

# MA4211 Functional Analysis

Thang Pang Ern, Loh Wei Xuan Ryan, and Malcolm Tan Jun Xi

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## References

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# Chapter 1

## Metric Spaces

### 1.1

#### Metric Spaces

Functional Analysis is essentially the study of infinite-dimensional Linear Algebra.

**Example 1.1 (Euclidean metric/distance).** Recall the familiar metric in Euclidean space  $\mathbb{R}$

$$d(x, y) = |x - y|.$$

We call this the Euclidean metric or Euclidean distance. Naturally, we can extend this to the Euclidean 2-space  $\mathbb{R}^2$ . Consider  $\mathbf{x} = (x_1, x_2)$  and  $\mathbf{y} = (y_1, y_2)$  in  $\mathbb{R}^2$ . Then,

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

We give the definition of a metric space (Definition 1.1).

**Definition 1.1 (metric space).** Let  $X$  be a set. A metric space is an ordered pair  $(X, d)$  equipped with a distance function  $d : X \times X \rightarrow \mathbb{R}$  such that the following properties are satisfied:

- (i) **Non-negativity:**  $d(x, y) \geq 0$
- (ii) **Positive-definiteness:**  $x = y$  if and only if  $d(x, y) = 0$
- (iii) **Symmetry:** for all  $x, y \in X$ , we have  $d(x, y) = d(y, x)$
- (iv) **Triangle inequality:** for all  $x, y, z$ , we have  $d(x, z) \leq d(x, y) + d(y, z)$

**Example 1.2 (MA4211 AY24/25 Sem 2 Tutorial 1).** Given two non-empty subsets  $A, B$  of a metric space  $(X, d)$ , their distance is defined as

$$D(A, B) = \inf_{a \in A, b \in B} d(a, b).$$

Consider the power set of  $X$  and the function  $D$ . Which of the axioms of a metric space does this pair satisfy?

*Solution.* We first claim that non-negativity is satisfied. For  $A, B \subseteq X$ , we have

$$d(a, b) \geq 0 \quad \text{which implies} \quad \inf_{a \in A, b \in B} d(a, b) \geq 0.$$

Next, symmetry is satisfied since

$$D(B, A) = \inf_{a \in A, b \in B} d(b, a) = \inf_{a \in A, b \in B} d(a, b)$$

which follows from the fact that  $d$  satisfies symmetry. Next, the triangle inequality is satisfied. To see why, let  $A, B, C \subseteq X$ . Then,

$$D(A, C) = \inf_{a \in A, c \in C} d(a, c) \leq \inf_{a \in A, b \in B} d(a, b) + \inf_{b \in B, c \in C} d(b, c) = D(A, B) + D(B, C)$$

We claim that the property  $D(A, B) = 0$  if and only if  $A = B$  is not satisfied. Recall from Definition 1.1 that this is called positive-definiteness. Anyway, to see why, let  $A = \{x\}$  and  $B = \{x_n\}_{n=1}^{\infty}$ , where  $x_n \rightarrow x$ . Then,  $D(A, B) = 0$  since  $x_n$  can be arbitrarily close to  $x$  for large  $n$ . However,  $A \neq B$ .  $\square$

**Example 1.3** (MA4211 AY24/25 Sem 2 Homework 1). Let  $(X, d)$  be a metric space. Define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that  $\rho$  is a metric on  $X$  and that the identity map on  $X$  is a homeomorphism (a continuous, bijective map with continuous inverse) between  $(X, d)$  and  $(X, \rho)$ . Thus, every metric space is homeomorphic to a space with a bounded metric.

*Solution.* We first show that  $\rho$  is a metric on  $X$ . Note that non-negativity, symmetry and homogeneity are obvious (as usual) so it suffices to only prove that it satisfies the triangle inequality. Let  $x, y, z \in X$  such that

$$d(x, z) \leq d(x, y) + d(y, z) \quad \text{since } d \text{ satisfies the triangle inequality.}$$

As such,

$$\rho(x, y) + \rho(y, z) - \rho(x, z) = \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} - \frac{d(x, z)}{1 + d(x, z)}$$

Let  $a = d(x, y)$ ,  $b = d(y, z)$ , and  $c = d(x, z)$ . Then, the above becomes

$$\begin{aligned} \frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} &= \frac{a(1+b)(1+c) + b(1+a)(1+c) - c(1+a)(1+b)}{(1+a)(1+b)(1+c)} \\ &= \frac{a + ab + ac + abc + b + ab + bc + abc - c - ac - bc - abc}{(1+a)(1+b)(1+c)} \\ &= \frac{a + b - c + 2ab + abc}{(1+a)(1+b)(1+c)} \end{aligned}$$

Since  $a + b - c \geq 0$  is equivalent to  $d(x, z) \leq d(x, y) + d(y, z)$ , then

$$\frac{a + b - c + 2ab + abc}{(1+a)(1+b)(1+c)} \geq \frac{2ab + abc}{(1+a)(1+b)(1+c)} \geq 0$$

and we conclude that  $\rho$  is indeed a metric on  $X$ .

For the second part, we wish to show that

$$\text{id}_X : (X, d) \rightarrow (X, \rho) \quad \text{where } x \mapsto x \text{ is a homeomorphism.}$$

Note that a map is continuous if it takes convergent sequences in the domain metric to convergent sequences in the codomain metric. Suppose  $x_n \rightarrow x$  in  $(X, d)$ . Then,  $d(x_n, x) = 0$ . As such,  $\rho(x_n, x) = 0$ . Similarly, we can consider the inverse map

$$\text{id}_X = (\cdot)^{-1} : (X, \rho) \rightarrow (X, d) \quad \text{where } x \mapsto x.$$

Again, it is clear that  $\rho(x_n, x) = 0$  implies  $d(x_n, x) = 0$ , so  $\text{id}_X$  is indeed a homeomorphism.  $\square$

**Definition 1.2** ( $\mathbb{R}^\infty$ ). Define  $\mathbb{R}^\infty$  to be the space of all infinite sequences of real numbers, i.e.  $(x_1, x_2, \dots)$  where  $x_1, x_2, \dots \in \mathbb{R}$ .

**Example 1.4** ( $\mathbb{R}^\infty$ ). We have the infinite sequences  $(0, 0, \dots)$  and  $(1, 2, 3, \dots, 100, \dots)$  in  $\mathbb{R}^\infty$ .

**Example 1.5.** For  $X = \mathbb{R}$ , we can define

$$d(x, y) = \min\{|x - y|, 1\} \quad \text{such that it is a metric.}$$

**Example 1.6.** For  $X = \mathbb{R}^\infty$ , let  $\mathbf{x} = (x_1, x_2, \dots)$  and  $\mathbf{y} = (y_1, y_2, \dots)$ , where each element is in  $\mathbb{R}$ . Then, one can check that

$$d(\mathbf{x}, \mathbf{y}) = \sup d(x_i, y_i) \quad \text{is a metric.}$$

**Example 1.7** ( $\ell^\infty$ ). We give an introduction to the sequence space  $\ell^\infty$ . This example gives one an impression of how surprisingly general the concept of a metric space is. We can define

$$X = \{\text{bounded sequences of complex numbers}\}.$$

So, every element of  $X$  is a complex sequence  $\xi_j$  such that for all  $j = 1, 2, \dots$ , we have

$$|\xi_j| \leq c_x \quad \text{where} \quad c_x \text{ is a real number which may depend on } x.$$

Then, the following is a metric:

$$d(\mathbf{x}, \mathbf{y}) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \quad \text{where} \quad \mathbf{y} = (\eta_1, \eta_2, \dots) \in X$$

**Definition 1.3** (translation invariant and homogeneous metrics). Let  $X$  be a normed space and  $d$  be a metric. We say that  $d$  is translation invariant if

$$\text{for all } x, y, z \in X \quad \text{we have} \quad d(x, y) = d(x + z, y + z).$$

Similarly,  $d$  is said to be homogeneous if

$$\text{for all } x, y \in X \text{ and } \alpha \in \mathbb{F} \quad \text{we have} \quad d(\alpha x, \alpha y) = |\alpha| d(x, y).$$

**Example 1.8** (MA4211 AY24/25 Sem 2 Tutorial 2). Let  $X$  be a normed space and  $d$  be a metric. We say that  $d$  is translation invariant if

$$\text{for all } x, y, z \in X \quad \text{we have} \quad d(x, y) = d(x + z, y + z).$$

Similarly,  $d$  is said to be homogeneous if

$$\text{for all } x, y \in X \text{ and } \alpha \in \mathbb{F} \quad \text{we have} \quad d(\alpha x, \alpha y) = |\alpha| d(x, y).$$

- (a) Show that the metric induced on  $X$  by its norm is translation invariant and homogeneous.
- (b) Show that if  $d$  is a translation invariant and homogeneous metric on  $X$ , then  $\|x\| = d(x, 0)$  defines a norm on  $X$ .

*Solution.*

- (a) We have

$$\|x - y\| = \|(x - z) + (z - y)\| = \|x + z\| + \|y + z\|$$

so  $d$  is translation invariant.

Next, we prove that  $d$  is homogenous. We have

$$\|\alpha x - \alpha y\| = |\alpha| \|x - y\|.$$

(b) Since  $d$  is non-negative, then  $\|\cdot\|$  is also non-negative. Positive-definiteness is also clear. Then,

$$d(\alpha x, 0) = \|\alpha x\| = |\alpha| \|x\| = |\alpha| d(x, 0)$$

so homogeneity holds. Lastly, we prove that  $\|\cdot\|$  satisfies the triangle inequality. We have

$$\begin{aligned} \|x - z\| &= d(x - z, 0) \\ &= d(x, z) \quad \text{since } d \text{ is translation invariant} \\ &\leq d(x, y) + d(y, z) \quad \text{since } d \text{ satisfies the triangle inequality} \\ &= d(x - y, 0) + d(y - z, 0) \quad \text{since } d \text{ is translation invariant} \\ &= \|x - y\| + \|y - z\| \end{aligned}$$

so  $\|\cdot\|$  defines a norm on  $X$ . □

**Definition 1.4 (function space).** A function space is a set of functions holding the properties of a vector space structure, norm, or inner product. In particular, it has either of the following properties:

- **Vector space structure:**
  - **Closure under addition:** for any  $f, g \in \mathcal{C}[a, b]$ , we have  $f + g \in \mathcal{C}[a, b]$
  - **Closure under scalar multiplication:** for any  $k \in \mathbb{R}$ ,  $kf \in \mathcal{C}[a, b]$  for  $f \in \mathcal{C}[a, b]$
- **Norm:** A norm  $\|\cdot\|$  is a function  $\|\cdot\| : \mathcal{C}[a, b] \rightarrow \mathbb{R}$  that satisfies the following:
  - **Non-negativity:**  $\|f\| \geq 0$  for all  $f \in \mathcal{C}[a, b]$ , and  $\|f\| = 0$  if and only if  $f = 0$
  - **Scalar multiplication:**  $\|kf\| = |k| \|f\|$  for any  $k \in \mathbb{R}$  and  $f \in \mathcal{C}[a, b]$
  - **Triangle inequality:**  $\|f + g\| \leq \|f\| + \|g\|$  for all  $f, g \in \mathcal{C}[a, b]$
- **Inner product:** An inner product  $\langle \cdot, \cdot \rangle$  is a function  $\langle \cdot, \cdot \rangle : \mathcal{C}[a, b] \times \mathcal{C}[a, b] \rightarrow \mathbb{R}$  that satisfies the following:
  - **Conjugate symmetry:**  $\langle f, g \rangle = \overline{\langle g, f \rangle}$
  - **Linearity in the first argument:**  $\langle kf + g, h \rangle = k \langle f, h \rangle + \langle g, h \rangle$  for any  $k \in \mathbb{R}$
  - **Positive-definiteness:**  $\langle f, f \rangle \geq 0$  for all  $f \in \mathcal{C}[a, b]$ , and  $\langle f, f \rangle = 0$  if and only if  $f = 0$

**Example 1.9.** Let

$\mathcal{C}[a, b]$  denote the set of continuous functions on  $[a, b]$ .

**Example 1.10 (function space).** Let  $X = \mathcal{C}[a, b]$ , for which we recall that this refers to the set of continuous functions on  $[a, b]$ . Let  $f, g \in \mathcal{C}[a, b]$ . Then,

$$d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \quad \text{and} \quad d(f, g) = \sqrt{\int_0^L |f(x) - g(x)|^2 dx} \quad \text{are metrics.}$$

**Example 1.11 (Hamming distance).** Consider the two English words ‘word’ and ‘wind’ of the same length for which the second and third letters differ. Since two letters differ, we say that their Hamming distance is 2. We write

$$d(\text{wind}, \text{word}) = 2.$$

In this case,  $d$  is a metric. The reader can read Kreyszig p. 9 Question 10 to prove that the Hamming distance is indeed a metric.

**Example 1.12 (Hamming metric; Kreyszig p. 9 Question 10).** Let  $X$  be the set of all ordered triples of zeros and ones. Show that  $X$  consists of eight elements and a metric  $d$  on  $X$  defined by

$$d(x, y) = \text{number of places where } x \text{ and } y \text{ have different entries.}$$

*Solution.* First, non-negativity clearly holds as it is impossible for two words of the same length to differ by a negative number of letters. So,  $d(x, y) \geq 0$ . Symmetry is also obvious.

For positive-definiteness, suppose  $x = y$ . Then,  $x$  and  $y$  are the same word. Hence, they are two words of the same length which differ by 0 letters. By definition of  $d$ , we have  $d(x, y) = 0$ . Similarly, if  $d(x, y) = 0$ , then  $x$  and  $y$  are two words of the same length that are of Hamming distance 0, which implies that they differ by 0 letters. As such,  $x = y$ .

Lastly, we prove that  $d$  satisfies the triangle inequality. Let  $x$ ,  $y$  and  $z$  be three words of length  $n$ . We can explicitly define  $d$  as follows:

$$d(x, y) = \sum_{i=1}^n \mathbf{1}_{\{x_i \neq y_i\}} \quad \text{where} \quad \mathbf{1}_{\{x_i \neq y_i\}} = \begin{cases} 1 & \text{if } x_i \neq y_i; \\ 0 & \text{if } x_i = y_i \end{cases}$$

We note that for each position  $1 \leq i \leq n$ , the inequality

$$\mathbf{1}_{\{x_i \neq z_i\}} \leq \mathbf{1}_{\{x_i \neq y_i\}} + \mathbf{1}_{\{y_i \neq z_i\}}.$$

To see why, if  $x_i = z_i$ , then  $\mathbf{1}_{\{x_i \neq z_i\}} = 0$  and the inequality holds trivially since  $\mathbf{1}_{\{x_i \neq y_i\}}$  and  $\mathbf{1}_{\{y_i \neq z_i\}}$  are non-negative. If  $x_i \neq z_i$ , then either  $x_i \neq y_i$ , or  $y_i \neq z_i$ , or both. Hence, at least one of  $\mathbf{1}_{\{x_i \neq y_i\}}$  or  $\mathbf{1}_{\{y_i \neq z_i\}}$  is 1, and the inequality holds. Hence,

$$\sum_{i=1}^n \mathbf{1}_{\{x_i \neq z_i\}} \leq \sum_{i=1}^n \mathbf{1}_{\{x_i \neq y_i\}} + \sum_{i=1}^n \mathbf{1}_{\{y_i \neq z_i\}} \quad \text{or equivalently} \quad d(x, z) \leq d(x, y) + d(y, z).$$

so the triangle inequality is satisfied. □

**Definition 1.5 ( $\ell^p$ -space).** Let  $p \geq 1$  be a fixed real number. Each element in the space  $\ell^p$  is a sequence  $(x_1, \dots)$  such that  $|x_1|^p + \dots$  converges. So,

$$\ell^p = \left\{ \mathbf{x} \in \mathbb{R}^\infty : \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty \right\}.$$

**Definition 1.6 ( $p$ -norm).** Every element in  $\ell^p$ -space is equipped with a norm, known as the  $p$ -norm. We define it as follows (will not be strict with the use of either  $x$  or  $\mathbf{x}$ ):

$$\text{if } x \in \ell^p \quad \text{then} \quad \|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

**Example 1.13 (MA4211 AY24/25 Sem 2 Tutorial 1).** Give an example of a sequence  $x_n$  such that

$$x_n \rightarrow 0 \quad \text{but} \quad x_n \notin \ell^p \text{ for any } 1 \leq p < \infty.$$

*Solution.* Consider

$$x_n = \frac{1}{\log(n+1)} \quad \text{for which} \quad \lim_{n \rightarrow \infty} x_n = 0.$$

Then, we have

$$\|x_n\|_p = \frac{1}{[\log(n+1)]^p} \quad \text{so} \quad \sum_{n=1}^{\infty} \|x_n\|_p = \sum_{n=1}^{\infty} \frac{1}{[\log(n+1)]^p} \text{ diverges.}$$

□

**Example 1.14** (MA4211 AY24/25 Sem 2 Homework 1). Recall that for  $1 \leq p < \infty$ , we defined

$$\ell^p = \left\{ x_i \in \mathbb{R}^\infty \mid \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \quad \text{with norm} \quad \|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

Show that if  $p \leq q$ , then  $\|x\|_q \leq \|x\|_p$  and hence  $\ell^p \subseteq \ell^q$ .

*Solution.* We have

$$\|x\|_q = \left( \sum_{i=1}^{\infty} |x_i|^q \right)^{1/q} = \left[ \left( \sum_{i=1}^{\infty} |x_i|^q \right)^{p/q} \right]^{1/p} \leq \left( \sum_{i=1}^{\infty} |x_i|^{q(p/q)} \right)^{1/p} = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

The inequality holds because

$$\left[ \left( \sum_{i=1}^{\infty} |x_i|^q \right)^{p/q} \right]^{1/p} \leq \left( \sum_{i=1}^{\infty} |x_i|^{q(p/q)} \right)^{1/p} \quad \text{is equivalent to} \quad \left( \sum_{i=1}^{\infty} |x_i|^q \right)^{p/q} \leq \sum_{i=1}^{\infty} |x_i|^{q(p/q)}$$

Let

$$a_i = |x_i|^q.$$

Then, it suffices to prove that

$$\left( \sum_{i=1}^{\infty} a_i \right)^r \leq \sum_{i=1}^{\infty} a_i^r \quad \text{where} \quad r = \frac{p}{q} \leq 1.$$

To see why this holds, note that for all  $a, b \geq 0$  and  $0 < r \leq 1$ , we have

$$(a+b)^r \leq a^r + b^r.$$

Let  $f(a) = a^r + b^r - (a+b)^r$ . Then,  $f'(a) = ra^{r-1} - r(a+b)^{r-1} \geq 0$ . Since  $f(0) = 0$ , then  $f(a) \geq 0$ . As such, the inequality  $(a+b)^r \leq a^r + b^r$  holds. We can extend it to any finite sum by induction; for an infinite sum, use a limiting argument. As such, for any sequence in  $\ell^p$ , it must be contained in  $\ell^q$ . □

**Definition 1.7** ( $L^p$ ). Define the  $L^p$  space as follows:

$$L^p[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{R} \quad \text{such that} \quad \int_a^b |f(x)|^p dx < \infty \right\}$$

**Theorem 1.1** (Young's inequality). Suppose  $\alpha, \beta > 0$ . Then,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

**Example 1.15.** Let  $t = 1/p$ . Then,

$$\begin{aligned}\ln(t\alpha^p + (1-t)\beta^q) &\geq t \ln(\alpha^p) + (1-t) \ln(\beta^q) \quad \text{since } \ln \text{ is concave down} \\ &= \frac{1}{p} \ln(\alpha^p) + \frac{1}{q} \ln(\beta^q) \\ &= \ln \alpha + \ln \beta \\ &= \ln \alpha \beta\end{aligned}$$

Taking exponentials on both sides yields the desired result.

**Theorem 1.2 (Hölder's inequality).** For  $1 \leq p, q < \infty$  such that  $1/p + 1/q = 1$ , we have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

**Theorem 1.3 (Minkowski's inequality).** For  $1 \leq p < \infty$ , we have

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Please refer to my MA4262 notes for proofs of Theorems 1.2 and 1.3. Minkowski's inequality (Theorem 1.3) looks like a generalisation of the triangle inequality to some  $\ell^p$  space, where  $1 \leq p < \infty$ . We will see in Example 1.16 that the inequality does not hold if  $0 < p < 1$  as it would violate the triangle inequality.

**Example 1.16 (MA4211 AY24/25 Sem 2 Tutorial 2).** Consider the space  $\ell^p$  for  $0 < p < 1$ , with  $\|\cdot\|_p$  defined as usual. Which of the properties of a norm does  $\|\cdot\|_p$  satisfy?

*Solution.* We claim that  $\|\cdot\|_p$  satisfies all properties of a norm in Definition 2.6 except the triangle inequality. Verifying the other properties is easy. We have

$$\|(1,0) + (0,1)\|_p = \|(1,1)\|_p = 2^{1/p} \geq 2 = 1 + 1 = \|(1,0)\|_p + \|(0,1)\|_p$$

so the triangle inequality is not satisfied. □

**Example 1.17 (MA4211 AY24/25 Sem 2 Tutorial 1).** Prove the reverse triangle inequality

$$|d(x,z) - d(z,y)| \leq d(x,y) \quad \text{for all } x, y, z \text{ in a metric space } (X, d).$$

*Solution.* Recall the triangle inequality which states that

$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{for } x, y, z \text{ in a metric space } (X, d).$$

So,

$$d(x,z) - d(z,y) \leq d(x,y) \quad \text{where we used the fact that the metric } d \text{ is symmetric.}$$

If  $d(x,z) - d(z,y) \geq 0$ , then the result follows by taking absolute value; otherwise, we now consider the case where  $d(x,z) < d(z,y)$ . So,

$$d(x,y) \geq d(y,z) - d(x,z) > 0.$$

Taking absolute value again yields the desired result. □

## 1.2

### Convergence, Completeness, and Topology



**Definition 1.8 (convergence of sequence).** Let  $x_n$  be a sequence in  $\mathbb{R}$ . We say that

$$x_n \rightarrow x \quad \text{if} \quad \text{there exists } x \in \mathbb{R} \text{ such that } \lim_{n \rightarrow \infty} d(x_n, x) = 0.$$

One should recall from MA2108 that this is equivalent to saying that

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \quad \text{such that for all } n \geq N \text{ we have } d(x_n, x) < \varepsilon.$$

**Definition 1.9 (continuous function).** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. We say that  $f$  is continuous at  $x_0 \in X$  if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \quad \text{such that } d_X(x, x_0) < \delta \text{ implies } d_Y(f(x), f(x_0)) < \varepsilon.$$

**Theorem 1.4.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $T : X \rightarrow Y$  be a map. Then,

$$T \text{ is continuous} \quad \text{if and only if} \quad x_n \rightarrow x \text{ implies } T(x_n) \rightarrow T(x).$$

**Definition 1.10 (isometry).** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and  $T : X \rightarrow Y$ . We say that

$$T \text{ is an isometry} \quad \text{if} \quad d_X(x_1, x_2) = d_Y(T(x_1), T(x_2)).$$

**Definition 1.11 (isometric spaces).** If there exists a bijective isometry  $T$  between two metric spaces  $X$  and  $Y$ , then we say that  $X$  and  $Y$  are isometric.

**Definition 1.12 (open and closed intervals).** Let

$$(a, b) \text{ and } [a, b] \quad \text{denote} \quad \text{the open interval and closed interval in } \mathbb{R} \text{ respectively.}$$

**Definition 1.13 (open and closed balls).** Define

$$B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\} \quad \text{denote} \quad \text{the open ball in } \mathbb{R}^n$$

$$\bar{B}(x, r) = \{y \in \mathbb{R}^n : d(x, y) \leq r\} \quad \text{denote} \quad \text{the closed ball in } \mathbb{R}^n$$

Here, each ball is centred at  $x$  and is of radius  $r$ .

**Definition 1.14 (open set).** A subset  $A \subseteq X$  is open if

$$\text{for all } x \in A \text{ there exists } r > 0 \quad \text{such that} \quad B(x, r) \subseteq A.$$

**Definition 1.15 (closed set).** A subset of a metric space  $S \subseteq X$  is closed if

$$\text{its complement } S^c \quad \text{is open.}$$

**Example 1.18 (MA4211 AY24/25 Sem 2 Tutorial 2).** Show that the closed unit ball of a normed linear space is closed.

*Solution.* Let  $(X, \|\cdot\|)$  be a normed space. Define the closed unit ball as follows:

$$\overline{B(0,1)} = \{x \in X : \|x - y\| \leq 1 \text{ where } y \in X\}$$

It suffices to show that  $\overline{B(0,1)}$  contains all of its limit points. Let  $x_n \in \overline{B(0,1)}$  be such that  $x_n \rightarrow x$ . Since  $x_n$  converges, then for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\text{for all } n \geq N \quad \text{we have} \quad \|x - x_n\| < \varepsilon$$

By the triangle inequality,

$$\|x - y\| \leq \|x - x_n\| + \|x_n - y\| < 1 + \varepsilon.$$

Since  $\varepsilon > 0$  can be made sufficiently small, we see that  $\|x - y\| \leq \|x_n - y\| \leq 1$ . We conclude that  $\|x\|$  is contained in  $\overline{B(0,1)}$ , or to be more explicit, the limit point of  $x_n$  is contained in the closed unit ball, making  $\overline{B(0,1)}$  a closed set.  $\square$

**Definition 1.16 (topology).** Given a set  $X$ , a topology  $\mathcal{T}$  on  $X$  is a collection of subsets of  $X$  satisfying the following properties:

- (i)  $\emptyset, X \in \mathcal{T}$
- (ii)  $\mathcal{T}$  is closed under arbitrary unions
- (iii)  $\mathcal{T}$  is closed under finite intersection

**Definition 1.17 (limit point).** In a topological space, a point  $x$  is a limit point of a sequence  $x_n$  if for every neighbourhood of  $x$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n$  belongs to that neighbourhood.

**Definition 1.18 (continuous function).** If  $f : X \rightarrow Y$  is a function between two topological spaces  $X$  and  $Y$ , then  $f$  is continuous if the pre-image of every open set in  $Y$  is open in  $X$ .

**Example 1.19 (MA4211 AY24/25 Sem 2 Homework 1).** Let  $X$  and  $Y$  be normed linear spaces over the same field  $\mathbb{F}$  and having norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  respectively. For  $(x, y) \in X \times Y$ , define

$$\|(x, y)\| = \|x\|_1 + \|y\|_2.$$

Show that this defines a norm on  $X \times Y$  and that the projection maps

$$\pi_1 : (x, y) \mapsto x \quad \text{and} \quad \pi_2 : (x, y) \mapsto y \quad \text{are continuous.}$$

*Solution.* Note that

$$\|(0, 0)\| = \|0\|_1 + \|0\|_2 = 0$$

and conversely, if  $\|(x, y)\| = 0$ , then  $\|x\|_1 + \|y\|_2 = 0$ . Since the 1-norm and 2-norm are non-negative, then  $\|x\|_1 = \|y\|_2 = 0$ , so  $\|\cdot\|$  is positive-definite.

Non-negativity and homogeneity of  $\|\cdot\|$  is clear. Lastly,

$$\begin{aligned} \|(x_1 + x_2, y_1 + y_2)\| &= \|x_1 + x_2\|_1 + \|y_1 + y_2\|_2 \\ &\leq \|x_1\|_1 + \|x_2\|_1 + \|y_1\|_2 + \|y_2\|_2 \quad \text{since } \|\cdot\|_1 \text{ and } \|\cdot\|_2 \text{ satisfy triangle inequality} \\ &= \|x_1\|_1 + \|y_1\|_2 + \|x_2\|_1 + \|y_2\|_2 \\ &= \|(x_1, y_1)\| + \|(x_2, y_2)\| \end{aligned}$$

so  $\|\cdot\|$  satisfies the triangle inequality. As such,  $\|\cdot\|$  defines a norm on  $X \times Y$ .

We wish to prove for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\|(x, y) - (x_0, y_0)\| < \delta \quad \text{implies} \quad \|\pi_1(x, y) - \pi_1(x_0, y_0)\| < \varepsilon.$$

We have

$$\|\pi_1(x, y) - \pi_1(x_0, y_0)\| = \|x - x_0\|_1 \leq \|x - x_0\| + \|y - y_0\|_2 < \delta$$

so  $\pi_1$  is a continuous map. Similarly,  $\pi_2$  is a continuous map. □

**Definition 1.19 (closure).** Let  $X$  be a topological space. For  $A \subseteq X$ , the closure of  $A$ , denoted by  $\text{cl}(A)$  or  $\bar{A}$ , is defined as follows:

$$\bar{A} = A \cup \{\text{limit points of } A\}$$

**Definition 1.20 (dense set).** Let  $X$  be a topological space. If  $D \subseteq X$  such that

$$\bar{D} = X \quad \text{then} \quad D \text{ is dense in } X.$$

**Example 1.20 (MA4211 AY24/25 Sem 2 Homework 1).** Suppose that  $(X, \rho)$  and  $(Y, d)$  are metric spaces and  $A$  is a dense subset of  $X$ . Show that if

$F : X \rightarrow Y$  and  $G : X \rightarrow Y$  are two continuous functions such that  $F = G$  on  $A$  then  $F = G$  on  $X$ .

*Solution.* Since  $A$  is a dense subset of  $X$ , we take some sequence  $x_n$  in  $A$  such that  $x_n \rightarrow x$  in  $(X, \rho)$ . As  $F$  and  $G$  are continuous, we have

$$x_n \xrightarrow{\rho} x \quad \text{implies} \quad F(x_n) \xrightarrow{d} F(x) \quad \text{and} \quad G(x_n) \xrightarrow{d} G(x).$$

Since  $x_n \in A$  for all  $n \in \mathbb{N}$ , then  $F(x_n) = G(x_n)$ , i.e. these sequences are the same. These two sequences converge in  $(Y, d)$ , i.e.

$$F(x) = \lim_{n \rightarrow \infty} F(x_n) \quad \text{and} \quad G(x) = \lim_{n \rightarrow \infty} G(x_n).$$

By the uniqueness of limits, we must have  $F(x) = G(x)$ . As this holds for all  $x \in X$ , then  $F = G$  on  $X$ . □

**Definition 1.21 (separable space).** A topological space is separable if it has a countable dense subset.

**Theorem 1.5.** Let  $1 \leq p < \infty$ . Then,  $\ell^p$  is separable.

*Proof.* Define  $X \subseteq \ell^p$  to be the collection of sequences of the form

$$(x_1, x_2, \dots, x_n, 0, 0, \dots) \quad \text{where } x_i \in \mathbb{Q}.$$

As  $X$  is a countable union of countable sets,  $X$  is countable. Let  $y \in \ell^p$  and  $\varepsilon > 0$  be arbitrary. That is,

$$\sum_{i=1}^{\infty} |y_i|^p < \infty.$$

Also, there exists  $n \in \mathbb{N}$  such that

$$\sum_{i=n+1}^{\infty} \frac{\varepsilon^p}{2}.$$

Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , we can choose  $x \in X$  such that

$$\sum_{i=1}^n |y_i - x_i|^p < \frac{\varepsilon^p}{2}.$$

Then,

$$\|y - x\|_p^p = \sum_{i=1}^n |y_i - x_i|^p + \sum_{i=n+1}^{\infty} |y_i|^p < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p.$$

Taking the  $p^{\text{th}}$  root, we obtain  $d(x, y) = \|x - y\|_p < \varepsilon$ . □

**Theorem 1.6.**  $\ell^\infty$  is not separable.

*Proof.* Let  $y \in \mathbb{R}^\infty$  be a sequence of 0s and 1s. Define  $z \in \mathbb{R}$  to be as follows:

$$z = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \dots \quad \text{so we infer that } 0 \leq z \leq 1.$$

Because there are uncountably many real numbers in  $[0, 1]$ , it follows there are uncountably many distinct sequences  $y \in \{0, 1\}^{\mathbb{N}}$ . Denote this uncountable family by  $\mathcal{Y} \subset \ell^\infty$ . Note that for any two distinct sequences

$$y = (y_1, y_2, y_3, \dots) \quad y' = (y'_1, y'_2, y'_3, \dots) \in \mathcal{Y},$$

there is at least one index  $i$  such that  $y_i \neq y'_i$ . Because each coordinate is either 0 or 1, at index  $i$ , we have  $|y_i - y'_i| = 1$ . Hence,

$$\|y - y'\|_\infty = \sup_{n \in \mathbb{N}} |y_n - y'_n| \geq 1.$$

In fact, it is exactly 1 if the two sequences differ in at least one place (and cannot exceed 1 because each coordinate difference is 0 or 1). Around each  $y \in \mathcal{Y}$ , consider the open ball  $B(y, 1/3)$  of radius  $1/3$ . Since any two distinct  $y, y'$  are at distance  $\|y - y'\|_\infty = 1$ , their balls  $B(y, 1/3)$  and  $B(y', 1/3)$  cannot overlap. In other words, these balls are pairwise disjoint.

Suppose on the contrary that  $D \subseteq \ell^\infty$  is a countable dense set. Then, for each  $y \in \mathcal{Y}$ ,  $B(y, 1/3)$  must contain at least one point of  $D$ . However, there are uncountably many such disjoint balls  $B(y, 1/3)$  since  $\mathcal{Y}$  is uncountable. A single countable set  $D$  cannot meet each of these uncountably many disjoint balls in a distinct point. This leads to a contradiction. □

**Definition 1.22 (Cauchy sequence).** A sequence  $x_n$  in a metric space  $X$  is Cauchy if

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that for all } m, n \geq N \text{ we have } d(x_m, x_n) < \varepsilon.$$

**Proposition 1.1.** Every convergent sequence is Cauchy.

**Remark 1.1.** The converse of Proposition 1.1 is not true, i.e. not every Cauchy sequence is convergent.

**Definition 1.23 (complete metric space).** A metric space is called complete if every Cauchy sequence in the space converges.

**Example 1.21.**  $\ell^p$  and  $\ell^\infty$  are complete.

**Example 1.22.** The space of continuous functions on  $[a, b]$

$\mathcal{C}[a, b]$  equipped with the norm  $d_\infty(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$  is complete.

However, for any  $1 \leq p < \infty$ ,

$\mathcal{C}[a, b]$  equipped with the norm  $d_p(f, g) = \left( \int_a^b |f(x) - g(x)|^p dx \right)^{1/p}$  is not complete.

Having said that, the completion of this is denoted by  $L^p[a, b]$ .

**Example 1.23 (MA4211 AY24/25 Sem 2 Tutorial 1).** Show that the limits of Cauchy sequences in a complete metric space are unique.

*Solution.* Let  $x_n$  be a Cauchy sequence in a complete metric space. Suppose  $x_n$  has two limits  $x$  and  $y$ . Then, for all  $\varepsilon > 0$ , there exist  $N_1, N_2 \in \mathbb{N}$  such that

$$\text{for all } n \geq N_1 \text{ we have } |x_n - x| < \frac{\varepsilon}{2} \quad \text{and} \quad \text{for all } n \geq N_2 \text{ we have } |x_n - y| < \frac{\varepsilon}{2}.$$

By the triangle inequality,

$$|x - y| \leq |x_n - x| + |x_n - y| < \varepsilon.$$

Since  $\varepsilon$  can be made arbitrarily small, then  $x = y$ , i.e. the limits of the Cauchy sequence are the same.  $\square$

**Example 1.24 (MA4211 AY24/25 Sem 2 Tutorial 1).** Let

$(X, d)$  and  $(Y, \rho)$  be metric spaces and suppose  $f : X \rightarrow Y$  is uniformly continuous.

Prove that under  $f$ , the image of every Cauchy sequence is a Cauchy sequence.

*Solution.* Since  $f : X \rightarrow Y$  is uniformly continuous, for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, y \in X$  such that

$$d(x, y) < \delta \quad \text{we have} \quad \rho(f(x), f(y)) < \varepsilon.$$

Let  $x_n$  be a Cauchy sequence in the metric space  $(X, d)$ . Then, for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have  $d(x_m, x_n) < \delta$ . Hence,  $\rho(f(x_m), f(x_n)) < \varepsilon$ .  $\square$

**Theorem 1.7 (completion of metric space).** Every metric space can be completed, and the completion is unique up to isometry.

We first state a rough proof sketch of Theorem 1.7 before formally proving it. Given a metric space  $(X, d)$ , we can construct a new metric space  $(\tilde{X}, \tilde{d})$ . Then, there exists  $W \subseteq \tilde{X}$  such that  $X$  is isometric to  $W$  and  $\overline{W} = \tilde{X}$ . So,  $\tilde{X}$  is complete and  $\tilde{X}$  is unique up to isometry.

*Proof.* Let  $X$  be a metric space. Suppose  $x_n$  and  $y_n$  be Cauchy sequences of  $X$ . We say that

$$x_n \sim y_n \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Define  $\tilde{X}$  to be the set of equivalence classes determined by  $\sim$ . We write  $x_n \in \tilde{X}$ . Then,

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \rightarrow \infty} d(x_n, y_n) \quad \text{where} \quad x_n, y_n \in X.$$

By applying the triangle inequality twice, we obtain

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x'_n) + d(x'_n, y_n) \\ &\leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \end{aligned}$$

Also, by applying the triangle inequality, we have

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n).$$

We claim that  $d(x_n, y_n)$  is a Cauchy sequence. To see why, let  $\varepsilon > 0$  be arbitrary. Recall that  $x_n$  and  $y_n$  are both Cauchy sequences. So, there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have

$$d(x_n, x_m) < \varepsilon \quad \text{and} \quad d(y_n, y_m) < \varepsilon.$$

Then,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_n) \\ &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \end{aligned}$$

Let  $x \in X$  and consider the infinite sequence  $(x, x, \dots)$ . This sequence belongs to some element of  $\tilde{X}$ , so we call it  $\tilde{x}$ . Define  $\varphi : X \rightarrow \tilde{X}$  via  $\varphi(x) = \tilde{x}$ . We also define  $W = \varphi(X)$ . Then,

$$\tilde{d}(\varphi(x), \varphi(y)) = \tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} d(x, y) = d(x, y).$$

Let  $\tilde{x} \in \tilde{X}$  and consider  $x_n \in \tilde{x}$ . Then, for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$d(x_n, x_N) < \frac{\varepsilon}{2} \quad \text{for all } n \geq N.$$

Now, consider the sequence  $(x_N, x_N, \dots)$ , so  $\tilde{x}_N \in W$ . As such,

$$\tilde{d}(\tilde{x}, \tilde{x}_N) = \lim_{n \rightarrow \infty} d(x_n, x_N) \leq \frac{\varepsilon}{2} < \varepsilon.$$

This shows that  $W$  is dense in  $\tilde{X}$ .

Next, let  $\tilde{x}_n$  be a Cauchy sequence of elements in  $\tilde{X}$ . Since  $W$  is dense in  $\tilde{X}$ , for each  $n \in \mathbb{N}$ , there exists  $\tilde{w}_n \in W$  such that  $\tilde{d}(\tilde{x}_n, \tilde{w}_n) < 1/n$ . Hence,

$$\begin{aligned} \tilde{d}(\tilde{w}_m, \tilde{w}_n) &\leq \tilde{d}(\tilde{w}_m, \tilde{x}_m) + \tilde{d}(\tilde{x}_m, \tilde{x}_n) + \tilde{d}(\tilde{x}_n, \tilde{w}_n) \\ &< \frac{1}{m} + \tilde{d}(\tilde{x}_m, \tilde{x}_n) + \frac{1}{n} \end{aligned}$$

Hence,  $\tilde{w}_n$  is Cauchy. For each  $\tilde{w}_n$ , define  $w_n \in X$  by  $w_n = \varphi^{-1}(\tilde{w}_n)$ . Since  $\varphi$  is an isometry, then  $w_n$  is Cauchy. Let  $w_n \in \tilde{X}$ . Then,

$$\begin{aligned} \tilde{d}(\tilde{x}_n, \tilde{x}) &\leq \tilde{d}(\tilde{x}_n, \tilde{w}_m) + d(\tilde{w}_m, \tilde{x}) \\ &< \frac{1}{n} + d(\tilde{w}_m, \tilde{x}) \end{aligned}$$

So,

$$\tilde{d}(\tilde{w}_m, \tilde{x}) = \lim_{n \rightarrow \infty} d(w_m, w_n).$$

As such, the sequence  $w_n$  is Cauchy, which implies

$$\lim_{n \rightarrow \infty} \tilde{d}(\tilde{x}_n, \tilde{x}) = 0 \quad \text{or equivalently} \quad \tilde{x}_n \rightarrow \tilde{x}.$$

Finally, we need to argue that  $Y$  is another complete metric space and there is an isometric embedding  $f : X \rightarrow Y$  with  $f(X)$  dense in  $Y$ , then  $Y$  is isometric to  $\tilde{X}$ . Concretely, each Cauchy sequence  $x_n$  in  $X$  gives a Cauchy sequence  $f(x_n)$  in  $Y$ . Since  $Y$  is complete, then  $f(x_n)$  converges to some point in  $Y$ . Define a map  $\psi : \tilde{X} \rightarrow Y$  by sending  $\tilde{x}$  to the limit of  $f(x_n)$  in  $Y$ .  $\psi$  is well-defined, isometric and surjective by the density of  $f(X)$  in  $Y$ . We conclude that  $\tilde{X}$  is unique up to isometry.  $\square$

**Example 1.25** (continuous extension theorem; MA4211 AY24/25 Sem 2 Tutorial 1). <sup>†</sup> Suppose  $W$  is a dense subset of a metric space  $X$  and  $f$  is a uniformly continuous function from  $W$  into some complete metric space  $Y$ . Prove that  $f$  has a unique continuous extension to  $X$ . That is,

there exists a unique continuous function  $F : X \rightarrow Y$  such that  $F(w) = f(w)$  for all  $w \in W$ .

*Solution.* Since  $W$  is dense in  $X$ , there exists a sequence  $\{w_n\}_{n=1}^{\infty} \subseteq W$  such that  $w_n \rightarrow x$  in  $X$ . We first show that  $\{f(w_n)\}$  is a Cauchy sequence in  $Y$ . Since  $f : W \rightarrow Y$  is uniformly continuous, then for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for all  $u, v \in W$ , we have

$$d_X(u, v) < \delta \quad \text{implies} \quad d_Y(f(u), f(v)) < \varepsilon.$$

For  $m, n$  large enough, suppose  $w_m$  and  $w_n$  are within  $\delta/2$  of  $x$ , so

$$d_X(w_m, w_n) \leq d_X(w_m, x) + d_X(w_n, x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad \text{by the triangle inequality.}$$

Hence,

$$d_Y(f(w_m), f(w_n)) < \varepsilon.$$

As such,  $\{f(w_n)\}$  is a Cauchy sequence. As  $Y$  is a complete metric space, then the limit of the sequence  $\{f(w_n)\}$  exists in  $Y$ . Now, we define

$$F : X \rightarrow Y \quad \text{such that} \quad F(x) = \lim_{n \rightarrow \infty} f(w_n).$$

Note that  $F$  extends  $f$ . To see why, if  $x \in W$ , we can take the constant sequence  $w_n = x$ , so  $w_n \rightarrow x$ , which implies

$$F(x) = \lim_{n \rightarrow \infty} f(w_n) = \lim_{n \rightarrow \infty} f(x) = f(x).$$

We then show that  $F$  is continuous at  $x$ . Suppose  $x, x' \in X$  such that  $d_X(x, x') < \delta/3$ . Then, choose sequences  $w_n \rightarrow x$  and  $w'_n \rightarrow x'$ . For large  $n$ , we have

$$d_X(w_n, x) < \frac{\delta}{3} \quad \text{and} \quad d_X(w'_n, x') < \frac{\delta}{3}.$$

<sup>†</sup>As remarked by Tianle, this was taken from Rudin's Principles of Mathematical Analysis p. 99 Question 11.

By the triangle inequality,  $d_X(w_n, w'_n) < \delta$ . As  $f : W \rightarrow Y$  is uniformly continuous, then  $d_Y(f(w_n), f(w'_n)) < \varepsilon$  for large  $n$ . As  $F$  extends  $f$ , it follows that  $d_Y(F(x), F(x')) < \varepsilon^\ddagger$ .

Lastly, we prove that the extension is unique. Suppose there exists another continuous map  $\tilde{F} : X \rightarrow Y$  such that  $\tilde{F}(w) = f(w)$  for all  $w \in W$ . Since  $\tilde{F}$  is continuous, then

$$\lim_{n \rightarrow \infty} \tilde{F}(w_n) = \tilde{F}\left(\lim_{n \rightarrow \infty} w_n\right) = \tilde{F}(x) \quad \text{but also} \quad \lim_{n \rightarrow \infty} \tilde{F}(w_n) = \lim_{n \rightarrow \infty} f(w_n) = F(x)$$

so  $\tilde{F} = F$ . The result follows. □

<sup>‡</sup>Here, we are proving a stronger result, which is that  $f$  has a unique uniformly continuous extension to  $X$ .



## Chapter 2

### Normed Spaces and Banach Spaces

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#### 2.1 Vector Spaces

**Definition 2.1 (vector space).** A vector space is a set  $V$  together with a field  $\mathbb{F}$  equipped with two operations (addition  $+$  and multiplication  $\cdot$ )

$$+ : V \times V \rightarrow V \quad \text{and} \quad \cdot : \mathbb{F} \times V \rightarrow V$$

satisfying the following properties

- (i)  $+$  is commutative
- (ii)  $+$  is associative
- (iii) There exists an additive identity
- (iv) There exists an additive inverse
- (v)  $\cdot$  is associative
- (vi) There exists a multiplicative identity
- (vii) The distributivity properties hold

In this course, we are interested in infinite-dimensional vector spaces. Next, we are interested in continuous functions over  $\mathbb{R}$ . Recall that the set of continuous functions on  $[a, b]$  over  $\mathbb{R}$  is denoted by  $\mathcal{C}[a, b]$ . We define the addition and scalar multiplication functions by the obvious way — that is for any  $f, g \in \mathcal{C}[a, b]$ , we have

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha \cdot f(x),$$

where  $\alpha \in \mathbb{R}$ .

**Example 2.1.**  $\ell^p$  for  $1 \leq p \leq \infty$  is a vector space.

**Example 2.2.**  $\mathbb{R}^\infty$  is a vector space.

**Definition 2.2 (linear combination and span).** A linear combination is a vector of the form

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n \quad \text{for all } \alpha_i \in \mathbb{F} \text{ and } \mathbf{v}_i \in V.$$

Given a set of vectors  $S$ , its span, denoted by  $\text{span}(S)$ , is the set of all linear combinations of its elements.

**Definition 2.3 (linear independence).** A set of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is linearly independent if for any  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ , we have

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0} \quad \text{implies} \quad \alpha_1 = \dots = \alpha_n = 0.$$

**Definition 2.4 (dimension).** Let  $V$  be a vector space. If there exists  $S \subseteq V$  such that  $|S| < \infty$  and  $\text{span}(S) = V$ , then  $V$  is finite-dimensional. Otherwise,  $V$  is infinite-dimensional.

**Example 2.3 (infinite-dimensional vector spaces).** We give some examples of infinite-dimensional vector spaces. For  $1 \leq p < \infty$ ,  $\ell^p(\mathbb{R})$ , which denotes the space of real sequences  $a_n$  such that

$$\sum_{n=1}^{\infty} |a_n|^p \text{ converges} \quad \text{is infinite-dimensional.}$$

$\ell^\infty(\mathbb{R})$ , which denotes the space of bounded real sequences equipped with the supremum norm, is also infinite-dimensional. Lastly, the space of polynomials with real coefficients, denoted by  $\mathbb{R}[x]$ , is infinite-dimensional.

**Definition 2.5 (basis).** If  $\mathcal{B}$  is an independent spanning set for a finite-dimensional vector space  $V$ , then we say that  $\dim(V) = |\mathcal{B}|$ . Such a set  $\mathcal{B}$  is a basis. More formally, we call it a Hamel basis.

In Definition 1.4 on the definition of a function space, we briefly discussed the definition of a norm. We also talked about the  $p$ -norm in Definition 1.6. Now, we will formally introduce  $p$ -norms.

**Definition 2.6 (norm).** Given a vector space  $V$ , a norm on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}$  satisfying the following properties for any  $\mathbf{u}, \mathbf{v} \in V$ :

- (i) **Non-negativity:**  $\|\mathbf{v}\| \geq 0$  for all  $\mathbf{v} \in V$
- (ii) **Positive-definiteness:**  $\|\mathbf{v}\| = 0$  if and only if  $\mathbf{v} = \mathbf{0}$
- (iii) **Homogeneity:** For any  $\alpha \in \mathbb{F}$ , we have  $\|\alpha\mathbf{v}\| = |\alpha| \|\mathbf{v}\|$
- (iv) **Triangle inequality:**  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

It is important to note that a metric space is a space where we wish to measure distance, but a normed space is a space where we wish to measure the Euclidean distance between two vectors. We then discuss some properties of normed spaces.

**Definition 2.7.** Let  $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{z}$  be vectors in a normed space and  $\alpha$  be a real-valued scalar. Then, we have the following:

- (i)  $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$ , which defines the metric induced by the norm, also known as the norm-induced metric
- (ii) We have the familiar  $p$ -norm

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

- (iii)  $d(\mathbf{x} + \mathbf{z}, \mathbf{y} + \mathbf{z}) = d(\mathbf{x}, \mathbf{y})$ , which denotes translation invariance
- (iv)  $d(\alpha\mathbf{x}, \alpha\mathbf{y}) = |\alpha| d(\mathbf{x}, \mathbf{y})$ , which denotes the homogeneity of the metric

**Definition 2.8 (Banach space).** A normed space where the space is complete with respect to the induced metric is called a Banach space.

**Example 2.4 (MA4211 AY24/25 Sem 2 Tutorial 2).** Consider the space of continuously differentiable functions  $\mathcal{C}^1([a, b])$  with the  $\mathcal{C}^1$ -norm

$$\|f\| = \sup_{a \leq x \leq b} |f(x)| + \sup_{a \leq x \leq b} |f'(x)|.$$

Prove that  $\mathcal{C}^1([a, b])$  is a Banach space with respect to the given norm.

*Solution.* We first show that  $\|\cdot\|$  is indeed a norm. As usual, non-negativity, positive-definiteness and homogeneity are obvious. It suffices to prove that it satisfies the triangle inequality. We have

$$\begin{aligned}\|f + g\| &= \sup_{a \leq x \leq b} |f(x) + g(x)| + \sup_{a \leq x \leq b} |f'(x) + g'(x)| \\ &\leq \sup_{a \leq x \leq b} |f(x)| + \sup_{a \leq x \leq b} |g(x)| + \sup_{a \leq x \leq b} |f'(x)| + \sup_{a \leq x \leq b} |g'(x)| \\ &= \sup_{a \leq x \leq b} |f(x)| + \sup_{a \leq x \leq b} |f'(x)| + \sup_{a \leq x \leq b} |g(x)| + \sup_{a \leq x \leq b} |g'(x)| \\ &= \|f\| + \|g\|\end{aligned}$$

Next, let  $f_n$  be a Cauchy sequence in  $\mathcal{C}^1([a, b])$ . Then, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ , we have

$$\|f_m - f_n\| = \sup_{a \leq x \leq b} |f_m(x) - f_n(x)| + \sup_{a \leq x \leq b} |f'_m(x) - f'_n(x)| < \varepsilon.$$

As  $\mathcal{C}^1([a, b])$  equipped with the supremum norm is a Banach space, then  $f_n$  and  $f'_n$  converge uniformly to some continuous functions  $f$  and  $g$  respectively. By the Fundamental Theorem of Calculus, we have

$$f_n(x) - f_n(a) = \int_a^x f'_n(t) dt.$$

Taking the limit as  $n \rightarrow \infty$ , we have

$$f(x) - f(a) = \int_a^x g(t) dt.$$

Since  $f_n \rightarrow f$  uniformly and  $f'_n \rightarrow g$  uniformly, then  $\|f_n - f\| \rightarrow 0$ , which implies every Cauchy sequence in  $\mathcal{C}^1([a, b])$  converges to a limit in  $\mathcal{C}^1([a, b])$ .  $\square$

**Theorem 2.1.** A subspace  $Y$  of a Banach space  $X$  is complete if and only if it is closed in  $X$ .

Theorem 2.1 can be reformulated as saying that a subspace of a Banach space  $X$  is a Banach space if and only if  $Y$  is closed.

*Proof.* We first prove the forward direction. Suppose  $Y$  is complete. Let  $y \in \overline{Y}$ . Then, there exists a subsequence  $y_n \subseteq Y$  such that

$$\lim_{n \rightarrow \infty} y_n = y.$$

So,  $y_n$  is Cauchy and in fact,  $y_n$  converges in  $Y$ . So,  $Y$  is closed in  $X$ .

For the reverse direction, suppose  $Y$  is closed in  $X$ . Let  $y_n$  be Cauchy in  $Y$ . Since  $Y \subseteq X$  and  $X$  is complete, then

$$\lim_{n \rightarrow \infty} y_n = y.$$

So,  $y \in \overline{Y} = Y$ .  $\square$

**Definition 2.9 (convergence).** If  $X$  is a normed space and  $x_n$  is a sequence of elements in  $X$ , then we can define

$$s_n = \sum_{k=1}^n x_k.$$

If there exists  $s \in X$  such that  $\lim_{n \rightarrow \infty} \|s_n - s\| = 0$ , then we write

$$s = \sum_{k=1}^{\infty} x_k.$$

Note that if  $\sum_{k=1}^{\infty} \|x_k\| < \infty$ , the series is said to be absolutely convergent.

**Example 2.5** (MA4211 AY24/25 Sem 2 Tutorial 2). Let  $Y$  be a closed subspace of a normed linear space  $(X, \|\cdot\|)$ . Let  $X/Y$  denote the quotient space (elements of  $X/Y$  are additive cosets). For  $x + Y \in X/Y$ , define the quotient norm  $\|\cdot\|_q$  by

$$\|x + Y\|_q = \inf_{y \in Y} \|x - y\|.$$

Show that  $\|\cdot\|_q$  is a norm on  $X/Y$ . Also, if  $X$  is a Banach space, show that  $X/Y$  is a Banach space under the quotient norm.

*Solution.* Clearly, the norm satisfies positive-definiteness, non-negativity and homogeneity. We prove that it satisfies the triangle inequality. We have

$$\begin{aligned} \|x + Z\|_q &= \inf_{z \in Z} \|x - z\| \\ &\leq \inf_{z \in Z} \|(x - y) + (y - z)\| \\ &= \|x - y\| + \inf_{z \in Z} \|y - z\| \\ &= \inf_{y \in Y} \|x - y\| + \inf_{z \in Z} \|y - z\| \\ &\leq \|x + Y\|_q + \|y + Z\|_q \end{aligned}$$

so  $\|\cdot\|_q$  is a norm on  $X/Y$ .

Next, suppose  $X$  is a Banach space. We wish to prove that  $X/Y$  is also a Banach space, i.e. every Cauchy sequence in  $X/Y$  converges to a limit in  $X/Y$  (norm on  $X/Y$  was established earlier). For each  $x \in X$ , let  $\hat{x} = x + Y \in X/Y$ . Consider  $\hat{x}_n$  such that

$$\sum_{n=1}^{\infty} \|\hat{x}_n\| \quad \text{converges.}$$

We have for every  $n \in \mathbb{N}$ , there exists  $x_n \in \hat{x}_n$  such that  $\|x_n\| \leq 2\|\hat{x}_n\|$ . Since the mentioned sum converges, then

$$\sum_{n=1}^{\infty} \|x_n\| \quad \text{converges.}$$

As  $X$  is Banach, then

$$\sum_{n=1}^{\infty} x_n \quad \text{converges in } X \quad \text{to some } x \in X.$$

From definition of the norm in  $X/Y$ , it follows that

$$\sum_{n=1}^{\infty} \hat{x}_n \quad \text{converges to } \hat{x} \quad \text{in } X/Y.$$

We conclude that  $X/Y$  is Banach as well. □

**Example 2.6** (MA4211 AY24/25 Sem 2 Homework 1). Much of our motivation comes from wanting to do Linear Algebra in infinite dimensions. In order to compute  $\mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is an infinite matrix and  $\mathbf{x}$  is an infinite sequence, we need expressions like

$$\sum_{k=1}^{\infty} a_k x_k \quad \text{to converge.}$$

Even better, we would like these to converge absolutely, i.e.

$$\sum_{k=1}^{\infty} |a_k x_k| < \infty.$$

In this problem, we investigate what conditions are needed to guarantee convergence.

(a) Give an example of  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\infty}$  such that

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{does not converge.}$$

(b) Give an example of  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\infty} \setminus \ell^{\infty}$  such that

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{does not converge.}$$

(c) Prove or disprove: If  $\mathbf{x} \in \mathbb{R}^{\infty} \setminus \ell^{\infty}$ , then there exists  $\mathbf{y} \in \ell^1$  such that

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{does not converge.}$$

(d) Recall Hölder's inequality (Theorem 1.2): if  $1 \leq p, q < \infty$  such that  $1/p + 1/q = 1$ , then

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q.$$

Prove that this inequality still holds if we take  $q = \infty$  and use the convention that  $\frac{1}{\infty} = 0$ .

(e) Prove or disprove: If  $p, q \in [1, \infty]$  such that  $1/p + 1/q \geq 1$  and  $\mathbf{x} \in \ell^p, \mathbf{y} \in \ell^q$ , then

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{converges.}$$

(f) Prove or disprove: If  $p, q \in [1, \infty]$  such that  $1/p + 1/q < 1$ , then there exist  $\mathbf{x} \in \ell^p, \mathbf{y} \in \ell^q$  such that

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{does not converge.}$$

*Solution.*

(a) Let  $\mathbf{x} = \mathbf{y} = (1, \dots)$  be two sequences of ones, so

$$\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} 1 \quad \text{which is divergent.}$$

(b) Let  $x_k = y_k = k$  for all  $k \in \mathbb{N}$ . Then,

$$\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} k^2 \quad \text{which does not converge.}$$

(c) Yes. Recall that  $\mathbb{R}^{\infty} \setminus \ell^{\infty}$  denotes the set of unbounded sequences. Let  $x_k = 2^k$  and  $y_k = 1/2^k$ , so

$$\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} 1 \quad \text{which does not converge.}$$

(d) If  $q = \infty$ , then  $p = 1$ , so we wish to prove that

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left( \sum_{k=1}^{\infty} |x_k| \right) \cdot \sup_{k \in \mathbb{N}} |y_k|.$$

This is obvious because

$$\sum_{k=1}^{\infty} |x_k y_k| = |x_1 y_1| + |x_2 y_2| + \dots \leq \left| x_1 \cdot \sup_{k \in \mathbb{N}} |y_k| \right| + \left| x_2 \cdot \sup_{k \in \mathbb{N}} |y_k| \right| + \dots = \sup_{k \in \mathbb{N}} |y_k| \sum_{k=1}^{\infty} |x_k|$$

(e) Recall Example 1.14, which mentioned that if  $1 \leq p \leq q$ , then  $\ell^p \subseteq \ell^q$ . Hence,  $\ell^1 \subseteq \ell^q$ . It suffices to prove that if  $\mathbf{x}, \mathbf{y} \in \ell^1$ , then

$$\sum_{k=1}^{\infty} |x_k y_k| \quad \text{converges.}$$

This holds because

$$\sum_{k=1}^{\infty} |x_k y_k| \leq \left( \sum_{k=1}^{\infty} |x_k| \right) \left( \sum_{k=1}^{\infty} |y_k| \right)$$

As such, the statement is true.

(f) The statement is true<sup>†</sup>. We shall consider three cases.

- **Case 1:** Suppose  $p = q = \infty$ . Then, we can take  $\mathbf{x} = \mathbf{y}$  to be the constant sequence 1. As such,

$$\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} 1 \quad \text{which diverges.}$$

- **Case 2:** Suppose exactly one of  $p$  or  $q$  is  $\infty$ . Say  $p = \infty$ , then  $1/q < 1$ . Let  $\mathbf{x}$  be the constant sequence 1 and  $\mathbf{y}$  be the sequence  $1/k$ . Then,  $\mathbf{x} \in \ell^\infty$  and

$$\sum_{k=1}^{\infty} \left| \frac{1}{k} \right|^p = \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

Since  $1/p < 1$ , then  $p > 1$  so the series converges. However,

$$\sum_{k=1}^{\infty} |x_k y_k| = \sum_{k=1}^{\infty} 1 \quad \text{which does not converge.}$$

- **Case 3:** Suppose  $1 \leq p, q < \infty$ . Since  $1/p + 1/q < 1$ , let

$$\varepsilon = \frac{1}{2} \left( 1 - \frac{1}{p} - \frac{1}{q} \right).$$

Define

$$x_k = \frac{1}{k^{\varepsilon+1/p}} \quad \text{and} \quad y_k = \frac{1}{k^{\varepsilon+1/q}}.$$

Hence,

$$\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^{\infty} \frac{1}{k^{\varepsilon+1/p}}.$$

As  $1 + \varepsilon p > 1$ , then  $\mathbf{x} \in \ell^p$ . A similar argument shows that  $\mathbf{y} \in \ell^q$ . However,

$$\sum_{k=1}^{\infty} |x_k y_k| = \frac{1}{k^{2\varepsilon+1/p+1/q}} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

This is precisely the harmonic series, which does not converge!.

□

<sup>†</sup>We credit the solution to Lou Yi.

**Definition 2.10 (Schauder basis).** If  $X$  is a normed space and  $e_k$  is a sequence of elements of  $x$  such that for all  $x \in X$ , there exists a unique sequence of scalars  $a_n$  such that

$$\lim_{n \rightarrow \infty} \left\| \sum_{k=1}^n a_k e_k - x \right\| = 0 \quad \text{then} \quad \{e_1, \dots, e_n\} \text{ is a Schauder basis.}$$

In a *slightly more general notion*, a Schauder basis is also called a countable basis.

**Definition 2.11 (partial order).** A partial order  $\leq$  on a set  $S$  is a binary relation that satisfies the following properties:

- (i) **Reflexivity:**  $a \leq a$
- (ii) **Transitivity:** If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$
- (iii) **Antisymmetry:** If  $a \leq b$  and  $b \leq a$ , then  $a = b$

**Definition 2.12 (total order).** A total order relation is a partial order in which every element of the set is comparable with every other element of the set, i.e. if  $\leq$  is a partial order on a set  $S$ , then

for any  $x, y \in S$  such that either  $x \leq y$  or  $y \leq x$  then  $\leq$  is a total order on  $S$ .

Note that every total order is a partial order, but the converse does not hold.

**Theorem 2.2 (Zorn's lemma).** If  $S$  is a partially ordered set such that every totally ordered subset of  $S$  has an upper bound in  $S$ , then

there exists an element  $m \in S$  such that for all  $a \in S$  we have  $m \geq a$ .

**Theorem 2.3.** Every vector space has a Hamel basis.

*Proof.* If  $V = \{\mathbf{0}\}$ , then  $\emptyset$  is its basis. Suppose  $V \neq \emptyset$ . Let  $P$  be the set of linearly independent subsets of  $V$  ordered by inclusion.

Let  $S_\alpha$  be a totally ordered subset of  $P$  and define

$$M = \bigcup_{\alpha} S_\alpha.$$

Suppose  $\mathbf{v}_1, \dots, \mathbf{v}_n \in M$  such that  $\alpha_1 \mathbf{v}_1 + \alpha_n \mathbf{v}_n = \mathbf{0}$  with not all  $\alpha_i = 0$ . Then, there exists an  $\alpha$  such that  $S_\alpha$  contains all  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . By Zorn's lemma (Theorem 2.2),  $S_\alpha$  has a maximal element, say  $\mathcal{B}$ .

Suppose on the contrary that  $\mathcal{B}$  does not span  $V$ . Then, there exists  $\mathbf{v} \in V$  such that  $\mathbf{v} \notin \text{span}(\mathcal{B})$ . So,  $\mathcal{B} \cup \{\mathbf{v}\}$  is linearly independent but  $\mathcal{B} \subseteq \mathcal{B} \cup \{\mathbf{v}\}$ . This is a contradiction. Hence,  $\mathcal{B}$  spans  $V$  and is a Hamel basis. Since  $V$  was an arbitrary vector space, we conclude that every vector space has a Hamel basis.  $\square$

**Lemma 2.1 (frame condition).** Let  $\mathbf{x}_1, \dots, \mathbf{x}_n$  be linearly vectors independent in  $X$ . The set  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is a frame in  $X$  if there exist constants  $0 < A \leq B$  such that for any scalars  $\alpha_1, \dots, \alpha_n$ , we have

$$A \sum_{i=1}^n |\alpha_i| \leq \left\| \sum_{i=1}^n \alpha_i \mathbf{x}_i \right\| \leq B \sum_{i=1}^n |\alpha_i|.$$

Note that one part of the frame condition in Lemma 2.1 is obvious — by the triangle inequality, we have

$$\left\| \sum_{i=1}^n \alpha_i \mathbf{x}_i \right\| = \sum_{i=1}^n |\alpha_i| \|\mathbf{x}_i\| \leq \max_{1 \leq i \leq n} \|\mathbf{x}_i\| \sum_{i=1}^n |\alpha_i|$$

As such,

setting  $B = \max_{1 \leq i \leq n} \|\mathbf{x}_i\|$  yields part of the desired result.

## 2.2 Compactness

We first define sequential compactness (Definition 2.13). Note that in metric spaces, compactness and sequential compactness are equivalent. This is a key property of metric spaces, where the structure of the metric allows for this equivalence to hold.

**Definition 2.13** (sequential compactness). Let  $X$  be a metric space. A subset  $K$  of  $X$  is said to be sequentially compact if every sequence of elements in  $K$  contains a subsequence that converges to an element in  $K$ .

**Theorem 2.4** (Heine-Borel theorem). A subset of  $\mathbb{R}^n$  is

compact if and only if it is closed and bounded.

**Remark 2.1.** In any metric space, a compact set is always closed and bounded.

**Theorem 2.5** (Bolzano-Weierstrass theorem). Every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.

*Proof.* Let  $x_n$  be a bounded sequence in  $\mathbb{R}^n$ . Then, there exists a positive constant  $M > 0$  such that the sequence is contained in  $[-M, M]^n = C_0$ . Then, partition  $C_0$  into  $2^n$  smaller cubes of side length  $M/2$ . By the pigeonhole principle, at least one of these subcubes, call it  $C_1$ , contains infinitely many points of the sequence.

From  $C_1$  (which has side length  $M/2$ ), divide it again into  $2^n$  subcubes of side length  $M/4$ . At least one of these subcubes contains infinitely many points of the sequence; call it  $C_2$ . Next, from  $C_2$  (side length  $M/4$ ), divide it into  $2^n$  subcubes of side length  $M/8$ . Again, at least one of these new subcubes contains infinitely many points; call it  $C_3$ .

Continue the above process inductively. At the  $k^{\text{th}}$  step, we will have a cube  $C_k$  of side length  $M/2^k$ . Subdivide  $C_k$  into  $2^n$  smaller cubes of side length  $M/2^{k+1}$ . At least one of these smaller cubes, denoted by  $C_{k+1}$ , contains infinitely many points of  $\{x_n\}$ .

Hence, we obtain a nested sequence of closed cubes as follows:

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots \supseteq C_k \supseteq \dots \quad \text{with side lengths decreasing to 0.}$$

As each cube  $C_k$  contains infinitely many terms of the sequence, we can choose at least one index  $n_k$  such that  $x_{n_k} \in C_k$ . We wish to construct a subsequence  $\{x_{n_k}\}$ . To ensure it is well-defined (i.e. so that  $n_1 < n_2 < n_3 < \dots$ ),



choose each index  $n_{k+1}$  to be strictly larger than  $n_k$ . This gives us a subsequence  $\{x_{n_k}\}$ .

Since

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots \supseteq C_k \supseteq \dots,$$

each  $C_k$  is a closed cube whose diameter goes to 0 as  $k \rightarrow \infty$ . More precisely,

$$\text{the diameter of } C_k \text{ is } \sqrt{n} \cdot \frac{M}{2^{k-1}} \text{ which tends to 0 as } k \text{ goes to infinity.}$$

As  $x_{n_k} \in C_k$  for each  $k$ , any two points  $x_{n_k}, x_{n_\ell}$  eventually lie in the same small cube for sufficiently large  $k, \ell$ . Hence,

$$\lim_{k, \ell \rightarrow \infty} \|x_{n_k} - x_{n_\ell}\| \rightarrow 0 \text{ which shows that } x_{n_k} \text{ is a Cauchy subsequence in } \mathbb{R}^n.$$

$$\|x_{n_k} - x_{n_\ell}\| \rightarrow 0 \text{ as } k, \ell \rightarrow \infty,$$

As  $\mathbb{R}^n$  is complete, then every Cauchy sequence converges to some limit point in  $\mathbb{R}^n$ . Therefore, the subsequence  $\{x_{n_k}\}$  converges and the proof is complete.  $\square$

**Theorem 2.6.** If  $f : K \rightarrow Y$  is continuous, then  $f(K)$  is compact.

**Theorem 2.7 (Heine-Cantor theorem).** If  $f : K \rightarrow Y$  is continuous, where  $K$  is compact, then  $f$  is uniformly continuous.

We now turn our attention to the extreme value theorem (Theorem 2.8), which extends the familiar result from Calculus and Real Analysis (MA2002 and MA2108, respectively). This generalisation, originally attributed to Weierstrass, provides a broader framework for understanding the attainment of extrema.

**Theorem 2.8 (extreme value theorem).** If  $f : K \rightarrow \mathbb{R}$  is continuous, where  $K$  is compact, then  $f$  attains its maximum and minimum on  $K$ .

**Lemma 2.2.** Let  $X$  be a finite-dimensional normed space and let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis for  $X$ . Then, there exist  $0 < A \leq B$  such that

$$\text{for all } \mathbf{x} \in X \text{ where } \mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i \text{ we have } A \sum_{i=1}^n |\alpha_i| \leq \|\mathbf{x}\| \leq B \sum_{i=1}^n |\alpha_i|.$$

*Proof.* Define

$$S = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : \sum_{i=1}^n |\alpha_i| = 1 \right\} \text{ which is closed and bounded, hence compact.}$$

Note that  $S$  denotes the unit sphere in  $\mathbb{R}^n$ , i.e. the boundary of the unit ball. Define

$$f : S \rightarrow X \text{ where } (\alpha_1, \dots, \alpha_n) \mapsto \sum_{i=1}^n \alpha_i \mathbf{e}_i.$$

Let  $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in S$ . Then,

$$\begin{aligned} \|f(\alpha_1, \dots, \alpha_n) - f(\beta_1, \dots, \beta_n)\| &= \left\| \sum_{i=1}^n \alpha_i \mathbf{e}_i - \sum_{i=1}^n \beta_i \mathbf{e}_i \right\| \\ &= \sum_{i=1}^n |\alpha_i - \beta_i| \|\mathbf{e}_i\| \\ &\leq \max_{1 \leq i \leq n} \|\mathbf{e}_i\| \sum_{i=1}^n |\alpha_i - \beta_i| \end{aligned}$$

Recall that  $\|\cdot\| : X \rightarrow \mathbb{R}$  is continuous. Since  $S$  is compact and  $f$  is continuous, the composition  $\|\cdot\| \circ f$  is a continuous real-valued function on the compact set  $S$ . By the extreme value theorem (Theorem 2.8), it achieves its minimum and maximum on  $S$ . Let  $A = \min \|f\|$  and  $B = \max \|f\|$ .

Then, let  $(\beta_1, \dots, \beta_n)$  be a point in  $S$  where  $\|f\|$  attains a minimum. Then,

$$\left\| \sum_{i=1}^n \beta_i \mathbf{e}_i \right\| = A.$$

Since we cannot have all  $\beta_i = 0$  and  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is linearly independent, then  $A > 0$ . The proof pretty much follows from here.  $\square$

**Theorem 2.9.** Every finite-dimensional normed space is a Banach space.

**Theorem 2.10 (equivalence of norms).** If  $X$  is a finite-dimensional vector space and  $\|\cdot\|_a$  and  $\|\cdot\|_b$  are norms on  $X$ , then there exist  $0 < A \leq B$  such that for all  $\mathbf{x} \in X$ , we have

$$A \|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b \leq B \|\mathbf{x}\|_a$$

*Proof.* Let  $\mathbf{x}_k$  be a Cauchy sequence in  $X$ , where  $X$  is a finite-dimensional vector space. Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be a basis for  $X$ . So, we can write

$$\mathbf{x}_k = \sum_{i=1}^n \alpha_{ik} \mathbf{e}_i.$$

Fix  $i$  and consider the sequence  $\alpha_{ik}$ , so  $k \in \mathbb{N}$ . Now,

$$\|\mathbf{x}_k - \mathbf{x}_j\| \geq A \sum_{i=1}^n |\alpha_{ij} - \alpha_{ik}| \geq A |\alpha_{ij} - \alpha_{ik}| \quad \text{which implies} \quad |\alpha_{ij} - \alpha_{ik}| \leq \frac{1}{A} \|\mathbf{x}_k - \mathbf{x}_j\|.$$

As such,  $\alpha_{ik}$  is a Cauchy sequence, hence convergent. Call its limit  $\alpha_i$ , i.e.

$$\lim_{k \rightarrow \infty} \alpha_{ik} = \alpha_i.$$

Define

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i.$$

Then,

$$\|\mathbf{x}_k - \mathbf{x}\| \leq B \sum_{i=1}^n |\alpha_{ik} - \alpha_i|.$$

Since  $\alpha_{ik}$  is a Cauchy sequence, then  $\mathbf{x}_k \rightarrow \mathbf{x}$  and the result follows.  $\square$

**Theorem 2.11 (equivalence of norms).** Let  $X$  be a finite-dimensional vector space and let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be norms on  $X$ . Then, there exist constants  $0 \leq A < B$  such that

$$A \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq B \|\mathbf{x}\|_2 \quad \text{for all } \mathbf{x} \in X.$$