Path following of car-like vehicles using dynamic inversion

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Aurelio Piazzi
Università di Parma

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Path following of car-like vehicles using dynamic inversion

LUCA CONSOLINI†, AURELIO PIAZZI†* and MARIO TOSQUES‡

This paper focuses on a special path following task arising from the needs of vision-based autonomous guidance: a given front point of a car-like vehicle that is within the look-ahead range of a stereo vision system must follow a prespecified Cartesian path. A solution to this path following problem is provided by a new feedforward/feedback control strategy where the feedforward is determined by a dynamic generator based on exact dynamic inversion over the nominal vehicle model and the feedback is mainly issued by correcting terms proportional to the tangential and normal errors determined with respect to the vehicle's ideal trajectory. A convergence analysis of the resulting dynamic inversion based controller is established versus a vehicle's uncertain model defined via equation errors. Simulation examples highlighting the controller's performances are included.

1. Introduction

Path following of car-like vehicles has been treated using various approaches in the literature. Focusing on motion planning, i.e. on methods to derive open-loop controls to steer the vehicle on desired Cartesian paths, a particularly relevant method is the differential flatness approach of Fliess and coworkers (Rouchon *et al.* 1993, Fliess *et al.* 1995, 1999). On the other hand, path following can be approached with feedback control and in many cases the feedback strategy is derived by reduction from a trajectory tracking methodology (Dickmanns and Zapp 1987, Sampei *et al.* 1991) or by extension from a point stabilization task (Sørdalen and de Wit 1993). When better performances are sought an integrated feedforward/feedback design is in order as also pointed out in the survey of De Luca *et al.* (1998).

This paper focuses on a special path following task arising from the needs of vision-based autonomous guidance (Broggi *et al.* 1999 b, Piazzi *et al.* 2002): a given front point of a car-like vehicle that is within the look-ahead range of a stereo vision system (Bertozzi and Broggi 1998) must follow a prespecified Cartesian path. Solution to this path following problem is provided by a new feedforward/feedback control strategy where the feedforward is determined by a *dynamic generator* based on exact dynamic inversion over the nominal vehicle model and the feedback is mainly issued by correcting terms proportional to the tangential and normal errors determined with respect to the vehicle's ideal trajectory. A distinguished feature of the proposed approach is the explicit use of a vehicle's uncertain model defined via equation errors (cf. § 3). Preliminarily results on this problem have been reported in Consolini *et al.* (2001, 2002).

The paper is organized as follows. Section 2 describes the motion planning of the vehicle using dynamic inversion and the related main result is given by Theorem 1. This provides a sufficient condition to ensure the desired path following in terms of a bound over the curvature of the path to follow. Moreover, the explicit dynamic inversion based equations that generate the open-loop steering control are given in (20). Section 3 exposes the inversion based controller structure characterizing the integrated feedfoward/feedback law and a convergence result (Theorem 2) is established vs. the uncertain vehicle model (27). Simulation examples are included in § 4 and final remarks are presented in § 5.

Notation: $\|\mathbf{P}\|$ and $\mathbf{P}^{\mathrm{T}}$ will denote the Euclidean norm and the transpose of a vector $\mathbf{P}$, respectively. If I is a real open interval, we denote by $C^k(I,\mathbb{R})$ the set of all differentiable functions up to the order k, by $L^1(I,\mathbb{R})$ the set of all (Lebesgue) mesurable functions such that $\int_I |f(t)|\,\mathrm{d}t < +\infty$. We recall that a function $y: I \to \mathbb{R}$ is called absolutely continuous if y is continuous, $\dot{y}$ exists (according to the Lebesgue measure) a.e. on I and $\forall t_1, t_2 \in I$ with $t_1 < t_2$

$$\dot{y} \in L^1(]t_1, t_2[, \mathbb{R}), \qquad y(t_2) - y(t_1) = \int_{t_1}^{t_2} \dot{y}(t)\,\mathrm{d}t$$

(therefore if $y \in C^1(I,\mathbb{R})$, y is absolutely continuous).

If $\boldsymbol{\gamma}: I \to \mathbb{R}^2$ is a curve we denote by Γ, the image of I under the vectorial map $\boldsymbol{\gamma}$, i.e. $\Gamma = \boldsymbol{\gamma}(I)$. We say that the curve $\boldsymbol{\gamma}$ is regular if there exists $\dot{\boldsymbol{\gamma}}(\lambda)$ and $\dot{\boldsymbol{\gamma}}(\lambda) \neq 0$ $\forall \lambda \in I$. We write that a curve $\boldsymbol{\gamma} \in C^k(I,\mathbb{R}^2)$ if both its coordinate functions belong to $C^k(I,\mathbb{R})$. Analogously we write that $\boldsymbol{\gamma} \in L^1(I,\mathbb{R}^2)$ if both its coordinate functions belong to $L^1(I,\mathbb{R})$. Associated to every point $\boldsymbol{\gamma}(\lambda)$ of a regular curve $\boldsymbol{\gamma}$ there is the orthonormal *moving* frame $\{\boldsymbol{\tau}(\lambda), \boldsymbol{\nu}(\lambda)\}$ where $\boldsymbol{\tau}(\lambda)$ is the unit tangent vector and $\boldsymbol{\nu}(\lambda)$ is the unit normal vector oriented in such a way

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* Author for correspondence. e-mail: aurelio@ce.unipr.it
† Dipartimento di Ingegneria dell'Informazione, Università di Parma, Parco Area delle Scienze 181A, 43100 Parma, Italy.
‡ Dipartimento di Ingegneria Civile, Università di Parma, Parco Area delle Scienze 181A, 43100 Parma, Italy.

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that $\{\boldsymbol{\tau}(\lambda), \boldsymbol{\nu}(\lambda)\}$ is congruent to the (x, y)-plane. Let $\gamma \in C^1(I, \mathbb{R}^2)$, we say that γ has arc-length parameterization if $\|\dot{\gamma}(\lambda)\| = 1 \ \forall \lambda \in I$, therefore $\boldsymbol{\tau}(\lambda) = \dot{\boldsymbol{\gamma}}(\lambda)$ and, if there exists $\dot{\boldsymbol{\tau}}(\lambda)$, as known from the Frenet formulae, $\dot{\boldsymbol{\tau}}(\lambda) = \kappa(\lambda)\boldsymbol{\nu}(\lambda)$ where $\kappa(\lambda)$ is the curvature.

2. The open-loop dynamic inversion based generator

Let the motion model of a car-like vehicle be given by the following simplified non-holonomic system:

$$\left.\begin{aligned} \dot{x}(t) &= v\cos\theta(t) \\ \dot{y}(t) &= v\sin\theta(t) \\ \dot{\theta}(t) &= (v/l)\tan\delta(t) \end{aligned}\right\} \tag{1}$$

where (see figure 1) x and y are the Cartesian coordinates of the rear axle middle point $\mathbf{P}$ whose velocity norm v is supposed constant (i.e. $\|\dot{\mathbf{P}}\|(t) = v$ for any t), θ is the vehicle's heading angle, l is the inter-axle distance, and δ, the front wheel angle, is the control variable to steer the vehicle. Let the initial conditions of the above model be given by

$$x(0) = x_0, \quad y(0) = y_0, \quad \theta(0) = \theta_0 \quad \text{and} \quad \delta(0) = \delta_0$$

remark that δ_0, being the vehicle steering angle, has to be such that

$$-\pi/2 < \delta_0 < \pi/2$$

A distinguished point of the model, denoted as the 'front point' of the vehicle, is $\mathbf{Q}$ which belongs to the vehicle's symmetry axis at a fixed distance d from $\mathbf{P}$, ahead of the vehicle. This point could be a physical point of the vehicle or a virtual one belonging to the road scene as viewed in the look-ahead range by the vehicle's vision system. At any time the coordinates of $\mathbf{Q}$ are given by

$$\mathbf{Q}(t) = (x_{\mathbf{Q}}, y_{\mathbf{Q}})^{\mathrm{T}} = \mathbf{P}(t) + d(\cos\theta(t), \sin\theta(t))^{\mathrm{T}} \tag{2}$$

and its motion is governed by the system

$$\left.\begin{aligned} \dot{x}_{\mathbf{Q}}(t) &= v\cos\theta(t) - (dv/l)\sin\theta(t)\tan\delta(t) \\ \dot{y}_{\mathbf{Q}}(t) &= v\sin\theta(t) + (dv/l)\cos\theta(t)\tan\delta(t) \\ \dot{\theta}(t) &= (v/l)\tan\delta(t) \end{aligned}\right\} \tag{3}$$

Let us introduce the orthonormal frame $\{\mathbf{w}(\theta), \mathbf{z}(\theta)\}$ as a function of the vehicle's heading angle

$$\mathbf{w}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \mathbf{z}(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{4}$$

This frame can be thought as attached to the vehicle's body and is congruent to the Cartesian $\{x, y\}$ plane. With notation (4), equation (2) can be rephrased as

$$\mathbf{Q}(t) = \mathbf{P}(t) + d\mathbf{w}(\theta(t)) \tag{5}$$

A first problem can be introduced as follows.

Motion planning problem: *Given a sufficiently smooth Cartesian curve* $\boldsymbol{\gamma}$ *find sufficient conditions on* $\boldsymbol{\gamma}$ *such that there exists a continuous steering control* $\delta(t)$ *in such a way that the motion path of the front point* $\mathbf{Q}$ *exactly matches the path* Γ.

For the degenerate case $d = 0$, i.e. $\mathbf{Q} = \mathbf{P}$, the above problem has been solved in Broggi *et al.* (1999 b) by means of a closed-form solution exploiting the curvature function along the curve $\boldsymbol{\gamma}$. For the non-degenerate case $d > 0$ we can state the following result.

Theorem 1: *Let be given a curve* $\boldsymbol{\gamma}\colon [0, a] \to \mathbb{R}^2$ $(0 < a < +\infty)$ *of class* C^1 *with arc-length parameterization.*

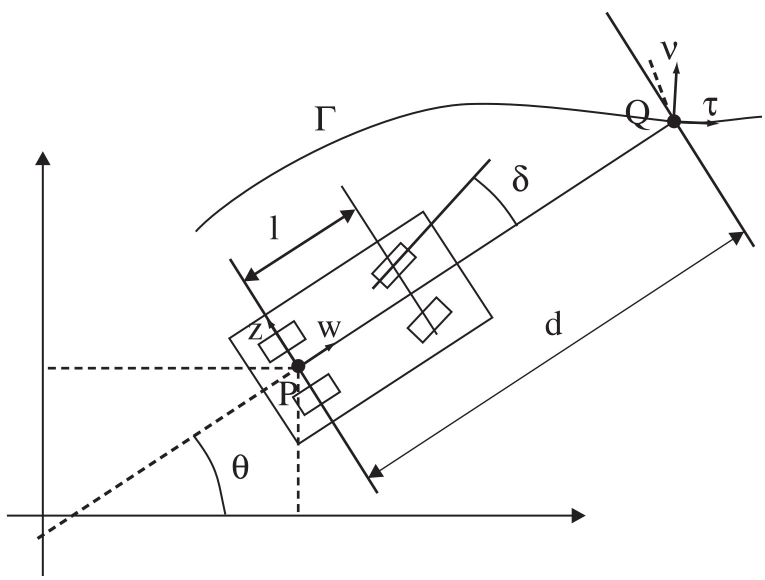


Figure 1. The car-like vehicle with the front point $\mathbf{Q}$.

Assume that $\boldsymbol{\gamma}$ *verifies the 'initial conditions'*

$$\mathbf{Q}(0) = \boldsymbol{\gamma}(0) \tag{6}$$

and

$$\arg(\dot{\boldsymbol{\gamma}}(0)) = \theta_0 + \arctan\left(\frac{d}{l}\tan(\delta_0)\right) \tag{7}$$

(remember that $-\pi/2 < \delta_0 < \pi/2$*).*

(a) *Then there exist a sufficiently small* $\bar{t} \in \mathbb{R}^+$ *and a steering function* $\delta \in C^0([0,\bar{t}\,],\mathbb{R})$ *such that the motion of the front point* $\mathbf{Q}$ *belongs to* $\Gamma = \gamma([0,a])$, *i.e.* $\mathbf{Q}(t) \in \Gamma\ \forall t \in [0,\bar{t}\,]$

(b) *Moreover suppose that* $\ddot{\boldsymbol{\gamma}}$ *there exists a.e. on* $[0,a]$ *and belongs to* $L^1(]0,a[,\mathbb{R}^2)$. *If there exists* $\epsilon > 0$ *such that the curvature* $\kappa(\lambda)$ *of* $\boldsymbol{\gamma}(\lambda)$ *satisfies the condition*

$$\sup_{\lambda\in[0,a]}\left\{\alpha_0^+\,\mathrm{e}^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^+(\lambda-s)\mathrm{e}^{-(2/\pi d)\lambda}\,\mathrm{d}s,\right.$$
$$\left. -\alpha_0^-\mathrm{e}^{-(2/\pi d)\lambda} - \int_0^\lambda \kappa^-(\lambda-s)\mathrm{e}^{-(2/\pi d)\lambda}\,\mathrm{d}s\right\} \le \frac{\pi}{2} - \epsilon \tag{8}$$

(where $\alpha_0 = \arg(\dot{\boldsymbol{\gamma}}(0)) - \theta_0$ *and* α_0^+, α_0^-, $\kappa^+(\lambda)$, $\kappa^-(\lambda)$ *are, respectively, the positive and negative parts of* α_0 *and the curvature* $\kappa(\lambda)$*), then there exist* $t_f \in \mathbb{R}^+$ *and a steering function* $\delta \in C^0([0,t_f],\mathbb{R})$ *such that the point* $\mathbf{Q}(t)$ *exactly covers the entire path* Γ, *i.e.* $\mathbf{Q}([0,t_f]) = \Gamma$.

In particular, if

$$|\kappa(\lambda)| \le 1/d, \quad (\textit{Lebesgue})\ \textit{a.e. on}\ [0,a] \tag{9}$$

then (8) *holds.*

The invertibility conditions appearing in the above theorem have a simple geometrical interpretation. In addition to the obvious necessary condition that the front point $\mathbf{Q}(0)$ has to coincide with the starting point of the curve $\boldsymbol{\gamma}$, the exact motion planning is possible, at least for a while, if at the initial time, the tangent to the curve has the same direction as the one that the vehicle would have, with δ_0 as steering angle. Moreover if condition (8) holds (this is the case, for instance, if the modulus of the curvature along $\boldsymbol{\gamma}$ is less than $1/d$), then the entire path Γ can be followed by a suitable steering input. This input will be determined by the dynamic inversion based generator exposed below, see equations (20).

Remark 1: The two terms inside the brackets of (8) may be interpreted as the outputs of a 1-pole low-pass filter (in the arc-length domain) with transfer function $1/(s+(2/\pi d))$, inputs $\kappa^+(\lambda)$, $\kappa^-(\lambda)$ respectively; this means that the absolute value of the curvature is allowed to be greater than $1/d$ but just for subsets of the arc-length variable of small measure.

A straightforward consequence of point (b) of Theorem 1 is expressed by the following corollary.

Corollary 1: *Let a curve* $\boldsymbol{\gamma}\colon [0,+\infty[\to \mathbb{R}^2$ *of class* C^1 *be given with arc-length parameterization such that*:

(α) *there exists an increasing sequence* $\{a_n\}$, *such that* $a_0 = 0 < a_1 < \cdots < a_n < \cdots$, $\lim_{n\to+\infty} a_n = +\infty$ *and* $\boldsymbol{\gamma} \in C^2(]a_{i-1},a_i[,\mathbb{R}^2)$, $\forall i = 1,\ldots,n,\ldots$, *(clearly this is the case if, for instance,* $\boldsymbol{\gamma} \in C^2([0,+\infty[,\mathbb{R}^2)$*).*

Assume that $\boldsymbol{\gamma}$ *verifies the initial conditions* (6) *and* (7). *If the curvature* $\kappa(\lambda)$ *satisfies the inequality*

$$|\kappa(\lambda)| \le \frac{1}{d}, \qquad \textit{a.e. on}\ [0,+\infty[$$

then there exists a steering function $\delta \in C^0([0,+\infty[,\mathbb{R})$ *such that the point* $\mathbf{Q}(t)$ *exactly covers the entire path* Γ, *i.e.* $\mathbf{Q}([0,+\infty)) = \Gamma$.

The proof of Theorem 1 needs the following lemma:

Lemma 1: *Let be given a curve* $\boldsymbol{\gamma}\colon [0,a] \to \mathbb{R}^2$ $(0 < a < +\infty)$ *of class* C^1 *with arc-length parameterization, i.e.* $\boldsymbol{\tau}(\lambda) = \dot{\boldsymbol{\gamma}}(\lambda)\ \forall\lambda \in [0,a]$. *Suppose there exist functions* $\sigma \in C^1(\mathcal{I},\mathbb{R})$ *and* $\mu \in C^1(\mathcal{I},[0,a])$ *where* $\mathcal{I}$ *is a suitable right (interval) neighbourhood of zero and such that for all* $t \in \mathcal{I}$

$$\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t)) > 0 \tag{10}$$

and

$$\left.\begin{aligned} \dot{\mu}(t) &= v\frac{1}{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))} \\ \dot{\sigma}(t) &= \frac{v}{d}\frac{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{z}(\sigma(t))}{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))} \end{aligned}\right\} \tag{11}$$

with initial conditions $\mu(0) = 0$ *and* $\sigma(0) = \theta_0$.

Assume also that relation (6) *holds, i.e.* $\boldsymbol{\gamma}(0) = \mathbf{P}(0) + d\mathbf{w}(\theta_0)$, *and the steering input of model* (1) *be given by*

$$\delta(t) = \arctan\left(\frac{l}{v}\dot{\sigma}(t)\right) \quad \forall t \in \mathcal{I}. \tag{12}$$

Then we have that

$$\mathbf{Q}(t) = \boldsymbol{\gamma}(\mu(t)) \qquad \forall t \in \mathcal{I}. \tag{13}$$

Proof: By the hypotheses, we get that

$$\mathbf{Q}\ \text{and}\ \boldsymbol{\gamma}\circ\mu \in C^1(\mathcal{I},\mathbb{R}^2), \qquad \mathbf{Q}(0) = \boldsymbol{\gamma}\circ\mu(0)$$

and $\forall t \in \mathcal{I}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\gamma}\circ\mu)(t) &= \boldsymbol{\tau}(\mu(t))\dot{\mu}(t) = [\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))\mathbf{w}(\sigma(t)) \\ &\quad + \boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{z}(\sigma(t))\mathbf{z}(\sigma(t))]\frac{v}{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))} \\ &= v\mathbf{w}(\sigma(t)) + \mathbf{z}(\sigma(t))\dot{\sigma}(t) = \dot{\mathbf{Q}}(t)\end{aligned}$$

remark that $\mathbf{Q(t)} = \mathbf{P(t)} + d\mathbf{w}(\sigma(t))$ since $\sigma(t) = \theta(t)$, $\forall t \in \mathcal{I}$. Therefore by the fundamental theorem of integral calculus, we obtain equation (13). □

Proof of Theorem 1: By the hypothesis (7) we have that

$$\boldsymbol{\tau}(0)^{\mathrm{T}}\mathbf{w}(0) > 0$$

since $\boldsymbol{\tau}(0)^{\mathrm{T}}\mathbf{w}(0) = \cos(\dot{\boldsymbol{\gamma}}(0) - \theta_0)$. Therefore there exists $\bar{\epsilon} > 0$ such that

$$\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\sigma) > 0, \qquad \forall(\lambda,\sigma) \in \mathcal{W} := [0,\bar{\epsilon}] \times [\theta_0 - \bar{\epsilon}, \theta_0 + \bar{\epsilon}].$$

Therefore the function

$$\mathbf{F}\colon \mathcal{W} \to \mathbb{R}^2, \text{ defined by}$$
$$\mathbf{F}(\lambda,\sigma) := \left(\frac{v}{\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\sigma)}, \frac{v}{d}\frac{\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{z}(\sigma)}{\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\sigma)}\right)$$

is a well defined Lipschitz map and $(0,\theta_0)$ belongs to $\mathcal{W}$. Consequently, by the theorem of local existence and uniqueness for ordinary differential equations, there exist $\bar{t} > 0$ and one and only one pair of functions $\mu \in C^1([0,\bar{t}\,],[0,a])$ and $\sigma \in C^1([0,\bar{t}\,],\mathbb{R})$ such that

$$\left.\begin{aligned}&(\dot{\mu}(t),\dot{\sigma}(t)) = \mathbf{F}(\mu(t),\sigma(t)) \quad \forall t \in [0,\bar{t}\,] \\ &(\mu(0),\sigma(0)) = (0,\theta_0)\end{aligned}\right\} \tag{14}$$

Therefore it is sufficient to apply Lemma 1 with $\mathcal{I} = [0,\bar{t}\,]$ to prove point (a).

Let $[0,t_f[$ be the maximum right set of existence of the solution pair (μ,σ) found before, in accordance with condition

$$\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t)) > 0 \qquad \forall t \in [0,t_f[\tag{15}$$

To prove part (b) it suffices to show that $\mu([0,t_f]) = [0,a]$ which holds if $\mu([0,t_f[) = [0,a[$, being μ a monotone increasing function since

$$\dot{\mu}(t) = \frac{v}{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))} > 0, \quad \forall t \in [0,t_f[$$

by condition (15). By a contradiction argument on the definition of t_f, it is sufficient to prove that

$$\begin{aligned}\exists c > 0 :\ &\sup_{0\le t<t_f}\left\{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))\right\} \\ &= \sup_{0\le\lambda<\bar{\lambda}}\left\{\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\mu^{-1}(\lambda))\right\} \ge c\end{aligned} \tag{16}$$

where $\bar{\lambda} = \sup\mu([0,t_f])$.

To this aim, set $\alpha(\lambda) = \beta(\lambda) - \sigma(\mu^{-1}(\lambda))$, $\forall\lambda \in [0,\bar{\lambda}[$, where $\beta(\lambda) := \arg(\boldsymbol{\tau}(\lambda))$; then $\dot{\beta}(\lambda) = \kappa(\lambda)$, the curvature of $\boldsymbol{\gamma}$ at λ. Since $\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\mu^{-1}(\lambda)) = \cos(\alpha(\lambda))$, to prove equation (16), it suffices to show that there exists $\bar{\alpha} : 0 < \bar{\alpha} < \pi/2$ such that $|\alpha(\lambda)| \le \bar{\alpha}\ \forall\lambda \in [0,\bar{\lambda}[$. Now by equations (11)

$$\begin{aligned}\dot{\alpha}(\lambda) &= \frac{\mathrm{d}\beta(\lambda)}{\mathrm{d}\lambda} - \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda} = -\frac{1}{d}\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{z}(\sigma(\mu^{-1}(\lambda))) + \kappa(\lambda) \\ &= -\frac{1}{d}\sin(\alpha(\lambda)) + \kappa(\lambda)\end{aligned}$$

almost everywhere in $[0,\lambda[$, since

$$\begin{aligned}\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{z}(\sigma(\mu^{-1}(\lambda))) &= [\cos(\beta(\lambda))\ \sin(\beta(\lambda))] \\ &\quad \times [-\sin(\sigma(\mu^{-1}(\lambda)))\ \cos(\sigma(\mu^{-1}(\lambda)))]^{\mathrm{T}} \\ &= \sin(\beta(\lambda) - \sigma(\mu^{-1}(\lambda))) \\ &= \sin(\alpha(\lambda))\end{aligned}$$

Therefore $\alpha(\lambda)$ verifies the initial value problem

$$\left.\begin{aligned}&\dot{\alpha}(\lambda) = -\frac{1}{d}\sin(\alpha(\lambda)) + \kappa(\lambda) \quad \text{a.e. on } [0,\bar{\lambda}[\\ &\alpha(0) = \beta(0) - \theta_0 = \alpha_0\end{aligned}\right\} \tag{17}$$

remark that $\alpha_0 = \arctan((d/l)\tan\delta_0)$.

Let $\alpha^+\colon [0,\bar{\lambda}[\to \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \ge 0\}$, $\alpha^+(\lambda) := \max\{\alpha(\lambda),0\}$, $\alpha_0^+ = \max\{\alpha(0),0\}$. The function α^+ is absolutely continuous and, by (17), the following differential inequality holds almost everywhere in $[0,\bar{\lambda}[$

$$\frac{\mathrm{d}\alpha^+(\lambda)}{\mathrm{d}\lambda} \le -\frac{2}{\pi d}\alpha^+(\lambda) + \kappa^+(\lambda)$$

and $\alpha^+(0) = \alpha_0^+$, where $\kappa^+(\lambda) = \max\{\kappa(\lambda),0\}$. Hence

$$\begin{aligned}\alpha^+(\lambda) &\le \alpha_0^+\mathrm{e}^{-(2/\pi d)\lambda} \\ &\quad + \int_0^\lambda \kappa^+(\lambda - s)\,\mathrm{e}^{-(2/\pi d)s}\mathrm{d}s, \quad \forall\lambda \in [0,\bar{\lambda}[\end{aligned} \tag{18}$$

Likewise we obtain

$$\alpha^-(\lambda) \ge \alpha_0^-\mathrm{e}^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^-(\lambda - s)\mathrm{e}^{-(2/\pi d)s}\mathrm{d}s, \quad \forall\lambda \in [0,\bar{\lambda}[\tag{19}$$

where $\alpha^-(\lambda) = \min\{\alpha(\lambda),0\}$, $\kappa^-(\lambda) = \min\{\kappa(\lambda),0\}$ and $\alpha_0^- = \min\{\alpha_0,0\}$. Obviously $\alpha^-(\lambda) \le \alpha(\lambda) \le \alpha^+(\lambda)$ so that, from (18) and (19), it follows that

$$\begin{aligned}&\alpha_0^-\mathrm{e}^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^-(\lambda - s)\mathrm{e}^{-(2/\pi d)s}\,\mathrm{d}s \\ &\quad \le \alpha(\lambda) \le \alpha_0^+\mathrm{e}^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^+(\lambda - s)\mathrm{e}^{-(2/\pi d)s}\,\mathrm{d}s.\end{aligned}$$

Therefore if we set

$$c_0 = \sup_{\lambda \in [0,\bar{\lambda}[} \left\{ \alpha_0^+ e^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^+(\lambda - s) e^{-(2/\pi d)s}\, ds, \right.$$
$$\left. -\alpha_0^- e^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^+(\lambda - s) e^{-(2/\pi d)s}\, \mathrm{d}s \right\}$$

we obtain by (8) that $0 \le c_0 < \pi/2$, hence $-c_0 < \alpha(\lambda) < c_0$, $\forall \lambda \in [0, \bar{\lambda}[$.

In conclusion equation (16) holds with $c = \cos(c_0)$.

Finally, if $|\kappa(\lambda)| \le 1/d$ almost everywhere on $[0, a]$, it follows that

$$\sup_{\lambda \in [0,\bar{\lambda}[} \left\{ \alpha_0^+ e^{-(2/\pi d)\lambda} + \int_0^\lambda \kappa^+(\lambda - s) e^{-(2/\pi d)s}\, \mathrm{d}s, \right.$$
$$\left. -\alpha_0^- e^{-(2/\pi d)\lambda} - \int_0^\lambda \kappa^-(\lambda - s) e^{-(2/\pi d)s}\, \mathrm{d}s \right\}$$
$$\le \frac{\pi}{2} - e^{-(2/\pi d)\bar{\lambda}} \left(\frac{\pi}{2} - |\alpha_0| \right) \le \frac{\pi}{2} - \epsilon$$

for a suitable $\epsilon > 0$, therefore (9) holds and this completes the proof of part (b). □

2.1. *Open-loop steering control*

Suppose, for simplicity, that the desired curve $\boldsymbol{\gamma} \in C^2([0, +\infty[, \mathbb{R}^2)$ with an arc-length parameterization for which at time $t = 0$

$$\boldsymbol{\gamma}(0) = \mathbf{Q}(0)$$

and

$$\arg(\dot{\boldsymbol{\gamma}}(0)) = \theta_0 + \arctan\left(\frac{d}{l} \tan(\delta_0) \right)$$

and the curvature satisfies $|\kappa(\lambda)| < 1/d \ \forall \lambda \in [0, +\infty[$.

Suppose that $\boldsymbol{\gamma}(\lambda) = (\xi(\lambda), \eta(\lambda))^{\mathrm{T}}$, $\forall \lambda \in [0, +\infty[$, then the previous results permit us to determine the open-loop control law to solve the posed motion planning problem by means of the following dynamic inversion based generator (see figure 2)

$$\left. \begin{aligned} \dot{\mu}(t) &= \frac{v}{\dot{\xi}(\mu(t))\cos(\sigma(t)) + \dot{\eta}(\mu(t))\sin(\sigma(t))}, \quad \mu(0) = 0 \\ \dot{\sigma}(t) &= \frac{v}{d} \frac{\dot{\eta}(\mu(t))\cos(\sigma(t)) - \dot{\xi}(\mu(t))\sin(\sigma(t))}{\dot{\xi}(\mu(t))\cos(\sigma(t)) + \dot{\eta}(\mu(t))\sin(\sigma(t))}, \quad \sigma(0) = \theta_0 \\ \delta(t) &= \arctan\left(\frac{l}{v} \dot{\sigma}(t) \right) \end{aligned} \right\} \tag{20}$$

Then the above steering command generator determines δ in such a way that $\mathbf{Q}(t) = \boldsymbol{\gamma}(\mu(t)) \ \forall t > 0$, i.e. the front point $\mathbf{Q}$ exactly and indefinitely follows the path Γ.

It is worth noting that the internal states of the generator (20) have a special meaning: $\mu(t)$ is the distance covered at time t by point $\mathbf{Q}$ along path $\boldsymbol{\gamma}$ and $\sigma(t) \equiv \theta(t)$ $\forall t > 0$. In such a way, the given generator can be interpreted as a control law based on a special open-loop observer of model (1).

The equations of generator (20) are particularly appropriate for real-time implementation; an alternative approach to obtain the open-loop steering $\delta(t)$ for a given curve $\gamma(\lambda)$ is the following procedure:

- Solve (17), i.e.

$$\dot{\alpha}(\lambda) = -\frac{1}{d}\sin(\alpha(\lambda)) + \kappa(\lambda), \quad \alpha(0) = \arg(\dot{\boldsymbol{\gamma}}(0)) - \theta_0, \tag{21}$$

  to obtain function $\alpha(\lambda)$.
- Find $\mu(t)$ by inverting the monotone function $t(\mu)$ defined by

$$t(\mu) = \int_0^\mu \frac{\cos(\alpha(\lambda))}{v} \mathrm{d}\lambda \tag{22}$$

  this equations comes from (11); note that $t(\mu)$ is evidently monotone since $\cos(\alpha(\lambda)) \ge 0$, being $-\pi/2 < \alpha(\lambda) < \pi/2$ (remark that $|\kappa(\lambda)| < 1/d$).
- Compute $\delta(t)$ according to

$$\delta(t) = \arctan\left(\frac{l}{d} \tan(\alpha(\mu(t))) \right) \tag{23}$$

  indeed from (11) we obtain $\dot{\sigma}(t) = (v/d) \times \tan(\alpha(\mu(t)))$ and then (23) from (12).

2.2. *Open-loop path following examples*

This section reports three examples of path following using open-loop control. They refer to a straight line, a circle arc in which different cases are considered according to its curvature, and a quintic spline whose solution is gained through numerical integration.

(1) The path to be followed is a straight line parallel to the x axis and, initially, the symmetry axis of the vehicle form the angle θ_0 with the path itself. In this case $\boldsymbol{\gamma}(\lambda) = (\lambda, 0)^{\mathrm{T}}$, $\dot{\boldsymbol{\gamma}}(\lambda) = (1, 0)^{\mathrm{T}}$,

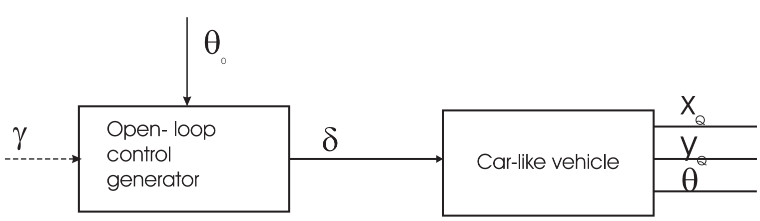
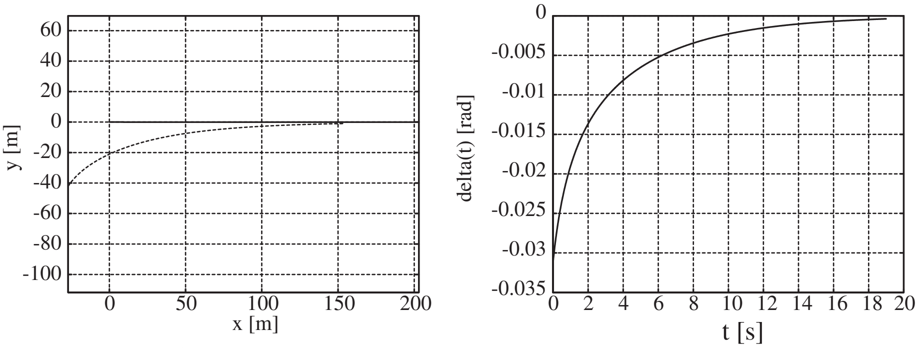


Figure 2. The dynamic inversion based generator.



Paths of points P(t) and Q(t)

Function $\delta(t)$

Figure 3. Straight line.

$\forall\lambda \geq 0$, hence the differential equations (20) become the decoupled system

$$\left.\begin{aligned} \dot{\mu}(t) &= v\frac{1}{\cos\sigma(t)}, & \mu(0) &= 0 \\ \dot{\sigma}(t) &= -\frac{v}{d}\frac{\sin\sigma(t)}{\cos\sigma(t)}, & \sigma(0) &= \theta_0 \end{aligned}\right\} \tag{24}$$

The solution of the second equation is

$$\sigma(t) = \arcsin(\sin\theta_0 \mathrm{e}^{-(v/d)t})$$

so that we obtain the steering law:

$$\delta(t) = \arctan\left(\frac{l}{v}\arcsin(\sin\theta_0\, \mathrm{e}^{-(v/d)t})\right)$$

In the first graph of figure 3 the continuous line represents the path that point $\mathbf{Q}(t)$ follows and the dashed line represents the path of point $\mathbf{P}(t)$. The second plot shows the steering function $\delta(t)$.

(2) We consider a circular path of constant curvature $\kappa > 0$ (and radius $= 1/\kappa$), infinite in the sense that the same shape is outlined over and over again. Initially the symmetry axis of the vehicle is parallel to the tangent at the point $\mathbf{Q}(0)$ of the circle, hence $\alpha_0 = 0$. In this case we follow the procedure outlined in (21)–(23). In (21), the sought function $\alpha(\lambda)$ represents the angle between the tangent to the circle at the point $\boldsymbol{\gamma}(\lambda)$ and the symmetry axis of the vehicle.

The solution of (21) can be found via the implicit formula

$$\int_0^{\alpha(\lambda)} \frac{1}{\kappa - (1/d)\sin\alpha}\,\mathrm{d}\alpha = \int_0^{\lambda} \mathrm{d}l \tag{25}$$

If the substitution $x = \tan(\alpha/2)$ is performed to evaluate the left-hand integral of (25) we obtain

$$\frac{2}{\kappa}\int_0^{\tan(\alpha(\lambda)/2)} \frac{1}{x^2 - (2/d\kappa)x + 1}\,\mathrm{d}x = \lambda \tag{26}$$

The following cases are considered:

($\kappa < 1/d$) In this case the existence of the steering function is guaranteed by Corollary 1. Let x_1, x_2, with $0 < x_1 < x_2$ be the real solutions of equation

$$x^2 - \frac{2}{d\kappa}x + 1 = 0$$

then we deduce

$$\alpha(\lambda) = 2\arctan\left(\frac{x_1x_2(1 - \mathrm{e}^{(\kappa/2)(x_2-x_1)\lambda})}{x_1 - x_2\mathrm{e}^{(\kappa/2)(x_2-x_1)\lambda}}\right)$$

The further steps to compute $\delta(t)$, cf. (22) and (23), are performed numerically. The results are depicted in figure 4. Note that $\lim_{\lambda\to+\infty}\alpha(\lambda) = 2\arctan(x_1) =: \bar{\alpha}$ and the vehicle point $\mathbf{P}$ converges to a circle of radius $r_\infty = (1/\kappa)\cos|\bar{\alpha}| = \sqrt{(1/\kappa^2) - d^2}$ when $t \to \infty$.

($\kappa = 1/d$) Solution to (21) is

$$\alpha(\lambda) = 2\arctan\left(\frac{\kappa\lambda}{2 + \kappa\lambda}\right)$$

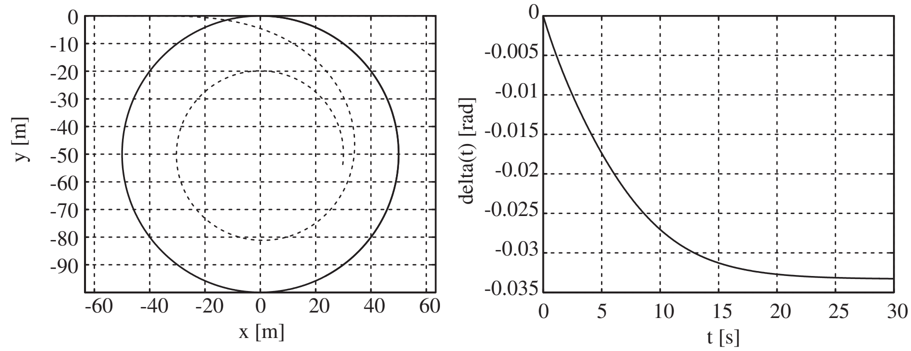
and note that $\lim_{\lambda\to+\infty}\alpha(\lambda) = \pi/2$. The path described by point $\mathbf{P}$ is a spiral converging to the centre of the circle as $\lambda \to +\infty$.

($\kappa > 1/d$) In this case the existence and uniqueness of the control function are guaranteed only in a neighbourhood of the initial point. We show that, actually, the function $\alpha(\lambda)$ exists only in an open interval. This means that the given path cannot be followed entirely. We obtain

$$\alpha(\lambda) = 2\arctan \times \frac{1 + \sqrt{\kappa^2d^2 - 1}\tan\left(\dfrac{\lambda\sqrt{\kappa^2d^2-1}}{d\kappa} - \arctan\dfrac{1}{\sqrt{\kappa^2d^2-1}}\right)}{d\kappa}$$

This equation has a singularity for

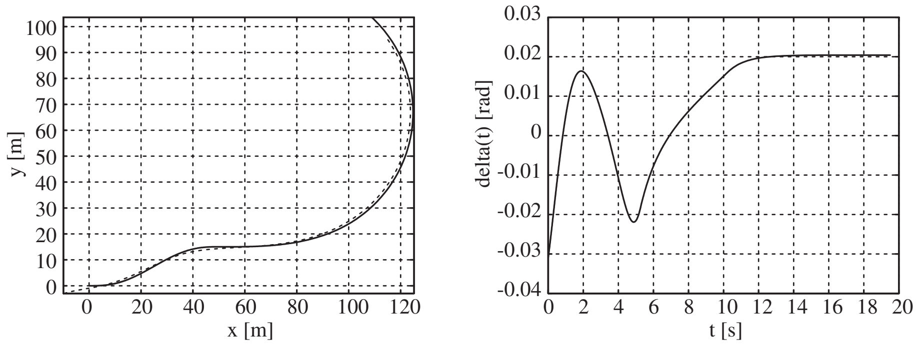
$$\bar{\lambda} = \frac{\pi}{2}\frac{d}{\sqrt{\kappa^2d^2 - 1}}$$



Paths of points P (t) and Q(t)

Function $\delta(t)$

Figure 4. Circle.



Paths of points P(t) and Q(t)

Function $\delta(t)$

Figure 5. Quintic spline.

and the function $\alpha(\lambda)$ exists only in the interval $[0, \bar{\lambda}[$.

(3) Although a closed form solution of (11) can be found only for very simple trajectories, such as the ones discussed in the first two examples, more complex cases can be solved through numerical integration. For instance, we show in figure 5 the graphs obtained for a quintic spline, a curve obtained by a fifth degree polynomial (Piazzi *et al.* 2002).

3. The dynamic inversion based controller

In this section, as usual, we suppose that $\boldsymbol{\gamma} \in C^1([0, +\infty[, \mathbb{R}^2)$ with arc-length parameterization and we denote by

$$\mathbf{E} = \mathbf{E}(\mathbf{Q}, \boldsymbol{\gamma}(\lambda)) = \mathbf{Q} - \boldsymbol{\gamma}(\lambda)$$

the 'error vector' of $\mathbf{Q}$ with respect to the point $\boldsymbol{\gamma}(\lambda)$. Therefore it is natural to name absolute error $\mathcal{E}(\mathbf{Q})$ the distance between $\mathbf{Q}$ and the path $\Gamma = \boldsymbol{\gamma}([0, +\infty[)$ in the following way

$$\mathcal{E}(\mathbf{Q}) = \inf_{\lambda \in [0,+\infty[} \|\mathbf{E}(\mathbf{Q}, \boldsymbol{\gamma}(\lambda))\|$$

Now we consider the following uncertain model for the car-like vehicle

$$\left.\begin{aligned} \dot{x}(t) &= v\cos\theta(t) + e_x(t) \\ \dot{y}(t) &= v\sin\theta(t) + e_y(t) \\ \dot{\theta}(t) &= (v/l)\tan\delta(t) + e_\theta(t) \end{aligned}\right\} \tag{27}$$

with initial conditions $x(0) = x_0$, $y(0) = y_0$, $\theta(0) = \theta_0$ and $\delta(0) = \delta_0$ where the functions $e_x, e_y, e_\theta \in C^1([0, +\infty[)$ and satisfy the following bounds

$$\|e_x\|_\infty \leq M_x, \quad \|e_y\|_\infty \leq M_y, \quad \|e_\theta\|_\infty \leq M_\theta \tag{28}$$

Since the coordinates of the front point $\mathbf{Q}$ are given by (2), the perturbed motion of $\mathbf{Q} = (x_{\mathbf{Q}}, y_{\mathbf{Q}})^{\mathrm{T}}$ is governed by the system

$$\left.\begin{aligned} \dot{x}_{\mathbf{Q}} &= v\cos\theta - (dv/l)\sin\theta\tan\delta + e_x - d(\sin\theta)e_\theta \\ \dot{y}_{\mathbf{Q}} &= v\sin\theta + (dv/l)\cos\theta\tan\delta + e_y + d(\cos\theta)e_\theta \\ \dot{\theta} &= (v/l)\tan\delta + e_\theta \end{aligned}\right\} \tag{29}$$

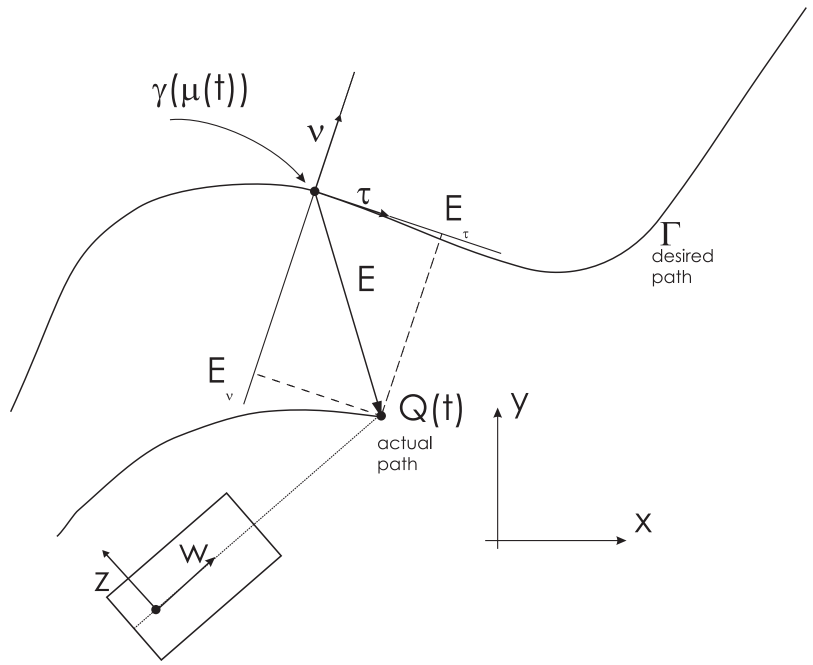


Figure 6. The error vector and its components.

Using the open-loop control strategy of the previous section, due to the modelling errors introduced in (27), the actual position $\mathbf{Q}(t)$ of the front point at time t may be different from the *estimated position* $\boldsymbol{\gamma}(\mu(t))$ as shown in figure 6. Therefore, in order to decrease the distance of $\mathbf{Q}(t)$ from $\boldsymbol{\gamma}(\mu(t))$, we modify the equations of the generator (20) by means of correcting feedback terms which use the error components $E_\tau(t)$ and $E_\nu(t)$ defined with respect to the moving frame $\{\boldsymbol{\tau}(\mu(t)), \mathbf{v}(\mu(t))\}$

$$E_\tau = E_\tau(t) = \mathbf{E}(\mathbf{Q}(t), \boldsymbol{\gamma}(\mu(t)))^{\mathrm{T}}\boldsymbol{\tau}(\mu(t))$$
$$E_\nu = E_\nu(t) = \mathbf{E}(\mathbf{Q}(t), \boldsymbol{\gamma}(\mu(t)))^{\mathrm{T}}\mathbf{v}(\mu(t))$$

The overall feedback strategy is then given by the dynamic inversion based controller

$$\left.\begin{aligned} \dot\mu &= \frac{v}{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{w}(\sigma)} + K_\tau E_\tau \\ \dot\sigma &= \frac{v}{d}\frac{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{z}(\sigma)}{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{w}(\sigma)} - K_\nu E_\nu + K_\theta(\theta-\sigma) \\ \delta &= \arctan\left(\frac{l}{v}\left(\frac{v}{d}\frac{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{z}(\sigma)}{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{w}(\sigma)} - K_\nu E_\nu\right)\right) \end{aligned}\right\} \tag{30}$$

with initial conditions $\mu(0)=0$, $\sigma(0)=\theta_0$, where K_τ, K_ν, K_θ are positive feedback-gain constants. The corresponding control architecture is depicted in figure 7.

The closed-loop system equations are then given by

$$\left.\begin{aligned} \dot x_{\mathbf{Q}} &= v\cos\theta - d(\sin\theta)u + e_x - d(\sin\theta)e_\theta \\ \dot y_{\mathbf{Q}} &= v\sin\theta + d(\cos\theta)u + e_y + d(\cos\theta)e_\theta \\ \dot\theta &= u + e_\theta \\ \dot\mu &= \frac{v}{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{w}(\sigma)} + K_\tau E_\tau \\ \dot\sigma &= u + K_\theta(\theta-\sigma) \\ u &= \frac{v}{d}\frac{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{z}(\sigma)}{\boldsymbol{\tau}(\mu)^{\mathrm{T}}\mathbf{w}(\sigma)} - K_\nu E_\nu \end{aligned}\right\} \tag{31}$$

with initial condition

$$x_{\mathbf{Q}}(0) = x_0 + d\cos\theta_0, \quad y_{\mathbf{Q}}(0) = y_0 + d\sin\theta_0,$$
$$\theta(0)=\theta_0, \quad \sigma(0)=\theta_0, \quad \mu(0)=0 \tag{32}$$

Furthermore we suppose that $\boldsymbol{\gamma}$ verifies the same property (α) of Corollary 1:

(α) there exists an increasing sequence $\{a_n\}$, such that $a_0 = 0 < a_1 < \cdots < a_n < \cdots$, $\lim_{n\to+\infty} a_n = +\infty$ and $\boldsymbol{\gamma} \in C^2(]a_{i-1}, a_i[, \mathbb{R}^2)$, $\forall i = 1,\ldots,n,\ldots$, (clearly this is the case if, for instance, $\boldsymbol{\gamma} \in C^2([0,+\infty[, \mathbb{R}^2))$.

Set

$$\bar\kappa = \inf\{\tilde\kappa \mid \text{such that } |\kappa(\lambda)| \le \tilde\kappa \text{ a.e. on }]0,+\infty[\}$$

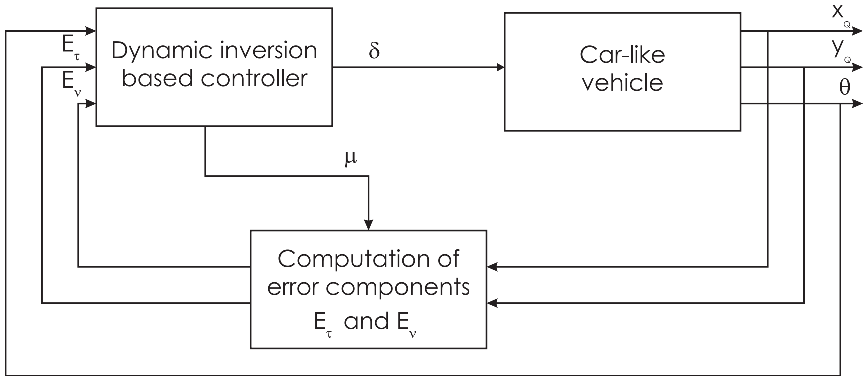


Figure 7. Control architecture.

the 'essential maximum' of the absolute value of curvature of $\boldsymbol{\gamma}$ and

$$M = \|(M_x, M_y)^{\mathrm{T}}\|$$

Theorem 2: *In the previous hypotheses and notations, assume that* $\boldsymbol{\gamma}$ *verifies the 'initial conditions'*

$$\boldsymbol{\gamma}(0) = \mathbf{Q}(0) \tag{33}$$

and

$$\arg(\dot{\boldsymbol{\gamma}}(0)) = \theta_0 + \arctan\left(\frac{d}{l}\tan(\delta_0)\right) \tag{34}$$

(remember that $-\pi/2 < \delta_0 < \pi/2$*).*

If the following inequalities hold

$$v > 2(M_\theta d + M), \qquad d\bar{\kappa} + \frac{4M_\theta d + 3M}{v - 2(M_\theta d + M)} < 1 \tag{35}$$

then we can find a suitable constant $\bar{K}_\theta > 0$ *such that* $\forall K_\tau > 0$, $\forall K_\nu > 0$ *and* $\forall K_\theta \geq \bar{K}_\theta$ *there exists one and only one solution defined on* $[0, +\infty[$ *of the closed-loop system* (31) *with initial conditions* (32) *and with the time-varying errors satisfying* (28), *i.e. the feedback control strategy* (30) *is well posed for the entire family of uncertain models* (27).

Moreover, for any given $\epsilon > 0$, *there exist suitable positive constants* $\bar{K}_\tau$ *and* $\bar{K}_\nu$ *such that, if* $K_\tau \geq \bar{K}_\tau$, $K_\nu \geq \bar{K}_\nu$ *and* $K_\theta \geq \bar{K}_\theta$ *then*

$$\mathcal{E}(\mathbf{Q}(t)) < \epsilon, \qquad \forall t \geq 0 \tag{36}$$

(Remark that we find again that $\mathcal{E}(\mathbf{Q}(t)) = 0\ \forall t \geq 0$, *if* $M_0 = M = 0$*.)*

Remark 2: The above theorem guarantees a robust path following stability of the proposed controller. Indeed, the front point $\mathbf{Q}$ remains arbitrarily near to the desired path Γ provided that the feedback gains are sufficiently high and the modelling errors obey to conditions (35); moreover it is not restrictive to suppose that $\gamma(0) = Q(0)$ (i.e. $\mathcal{E}(Q(0)) = 0$) because in the case of the car-like vehicle is always possible to plan a trajectory γ whose starting point is exactly the initial position of the vehicle.

Remark 3: It is interesting to note that the proposed inversion based control architecture combines the (feedforward) dynamic inversion and the feedback action in a novel manner. Indeed, the proposed control strategy uses feedback corrections on the equations of the dynamic inversion generator directly, whereas in analogous control scheme known in the literature (Devasia *et al.* 1996, Hunt and Meyer 1997) the inversion based generator is not affected by the feedback.

In the proof of Theorem 2 we use the following lemma.

Lemma 2: *Let it be given a solution of system* (31) *defined in a closed interval* $[0, \bar{t}\,]$, *with the error functions satisfying bounds* (28) *and R a positive constant such that* $\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t)) \geq R\ \forall t \in [0, \bar{t}\,]$. *Provided that* $K_\theta > M_\theta/R$ *it follows that* $\forall t \in [0, \bar{t}\,]$

$$|E_\tau(t)| \leq \frac{1}{K_\tau}\left(\frac{vM_\theta}{K_\theta}\left(1 + \frac{1}{R}\right) + M_\theta d + M\right) \times \left(1 + \frac{(dM_\theta/K_\theta) + 1}{R - (M_\theta/K_\theta)}\right)$$

$$|E_\nu(t)| \leq \frac{1}{K_\nu}\left(\frac{vM_\theta}{K_\theta}\left(1 + \frac{1}{R}\right) + M_\theta d + M\right) \times \frac{1}{d(R - (M_\theta/K_\theta))} \tag{37}$$

Proof: First of all we note that from equations (31), we can write

$$\frac{\mathrm{d}(\theta - \sigma)}{\mathrm{d}t} = e_\theta - K_\theta(\theta - \sigma)$$

which implies, being $|e_\theta| \leq M_\theta$, that $|\theta - \sigma| \leq M_\theta/K_\theta$.

Now we write the first two equations of system (31) as a vector equation

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = v\mathbf{w}(\theta) + d\mathbf{z}(\theta)(u + e_\theta) + \boldsymbol{e}$$

where $\boldsymbol{e} = (e_x, e_y)^{\mathrm{T}}$. From the definition of error vector, $\mathbf{E} = \mathbf{Q} - \gamma(\mu)$, we get

$$
\begin{aligned}
\dot{\mathbf{E}} &= \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} - \boldsymbol{\tau}(\mu)\dot{\mu} \\
&= v\mathbf{w}(\theta) + d\mathbf{z}(\theta)\left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} - K_\nu E_\nu + e_\theta\right) + \boldsymbol{e} - \boldsymbol{\tau}(\mu)\dot{\mu} \\
&= v[\mathbf{w}(\theta) + \mathbf{w}(\sigma) - \mathbf{w}(\sigma)] \\
&\quad + d\left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} - K_\nu E_\nu + e_\theta\right)[\mathbf{z}(\theta) + \mathbf{z}(\sigma) - \mathbf{z}(\sigma)] \\
&\quad + \boldsymbol{e} - \boldsymbol{\tau}(\mu)\left[\frac{v}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} + K_\tau E_\tau\right] \\
&= v\left(\frac{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)\mathbf{w}(\sigma) + \mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)\mathbf{z}(\sigma) - \boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}\right) \\
&\quad + v[\mathbf{w}(\theta) - \mathbf{w}(\sigma)] + d[\mathbf{z}(\theta) - \mathbf{z}(\sigma)]\left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} - K_\nu E_\nu\right) \\
&\quad - d\mathbf{z}(\sigma)(K_\nu E_\nu) + de_\theta[\mathbf{z}(\theta)] + \boldsymbol{e} - \boldsymbol{\tau}(\mu)K_\tau E_\tau \\
&= v[\mathbf{w}(\theta) - \mathbf{w}(\sigma)] + d[\mathbf{z}(\theta) - \mathbf{z}(\sigma)] \\
&\quad \times \left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} - K_\nu E_\nu\right) - d\mathbf{z}(\sigma)K_\nu E_\nu \\
&\quad + de_\theta\mathbf{z}(\theta) + \boldsymbol{e} - \boldsymbol{\tau}(\mu)K_\tau E_\tau
\end{aligned}
$$

Remark that $\boldsymbol{\tau}(\mu) = \mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)\mathbf{w}(\sigma) + \mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)\mathbf{z}(\sigma)$ holds being $\mathbf{w}(\sigma)$ and $\mathbf{z}(\sigma)$ orthonormal vectors.

We calculate the error vector components as

$$
\begin{aligned}
\dot{E}_\nu = \mathbf{E}^{\mathrm{T}}\boldsymbol{\nu}(\mu) &= v[\mathbf{w}(\theta) - \mathbf{w}(\sigma)]^{\mathrm{T}}\boldsymbol{\nu}(\mu) \\
&\quad + d[\mathbf{z}(\theta) - \mathbf{z}(\sigma)]^{\mathrm{T}}\boldsymbol{\nu}(\mu)\left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} - K_\nu E_\nu\right) \\
&\quad - d\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\nu}(\mu)K_\nu E_\nu + de_\theta\mathbf{z}(\theta)^{\mathrm{T}}\boldsymbol{\nu}(\mu) + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\nu}(\mu)
\end{aligned}
$$

Being

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{\nu}(\mu) \le \|e\| \le M$$

$$[\mathbf{w}(\theta) - \mathbf{w}(\sigma)]^{\mathrm{T}}\boldsymbol{\nu}(\mu) \le \|\mathbf{w}(\theta) - \mathbf{w}(\sigma)\| \le |\theta - \sigma| \le \frac{M_\theta}{K_\theta}$$

$$[\mathbf{z}(\theta) - \mathbf{z}(\sigma)]^{\mathrm{T}}\boldsymbol{\nu}(\mu) \le \|\mathbf{z}(\theta) - \mathbf{z}(\sigma)\| \le |\theta - \sigma| \le \frac{M_\theta}{K_\theta}$$

and

$$\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\nu}(\mu) = 1 - \mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu) = \mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu) \ge R,$$

we can write, for $E_\nu(t) \ge 0$

$$
\begin{aligned}
\dot{E}_\nu &\le v\frac{M_\theta}{K_\theta} + d\frac{M_\theta}{K_\theta}\left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} + K_\nu E_\nu\right) \\
&\quad - d\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\nu}(\mu)K_\nu E_\nu + M_\theta d + M \\
&\le v\frac{M_\theta}{K_\theta} + d\frac{M_\theta}{K_\theta}\left(\frac{v}{dR} + K_\nu E_\nu\right) \\
&\quad - dRK_\nu E_\nu + M_\theta d + M
\end{aligned}
$$

therefore, using Lemma 3, since $E_\nu(0) = 0$, we have that

$$E_\nu(t) \le \frac{(vM_\theta/K_\theta)(1 + 1/R) + M_\theta d + M}{dK_\nu(R - (M_\theta/K_\theta))}, \qquad \forall t \in [0, \bar{t}\,].$$

Analogously we can deduce a similar lower bound for $E_\nu(t) \le 0$, therefore we can obtain the final estimate for $|E_\nu(t)|$.

We proceed likewise for the tangential error component

$$
\begin{aligned}
\dot{E}_\tau = \mathbf{E}^{\mathrm{T}}\boldsymbol{\tau}(\mu) &= v[\mathbf{w}(\theta) - \mathbf{w}(\sigma)]^{\mathrm{T}}\boldsymbol{\tau}(\mu) + d[\mathbf{z}(\theta) - \mathbf{z}(\sigma)]^{\mathrm{T}}\boldsymbol{\tau}(\mu) \\
&\quad \times \left(\frac{v}{d}\frac{\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)}{\mathbf{w}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)} - K_\nu E_\nu\right) - d\mathbf{z}(\sigma)^{\mathrm{T}}\boldsymbol{\tau}(\mu)K_\nu E_\nu \\
&\quad + de_\theta[\mathbf{z}(\sigma)]^{\mathrm{T}}\tau(\mu) + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\tau}(\mu) - K_\tau E_\tau
\end{aligned}
$$

and we can write, for $E_\tau(t) \ge 0$

$$
\begin{aligned}
\dot{E}_\tau &\le v\frac{M\theta}{K_\theta}\left(1 + \frac{1}{R}\right) + d\frac{M_\theta}{K_\theta}(K_\nu E_\nu) \\
&\quad + dK_\nu E_\nu + dM_\theta + M - K_\tau E_\tau \\
&\le \left(\frac{vM_\theta}{K_\theta}\left(1 + \frac{1}{R}\right) + M_\theta d + M\right) \\
&\quad \times \left(1 + \frac{(dM_\theta/K_\theta) + 1}{R - (M_\theta/K_\theta)}\right) - K_\tau E_\tau
\end{aligned}
$$

which leads to

$$
\begin{aligned}
E_\tau(t) &\le \frac{1}{K_\tau}\left(\frac{vM_\theta}{K_\theta}\left(1 + \frac{1}{R}\right) + M_\theta d + M\right) \\
&\quad \times \left(1 + \frac{(dM_\theta/K_\theta) + 1}{R - (M_\theta/K_\theta)}\right), \quad \forall t \in [0, \bar{t}\,]
\end{aligned}
$$

analogously we can deduce a similar lower bound for $E_\tau(t) \le 0$ which implies the estimate for $|E_\tau(t)|$. □

Proof of Theorem 2: Since $\boldsymbol{\tau}(\mu(0))^{\mathrm{T}}\mathbf{w}(\sigma(0)) > 0$ by the given initial conditions, it is easy to see that there exists a unique local solution of system (31) and let $[0, \bar{t}\,[$ be its maximum interval of existence; clearly, if we show that

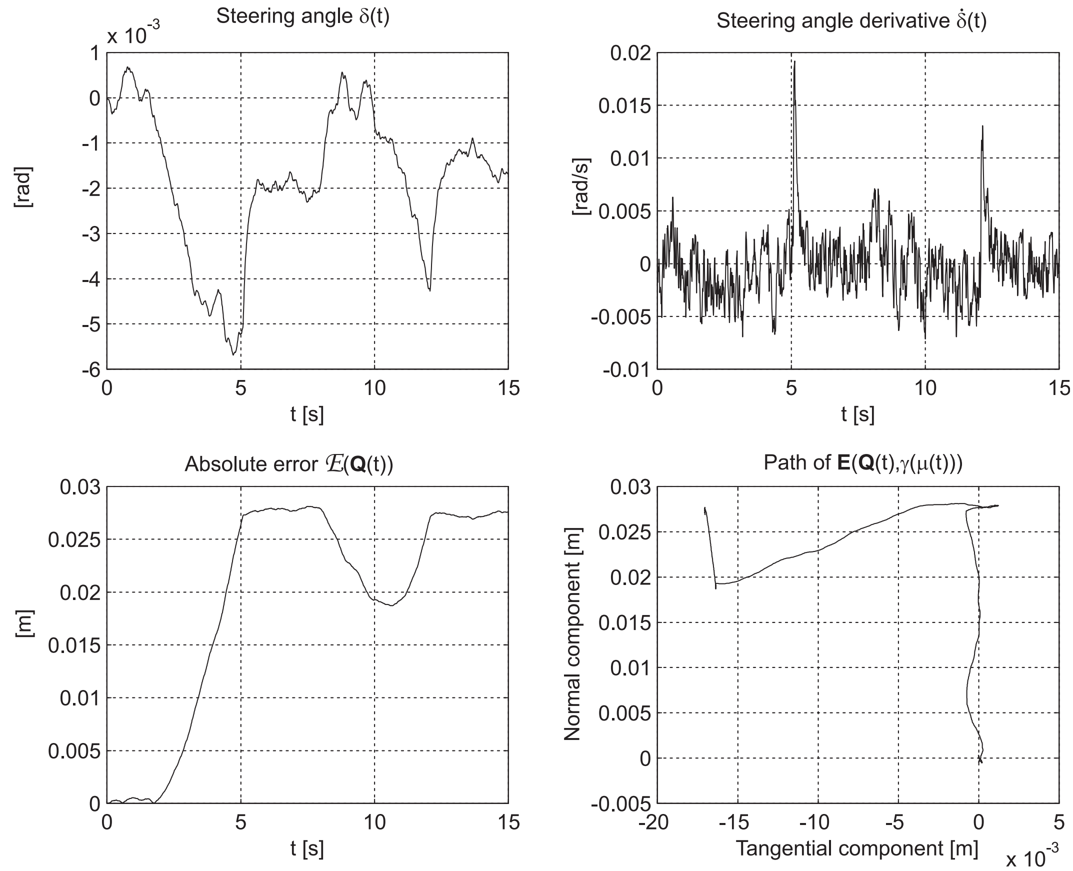


Figure 8. Relevant plots for the path following of a straight line.

$\bar{t} = +\infty$, the first part of Theorem 2 is proved. As in the proof of (b) in Theorem 1, this holds if we obtain that

$$\left.\begin{array}{l} \exists \bar{K}_\theta > 0, \text{ such that } \forall K_\theta \geq \bar{K}_\theta,\ \forall K_\tau > 0,\ \forall K_\nu > 0,\\ \text{we have that:}\\ \inf\limits_{0\leq t<\bar{t}}\left\{\boldsymbol{\tau}(\mu(t))^{\mathrm{T}}\mathbf{w}(\sigma(t))\right\}\\ \quad = \inf\limits_{0\leq\lambda<\bar{\lambda}}\left\{\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\sigma(\mu^{-1}(\lambda)))\right\}>0 \end{array}\right\} \tag{38}$$

where $\bar{\lambda} = \sup \mu([0, \bar{t}[)$ (we recall that $\boldsymbol{\tau}(\lambda) = \dot{\boldsymbol{\gamma}}(\lambda)$).

To this aim, set

$$\alpha(\lambda) = \beta(\lambda) - \sigma(\mu^{-1}(\lambda)), \quad \text{where } \beta(\lambda) = \arg(\boldsymbol{\tau}(\lambda)),$$

and

$$\alpha_M = \max\left\{|\alpha(0)|,\ \arcsin\left(d\bar{\kappa} + \frac{4M_\theta d + 3M}{v - 2(M_\theta d + M)}\right)\right\}$$

remark that $0 \leq \alpha_M < \pi/2$, by (35) and since $|\alpha(0)| < \pi/2$ by (34). Set

$$\lambda' = \sup\left\{s < \bar{\lambda} \,|\, \cos(\alpha(\lambda)) \geq \frac{\cos(\alpha_M)}{2},\ \forall \lambda \in [0, s]\right\}$$

clearly $\lambda' > 0$; we have to show that $\lambda' = \bar{\lambda}$. Suppose, by contradiction, that $\lambda' < \bar{\lambda}$, then it must be

$$\begin{aligned} R &= \inf_{0\leq\lambda<\lambda'}\{\boldsymbol{\tau}(\lambda)^{\mathrm{T}}\mathbf{w}(\sigma(\mu^{-1}(\lambda)))\}\\ &= \inf_{0\leq\lambda<\lambda'}\{\cos(\alpha(\lambda))\ \} = \frac{\cos\alpha_M}{2} \end{aligned} \tag{39}$$

As we did for the open-loop case, we find the expression for $\mathrm{d}\alpha/\mathrm{d}\lambda$ in the case of the feedback controller, a.e. on $]0, \bar{\lambda}[$

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}\lambda} &= \frac{\mathrm{d}\beta}{\mathrm{d}\lambda} - \frac{\mathrm{d}\sigma}{\mathrm{d}t}\frac{\mathrm{d}\mu^{-1}}{\mathrm{d}\lambda}\\ &= \kappa(\lambda) - \frac{(v/\mathrm{d})\tan(\alpha) - K_\nu E_\nu + K_\theta(\theta - \sigma)}{(v/\cos(\alpha)) - K_\tau E_\tau}\\ &= -\frac{1}{d}\sin(\alpha) + \cos(\alpha)\\ &\quad \times \frac{\sin(\alpha)K_\tau E_\tau + dK_\nu E_\nu - dK_\theta(\theta - \sigma)}{dv + K_\tau E_\tau d\cos(\alpha)} + \kappa(\lambda) \end{aligned}$$

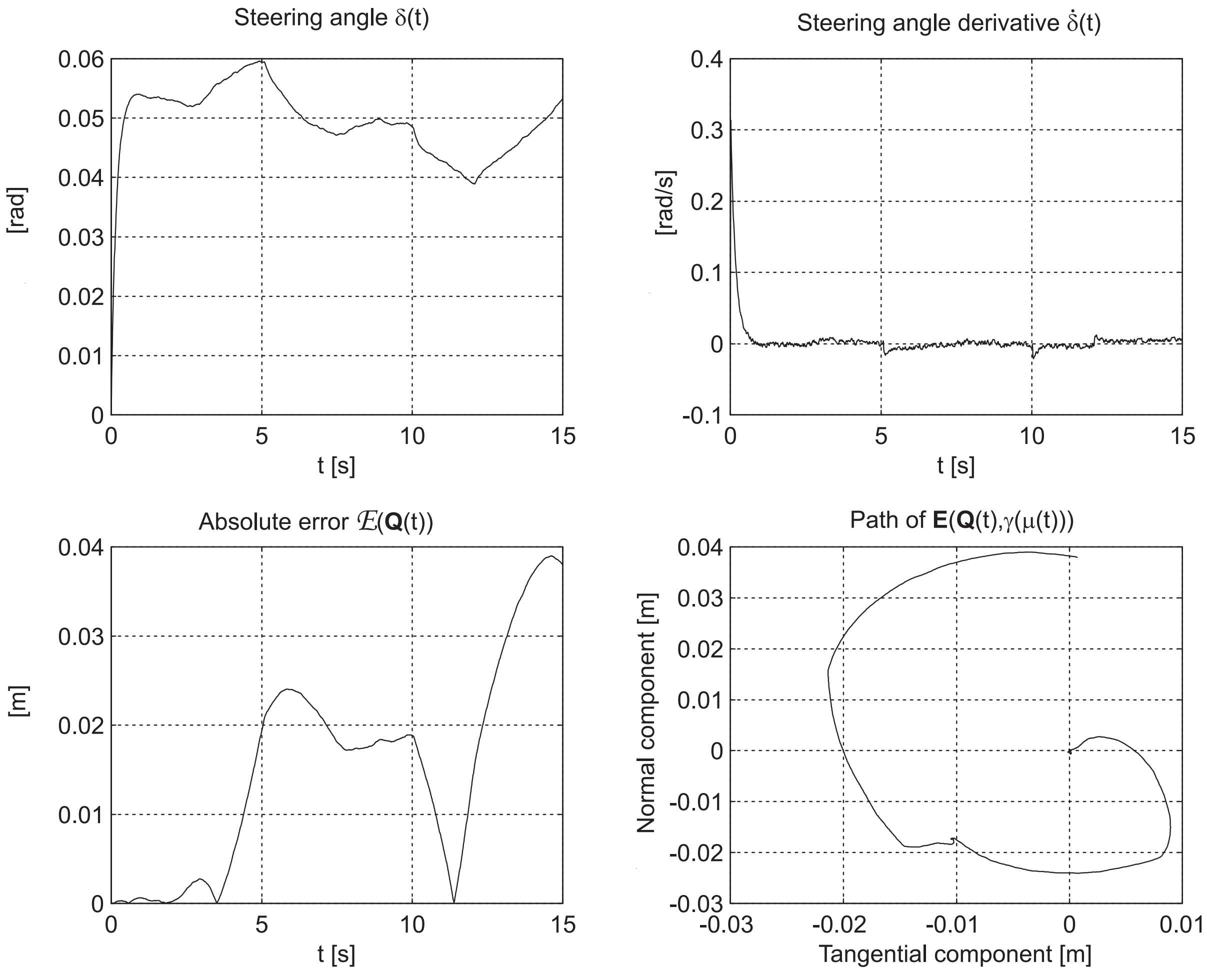


Figure 9. Relevant plots for the path following of a circle.

We set $M_\theta/\kappa_\theta = hR$, thanks to (35). Applying Lemma 2 the following inequality holds (remark that $R \le 1$ by (39)), a.e. on $]0, a[$

$$
\begin{aligned}
\frac{\mathrm{d}\alpha}{\mathrm{d}\lambda} &\le -\frac{1}{d}\sin(\alpha) + \cos(\alpha) \\
&\quad \times \frac{(vh(R+1) + M_\theta d + M)\left(1 + \dfrac{dhR+2}{R-Rh}\right) + M_\theta d}{vd - d\cos(\alpha)(vh(R+1) + M_\theta d + M)\left(1 + \dfrac{dhR+1}{R-hR}\right)} + \bar{\kappa} \\
&\le -\frac{1}{d}\sin(\alpha) + \cos(\alpha) \\
&\quad \times \frac{(2vh + M_\theta d + M)\left(\dfrac{dh+3}{R-Rh}\right) + M_\theta d}{vd - d\cos(\alpha)(2vh + M_\theta d + M)\left(\dfrac{(1-h)+dh+1}{R-hR}\right)} + \bar{\kappa} \\
&\le -\frac{1}{d}\sin(\alpha) + \frac{(2vh + M_\theta d + M)\left(\dfrac{dh+3}{R-Rh}\right) + M_\theta d}{vd - d(2vh + M_\theta d + M)\left(\dfrac{(1-h)+dh+1}{R-hR}\right)} + \bar{\kappa}
\end{aligned}
$$

for every h sufficiently small, that is, for K_θ sufficiently big such that

$$
\frac{(2vh + M_\theta d + M)\left(\dfrac{dh+3}{R-Rh}\right) + M_\theta d}{vd - d(2vh + M_\theta d + M)\left(\dfrac{(1-h)+dh+1}{R-hR}\right)} + \bar{\kappa} < \frac{1}{d}
$$

(this is possible by (35)). Now, applying Lemma 3 in the Appendix, we get that $\alpha(\lambda) \le \alpha_h \quad \forall \lambda \in [0, \lambda'[$ where

$$
\alpha_h = \max\left\{ |\alpha(0)|, \arcsin\left(\frac{(2vh + M_\theta d + M)\left(\dfrac{dh+3}{1-h}\right) + M_\theta d}{v - (2vh + M_\theta d + M)\left(\dfrac{(1-h)+dh+1}{1-h}\right)} + d\,\bar{\kappa} \right) \right\}
$$

In the same way we can prove that $\alpha(\lambda) \ge -\alpha_h$, $\forall \lambda \in [0, \lambda'[$ provided that h is sufficiently small. Since $\lim_{h\to 0} \cos\alpha_h = \cos\alpha_M$, we can find an $\bar{h}$ such that

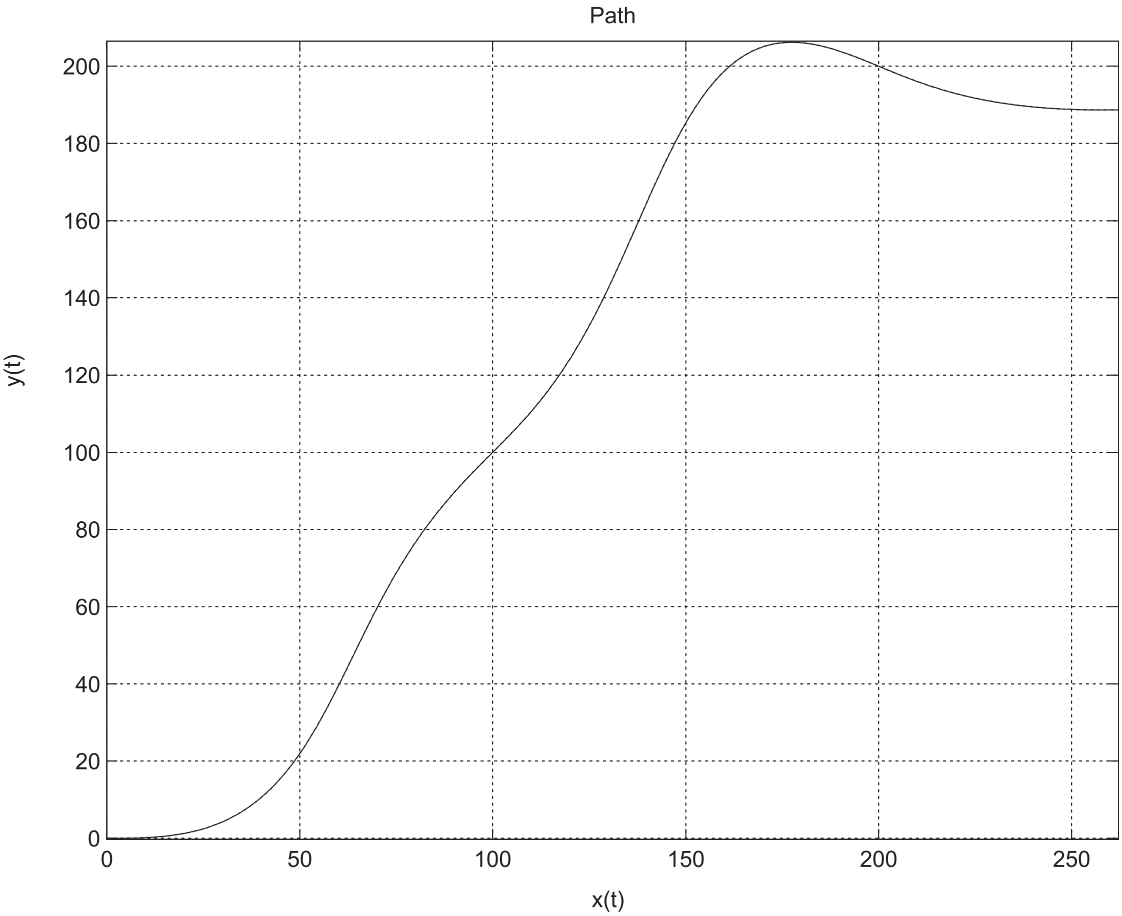


Figure 10. Path following of G^2-quintic splines.

$\forall h \in]0, \bar{h}[, \ \forall \lambda \in [0, \lambda'[, \ \cos\alpha(\lambda) \geq \cos\alpha_h \geq 2\cos(\alpha_M)/3$, therefore $R > \cos(\alpha_M)/2$ which contradicts (39).

This concludes the the proof of the first part which implies also the proof of (36) by Lemma 2 and (37). Finally we remark that $\mathcal{E}(\mathbf{Q}(t)) = 0 \ \forall t \geq 0$, if $M_0 = M = 0$ by (37). $\square$

Corollary 2 (Controller's design procedure)**:** *Relying on the provided constructive proofs of Theorem 2 and Lemma 2, the following design procedure can be accordingly introduced to set the controller's feedback gains.*

Let the error bounds M_θ and M for the uncertain model of the car-like vehicle satisfy the inequalities

$$M_\theta d + M < v/2, \qquad d\bar{\kappa} + \frac{4M_\theta d + 3M}{v - 2(M_\theta d + M)} < 1 \quad (40)$$

Then:

(1) *take a sufficiently small $h > 0$ for which*

$$\frac{(2vh + M_\theta d + M)\left(\frac{h+3}{1-h}\right) + M_\theta d}{v - (2vh + M_\theta d + M)\left(\frac{dh+2}{1-h}\right)} + d\bar{\kappa} < 1 \qquad (41)$$

clearly the existence of such $h > 0$ is guaranteed by conditions (40);

(2) *calculate R by means of the expression*

$$R = \sqrt{1 - \left(\frac{(2vh + M_\theta d + M)\left(\frac{dh+3}{1-h}\right) + M_\theta d}{v - (2vh + M_\theta d + M)\left(\frac{(1-h) + dh + 1}{1-h}\right)} + d\bar{\kappa}\right)^2}$$

(3) *and for any given $\epsilon > 0$, set*

$$\left.\begin{aligned} \bar{K}_\theta &= \frac{M_\theta}{hR} \\ \bar{K}_\tau &= \frac{\sqrt{2}}{\epsilon}\left((vh(1+R) + M_\theta d + M)\left(1 + \frac{1 + dhR}{R(1-h)}\right)\right) \\ \bar{K}_v &= \frac{\sqrt{2}}{\epsilon}(vh(1+R) + M_\theta d + M)\frac{1}{dR(1-h)} \end{aligned}\right\} \qquad (42)$$

If we design the dynamic inversion-based controller (30) *in such a way that the feedback gains K_τ, K_v and K_θ verify*

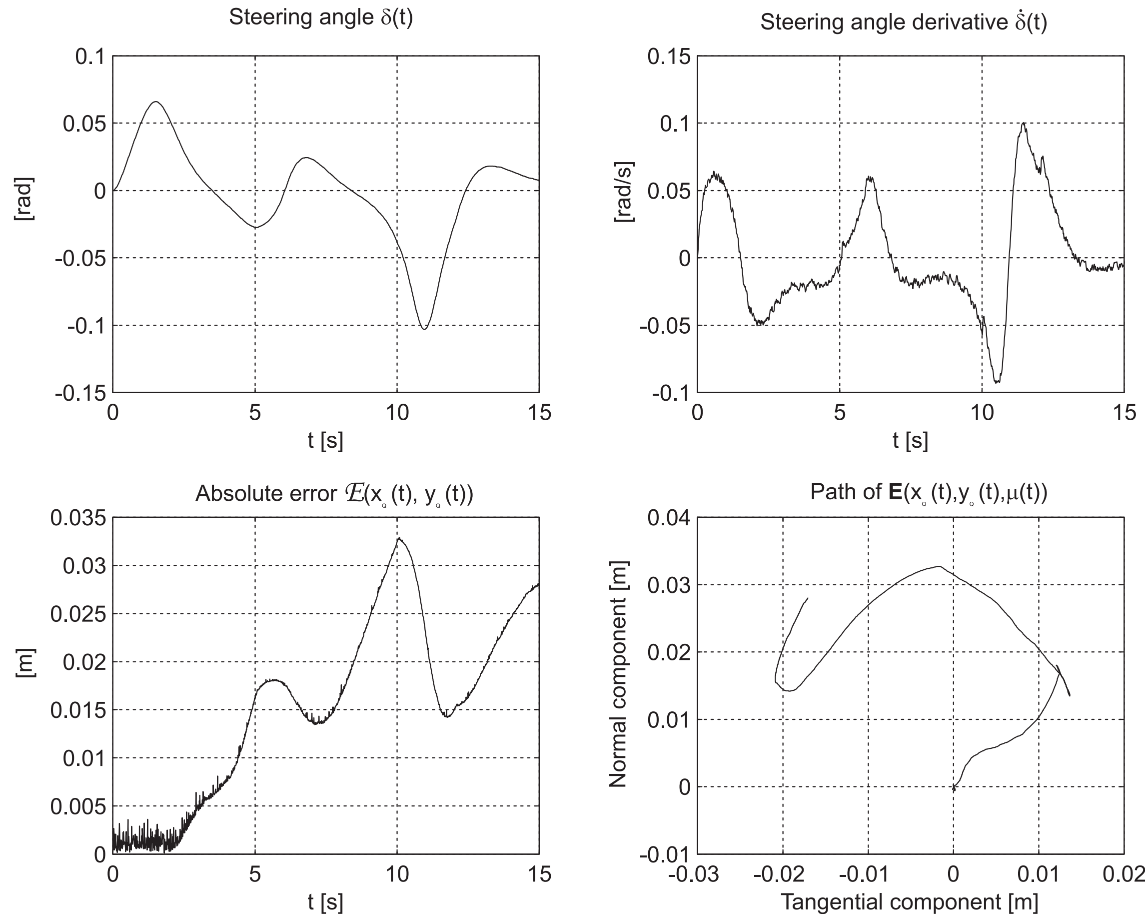


Figure 11. Relevant plots for the path following of G^2-quintic splines.

the inequalities

$$K_\tau \geq \bar{K}_\tau, \quad K_\nu \geq \bar{K}_\nu, \quad K_\theta \geq \bar{K}_\theta,$$

then the path following of Γ is achieved within the prefixed error ϵ, that is

$$\mathcal{E}(\mathbf{Q}(t)) < \epsilon, \quad \forall t \geq 0.$$

4. Examples and simulations

In the following simulations we consider the uncertain model given by equations (29) with the following parameters (the technical data are taken from the ARGO car (Broggi *et al.* 1999 a):

$$v = 25\,\mathrm{m/s}, \quad l = 2.67\,\mathrm{m}, \quad d = 4\,\mathrm{m}$$

and the perturbation functions are bounded as

$$\|e_x\|_\infty = 2\,\mathrm{m/s}, \quad \|e_y\|_\infty = 2\,\mathrm{m/s}, \quad \|e_\theta\|_\infty = 2\,\mathrm{deg/s}$$
$$\|\dot{e_x}\|_\infty = 1\,\mathrm{m/s^2}, \quad \|\dot{e_y}\|_\infty = 1\,\mathrm{m/s^2}, \quad \|\dot{e_\theta}\|_\infty = 2\,\mathrm{deg/s^2}$$

We present three simulations: a straight line path, a circular path of radius 50 m and a composite path modelled by G^2-quintic splines (Piazzi *et al.* 2002) (see figure 10).

For all the considered paths we set $\epsilon = 0.10\,\mathrm{m}$ as the maximum tolerable absolute error on the path (see (36) of Theorem 2).

The feedback gains of the dynamic inversion based controller have been determined according to the procedure described in § 3. For all the considered examples the set of values

$$K_\tau = 19.4, \quad K_\nu = 127, \quad K_\theta = 5.6$$

satisfies condition (42) (where we have set parameter $h = 0.01$ satisfying (41) for all cases).

Figures 8, 9 and 11 report the results of the simulations. For each case we have plotted the steering angle $\delta(t)$ and its derivative $\dot{\delta}(t)$, the absolute error of the front point with respect to the desired path $\mathcal{E}(\mathbf{Q}(t))$ and the plot of the vector error $\mathbf{E}(\mathbf{Q}(t), \boldsymbol{\gamma}(\mu(t)))$.

5. Conclusions

Dynamic inversion is, essentially, a methodology for dynamical systems to obtain input signals which will give desired output functions or output path planning. For the latter case we have exploited a path following controller which effectively combines, we think in a new way, the action of a feedforward dynamic inversion with a feedback action. A theoretical key point of the paper is the convergence result (Theorem 2) ensuring an arbitrarily good path following for an entire family of vehicle's uncertain models.

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Appendix

It is easy to prove the following lemma.

Lemma 3: *Let* $f: [0, \delta[\to \mathbb{R}$ *be a function such that there exist* $\bar{y} \in [0, \delta[$ *and* $f(y) \le 0, \quad \forall y \in [\bar{y}, \delta[$. *Let* $y: I \to \mathbb{R}$ *be an absolutely continuous function, where* I *is a real interval, (for instance if* $y \in C^1(I, \mathbb{R})$*). Suppose that* $\dot{y} \le f(y(t))$, *a.e. on* $\{t \in I \mid \bar{y} < y(t) < \delta[\}$. *If there exists* $t_0 \in I$ *such that* $y(t_0) \le \bar{y}$, *then* $y(t) \le \bar{y}$, $\forall t \in I$, *with* $t \ge t_0$.

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