MA3211 MA3211S MA4247 MA5217 Complex Analysis Notes

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Reference books:

(1) Sasane, S. M. and Sasane, A. (2014). *A Friendly Approach to Complex Analysis*. World Scientific.

(2) Needham, T. (2023). *Visual Complex Analysis: 25th Anniversary Edition*. Oxford.

(3) Conway, J. B. (1973). *Functions of One Complex Variable I*. Springer.

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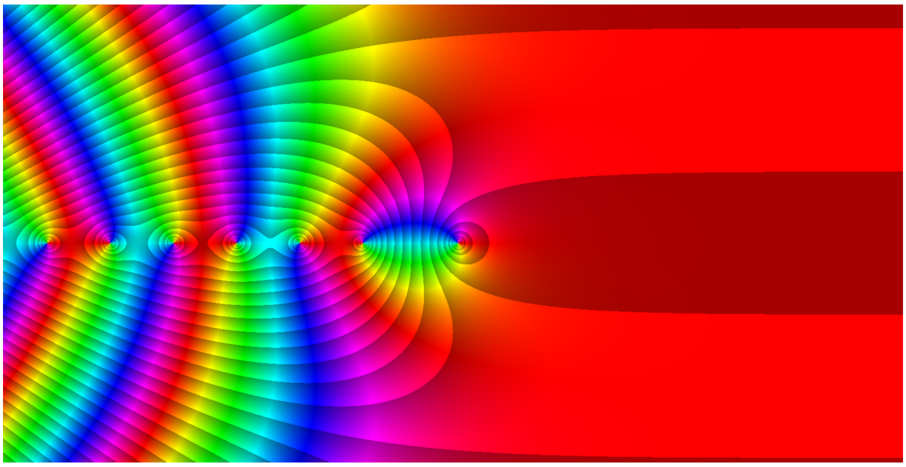


1. Complex Numbers

1.1. *Some Background Knowledge*

First, we define

$$\mathbb{R}[x] \quad \text{to be} \quad \text{the set of polynomials with real coefficients.}$$

The polynomial $x^2+1 \in \mathbb{R}[x]$ of degree 2 over $\mathbb{R}$ has no solution in $\mathbb{R}$ since for all $\alpha \in \mathbb{R}$, we have $\alpha^2+1>0$, so x^2+1 is irreducible over $\mathbb{R}[x]$. For those who have prior knowledge on Abstract Algebra, since $\mathbb{R}[x]$ is a principal ideal domain (PID)[†], then

$$\left(x^2+1\right)\mathbb{R}[x] \subseteq \mathbb{R}[x] \quad \text{is} \quad \text{a maximal ideal.}$$

As such, we are now in position to define the complex numbers $\mathbb{C}$.

Definition 1.1 (complex numbers)**.** Define

$$\mathbb{C} = \mathbb{R}[x] / \left(x^2+1\right)\mathbb{R}[x]$$

to be the quotient ring of $\mathbb{R}[x]$ modulo the maximal ideal $\left(x^2+1\right)\mathbb{R}[x]$. This is a field, known as the field of complex numbers.

Proposition 1.1. The image of

$$x \in \mathbb{R}[x] \text{ in } \mathbb{C} \quad \text{is denoted by} \quad i \in \mathbb{C},$$

called the imaginary unit. i has the property that $i^2=-1$.

Proposition 1.2 (field extension)**.** The composite of the canonical ring homomorphisms

$$\mathbb{R} \hookrightarrow \mathbb{R}[x] \twoheadrightarrow \mathbb{C} \quad \text{where} \quad x \mapsto i$$

is an inclusion of fields $\mathbb{R} \hookrightarrow \mathbb{C}$ so $\mathbb{C}$ is a field extension of $\mathbb{R}$.

Proposition 1.3. As an $\mathbb{R}$-vector space, $\mathbb{C}$ has dimension 2 with standard ordered $\mathbb{R}$-basis $\{1,i\}$.

Definition 1.2. The $\mathbb{R}$-linear projection maps

$$\mathrm{Re}:\mathbb{C}\to\mathbb{R} \text{ where } z\mapsto x \quad \text{and} \quad \mathrm{Im}:\mathbb{C}\to\mathbb{R} \text{ where } z\mapsto y$$

are called the real part and imaginary part of $z \in \mathbb{C}$. So,

$$\text{for all } z\in\mathbb{C} \quad \text{one has } z=\mathrm{Re}\,z+i\,\mathrm{Im}\,z \text{ in } \mathbb{C}.$$

[†]Recall from MA3201 that if F is a field, then $F[x]$ is a Euclidean domain. In fact, recall the chain of inclusions ED $\subseteq$ PID $\subseteq$ UFD, where ED and UFD denote Euclidean domain and unique factorisation domain respectively. I recall in one of Sadhukhan's MA2101S that one student asked whether F is a field implies $F[x]$ is also a field. Clearly, this is wrong and Sadhukhan mentioned that $F[x]$ is a UFD. It was only when I crashed one of Bao Hauncheng's MA4203 lectures (first lecture actually) where I learnt that the stronger statement $F[x]$ is an ED holds.

Proposition 1.4 (field operations). The field operations of $\mathbb{C}$, expressed in terms of the real/imaginary parts, are:

(i) Addition/Subtraction:

$$(a+ib)\pm(c+id)=(a\pm c)+i(b\pm d)$$

(ii) Multiplication:

$$(a+ib)\cdot(c+id)=(ac-bd)+i(ad+bc)$$

(iii) Division:

$$\frac{(a+ib)}{(c+id)}=\frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

(iv) Multiplicative inverse:

$$(c+id)^{-1}=\frac{c-id}{c^2+d^2}$$

Definition 1.3 (complex conjugation). The $\mathbb{R}$-linear map

$$\overline{(\cdot)}:\mathbb{C}\to\mathbb{C} \quad \text{where} \quad z=x+iy\mapsto\bar{z}=x-iy$$

is called complex conjugation.

Proposition 1.5. We say that complex conjugation is an automorphism of $\mathbb{C}$ as a field over $\mathbb{R}$. The automorphism group $\operatorname{Aut}(\mathbb{C}/\mathbb{R})$ is of order 2. That is to say,

$$\bar{\bar{z}}=z.$$

Proposition 1.6. The following properties hold for all $z,w\in\mathbb{C}$:

(i) $\overline{z+w}=\bar{z}+\bar{w}$ and $\overline{zw}=\bar{z}\bar{w}$

(ii) $\operatorname{Re}z=\frac{1}{2}(z+\bar{z})$ and $\operatorname{Im}z=\frac{1}{2i}(z-\bar{z})$

Definition 1.4 (absolute value). The absolute value of a complex number is the map

$$|\cdot|_{\mathbb{C}}:\mathbb{C}\to\mathbb{R}_{\geq 0} \text{ where } z\mapsto|z|_{\mathbb{C}} \quad \text{given by} \quad |z|_{\mathbb{C}}=\sqrt{(\operatorname{Re}z)^2+(\operatorname{Im}z)^2}=\sqrt{z\bar{z}}.$$

As such, if $z=x+iy$ (where $x,y\in\mathbb{R}$), we have

$$|z|_{\mathbb{C}}^2=x^2+y^2=z\bar{z}.$$

Proposition 1.7. For any $a\in\mathbb{R}\subseteq\mathbb{C}$, we have $|a|_{\mathbb{C}}=|a|_{\mathbb{R}}$.

Lemma 1.1. For any $z,w\in\mathbb{C}$, we have

(i) Positive-definiteness: $|z|_{\mathbb{C}}=0$ in $\mathbb{R}_{\geq 0}$ if and only if $z=0$ in $\mathbb{C}$

(ii) $|\bar{z}|_{\mathbb{C}}=|z|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$

(iii) Multiplicativity: $|zw|_{\mathbb{C}}=|z|_{\mathbb{C}}|w|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$

(iv) $|\operatorname{Re}z|_{\mathbb{R}},|\operatorname{Im}(z)|_{\mathbb{R}}\leq|z|_{\mathbb{C}}$ in $\mathbb{R}_{\geq 0}$

Proof. **(i)** and **(ii)** are trivial. To prove **(iii)**, we have

$$|zw|_{\mathbb{C}}^2 = zw\overline{zw} = z\overline{z}\cdot w\overline{w} = |z|_{\mathbb{C}}^2\,|w|_{\mathbb{C}}^2\,.$$

Taking square roots on both sides, **(iii)** follows.

For **(iv)**, let $z = x+iy$, where $x,y\in\mathbb{R}$. Then, $x^2, y^2 \le x^2+y^2$, so $|x|_{\mathbb{R}} \le |z|_{\mathbb{C}}$ and $|y|_{\mathbb{R}} \le |z|_{\mathbb{C}}$. □

Lemma 1.2 (triangle inequality)**.** For any $z,w\in\mathbb{C}$, we have

$$|z+w|_{\mathbb{C}} \le |z|_{\mathbb{C}} + |w|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\ge 0}.$$

Proof. We have

$$\begin{aligned} |z+w|_{\mathbb{C}}^2 = (z+w)\left(\overline{z+w}\right) &= z\overline{z} + w\overline{w} + (z\overline{w} + \overline{z}w) \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2\,\mathrm{Re}\,(z\overline{w}) \\ &\le |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2\,|z\overline{w}|_{\mathbb{C}} \quad \text{by } \textbf{(iv)} \text{ of Lemma 1.1} \\ &= |z|_{\mathbb{C}}^2 + |w|_{\mathbb{C}}^2 + 2\,|z|_{\mathbb{C}}\,|w|_{\mathbb{C}} \\ &= (|z|_{\mathbb{C}} + |w|_{\mathbb{C}})^2 \end{aligned}$$

Taking square roots on both sides, the result follows. □

By **(i)** and **(iii)** of Lemma 1.1 on the positive-definiteness and multiplicativity, as well as Lemma 1.2 on the triangle inequality, we infer that

$$|\cdot|_{\mathbb{C}} \text{ is an absolute value of } \mathbb{C} \quad \text{in the abstract sense.}$$

Corollary 1.1. We say that

$$\mathbb{C} \text{ equipped with the absolute value function } |\cdot|_{\mathbb{C}} \quad \text{as a normed } \mathbb{R}\text{-vector space}$$

is isomorphic to $\mathbb{R}^2$ with the standard Euclidean norm $\|\cdot\|_2$, so $\mathbb{C}$ is said to be *Cauchy complete*.

Corollary 1.2 (generalised triangle inequality)**.** For any $z_1, z_2, \ldots, z_n \in \mathbb{C}$, we have

$$|z_1 + \ldots + z_n|_{\mathbb{C}} \le |z_1|_{\mathbb{C}} + \ldots + |z_n|_{\mathbb{C}} \quad \text{in } \mathbb{R}_{\ge 0}.$$

Proof. Consider the triangle inequality (Lemma 1.2) and use induction. □

Theorem 1.1 (Cauchy-Schwarz inequality for $\mathbb{R}^2$)**.** For any $z,w\in\mathbb{C}$, we have

$$|\langle z,w\rangle_{\mathbb{R}^2}|_{\mathbb{R}} \le |z|_{\mathbb{C}}\,|w|_{\mathbb{C}} \quad \text{with equality} \quad \text{if and only if} z \text{ and } w \text{ are } \mathbb{R}\text{-linearly dependent.}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the inner product of the two inputs. That is to say,

$$z = x+iy \text{ and } w = u+iv \quad \text{implies} \quad \langle z,w\rangle_{\mathbb{R}^2} = xu+yv.$$

Proof. The trick is as follows:

$$\begin{aligned}\langle z,w\rangle_{\mathbb{R}^2}^2+\langle iz,w\rangle_{\mathbb{R}^2}^2&=(xu+yv)^2+(-yu+xv)^2\\&=x^2u^2+y^2v^2+2xuyv+y^2u^2+x^2v^2-2yuxv\\&=\left(x^2+y^2\right)\left(u^2+v^2\right)\\&=|z|_{\mathbb{C}}^2\,|w|_{\mathbb{C}}^2\end{aligned}$$

which implies $\langle z,w\rangle_{\mathbb{R}^2}\leq|z|_{\mathbb{C}}\,|w|_{\mathbb{C}}$. Equality holds if and only if $\langle iz,w\rangle_{\mathbb{R}^2}=0$, or equivalently, $-yu+xv=0$, i.e. z and w are $\mathbb{R}$-linearly dependent. Well, to be more explicit, we recall that

$$z=\begin{bmatrix}x\\y\end{bmatrix}\quad\text{and}\quad w=\begin{bmatrix}u\\v\end{bmatrix}\quad\text{as } \textit{vectors} \text{ in } \mathbb{R}^2.$$

If z and w are linearly dependent, there exists $k\in\mathbb{R}$ such that $(x,y)=k(u,v)$, so $x=ku$ and $y=kv$. As such, $-yu+xv=0$. □

We can generalise Theorem 1.1 to the Cauchy-Schwarz inequality for $\mathbb{C}^n$ (Theorem 1.2).

Theorem 1.2 (Cauchy-Schwarz inequality for $\mathbb{C}^n$)**.** For any $z_1,\ldots,z_n,w_1,\ldots,w_n\in\mathbb{C}$, we have

$$|z_1w_1+\ldots+z_nw_n|_{\mathbb{C}}^2\leq\left(|z_1|_{\mathbb{C}}^2+\ldots+|z_n|_{\mathbb{C}}^2\right)\left(|w_1|_{\mathbb{C}}^2+\ldots+|w_n|_{\mathbb{C}}^2\right)$$

and

$$\text{equality holds}\quad\text{if and only if}\quad\begin{bmatrix}z_1\\\vdots\\z_n\end{bmatrix}\text{ and }\begin{bmatrix}\overline{w_1}\\\vdots\\\overline{w_n}\end{bmatrix}\text{ are }\mathbb{C}\text{-linearly dependent over }\mathbb{C}^n.$$

Equivalently, this means that there exist $\lambda,\mu\in\mathbb{C}$ which are both non-zero such that

$$\text{for all }1\leq j\leq n\quad\text{we have}\quad\lambda z_j=\mu\overline{w_j}\quad\text{in }\mathbb{C}.$$

Proof. We have

$$\begin{aligned}0&\leq\sum_{i<j}\left|z_i\overline{w_j}-z_j\overline{w_i}\right|_{\mathbb{C}}^2\\&=\sum_{i<j}\left(z_i\overline{w_j}-z_j\overline{w_i}\right)\left(\overline{z_i\overline{w_j}-z_j\overline{w_i}}\right)\\&=\sum_{i<j}|z_i|^2\left|w_j\right|^2+\left|z_j\right|^2|w_i|^2-2\,\mathrm{Re}\left(z_i\overline{z_j}w_i\overline{w_j}\right)\end{aligned}$$

We now add the following term to both sides of the inequality:

$$\left|\sum_{i=1}^n z_iw_i\right|^2=\sum_{i=1}^n|z_i|^2|w_i|^2+\sum_{i<j}\left(z_iw_i\overline{z_jw_j}+\overline{z_iw_i}z_jw_j\right)$$

for which it follows that

$$\begin{aligned}\left|\sum_{i=1}^n z_iw_i\right|^2&\leq\sum_{i=1}^n|z_i|^2|w_i|^2+\sum_{i<j}\left(|z_i|^2\left|w_j\right|^2+\left|z_j\right|^2|w_i|^2\right)\\&=\left(\sum_{i=1}^n|z_i|^2\right)\left(\sum_{i=1}^n|w_i|^2\right)\end{aligned}$$

Equality holds if and only if

$$\sum_{i<j} \left| z_i \overline{w_j} - z_j \overline{w_i} \right|_{\mathbb{C}}^2 = 0.$$

This holds if and only if for all $i < j$, one has $z_i \overline{w_j} = z_j \overline{w_i}$. □

Example 1.1 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all solutions of the equation $e^{e^z} = 1$.

Solution. Note that $1 = e^{2k\pi i}$ for all $k \in \mathbb{Z}$. Since the exponential function is injective, we have $e^z = 2k\pi i$. Hence, $z = \ln|2k\pi| + i\pi/2$. □

1.2. *Complex-Valued Functions*

Let X be any set. Then, we have the following:

$$\begin{aligned} \text{Maps}(X, \mathbb{R}) &= \{\text{all } \mathbb{R}\text{-valued functions on } X\} \quad \text{is an } \mathbb{R}\text{-vector space} \\ \text{Maps}(X, \mathbb{C}) &= \{\text{all } \mathbb{C}\text{-valued functions on } X\} \quad \text{is a } \mathbb{C}\text{-vector space} \end{aligned}$$

Proposition 1.8. The $\mathbb{R}$-basis $\{1, i\}$ of $\mathbb{C}$ gives an $\mathbb{R}$-linear decomposition:

$$\text{Maps}(X, \mathbb{C}) \cong \text{Maps}(X, \mathbb{R}) \oplus i \cdot \text{Maps}(X, \mathbb{R}) \quad \text{where} \quad f \mapsto \text{Re} f + i \cdot \text{Im} f.$$

This is such that for any $x \in X$,

$$\text{Re}(f)(x) = \text{Re}(f) \in \mathbb{R}, \quad \text{Im}(f)(x) = \text{Im}(f) \in \mathbb{R}.$$

Proposition 1.9. The $\mathbb{R}$-automorphism $(\bar{\cdot})$ of $\mathbb{C}$ also gives an $\mathbb{R}$-linear automorphism:

$$(\bar{\cdot}) : \text{Maps}(X, \mathbb{C}) \to \text{Maps}(X, \mathbb{C}) \quad \text{where} \quad f \mapsto \overline{f}.$$

This is such that for any $x \in X$,

$$\overline{f}(x) = \overline{f(x)} \quad \text{in } \mathbb{C}.$$

Proposition 1.10. One has the following decomposition:

$$\text{Re} f = \frac{f + \overline{f}}{2}, \quad \text{Im} f = \frac{f - \overline{f}}{2i}.$$

2. Holomorphic and Analytic Functions

2.1. *Holomorphic Functions*

Definition 2.1. Let

$$\begin{aligned} \Omega \subseteq \mathbb{C} \quad & \text{be an open and connected set in } \mathbb{C} \\ H(\Omega) \quad & \text{be the set of holomorphic functions in } \Omega \end{aligned}$$

Definition 2.2 (holomorphic function)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set. A function $f : \Omega \to \mathbb{C}$ is holomorphic at a or $\mathbb{C}$-differentiable at a (Proposition 2.3) if and only if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists in } \mathbb{C}.$$

In this case, the limit, which is uniquely determined by f and a, is called the holomorphic derivative of f at a, denoted by

$$\frac{df}{dz}(a) = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{in } \mathbb{C}.$$

As such,

$$f : \Omega \to \mathbb{C} \text{ is holomorphic on } G \quad \text{if and only if} \quad \text{for all } a \in G,\ f \text{ is holomorphic at } a.$$

Proposition 2.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f, g : \Omega \to \mathbb{C}$ be functions holomorphic at a. Then, the following hold:

(i) **$\mathbb{C}$-linearity:** for all $c, d \in \mathbb{C}$,

$$\text{the function } cf + dg : \Omega \to \mathbb{C} \quad \text{is} \quad \text{also holomorphic at } a$$

equipped with

$$\text{its holomorphic derivative} \quad (cf + dg)'(a) = c \cdot f'(a) + d \cdot g'(a) \text{ in } \mathbb{C}$$

(ii) **Product rule:** the function $f \cdot g : \mathbb{C} \to \mathbb{C}$ is also holomorphic at a equipped with its holomorphic derivative

$$(fg)'(a) = f'(a)\, g(a) + g'(a)\, f(a) \quad \text{in } \mathbb{C}$$

Remark 2.1. Recall Definition 2.1, which mentioned that $H(\Omega)$ denotes the set of all functions $f : \Omega \to \mathbb{C}$ which are holomorphic on Ω. We say that

$$H(\Omega) \quad \text{is} \quad \text{a } \mathbb{C}\text{-algebra under pointwise } \pm, \times \text{ of functions.}$$

Note that for any point $a \in \Omega$, we have the evaluation at a map, i.e.

$$\mathrm{ev}_a : H(\Omega) \to \mathbb{C} \quad \text{where} \quad f \mapsto f(a),$$

which is a $\mathbb{C}$-algebra homomorphism.

Also, Proposition 2.1 says that the derivative at a map

$$H(\Omega) \to \mathbb{C} \quad \text{where} \quad f \mapsto f'(a)$$

is a $\mathbb{C}$-linear derivative of $H(\Omega)$ to the $H(\Omega)$-module $\mathbb{C}$ via ev_a.

Example 2.1 (identity map)**.** For any open $\Omega \subset \mathbb{C}$, the identity map id is holomorphic with derivative

$$\mathrm{id}'(a) = \frac{dz}{dz}(a) = 1 \quad \text{for all } a \in G.$$

Hence, $z \in H(\Omega)$. In fact, for any polynomial $f \in \mathbb{C}[z]^{\dagger}$, the function $z \mapsto f(z)$ is also $H(G)$.

Example 2.2. For any open $G = \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$, the reciprocal function z^{-1} is holomorphic with derivative

$$\frac{dz^{-1}}{dz}(a) = -\frac{1}{a^2} \quad \text{for all } a \in G.$$

Hence, $z^{-1} \in H(\Omega)$. Moreover, for any Laurent polynomial $f \in \mathbb{C}\left[z, z^{-1}\right]$ (we will only discuss this when formally defining Laurent series/polynomials in Theorem 5.1), the function $z \mapsto f(z)$ is also in $H(\Omega)$.

Proposition 2.2 (chain rule)**.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be open sets. Let

$$f : \Omega_1 \to \mathbb{C} \text{ and } g : \Omega_2 \to \mathbb{C} \quad \text{such that} \quad f(\Omega_1) \subseteq \Omega_2$$

so $g \circ f : \Omega_1 \to \mathbb{C}$ is defined. If f is holomorphic at a and g is holomorphic at $f(a)$, then $g \circ f$ is holomorphic at a, equipped with its holomorphic derivative

$$(g \circ f)'(a) = g'(f(a))\, f'(a).$$

Proof. Let $b = f(a) \in \Omega_2$. Define the functions $\xi : \Omega_1 \to \mathbb{C}$ and $\eta : \Omega_2 \to \mathbb{C}$ by setting

$$\xi(z) = \begin{cases} \dfrac{f(z) - f(a)}{z - a} - f'(a) & \text{if } z \in \Omega_1 \setminus \{a\} \\ \text{any value} & \text{if } z = a \end{cases} \quad \text{and} \quad \eta(w) = \begin{cases} \dfrac{g(w) - g(b)}{w - b} - g'(b) & \text{if } w \in \Omega_2 \setminus \{b\} \\ \text{any value} & \text{if } w = b. \end{cases}$$

Then, for all $z \in \Omega_1$ and $w \in \Omega_2$, we have the following in $\mathbb{C}$:

$$\begin{aligned} f(z) - f(a) &= \left[f'(a) + \xi(z)\right](z - a) \\ g(w) - g(b) &= \left[g'(b) + \eta(w)\right](w - b) \end{aligned}$$

Thus, for all $z \in \Omega_1$, we have

$$\begin{aligned} g(f(z)) - g(f(a)) &= \left(g'(f(a)) + \eta(f(z))\right)\left(f(z) - f(a)\right) \\ &= \left(g'(f(a)) + \eta(f(z))\right)\left(f'(a) + \xi(z)\right)(z - a) \end{aligned}$$

so for all $z \in \Omega_1 \setminus \{a\}$, we have

$$\frac{g(f(z)) - g(f(a))}{z - a} = \left(g'(f(a)) + \eta(f(z))\right)\left(f'(a) + \xi(z)\right).$$

†Here, one should perhaps recall from MA3201 that $\mathbb{C}[z]$ denotes the set of all polynomials in z with complex coefficients. That is, $\mathbb{C}[z] \ni f(z) = a_0 + a_1 z + \ldots + a_n z^n \quad \text{where } a_0, a_1, \ldots, a_n \in \mathbb{C}$.

Since

$$\begin{aligned} f \quad & \text{is holomorphic at } a \in \Omega_1 \quad \text{and} \\ g \quad & \text{is holomorphic at } b \in \Omega_2 \end{aligned}$$

then

$$\lim_{z \to a} \xi(z) = 0 \text{ and } \lim_{w \to b} \eta(w) = 0.$$

Also,

$$f \text{ is continuous at } a \quad \text{implies} \quad \lim_{z \to a} f(z) = f(a) = b.$$

Hence,

$$\lim_{z \to a} \frac{g(f(z)) - g(f(a))}{z - a} \text{ exists in } \mathbb{C} \quad \text{and} \quad \text{equals } g'(f(a)) f'(a).$$

□

Next, recall Definition 2.3 on $\mathbb{R}$-differentiability from MA3210.

Definition 2.3 ($\mathbb{R}$-differentiability)**.** We say that f is $\mathbb{R}$-differentiable at a if and only if there exists an $\mathbb{R}$-linear map $(Df)(a) : \mathbb{C} \to \mathbb{C}$ such that

$$\text{for all } \varepsilon \in \mathbb{R}_{>0}, \text{ there exists } \delta \in \mathbb{R}_{>0}$$

such that

$$\text{for all } z \in G \text{ with } 0 \leq \|z - a\| < \delta \quad \text{we have} \quad \|f(z) - f(a) - (Df)(a)(z - a)\| \leq \varepsilon \cdot \|z - a\|.$$

When this holds, the $\mathbb{R}$-linear map $(Df)(a)$ is uniquely determined by f and a and we call this the derivative of f at a.

Proposition 2.3 ($\mathbb{C}$-differentiability)**.** If f is holomorphic at a ($\mathbb{C}$-differentiable at a), then f is $\mathbb{R}$-differentiable at a and

$$(Df)(a) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{is the image of} \quad f'(a) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

under the following canonical inclusion:

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \quad \text{where} \quad z \mapsto \text{multiplication by } x.$$

Corollary 2.1. Suppose f is holomorphic on Ω and for all $a \in G$, we have $f'(a) = 0$ in $\mathbb{C}$. Then, f is locally constant on Ω.

Proof. Let $a \in \Omega$ be an arbitrary point. Choose $r \in \mathbb{R}_{>0}$ be sufficiently small such that $B(a, r) \subseteq G$, where

$$B(a, r) \quad \text{is} \quad \text{the open ball in } \mathbb{C} \text{ centred at } a \text{ of radius } r.$$

By the mean-value inequality, for any $z \in B(a, r)$, there exists $\xi \in [a, z] \subseteq B(a, r)$ such that

$$\|f(z) - f(a)\| \leq \|f'(\xi)\| \, \|z - a\|$$

Since $f'(\xi) = 0$, then f is constant of value $f(a)$ on $B(a, r)$. □

Remark 2.2. Throughout this set of notes, we will generally use the terms open ball $B(a,r)$ and open disc $D(a,r)$ interchangeably. Also, the same can be said for closed balls and closed discs.

Now, identify $\mathbb{C}$ with the standard $\mathbb{R}$-basis $\{1,i\}$. Then, consider the following comparison:

$$\mathbb{R}^2 \xleftrightarrow{1,i} \mathbb{C} \xrightarrow{z \mapsto \text{multiplication by } z} \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \xleftrightarrow{1,i} \mathcal{M}_{2\times 2}(\mathbb{R})$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto a+bi \mapsto (x+yi \mapsto (a+bi)(x+yi) = (ax-by)+i(bx+ay)) \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

We infer that via 1 and i, the matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$$

corresponds to the $\mathbb{R}$-linear map $\mathbb{C} \to \mathbb{C}$ given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad x+yi \mapsto (px+qy)+i(rx+sy).$$

This $\mathbb{R}$-linear map is $\mathbb{C}$-linear if and only if $p=s$ and $q=-r$ in $\mathbb{R}$. As such, we can set $a=p$ and $q=-b$.

Now, again via 1 and i, write

$$f:\Omega \to \mathbb{C} \quad \text{as} \quad x+iy \mapsto f(x+iy) = u(x,y)+iv(x,y).$$

Suppose f is $\mathbb{R}$-differentiable at a. Then,

$$(Df)(a) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \quad \text{corresponds to} \quad \begin{bmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial u}{\partial y}(a) \\ \frac{\partial v}{\partial x}(a) & \frac{\partial v}{\partial y}(a) \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}).$$

Hence, $(Df)(a)$ lies in the image of $\mathbb{C} \hookrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ if and only if

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \text{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).$$

This is precisely the Cauchy-Riemann equations (will formally introduce in Theorem 2.1).

2.2. *The Cauchy-Riemann Equations*

Theorem 2.1 (Cauchy-Riemann equations)**.** Let $\Omega \subseteq \mathbb{C}$ be an open set. Let $f:\Omega \to \mathbb{C}$ be a function written as

$$x+iy \mapsto f(x+iy) = u(x,y)+iv(x,y).$$

Suppose f is $\mathbb{R}$-differentiable at a. Then,

$$f \text{ is holomorphic at } a \quad \text{if and only if} \quad \frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \text{ and } \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a) \text{ are satisfied.}$$

Theorem 2.2 (polar form of CR equations)**.** If u and v are expressed in terms of polar coordinates (r,θ), then

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Proof. Using the substitution $z=re^{i\theta}$, we have $x=r\cos\theta$ and $y=r\sin\theta$. Since $f(z)=u(x,y)+iv(x,y)$, we will now perform change of variables from (x,y) to (r,θ). By the chain rule for partial derivatives, to compute $\partial u/\partial r$,

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta.$$

By the CR equations (Theorem 2.1),

$$\frac{\partial u}{\partial r}=\frac{\partial v}{\partial y}\cos\theta-\frac{\partial v}{\partial x}\sin\theta.$$

To compute $\partial v/\partial \theta$,

$$\frac{\partial v}{\partial \theta}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial v}{\partial x}(-r\sin\theta)+\frac{\partial v}{\partial y}(r\cos\theta).$$

It is thus clear that the first equation of the theorem holds true. The proof of the second theorem is left as an exercise. □

Theorem 2.3. Let $f(z)=u(x,y)+iv(x,y)$. Suppose the first-order partial derivatives of u and v (u_x,u_y,v_x and v_y) exist in a neighbourhood of z. If they are continuous at z and the CR equations hold, then f is differentiable at z.

Example 2.3. Suppose

$$f(z)=\begin{cases}(\overline{z})^2/z & \text{if } z\neq 0;\\ 0 & \text{if } z=0.\end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at the point $z=0$ but the derivative of f fails to exist at $z=0$.

Solution. We let $z=x+iy$, where $x,y\in\mathbb{R}$. Then, for $z\neq 0$,

$$f(z)=\frac{(x-iy)^2}{x+iy}=\frac{(x-iy)^3}{x^2+y^2}=\frac{x^3-3xy^2}{x^2+y^2}+i\left(\frac{-3x^2y+y^3}{x^2+y^2}\right)$$

which is of the form $f(z)=u(x,y)+iv(x,y)$. The reader can check that at $(0,0)$, u_x,u_y,v_x,v_y are all zero, so the CR equations are satisfied. Next, we consider the following limit:

$$L=\lim_{h\to 0}\frac{(\overline{h})^2/h-0}{h}=\lim_{h\to 0}\left(\frac{\overline{h}}{h}\right)^2=\lim_{(x,y)\to(0,0)}\left(\frac{x-iy}{x+iy}\right)^2.$$

Say we approach along the real axis. Then, $L=1$. However, if we approach along the line $y=x$,

$$L=\lim_{(x,x)\to(0,0)}\left[\frac{x(1-i)}{x(1+i)}\right]^2=-1$$

so we conclude that $f'(0)$ does not exist. □

Example 2.4. Let

$$f(z)=f(x,y)=\begin{cases}\dfrac{xy(x+iy)}{x^2+y^2} & z\neq 0,\\ 0 & z=0.\end{cases}$$

Show that the Cauchy-Riemann equations are satisfied at $z=0$ but f is not differentiable at $z=0$.

Solution. We let the reader verify that the CR equations are satisfied at $z=0$. As for differentiability, let $h=a+ib$, where $a,b\in\mathbb{R}$. Then consider

$$\frac{f(h)-f(0)}{h}=\frac{ab(a+ib)}{(a^2+b^2)(a+ib)}=\frac{ab}{a^2+b^2}.$$

We need to prove that as $(a,b)\to(0,0)$, the limit L does not exist. Suppose we approach along the x-axis, then $L=0$. However, if we approach along the line $y=x$, we have

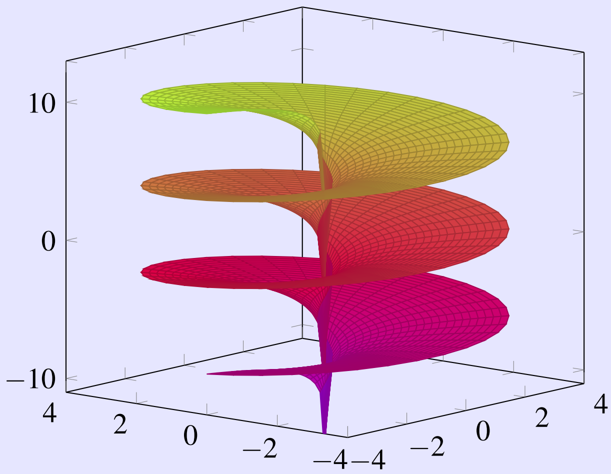
$$L=\lim_{a\to 0}\frac{a^2}{a^2+a^2}=\frac{1}{2}.$$

As such, the limit L does not exist so we can conclude that $f'(0)$ does not exist. □

Definition 2.4 (principal logarithm)**.** Define

$$\operatorname{Log} z=\ln|z|+i\operatorname{Arg} z.$$

Note that $\operatorname{Log} z$ is a single-valued function defined on $\mathbb{C}\setminus\{0\}$.



2.3. *Analytic Functions and Entire Functions*

Definition 2.5 (power series)**.** A power series over $\mathbb{C}$ in the variable z centred at $a\in\mathbb{C}$ is a formal sum

$$\sum_{n=0}^{\infty}a_n(z-a)^n\quad\text{for all } n\in\mathbb{N}\text{ and } a_n\in\mathbb{C}.$$

Definition 2.6 (different types of convergence)**.** Let

$$\sum_{n=0}^{\infty}a_n(z-a)^n\quad\text{with } z\in\mathbb{C}\quad\text{be a power series over }\mathbb{C}.$$

We say that

(i) the series converges at $z\in\mathbb{C}$ if and only if

$$\lim_{N\to\infty}\sum_{n=0}^{N}a_n(z-a)^n\quad\text{exists in }\mathbb{C};$$

(ii) the series converges absolutely at $z\in\mathbb{C}$ if and only if

$$\sum_{n=0}^{\infty}|a_n(z-a)^n|<\infty\quad\text{in }\mathbb{R}_{\geq 0};$$

(iii) the series converges normally on some compact $D \subseteq \mathbb{C}$ if and only if

$$\sum_{n=0}^{\infty} \sup_{z \in D} |a_n (z-a)^n| < \infty \quad \text{in } \mathbb{R}_{\geq 0};$$

(iv) the series converges locally normally on some open $U \subseteq \mathbb{C}$ if and only if for all $a \in U$, there exists a neighbourhood $D \subseteq U$ such t hat

$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{converges normally on } D$$

Example 2.5. We have the classic example of the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \quad \text{in } \mathbb{C}.$$

this series converges absolutely for all $z \in \mathbb{C}$ with $|z| < 1$ to $1/(1-z) \in \mathbb{C}$ and it does not converge for all $z \in \mathbb{C}$ with $|z| > 1$. Also, for all $r \in (0,1)$, the series converges normally on $\overline{B(0,r)}$ and it converges locally normally on $B(0,1)$.

Lemma 2.1. Let

$$S = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{with } z \in \mathbb{C} \quad \text{be a power series over } \mathbb{C}.$$

Then, the following hold:

(i) If S converges absolutely at $z_0 \in \mathbb{C}$, then it converges normally on the compact set $\overline{B}(a, |z_0 - a|)$

(ii) If S converges at $z_0 \in \mathbb{C}$, then it converges locally normally on the open set $B(a, |z_0 - a|)$

Definition 2.7 (radius of convergence)**.** The radius of convergence of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

is given by

$$\begin{aligned} R &= \sup \{r \in \mathbb{R}_{\geq 0} : f(z) \text{ converges at all points in } B(a,r)\} \\ &= \sup \left\{r \in \mathbb{R}_{\geq 0} : f(z) \text{ converges absolutely at all points in } \overline{B(a,r)}\right\} \end{aligned}$$

We note that $R \in \mathbb{R}_{\geq 0}$.

Proposition 2.4 (Cauchy-Hadamard formula)**.** There is a nice formula on the radius of convergence of a power series over $\mathbb{C}$ which is given by

$$\frac{1}{R} = \limsup_{n} |a_n|^{1/n}.$$

One notes that the Cauchy-Hadamard formula in Proposition 2.4 can be easily deduced from the root test.

Definition 2.8 (analytic function). Let $U \subseteq \mathbb{C}$ be an open set and $a \in U$ be a point. A $\mathbb{C}$-valued function $\varphi : U \to \mathbb{C}$ on U is analytic at $a \in U$ if and only if there exists a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ centred at } a \text{ with positive radius of convergence } R$$

such that for all $z \in U \cap B(a,R)$, one has

$$\varphi(z) = f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{in } \mathbb{C}.$$

Then,

$$\varphi : U \to \mathbb{C} \text{ is an analytic function (on } U) \quad \text{if and only if} \quad \text{for all } a \in U,\ \varphi \text{ is analytic at } a.$$

Proposition 2.5. Let

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{be a power series centred at 0} \quad \text{with positive radius of convergence } R.$$

Write $f : B(0,R) \to \mathbb{C}$ for the $\mathbb{C}$-valued function it represents. Then, f is analytic on $B(0,R)$.

We will see an alternative and more rigorous way of formulating Proposition 2.5 in Proposition 2.6[†].

Example 2.6. Show that there are no analytic functions $f = u + iv$ such that $u(x,y) = x^2 + y^2$.

Solution. Suppose on the contrary that there exists some analytic function f. Then, $u_x = 2x$ and $u_y = 2y$, so by the CR equations, $v_y = 2x$ and $v_x = -2y$. $v_y = 2x$ implies that $v(x,y) = 2xy + g(x)$. Taking the partial with respect to x and substituting it into $v_x = -2y$, we have $2y + g'(x) = -2y$. As such, $g'(x) = -4y$, so $g(x) = -4xy + c$, where c is an arbitrary constant. Putting everything together,

$$f(x,y) = x^2 + y^2 + i(-2xy + c).$$

However, this does not satisfy $u_x = v_y$ in the CR equations. So, such an f does not exist. □

Example 2.7. Suppose f is analytic and real-valued in a domain D. Prove that f is constant in D.

Solution. Suppose $f(z) = u + iv$. We have $\mathrm{Im}(f) = 0$ so by the CR equations, $u_x = 0$ and $u_y = -v_x = 0$. This implies that $f'(z) = u_x + iv_x = 0$ so f is constant in D. □

Example 2.8. Suppose f and $\overline{f}$ are analytic in a domain D. Show that f is constant in D.

Solution. Observe that $\mathrm{Re}(f) = \left(f + \overline{f}\right)/2$ which is real-valued and analytic if both f and $\overline{f}$ are analytic. By Example 2.7, $\mathrm{Re}(f)$ is constant, so f is constant. □

Proposition 2.6. For any $a \in B(0,R)$ and $k \in \mathbb{N}$, define

$$d_k = \sum_{n=k}^{\infty} \binom{n}{k} c_n a^{n-k}.$$

Then, the following properties hold:

(i) For all $k \in \mathbb{N}$, the series d_k converges absolutely in $\mathbb{C}$

[†]As you will see in Proposition 2.6, the latter is indeed more rigorous. Also, I think Prof. Chin Chee Whye set *something related* for an iteration of his MA2108S finals.

(ii) The power series

$$g(z) = \sum_{k=0}^{\infty} d_k (z-a)^k \quad \text{has positive radius of convergence } r \geq R - |a| > 0$$

(iii) For all $z \in B(0,R) \cap B(a,r)$, we have $f(z) = g(z)$

Proof. We first prove **(i)**. Fix $\rho \in \mathbb{R}_{\geq 0}$ with $|a| < \rho < R$. Then,

$$\begin{aligned}
\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}\binom{n}{k}|c_n|\,|a|^{n-k}\right)(\rho-|a|)^k &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{n}{k}|c_n|\,|a|^{n-k}(\rho-|a|)^k \\
&= \sum_{n=0}^{\infty}|c_n|\left[\sum_{k=0}^{n}\binom{n}{k}|a|^{n-k}(\rho-|a|)^k\right] \\
&= \sum_{n=0}|c_n|\,(|a|+\rho-|a|)^n \\
&= \sum_{n=0}^{\infty}|c_n|\,\rho^n
\end{aligned}$$

which is $< \infty$ by the choice of ρ. Hence, the series defining d_k converges absolutely, proving **(i)**.

Next, we take a look at **(ii)**. As the power series

$$\sum_{k=0}^{\infty} |d_k|(\rho-|a|)^k \quad \text{is finite,}$$

then the power series $g(z)$ converges normally on the compact set $\overline{B(a,\rho-|a|)}$ so it has a radius of convergence r with $r \geq \rho - |a|$ for any $|a| < \rho < R$. As such, $r \geq R - |a|$, which is positive. This proves **(ii)**.

Lastly, we prove **(iii)**. For all $z \in B(0,R) \cap B(a,r)$, we have

$$\begin{aligned}
f(z) = \sum_{n=0}^{\infty} c_n z^n &= \sum_{n=0}^{\infty} c_n (a+z-a)^n \\
&= \sum_{n=0}^{\infty} c_n \left[\sum_{k=0}^{n}\binom{n}{k} a^{n-k}(z-a)^k\right] \quad \text{by the biomial theorem} \\
&= \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}\binom{n}{k} c_n a^{n-k}\right)(z-a)^k \\
&= g(z)
\end{aligned}$$

and the result follows. □

Definition 2.9 (convolution of series)**.** Let

$$\sum_{n\in\mathbb{Z}} a_n \text{ and } \sum_{n\in\mathbb{Z}} b_n \quad \text{be} \quad \text{two series in } \mathbb{C} \text{ indexed by } \mathbb{Z}.$$

Their convolution is the double series

$$\sum_{n\in\mathbb{Z}} c_n \quad \text{defined by} \quad \text{for all } n \in \mathbb{Z} \text{ we have } c_n = \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} a_k b_l = \sum_{k\in\mathbb{Z}} a_k b_{n-k}.$$

In Definition 2.9, we can also write

$$\sum_{k+l=n} a_k b_l \quad \text{in place of} \quad \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} a_k b_l.$$

Proposition 2.7 (convolution). Suppose

$$\sum_{n\in\mathbb{Z}} a_n \text{ and } \sum_{n\in\mathbb{Z}} b_n \quad \text{are} \quad \text{absolutely convergent series in } \mathbb{C}.$$

Also, we define

$$c_n = \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} a_k b_l.$$

Then, the following hold:

(i) For all $n \in \mathbb{Z}$, the series c_n converges absolutely in $\mathbb{C}$

(ii) The series $\sum_{n\in\mathbb{Z}} c_n$ converges absolutely in $\mathbb{C}$

(iii) We have

$$\left(\sum_{n\in\mathbb{Z}} a_n\right)\left(\sum_{n\in\mathbb{Z}} b_n\right) = \sum_{n\in\mathbb{Z}} c_n = \sum_{n\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} a_k b_l \quad \text{in } \mathbb{C}$$

Proof. We first prove **(i)**. Consider the double series

$$\sum_{n\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} |a_k||b_l| = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} |a_k||b_l| = \sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}} |a_k||b_l|\right) = \left(\sum_{k\in\mathbb{Z}} |a_k|\right)\left(\sum_{l\in\mathbb{Z}} |b_l|\right)$$

which is the product of two series with finite value. Hence, c_n converges absolutely in $\mathbb{C}$. This proves **(i)**. As a consequence, **(ii)** follows from the triangle inequality for series (see it as an application of Corollary 1.2).

To prove **(iii)**, we start with the RHS. So,

$$\sum_{n\in\mathbb{Z}} \sum_{\substack{k,l\in\mathbb{Z}\\ k+l=n}} a_k b_l = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} a_k b_l = \sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}} a_k b_l\right) = \left(\sum_{k\in\mathbb{Z}} a_k\right)\left(\sum_{l\in\mathbb{Z}} b_l\right).$$

Since k and l are dummy variables, the result follows. □

Theorem 2.4 ($\mathbb{C}$-differentiability of analytic functions). Let $a \in \mathbb{C}$ and

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{be} \quad \text{a power series with strictly positive radius of convergence } R.$$

Then, the following hold:

(i) The termwise differentiated power series

$$\sum_{n=1}^{\infty} n a_n (z-a)^{n-1} \quad \text{has} \quad \text{the same radius of convergence } R$$

(ii) The $\mathbb{C}$-valued function $f : B(a,R) \to \mathbb{C}$ represented by the power series is $\mathbb{C}$-differentiable on $B(a,R)$

(iii) The $\mathbb{C}$-derivative $f' : B(a,R)$ is represented by the power series

$$g(z) = \sum_{n=1}^{\infty} na_n (z-a)^{n-1}$$

We will only prove **(i)** as the proofs of **(ii)** and **(iii)** are pretty long.

Proof. Without loss of generality, we may assume that $a = 0$ throughout the proof. For **(i)**, by the Cauchy-Hadamard formula (Proposition 2.4), it suffices to show that

$$\limsup_{n\to\infty} (n \cdot |a_n|)^{1/(n-1)} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

We will prove that

$$\lim_{n\to\infty} (n+1)^{1/n} = 1.$$

For $n \geq 1$, we can write $(n+1)^{1/n} = 1 + \delta_n$ for some $\delta n > 0$. Then,

$$\begin{aligned} n+1 = (1+\delta_n)^n &= 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \ldots + \delta_n^n \\ &> 1 + \frac{n(n-1)}{2}\delta_n^2 \quad \text{when } n \geq 2 \end{aligned}$$

so

$$\delta_n^2 < \frac{2}{n-1} \quad \text{which implies} \quad \lim_{n\to\infty} \delta_n^2 = 0.$$

This proves **(i)**. □

For any open set $U \subseteq \mathbb{C}$, we let

$$\begin{aligned} &\mathcal{C}^{\omega}(U) \quad \text{denote} \quad \text{the set of analytic functions on } U \quad \text{and} \\ &\mathcal{C}^{\infty}(U) \quad \text{denote} \quad \text{the set of smooth functions on } U \end{aligned}$$

We note that $\mathcal{C}^{\omega}(U) \subseteq \mathcal{C}^{\infty}(U)$, i.e. analytic functions are smooth, with derivatives of all orders.

Corollary 2.2 (Taylor's theorem). Let $U \subseteq \mathbb{C}$ be an open set and $f \in \mathbb{C}^{\omega}(U)$ be an analytic function on U. Let $a \in U$ and

$$\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{be a power series with positive radius of convergence.}$$

Then, for all $n \in \mathbb{N}$, we have

$$a_n = \frac{1}{n!} f^{(n)}(a) \quad \text{in } \mathbb{C}.$$

In particular, the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n \quad \text{must have positive radius of convergence.}$$

Corollary 2.3 (uniqueness of power series)**.** If two power series with the same centre a converge to the same function on a disc of positive radius centred at a, then the two power series are the same, i.e. have the same coefficients.

Definition 2.10 (entire function)**.** A function f which is analytic on the whole of $\mathbb{C}$ is entire.

Example 2.9. Let

$$f(z) = x^3 - 3xy^2 + x^2 - y^2 + x + 1 + i(3x^2y - y^3 + 2xy + y).$$

(a) Show that $f(z)$ is entire.

(b) Express $f(z)$ as a function of z.

Solution.

(a) This is a very simple exercise using the CR equations.

(b) Recall the binomial theorem and see that

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i\left(3x^2y - y^3\right) + x^2 - y^2 + x + 1 + i(2xy + y) \\ &= x^3 - 3xy^2 + i\left(3x^2y - y^3\right) + x^2 - y^2 + 2ixy + x + iy + 1 \\ &= (x+iy)^3 + (x+iy)^2 + x + iy + 1 \\ &= z^3 + z^2 + z + 1 \end{aligned}$$

So, $f(z) = z^3 + z^2 + z + 1$. □

Example 2.10. Find an entire function f such that $\mathrm{Re}(f) = x^2 - 3x - y^2$ or explain why there is no such function.

Solution. Write $f = u + iv$, where u and v are real-valued functions. Given that $u = x^2 - 3x - y^2$, we apply the CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 3 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y.$$

Solving the first equation yields $v = 2xy + g(y)$, where $g(y)$ is a function in terms of y. Then, $2x + g'(y) = 2x - 3$, which implies that $g(y) = -3y + c$ for some constant c.

Now, we have $v = 2xy - 3y + c$. We conclude that the following function satisfies the hypotheses:

$$\begin{aligned} f(z) &= x^2 - 3x - y^2 + i(2xy - 3y + c) \\ &= x^2 - y^2 + 2ixy - 3x - 3iy + ic \\ &= z^2 - 3z + ci \end{aligned}$$

So, $f(z) = z^2 - 3z + ci$. □

Example 2.11 (Dinh's 70 problems)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(0) = f'(0) = 0 \quad \text{and} \quad \mathrm{Re}\left(f'\right) = x^2 - y^2 + 6xy.$$

Find f.

Solution. Let $z = x + iy$, so $z^2 = x^2 - y^2 + 2xyi$. As such,

$$x^2 - y^2 + 6xy = \mathrm{Re}\left(z^2 - 3iz^2\right)$$

Since $f'(0) = 0$, then $f'(z) = z^2 - 3iz^2$. It follows that $f(z) = z^3/3 - iz^3$ as $f(0) = 0$. □

2.4. *The Exponential Function*

Recall from Real Analysis (MA2108) that e can be defined to be the following infinite series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

This can be deduced from the Maclaurin expansion of e^x, which is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{which has radius of convergence } R = \infty.$$

Definition 2.11 (complex exponential function)**.** The complex exponential function is the function $\exp : \mathbb{C} \to \mathbb{C}$ defined by the power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By the ratio test, the power series representing the complex exponential function converges absolutely for all $z \in \mathbb{C}$, which implies that the radius of convergence R is ∞. This implies

$$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0.$$

Alternatively, one can directly deduce the value of this lim sup using Stirling's formula, which states that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} \quad \text{or} \quad \text{the alternative asymptotic relation } n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n.$$

Proposition 2.8. For any $z, w \in \mathbb{C}$, we have

$$\exp(z+w) = \exp(z) \cdot \exp(w) \quad \text{in } \mathbb{C}.$$

Proof. The power seires for $\exp(z)$ and $\exp(w)$ converge absolutely, so by Proposition 2.7 (an important proposition on convolution), we have the following:

(i) For all $n \in \mathbb{Z}_{\geq 0}$, the series

$$c_n = \sum_{\substack{k,\ell \in \mathbb{Z}_{\geq 0} \\ k+\ell = n}} \frac{z^k}{k!} \cdot \frac{w^\ell}{\ell!} \quad \text{converges absolutely in } \mathbb{C}$$

(ii) The series

$$\sum_{n \in \mathbb{Z}_{\geq 0}} c_n = \sum_{n \in \mathbb{Z}} \frac{(z+w)^n}{n!} \quad \text{converges in } \mathbb{C}$$

(iii) One has $\exp(z) \cdot \exp(w) = \exp(z+w)$ in $\mathbb{C}$

By considering **(iii)**, we see that the result follows. □

From the lens of Group Theory, we say that the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a continuous group homomorphism from

$$\text{the additive group } \mathbb{C} \quad \text{to} \quad \text{the multiplicative group } \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

To see why, we have $\exp(0) = 0^0/0! = 1$. Then, for all $z \in \mathbb{C}$, we have

$$1 = \exp(0) = \exp(z) \cdot \exp(-z) \quad \text{so} \quad \exp(z) \in \mathbb{C}^\times.$$

Proposition 2.8 shows that exp is a group homomorphism from $\mathbb{C}$ to $\mathbb{C}^\times$. Since exp is a function defined by a convergent power series, we conclude that it is continuous.

Remark 2.3. $\mathbb{C} = \mathbb{R} \times i\mathbb{R}$ as groups.

Theorem 2.5. exp restricts to an isomorphism $\exp : \mathbb{R} \to \mathbb{R}^{\times}_{>0}$.

Proof. It is clear from the power series definition that $\exp : \mathbb{R} \to \mathbb{R}$. Also, note that

$$\exp : \mathbb{R} \to \mathbb{C}^{\times} \cap \mathbb{R} = \mathbb{R}^{\times} = \mathbb{R}^{\times}_{>0} \sqcup \mathbb{R}_{<0}.$$

Since exp is continuous and $\mathbb{R}$ is connected, we must have $\exp(\mathbb{R})$ being connected in $\mathbb{R}^{\times}$, where $\exp(\mathbb{R}) \subseteq \mathbb{R}^{\times}_{>0}$. For $x \in \mathbb{R}_{\geq 0}$, we have $\exp(x) \geq 1 + x$ is not bounded above so $[1, \infty) \subseteq \exp(\mathbb{R})$ and $\ker(\exp) \cap \mathbb{R}_{\geq 0} = \{0\}$. Then from $\exp(-x) = [\exp(x)]^{-1}$, we have $(0, 1] \subseteq \exp(\mathbb{R})$ and $\ker(\exp) \cap \mathbb{R}_{\geq 0} = \{0\}$. □

Lemma 2.2. For $z \in \mathbb{C}$, we have $\exp(\overline{z}) = \overline{\exp(z)}$.

Proof. We note that

$$\exp(\overline{z}) = \sum_{n=0}^{\infty} \frac{(\overline{z})^n}{n!} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \overline{\exp(z)}.$$

□

Definition 2.12 (circle group)**.** Let

$$\mathbb{T} = \left\{z \in \mathbb{C}^{\times} : |z|_{\mathbb{C}} = 1\right\} \leq \mathbb{C}^{\times} \quad \text{denote} \quad \text{the circle group.}$$

Proposition 2.9. For any $t \in \mathbb{R}$, we have $|\exp(it)|_{\mathbb{C}} = 1$. In other words, exp maps $iR \subseteq \mathbb{C}$ into $\mathbb{T} \subseteq \mathbb{C}^{\times}$.

Proof. We have

$$\begin{aligned} |\exp(it)|_{\mathbb{C}}^2 &= \exp(it)\overline{\exp(it)} \\ &= \exp(it)\exp\left(\overline{it}\right) \quad \text{by Lemma 2.2} \\ &= \exp(it)\exp(-it) \end{aligned}$$

which is equal to $\exp 0 = 1$. □

Corollary 2.4. For any $z \in \mathbb{C}$, we have

$$|\exp(z)|_{\mathbb{C}} = \exp(\operatorname{Re}(z)) \quad \text{in } \mathbb{R}_{>0}.$$

Proof. We have $z = \operatorname{Re}(z) + i\operatorname{Im}(z)$ implies $\exp(z) = \exp(\operatorname{Re}(z)) \cdot \exp(i\operatorname{Im}(z))$. □

Theorem 2.6. For any $z \in \mathbb{C}$, we have

$$\exp(z) \in \mathbb{T} \quad \text{if and only if} \quad z \in i\mathbb{R}.$$

Proof. We have $\exp(z) \in \mathbb{T}$ if and only if $\exp(\operatorname{Re}(z)) = 1$, or equivalently $\operatorname{Re}(z) = 0$. □

For this set of notes, we let

$$\mathbb{D} = B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$$

denote the open unit ball centred at 0 in $\mathbb{C}$.

Definition 2.13 (logarithmic function)**.** The logarithmic series $\lambda : \mathbb{D} \to \mathbb{C}$ is the power series

$$\log(1+z) = \lambda(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{z^n}{n!} = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

Proposition 2.10. For any $z \in \mathbb{D}$, one has $\exp(\lambda(z)) = 1+z$.

Lemma 2.3. The series defining $\lambda(z)$ has radius of convergence 1.

Proof. As $z \in \mathbb{D}$ (open unit disc centred at 0), the series converges absolutely by the ratio test, i.e.

$$\left| \frac{z^{n+1}/(n+1)}{z^n/n} \right| = \frac{n}{n+1}|z| \quad \text{which is } < 1.$$

$\square$

Theorem 2.7. The function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is surjective.

Theorem 2.8. $\ker(\exp) \subseteq \mathbb{C}$ is a non-trivial, discrete subgroup contained in $i\mathbb{R} \subseteq \mathbb{C}$.

Proof. By surjectivity (Theorem 2.7), there exists $z \in \mathbb{C}$ such that $\exp(z) = -1$ in $\mathbb{C}^\times$. Then, $z \neq 0$ in $\mathbb{C}$ since $\exp(0) = 1 \neq -1$ so $2z \neq 0$ in $\mathbb{C}$. However,

$$\exp(2z) = \exp(z+z) = [\exp(z)]^2 = (-1)^2 = 1$$

so $\ker(\exp)$ is a non-trivial subgroup of $\mathbb{C}$.

We then prove that $\ker(\exp)$ is contained in $i\mathbb{R}$. Note that

$$\begin{aligned} \ker(\exp) &= \{z \in \mathbb{C} : \exp(z) = 1\} \\ &\subseteq \{z \in \mathbb{C} : |\exp(z)|_{\mathbb{C}} = 1\} \end{aligned}$$

which is equal to $\exp^{-1}(\mathbb{T}) = i\mathbb{R}$.

Lastly, we prove that $\ker(\exp)$ is a discrete subgroup of $\mathbb{C}$. Note that for $z \in \mathbb{C} \setminus \{0\}$, we have the following[†]:

$$\frac{\exp(z)-1}{z} = \frac{1}{z}\sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \quad \text{so} \quad \lim_{z \to 0} \frac{\exp(z)-1}{z} = 1$$

Thus, the function

$$g : \mathbb{C} \to \mathbb{C} \quad \text{where} \quad g(z) = \begin{cases} \dfrac{\exp(z)-1}{z} & \text{if } z \neq 0; \\ 1 & \text{if } z = 0 \end{cases} \quad \text{is continuous.}$$

[†]Here is an interesting fact: the function $x/(e^x - 1)$ appears in the definition of Bernoulli numbers. This pops up in Combinatorics and Analytic Number Theory.

As such, there exists an open subset $U \subseteq \mathbb{C}$ with $0 \in U$ such that $0 \notin g(U)$. Equivalently, $g^{-1}(0) \cap U \neq \emptyset$. Then, for all $z \in U$, $\exp(z) = 1$ if and only if $z = 0$, so $\ker(\exp) \cap U = \{0\}$. As such, for all $w \in \ker(\exp)$, we have $\ker(\exp) \cap (w+U) = \{w\}$, so every point in $\ker(\exp)$ is isolated. □

Now, *we will define* π!

Definition 2.14. We define π to be the following:

$$\begin{aligned}\pi &= \inf\{t \in \mathbb{R}_{>0} : \exp(2it) = 1\} \\ &= \inf\left\{\frac{1}{2i}\ker(\exp) \cap \mathbb{R}_{>0}\right\}\end{aligned}$$

In Theorem 2.8, we mentioned that

$$\frac{1}{2\pi}\ker(\exp) \cap \mathbb{R}_{>0} \quad \text{is} \quad \text{non-empty and discrete.}$$

As such, π is a positive real number!

Proposition 2.11. $\ker(\exp) = 2\pi i\mathbb{Z} \subseteq i\mathbb{R}$

Proof. The reverse inclusion $\supseteq$ is obvious. For the forward inclusion, suppose $z \in \ker(\exp)$. Then, write

$$z = 2i\pi(n\pi + t) \quad \text{where } n \in \mathbb{Z}, 0 \leq t < \pi.$$

So, $\exp(2it) = 1$ and the result follows. □

Corollary 2.5 (Euler's identity)**.** $e^{\pi i} + 1 = 0$

Proof. Note that $w = e^{\pi i}$ in $\mathbb{C}$ satisfies $w^2 = e^{2\pi i} = 1$. So, $w = \pm 1$ in $\mathbb{C}$. Since $\pi i \notin 2\pi i\mathbb{Z}$, then $w \neq 1$, so $w = -1$. □

Theorem 2.9 (de Moivre's theorem)**.** For $n \in \mathbb{Z}$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta.$$

Proof. By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. In de Moivre's theorem, the left side of the equation is $e^{in\theta}$ by raising both sides to the power of n. The result follows by using Euler's formula on $e^{in\theta}$. □

Definition 2.15 (topological group)**.** A topological group is a group G equipped with a topology such that we have the following:

(i) G is a topological space

(ii) The group operation $\cdot : G \times G \to G$, given by $(g,h) \mapsto g \cdot h$, is continuous with respect to the product topology on $G \times G$

(iii) The inverse function $(\cdot)^{-1} : G \to G$ given by $g \mapsto g^{-1}$ is continuous

In summary

$$\exp : \mathbb{C} \to \mathbb{C}^\times \quad \text{is} \quad \text{a continuous, surjective homomorphism of topological groups (Definition 2.15).}$$

Its kernel is $\ker(\exp) = 2\pi i\mathbb{Z} \subseteq \mathbb{C}$. Hence, it induces an isomorphism of topological groups

$$\mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^\times \quad \text{where} \quad z + 2\pi i\mathbb{Z} \mapsto e^z.$$

Restricting to the real axis yields

$$\mathbb{R} \xrightarrow{\sim} \mathbb{R}_{>0} \quad \text{where} \quad x \mapsto e^x,$$

while restricting to purely imaginary parts modulo $2\pi i$ yields

$$i\mathbb{R}/2\pi i\mathbb{Z} \xrightarrow{\sim} \mathbb{T} \quad \text{where} \quad iy + 2\pi i\mathbb{Z} \mapsto e^{iy}.$$

Using polar coordinates, we obtain an isomorphism

$$\mathbb{R}_{>0} \times \mathbb{T} \xrightarrow{\sim} \mathbb{C}^\times, \quad (r,\theta) \mapsto r\theta \quad \text{whose inverse is} \quad z \mapsto \left(|z|, \frac{z}{|z|}\right).$$

On the additive side, we note that

$$\mathbb{R} \oplus i\mathbb{R} \cong \mathbb{C} \quad \text{which is given by the map} \quad (x, iy) \mapsto x + iy.$$

2.5. *Harmonic Functions*

Definition 2.16 (harmonic function)**.** A real-valued function $h(x,y)$ is said to be harmonic if it is twice continuously differentiable and satisfies Laplace's equation. That is,

$$h_{xx} + h_{yy} = 0 \quad \text{or} \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Example 2.12. Show that u^2 cannot be harmonic for any non-constant harmonic function u.

Solution. Let u be a non-constant harmonic function. Then,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Also, we have

$$\frac{\partial^2 (u^2)}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 \quad \text{and} \quad \frac{\partial^2 (u^2)}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2.$$

However,

$$\frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2 \neq 0,$$

which concludes the proof. □

Definition 2.17 (harmonic conjugate)**.** Let u be a harmonic function. If v is a harmonic function satisfying the Cauchy-Riemann equations, then v is a harmonic conjugate of u.

Example 2.13 (MA5217 AY24/25 Sem 1 Homework 1)**.** Show that the function

$$u(x,y) = e^{x-y}\cos(x+y) + e^{x+y}\cos(x-y)$$

is harmonic in $\mathbb{C}$ and find a harmonic conjugate of u.

Solution. By definition, we need to show that u satisfies Laplace's equation, i.e.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Let $s=x+y$ and $t=x-y$, so

$$u\left(\frac{s+t}{2},\frac{s-t}{2}\right)=e^t\cos s+e^s\cos t$$

Hence,

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=e^s\cos t-e^t\cos s+e^t\cos s-e^s\cos t=0$$

so u is harmonic. Finding a harmonic conjugate is trivial. □

Example 2.14 (Dinh's 70 problems)**.** Find all harmonic functions $u(x,y)$ in $\mathbb{C}$ such that

$$\left(x^2-y^2\right)u\left(x,y\right)\quad\text{is harmonic in }\mathbb{C}.$$

Solution. Let $f(x,y)=(x^2-y^2)u(x,y)$. Then,

$$f_{xx}=(x^2-y^2)u_{xx}+4xu_x+2u\quad\text{and}\quad f_{yy}=(x^2-y^2)u_{yy}-4yu_y-2u.$$

As such,

$$f_{xx}+f_{yy}=4(xu_x-yu_y),$$

where we used the fact that u is harmonic (i.e. $u_{xx}+u_{yy}=0$). For f to be harmonic, $xu_x=yu_y$. One can use techniques taught to solve partial differential equations to deduce that $u(x,y)=g(xy)$, where $g:\mathbb{R}\to\mathbb{R}$. Therefore, $g''(xy)=0$, so $g(t)=at+b$, where $a,b\in\mathbb{R}$. Hence, $u(x,y)=axy+b$. □

3. Line Integrals

3.1. *Fundamental Results*

Theorem 3.1 (fundamental theorem of line integrals)**.** Suppose C is a smooth curve given by $z(t):$ $a \le z \le b$ and $F'(z) = f(z)$. Then,

$$\int_C f(z)\,dz = F(z(b)) - F(z(a)).$$

Lemma 3.1 (triangle inequality)**.** Suppose f is a continuous complex-valued function of t. Then,

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

Lemma 3.2 (ML inequality/estimation lemma)**.** If C is a smooth curve of length L, f is continuous on C and $|f| \le M$ on C, then

$$\left|\int_C f(z)\,dz\right| \le ML \quad \text{where} \quad M = \sup_{z\in C} |f(z)| \text{ alternatively.}$$

Example 3.1. Let γ be the contour given by $\gamma(t) = 3e^{it}$, where $0 \le t \le \pi$. Prove that

$$\left|\int_\gamma \frac{\overline{ze^{iz}}}{z^2 - 11z + 30}\,dz\right| \le 5.$$

Solution. Obviously, $L = 3\pi$ since $\gamma(t) = 3e^{it}$, where $0 \le t \le \pi$ is the equation of the upper half of a circle of radius 3 centred at the origin, so its arc length is 3π. Now, we need to justify that $M \le 5/3\pi$. Let $z = x + iy$.

We have
$$\left|\frac{\overline{ze^{iz}}}{z^2 - 11z + 30}\right| = \left|\frac{\overline{z}\cdot\overline{e^{iz}}}{(z-5)(z-6)}\right| = \frac{|\overline{z}|\,e^{-y}}{|z-5|\,|z-6|} = \frac{|z|\,e^{-y}}{|z-5|\,|z-6|}.$$

Since $|z| \le 3$ and applying the triangle inequality, we see that

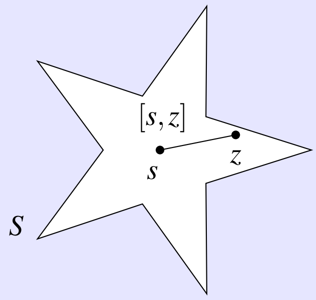
$$\frac{|z|\,e^{-y}}{|z-5|\,|z-6|} \le \frac{3\cdot 1}{||z|-5|\,||z|-6|} \le \frac{3}{|3-5|\,|3-6|} = \frac{1}{2}$$

so $M = 1/2$. It is clear that $1/2 < 5/3\pi$ so we conclude that $M \le 5/3\pi$. □

4. The Cauchy-Goursat Theorem

4.1. *Some Results in Topology*

Definition 4.1 (star-shaped set)**.** A set S is star-shaped if it has a point s, known as the star centre, so that for each $z \in S$, the segment $[s,z]$ lies in S.



Remark 4.1. Note that a star domain is not necessarily convex.

Example 4.1. A cross-shaped figure is a star domain but is not convex.

Theorem 4.1. Let S be an open star-shaped region and f continuous on S. Let T be a closed triangular region and ∂T be the boundary of the triangle traversed in the anticlockwise direction. Suppose

$$\int_{\partial T} f(z)\, dz = 0$$

for every T in S, then f has an anti-derivative, F, in S.

Definition 4.2 (open ball)**.** Define $B_r(w)$ or $B(w,r)$ to be

$$\text{the open ball of} \quad \text{radius } r \text{ and centre } w.$$

Throughout this set of notes, we will be using these two representations interchangeably.

Definition 4.3 (boundary point)**.** A point $w \in \mathbb{C}$ is a boundary point of S if

$$\text{for every } r \in \mathbb{R}^+ \quad \text{we have} \quad B_r(w) \cap S \neq \emptyset.$$

Definition 4.4 (closure)**.** Denote the set of boundary points by ∂S. Given a set S, the closure of S, denoted by $\overline{S}$, is defined by

$$\overline{S} = S \cup \partial S.$$

Theorem 4.2. A set G is closed if and only if $G = \overline{G}$.

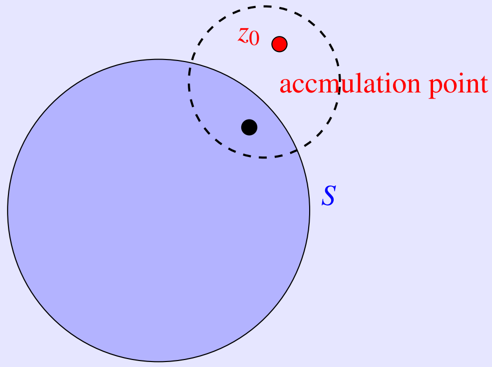
Proof. For the forward direction, suppose G is closed. We wish to prove that $G = G \cup \partial G$, or equivalently, $\partial G \subseteq G$. Suppose on the contrary that $\partial G \not\subseteq G$. Then, there exists $w \in \partial G \setminus G$. For every $\varepsilon > 0$, we have

$$B_\varepsilon(w) \cap G \neq \emptyset \text{ and } B_\varepsilon(w) \cap G' \neq \emptyset \quad \text{which implies} \quad B_\varepsilon(w) \cap G \neq \emptyset.$$

However, $w \notin G$, so $w \in G'$. As G is closed, then G' is open, so there exists $\varepsilon' > 0$ such that $B(w,\varepsilon') \subseteq G'$. Hence, $B(w,\varepsilon') \cap G = \emptyset$ and this is a contradiction, so $\partial G \subseteq G$.

We then prove the reverse direction. Suppose $G = G \cup \partial G$. We wish to prove that G' is open. Let $x \in G'$. As $\partial G \subseteq G$, then $G' \cap \partial G = \emptyset$. There exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap G = \emptyset$ or $B(x,\varepsilon) \cap G = \emptyset$. As $x \in G'$, then $B(x,\varepsilon) \cap G' \neq \emptyset$. Therefore, $B(x,\varepsilon) \cap G = \emptyset$ or $B(x,\varepsilon) \subseteq G'$, which is the definition of G' being open. □

Definition 4.5 (accumulation point)**.** A point z_0 is an accumulation point of a set S if each neighbourhood of z_0 contains at least one point of S distinct from z_0.



Remark 4.2. The accumulation point of a set S does not have to be an element of that set.

Example 4.2. Prove that a set S is closed if and only if S contains all its accumulation points.

Solution. For the forward direction, we proceed with contradiction. Let y be an accumulation of S which is not in S. Then, $y \in S'$. As S' is an open set, there exists $\delta > 0$ such that $B_\delta(y) \subseteq S'$. As such, $B_\delta(y) \cap S = \emptyset$, contradicting the assumption that y is an accumulation point for S.

For the reverse direction, suppose S contains all its accumulation points. We need to show that S is closed. It suffices to show that S' is open. Let $x \in S'$. Then, x is not an accumulation of S since S already contains all its accumulation points. So, there exists $\delta > 0$ such that

$$B_\delta(x) \setminus (\{x\} \cap S) = B_\delta(x) \cap S = \emptyset.$$

We conclude that $B_\delta(x) \subseteq S'$, so S' is open. □

4.2. *Cauchy-Goursat Theorem*

Theorem 4.3 (Cauchy-Goursat theorem/Cauchy integral theorem)**.** Suppose f is analytic on a star-shaped region S. Then, for every simple closed path C in S traversed in the anticlockwise direction,

$$\int_C f(z)\, dz = 0.$$

Example 4.3. Let $f(z) = \operatorname{Log}(z+2)$ and the contour γ be the circle $|z| = 1$ oriented in the anticlockwise direction. Use the Cauchy-Goursat theorem to prove that

$$\int_\gamma f(z)\, dz = 0.$$

Solution. Recall that $\operatorname{Log} z$ is analytic on $\mathbb{C}\setminus(-\infty,0]$. Thus, $f(z)=\operatorname{Log}(z+2)$ is analytic on $\mathbb{C}\setminus(-\infty,-2]$. However, $(-\infty,-2]$ lies outside the circle $|z|=1$. Thus, $f(z)$ is analytic inside and on the circle $|z|=1$, which is a simple closed contour. The result follows by the Cauchy-Goursat theorem. □

Theorem 4.4. Let f be continuous on a star-shaped region S, and analytic on $S\setminus\{z_0\}$, i.e. the set S but excluding the point z_0. Then, f has an anti-derivative on S, and consequently,

$$\int_C f(z)\,dz=0 \quad \text{for every simple closed curve } \in S \text{ traversed anticlockwise.}$$

4.3. *Cauchy's Integral Formula*

Theorem 4.5 (Cauchy's integral formula). Let f be analytic everywhere within and on a simple closed contour C traversed in the anticlockwise direction. If a is interior to C, then

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

Proof. Define

$$g(z)=\frac{f(z)-f(a)}{z-a} \quad \text{which is analytic everywhere except at } z=a.$$

Since the derivative of f exists at a, then by the first principles of differentiation,

$$\lim_{z\to a} g(z)=f'(a)\,.$$

Using the Cauchy-Goursat formula applied to a star-shaped region excluding a (since g is analytic everywhere except a), then

$$\int_C g(z)\,dz=0 \quad \text{which implies} \quad \int_C \frac{f(z)-f(a)}{z-a}\,dz=0.$$

So,

$$\int_C \frac{f(z)}{z-a}\,dz=f(a)\int_C \frac{1}{z-a}\,dz=f(a)\cdot 2\pi i,$$

where the last equality follows since we are taking the contour integral on a loop around a. □

Example 4.4. Let $z_0\in\mathbb{C}$ and γ be a simple closed contour enclosing z_0 with positive orientation. Without using Cauchy's integral formula, and using only the fact that

$$\int_\gamma \frac{1}{z-z_0}\,dz=2\pi i,$$

show that

$$\text{if } p(z)=z_0+z_1z+\ldots+a_{n-1}z^{n-1}+a_nz^n \text{ is a polynomial} \quad \text{then} \quad \int_\gamma \frac{p(z)}{z-z_0}=p(z_0)\cdot 2\pi i.$$

Solution. By the division algorithm for polynomials, there exist polynomials $f(z)$ and r such that $p(z)=(z-z_0)\,f(z)+r$. So, $p(z_0)=r$.

Hence, $p(z)=(z-z_0)f(z)+p(z_0)$ and we have

$$\int_\gamma \frac{p(z)}{z-z_0}\,dz=\int_\gamma f(z)+\frac{p(z_0)}{z-z_0}\,dz=\int_\gamma f(z)\,dz+p(z_0)\int_\gamma \frac{1}{z-z_0}\,dz=p(z_0)\cdot 2\pi i.$$

Note that the integral

$$\int_\gamma f(z)\,dz=0$$

by the Cauchy-Goursat theorem. □

Example 4.5. Let C be the circle $|z| = 2$ oriented in the anticlockwise direction. Evaluate

$$\int_C \frac{1}{|z-i|^2}\, dz.$$

Solution. We use the identity $|z|^2 = z\bar{z}$, so $|z-i|^2 = (z-i)(\bar{z}+i)$. Since $|z| = 2$, then $\bar{z} = 4/z$, so

$$(z-i)(\bar{z}+i) = z\bar{z} + i(z-\bar{z}) + 1 = 5 + i\left(z - \frac{4i}{z}\right) = \frac{iz^2+5z+4}{z} = \frac{(iz+1)(z-4i)}{z}.$$

Hence, the contour integral is equivalent to

$$\int_C \frac{z}{(iz+1)(z-4i)}\, dz = \int_C \frac{f(z)}{z-i}\, dz \quad \text{where } f(z) = -\frac{iz}{z-4i}.$$

By Cauchy's integral formula, the integral is equivalent to $2\pi i f(i) = -2\pi/3$. □

Corollary 4.1 (Cauchy's differentiation formula)**.** If f is analytic at a point a, then $f^{(n)}(a)$ exists for $n = 1, 2, \ldots$ and are also analytic z_0, and

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz,$$

where C is a simple closed curve traversed in the anticlockwise direction that encloses a.

Proof. Apply induction on Cauchy's integral formula. □

Theorem 4.6 (Liouville's theorem)**.** If f is a bounded and entire function, then f is a constant.

Proof. Since f is entire, we can represent it using a Taylor series about $z = 0$, so

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

By Cauchy's integral formula,

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz,$$

where C is a circle of radius r centred at the origin. Since f is bounded, then $|f(z)| \leq M$ for some constant M and for all $z \in \mathbb{C}$. We have

$$|a_n| = \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \int_C \left| \frac{f(z)}{z^{n+1}} \right| |dz| \leq \frac{1}{2\pi} \int_C \frac{M}{|z^{n+1}|} |dz| \leq \frac{M}{2\pi r^{n+1}} \int_C |dz| = \frac{M}{2\pi r^{n+1}} \cdot 2\pi r = \frac{M}{r^n}$$

Now, as $|z| = r$ on the circle C, by setting $r > 0$ to be arbitrary, as r tends to infinity, $a_n = 0$ for all $n \geq 1$. This is because f is entire. Hence, $f(z) = a_0 = M/r$ which is a constant. □

Example 4.6. Find all entire functions $f(z)$ with $f(0) = 2$ and $|f(z) - e^z| \geq 1$ for all $z \in \mathbb{C}$.

Solution. We note that

$$\frac{1}{|f(z)-e^z|} \leq 1 \quad \text{where} \quad f(z) - e^z \neq 0.$$

So, $1/(f(z) - e^z)$ is bounded and entire. By Liouville's theorem,

$$\frac{1}{f(z)-e^z} = c,$$

where c is a constant. Since $f(0) = 2$, then $c = 1$. As such, $f(z) = e^z + 1$. □

Example 4.7. Let g be an entire function such that $|g'(z)| < |g'(z)+i|$ for all complex numbers z. Show that there exist $\alpha,\beta \in \mathbb{C}$ such that $g(z) = \alpha z+\beta$ for all $z \in \mathbb{C}$.

Solution. Since g is entire, then g' is also entire. Let

$$h(z) = \frac{g'(z)}{g'(z)+i}.$$

Then h is the quotient of two entire functions such that the denominator is not equal to zero at each $z \in \mathbb{C}$, hence h is entire. It is clear that for all $z \in \mathbb{C}$, $|h(z)| < 1$, so h is bounded on $\mathbb{C}$. By Liouville's theorem, $h(z) = c$, where c is a constant, so $g'(z) = cg'(z)+ci$. We have

$$g'(z) = \frac{ic}{1-c} = \alpha.$$

Hence, $g(z) = \alpha z+\beta$. □

Example 4.8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$\lim_{z\to\infty} f(z) = \infty.$$

Show that f has at least one zero in $\mathbb{C}$.

Solution. Suppose on the contrary f has no zeros in $\mathbb{C}$. Consider

$$g(z) = \frac{1}{f(z)}.$$

Note that g is entire. Using the given limit, there exists $R > 0$ such that for all $|z| > R$, $|f(z)| > 1$. This implies that $|g(z)| < 1$ but since g is continuous, it obtains a maximum M on the compact set $\overline{D(0,R)}$. Hence, for all $z \in \mathbb{C}$, $|g(z)| \leq \max\{1,M\}$, so by Liouville's theorem, g is a constant, implying that f is a constant, which is a contradiction. □

Example 4.9. Find all entire functions $f(z)$ such that

$$|f(z)| \leq \frac{1}{1+x^2+2y^2} \quad \text{for all } z = x+iy \in \mathbb{C}.$$

Solution. Since $x^2, y^2 \geq 0$, then $|f(z)| \leq 1$. By Liouville's theorem, f is a constant, say c. Then,

$$c \leq \frac{1}{1+|z|^2+y^2}.$$

It is clear that

$$\lim_{z\to\infty} f(z) = 0$$

so $c = 0$. Hence, the only function satisfying the hypothesis is $f(z) = 0$. □

Example 4.10 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all entire functions f satisfying $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for every $z \in \mathbb{C}$.

Solution. By an inductive argument, for all $n \in \mathbb{Z}$, we have

$$f(z+n) = f(z) \quad \text{and} \quad f(z+ni) = f(z).$$

Hence, it suffices to consider the behaviour of f on the unit square $[0,1]\times[0,1]$. Since the unit square is a compact set, it is bounded by the Heine-Borel theorem. Hence, $f(x+iy)$ is bounded for all $x,y \in \mathbb{R}$. Since f is a bounded function, it is constant (follows by Liouville's theorem where we assumed that f is entire). So, $f(z) = c$ for some $c \in \mathbb{R}$. □

Example 4.11 (Dinh's 70 problems). Let $f = u + iv$ be an entire function. Show that if $u^2(z) \geq v^2(z)$ for all $z \in \mathbb{C}$, then f must be a constant.

Solution. We have $f^2 = u^2 - v^2 + 2uvi$. Consider

$$g = e^{-f^2} = e^{v^2-u^2}e^{-2uvi} \quad \text{which is entire} \quad \text{and} \quad |g| \leq \frac{1}{e}.$$

By Liouville's theorem, g is a constant. So, $e^{-f^2} = k$ for some constant k. Thus, f is a constant. $\square$

Theorem 4.7 (fundamental theorem of algebra). Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.

4.4. *Applications of Cauchy's Integral Formula*

Theorem 4.8 (Morera's theorem). Let f be a continuous function on D. Let T be a closed triangle in D and ∂T be the boundary of T traversed in the anticlockwise direction. Then,

$$\int_{\partial T} f(z)\, dz = 0.$$

Theorem 4.9. Let f be an entire function. Define $g(z) = f'(a)$ if $z = a$ and

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

if $z \neq a$. Then, g is also entire.

Theorem 4.10 (extended Liouville's theorem). If f is entire and if for some $k \in \mathbb{N}$, there exists constants $A, B > 0$ such that

$$|f(z)| \leq A + B|z|^k,$$

then f is a polynomial of degree at most k.

Example 4.12 (Dinh's 70 problems). Let u be a real-valued harmonic function in the complex plane such that

$$u(z) \leq a\,|\ln|z|| + b$$

for all z, where a and b are positive constants. Prove that u is constant.

Solution. By Liouville's theorem, since u is harmonic, it suffices to show that u is bounded. Let $f(z) = a\,|\ln|z|| + b$. Then, by Cauchy's integral formula,

$$|u'(k)| = \left| \frac{1}{2\pi i} \int_{\gamma:|z|=R} \frac{f(z)}{(z-k)^2}\, dz \right| \leq R \cdot \frac{a\,|\ln R| + b}{|R - |k||^2},$$

where we have considered γ to be the circle of radius R centred at the origin and naturally, the path is taken to be positively-oriented. To establish the upper bound for $|u'(k)|$, the triangle inequality and reverse triangle inequality are used. Now, note that

$$\lim_{R\to\infty} R \cdot \frac{a\,|\ln R| + b}{|R - |k||^2}, = 0$$

which implies that $|u'(k)| = 0$, or rather, $u'(k) = 0$. So, $u(k)$ is a constant for all $k \in \mathbb{R}$. $\square$

Theorem 4.11 (Gauss' mean value theorem)**.** If f is analytic in D and $\alpha \in D$, then

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta.$$

Proof. By Cauchy's integral formula, for $a \in D$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz.$$

Let C be a circle of radius r centred at a. Then, our parameterisation is $z = a + re^{i\theta}$, so $dz/d\theta = ire^{i\theta}$. Hence,

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{a + re^{i\theta} - a} \cdot ire^{i\theta}\, d\theta$$

and the result follows with some simple cancellation. □

Theorem 4.12 (maximum modulus theorem for open balls)**.** Suppose $f(z)$ is analytic throughout a neighbourhood $|z - z_0| < R$ of a point z_0. If $|f(z)| \leq |f(z_0)|$ for each z in the neighbourhood, then $f(z)$ attains a constant value $f(z_0)$ throughout the neighbourhood.

Theorem 4.13 (maximum modulus principle)**.** If f is analytic in D and

$$|f(z)| \leq |f(z_0)| \text{ for all } z \in D \quad \text{then} \quad f(z) \text{ is a constant.}$$

Example 4.13 (Dinh's 70 problems)**.** Let $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a complex polynomial of degree $n > 0$. Prove that

$$\frac{1}{2\pi i} \int_{|z|=R} z^{n-1} |f(z)|^2\, dz = a_0 \overline{a_n} R^{2n}.$$

Solution. Note that $|f(z)|^2 = f(z) \cdot \overline{f(z)}$. Setting $z = Re^{i\theta}$, the integral becomes

$$\frac{1}{2\pi} \int_0^{2\pi} R^n e^{in\theta} \left(a_0 + a_1 Re^{i\theta} + \ldots + a_n R^n e^{in\theta}\right) \left(\overline{a_0} + \overline{a_1} Re^{-i\theta} + \ldots + \overline{a_n} R^n e^{-in\theta}\right) d\theta.$$

Since

$$\int_0^{2\pi} e^{ik\theta}\, d\theta = 0 \quad \text{for all } k \neq 0,$$

upon multiplying the polynomials $a_0 + a_1 Re^{i\theta} + \ldots + a_n R^n e^{in\theta}$ and $\overline{a_0} + \overline{a_1} Re^{-i\theta} + \ldots + \overline{a_n} R^n e^{-in\theta}$, we wish to extract the coefficient of $e^{-in\theta}$. So, the integral becomes

$$\frac{1}{2\pi i} \int_0^{2\pi} R^n e^{in\theta} a_0 \overline{a_n} R^n e^{-in\theta}\, d\theta$$

and the result follows. □

Example 4.14 (Dinh's 70 problems)**.** Suppose $u(z)$ is harmonic on $D(0,r)$, where $r > 1$. Prove that

$$\int_0^{2\pi} u(e^{it}) \cos^2\left(\frac{t}{2}\right) dt = \pi u(0) + \frac{\pi}{2} u'(0) \quad \text{and} \quad \int_0^{2\pi} u(e^{it}) \sin^2\left(\frac{t}{2}\right) dt = \pi u(0) - \frac{\pi}{2} u'(0),$$

where $u'(0) = u_x(0)$.

Solution. Let I_1 and I_2 denote the two integrals respectively. We have

$$I_1 + I_2 = \int_0^{2\pi} u(e^{it})\, dt \text{ and } I_1 - I_2 = \int_0^{2\pi} u(e^{it}) \cos t\, dt.$$

We parametrise each integral using $z = e^{it}$ so $dz/dt = ie^{it}$. Also, recall that $\cos t = \left(z + z^{-1}\right)/2$. So,

$$I_1 + I_2 = \frac{1}{i} \int_{|z|=1} \frac{u(z)}{z}\, dz = \pi u(0),$$

where we used Cauchy's integral formula. Also,

$$I_1 - I_2 = \frac{1}{2i} \int_{|z|=1} u(z) + \frac{u(z)}{z^2}\, dz = \frac{1}{2i} \int_{|z|=1} \frac{u(z)}{z^2}\, dz = \pi u'(0),$$

where we used Cauchy's integral formula and the fact that $u(z)$ is analytic on $D(0,r)$ (since $u(z)$ is harmonic on $D(0,r)$). □

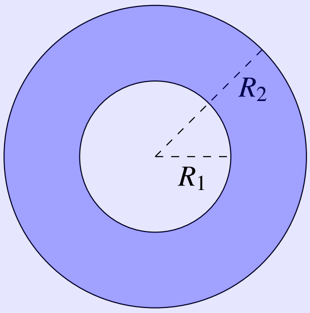
5. Series

5.1. *Laurent Series*

Definition 5.1 (annulus). Define

$$\mathrm{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\}$$

to be the shaded region as follows:



Theorem 5.1 (Laurent expansion). If f is analytic in the annulus

$$\mathrm{Ann} = \{z \in \mathbb{C} \mid R_1 < |z| < R_2\},$$

then it has a Laurent expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz.$$

Here, C is a circle of radius R centred at the origin with $R_1 < R < R_2$.

Example 5.1.

(a) Consider the function

$$f(z) = \frac{5z-3}{(z+1)(z-3)}.$$

Find the Laurent series of $f(z)$ for the annular domain $1 < |z| < 3$.

(b) Find the value of the contour integral

$$\int_C \frac{5z-3}{z^5(z+1)(z-3)}\, dz,$$

where C denotes the circle $|z| = 2$ oriented in the anticlockwise direction.

(c) Find the Laurent series of the function

$$\frac{10z^6 - 6z^4}{(z^2+1)(z^2-3)}$$

in the annular domain $1 < |z| < \sqrt{3}$.

Solution.

(a) We see that

$$\begin{aligned}\frac{5z-3}{(z+1)(z-3)} &= \frac{2}{z+1}+\frac{3}{z-3}\\ &= \frac{2}{z}\cdot\frac{1}{1+1/z}-\frac{1}{1-z/3}\\ &= \frac{2}{z}\sum_{n=0}^{\infty}(-1)^n(-z)^n-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\end{aligned}$$

We note that the first summation is valid for $|1/z|<1$ while the second summation is valid for $|z/3|<1$.

(b) We see that the contour integral is equivalent to

$$\int_C \frac{f(z)}{z^5}\,dz = 2\pi i\left(-\frac{1}{3^4}\right) = -\frac{2\pi i}{81}.$$

(c) Let us make a comparison. Perhaps we can consider $f(z^2)$. Note that

$$f(z^2)=\frac{5z^2-3}{(z^2+1)(z-3)}.$$

Hence, it is clear that the function in **(c)** is $2z^4 f(z^2)$. Recall that the Laurent series of f in the annulus $1<|z|<3$ is

$$2\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

so the required answer is

$$4z^4\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}-2z^4\sum_{n=0}^{\infty}\left(\frac{z^2}{3}\right)^n = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n-2}}-2\sum_{n=0}^{\infty}\frac{z^{2n+4}}{3^n}$$

in the annular domain $1<|z|<\sqrt{3}$. □

Example 5.2. Suppose $f(z)$ is entire and $|f(z)|>1$ when $|z|>1$. Prove that $f(z)$ is a polynomial.

Solution. Since f is entire, then in the closed unit disk, it has a finite number of zeros. Say the zeros are $z_1,\ldots,z_m$. So, we can write

$$f(z)=(z-z_1)\ldots(z-z_m)g(z)=p(z)g(z),$$

where g is entire with no zeros and $p(z)$ is a polynomial of degree m. It suffices to show that g is a constant. Let $h(z)=1/g(z)$ so we shall write h as the following Laurent series:

$$h(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{where } a_n=\frac{1}{2\pi i}\int_\gamma\frac{h(z)}{z^{n+1}}\,dz$$

Here, we let γ be $|z|=R$, i.e. the circle of radius R centred at the origin. Letting $z=Re^{i\theta}$, the contour integral becomes

$$\int_0^{2\pi}\frac{iRe^{i\theta}h\left(Re^{i\theta}\right)}{R^{n+1}e^{i(n+1)\theta}}\,d\theta.$$

Let $h\left(Re^{i\theta}\right)\le kR^m$ so it is clear that for all $n>m$,

$$\lim_{R\to\infty}|a_n|\le\lim_{R\to\infty}\frac{kR^m}{R^n}=0.$$

As such, $h(z)$ is a constant, and $g(z)$ is a constant. □

6. Residue Theory

6.1. *Introduction*

We adopt an alternative representation for the annulus $\text{Ann}(z_0, R_1, R_2)$, so if $f(z)$ is analytic in this annulus,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{n+1}}\, ds \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z_0)^{-n+1}}\, ds$$

and C is any positively oriented simple closed contour around z_0 lying inside $\text{Ann}(z_0, R_1, R_2)$.

Definition 6.1 (principal part of Laurent series)**.** The sum

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{is} \quad \text{the principal part of } f(z) \text{ at } z_0.$$

Theorem 6.1 (removable singularity)**.** If $b_n = 0$ for all $n \in \mathbb{N}$, then z_0 is a point of removable singularity of $f(z)$. Thus, the Laurent series of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{where } 0 < |z - z_0| < R.$$

Example 6.1. The singular point $z = 0$ of $\sin z / z$ is a removable singularity. We have

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

where $0 < |z| < \infty$. This asserts that our claim is true.

Example 6.2 (Dinh's 70 problems)**.** Let $f(z)$ be holomorphic in $\mathbb{C} \setminus \{0\}$ and suppose that

$$\int_{|z|=1} z^n f(z)\, dz = 0 \quad \text{for any } n \in \mathbb{Z}_{\geq 0}.$$

Show that f has a removable singularity at $z = 0$.

Solution. f has a Laurent series representation around $z = 0$. Write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$$

so the integral becomes

$$\int_{|z|=1} z^n \sum_{k=-\infty}^{\infty} a_k z^k\, dz = 0$$

$$\sum_{k=-\infty}^{\infty} \int_{|z|=1} a_k z^{n+k}\, dz = 0$$

since that the series converges uniformly on compact sets away from the singularity. Note that

$$\int_{|z|=1} z^k\, dz = 0 \quad \text{for all } k \neq -1.$$

As such, $n+k=-1$. Since $n \geq 0$, it forces the inequality $k \leq -1$, which implies that $a_k = 0$ for all $k \leq -1$, i.e.

$$\sum_{k=-1}^{\infty} \int_{|z|=1} a_k z^{n+k}\, dz = 0.$$

It is clear that $a_{-1} = 0$. With all coefficients of negative powers being zero, it shows that $f(z)$ has a removable singularity at $z=0$. □

Definition 6.2 (essential singularity)**.** If $b_n \neq 0$ for infinitely many n, then z_0 is a point of essential singularity of $f(z)$. In this case, some of the b_n's may be zero.

Example 6.3. The point $z=0$ of $\exp(1/z)$ is an essential singularity as

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + z^{-1} + \frac{1}{2!}z^{-2} + \frac{1}{3!}z^{-3} + \ldots$$

where $0 < |z| < \infty$.

Definition 6.3 (pole)**.** If there exists $m \in \mathbb{N}$ such that $b_m \neq 0$ but $b_n = 0$ for all $n > m$ so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n},$$

then z_0 is a pole of order m of $f(z)$. If $m=1$, z_0 is a simple pole of $f(z)$; if $m=2$, z_0 is a double pole of $f(z)$.

Example 6.4. Consider the point $z=1$ of

$$f(z) = \frac{1}{(z-1)^2} + z.$$

We can rewrite it as

$$f(z) = \frac{1}{(z-1)^2} + 1 + (z-1)$$

Hence, $z=1$ is a double pole.

Example 6.5 (MA5217 AY24/25 Sem 1 Homework 1)**.** Find all the singularities in $\mathbb{C}$ of the following function $f(z)$ and their types where

$$f(z) = \frac{z^2+3z+2}{z(z^4-1)} e^{1/z^2}.$$

Solution. Consider the term z^4-1 in the denominator of $f(z)$. Then, $z^4-1 = (z^2+1)(z^2-1) = (z^2+1)(z+1)(z-1)$. Also, the numerator can be factorised as $(z+2)(z+1)$. Also, consider

$$\frac{e^{1/z^2}}{z} = \sum_{n=0}^{\infty} \left(\frac{1}{z^2}\right)^n \frac{1}{n!} \cdot \frac{1}{z} = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1} n!}.$$

So, $f(z)$ has simple poles at $z=1, z=i, z=-i$, a removable singularity at $z=-1$, and an essential singularity at $z=0$. □

Theorem 6.2 (residue theorem)**.** Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ interior to C. Let

$\text{Res}(f, a_k)$ denote the residue of f at a_k, for all $1 \leq k \leq n$. Then,

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, a_k).$$

Theorem 6.3. If f is analytic everywhere on the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)\, dz = 2\pi i \text{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

Example 6.6 (Dinh's 70 problems)**.** Evaluate the integral

$$\int_{C^+(0,2)} e^{e^{1/z}}\, dz.$$

Solution. Let $w = 1/z$ so $dw/dz = -w^2$. The integral becomes

$$\int_{C^+(0,1/2)} e^{e^w} \cdot \frac{dw}{w^2}.$$

Let $f(w) = e^{e^w}$. By the residue theorem,

$$\int_{C^+(0,1/2)} \frac{f(w)}{w^2}\, dw = 2\pi i \text{Res}(f(w), 0) = 2\pi i e$$

and we are done. □

6.2. *Residue Computation Methods*

There are three methods for computing residues.

Theorem 6.4 (method 1)**.** Suppose for z near z_0, $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{z - z_0},$$

where $\phi(z)$ is analytic at z_0 and f has a simple pole or a removable singularity at z_0. Then,

$$\text{Res}_{z=z_0} f(z) = \phi(z_0).$$

Proof. Since $\phi(z)$ is analytic at z_0, then by Taylor's theorem, for z near z_0,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z - z_0) + \ldots$$

so the Laurent series of $f(z)$ at z_0 is

$$f(z) = \frac{\phi(z)}{z - z_0} = \frac{\phi(z_0) + \phi'(z_0)(z - z_0) + \ldots}{z - z_0} = \frac{\phi(z_0)}{z - z_0} + \phi'(z_0) + \ldots$$

and the result follows. □

Theorem 6.5 (method 2)**.** Suppose for z near z_0, $f(z)$ can be written as

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where $\phi(z)$ is analytic at z_0 and $m \geq 1$. Then,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

Proof. It is inferred that f has a pole of order less than or equal to m or a removable point of singularity at z_0. Observe that when $m=1$, it is just method 1 (recall Theorem 6.4). Using Taylor's theorem again, the series expansion of $\phi(z)$ is the same as before. That is,

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z-z_0) + \ldots$$

so

$$\begin{aligned} f(z) &= \frac{\phi(z)}{(z-z_0)^m} \\ &= \frac{1}{(z-z_0)^m}\left[\phi(z_0) + \ldots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \ldots\right] \\ &= \frac{\phi(z_0)}{(z-z_0)^m} + \ldots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \cdot \frac{1}{z-z_0} + \ldots \end{aligned}$$

The result follows. □

Theorem 6.6 (method 3)**.** If $p(z)$ and $q(z)$ are analytic at z_0 and $q(z)$ has a simple zero at z_0 (i.e. $q(z_0) = 0$ but $q'(z_0) \neq 0$), then

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$

Theorem 6.7 (method 4)**.** If all the above methods fail, use the Laurent series of $f(z)$ and read b_1.

Example 6.7. For the following function $f(z)$, find all of its singularities in $\mathbb{C}$, their types and residues at these points:

$$f(z) = \frac{z^2+1}{z^6+1}.$$

Solution. The singularities of $f(z)$ are the zeros of the denominator z^6+1, that is the 6 points

$$z_k = \exp\left(\frac{i\pi}{6} + \frac{k\pi i}{3}\right),$$

where $0 \leq k \leq 5$. These points are simple zeros of $z^6+1=0$. The points $z_1 = i$ and $z_4 = -i$ are the roots of the equation $z^2+1=0$ (refer to the numerator). Thus, z_1, z_4 are removable and z_0, z_2, z_3, z_5 are simple poles of f.

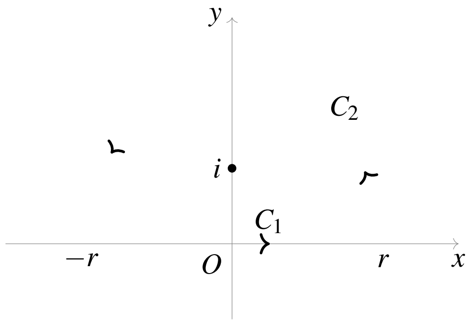
So, the residues of f at z_1, z_4 are 0, whereas the residue of f at z_k for $k = 0,2,3,5$ is equal to $(z_k^2+1)/6z_k^5$. □

Example 6.8 (classic result)**.** Prove that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e}.$$

Solution. We consider
$$f(z) = \frac{e^{iz}}{z^2+1}$$
so the integral $\mathrm{Re}(f(z))$ over the real numbers is the required answer. Let C be the path $C_1 + C_2$, where C_1 and C_2 are parametrised as follows:

$$C_1(t) = t, \text{ where } t \in [-r, r]$$
$$C_2(t) = re^{it}, \text{ where } t \in [0, \pi]$$



By Cauchy's residue theorem,
$$\sum \mathrm{Res}(f(z)) = \frac{1}{2\pi i}\int_C f(z)\,dz.$$
Only one of the two poles of $f(z)$, $z = i$, is inside C as we are considering the upper half of the circle centred at the origin. We have
$$\int_C f(z)\,dz = \int_{C_1} \frac{e^{iz}}{z^2+1}\,dz + \int_{C_2} \frac{e^{iz}}{z^2+1}\,dz.$$
For the integral over C_1, applying the parametrisation,
$$\int_{C_1} \frac{e^{iz}}{z^2+1}\,dz = \int_{-r}^{r} \frac{e^{it}}{t^2+1}\,dt = \int_{-r}^{r} \frac{\cos t}{t^2+1}\,dt + i\int_{-r}^{r} \frac{\sin t}{t^2+1}\,dt.$$
Since $\sin t$ is an odd function, then the integral of $\sin t/(t^2+1)$ is zero. Hence,
$$\int_{C_1} \frac{e^{iz}}{z^2+1}\,dz = \int_{-r}^{r} \frac{\cos t}{t^2+1}\,dt.$$
As for the integral over C_2, applying the parametrisation,
$$\int_{C_2} \frac{e^{iz}}{z^2+1}\,dz = \int_0^{\pi} \frac{\exp\left(ire^{it}\right)}{r^2e^{i2t}+1} \cdot ire^{it}\,dt.$$
By applying Euler's Formula,
$$\begin{aligned}
\int_0^{\pi} \frac{\exp\left(ire^{it}\right)}{r^2e^{i2t}+1} \cdot ire^{it}\,dt &= ir\int_0^{\pi} \frac{e^{i(t+r\cos t)}e^{-r\sin t}}{r^2e^{i2t}+1}\,dt \\
\left|\int_0^{\pi} \frac{\exp\left(ire^{it}\right)}{r^2e^{i2t}+1} \cdot ire^{it}\,dt\right| &= r\int_0^{\pi} \frac{e^{-r\sin t}}{|r^2e^{i2t}+1|}\,dt \\
&\leq \frac{r}{r^2-1}\int_0^{\pi} e^{-r\sin t}\,dt
\end{aligned}$$
Let the radius r of the semicircle tend to infinity so it is then clear that
$$\int_{C_2} \frac{e^{iz}}{z^2+1}\,dz = 0.$$

Therefore, by Cauchy's residue theorem (rearrange the equation at the start of our solution),

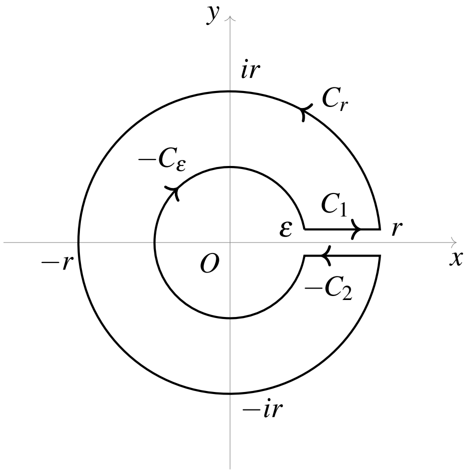
$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt = 2\pi i \cdot \frac{e^{i^2}}{2i} = \frac{\pi}{e}.$$

□

Example 6.9 (branch cut). Prove that

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2+5x+6}\,dx = \pi(\sqrt{3}-\sqrt{2}).$$

Solution. Note that 0 is a branch point of $\sqrt{z}$. So, $\sqrt{z}$ has a branch cut along the positive real axis, i.e. $[0,\infty)$. Hence, $\sqrt{z}$ is analytic on $\mathbb{C}\setminus[0,\infty)$. We adopt the following keyhole contour.



One should think of the above contour as having ε so small that C_1 and C_2 are essentially on the x-, or rather, real axis. Let the region the contour encloses be D. Then, we shall consider the integral over the boundary (this is denoted by ∂D). That is,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2+5z+6}\,dz.$$

By Cauchy's residue theorem,

$$\int_{\partial D} \frac{\sqrt{z}}{z^2+5z+6}\,dz = 2\pi i \left[\left.\frac{\sqrt{z}}{2z+5}\right|_{z=-3} + \left.\frac{\sqrt{z}}{2z+5}\right|_{z=-2} \right] = 2\pi\left(\sqrt{3}-\sqrt{2}\right).$$

We now evaluate the contour integral by considering the different *pieces*.

$$\begin{aligned}\int_{\partial D} \frac{\sqrt{z}}{z^2+5z+6}\,dz &= \int_{C_r} - \int_{C_\varepsilon} + \int_{C_1} - \int_{C_2} \\ &= \int_{C_r} - \int_{C_\varepsilon} + 2\int_\varepsilon^r \frac{\sqrt{x}}{x^2+5x+6}\,dx\end{aligned}$$

By the estimation lemma,

$$\left| \int_{C_r} f(z)\,dz \right| \le 2\pi r \cdot \frac{\sqrt{r}}{r^2-5r-6}$$

which tends to 0 as r tends to infinity. In a similar fashion, one can prove that

$$\left| \int_{C_\varepsilon} f(z)\,dz \right| \le 2\pi\varepsilon \cdot \frac{\sqrt{\varepsilon}}{6-5\varepsilon-\varepsilon^2}$$

which tends to 0 too as ε tends to 0. As such,

$$2\pi\left(\sqrt{3}-\sqrt{2}\right) = 2\int_0^{\infty} \frac{\sqrt{x}}{x^2+5x+6}\,dx$$

and the result follows. □

Example 6.10 (pizza contour). Prove that for $n \geq 2$,

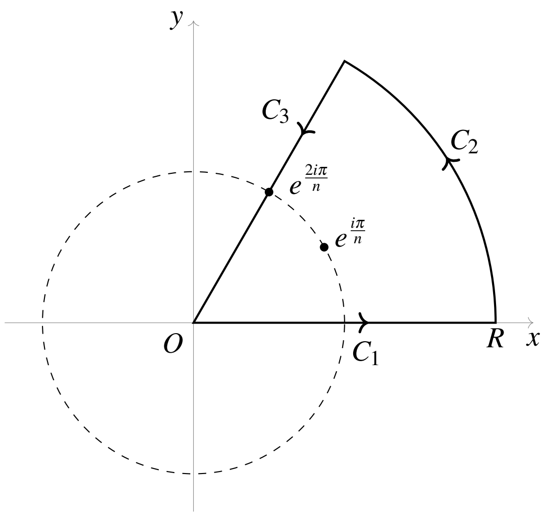
$$\int_0^\infty \frac{1}{x^n+1}\,dx = \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}.$$

Solution. Let

$$f(z) = \frac{1}{z^n+1}$$

and the required integral to be I. Consider the following parametrisation (informally known as the *pizza contour*):

$$\begin{aligned} C_1(t) &:= t, \text{ where } 0 \leq t \leq R \\ C_2 &: |z| = R \text{ (note that the angle subtended by the arc does not matter)} \\ C_3(t) &:= (R-t)\exp\left(\frac{2\pi i}{n}\right), \text{ where } 0 \leq t \leq R \end{aligned}$$



By defining C to be the contour, it is clear that

$$\int_C \frac{1}{z^n+1}\,dz = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

Only the pole $z = e^{i\pi/n}$ is in C so by the residue theorem,

$$\int_C \frac{1}{z^n+1}\,dz = 2\pi i \operatorname*{Res}_{z=e^{i\pi/n}} f(z) = -\frac{2\pi i}{n}\exp\left(\frac{i\pi}{n}\right).$$

We focus on C_1.

$$\int_{C_1} \frac{1}{z^n+1}\,dz = \int_0^R \frac{1}{t^n+1}\,dt.$$

Letting R tend to infinity, and since t is a dummy variable, it is easy to see that

$$\int_{C_1} \frac{1}{z^n+1}\,dz = \int_0^\infty \frac{1}{x^n+1}\,dx = I.$$

For C_2, by the triangle inequality, $|z^n+1| \geq ||z^n|-|-1|| = |R^n-1|$. Hence,

$$\left|\int_{C_2} \frac{1}{z^n+1}\,dz\right| \leq \int_{C_2} \frac{1}{R^n-1}\,dz = \frac{c\pi R}{R^n-1}.$$

Letting R tend to infinity, we see that the integral over C_2 is zero. Earlier, we mentioned that the angle subtended by the arc does not matter and we affirm this statement here.

The integral over C_3 is more complicated. Using the substitution

$$z = (R-t)\exp\left(\frac{2\pi i}{n}\right),$$

we see that

$$\int_{C_3} \frac{1}{z^n+1}\,dz = -\exp\left(\frac{2\pi i}{n}\right)\int_0^R \frac{1}{(R-t)^n+1}\,dt.$$

This calls for a substitution, say $u = R-t$. Hence, the integral over C_3 becomes

$$-\exp\left(\frac{2\pi i}{n}\right)\int_0^R \frac{1}{u^n+1}\,du \xrightarrow{R\to\infty} -\exp\left(\frac{2\pi i}{n}\right)\int_0^\infty \frac{1}{x^n+1}\,dx = -I\exp\left(\frac{2\pi i}{n}\right).$$

To conclude,

$$\begin{aligned}
-\frac{2\pi i}{n}\exp\left(\frac{i\pi}{n}\right) &= I\left[1-\exp\left(\frac{2\pi i}{n}\right)\right] \\
I &= -\frac{2\pi i}{n}\cdot\frac{\exp\left(\frac{i\pi}{n}\right)}{1-\exp\left(\frac{2\pi i}{n}\right)} \\
&= -\frac{2\pi i}{n}\cdot\frac{\exp\left(\frac{i\pi}{n}\right)}{\exp\left(\frac{i\pi}{n}\right)\exp\left(-\frac{i\pi}{n}\right)-\exp\left(\frac{i\pi}{n}\right)\exp\left(\frac{i\pi}{n}\right)} \\
&= \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}
\end{aligned}$$

so we have finally derived this beautiful result. □

Example 6.11. Prove that

$$\int_0^{2\pi} \frac{1}{5+3\sin\theta}\,d\theta = \frac{\pi}{2}.$$

Solution. Set $z = e^{i\theta}$ so $\sin\theta = (z-z^{-1})/2i$. The integral becomes

$$\int_{|z|=1} \frac{1}{5+3\left(\frac{z-z^{-1}}{2i}\right)}\cdot\left(-\frac{i}{z}\right)\,dz = 2\int_{|z|=1}\frac{1}{3z^2+10iz-3}\,dz.$$

Let

$$f(z) = \frac{1}{3z^2+10iz-3}.$$

It has two simple poles $z_1 = -i/3$ and $z_2 = -3i$. The first one is interior to the circle $|z| = 1$ so we shall consider this. By the residue theorem, the answer is

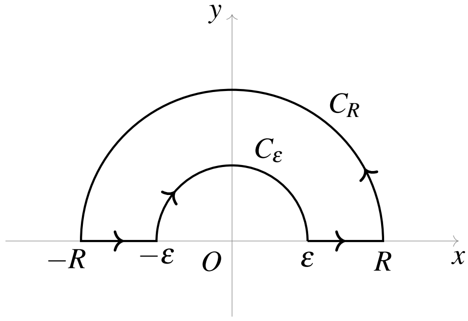
$$2\cdot 2\pi i\cdot\frac{1}{3(z_1+3i)} = \frac{\pi}{2}.$$

□

Example 6.12. Prove that

$$\int_0^\infty \frac{(\log x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}.$$

Solution. We consider the following contour.



Define

$$f(z) = \frac{(\log z)^2}{z^2+1}$$

and in our contour, say C, we let $0 < \varepsilon < 1 < R$. $\log z$ denotes the branch of the logarithm function defined on $\{z \in \mathbb{C} : -\pi/2 < \arg z < 3\pi/2\}$. Hence, it is clear that

$$\int_C = \int_{C_R} + \int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_\varepsilon^R.$$

By the residue theorem,

$$\int_C \frac{(\log z)^2}{z^2+1}\,dz = \frac{(\log i)^2}{2i} = -\frac{\pi^3}{4}.$$

Now, let us focus on C_R. We use the estimation lemma to help us.

$$\left|\int_{C_R}\right| \leq \pi R \cdot \frac{(\log R + i\theta)^2}{R^2-1}$$

which tends to 0 as R tends to infinity. In a similar fashion, one can show that

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} = 0.$$

As such,

$$\begin{aligned}
\int_C \frac{(\log z)^2}{z^2+1}\,dz &= \int_{-R}^{-\varepsilon} \frac{(\log z)^2}{z^2+1}\,dz + \int_\varepsilon^R \frac{(\log z)^2}{z^2+1}\,dz \\
&= \int_\varepsilon^R \frac{(\log(-z))^2}{z^2+1}\,dz + \int_\varepsilon^R \frac{(\log z)^2}{z^2+1}\,dz \\
&= \int_\varepsilon^R \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2+1}\,dz
\end{aligned}$$

Now we set R to tend to infinity and ε to tend to 0. Also, we computed the value of the integral over C earlier so putting everything together,

$$\begin{aligned}
-\frac{\pi^3}{4} &= \int_0^\infty \frac{(i\pi + \log z)^2 + (\log z)^2}{z^2+1}\,dz \\
&= -\pi^2 \int_0^\infty \frac{1}{z^2+1}\,dz + 2i\pi \int_0^\infty \frac{\log z}{z^2+1}\,dz + 2\int_0^\infty \frac{(\log z)^2}{z^2+1}\,dz \\
&= -\frac{\pi^3}{2} + 2i\pi \int_0^\infty \frac{\log z}{z^2+1}\,dz + 2\int_0^\infty \frac{(\log z)^2}{z^2+1}\,dz
\end{aligned}$$

Lastly, we will show that

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx = 0.$$

Using the substitution $u = 1/x$,

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx = \int_0^\infty \frac{-\log u}{(1/u)^2+1} \cdot \left(-\frac{1}{u}\right)^2 du = -\int_0^\infty \frac{\log u}{u^2+1}\,du$$

and the result follows. $\square$

We have the following beautiful corollary:

Corollary 6.1. Let

$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx.$$

Then for all $n \geq 1$, I_{2n} satisfies the recurrence relation

$$I_{2n} = \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}} - \frac{1}{2}\sum_{k=1}^{n} \binom{2n}{2k} (-1)^k \pi^{2k} I_{2n-2k}.$$

It is not surprising that we only discuss the integrals I_{2n} instead of I_{2n+1} because

$$\int_0^\infty \frac{(\log x)^{2n+1}}{x^2+1}\,dx = 0$$

for all $n \geq 0$ by performing the substitution $u = 1/x$.

The above formula is also equivalent to the following by using the Dirichlet beta function:

Definition 6.4 (Dirichlet beta function)**.** Define the Dirichlet beta function to be

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

Corollary 6.2. Let

$$I_{2n} = \int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx.$$

Then for all $n \geq 1$, $I_{2n} = 2(2n)!\beta(2n+1)$.

Proof.

$$\begin{aligned}
\int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx &= \int_0^1 \frac{(\log u)^{2n}}{u^2+1}\,du \quad \text{using } u = \frac{1}{x} \\
\int_0^\infty \frac{(\log x)^{2n}}{x^2+1}\,dx &= 2\int_0^1 \frac{(\log x)^{2n}}{x^2+1}\,dx \\
&= 2\int_0^1 (-1)^k \sum_{k=0}^{\infty} (\log x)^{2n} x^{2k}\,dx \quad \text{using integration by parts} \\
&= 2(2n)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}
\end{aligned}$$

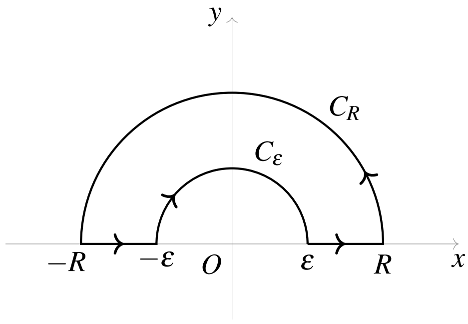
and the result follows. □

Example 6.13 (Dinh's 70 problems)**.** Show that

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)^2}\,dx = \frac{\pi(1-\alpha)}{2\cos\left(\frac{\pi\alpha}{2}\right)}$$

for $-1 < \alpha < 3$, $\alpha \neq 1$. What happens if $\alpha = 1$?

Solution. We consider the following contour.



Here, $0<\varepsilon<R$ and C_R and C_ε denote the upper-half of the semicircle of radius R and ε respectively. So,

$$\int_C f(z)\,dz=\int_{C_R}+\int_{-R}^{-\varepsilon}+\int_{C_\varepsilon}+\int_{\varepsilon}^{R}.$$

By the residue theorem, it is clear that

$$\int_C f(z)\,dz=\frac{i\pi e^{ia\pi/2}}{2}.$$

It is clear that

$$\lim_{R\to\infty}\int C_R=0 \text{ and } \lim_{\varepsilon\to 0}\int C_\varepsilon=0.$$

So,

$$\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}=\int_C f(z)\,dz$$

and the result follows. □

Example 6.14 (MA5217 AY24/25 Sem 1 Homework 1)**.** Compute the following integrals using the residue formula:

$$\int_{-\infty}^{\infty}\frac{x-4}{(x^2-4x+5)(x^2+4)}\,dx \quad\text{and}\quad \int_0^{\infty}\frac{x^2}{(x^2+4)^2}\,dx$$

Solution. We deal with the first integral. Note that $z=2+i, z=2-i, z=2i, z=-2i$ are simple poles of the integral. Let $C=C_1+C_2$ be the upper half of the semicircle of radius R centred at the origin on the complex plane, where C_1 is the diameter and C_2 is the arc.

So,

$$\begin{aligned} C_1&=\{z=x+iy\in\mathbb{C}:-R\le x\le R\}\\ C_2&=\left\{z=x+iy\in\mathbb{C}:z=Re^{i\theta},0\le\theta\le\pi\right\}\end{aligned}$$

Let $f(z)$ denote the integrand. We are only interested in the poles interior and on the boundary of C. By the residue theorem,

$$\int_C f(z)\,dz=2\pi i\sum\operatorname{Res}(f(z),z=z_k)=2\pi i\left(\frac{2i}{13}\right)=-\frac{4\pi}{13}$$

Hence,

$$\int_{C_1} f(z)\,dz=\int_{-R}^{R}\frac{x-4}{(x^2-4x+5)(x^2+4)}\,dx.$$

Letting $R \to \infty$, we see that we obtain the original integral. Also,

$$\left|\int_{C_2} f(z)\,dz\right| = \left|\int_0^\pi \frac{Re^{i\theta}\cdot iRe^{i\theta}}{(R^2e^{2i\theta}-4Re^{i\theta}+5)(R^2e^{2i\theta}+4)}\,d\theta\right|$$
$$= \left|\int_0^\pi \frac{R^2}{(R^2e^{2i\theta}-4Re^{i\theta}+5)(R^2e^{2i\theta}+4)}\,d\theta\right|$$

which is equal to 0 by the triangle inequality. Hence, the answer is $-4\pi/13$.

For the second integral, we note that the function is even. Letting g denote the integrand, we have

$$\int_0^\infty g(z)\,dz = \frac{1}{2}\int_{-\infty}^\infty g(z)\,dz.$$

We consider the same contour as the previous part, acknowledging that $z = \pm 2i$ are double poles of g. So, it follows that the sum of residues is $-\pi/8$, and by some tedious computation, the integral evaluates to $\pi/8$.

To compute the residue of the double pole $z = 2i$, we use the formula

$$\lim_{z\to 2i}\frac{d}{dz}\left((z-2i)^2 g(z)\right)$$

which is quite easy. □

Example 6.15 (Dinh's 70 problems)**.** Evaluate

$$\int_{-\infty}^\infty \frac{x\sin x}{(1+x^2)^2}\,dx.$$

Solution. Let $f(z) = \dfrac{ze^{iz}}{(1+z^2)^2}$. Define C_1 to be the upper half of the semicircle of radius R centred at the origin and C_2 to be the real axis bounded by $\pm R$. So, C_1 can be parametrised using $z = Re^{it}$ for $t \in [0,\pi]$, whereas C_2 can be parametrised using $z = t$ for $t \in [-R,R]$. Let $C = C_1 \cup C_2$. By the residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,\mathrm{Res}(f(z),i).$$

Note that

$$\mathrm{Res}(f(z),i) = \lim_{z\to i}\frac{d}{dz}\left(\frac{ze^{iz}}{(z+i)^2}\right) = \frac{1}{4e}.$$

Hence,

$$\int_C f(z)\,dz = \frac{i\pi}{2e}.$$

Now,

$$\lim_{R\to\infty}\left|\int_{C_1} f(z)\,dz\right| = \lim_{R\to\infty}\left|R^2\int_0^\pi \frac{1}{\left(1+R^2e^{2i\theta}\right)^2}\,d\theta\right| = 0.$$

Lastly, we work with C_2. So, we have

$$\lim_{R\to\infty}\int_{C_2} f(z)\,dz = \lim_{R\to\infty}\int_{-R}^R \frac{t\sin t}{\left(1+t^2\right)^2}\,dt = \int_{-\infty}^\infty \frac{x\sin x}{\left(1+x^2\right)^2}\,dt.$$

It follows that the answer is $\pi/2e$. □

Example 6.16 (Dinh's 70 problems)**.** Show that for any $0 < a < 1$,

$$\int_0^\infty \frac{x^a}{x(1+x)}\,dx = \frac{\pi}{\sin(a\pi)}.$$

Solution. Let $t = x/(1+x)$, so

$$x = \frac{t}{1-t} \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{(1-t)^2}.$$

The integral becomes

$$\begin{aligned}
\int_0^1 t^{a-1}(1-t)^{-a}\,dt &= B(a, 1-a) && \text{by definition of beta function} \\
&= \frac{\Gamma(a)\Gamma(1-a)}{\Gamma(1)} && \text{by relationship with gamma function} \\
&= \Gamma(a)\Gamma(1-a)
\end{aligned}$$

and the result follows by Euler's reflection formula. □

7. Further Properties of Holomorphic Functions

7.1. *Properties of Holomorphic and Harmonic Functions*

Definition 7.1 (extended complex plane)**.** Define

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\} \quad \text{to be the extended complex plane.}$$

Theorem 7.1 (Cauchy's estimate)**.** Let $f \in H(\Omega)$ and let $\overline{D}(z_0, r) \subseteq \Omega$. Then, for all $n = 0, 1, 2, \ldots$,

$$|a_n| \leq r^{-n} \sup_{|z-z_0|=r} |f(z)|.$$

Example 7.1 (Dinh's 70 problems)**.** Suppose $f(z)$ is an odd function and holomorphic in $\mathbb{C} \setminus \{0\}$ and satisfies

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^2} \text{ for all } z \neq 0.$$

Prove that

$$f(z) = \frac{a_{-1}}{z} + a_1 z \quad \text{for all } z \in \mathbb{C} \setminus \{0\} \text{ where } a_{-1}, a_1 \in \mathbb{C}.$$

Solution. Since f is holomorphic in $\mathbb{C} \setminus \{0\}$, its Laurent series representation about $z = 0$ is

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

f is odd implies $f(-z) = -f(z)$, so

$$f(z) = \ldots + \frac{a_{-3}}{z^3} + \frac{a_{-1}}{z} + a_1 z + a_3 z^3 + \ldots.$$

Note that for $|z| \leq 1$, we have $|z^2 f(z)| \leq |z|^4 + 1 \leq 2$ and

$$z^2 f(z) = \ldots + \frac{a_{-3}}{z} + a_{-1} z + a_1 z^3 + a_3 z^5 + \ldots$$

so it forces $a_{-3}, a_{-5}, \ldots, a_5, a_7, \ldots = 0$. The result follows. □

Here is an alternative solution.

Solution. Again, write

$$f(z) = \sum_{k \in \mathbb{Z}} a_k z^k.$$

By Cauchy's estimate (Theorem 7.1), if $|f(z)| \leq M$, we have

$$|f^{(k)}(a)| \leq \frac{k! M}{R^k}.$$

So, for $|z| \leq R$, we have

$$|a_k| \leq \frac{1}{R^k} \left(\frac{1}{R^2} + R^2 \right).$$

For $k \geq 3$, $\lim_{r \to \infty} |a_k| = 0$ and for $k \leq -3$, $\lim_{r \to 0} |a_k| = 0$. So, $a_k = 0$ for all $|k| \geq 3$. Hence,

$$f(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2.$$

Using the fact that f is odd, the result follows. □

Example 7.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in $D(0,1)$ and assume that the integral

$$A := \iint_{D(0,1)} |f'(z)|^2 \, dxdy < \infty.$$

(a) Express A in terms of the coefficients a_n.

(b) Prove that

$$|f(z) - f(0)| \leq \sqrt{\frac{A}{\pi} \ln\left(\frac{1}{1-|z|^2}\right)}$$

for all $z \in D(0,1)$.

Solution.

(a) Note that

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

We shall parametrise z using polar coordinates. Let $z = re^{i\theta}$. As such,

$$\begin{aligned}
\iint_{D(0,1)} \left|f'(z)\right| \, dxdy &= \int_0^1 \int_0^{2\pi} \left|\sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}\right|^2 r \, dr d\theta \\
&= \int_0^1 \int_0^{2\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn a_m a_n r^{m+n-1} e^{i(m+n-2)\theta} \, dr d\theta \\
&= 2\pi \int_0^1 \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} \, dr \\
&= \pi \sum_{n=1}^{\infty} n|a_n|^2
\end{aligned}$$

(b) Note that $f(0) = 0$ and the RHS can be written as

$$\sqrt{\ln\left(\frac{1}{1-|z|^2}\right) \sum_{n=1}^{\infty} n|a_n|^2}.$$

Starting with the LHS,

$$|f(z)| = \left|\sum_{n=0}^{\infty} a_n z^n\right| = \left|\sum_{n=1}^{\infty} a_n z^n\right| = \left|\sum_{n=1}^{\infty} (\sqrt{n} a_n)\left(\frac{z^n}{\sqrt{n}}\right)\right| \leq \sqrt{\left(\sum_{n=1}^{\infty} n\left|a_n^2\right|\right)\left(\sum_{n=1}^{\infty} \frac{z^{2n}}{n}\right)}$$

where we applied the Cauchy-Schwarz inequality at the end. The result follows. □

Theorem 7.2 (identity theorem). If two holomorphic functions f and g coincide on some set $E \subseteq D$ containing at least one limit point in D, then $f(z) = g(z)$ everywhere in D.

Example 7.3. Does there exist an entire function with the property that for $n \in \mathbb{N}$,

$$f\left(\frac{1}{n}\right) = \frac{n^4}{1+n^4}?$$

Solution. Replacing n with $1/z$, we consider the function

$$g(z) = \frac{1}{z^4+1}.$$

Note that the roots of the equation $z^4+1=0$ can be found as follows. As $z^4=-1=e^{i\pi+2k\pi i}$, then

$$z=\exp\left(i\pi\cdot\frac{2k+1}{4}\right),$$

where $k=0,1,2,3$. We denote the roots by p_n, where $0\le n\le 3$. Obviously, $g(z)$ is holomorphic outside the 4 points p_n. By our hypothesis, $f(z)=g(z)$ for $z=1,1/2,1/3,\ldots$ and both f and g are defined on $\Omega=\mathbb{C}\setminus\{p_0,p_1,p_2,p_3\}$. The sequence $1,1/2,1/3,\ldots$ converges to 0 which is inside Ω, so this sequence is not discrete in Ω. We conclude that $f=g$ in Ω by the identity theorem.

On the other hand, the function f is entire and bounded near p_n but g is not bounded near these points. We have obtained a contradiction so such a function f does not exist. $\square$

Example 7.4. Do there exist functions f and g that are holomorphic at $z=0$ and that satisfy

(a) $f(1/n)=f(-1/n)=1/n^2$, where $n\in\mathbb{N}$;

(b) $g(1/n)=g(-1/n)=1/n^3$, where $n\in\mathbb{N}$?

Solution.

(a) Yes, $f(z)=z^2$.

(b) We prove that such a function g does not exist in a neighbourhood of 0. Suppose on the contrary that g exists. Define $h(z)=z^3$ and $l(z)=-z^3$. We have $g(z)=h(z)$ on a non-discrete sequence $z=1,1/2,1/3,\ldots$ which converges to 0, and 0 is in the domain of g. By the identity theorem, $g(z)=h(z)$. In a similar fashion, by considering the sequence $z=-1,-1/2,-1/3,\ldots$, we obtain $g(z)=l(z)$. Hence, $h(z)=l(z)$, implying that $z^3=-z^3$, so $z^3=0$. However, this is a contradiction. $\square$

Example 7.5. Show that there is no holomorphic function f in $\mathbb{C}$ such that

$$f\left(\frac{1}{n}\right)=\frac{ne^{-2/n}}{n+1}\quad\text{for all } n\in\mathbb{N}.$$

Solution. Suppose on the contrary that such a function exists. Consider

$$g(z)=\frac{e^{-2z}}{z+1}.$$

This function is defined for all $z\in\mathbb{C}$ except at $z=-1$. By the hypothesis, this function is equal to f on the sequence $1/n$ which is not discrete on $\mathbb{C}\setminus\{-1\}$ and so, $f=g$ on $\mathbb{C}\setminus\{-1\}$. However, this is a contradiction. $\square$

Example 7.6 (MA5217 AY24/25 Sem 1 Homework 1)**.** Show that the function $h(z)=\sin(\sin z)+\sin|z|^2$ is not holomorphic in any domain of $\mathbb{C}$.

Solution. Note that

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}.$$

We let $z=x+iy, x,y\in\mathbb{R}$ and note that $|z|^2=x^2+y^2$. Hence,

$$h(z)=\sin\left(\frac{e^{iz}}{2i}\right)\cos\left(\frac{e^{-iz}}{2i}\right)-\cos\left(\frac{e^{iz}}{2i}\right)\sin\left(\frac{e^{-iz}}{2i}\right)+\sin\left(|z|^2\right)$$

By the Looman-Menchoff theorem, it suffices to prove that h does not satisfy the Cauchy-Riemann equations, i.e.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

The computation is tedious so we skip the details. $\square$

Example 7.7 (MA5217 AY24/25 Sem 1 Homework 1). Find all holomorphic functions $f(z)$ in $\mathbb{C}\setminus\{1\}$ such that

$$\operatorname{Res}(f,1)=1,\quad \lim_{z\to\infty}(f(z)-z)=2,\quad \lim_{z\to 1}|z-1|^{4/3}f(z)=0.$$

Solution. We claim that

$$f(z)=z+2+\frac{1}{z-1}.$$

By the second condition, we infer that

$$f(z)=z+2+\sum_{n=1}^{\infty}\frac{1}{(az+b)^n}.$$

By the third condition, we infer that

$$\lim_{z\to 1}(z-1)^{4/3}\sum_{n=1}^{\infty}\frac{1}{(az+b)^n}=0$$

which implies we have to restrict the index of the infinite sum to $n=1$ instead of $nin\mathbb{N}$. Hence,

$$f(z)=z+2+\frac{1}{az+b}.$$

We see that $1/(az+b)$ has a simple pole at $z=-b/a$ but the first condition implies that $z=1$ is a pole, so $a=-b$. Since the value of the residue at $z=1$ is 1, then $a=1$, so

$$f(z)=z+2+\frac{1}{z-1}.$$

□

Example 7.8. Let f and g be entire functions and suppose that $|f(z)|\le|g(z)|$ for all $z\in\mathbb{C}$. Show that $f(z)=cg(z)$ for some constant $c\in\mathbb{C}$.

Solution. First, we assume that g is identically equal to zero. Then, the result immediately follows. Now, we consider the case where g is not identically equal to zero. Define $h(z)=f(z)/g(z)$ on $\mathbb{C}$ excluding the set of zeros of g. As such, h is holomorphic outside the zeros of g and $|h(z)|\le 1$. As h is bounded and entire, the result follows by Liouville's theorem. □

Definition 7.2 (analytic function). $f:\Omega\to\mathbb{C}$ is analytic if for any $z_0\in\Omega$, we can write

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,$$

where $a_n\in\mathbb{C}$ for all $n\in\mathbb{Z}_{\ge 0}$ and the series is convergent to $f(z)$ for z in a neighbourhood of z_0.

Example 7.9 (Dinh's 70 problems). Suppose f is entire and $f(z)$ is real iff z is real. Prove that f has at most one zero.

Solution. Suppose $f(z)=0$, then it implies that z is real. Suppose on the contrary that f has a zero $x_0\in\mathbb{R}$ with a multiplicity $m\ge 2$. Then, we can write $f(z)$ as the following power series:

$$f(z)=(z-x_0)^m(a_0+a_1(z-x_0)+a_2(z-x_0)^2+\ldots)$$

Here, $a_0\ne 0$. Note that

$$a_0=\lim_{z\to x_0}\frac{f(z)}{(z-x_0)^m}$$

so for any $z \in \mathbb{R} \setminus \{x_0\}$, we have $\dfrac{f(z)}{(z-x_0)^m} \in \mathbb{R}$. Hence, $a_0 \in \mathbb{R}$. Now, write $z = x_0 + \varepsilon e^{i\theta}$, so we are considering the general case when $z \in \mathbb{C}$. It is clear that

$$f(z) = \varepsilon^m e^{mi\theta}(a_0 + a_1\varepsilon e^{i\theta} + a_2\varepsilon^2 e^{2i\theta} + \ldots).$$

Define $g(\theta) = \mathrm{Im}(e^{mi\theta}(a_0 + \varepsilon u(\theta,\varepsilon) + i\varepsilon v(\theta,\varepsilon)))$, where u, v are real and continuous functions and ε is sufficiently small. Note that $g(\pi/2m)g(3\pi/2m) < 0$ so by the intermediate value theorem, there exists $\theta' \in (\pi/2m, 3\pi/2m)$ such that $g(\theta_0) = 0$. So, $f(x_0 + \varepsilon e^{i\theta_0}) \in \mathbb{R}$. But because $m \geq 2$, it implies that $x_0 + \varepsilon e^{i\theta_0} \notin \mathbb{R}$, so we reached a contradiction. The result follows. □

The next theorem summarises a list of important properties regarding holomorphic functions.

Theorem 7.3. Let Ω be an open and simply-connected domain in $\mathbb{C}$ and let $f \in H(\Omega)$. Then,

- $f \in C^\infty(\Omega)$ and f satisfies the Cauchy-Riemann equations on Ω.
- **Cauchy-Goursat theorem:** If γ is a piecewise differentiable simple closed curve in Ω, then

$$\int_\gamma f(z)\,dz = 0.$$

- **Cauchy's integral formula:** If γ is an anticlockwise oriented and piecewise differentiable simple closed curve in Ω, then for any a interior to γ,

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz.$$

  Moreover, Cauchy's differentiation formula applies here.
- f is analytic, i.e. for any $z_0 \in \Omega$, one can write for $z \in D(z_0, r)$ with $\overline{D(z_0,r)} \subseteq \Omega$,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

  where

$$c_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz,\ n = 0,1,2,\ldots$$

Example 7.10. Let f be a holomorphic function in the unit disc $\mathbb{D}$ such that $|f(z)| < 1$ for $z \in \mathbb{D}$. Show that $|f''(0)| \leq 2$. Give an example of such a map with $f''(0) = 2$.

Solution. We use Cauchy's Differentiation Formula. Note that

$$f''(0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{z^3}\,dz.$$

Here, we let C be such that $|z| = r$, where $r < 1$ (i.e. C contains all points interior to the circle of radius 1 centred at the origin). Using the parametrisation $z = re^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we see that

$$|f''(0)| \leq \frac{1}{\pi}\left|\int_0^{2\pi} \frac{f(re^{i\theta})}{r^3 e^{3i\theta}} \cdot ire^{i\theta}\,d\theta\right| \leq \frac{1}{\pi r^2}\left|\int_0^{2\pi} f(re^{i\theta})\,d\theta\right| \leq \frac{2}{r^2}.$$

Hence, letting r tend to 1, the result follows.

For the later part of the question, we need to find a map such that $f''(0) = 2$. Well, consider

$$\int_{|z|=r} \frac{f(z)}{z^3}\,dz = 2\pi i$$

for which an obvious answer is $f(z) = z^2$. □

Example 7.11 (Dinh's 70 problems). Determine all complex holomorphic functions f defined on the unit disk which satisfy

$$f''\left(\frac{1}{n}\right)+f\left(\frac{1}{n}\right)=0$$

for $n=2,3,4,\ldots$

Solution. Let $g(z)=f''(z)+f(z)$, so g is holomorphic on $\mathbb{D}$. We have $g(1/n)=0$ for all $n=2,3,4,\ldots$ and since $\lim_{n\to\infty} 1/n=0\in\mathbb{D}$, it follows that $g(z)=0$ on $\mathbb{D}$. As such, $f(z)=-f''(z)$ on $z\in\mathbb{D}$. One can use Maclaurin Series to dedcue that $f(z)=f(0)\cos z+f'(0)\sin z$. □

Theorem 7.4 (Casorati-Weierstrass theorem). Let f have an isolated essential singularity at z_0. Then, for any $w\in\mathbb{C}$, $f(z)$ comes arbitrarily close to w in every deleted neighbourhood of z_0. That is, for any $\delta>0$, $f(D'(z_0,\delta))$ is a dense subset of $\mathbb{C}$.

Proof. Suppose on the contrary that for some $\delta>0$, $f(D'(z_0,\delta))$ is not dense in $\mathbb{C}$. Then, there exists $w\in\mathbb{C}$ and $\varepsilon>0$ such that

$$D(w,\varepsilon)\cap f(D'(z_0,\delta))=\varnothing.$$

For $z\in D'(z_0,\delta)$, write

$$g(z)=\frac{1}{f(z)-w}.$$

Then, g is bounded and holomorphic on $D'(z_0,\delta)$, so g has a removable singularity at z_0. Let m be the order of the zero of g at z_0. If $g(z_0)\neq 0$, set $m=0$. Otherwise, write $g(z)=(z-z_0)^m g_1(z)$, where g_1 is holomorphic and does not vanish on $D(z_0,\delta)$. Hence,

$$(z-z_0)^m g_1(z)=\frac{1}{f(z)-w}.$$

Thus, we can write $f(z)$ as

$$f(z)=w+\frac{g_2(z)}{(z-z_0)^m},$$

where $g_2(z)=1/g_1(z)$ is a holomorphic function on $D(z_0,\delta)$. Thus, f has a removable singularity ($m=0$) or a pole ($m\neq 0$) at z_0, which is a contradiction. □

Definition 7.3. A meromorphic function in D is holomorphic on all D, except on a set of isolated points which are poles. Also, they can be written in the form $f=u/v$, where $u,v\in H(D)$ and $v\neq 0$, and they do not have a common zero.

7.2. *The Argument Principle and Rouché's Theorem*

Theorem 7.5 (argument principle). Let $f\in H(\Omega)$ and γ be a positively oriented, piecewise differentiable, simple closed contour in Ω such that all points interior to γ belong to Ω. Suppose f has no zero on γ. The zeros of f inside γ are $a_1,a_2,\ldots,a_n$ and $\alpha_1,\alpha_2,\ldots,\alpha_n$ are their respective multiplicites. Then,

$$\sum_{j=1}^{n}\alpha_j=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

Example 7.12 (Dinh's 70 problems). Evaluate the integral

$$\int_{|z|=2}\frac{f'(z)}{f(z)}\,dz,$$

where $f(z) = \dfrac{\sin z \cos z}{z^7 - z^5 + z^3 - z}$, and $|z| = 2$ is positively oriented.

Solution. We use the argument principle. The answer is $2\pi i(Z-P)$, where Z and P are to be determined. Here, Z and P refer to the respective number of zeros and poles in the circle $|z| = 2$. To calculate Z, set $\sin z \cos z = 0$, so $z = -\pi/2, 0, \pi/2$. Hence, $Z = 3$. To calculate P, set $z^7 - z^5 + z^3 - z = 0$, so $z(z^4+1)(z+1)(z-1) = 0$. The solutions to $z^4 + 1 = 0$ are $z = e^{i\pi/4}, e^{-i\pi/4}, e^{3i\pi/4}, e^{-3i\pi/4}$. As such, $P = 7$, so the required answer is $-8\pi i$. □

Theorem 7.6 (Rouché's theorem). Let $f, g \in H(\Omega)$ and γ be a piecewise differentiable simple closed curve such that all the points interior to γ are contained in Ω. Assume that

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \gamma$. Then, f and g have the same number of zeros (counting multiplicity) inside γ.

Example 7.13 (Dinh's 70 problems). Determine the number of zeros of $e^{z^2} - 3z^4$ in the unit disk.

Solution. For $|z| = 1$ (i.e. on the boundary of the unit disk), $|e^{z^2}| \le e \le 3 = 3|z^4|$ so it follows by Rouché's theorem that there are 4 zeros. □

Example 7.14 (Dinh's 70 problems). Let N_k be the number of roots (counting multiplicity) in the disk $D(0,k) = \{|z| < k\}$ of the equation

$$z^6 - 5z^2 + 10 = 0.$$

For each positive integer k, determine N_k.

Solution. $N_1 = 0$; now consider the case when $k \ge 2$. On $|z| = 2$, $|5z^2 - 10| \le 5|z|^2 + 10 = 30 \le 2^6 = |z|^6$, so by Rouché's's theorem, $N_k = 6$ for $k \ge 2$. □

Example 7.15. Let $r > 0$. Prove that for n sufficiently large, the polynomial

$$1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!}$$

has no root in $D(0,r)$.

Solution. Fix an $r > 0$. Define $f(z) = e^z$ and $g_n(z)$ to be the polynomial above. For z on $\overline{D(0,r)}$, i.e. $|z| = r$,

$$|f(z) - g_n(z)| = \left| \sum_{k \ge n+1} \frac{z^k}{k!} \right| \le \sum_{k \ge n+1} \frac{r^k}{k!}.$$

We note that as $n \to \infty$, the sum on the right tends to zero. For n large enough, the last sum on the right is smaller than e^{-r}. On the other hand, by setting $z = x + iy$, where $x, y \in \mathbb{C}$, we see that $|f(z)| = e^x \ge e^{-|z|} = e^{-r}$. Therefore, for z on $\overline{D(0,r)}$, we have

$$f(z) - g_n(z)| < |f(z)|.$$

By Rouché's theorem, f and g_n have the same number of zeros inside $D(0,r)$. However, f vanishes nowhere so we can conclude that g_n does not vanish in $D(0,r)$. □

Example 7.16. Find the number of zeros (counting multiplicity) of the function $z^5 + 6z^3 + 11$ in the annulus $2 < |z| < 3$.

Solution. Let $f(z) = z^5 + 6z^3 + 11$. On the circle $|z| = 3$, $|f(z) - z^5| = |6z^3 + 11| \leq 6|z^3| + 11 = 173 < 243 = |z^5|$ so by Rouché's theorem, the number of zeros of $f(z)$ in the region $0 < |z| < 3$ is equal to that of z^5, which is 5.

On the circle $|z| = 2$, we have $|f(z) - 6z^3| = |z^5 + 11| \leq 2^5 + 11 = 43 < 48 = |6z^3|$ so the number of zeros of $f(z)$ in the region $0 < |z| < 2$ is equal to that of $6z^3$, which is 3. Therefore, f has exactly $5 - 3 = 2$ zeros in the annulus $2 < |z| < 3$. □

Example 7.17 (Dinh's 70 problems)**.**

(a) For each integer $n \geq 1$, find the number of zeros (counting multiplicity) in the disk $D(0, n)$ of the polynomial $z^7 + 5z^3 - z - 2$.

(b) Prove that the function $u(x, y) = \sinh x \sin y$ is harmonic and find its harmonic conjugates.

Solution.

(a) Let N_n be the number of zeros. We first show that $N_1 = 3$. Note that on $|z| = 1$, we have

$$|z^7 + 5z^3 - z - 2 - 5z^3| = |z^7 - z - 2| \leq |z|^7 + |z| + 2 = 4 \leq 5 = 5|z|^3.$$

By Rouché's's theorem, $N_1 = 3$.

For $n \geq 2$, we show that $N_n = 7$. Note that on the boundary $|z| = n$, we have

$$|z^7 + 5z^3 - z - 2 - z^7| \leq 5|z|^3 + |z| + 2 = 5n^3 + n + 2 \leq n^7 = |z|^7.$$

The result follows by Rouché's's theorem.

(b) Trivial. Show that u satisfies Laplace's equation, then to find its harmonic conjugates, use the Cauchy-Riemann equations. □

Example 7.18. Let $a_1, \ldots, a_n \in D(0, 1)$ and

$$f(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - \overline{a_k} z}.$$

Prove that for each $b \in D(0, 1)$, $f(z) = b$ has exactly n roots in $D(0, 1)$, counting multiplicity.

Solution. For $|z| = 1$, we have $\overline{z} = 1/z$. Hence,

$$\left| \frac{a_k - z}{1 - \overline{a_k} z} \right| = \frac{|a_k - z|}{|z|\,|1/z - \overline{a_k}|} = \frac{|a_k - z|}{|\overline{z} - \overline{a_k}|} = 1.$$

We infer that for $|z| = 1$, $|f(z)| = 1$. We deduce that for $b \in D(0, 1)$,

$$|(f(z) - b) - f(z)| = |b| < 1 = |f(z)|.$$

By Rouché's theorem, $f(z) - b$ and $f(z)$ have the same number of zeros in $D(0, 1)$. The roots of $f(z) = 0$ are $a_1, \ldots, a_n$ (so there are n roots), as such, the result follows. □

Example 7.19 (Dinh's 70 problems)**.** Show that if the integer n is sufficiently large, the equation

$$z = 1 + \left(\frac{z}{2}\right)^n$$

has exactly one solution in the disk $|z| < 2$.

Solution. Let

$$f_n(z) = z - 1 - \left(\frac{z}{2}\right)^n \quad \text{and} \quad f(z) = z - 1.$$

For arbitrary $\varepsilon > 0$, consider the boundary of $C(0, 2-\varepsilon)$, we have

$$|f_n(z) - f(z)| = \left|\frac{z}{2}\right|^n = \left(\frac{2-\varepsilon}{2}\right)^n = \left(1 - \frac{\varepsilon}{2}\right)^n.$$

Also,

$$|f(z)| = |z-1| \geq |z| - 1 = 1 - \varepsilon \quad \text{by the reverse triangle inequality.}$$

By Rouché's's theorem, we need $|f_n - f| \leq |f|$, i.e.

$$\left(1 - \frac{\varepsilon}{2}\right)^n \leq 1 - \varepsilon.$$

So, we choose

$$n \geq \frac{\ln(1-\varepsilon)}{\ln(1-\varepsilon/2)} \quad \text{where } n \in \mathbb{N} \text{ and } 0 < \varepsilon \leq \frac{1}{2}.$$

The number of zeros of f_n in $D(0, 2-\varepsilon)$ is 1. Letting $\varepsilon \to 0$, the result follows. □

Theorem 7.7 (Hurwitz's theorem). Let $f_n : \Omega \to \mathbb{C}$, where $n \in \mathbb{N}$, be a sequence of holomorphic functions that converges locally uniformly to a function $f : \Omega \to \mathbb{C}$. Let γ be a piecewise differentiable, simple closed contour in Ω such that all points interior to γ are contained in Ω. Assume that f has no zero on γ. Then,

$$\text{there exists } N \in \mathbb{N} \text{ such that for all } n > N \quad f_n \text{ and } f \text{ have the same number of zeros inside } \gamma.$$

Example 7.20. Assume that f is holomorphic in a neighbourhood of $\overline{D(0,1)}$ and that $f'(z)$ has no zero on $\partial D(0,1)$. Prove that for n sufficiently large,

$$F_n(z) = f\left(z + \frac{1}{n}\right) - f(z)$$

has the same number of zeros in $D(0,1)$ as $f'(z)$.

Solution. We consider the function $g_n(z) = nF_n(z)$. Note that

$$g_n(z) = n\left[f\left(z + \frac{1}{n}\right) - f(z)\right],$$

so

$$\lim_{n\to\infty} g_n(z) = \lim_{n\to\infty} \frac{f(z+1/n) - f(z)}{1/n} = f'(z).$$

By the Fundamental Theorem of Calculus,

$$g_n(z) = \frac{f(z+1/n) - f(z)}{1/n} = \int_0^1 f'\left(z + \frac{t}{n}\right)\, dt.$$

Hence, g_n converges locally and uniformly in a neighbourhood to f'. By Hurwitz's Theorem, g_n has the same number of zeros as f' in $D(0,1)$ when n is sufficiently large. Therefore, F_n satisfies the same property. □

7.3. *Open Mapping Theorem and the Maximum Modulus Principle*

Theorem 7.8 (open mapping theorem). Let f be a non-constant holomorphic function on an open connected set Ω. Then, f is open, i.e. for any open set $U \subseteq \Omega$, we have $f(U)$ is open.

Theorem 7.9 (maximum modulus principle). Suppose f is a non-constant holomorphic function defined on a domain Ω. Then, $|f|$ does not attain the maximum value in Ω.

Example 7.21. Suppose f is holomorphic on a neighbourhood of the unit disc $\overline{D(0,1)}$ and satisfies $f(0) = 3+4i$, $|f(z)| \leq 5$ if $|z| = 1$. Find $f'(0)$.

Solution. We prove that f is constant. Suppose on the contrary that f is not constant, then by the maximum modulus principle,

$$5 = f(0) < \max_{|z|=1} |f(z)| \leq 5.$$

This is a contradiction, so $f'(0) = 0$. □

Example 7.22. Let f be a continuous function on $\overline{A} = \{1 \leq |z| \leq 4\}$ and holomorphic on $A = \{1 < |z| < 4\}$. Assume that

$$\max_{|z|=1} |f(z)| = 5 \text{ and } \max_{|z|=4} |f(z)| = 20.$$

(i) Show that $|f(2)| \leq 10$.
(ii) Find all functions f such that $f(2) = 10$.

Solution. Let us discuss the solutions.

(i) Define $g(z) = f(z)/z$. Then,

$$\max_{|z|=1} |g(z)| = \max_{|z|=4} |g(z)| = 5.$$

By the maximum modulus principle, $|g(z)| \leq 5$ for $z \in A$. Setting $z = 2$, we have $f(2) \leq 10$.

(ii) $g(2) = 5$. By the maximum modulus principle, g is a constant, so $g(z) = 5$. Hence, $f(z) = 5z$.

□

Corollary 7.1. Let Ω be a domain in $\mathbb{C}$ and f be holomorphic in Ω.

(i) If $|f|$ assumes a local maximum at some point in Ω, then f is constant in Ω.
(ii) If Ω is bounded and f is continuous up to the boundary $b\Omega$, then,

$$\max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in b\Omega} |f(z)|.$$

We obtain the next corollary on the minimum modulus principle by switching to the reciprocal $1/f(z)$.

Corollary 7.2 (minimum modulus principle). Let Ω be a domain in $\mathbb{C}$ and f be holomorphic but never zero in Ω.

- If $|f|$ assumes a local minimum at some point in Ω, then f is constant on Ω.
- If Ω is bounded and f is continuous up to the boundary of Ω and never vanishes in $\overline{\Omega}$, then

$$\min_{z \in \overline{\Omega}} |f(z)| = \min_{z \in b\Omega} |f(z)|.$$

Example 7.23. Suppose f is holomorphic on a neighbourhood of $\overline{D(0,1)}$, $f(0) = i$ and $|f(z)| > 1$ whenever $|z| = 1$. Prove that f has a zero in $D(0,1)$.

Solution. Suppose on the contrary that f does not have a zero in $D(0,1)$. Then, $g(z) = 1/f(z)$ would be holomorphic in a neighbourhood $\overline{D(0,1)}$. Moreover, we have $|g(0)| = 1$ and $|g(z)| < 1$ when $|z| = 1$. This contradicts the maximum modulus principle. □

Theorem 7.10 (maximum and minimum principle for harmonic functions)**.** Let Ω be a domain in $\mathbb{C}$ and u be a real-valued harmonic in Ω.

(i) If u has either a local maximum or a local minimum at some point of Ω, then u is a constant on Ω.

(ii) If Ω is bounded and f is continuous up to the boundary of Ω, then

$$\max_{z\in\overline{\Omega}} u(z) = \max_{z\in b\Omega} u(z) \text{ and } \min_{z\in\overline{\Omega}} u(z) = \min_{z\in b\Omega} u(z).$$

Example 7.24. Find the maximal value of $\mathrm{Re}(z^3)$ for $z \in [0,1]\times[0,1]$.

Solution. Note that $\mathrm{Re}(z^3)$ is harmonic as it is the real part of a holomorphic function. Hence, it achieves its maximal value on the boundary of the unit square. Throughout this problem, $a \in \mathbb{R}$ and $a \in [0,1]$.

- **Case 1 (bottom edge of square):** $z = a$. Then, $\mathrm{Re}(z^3) = a^3$, whose maximum is 1.
- **Case 2 (top edge of square):** $z = a+i$. Then, $\mathrm{Re}(z^3) = a^3 - 3a$. The maximum here is 0.
- **Case 3 (left edge of square):** $z = ai$. Then, $\mathrm{Re}(z^3) = 0$.
- **Case 4 (right edge of square):** $z = 1+ai$. Then, $\mathrm{Re}(z^3) = 1-3a^2$. The maximum here is 1.

Overall, the maximum value is 1 which is achieved when $z = 1$. □

Example 7.25 (Dinh's 70 problems)**.** Let $a \in \mathbb{C}$, $|a| \le 1$, and consider the polynomial

$$P(z) = \frac{a}{2} + (1-|a|^2)z - \frac{\overline{a}}{2}z^2.$$

Prove that $|P(z)| \le 1$ whenever $|z| \le 1$.

Solution. Note that $z\overline{z} = 1$ on $|z| = 1$. Consider

$$\frac{P(z)}{z} = \frac{a}{2z} - \frac{\bar{a}z}{2} + 1 - |a|^2.$$

We have

$$\frac{a}{2z} - \frac{\bar{a}z}{2} = \frac{1}{2}\left(\frac{a}{z} - \overline{a/z}\right).$$

Let $\lambda = a/z \in \mathbb{C}$. Then, $\lambda - \overline{\lambda} = 2i\,\mathrm{Im}(\lambda)$, so

$$\frac{a}{2z} - \frac{\bar{a}z}{2} = i\,\mathrm{Im}\left(\frac{a}{z}\right) = i\,\mathrm{Im}(a\overline{z}).$$

Hence,

$$\begin{aligned}
\left|\frac{P(z)}{z}\right| &\le \left|i\,\mathrm{Im}(a\overline{z}) + 1 - |a|^2\right| \\
|P(z)| &\le \left|i\,\mathrm{Im}(a\overline{z}) + 1 - |a|^2\right| \text{ since } |z| \le 1 \\
|P(z)|^2 &\le |\mathrm{Im}(a\overline{z})|^2 + \left(1-|a|^2\right)^2
\end{aligned}$$

We bluntly state that $|\mathrm{Im}(a\overline{z})|^2 \le |a|^2$, so $|P(z)|^2 \le 1 - |a|^2 + |a|^4 \le 1$ since $|a| \le 1$. By the maximum modulus principle, whenever $|z| \le 1$, we have $|P(z)| \le 1$.

Now, we justify that $|\operatorname{Im}(a\bar{z})|^2 \le |a|^2$. Let $z = x+iy$ and $a = \alpha+i\beta$, where $x, y, \alpha, \beta \in \mathbb{R}$ such that $x^2+y^2 \le 1$ and $\alpha^2+\beta^2 \le 1$. We have $a\bar{z} = (\alpha+i\beta)(x-iy) = \alpha x - \beta y + i(\beta x - \alpha y)$ so $\operatorname{Im}(a\bar{z}) = \beta x - \alpha y$. It suffices to prove that $(\beta x - \alpha y)^2 \le \alpha^2+\beta^2$. In other words, $\alpha^2(1-y^2)+\beta^2(1-x^2)+2\alpha\beta xy \ge 0$. Let $x = \cos\theta$ and $y = \sin\theta$ so $\alpha^2\sin^2\theta+\beta^2\cos^2\theta+2\alpha\beta\cos\theta\sin\theta \ge 0$. This inequality is obviously true since $(\alpha\sin\theta+\beta\cos\theta)^2 \ge 0$, or equivalently $(\alpha y+\beta x)^2 \ge 0$. □

We then introduce the Schwarz-Pick lemma (Lemma 7.1), which is also known as the Schwarz lemma.

Lemma 7.1 (Schwarz-Pick Lemma). Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function with $f(0) = 0$ and $|f(z)| \le 1$ for each $z \in D(0,1)$. Then,

$$|f(z)| \le |z| \text{ and } |f'(0)| \le 1.$$

Moreover, if $|f(z)| = |z|$ for some $z \in D(0,1)\setminus\{0\}$ or if $|f'(0)| = 1$, then f is a rotation of $D(0,1)$; that is, there exists a constant $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta}z \text{ for all } z \in D(0,1).$$

Example 7.26 (Dinh's 70 problems). Does there exist a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Solution. Yes, this is simply proven using the Schwarz-Pick lemma since $f'(1/2) \le 7/12 < 2/3$. □

Example 7.27 (Dinh's 70 problems). Let f be a holomorphic function from the unit disk $D(0,1)$ to itself. Assume that there is a point $z_0 \in D(0,1)$ such that $f(z_0) = z_0$. Prove that $|f'(z_0)| \le 1$.

Solution. We use the Schwarz-Pick Lemma, which says that for $a, b \in \mathbb{D}$, a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ satisfies $f(a) = b$ and $|f'(a)| \le \dfrac{1-|b|^2}{1-|a|^2}$. So, we set $a = b = z_0$. The result follows. □

Example 7.28. Is there a holomorphic function of $D(0,1)$ onto itself such that $f(0) = 0$ and $f(i/4) = i/3$? Justify.

Solution. We will show that there is no such function. Suppose on the contrary that there exist such a function. By the Schwarz Lemma, as $|f(z)| \le |z|$ for $z \in \mathbb{D}$, we have $|f(i/4)| \le |i/4| = 1/4$, which is a contradiction. □

7.4. *Winding Numbers*

Definition 7.4 (winding number). Let $\gamma : [a,b] \to \mathbb{C}\setminus\{z_0\}$ be a closed curve that does not pass through z_0. Given an argument θ_a for $\gamma(a) - z_0$,

$$\text{there exists a unique continuous function } \theta : [a,b] \to \mathbb{R}$$

such that for each $t \in [a,b]$, $\theta(t)$ is an argument of $\gamma(t) - z_0$ and such that $\theta(a) = \theta_a$. Define

$$n(\gamma, z_0) = \frac{\theta(b)-\theta(a)}{2\pi} \quad \text{to be} \quad \text{the winding number of } \gamma \text{ around } z_0.$$

Sometimes, we also refer it to the index of z_0 with respect to γ.

Theorem 7.11. $n(\gamma, z_0) \in \mathbb{Z}$

Theorem 7.12. Let γ be a closed contour piecewise differentiable and $z_0 \in \gamma$. Then,

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\, dz.$$

Corollary 7.3. Let f be holomorphic on an open set Ω containing γ and $z_0 \in f(\gamma)$. Then,

$$n(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - z_0}\, dz.$$

Example 7.29 (Dinh's 70 problems)**.** Let C be the unit circle $|z| = 1$, anti-clockwise oriented, and let $f(z) = z^3$. How many times does the curve $f(C)$ wind around the origin? Explain.

Solution. We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i} \int_C \frac{3}{z}\, dz = 3.$$

□

Example 7.30 (Dinh's 70 problems)**.** Let C be the unit circle $|z| = 1$, anti-clockwise oriented, and let $f(z) = \left(z^2 + 2\right)/z^3$. How many times does the curve $f(C)$ wind around the origin? Explain.

Solution. We have

$$n(f \circ C, 0) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = -\frac{1}{2\pi i} \int_C \frac{z^2 + 6}{z(z^2 + 2)}\, dz.$$

The residue at $z = 0$ is 3, so by Cauchy's residue theorem, the answer is -3. □

Theorem 7.13 (generalised Cauchy's integral formula)**.** Suppose f is a holomorphic function in a simply connected domain Ω. Then for any piecewise differentiable closed contour γ in Ω, if $a \notin \gamma$,

$$n(\gamma, z) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz$$

Theorem 7.14 (generalised residue theorem)**.** Let Ω be a simply connected domain in $\mathbb{C}$. Suppose f is holomorphic outside a finite number of points $z_1, \ldots, z_N$ in Ω. Then, for any piecewise differentiable closed contour γ in Ω which does not pass through $z_1, \ldots, z_N$,

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{N} n(\gamma, z_k) \operatorname{Res}(f, z_k).$$

8. Conformal Mappings and Möbius Transformations

8.1. *Univalent Functions*

Definition 8.1 (univalent function)**.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then, f is univalent if it is injective, i.e.

$$f(z_1) = f(z_2) \quad \text{implies} \quad z_1 = z_2.$$

f is locally univalent if for each $z_0 \in \Omega$, there exists a neighbourhood U of z_0 such that $f|_U \to \mathbb{C}$ is injective.

Theorem 8.1. A holomorphic function $f : \Omega \to \mathbb{C}$

$$\text{locally univalent at } z_0 \quad \text{if and only if} \quad f'(z_0) \neq 0.$$

Corollary 8.1 (inverse function theorem)**.** If $f : \Omega \to \mathbb{C}$ is a univalent holomorphic function, then its inverse f^{-1} is also holomorphic defined on $f(\Omega)$. Moreover, for each $z \in \Omega$,

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

Definition 8.2. Suppose two curves γ and η intersect at z_0 and α is the oriented angle between the tangent vectors to these curves at z_0. A holomorphic map f preserves angles at z_0 if the image curves $f \circ \gamma$ and $f \circ \eta$ intersect at $f(z_0)$ and their tangent vectors at $f(z_0)$ form an angle equal to α.

Theorem 8.2. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$. Then, f preserves angles at z_0.

Definition 8.3 (conformal map and automorphism group)**.** A bijective holomorphic function $f : U \to V$ is a conformal map or a biholomorphism. A conformal map from a domain $\Omega \to \Omega$ is a conformal automorphism of Ω. Define $\mathrm{Aut}(\Omega)$ to be the set of conformal automorphisms of Ω.

Theorem 8.3. If f and g are automorphisms of Ω, then $f \circ g$ is also an automorphism.

8.2. *Automorphisms of the Complex Plane* $\mathbb{C}$

Example 8.1. Translations, rotations, and dilations are examples of automorphims of the complex plane. We only discuss translations here. Suppose $h \in \mathbb{C}$. Then, the translation

$$z \mapsto z + h \text{ is a conformal map } \mathbb{C} \to \mathbb{C} \quad \text{whose inverse is} \quad w \mapsto w - h.$$

Moreover, if $h \in \mathbb{R}$, this translation is also a conformal map from the upper half-plane $\mathbb{H}$ to itself.

Theorem 8.4. Let f be a conformal map from $\mathbb{C}$ to itself. Then, there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $f(z) = az + b$ for $z \in \mathbb{C}$. In particular, we have

$$\mathrm{Aut}(\mathbb{C}) = \{az + b : a, b \in \mathbb{C}, a \neq 0\}.$$

8.3. *Automorphisms of the Unit Disc $\mathbb{D}$*

Definition 8.4 (unit disc)**.** Define $\mathbb{D}$ to be the unit disc. This is sometimes denoted by $D(0,1)$ which represents

$$\text{the open disc} \quad \text{of} \quad \text{radius 1 centred at } 0.$$

Example 8.2. Any rotation by an angle $\theta \in \mathbb{R}$, i.e. $\rho_\theta(z) = e^{i\theta}z$, is an automorphism of $\mathbb{D}$ whose inverse is $e^{-i\theta}z$.

We can generalise the previous example to the following lemma:

Lemma 8.1 (Blaschke factor)**.** For any $a \in \mathbb{D}$, the map

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z} \quad \text{is a conformal automorphism of } \mathbb{D} \quad \text{with inverse } \varphi_a^{-1} = \varphi_a.$$

The transformation ϕ_a is known as the Blaschke factor.

Theorem 8.5. If $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism and $f^{-1}(0) = a$, then there exists $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta}\frac{a-z}{1-\bar{a}z}.$$

Hence,

$$\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta}\frac{a-z}{1-\bar{a}z} : \theta \in \mathbb{R}, a \in \mathbb{D} \right\}.$$

Example 8.3. Let f be a holomorphic function on $\mathbb{D}$ such that $|f(z)| \le 1$ when $|z| < 1$. Prove that

$$\frac{|f(0)|-|z|}{1+|f(0)||z|} \le |f(z)| \le \frac{|f(0)|+|z|}{1-|f(0)||z|} \quad \text{for all } |z| < 1.$$

Solution. We first consider the case where $|f(z)| = 1$ for some $z \in \mathbb{D}$. By the maximum modulus principle, f is constant and so $|f(z)| = 1$ for all $z \in \mathbb{D}$. The above inequality is equivalent to

$$|f(0)| - |z| \le 1 + |f(0)||z| \text{ and } 1 - |f(0)||z| \le |f(0)| + |z|,$$

so $1 - |z| \le 1 + |z|$, which holds.

Now, consider the case where $|f(z)| < 1$ for all $z \in \mathbb{D}$. Let $f(0) = a \in \mathbb{D}$. Note that

$$\phi(z) = \frac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D}).$$

As such, $g = \phi \circ f$ is a holomorphic function from $\mathbb{D}$ to itself. Moreover, $g(0) = \phi(f(0)) = \phi(a) = 0$. By the Schwarz Lemma, $|g(z)| \le |z|$ for all $z \in \mathbb{D}$. Since $\phi^{-1} = \phi$, then

$$f(z) = (\phi^{-1} \circ g)(z) = \frac{a-g(z)}{1-\bar{a}g(z)}.$$

As such,

$$\left| \frac{a-g(z)}{1-\bar{a}g(z)} \right| \le 1 \Rightarrow 1 - |f(0)||z| \le 1 - |\bar{a}||g(z)| \le |a-g(z)| \le |1-\bar{a}g(z)| \le 1 + |\bar{a}||g(z)| \le 1 + |f(0)||z|$$

and in a similar fashion, we can deduce that

$$|f(0) - |z| \le |a| - |g(z)| \le |a-g(z)| \le |a| + |g(z)| \le |f(0)| + |z|.$$

Hence, we have shown that

$$1-|f(0)|z|\le|1-\bar{a}g(z)|\le 1+|f(0)||z| \text{ and } |f(0)-|z|\le|a-g(z)|\le|f(0)|+|z|.$$

The desired inequality is thus proven. $\square$

Example 8.4. Find a conformal map $T:D(0,1)\to D(1,2)$ such that $T(0)=1+i$ and $T(1)=1-2i$. Is the transformation unique?

Solution. Let $S(z)=2z+1$, which maps $D(0,1)$ to $D(1,2)$ conformally. Define $f=S^{-1}\circ T$, which is an automorphism of the unit disc. We have $S^{-1}(z)=(z-1)/2$. So, the conditions $T(0)=1+i$ and $T(1)=1-2i$ are equivalent to $f(0)=i/2$ and $f(1)=-i$. To find such a map f, consider

$$g(z)=-i\cdot\frac{\frac{i}{2}-z}{1+\frac{i}{2}z}$$

which is a conformal automorphism of $D(0,1)$ such that $g(i/2)=0$ and $g(-i)=1$. Thus,

$$f(z)=\frac{i(1-2z)}{2-z}.$$

We conclude that

$$T(z)=\frac{2(1+i)-(1+4i)z}{2-z}$$

is the required conformal map satisfying the conditions.

Suppose $\widetilde{T}$ also satisfies the requirements. Then, $R=T^{-1}\circ\widetilde{T}$ is a conformal automorphism of $D(0,1)$ satisfying $R(0)=0$ and $R(1)=1$. It is known that all automorphisms of the unit disc which fix 0 are rotations. Hence, R is the identity function so we conclude that $\widetilde{T}=T$. $\square$

Example 8.5. Let $f:\mathbb{D}\to\mathbb{C}$ be a holomorphic function. Suppose $f(0)=0$ and there exists a constant $A>0$ such that $\mathrm{Re}(f(z))\le A$ for $z\in\mathbb{D}$. Prove that for $z\in\mathbb{D}$,

$$|f(z)|\le\frac{2A|z|}{1-|z|}.$$

Solution. Since $f(0)=0$, then f is identically 0 or f is not identically constant. If $f(z)=0$ for all $z\in\mathbb{D}$, the inequality is obvious. Suppose f is not identically constant. Consider

$$\phi_1(z)=-\frac{z}{A}+1,\ \phi_2(z)=\frac{1-z}{1+z} \text{ and } \phi(z)=(\phi_2\circ\phi_1)(z)=\frac{z}{2A-z}.$$

Note that ϕ_1 is a conformal map from $\{\mathrm{Re}(z)<A\}$ to $\{\mathrm{Re}(z)>0\}$ and sends 0 to 1; ϕ_2 is a conformal map from $\{\mathrm{Re}(z)>0\}$ to the unit disc and sends 1 to 0. Hence, ϕ is a conformal map from $\{\mathrm{Re}(z)<A\}$ to the unit disc and sends 0 to 0. As such, $F=\phi\circ f$ is a holomorphic map from $\mathbb{D}$ to itself and $F(0)=0$.

By the Schwarz Lemma, note that the conditions $F(0)=0$ and $|F(z)|\le 1$ are satisfied since $z\in\mathbb{D}$. Hence, $|F(z)|\le|z|$. That is to say,

$$|z|\ge|\phi(f(z))|=\left|\frac{f(z)}{2A-f(z)}\right|.$$

The desired inequality follows with some simple algebraic manipulation. $\square$

Example 8.6 (Dinh's 70 problems). Suppose that f is holomorphic on the open set containing $\mathbb{D}$, $|f(z)| \leq 4$ if $|z| = 1$ and $f(i/2) = 0$. Show that for all $|z| \leq 1$,

$$|f(z)| \leq 4\left|\frac{z - i/2}{1 + i/2 \cdot z}\right|.$$

Solution. Note that $g(z) = \dfrac{a-z}{1-\bar{a}z}$ is an automorphism of $\mathbb{D}$, so we set $f(z) = 4g(z)$ and $a = i/2$. The result follows. □

Example 8.7 (Dinh's 70 problems). Show that if $D(0,R) \to \mathbb{C}$ is holomorphic with $|f(z)| < M$ for some $M > 0$, then

$$\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)}f(z)}\right| \leq \frac{|z|}{MR}.$$

Solution. Let $a = f(0)$. We wish to prove

$$\left|\frac{a - f(z)}{M^2 - \bar{a}f(z)}\right| \leq \frac{|z|}{MR}.$$

Define $\phi : \mathbb{D} \to \mathbb{D}$ via

$$\phi(z) = \frac{a/M - z}{1 - (\bar{a}/M)z}.$$

Note that $\overline{a/M} = \bar{a}/M$ since $M \in \mathbb{R}$. So, define $g = \phi \circ \dfrac{f(Rz)}{M}$. It is clear that $g(0) = 0$ and $g : \mathbb{D} \to \mathbb{D}$. By the Schwarz Lemma, $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Hence,

$$\begin{aligned} |g(z)| &\leq |z| \\ \frac{1}{M}\left|\frac{a - f(Rz)}{1 - \bar{a}f(Rz)/M^2}\right| &\leq |z| \\ \frac{M^2}{M}\left|\frac{a - f(Rz)}{M^2 - \bar{a}f(Rz)}\right| &\leq |z| \\ \left|\frac{a - f(z)}{M^2 - \bar{a}f(z)}\right| &\leq \frac{|z|}{MR} \end{aligned}$$

and we are done. □

Lemma 8.2 (Schwarz-Pick lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, $a \in \mathbb{D}$ and $f(a) = b$. Then,

(i) for each $z \in \mathbb{D}$, $|\varphi_b(f(z))| \leq |\varphi_a(z)|$

(ii) $|f'(a)| \leq \dfrac{1-|b|^2}{1-|a|^2}$

If equality holds in **(ii)** or if we have equality in **(i)** for some $z \neq a$, then $f \in \text{Aut}(\mathbb{D})$.

8.4. *Maps from the Upper Half-Plane $\mathbb{H}$ to the Unit Disc $\mathbb{D}$*

Definition 8.5 (upper half-plane). Define $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ to be the upper half-plane.

Lemma 8.3. Let

$$F(z) = \frac{i-z}{i+z} \quad \text{and} \quad G(w) = i \cdot \frac{1-w}{1+w}.$$

Then, $F : \mathbb{H} \to \mathbb{D}$ is a conformal map with inverse $G : \mathbb{D} \to \mathbb{H}$.

Theorem 8.6. All conformal mappings from $\mathbb{H}$ to $\mathbb{D}$ take the form

$$\left\{e^{i\theta}\frac{z-\beta}{z-\overline{\beta}} : \theta \in \mathbb{R}, \beta \in \mathbb{H}\right\}.$$

8.5. *Automorphisms of the Upper Half-Plane* $\mathbb{H}$

Theorem 8.7.

$$\mathrm{Aut}(\mathbb{H}) = \left\{\frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1\right\}$$

Proof. Let $a,b,c,d \in \mathbb{R}$ and $ad-bc>0$. Define a',b',c',d' to be as follows:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \sqrt{ad-bc},$$

where $a',b',c',d' \in \mathbb{R}$ and $a'd'-b'c'=1$. As such,

$$\mathcal{G} = \left\{\frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1\right\} = \left\{\frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R} \text{ and } ad-bc>0\right\}.$$

We shall prove that $\mathcal{G} \subseteq \mathrm{Aut}(\mathbb{H})$. Let

$$f(z) = \frac{az+b}{cz+d} \in \mathcal{G}.$$

Then, $f:\mathbb{R}\to\mathbb{R}$. If we let $z=x+iy$, where $x,y\in\mathbb{R}$, Since $a,b,c,d\in\mathbb{R}$, then

$$\begin{aligned}
\mathrm{Im}(f(z)) &= \mathrm{Im}\left[\frac{a(x+iy)+b}{c(x+iy)+d}\right] = \mathrm{Im}\left[\frac{ax+b+i(ay)}{cx+d+i(cy)}\cdot\frac{cx+d-i(cy)}{cx+d-i(cy)}\right] \\
&= \mathrm{Im}\left[\frac{ac\left(x^2+y^2\right)+bcx+adx+bd+iy(ad-bc)}{c^2\left(x^2+y^2\right)+2cdx+d^2}\right] \\
&= \mathrm{Im}(z)\cdot\frac{ad-bc}{c^2|z|^2+2cdx+d^2} \\
&= \mathrm{Im}(z)\cdot\frac{ad-bc}{|cz+d|^2}
\end{aligned}$$

which shows $f:\mathbb{H}\to\mathbb{H}$. It is clear that

$$g(z) = \frac{-dz+b}{cz-a} \in \mathcal{G}$$

and $g\circ f = \mathrm{id}$. Hence, $f\in\mathrm{Aut}(\mathbb{H})$ and $\mathcal{G}\subseteq\mathrm{Aut}(\mathbb{H})$.

Conversely, let f be an arbitrary map in $\mathrm{Aut}(\mathbb{H})$. We will show that $f\in\mathcal{G}$. Define

$$F(z) = \frac{i-z}{i+z}$$

which is a conformal map from $\mathbb{H}$ to $\mathbb{D}$ with inverse

$$F^{-1}(z) = i\cdot\frac{1-z}{1+z}$$

and this maps from $\mathbb{D}$ to $\mathbb{H}$. Hence, $h=F\circ f$ is a conformal map from $\mathbb{H}$ to $\mathbb{D}$. All such a map h must be of the form

$$e^{2i\theta}\frac{z-\beta}{z-\overline{\beta}}$$

with $\beta \in \mathbb{H}$ and $\theta \in \mathbb{R}$. We let the reader prove that

$$f(z) = F^{-1}\left(e^{2i\theta}\frac{z-\beta}{z-\overline{\beta}}\right) = \frac{az+b}{cz+d}$$

and $ad - bc = \operatorname{Im}(\beta) > 0$ which would show that $f \in \mathcal{G}$, so $\operatorname{Aut}(\mathbb{H}) \subseteq \mathcal{G}$. □

Example 8.8 (Dinh's 70 problems). Find a conformal map from

$$H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$$

onto

$$A = \{z \in \mathbb{C} : |z-2| < 3, |z| > 1\}.$$

You may leave your answer as a composition of conformal mappings.

Solution. We find a conformal map from A to H first.

- Let $\phi_1(z) = 1/(z+1)$. Let us figure out what A gets mapped to via ϕ_1. So, if we let $w = 1/(z+1)$, we have $z = 1/w - 1$. Consider the annulus $|z-2| < 3$, so after the transformation, we have $w > 1/6$. For the region $|z| > 1$, we have $1/w > 2$. So, ϕ_1 maps A to A_1, where $A_1 = \{z \in \mathbb{C} : 1/6 < \operatorname{Re}(z) < 1/2\}$.
- Let $\phi_2(z) = z - \dfrac{1}{6}$. So, ϕ_2 maps A_1 to $A_2 = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1/3\}$.
- Let $\phi_3(z) = \tan\left(\dfrac{3\pi z}{2}\right)$, which maps A_2 to H.

As such, the required conformal map from H to A is $\phi_3^{-1} \circ \phi_2^{-1} \circ \phi_1^{-1}$. □

Example 8.9 (Dinh's 70 problems). Let $f : D(0,1) \to \mathbb{C}$ be a holomorphic function such that $\operatorname{Re}(f(z)) > 0$ for each $z \in D(0,1)$ and such that $f(0) = 1$.

(a) Prove that $|f'(0)| \le 2$.

(b) Assume that $|f'(0)| = 2$. Determine all possible forms of f.

Solution.

(a) We first find a holomorphic map from $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ to $\mathbb{D}$. To do this, let $\phi_1(z) = iz$, which maps the right half of the complex plane to the upper half, $\mathbb{H}$. Then, recall that $\phi_2(z) = \dfrac{z-i}{z+i}$ maps $\mathbb{H}$ to $\mathbb{D}$. So, $\phi = \phi_2 \circ \phi_1$ is the required holomorphic map. We have

$$\phi(z) = \frac{iz-i}{iz+i} = \frac{z-1}{z+1}.$$

Define $F = \phi \circ f$ so $F : \mathbb{D} \to \mathbb{D}$, i.e. F is an automorphism of the unit disk and $F(0) = 0$. By the Schwarz Lemma, $|F'(0)| \le 1$, so $|\phi'(1)f'(0)| \le 1$. Since $\phi'(1) = 1/2$, the result follows.

(b) Suppose equality holds. Then, $F'(0) = 1$, where

$$F(z) = \frac{f(z)-1}{f(z)+1}.$$

Then, $F(z) = ze^{i\theta}$ (recall that this is just rotating some point in the unit disk about the origin) for $\theta \in \mathbb{R}$. One can work out that $f = \phi^{-1} \circ F$ and find an explicit expression for it. □

8.6. *Möbius Transformations*

Definition 8.6 (Möbius transformation). A transformation of the form

$$T(z) = \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{C}$, is a linear fractional transformation (LFT). If $ad-bc \neq 0$, then T is a Möbius transformation.

Note that the condition $ad-bc \neq 0$ is equivalent to saying T is not constant. Consider

$$T(z) = \frac{az+b}{cz+d},$$

where $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$. If $c=0$, then $T : \mathbb{C} \to \mathbb{C}$. If $c \neq 0$, then $T : \mathbb{C} \setminus \{-d/c\} \to \mathbb{C}$.

Definition 8.7. The Möbius transformation $T : \mathbb{C}^* \to \mathbb{C}^*$ associated with a,b,c,d is defined by

$$T(z) = \begin{cases} \dfrac{az+b}{cz+d} & z \neq \infty, z \neq -d/c; \\ a/c & z = \infty; \\ \infty & z = -d/c. \end{cases}$$

Moreover, if $c=0$, then $a \neq 0$ and $d \neq 0$ so that the usual agreements regarding ∞ can be made. That is, $T(\infty) = \infty$.

The map $T : \mathbb{C}^* \to \mathbb{C}^*$ is injective.

Remark 8.1. T is holomorphic on $\mathbb{C}^* \setminus \{-d/c\}$ with a simple pole at the point $\{-d/c\}$.

Proposition 8.1. A Möbius transformation is a composition of transformations of the following forms:

(i) translation: $z \mapsto z+b$, $b \in \mathbb{C}$;

(ii) rotation and dilation: $z \mapsto \lambda z$, $\lambda \in \mathbb{C} \setminus \{0\}$;

(iii) reciprocation: $z \mapsto 1/z$

Definition 8.8. Let $\operatorname{Aut}(\mathbb{C}^*)$ be the set of meromorphic automorphisms of $\mathbb{C}^*$.

Theorem 8.8. A Möbius transformation $T(z) = \dfrac{az+b}{cz+d}$ is such that $T \in \operatorname{Aut}(\mathbb{C}^*)$ with

$$T^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Conversely,

$$\operatorname{Aut}(\mathbb{C}^*) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0 \right\}.$$

Definition 8.9. Define a line l in $\mathbb{C}^*$ to be the union of a line in $\mathbb{C}$ with $\{\infty\}$.

Lemma 8.4. Let

$$L=\left\{z\in\mathbb{C}^*:\alpha z\overline{z}+\beta z+\overline{\beta}\overline{z}+\gamma=0,\text{ where }\alpha,\gamma\in\mathbb{R},\beta\in\mathbb{C}\text{ and }\beta\overline{\beta}-\alpha\gamma>0\right\}.$$

(i) If $\alpha\neq 0$, then L is a circle;
(ii) if $\alpha=0$, then L is a line

Conversely, each line or circle can be expressed as one of the set L for appropriate α,β,γ.

Theorem 8.9. Suppose L is a line or a circle and T is a Möbius transformation. Then, $T(L)$ is a line or a circle.

Note that a Möbius transformation does not necessarily map circles to circles and lines to lines; even if it maps a circle to another circle, it does not necessarily map the first circle's centre to the second circle's centre.

Example 8.10. For $a\in\mathbb{D}$,

$$T_a:\mathbb{D}\to\mathbb{D}\quad\text{where}\quad T_a(z)=\frac{-z+a}{1-\overline{a}z}\text{ and }T_a(0)=a.$$

To see why, note that T is a holomorphic function on $\mathbb{C}\setminus\{1/\overline{a}\}$, so it is defined in a neighbourhood of $\overline{\mathbb{D}}$. For z in the boundary of $\mathbb{D}$, we have $|z|=1$ and $z\overline{z}=1$. It is easy to see that $|T_a(z)=1$. By the maximum modulus principle, when $|z|<1$, we have $|T_a(z)|<1$. Hence, $T_a(z)$ is a conformal automorphism of $\mathbb{D}$.

Also, $T_a(0)=a$ is obvious.

Example 8.11.

$$T(z)=i\cdot\frac{z-1}{z+1}$$

maps the real line to the imaginary line and $T(-1)=\infty$.

To see why, let $z=a$, where $a\in\mathbb{R}$. Then,

$$T(z)=\frac{i(a-1)}{a+1},$$

which is purely imaginary. It is also clear that $T(-1)=\infty$.

Example 8.12.

$$T(z)=\frac{i-z}{i+z}$$

maps the real line to the unit circle and $T(\infty)=-1$.

To see why, let $z=a$, where $a\in\mathbb{R}$. It suffices to show that $|T(a)|=1$, i.e.

$$\left|\frac{i-a}{i+a}\right|=1.$$

This is obvious.

Example 8.13.

$$T(z)=i\cdot\frac{1-z}{1+z}$$

maps the unit circle to the real line and $T(-1)=\infty$.

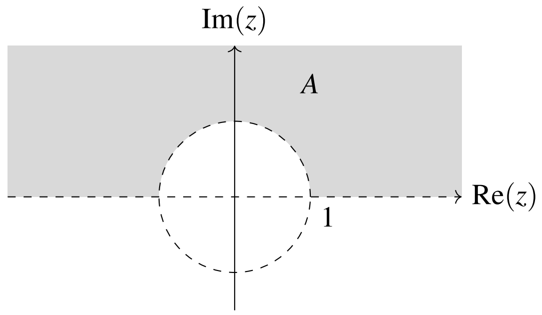
To see why, let $z=e^{i\theta}$. Then, we need to show that $T(z)\in\mathbb{R}$.

$$T(e^{i\theta})=\frac{i(1-e^{i\theta})}{1+e^{i\theta}}=\tan\left(\frac{\theta}{2}\right),$$

which is real. Also, $T(-1)$ can be attained by setting $\theta=(2k+1)\pi$ for $k\in\mathbb{Z}$, which implies $\tan(\theta/2)=\infty$.

Example 8.14. Find a conformal map f from $A=\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)>0,|z|>1\}$ onto the unit disc.

Solution. The locus A represents the intersection of the upper half-plane and the points exterior to the circle of radius 1 centred at the origin.



Consider the Cayley transform given by $f(z)=(z-i)/(z+i)$. □

Example 8.15. Let

$$A=\left\{z\in\mathbb{C}:|z|<1\text{ and }\mathrm{Im}(z)>\frac{1}{2}\right\}\text{ and }B=\left\{\frac{2\pi}{3}<\arg(z)<\pi\right\}.$$

Find a conformal map f from A to B.

Solution. We first find the intersection of $|z|=1$ and $\mathrm{Im}(z)=1/2$. Consider $x^2+y^2=1$ and $y=1/2$. Solving yields $x=-\sqrt{3}/2$. Hence, $z=e^{i\pi/6}$ or $z=e^{5\pi i/6}$.

Note that

$$f(z)=\frac{z-e^{i\pi/6}}{z-e^{5\pi i/6}}$$

is an example of such a conformal map.

To see why, we note that it is a Möbius transformation so it sends lines and circles to lines and circles. Note that $f(e^{5\pi i/6})=\infty$ and $f(e^{i\pi/6})=0$. Hence, the boundary of A is sent to the union of two half-lines which form an angle at the origin. For z in the interval joining $e^{5\pi i/6}$ and $e^{i\pi/6}$ (along the line $\mathrm{Im}(z)=1/2$), note that $z-e^{i\pi/6}\in\mathbb{R}_{<0}$ and $z-e^{5\pi i/6}\in\mathbb{R}_{>0}$, so $f(z)\in\mathbb{R}_{<0}$.

The angle at $e^{i\pi/6}$ between this interval and the rest of the boundary of A forms an angle of $-\pi/3$. Since f is conformal at $e^{i\pi/6}$, we conclude that the boundary of A is sent to the union of $\mathbb{R}_{<0}$ with the half line $e^{2\pi i/3}\mathbb{R}_{>0}$. We deduce that A is sent to B. □

Example 8.16. Find a Möbius transformation mapping the upper half-plane onto the unit disc and mapping a given point z_0 in the upper half-plane to 0.

Solution. Note that T maps the real line to the unit disc. Since z_0 and $\overline{z_0}$ are symmetric about the real axis, then $T(\overline{z_0})$ and $T(z_0)=0$ are symmetric with respect to the unit circle. Hence, $T(\overline{z_0})=\infty$. As such,

$$T(z)=\lambda\cdot\frac{z-z_0}{z-\overline{z_0}}$$

for some $\lambda \in \mathbb{C} \setminus \{0\}$. Since $|T(0)| = 1$, then $|\lambda| = 1$. Hence,

$$T(z) = e^{i\theta} \cdot \frac{z - z_0}{z - \overline{z_0}}$$

for some $\theta \in \mathbb{R}$. □

Example 8.17. Find a Möbius transformation that maps from

$$D = \{z : |z| > 1, |z-1| < 2\} \text{ to } G = \{w : 0 < \mathrm{Re}(w) < 1\}.$$

Solution. Observe that the region D is bounded by two circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 4$. The tangent to these circles is $x = -1$. We consider the conformal map $T(z) = 1/(z+1)$. Since $T(\mathbb{R}) = \mathbb{R}$ and C_1 and C_2 are perpendicular to $\mathbb{R}$, it follows that $T(C_1)$ and $T(C_2)$ are perpendicular to $\mathbb{R}$.

Hence, $T(C_1) = \{z : \mathrm{Re}(z) = 1/2\}$ and $T(C_2) = \{z : \mathrm{Re}(z) = 1/4\}$. So, $T(D)$ is bounded by these lines. Let $S(w) = 4w - 1$. Then, $S \circ T = (3-z)/(1-z)$ maps D onto G conformally. □

Example 8.18 (Dinh's 70 problems)**.** Let $T(z) = \dfrac{az+b}{cz+d}$ be a Möbius transformation.

(i) Assume that $z_1, z_2 \in \mathbb{C}$ are two distinct fixed points for T, i.e. $T(z_j) = z_j$, $j = 1, 2$. Show that there exists a constant λ such that
$$\frac{T(z) - z_1}{T(z) - z_2} = \lambda \cdot \frac{z - z_1}{z - z_2}.$$

(ii) Let $T^1(z) := T(z)$, $T^{n+1}(z) := T(T^n(z))$, $n = 1, 2, 3, \ldots$. Use **(i)** to find an expression for T^n, $n = 1, 2, 3, \ldots$, if
$$T(z) = \frac{1 - 3z}{z - 3}.$$

Solution.

(i) We have

$$\begin{aligned}
\frac{(T(z) - z_1)(z - z_2)}{(T(z) - z_2)(z - z_1)} &= \frac{\left(\frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d}\right)(z - z_2)}{\left(\frac{az+b}{cz+d} - \frac{az_2+b}{cz_2+d}\right)(z - z_1)} \\
&= \frac{((az+b)(cz_1+d) - (az_1+b)(cz+d))(cz_2+d)(z-z_2)}{((az+b)(cz_2+d) - (az_2+b)(cz+d))(cz_1+d)(z-z_1)} \\
&= \frac{cz_2+d}{cz_1+d}
\end{aligned}$$

so $\lambda = \dfrac{cz_2+d}{cz_1+d}$.

(ii) We first find the fixed points of T. Set $\dfrac{-3z+1}{z-3} = z$, so $z = \pm 1$. We can take $z_1 = -1$ and $z_2 = 1$, so by repeatedly applying **(i)**, we have
$$\frac{T^n(z) + 1}{T^n(z) - 1} = \left(\frac{1}{2}\right)^n \cdot \frac{z+1}{z-1}.$$
□

8.7. *Cross Ratio*

Definition 8.10 (cross ratio)**.** The cross ratio of a 4-tuple of points $z_0, z_1, z_2, z_3 \in \mathbb{C}^*$ is defined to be
$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$
When one of the z_j is ∞, the RHS is understood as the limit as $z \to \infty$.

Example 8.19.

$$(\infty, z_1, z_2, z_3) = \frac{z_1 - z_3}{z_1 - z_2}.$$

Proposition 8.2. A Möbius transformation T preserves cross ratios. That is,

$$(T(z_0), T(z_1), T(z_2), T(z_3)) = (z_0, z_1, z_2, z_3)$$

Lemma 8.5. Given three distinct points $z_1, z_2, z_3 \in \mathbb{C}^*$, let $T(z) = (z, z_1, z_2, z_3)$. Then, T is a Möbius transformation and

$$T(z_1) = 1, \ T(z_2) = 0 \text{ and } T(z_3) = \infty.$$

In fact, T is the unique Möbius transformation such that the above holds.

Theorem 8.10. Given two sets of three distinct points $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$, there exists a unique Möbius transformation T such that $T(z_j) = w_j$ for $j = 1, 2, 3$.

Corollary 8.2. Let z_0, z_1, z_2, z_3 be distinct points in $\mathbb{C}^*$. Then, they lie in a circle or a line in $\mathbb{C}^*$ if and only if $(z_0, z_1, z_2, z_3) \in \mathbb{R}$.

Example 8.20. Find a Möbius transformation f that maps $\mathbb{H}$ bijectively to the disc $D(0,2)$ such that $f(i) = 1$ and $f(1) = -2$.

Solution. A Möbius transformation preserves points of symmetry so $f(-i)$ is symmetric to $f(i) = 1$ with respect to $C(0,2)$. Hence, $f(-i) = 4$. Since the Möbius transformation f preserves cross ratios, then

$$\begin{aligned}
(f(z), f(1), f(i), f(-i)) &= (z, 1, i, -i) \\
(f(z), -2, 1, 4) &= (z, 1, i, -i) \\
\frac{f(z) - 1}{f(z) - 4} \cdot \frac{-6}{-3} &= \frac{z - i}{z + i} \cdot \frac{1 + i}{1 - i}
\end{aligned}$$

Finding $f(z)$ is left as a simple algebraic exercise. Note that $f(-1) = 2$. □

9. Harmonic Functions

9.1. *Basic Properties of Harmonic Functions*

Recall that a real-valued function u is defined on a domain $\Omega \subseteq \mathbb{C}$ is harmonic if it belongs to $\mathcal{C}^2$ (second derivative of f is continuous on Ω) and $\Delta u = 0$. The real and imaginary parts of a holomorphic function are harmonic.

Proposition 9.1. Let Ω be a simply connected domain in $\mathbb{C}$. A function $u : \Omega \to \mathbb{R}$ is harmonic if and only if u is the real part of some holomorphic function on Ω.

The above proposition implies that for any domain Ω, u is harmonic if and only if it is locally the real (or imaginary) part of a holomorphic function. In particular, harmonic functions belong to $\mathcal{C}^\infty$.

Example 9.1. Consider the function

$$u(x,y) = \frac{1}{2}\log(x^2+y^2)$$

on the annulus $\Omega = \{0 < r < |z| < R\}$. This is not a simply connected domain, which means that not all simple closed curves in Ω can be shrunk to a point while remaining in Ω. One can establish that u is harmonic but there is no holomorphic function on Ω whose real part is equal to u.

Showing that u is harmonic, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is simple

Example 9.2. Prove that the function

$$u(x,y) = \frac{\sin x}{\cos x + \cosh y}$$

is harmonic in

$$\Omega = \{x+iy : -\pi < x < \pi \text{ and } y \in \mathbb{R}\}.$$

Solution. One can see that $\cosh y = \cos(iy)$, so by setting $z = x+iy$, where $-\pi < x < \pi$ and $y \in \mathbb{R}$, it is clear that

$$u(x,y) = \operatorname{Re}\left(\tan\left(\frac{z}{2}\right)\right).$$

□

Example 9.3 (Dinh's 70 problems)**.** Let $f = u+iv$ be a holomorphic function in an open set Ω. Define

$$U := e^{u^2-v^2}\cos(2uv) \text{ and } V := e^{u^2-v^2}\sin(2uv).$$

Show that U is harmonic and V is a harmonic conjugate of U.

Solution. To show that U is harmonic, we need to show that it satisfies Laplace's Equation, i.e. $U_{uu} + U_{vv} = 0$. This is trivial. Next, one of the Cauchy-Riemann Equations states that $U_u = V_v$, so

$$V_v = -2e^{u^2-v^2}(v\sin(2uv) - u\cos(2uv)).$$

Using integration by parts or Euler's Formula, it can be shown that $\displaystyle\int V_v\,dv = V + c$, where c is an arbitrary constant. This shows that V is a harmonic conjugate of U. □

Theorem 9.1 (maximum-minimum principle)**.** If u is a real-valued non-constant harmonic function on a domain Ω, then u has no local maximum and no local minimum on Ω.

9.2. *Dirichlet Problem and Poisson Kernel*

Theorem 9.2 (Dirichlet problem)**.** Let Ω be a bounded domain in $\mathbb{C}$. Given a function $h : \partial\Omega \to \mathbb{R}$, is there a unique continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega; \\ u = h & \text{on } \partial\Omega? \end{cases}$$

In layman's terms, think of u being harmonic on the interior and $u = h$ on the boundary.

Definition 9.1 (Poisson kernel)**.** Define the Poisson kernel of the unit disc to be

$$P(a, e^{i\theta}) = \frac{1}{2\pi} \cdot \frac{1 - |a|^2}{|e^{i\theta} - a|^2}.$$

We shall consider the case where Ω is the unit disc $\mathbb{D}$. The following theorem gives the uniqueness of the solution to the Dirichlet problem.

Theorem 9.3. Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be a continuous function which is harmonic in $\mathbb{D}$. Then, for each $a \in \mathbb{D}$,

$$u(a) = \int_0^{2\pi} P(a, e^{i\theta}) u(e^{i\theta})\, d\theta.$$

Proof. Consider the automorphism of $\mathbb{D}$, which is

$$f(z) = \frac{a - z}{1 - \overline{a}z.}$$

Note that $f(0) = a$ and f is self-inverse. Find f' and f'/f, then use Gauss' mean value theorem to prove the result. □

Corollary 9.1 (Harnack's inequality)**.** Let u be a harmonic function in a neighborhood of $\overline{\mathbb{D}}$. Assume that $u \geq 0$ on $\{|z| = 1\}$. Then,

$$\frac{1 - |z|}{1 + |z|} u(0) \leq u(z) \leq \frac{1 + |z|}{1 - |z|} u(0)$$

for $|z| < 1$.

Proof. Apply the Poisson kernel formula. Consider the region $|z| < 1$ and the identity $1 - |z|^2 = (1 + |z|)(1 - |z|)$. □

10. Analytic Continuation

10.1. *Analytic Continuation*

Definition 10.1 (analytic continuation). Let f be a holomorphic function defined on a domain Ω. If there exists a domain $\Omega \subseteq \Omega'$ and a holomorphic function $F : \Omega' \to \mathbb{C}$ such that $F(z) = f(z)$ for each $z \in \Omega$, then F is an analytic continuation of f on Ω'.

Example 10.1. The power series

$$f(z) = 1 + z + z^2 + \dots$$

has a radius of convergence $R = 1$ and so $f(z)$ is a holomorphic function on the unit disc $\mathbb{D}$. On the other hand, one can see that

$$f(z) = \frac{1}{1-z} \text{ for } |z| < 1$$

but $g(z) = 1/(1-z)$ is holomorphic on $\mathbb{C} \setminus \{1\}$. Thus, g is an analytic continuation of f to the much bigger domain $\mathbb{C} \setminus \{1\}$.

Lemma 10.1. Let $\Omega \subseteq \Omega'$ be domains in $\mathbb{C}$. Let F_1 and F_2 be analytic continuations of a holomorphic function $f : \Omega \to \mathbb{C}$ to a domain Ω'. Then,

$$F_1(z) = F_2(z) \quad \text{for all } z \in \Omega'.$$

Lemma 10.2. Let $f_j : \Omega_j \to \mathbb{C}$ be holomorphic functions such that $f_1(z) = f_2(z)$ for $z \in \Omega_1 \cap \Omega_2$. Then,

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in \Omega_1; \\ f_2(z) & \text{if } z \in \Omega_2 \setminus \Omega_1. \end{cases}$$

10.2. *Schwarz Reflection Principle*

We say that a region Ω is symmetric with respect to the real axis if $z \in \Omega$ implies $\bar{z} \in \Omega$. We consider here an important particular case of analytic continuation.

Theorem 10.1 (reflection principle for holomorphic functions). Define Ω^+, Ω^-, L as the intersections of Ω with the upper half-plane, lower half-plane, and the real axis respectively. If f is a continuous complex-valued function on $\Omega^+ \cup L$, which is analytic on Ω^+ and real on L, then

$$f \text{ admits a unique extension} \quad \text{to a holomorphic function } F \text{ on } \Omega.$$

Moreover, the extension is given by

$$F(z) = \begin{cases} f(z) & \text{for } z \in \Omega^+ \cup L; \\ \overline{f(\bar{z})} & \text{for } z \in \Omega^-. \end{cases}$$

In particular, $F(\bar{z}) = \overline{F(z)}$ for all $z \in \Omega$.

Example 10.2 (MA5217 Lecture Notes). Suppose f is holomorphic on $\mathbb{H}$ and continuous on $S = \mathbb{H} \cup (0,1)$. Assume $f(x) = x^4 - 2x^2$ for all $x \in (0,1)$. Find $f(i)$.

Solution. We have $f(i) = i^4 - 2i^2 = 1 + 2 = 3$. $\square$