using the concept of resonance. For example, giant tides (Figure 6). This can be very dangerous! However, in reality resonance does not get completely out of control, because we cannot really ignore friction (or *resistance* in the case of an electrical circuit).

Figure 6: The Seven Sisters in England

So we should really solve

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\alpha t)$$

or

$$m\ddot{z} + b\dot{z} + kz = F_0 e^{i\alpha t}.$$

Set $z = ce^{i\alpha t}$ and take the real part at the end. One can solve for c. Similarly,

$$\frac{F_0[k - m\alpha^2 - ib\alpha]}{(k - m\alpha^2)^2 + b^2\alpha^2} \times [\cos(\alpha t) + i \sin(\alpha t)]$$

has real part

$$x(t) = \frac{F_0(k - m\alpha^2)\cos(\alpha t) + F_0 b\alpha \sin(\alpha t)}{(k - m\alpha^2)^2 + b^2\alpha^2}.$$

To this we should add the general solution to $m\ddot{x} + b\dot{x} + kx = 0$. However, we already know that looks like — whether overdamped or underdamped, the solution rapidly (exponentially) tends to zero. We call it the *transient*. So after the transient dies off, we are left with this expression. Recall that any expression of the form $C \cos x + D \sin x$ can be written as

$$C \cos x + D \sin x = \sqrt{C^2 + D^2} \cos(x - \gamma) \quad \text{where } \tan \gamma = \frac{D}{C}.$$

So, here we have

$$x(t) = \frac{1}{m} \frac{F_0 \cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}.$$

As such, the system eventually settles down into a steady oscillation, but at a frequency α and not ω. This means we can steer the system away from its natural frequency towards any other frequency we want. Also, the *amplitude* of this oscillation is a function of α. That is,

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}.$$

The graph of this is called the amplitude response curve. The amplitude (usually) has a maximum at a certain value of α, found by minimising

$$(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2.$$

If we are at this maximum, we say we are at resonance in this case. Notice that if $b = 0$, then this function is minimised by choosing $\alpha \to \omega$, which implies $A(\alpha) \to \infty$, taking us back to the discussion of resonance in the zero–friction case. This is why we call this resonance in the frictional case.

Example 2.10 (MA3264 AY25/26 Sem 1 Tutorial 3)**.** Consider a forced damped harmonic oscillator, which is modelled by the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

as in (2.1). It is known that the amplitude response function A is a function of α, the input frequency. That is,

$$A(\alpha) = \frac{F_0/m}{\sqrt{\left(\omega^2 - \alpha^2\right)^2 + \frac{b^2}{m^2}\alpha^2}}.$$

Find the maximum of this function. Note that there are different answers when the frictional constant b is large and when it is small. Show that, when b is so small that the dimensionless quantity $b^2/m^2\omega^2$ can be neglected, the maximal amplitude is proportional to $1/b$.

Solution. Differentiating A with respect to α and setting the derivative equal to zero, we get

$$4\alpha\left(\alpha^2 - \omega^2\right) + \frac{2b^2\alpha}{m^2} = 0 \quad \text{so} \quad \alpha^2 = \omega^2 - \frac{b^2}{2m^2}.$$

Of course, the left side cannot be negative, so if $b \geq \sqrt{2m\omega}$, then there is no resonance; this is the situation described above. In that case, the maximum amplitude is at $\alpha = 0$ and is given by $\frac{F_0}{m\omega^2}$. Otherwise, the maximal value of the amplitude is obtained by substituting this value of α into $A(\alpha)$. With some algebraic manipulation, we have

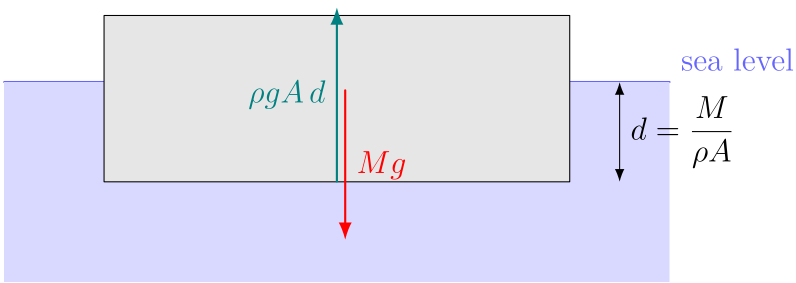
$$A_{\text{resonance}} = \frac{F_0/b\omega}{\sqrt{1 - (b^2/4m^2\omega^2)}}.$$

If $\frac{b^2}{m^2\omega^2}$ is negligible, then this is approximately $\frac{F_0}{b\omega}$. That is, the resonance amplitude grows without limit as b becomes smaller. □

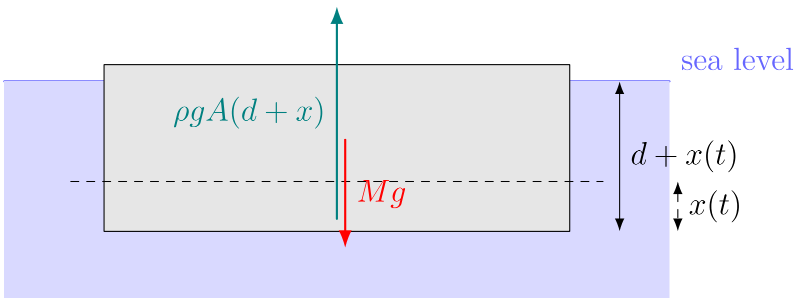
Example 2.11 (MA3264 AY25/26 Sem 1 Tutorial 4)**.** A fully loaded large oil tanker can be modelled as a solid object with perfectly vertical sides and a perfectly horizontal bottom, so all horizontal cross-sections have the same area, equal to A. Archimedes' principle states that the upward force exerted on a ship by the sea is equal to the weight of the water pushed aside by the ship. Let ρ be the mass density of seawater, and let M be the mass of the ship, so that its weight is Mg, where g is 9.8 m/s^2. When the ship is at rest, find the distance d from sea level to the bottom of the ship. This is called the *draught* of the ship.

Suppose now that the ship is not at rest; instead it is moving in the vertical direction. Let $d + x(t)$ be the distance from sea level to the bottom of the ship, where d is the draught as above. Show that, if gravity and buoyancy are the only forces acting on the ship, it will bob up and down with an angular frequency given by $\omega = \sqrt{\rho A g/M}$.

(a) Equilibrium



(b) Displaced



Next, suppose that waves from a storm strike the ship, which is initially at rest with $x(0) = 0$, and exert a vertical force $F_0 \cos(\omega t)$ on the ship, where F_0 is the amplitude of the wave force. Let H be the height of the deck of the ship above sea level when the ship is at rest. We assume that the ship is heavily loaded, so H is much less than d. Write down a formula which allows you to compute when the ship sinks. That is, find an equation satisfied by t_{sink}, the time at which the ship's deck first goes under water. You do not need to solve this equation.

Solution. When the ship is at rest, the part of it which is under sea level has a volume of Ad, where A denotes the cross-sectional area and d denotes the distance from sea level to the bottom of the ship. This is the volume of seawater that has been displaced by the ship. Suppose the density of the seawater is ρ. Then, the mass of water that has been displaced by the ship is $\rho \cdot Ad = \rho Ad$. So, the weight is ρAdg. This upward force balances the weight of the ship, so

$$\rho Adg = Mg \quad \text{which implies} \quad d = \frac{M}{\rho A}.$$

Now, if the ship is moving and the distance from sea level to the bottom of the ship is $d + x$, where x is a function of time. Recall from Newton's second law that force is the product of mass and acceleration. Taking the downwards direction to be positive, the buoyancy force is

$$-\rho A \left(d + x\right) g.$$

So, we have

$$M\ddot{x} = Mg - \rho A \left(d + x\right) g.$$

Using our formula for d, we have

$$\ddot{x} = -\frac{\rho A g}{M} x.$$

This represents simple harmonic motion with angular frequency $\omega = \sqrt{\rho A g / M}$ as claimed. The ship will bob up and down at this frequency. Note the inverse dependence on M, which is to be expected, but also that the frequency increases if A is large, which is not so obvious.

Lastly, suppose waves from a storm strike the ship. Taking into account the force acting on the waves and taking downwards to be negative, we have

$$M\ddot{x} = Mg - \rho A \left(d + x\right) g + F_0 \cos\left(\omega t\right).$$

Recall that at equilibrium (i.e. $x = 0$), we have $Mg = \rho Agd$. Substituting $Mg = \rho Agd$ into the equation of motion simplifies it to

$$M\ddot{x} + \rho A g x = F_0 \cos\left(\omega t\right).$$

At rest, the deck is at height H above sea level. If the bottom of the ship is at depth $d + x$, then the deck (which is a height H above that bottom) is located above sea level by the amount $H - x$. The deck is above water so long as $H - x > 0$. The instant the deck just touches the water surface is when $H - x = 0$, i.e. $x = H$. So, the sinking time t_{sink} is the first time for which this happens. Let $x\left(t\right)$ be the solution to

$$M\ddot{x} + \rho A g\, x = F_0 \cos(\omega t)$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Then the time at which the deck first goes underwater is defined implicitly by $x\left(t_{\text{sink}}\right) = H$. □

2.5 The Phase Plane Method

The phase plane method is a geometrical way of analysing second-order differential equations by turning them into first-order systems and studying their behaviour graphically — not as functions of time, but as trajectories in a plane whose axes are the variable and its derivative. Recall Definition 2.4 on the differential equation governing simple harmonic oscillation, which is

$$m\frac{d^2x}{dt^2} + kx = 0.$$

As such, we can rewrite it as follows:

$$m\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = -kx.$$

Integrating both sides yields

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = -\frac{1}{2}kx^2 + E,$$

where E is a constant. In fact this is not surprising as E is the total energy of the system! Since dx/dt denotes velocity, one would know that

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \text{ denotes the kinetic energy} \quad \text{and} \quad \frac{1}{2}kx^2 \text{ denotes the potential energy}$$

of the oscillator respectively. One should recall that the fact that E is constant is known as the conservation of energy.

This idea of turning time derivatives into space derivatives can be very useful when studying certain kinds of second-order non-linear differential equations. For example, we recall the pendulum problem (Definition 2.2) which is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0 \quad \text{with initial conditions} \quad \theta(0) = 1 \text{ and } \dot{\theta}(0) = 1.$$

Here, we have taken $g/L = 1$. Although we cannot find elementary solutions for this differential equation (recall that we can do so but the solution involving elliptic integrals would be non-elementary), we can still can gain some insights such as determining the maximum value of θ. This is simple as we have $\dot{\theta}(t) = 0$. Solving yields $\theta_{\text{max}} \approx 1.53$.

In fact, there is a nice way of thinking about what we did. One can look at the equation involving $\dot{\theta}$ and use it to think of $\dot{\theta}$ as a function of θ. If we graph that function, we can see that the graph is a closed curve. As time goes by, the point $\left(\theta, \dot{\theta}\right)$ moves around and around the closed curve. As such, the solution must be a periodic function of time. This makes sense as the physical system is a pendulum. We call this the phase plane method.

Example 2.12. We analyse the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{2}\cos y = 0 \quad \text{with initial conditions} \quad y(0) = 0 \text{ and } \dot{y}(0) = 1.$$

Note that the equation describes a non-linear oscillator, which should still typically produce bounded and oscillatory motion. In fact, for large t, $y(t)$ tends to infinity! What is really happening here? We note that we can write the differential equation as

$$\begin{aligned}
\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dy}{dt}\right)^2\right] + \frac{d}{dy}\left(\frac{1}{2}\sin y\right)\frac{dy}{dt} &= 0 \\
\frac{d}{dt}\left[\left(\frac{dy}{dt}\right)^2 + \sin y\right] &= 0 \\
\left(\frac{dy}{dt}\right)^2 + \sin y &= c
\end{aligned}$$

Substituting the initial conditions yields $c = 1$, so

$$\left(\frac{dy}{dt}\right)^2 + \sin y = 1,$$

which is the phase plane equation for the differential equation. On the $(y, \dot{y})$ phase plane, as the system moves from the point $(0,1)$ to the point $(\pi/2, 0)$, it actually never gets there! One can use the method of separation of variables to obtain

$$t = \int_0^{\pi/2} \frac{1}{\sqrt{1-\sin y}}\, dy = \int_0^{\pi/2} \frac{\sqrt{1+\sin y}}{\cos y}\, dy \geq \int_0^{\pi/2} \sec y\, dy$$

which is infinite. As such, the *correct* graph is not the one produced by Wolfram Mathematica (for instance), but rather the one in which y asymptotically approaches $\pi/2$. As such, the phase plane method helps us spot this error made by computers.

Example 2.13 (MA3264 AY25/26 Sem 1 Tutorial 4)**.** Use the phase plane method to find the largest and smallest possible values of $x(t)$ if $x(0) = 0$, $\dot{x}(0) = 1$, and $x(t)$ satisfies $\ddot{x} = \cos x$.

Solution. We have

$$\frac{d^2x}{dt^2} - \cos x = 0.$$

Let $v = \dot{x}$. Recall that

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right] = \frac{d^2x}{dt^2}.$$

Hence,

$$\frac{d}{dt}\left(\frac{1}{2}v^2 - \sin x\right) = 0 \quad \text{so} \quad \frac{1}{2}v^2 - \sin x = c.$$

Since $x(0) = 0$ and $v(0) = 1$, then $c = \frac{1}{2}$. As such, $v^2 = 2\sin x + 1$. Since $v \geq 0$, then $2\sin x + 1 \geq 0$. The turning points occur at $v = 0$, so we must have $2\sin x + 1 = 0$. One can then easily find the largest and smallest possible values of x. $\square$

Example 2.14 (MA3264 AY25/26 Sem 1 Tutorial 5)**.** Consider the differential equation

$$\frac{d^2y}{dx^2} + y = \frac{1}{2}\cosh y,$$

with $y(0) = 0,\ y'(0) = \sqrt{0.3}$. Show that the equivalent first–order equation is

$$\left(\frac{dy}{dx}\right)^2 + y^2 = \sinh y + 0.3,$$

and hence find the maximum value of y.

Solution. We have

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]+\frac{1}{2}\frac{d}{dx}\left(y^2\right)=\frac{1}{2}\frac{d}{dx}(\sinh y)$$

so

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2+y^2=\sinh y+c.$$

When $x=0$, $y=0$ and $y'=\sqrt{0.3}$. One can deduce that $c=0.3$ so the equivalent first-order equation is

$$\frac{1}{2}\left(\frac{dy}{dx}\right)^2+y^2=\sinh y+0.3.$$

To find the maximum value of y, we set the derivative to be zero, so $y^2=\sinh y+0.3$. One can use a computer to find the maximum value of y. $\square$

Chapter 3

Population Models

3.1 The Malthusian Model for Population Growth

Population modelling is a crucial area of applied mathematics that uses differential equations to understand the dynamics of populations. These models can reveal surprising and sometimes counterintuitive behaviors in various species, from fish to humans.

The total population of a country, denoted as $N(t)$, is clearly a function of time. For simplicity, N can be measured in millions, meaning values less than 1 are still meaningful. Given the current population, can we predict how it will change? To begin, consider the per capita birth rate, B, which represents the number of babies born per second, divided by the total population at that moment. The value of B varies — it might be small in a populous country or large in a smaller one, depending on societal factors like cultural attitudes toward marriage and children. B could depend on time t and the current population N.

For simplicity, we assume that B is constant, i.e. people will always have as many children as possible, regardless of time or population size. In this case, the number of births over a small time interval δt is given by $BN\delta t$.

Similarly, consider the per capita death rate D, which also depends on t (i.e. better healthcare) or N (i.e. overcrowding). Assuming D is constant, the number of deaths over δt is $DN\delta t$.

Assuming no immigration or emigration, the change in population, δN, over δt is simply the difference between births and deaths. That is,

$$\delta N = \text{births} - \text{deaths} = (B - D)\, N\delta t.$$

Recall from MA2002 that we can divide throughout by δt and take the limit as $\delta t \to 0$. We then obtain the differential equation

$$\frac{dN}{dt} = (B - D)\, N = kN \quad \text{where } k \text{ is the net growth rate.}$$

This simple model was first proposed by Thomas Malthus in 1798, laying the foundation for what is now known as Malthusian population growth (Definition 3.1).

Definition 3.1 (Malthusian growth model). Let N denote the current population, B and D denote the birth rate and death rate respectively. Then,

$$\frac{dN}{dt} = (B - D)\,N.$$

The Malthusian model predicts exponential growth if $k > 0$ or exponential decay if $k < 0$, assuming constant birth and death rates. To see why, one can easily solve the differential equation in Definition 3.1 to deduce that

$$N = N_0 e^{kt} \quad \text{where } k = B - D.$$

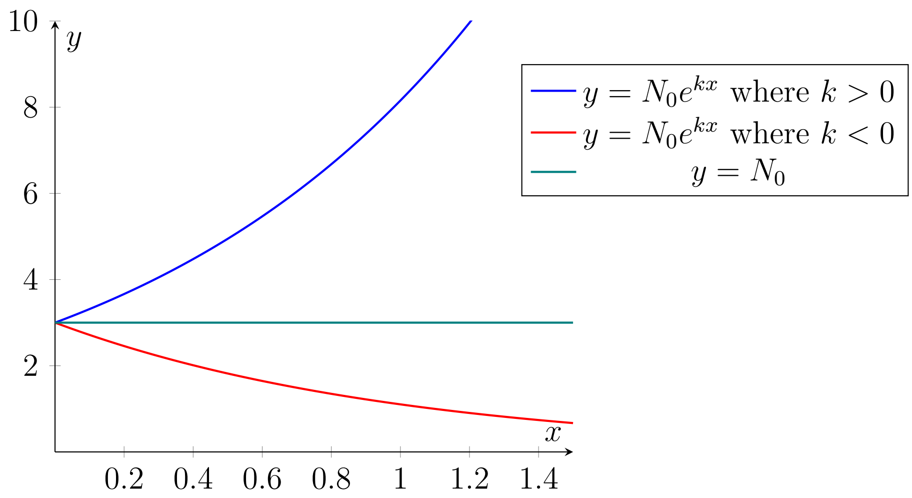


Figure 7: Interpretation of the Malthusian growth model

Malthus' model is interesting as it shows that static behaviour patterns can lead to disaster. As e^{kt} grows so quickly for $k > 0$, Malthus' assumptions must eventually go wrong — obviously there is a limit to the possible population. Eventually, if we do not control B, D will have to increase, so we have to assume that D is a function of N. Hence, we turn to Verhulst's model, which will be discussed in Chapter 3.2.

Example 3.1 (MA3264 AY25/26 Sem 1 Tutorial 5)**.** The bacteria in a certain culture number 10000 initially. Two and a half hours later, there are 11000 of them. Assuming a Malthus model, how many bacteria will there be 10 hours after the start of the experiment? How long will it take for the number to reach 20000?

Solution. Assume that the rate of growth of the population is proportional to the population. That is,

$$\frac{dN}{dt} = kN.$$

Solving the differential equation yields $N = Ce^{kt}$. Since $N(0) = 10000$ and $N(2.5) = 11000$, we have $C = 10000$ and $k = 0.4 \ln 1.1$. Hence,

$$N = 10000e^{0.4 \ln 1.1t}.$$

We leave the remaining computations as an exercise. □

Example 3.2 (MA3264 AY25/26 Sem 1 Tutorial 5)**.** On the island of *Orpsengia*, the human birth and death rates per capita are constant, and the population of the island has been doubling every 20 years. However, one day, several pirate ships arrive. All of the island women under the age of 50 decide to elope with the glamorous pirates, taking their children with them. After that, the remaining population of Orpsengia declines by half over the next ten years. What was the original birth rate per capita on Orpsengia? You will have to make several simplifying assumptions to solve this problem; that is ok as long as you list your assumptions carefully!

Solution. Suppose the original birth rate and death rate are B and D respectively. By assuming a Malthusian population, we have

$$\frac{dN}{dt} = (B - D) N \quad \text{so} \quad N = N_0 e^{(B-D)t}.$$

As the population doubles in 20 years, when $t = 20$, we have $N = 2N_0$, so $2N_0 = N_0 e^{20(B-D)}$. Hence,

$$B - D = \frac{\ln 2}{20}.$$

When the women depart, the new population model admits the differential equation

$$\frac{dN}{dt} = (B' - D) N.$$

It has solution $N = N_0 e^{B'-Dt}$. As the population declines by half over the next 10 years, when $t = 10$, $N = N_0/2$ so $10(B' - D) = \ln 2$. Hence, $B' = 0.104$. Hence, the original birth rate per capita is about 10.4%.

The assumptions made are as follows. Men and old women have same death rate as young women, which is not true in reality because men smoke, get into fights etc while on the other hand women naturally have longer life spans; the death rate of the remaining population is not changed by the departure of the girls. □

3.2 Verhulst's Model of Population Growth

Previously, we mentioned that the death rate D, should depend on N. A natural starting point is the simplest possible choice, i.e.

$$D = sN \quad \text{where } s \text{ is a constant.}$$

This assumption is often referred to as the logistic assumption. It captures the idea that finite resources in the environment lead to higher death rates as the population increases due to factors like starvation and disease. Hence, we obtain Verhulst's logistic growth model, which was proposed by Pierre-François Verhulst in the 19th century.

Definition 3.2 (Verhulst's logistic growth model)**.** Again, let N denote the current population, B and D denote the birth rate and death rate respectively. Then, we can write

$$\frac{dN}{dt} = BN - DN = BN - sN^2 = BN\left(1 - \frac{sN}{B}\right).$$

We shall analyse Verhulst's growth model. Suppose the initial population N_0 is small. Then, $N(t)$ will remain small as well. Since N^2 becomes negligible compared to N, the equation simplifies to

$$\frac{dN}{dt} \approx BN \quad \text{which has solution} \quad N = N_0 e^{Bt.}$$

Thus, for small populations, the growth is approximately exponential, as predicted by Malthus.

As the population grows, the quadratic term sN^2 dominates as N^2 increases much faster than N. At some point, the terms BN and sN^2 balance, i.e. $BN = sN^2$. This happens when

$$N \approx \frac{B}{s}.$$

At this population size, the growth rate dN/dt becomes zero, indicating that the population stabilises. As such, the quantity B/s would be of interest.

Definition 3.3 (carrying capacity)**.** In Verhulst's growth model, the value B/s is called the carrying capacity of the environment, representing the maximum sustainable population under the given conditions.

Note that Verhulst's equation can be easily solved by partial fraction decomposition (see Figure 8 for the graph of the logistic curve). Here, we consider the possibility that we begin with a small population, i.e. $N_0 < B/s$[†].

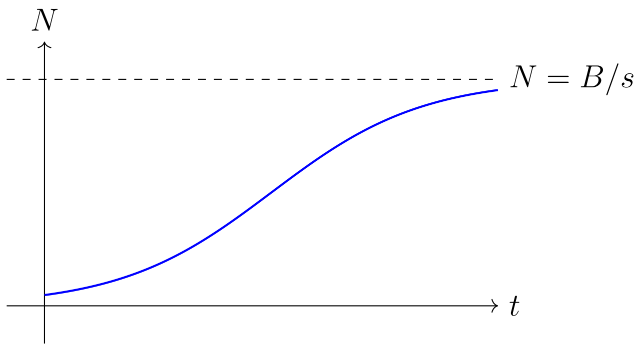


Figure 8: Graph of the logistic function

Example 3.3 (MA3264 AY25/26 Sem 1 Tutorial 5)**.** You have 200 bugs in a bottle. Every day you supply them with food and count them. After two days you have 360 bugs. It is known that the birth rate for this kind of bug is 150% per day. Assuming that the population is given by a logistic model, find the number of bugs after 3 days. Predict how many bugs you will have eventually.

[†]The case where $N_0 > B/s$ will not be discussed. In this other scenario, we assume that we begin with a large population, i.e. $N_0 > B/s$. Then, the solution is monotonically decreasing, but again, the asymptotic value is the same.

Solution. Assume that
$$\frac{dN}{dt} = BN\left(1 - \frac{sN}{B}\right)$$
where s is a constant. Let $B = 1.5$, so
$$\frac{dN}{dt} = 1.5N\left(1 - \frac{sN}{1.5}\right).$$
Solving the differential equation yields
$$N(t) = \frac{1.5/s}{1 + \left(\frac{1.5}{sN_0} - 1\right)e^{-1.5t}}.$$
When $t = 0$, $N = 200$ and when $t = 2$, $N = 360$. Substituting these yields
$$N(t) = \frac{375}{1 + 0.875e^{-1.5t}}.$$
Then, find $N(3)$ and the limit of N as t approaches infinity. We omit the calculations because they are too simple. □

3.3 Harvesting

One major application of mathematical modelling is in dealing with populations of animals. We wish to know how many we can eat (say fish). We will build on Verhulst's model, i.e. assume that the fish population would follow that model if we did not catch any. Next, we assume that we catch H fish per unit time (say year). Then, the new differential equation representing a basic harvesting model can be written as
$$\frac{dN}{dt} = bN - sN^2 - H. \tag{3.1}$$
Again, one can use partial fraction decomposition to determine the solution to the differential equation.

Example 3.4. Recall Example 3.3, but now we assume that you are keeping the bugs not as a hobby, but because you are developing a new insecticide. Suppose that you remove 80 bugs per day from the bottle, and that all of these bugs die a painful but well-deserved death as a result of being sprayed with this insecticide.

(a) What is the limiting population in this case?

(b) What is the maximum number of bugs you can put to death per day without causing the population to die out?

Solution.

(a) Recall that the birth rate for this kind of bug is 150% per day. By considering the harvesting equation (3.1), where $D = sN$ and $s \approx 0.00400$ we have

$$\frac{dN}{dt} = 1.5N - 0.004N^2 - 80.$$

The solution to the differential equation is

$$N(t) = \frac{310.61 - 64.39Ae^{-0.986t}}{1 - Ae^{-0.986t}} \quad \text{where} \quad A = \frac{N_0 - 310.61}{N_0 - 64.39}.$$

Recall that $N_0 = 200$ from Example 3.3. Hence, the limiting population is about 311.

(b) The harvesting equation is now

$$\frac{dN}{dt} = 1.5N - 0.004N^2 - H.$$

We regard the right as a quadratic polynomial in terms of N, which has discriminant $2.25 - 0.016H$. We must have this to be ≥ 0. If the discriminant was negative, then there are no real roots so no equilibria at all, meaning the population growth rate would always stay strictly positive or strictly negative regardless of how large or small N is. This is unrealistic biologically. Hence, to have realistic population dynamics with biologically meaningful equilibrium states, we require the discriminant to be ≥ 0, so the maximum value of H is about 141. □

Example 3.5 (MA3264 AY25/26 Sem 1 Tutorial 6)**.** Suppose that Peruvian fishermen take a fixed number of anchovies per year from an anchovy stock which would otherwise behave logistically, apart from occasional natural disasters. Say any fishing rate that is $\geq B^2/4s$, where $D = sN$, will be disastrous. Call this number E^*.

The fishermen want to take as many anchovies as they safely can, meaning that they want the fish to be able to bounce back from a natural disaster that pushes their population down by 10%. Advise them. That is, tell them the maximum number of fish they can take, expressed as a percentage of E^*. A hint is to assume that you start with the stable equilibrium population β_2, and compute the value of E, the harvesting rate, such that β_1, the unstable equilibrium population, becomes 90% of β_2.

Solution. We must have $\beta_1 = 0.9\beta_2$. The differential equation modelling the above-mentioned setup is

$$\frac{dN}{dt} = BN - sN^2 - E,$$

where E denotes the harvesting rate. We can write $BN - sN^2 - E$ as $-\left(sN^2 - BN + E\right)$, which is a quadratic polynomial with roots

$$\beta_1 = \frac{B - \sqrt{B^2 - 4sE}}{2s} \quad \text{and} \quad \beta_2 = \frac{B + \sqrt{B^2 - 4sE}}{2s}.$$

So,

$$\frac{\beta_1}{\beta_2} = \frac{B - \sqrt{B^2 - 4sE}}{B + \sqrt{B^2 - 4sE}} = 0.9.$$

Making E the subject of the equation yields

$$E = \frac{360}{361} \cdot \frac{B^2}{4s}$$

So, the maximum safe constant is 360/361 times of E^*. □

Example 3.6 (MA3264 AY25/26 Sem 1 Tutorial 5)**.** The sandhill crane is a beautiful Canadian bird (Figure 9) with an unfortunate liking for farm crops.

Figure 9: Sandhill cranes

For many years the cranes were protected by law, and eventually they settled down to a logistic equilibrium population of 194,600 with birth rate per capita 9.866% per year. Eventually the patience of the farmers was exhausted and they managed to have the hunting ban lifted. The farmers happily shot 10000 cranes per year, which they argued was reasonable enough since it only represents about 5% of the original population.

(a) Show that the sandhill crane is doomed.

(b) How long will it take, from the legalisation of hunting, to exterminate them?

Solution.

(a) We have $B = 0.09866$. The unharvested model admits the differential equation

$$\frac{dN}{dt} = 0.09866N\left(1 - \frac{N}{194,600}\right).$$

One hunting is legalised, the harvested model admits the differential equation

$$\frac{dN}{dt} = 0.09866N\left(1 - \frac{N}{194,600}\right) - 10000.$$

We regard the quadratic polynomial on the right as a function in terms of N, which has negative discriminant. Then, the harvested differential equation does not have a real positive equilibrium. Therefore, there is no sustainable population size, and the crane population will decline to extinction.

(b) Solve the harvesting differential equation. When $t = 0$, we have $N = 194,600$. Then, find the minimum value of t such that $N(t) < 0$. □

Example 3.7 (MA3264 AY25/26 Sem 1 Tutorial 6)**.** In harvesting models, the population will rebound if all harvesting is stopped in time. Unhappily, this is not always true: for some animals, if you drive their population down too low, they will have trouble finding mates, or they will be forced to breed with relatively close kin, which reduces genetic variability and hence their ability to resist disease. For such animals, extinction will result if the population falls too low, even if all harvesting is forbidden. Biologists call this *depensation*.

(a) Show that this situation can be modelled by the ordinary differential equation

$$\frac{dN}{dt} = -aN^3 + bN^2 - cN \tag{3.2}$$

where N is the population and a, b, c are positive constants such that $b^2 > 4ac$.

(b) Find the population below which extinction will occur.

Solution.

(a) Note that the cubic polynomial $-aN^3 + bN^2 - cN$ passes through the origin. Since $a > 0$, then for large N, the graph tends to $-\infty$. In fact, the slope is negative on the right of the origin, so the derivative is < 0 for small values of N. As such, the differential equation (3.2) indeed models the mentioned situation.

(b) We have

$$\frac{dN}{dt} = -N\left(aN^2 - bN + c\right).$$

The roots of the cubic equation $-N\left(aN^2 + bN - c\right)$ are

$$N = 0 \quad N = \frac{b \pm \Delta}{2a} \text{ where } \Delta = \sqrt{b^2 - 4ac}.$$

Note that

$$N = \frac{b - \Delta}{2a}$$

is an unstable equilibrium. It is unstable because a population slightly above that will grow away from it, while a population slightly below it will drop towards zero. We see that the tigers will become extinct if the population ever falls below that value. □

Example 3.8 (MA3264 AY25/26 Sem 1 Tutorial 6)**.** A constant harvesting rate E can lead to disastrous results. Perhaps it would be better to use *flexible* harvesting, where for example the number of fish you catch depends on how many fish there are: you catch many when there are many fish, few when there are few fish. Suppose we replace $E =$ constant with a harvesting rate $E = \alpha N^2$, where α is a positive constant with the right units.

(a) Show that the fish population will eventually settle down to a stable equilibrium with this policy.

(b) Calculate the harvesting rate R when the population is at equilibrium.

Notice that we have to choose α carefully — if you graph R as a function of α, you will find that it is small when α is small, but *also* when α is big.

(c) Find the optimal value of α.

(d) What is the harvesting rate when that optimal value is chosen? Have you seen that number before?

Solution.

(a) The differential equation is

$$\frac{dN}{dt} = bN - sN^2 - \alpha N^2 = bN - (s + \alpha) N^2.$$

There is a stable equilibrium population at

$$\frac{B}{s + \alpha}.$$

Hence, every solution with $N(0) > 0$ tends to the positive equilibrium, so we conclude that the policy yields a stable equilibrium.

(b) The harvesting rate is αN^2. At equilibrium, we have $N = \frac{B}{S+\alpha}$ so the harvesting rate is

$$\alpha \left(\frac{B}{S + \alpha} \right)^2 = \frac{\alpha B^2}{(S + \alpha)^2}.$$

(c) Let

$$f(\alpha) = \frac{\alpha B^2}{(S + \alpha)^2}.$$

Using differentiation, one can see that the unique maximum point is at $\alpha = s$.

(d) At the optimal $\alpha = s$, the harvesting rate is $B^2/4s$. This constant is precisely the critical value that appeared as the unsafe threshold under *constant* harvesting (recall Example 3.5). □

Chapter 4

Systems of First-Order Differential Equations

4.1 Solving Systems of Ordinary Differential Equations

Relationships often go through ups and downs. We shall explore a mathematical model to capture this phenomenon. Romeo loves Juliet, but Juliet has a subtler response. When Romeo shows strong affection, Juliet finds his enthusiasm overwhelming, making her feelings for him cool down. However, when Romeo becomes indifferent, Juliet finds him mysteriously attractive. Romeo, on the other hand, reacts more directly: his love for Juliet increases when she is warm and decreases when she is cold.

Let $R(t)$ and $J(t)$ denote Romeo's and Juliet's feelings over time. These feelings can be modelled using the system of first-order linear ordinary differential equations as follows:

$$\frac{dR}{dt} = aJ \text{ and } \frac{dJ}{dt} = -bR \quad \text{where} \quad R(0) = \alpha \text{ and } J(0) = \beta.$$

Here, $a, b > 0$ are positive constants and α, β are initial feelings at $t = 0$. This system describes the interaction between their feelings.

We propose solutions of the form

$$R = Ae^{\lambda t} \quad \text{and} \quad J = Be^{\lambda t}.$$

Note that these can be obtained by transforming the system into two separate second-order linear differential equations, and then construct the characteristic equation to find the solution. Anyway, returning to the Romeo and Juliet problem, substituting R and J into the differential equations yields

$$A\lambda = aB \quad \text{and} \quad B\lambda = -bA.$$

Eliminating A and B, we have $\lambda^2 = -ab$. Since $\lambda^2 < 0$, the solutions are complex, i.e. $\lambda = \pm i\sqrt{ab}$. As such, the general solution can be expressed as a linear combination of sin and cos as follows:

$$R = C\cos\left(\sqrt{ab}t\right) + D\sin\left(\sqrt{ab}t\right) \quad \text{and} \quad J = E\cos\left(\sqrt{ab}t\right) + F\sin\left(\sqrt{ab}t\right)$$

All that is left is to find C, D, E, F. This can be done so by considering the initial conditions. As such,

$$R = \alpha\cos\left(\sqrt{ab}t\right) + \beta\sqrt{\frac{a}{b}}\sin\left(\sqrt{ab}t\right) \quad \text{and} \quad J = \beta\cos\left(\sqrt{ab}t\right) - \alpha\sqrt{\frac{b}{a}}\sin\left(\sqrt{ab}t\right).$$

Motivated by the above, we consider a more general system, i.e.

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

We can write this as a matrix equation, which is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

We consider the solution

$$\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} = e^{rt}\mathbf{u}_0 \quad \text{where} \quad \mathbf{u}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

As such,

$$re^{rt}\mathbf{u}_0 = \mathbf{B}e^{rt}\mathbf{u}_0 \quad \text{where} \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

or equivalently,

$$\mathbf{B}\mathbf{u}_0 = r\mathbf{u}_0.$$

This is analogous to the matrix equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, where λ and $\mathbf{v}$ are an eigenvalue and corresponding eigenvector of the matrix $\mathbf{A}$! As such, the possibilities of r are given by the eigenvalues of $\mathbf{B}$. We have

$$\left(\mathbf{B} - r\mathbf{I}\right)\mathbf{u}_0 = \mathbf{0}$$

so non-trivial solutions exist if $\det(\mathbf{B} - r\mathbf{I}) = 0$, i.e. if $(a-r)(d-r) - bc = 0$. Except the case where this quadratic polynomial in r has two repeated roots (i.e. discriminant zero), we must have two solutions r_1 and r_2, which implies

$$\mathbf{u} = c_1 e^{r_1 t}\mathbf{u}_1 + c_2 e^{r_2 t}\mathbf{u}_2,$$

where c_1 and c_2 are constants and $\mathbf{u}_1$ and $\mathbf{u}_2$ are the eigenvectors of r_1 and r_2 respectively. Naturally, r_1 and r_2 might be complex so we might have to interpret the exponential functions in terms of sine and cosine.

Example 4.1. Solve

$$\frac{dx}{dt} = -4x + 3y$$
$$\frac{dy}{dt} = -2x + y$$

Solution. In fact, such questions are also covered in MA3220. The matrix representation is

$$\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \quad \text{which} \quad \text{has eigenvalues } -1 \text{ and } -2.$$

The eigenspaces are

$$E_{-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad E_{-2} = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}.$$

The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

In other words,

$$x = c_1 e^{-t} + 3c_2 e^{-2t} \quad \text{and} \quad y = c_2 e^{-t} + 2c_2 e^{-2t}.$$

One can deduce the values of c_1 and c_2 given the values of $x(0)$ and $y(0)$. $\square$

4.2 Classification using a Phase Plane

Consider a linear system of the form

$$\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u} \quad \text{where} \quad \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We classify the phase portrait by analysing the eigenvalues of the matrix $\mathbf{B}$. These are given by

$$r = \frac{1}{2}\left(\operatorname{tr}(\mathbf{B}) \pm \sqrt{(\operatorname{tr}(\mathbf{B}))^2 - 4\det(\mathbf{B})}\right).$$

Here, $\operatorname{tr}(\mathbf{B}) = a + d$ is the trace and $\det(\mathbf{B}) = ad - bc$ is the determinant. We now classify linear systems via their eigenvalues.

(i) Real and distinct eigenvalues

- Both positive implies we have a nodel source, i.e. trajectories move inwards, not spiralling
- Both negative implies we have a nodal sink, i.e. trajectories move outwards, not spiralling
- One positive and one negative eigenvalue implies we have a saddle point, i.e. trajectories approach along one eigenvector and escape along another

(ii) Complex eigenvalues. We write $r = \alpha \pm i\beta$ with $\alpha = \frac{1}{2}\operatorname{tr}(\mathbf{B})$.

- If $\alpha < 0$, we have a spiral sink so trajectories spiral inwards
- If $\alpha > 0$, we have a spiral soruce so trajectories spiral outwards
- If $\alpha = 0$, we have a centre, i..e closed orbits with no decay or growth

We recall the Romeo and Juliet model in Chapter 4.1. Note that

$$\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix}$$

where $a, b > 0$. Here, $\operatorname{tr}(\mathbf{B}) = 0$, $\det(\mathbf{B}) = ab > 0$, and the *discriminant* is $-4ab < 0$. Thus, the eigenvalues are purely imaginary. That is, $r = \pm i\sqrt{ab}$. Hence, the phase portrait is a centre — we have closed orbits in a clockwise direction, because along the positive R-axis, we have

$$\frac{dJ}{dt} = aR > 0.$$

Introducing constant reinforcements (g, m), we obtain the differential equation

$$\frac{d\mathbf{r}}{dt} - \mathbf{B}\mathbf{r} = \mathbf{F}$$

whose equilibrium point is shifted to

$$\mathbf{r}_{\text{equilibrium}} = -\mathbf{B}^{-1}\mathbf{F}.$$

The phase portrait is translated by this vector, but its qualitative shape remains exactly the same. Thus, by choosing reinforcement values cleverly, one may shift the saddle point so that the initial state lies on the favourable side of the separatrix.

Chapter 5

Modelling with Non-Linear Systems

5.1 The Lotka-Volterra Model

Lions like to eat zebras, and depend on them. That is, the lion population goes up if there are many zebras. However, if there were no zebras, then the lions would die out. The zebras eat grass, and would get along just fine if there were no lions. Their population tends to go down when there are lions about but when left to themselves, their population goes up.

Suppose at time t, there are $L(t)$ lions and $Z(t)$ zebras, and assume a Malthusian model for both the lions and zebras in the absence of the other. We shall assume that there is a stab equilibrium at populations (L_0, Z_0). This suggests that we can devise the following model:

$$\begin{aligned} \frac{dL}{dt} &= -(L - L_0) + 2(Z - Z_0) \\ \frac{dZ}{dt} &= -2(L - L_0) + \frac{1}{2}(Z - Z_0) \end{aligned}$$

This resembles the models mentioned in the previous chapter, except that the equilibrium has been shifted from $(0, 0)$ to a point in the first quadrant. One can verify that in this case, the equilibrium point is indeed (L_0, Z_0). In fact, it is a type of stable equilibrium — it is a spiral sink. If there is some kind of disturbance, the populations of lions and zebras fluctuate up and down for a while but they eventually get close to equilibrium.

We shall further analyse this model. Suppose $L(0) = 0$ and $Z(0) = 0$. Then, when $t = 0$, $dL/dt = L_0 - 2Z_0$ which is non-zero. The same can be said for dZ/dt when evaluated at $t = 0$. This means that lions and zebras are coming into existence out of nothing or that we will immediately have negative numbers of animals! Moreover, the system we are trying to model has two equilibria — other than (L_0, Z_0), we also have $(0, 0)$. That is, it must always be possible to have no lions and no zebras. However, this is not what we will get or with any linear model as such systems only ever have one equilibrium point.

We shall construct a different mathematical model for the lion and zebra situation. This is similar to the logistic model we discussed previously in a sense that the death rate per capita of zebras, D_Z, is not fixed. Here, D_Z depends on the number of lions, so suppose

$$D_Z = sL \quad \text{where} \quad s \text{ is a positive constant.}$$

The constant s tells us something about the relationship between lions and zebras. We continue to assume a Malthusian model for the zebra birth rate per capita, B_Z. As such, we have

$$\frac{dZ}{dt} = B_Z Z - sLZ.$$

What about the lions? When there is a shortage of zebras, they can eat other animals so they will not really starve. However, zebras are *nice and fat*, so the ones which really suffer from a shortage of zebras is not the adult lions but rather, the baby lions. This is because if there is an insufficient number of *nice and fat* zebras around, then the mother lion cannot produce enough milk for the young, and the latter will die. As such, the effect of a shortage of zebras is to reduce the effective birth rate of the lions. Hence, we use a Malthusian model for the death rate of the lions. We write

$$\frac{dL}{dt} = uZL - D_L L.$$

The pair of equations

$$\frac{dZ}{dt} = B_L Z - sLZ$$
$$\frac{dL}{dt} = uZL - D_L L$$

givesa famous model of such populations known as the Lotka-Volterra model, or the predator-prey model. One verifies that $(L, Z) = (B_Z/s, D_L/u)$ is an equilibrium point, and so is $(L, Z) = (0, 0)$! As such, we are on the right track. However, the Lotka-Volterra eqautions are non-linear!

5.2 Linearisation

Recall the Taylor series expansion for functions of several variables from MA2104/MA3210. Now, suppose that we have a pair of non-linear simultaneous first-order ODEs governing a pair of functions of time $(x(t), y(t))$ of the ofrm

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

and suppose that this system is known to have an equilibrium point at $(x,y) = (a,b)$. This implies $f(a,b) = g(a,b) = 0$. As such, when we obtain the Taylor series expansion for these two functions around the point (a,b), the constant term vanishes! Keeping the linear terms and discarding terms of higher order, we obtain the following equations:

$$\begin{aligned}\frac{dx}{dt} &\approx f_x(a,b)(x-a) + f_y(a,b)(y-b)\\ \frac{dy}{dt} &= g_x(a,b)(x-a) + g_y(a,b)(y-b)\end{aligned}$$

which we can write as

$$\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b)\\ g_x(a,b) & g_y(a,b)\end{bmatrix}\begin{bmatrix} x-a\\ y-b\end{bmatrix}$$

and now, the equations have become linear! One recalls from MA2104/ST2131 that the matrix of partial derivatives here is known as the Jacobian matrix. In summary, near an equilibrium point, a non-linear system can be approximated by a certain linear system with a matrix given by the Jacobian of the original system. This new system is called the linearisation of the original system. Indeed, this is a good piece of news as we can now apply knowledge from the previous chapter to solve our new pair of differential equations! In particular, the classification theorem of equilibrium points (recall MA3220) now applies. As such, we still have a good idea of what is happening in the phase diagram near to those points — we just have to compute the Jacobian there.

Example 5.1 (lions and zebras). Now, recall the Lotka-Volterra equations for the lion and zebra problem. That is,

$$\begin{aligned}\frac{dL}{dt} &= uZL - D_L L = f(L,Z)\\ \frac{dZ}{dt} &= B_Z Z - sLZ = g(L,Z)\end{aligned}$$

The Jacobian matrix here is

$$\mathbf{J}(L,Z) = \begin{bmatrix} uZ - D_L & uL \\ -sZ & B_Z - sL \end{bmatrix}$$

and we wish to evaluate it at the two equilibrium points.

The first is $(L,Z) = (0,0)$ and so

$$\mathbf{J}(0,0) = \begin{bmatrix} -D_L & 0 \\ 0 & B_Z \end{bmatrix}.$$

It is easy to see that this is a saddle point. Since the matrix $\mathbf{J}(0,0)$ is diagonal, the eigenvalues are the diagonal entries, namely $-D_L$ and B_Z, with corresponding eigenvectors $(1,0)$ and $(0,1)$ respectively. One checks that this makes sense because in the phase plane, everything is rushing towards the origin along the lion axis and away from the origin along the zebra axis. We can expect that if we start nearer to the L-axis, then the lion population will decrease greatly until the zebra populaltion increases rapdily, which makes sense!

The other equilibrium point is $(L,Z) = (B_Z/s, D_L/u)$, and one checks that

$$\mathbf{J}\left(\frac{B_Z}{s}, \frac{D_L}{u}\right) = \begin{bmatrix} 0 & uB_Z/s \\ -sD_L/u & 0 \end{bmatrix}$$

which we recognise as a centre. As such, in the middle of the phase diagram corresponding to large numbers of both lions and zebras, we expect to see the swirling motion. In fact, the direction of motion is clockwise.

Now, take the two Lotka-Volterra equations, multiply the dL/dt equation by $B_Z/L - s$, and multiply the dZ/dt equation by $D_L/Z - u$. Adding the resulting equations, we have

$$\left(\frac{B_Z}{L} - s\right)\frac{dL}{dt} + \left(\frac{D_L}{Z} - u\right)\frac{dZ}{dt} = 0.$$

Hence,

$$B_Z \ln L - sL + D_L \ln Z - uZ = c,$$

where c is an arbitrary constant. This is an exact relation between Z and L although the equation cannot be explicitly solved. One way to draw the graphs is to define a function $F(L,Z)$ on the phase plane by

$$F(L,Z) = B_Z \ln L - sL + D_L \ln Z - uZ.$$

One checks thatt this function has a global minimum at the point $(L,Z) = (B_Z/s, D_L/u)$, with contour curves around that point which are all closed. Since the paths in the phase plane are all closed curves, we see that all solutions of the Lotka-Volterra equations are periodic.

The Lotka-Volterra model can be used to understand an interesting paradox known as the paradox of pesticides. This is the strange observation that when a certain pest has a predator, using pesticides can actually leads to more pests than we had initially!

5.3 Logistic Lotka-Volterra Model

The Lotka-Volterra model assumes that the zebra population grows according to a Malthusian model when there are no lions. We know that this is not realistic so we should use something like the logistic model for them, while keeping the old equation for the lions. As such, we obtain the following pair of differential equations:

$$\begin{aligned}\frac{dL}{dt} &= uZL - D_L L \\ \frac{dZ}{dt} &= B_Z Z - pZ^2 - sLZ\end{aligned}$$

Here, p is the logistic constant, so the equilibrium population of zebras would be B_Z/p in the complete absence of lions.

In this model, there are actually three equilibrium points in the phase diagram. The first is the obvious one $(0,0)$. The second is almost as obvious, i.e. if there are no lions, then the zebras will approach a logistic equilibrium along the Z-axis, i.e. the point $(0, B_Z/p)$. The third and most interesting one is at

$$\left(\frac{B_Z - pD_L/u}{s}, \frac{D_L}{u}\right).$$

We omit the remaining details.

Chapter 6

Modelling with Partial Differential Equations

6.1 Introduction

Definition 6.1 (partial differential equation). A partial differential equation (PDE) is an equation containing an unknown function $u(x, y, \ldots)$ of two or more independent variables $x, y, \ldots$ and its partial derivatives with respect to these variables. We call u the dependent variable.

PDEs allow us to deal with situations where something depends on space as well as time. So far, all the models that we studied so far have only involved variations with time.

We discuss a method to solve PDEs known as the separation of variables. This method can be used to solve PDEs involving two independent variables say x and y that can be separated from each other in the PDE. There are similarities between this method and the technique of separating variables for ODEs in the first chapter.

We make the following observation. Suppose

$$u(x, y) = X(x) Y(y).$$

Then,

$$\begin{aligned} u_x &= X'(x) Y(y) \\ u_y &= X(x) Y'(y) \\ u_{xx} &= X''(x) Y(y) \\ u_{yy} &= X(x) Y''(y) \\ u_{xy} &= X'(x) Y'(y) \end{aligned}$$

Note that each derivative of u remains separated as a product of a function of x and a function of y. We can exploit this feature. Consider a PDE of the form

$$u_x = f(x)\, g(y)\, u_y.$$

If a solution of the form $u(x,y) = X(x)\, Y(y)$ exists, then one can deduce that

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = g(y) \cdot \frac{Y'(y)}{Y(y)}.$$

The important observation here is that the LHS is a function of x whereas the RHS is a function of y. We conclude that the LHS and RHS both equate to some constant k. As such, we obtain the two ODEs as follows:

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = k \quad \text{and} \quad g(y) \cdot \frac{Y'(y)}{Y(y)} = k.$$

In fact, it is easy to solve this pair of differential equations!

Example 6.1. Solve $u_x + xu_y = 0$.

Solution. Suppose a solution of the form $u(x,y) = X(x)\, Y(y)$ exists. Then, we deduce that

$$\begin{aligned} X'(x)\, Y(y) + xX(x)\, Y'(y) &= 0 \\ \frac{1}{x} \cdot \frac{X'(x)}{X(x)} &= -\frac{Y'(y)}{Y(y)} \end{aligned}$$

As such,

$$\frac{1}{x} \cdot \frac{X'(x)}{X(x)} = k \quad \text{and} \quad -\frac{Y'(y)}{Y(y)} = k.$$

This implies that $X(x) = ae^{kx^2/2}$ and $Y = be^{-ky}$ for some constants aA and b. As such, the general solution is

$$u(x,y) = X(x)\, Y(y) = ce^{kx^2/2 - ky}.$$

Here, $c = ab$ is also a constant. $\square$

6.2 The Wave Equation

Consider a flexible string that lies stretched tightly (another word would be 'taut') along the x-axis and has its ends fixed at $x = 0$ and $x = \pi$. We pull it along the u-axis so that it is stationary and has some specific shape $u = f(x)$ at time $t = 0$. Consequently, $f(0) = 0$ and $f(\pi) = 0$. We can assume that $f(x)$ is continuous and bounded. When we let go of the string, it will move. We assume that the only forces acting are those due to the tension in the string and that the pieces of the string will only move along the u-axis.

Now, the u-coordinate of any point on the string will become a function of time as well as a function of x. So, it becomes a function $u(t, x)$ of both t and x. Note that this function satisfies the boundary conditions

$$u(t, 0) = 0 \quad \text{and} \quad u(t, \pi) = 0$$

for all t as the ends are nailed down. Also, the initial condition

$$u(0, x) = f(x) \quad \text{is satisfied.}$$

Also, since the string is initially stationary, then

$$\frac{\partial u}{\partial t}(0, x) = 0.$$

We now introduce the wave equation.

Definition 6.2 (wave equation)**.** Let c be a fixed non-negative real constant representing the propagation speed of the wave. Then,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Also, $u(t, 0) = u(t, \pi) = 0$, $u(0, x) = f(x)$ and u_t evaluated at $t = 0$ gives 0.

We note that the function

$$u(t, x) = \frac{f(x + ct) + f(x - ct)}{2} \quad \text{is a solution to the wave equation.}$$

More generally, we have D'Alembert's formula (Theorem 6.1). One should check that the above equation indeed satisfies the wave equation. Moreover, the four conditions should be satisfied.

Theorem 6.1 (D'Alembert's solution to the wave equation)**.** The function

$$u(t,x) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\ d\xi$$

is a solution to the wave equation (Definition 6.2).

Initially, $f(x)$ was only defined between $x = 0$ and $x = \pi$, but the interpretation of D'Alembert's solution is that we can extend $f(x)$ to be an odd, periodic function of period 2π, for which one can then verify that $u(t,0) = u(t,\pi) = 0$.

Consider a function $f(x)$ defined on $[0,\pi]$. We aim to extend f to an odd periodic function with periodic 2π, covering $\mathbb{R}$. First, we define f on $[-\pi,\pi]$ as an odd function, i.e. for any $x \in [-\pi, 0]$, define $f(-x) = -f(x)$. This extension makes f an odd function on $[-\pi,\pi]$. Next, we extend f periodically across the real line, i.e. for any x outside $[-\pi,\pi]$, define

$$f(x) = f(x - 2n\pi) \quad \text{where } n \in \mathbb{Z} \text{ and } x - 2n\pi \in [-\pi,\pi].$$

This creates a periodic function with period 2π.

We then consider the function $f(x-ct)$, where $c > 0$. Note that $f(x-1)$ is the same as $f(x)$, just shifted 1 unit to the right. Similarly, $f(x-ct)$ represents the same shape as $f(x)$ but shifted to the right by ct units. The function $f(x+ct)$ represents $f(x)$ shifted to the left by ct, moving in the opposite direction with the same speed.

Geometrically, D'Alembert's solution describes the solution to the wave equation as a combination of two travelling waves, where the term

$$\begin{aligned} &f(x-ct)/2 \quad \text{representsa wave traveling to the right at speed } c \quad \text{and} \\ &f(x+ct)/2 \quad \text{representsa wave traveling to the left at speed } c \end{aligned}$$

Each piece maintains the shape of the original function $f(x)/2$ and moves without distortion.

We can also solve the wave equation using the method f separation of variables.

Suppose we wish to solve

$$u_{tt} = c^2 u_{xx}$$

subjected to the following conditions:

$$u(t,0) = u(t,\pi) = 0 \quad u(0,x) = f(x) \quad u_t(0,x) = 0$$

We separate the variables, i.e.

$$u(t,x) = v(x)\, w(t)$$

and obtain

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = -\lambda.$$

The usual separation argument from before implies λ is a constant, so we obtain the following pair of ODEs:

$$v'' + \lambda v = 0 \quad \text{and} \quad w'' + \lambda c^2 w = 0$$

Let us force $v(x)$ to vanish at $x = 0$ and $x = \pi$, so we can set

$$u(t,0) = v(0)\, w(t) = 0 \quad \text{and} \quad u(t,\pi) = v(\pi)\, w(t) = 0.$$

This is somewhat different from the usual scenario of solving second-order ODEs. Normally, we give some information about the function at one point, i.e. we might ask for solutions to the ODE $y'' + \lambda y = 0$ where $y(0)$ and $y'(0)$ are given. However, we are now giving the information from two different points.

If $\lambda < 0$, then $u(0) = 0$ implies that all solutions to the equation $v'' + \lambda v = 0$ are proportional to $\sinh x$, and such a function cannot intersect the x-axis twice. As such, λ cannot be negative. If $\lambda = 0$, then $v(x)$ is a straight line function which cannot intersect the x-axis twice. As such, $\lambda > 0$. We write $\lambda = n^2$ for some $n > 0$. As such,

$$v(x) = C \cos nx + D \sin x \quad \text{for some constants } C \text{ and } D.$$

Since $v(0) = 0$, then $C = 0$ and so $v(x) = D \sin nx$. If we want $v(\pi) = 0$, then it implies $\sin n\pi = 0$. As such, $n \in \mathbb{Z}$ so we also introduce this constraint. Earlier, we mentioned that $n > 0$. Combining both properties, we conclude that $n \in \mathbb{Z}^+$.

Solving the other equation for $w(t)$, we obtain

$$w(t) = A\cos nct + B\sin nct \quad \text{for some constants } A \text{ and } B.$$

We force $w(t)$ to satisfy $w'(0) = 0$ since we want $u_t(0, x) = u(x)\, v'(0) = 0$. So, now $B = 0$ and we are left with $w(t) = A\cos nct$. As such, our complete solution is

$$u(t, x) = b_n \sin nx \cos nct.$$

Here, b_n is an arbitrary constant and again, recall that n is a positive integer. This satisfies three of the four conditions in Definition 6.2. So, the only condition that is not yet satisfied is $u(0, x) = f(x)$.

We recall some concepts from MA2101. Think about the set of all continuous functions on $[0, \pi]$. It is a vector space over $\mathbb{R}$ (an obvious fact). What is a possible basis for it? Well, an example of a basis is given by the following set:

$$\left\{\sin nx : n \in \mathbb{Z}^+\right\}$$

In other words, any continuous function g on $[0, \pi]$ can be expressed as the following:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

This holds for certain real numbers b_n. In particular, the formula for b_n is

$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x) \sin nx \; dx.$$

That is, $2/\pi$ times the integral plays the role of the scalar product here. The series

$$\sum_{n=1}^{\infty} b_n \sin x \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx \; dx$$

is known as the Fourier series of $g(x)$. There is an amazing fact that the Fourier series allows us to express any function on this interval as the components — b_n! We now return to the problem of solving the wave equation. Recall that we have extended $f(x)$ to be an odd function of period 2π. As such, it has a Fourier sine series, and since f is continuous and has only a finite number of sharp corners, we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Now, consider the series

$$\sum_{n=1}^{\infty} b_n \sin nx \cos nct \quad \text{where} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \; dx.$$

First, observe that if we substitute $t = 0$ in this series, then we obtain $f(x)$ expressed as its Fourier sine series. Next, since the wave equation is linear and each term in this series is a solution to the wave equation, then this series is also a solution to the wave equation.

To summarise, the solution to the wave equation is

$$u(t,x) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \; dx \right) \sin nx \cos nct$$

We have done everything for the interval $[0, \pi]$. For a general interval $[0, L]$ of any length L, it is easy to obtain a solution to the modified wave equation. The basis functions are now $\sin(n\pi x/L)$ which are periodic with period $2L$ instead of 2π like before. The Fourier series formulae are

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad b_n = \frac{2}{L} \int_0^{L} g(x) \sin\left(\frac{n\pi x}{L}\right) \; dx.$$

f will now be a function that vanishes at 0 and L!

6.3 The Heat Equation

Consider the temperature in a long thin bar or wire of constant cross-section and homogeneous material which is oriented along the x-axis and is perfectly insulated laterally, so that heat only flows in the x-direction. Then the temperature u depends only on x and t and is given by the one-dimensional heat equation.

Definition 6.3 (heat equation). The heat equation states that

$$u_t = c^2 u_{xx},$$

where c^2 is a positive constant called the thermal diffusivity (sometimes this is denoted by κ). It measures how quickly heat moves through the bar and depends on what it is made of.

Let us assume that the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the following boundary conditions:

$$u(0, t) = 0 \quad u(L, t) = 0 \quad \text{for all } t,$$

and the initial temperature of the bar is $f(x)$, so that we have the initial condition

$$u(x, 0) = f(x).$$

Here, we will assume that when $f(x)$ is extended to be an odd function, it equals its Fourier sine series everywhere. Remember that this can happen, even if $f(x)$ is discontinuous at some points.

Notice that, unlike the wave equation, which needs four pieces of data, here we only need three, which matches the fact that the heat equation only involves a total of three derivatives (two in the spatial direction, but only one in the time direction).

The heat equation is particularly useful in modeling for the following reason. Think of an ordinary function, $g(x)$. We can think of its second derivative $g''(x)$ as a measure of the extent to which its graph is not a straight line (recall that the second derivative is zero everywhere if and only if $g(x)$ is a linear function). We say that $g''(x)$ measures the curvature of the graph.

The heat equation says that the second spatial derivative of u is equal to its time derivative. So as time goes by, if the graph of u as a function of x is concave up, then u will increase; whereas if the graph is concave down, then it tends to decrease. The effect in both cases is to reduce the curvature. So we can picture the equation as something that, given an initial shape described by $f(x)$, tries to "straighten it out." And of course, that is how we expect heat to behave, i.e. heat flows from a hotter region to a cooler region, trying to even out its distribution.

It turns out that the solution of the one-dimensional heat equation looks like this.

$$u\left(x,t\right)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2c^2}{L^2}t\right),$$

where the b_n are just the Fourier sine coefficients of $f(x)$.

Notice that we get exponentials here instead of sines. That is because the separated equation for the function of t is first-order (for obvious reasons), and as we know, first-order ordinary differential equations tend to have exponential solutions. Because of this, the solutions to the heat equation depend on the direction of time. This means that this PDE is useful for modeling situations involving irreversible time evolution.

6.4 Fisher's Equation

Life on dry land took a long time to evolve: animals and plants had lived in the sea for hundreds of millions of years before that happened, roughly 450 million years ago. Of course, it must have started along the sea shore, that is, along a line. There must have been some kind of marine plant growing along the shore line; a mutation occurred (helped by the extreme exposure to sunlight) which made one of them, at some particular time and place, better able to tolerate drying out. The descendants of that individual had a tremendous advantage over the non-mutated neighbours because sometimes there is a succession of exceptionally low tides which leave the plants dry for a long time. So they would have outcompeted their neighbours, and the mutation would have spread along the shoreline like a wave. Eventually, the result would be a plant that could survive out of the water full-time.

The process of spreading along the shoreline is clearly irreversible, so we need an equation like the heat equation, not the wave equation: we need a heat equation with a wave-like solution! On the other hand, we do not want the effect to go away, like the temperature going down as heat dissipates. What we need is a combination of the Heat Equation with our model of the spread of a rumour. In 1937, Ronald

Fisher proposed the following equation to model this situation:

$$u_t = \alpha u_{xx} + \beta u(1-u)$$

where $u(x,t)$ is the fraction of the plants at any given place and time which have mutated (so $1 - u(x,t)$ is the fraction which haven't). This is indeed a combination of the Heat Equation with the rumour equation! The constant α tells you how quickly the mutation tends to spread in space, while β measures how quickly it grows in time at a specific point in space (they have different units, of course).

This is a non-linear partial differential equation, and finding all of its solutions is very difficult. But it is important because it has many other applications, for example to the theory of how flames move and to the theory of how nuclear reactors work. To solve this equation, we specify some initial function $f(x) = u(x,0)$ and then try to evolve it forward in time. A good model for $f(x)$ would be a delta function.

We seek a wave solution of the form

$$u(x,t) = U(x - ct)$$

where $U(s)$, $s = x - ct$, describes the wave moving to the right at constant speed c. Substituting this into Fisher's equation gives

$$\alpha U'' + cU' + \beta U - \beta U^2 = 0.$$

This ODE has two equilibria: $(U, U') = (0,0)$ and $(U, U') = (1,0)$. The Jacobian at these equilibria determines their stability. We have

$$\mathbf{J}(U, U') = \begin{bmatrix} 0 & 1 \\ \frac{\beta}{\alpha}(2U-1) & -\frac{c}{\alpha} \end{bmatrix}.$$

We omit the remaining details.