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An analysis of three and four dimensional lorenz systems

# Abstract

This paper covers the history of Chaos Theory including the background of Edward Lorenz and his system, as well as an analysis of a four dimensional system derived using the Lorenz System and that exhibits hyper chaos.

# 1. Introduction

This document is intended for an audience with background in mathematics, namely dynamic systems, and covers the history of Chaos Theory, the history of Edward Lorenz, the analysis of a four dimensional system that exhibits hyperchaos, and an addition of an adaptive controller.

# 2. History

## 2.1. Chaos Theory

The forefather of dynamical chaos was Jules Henri Poincaré, and the idea of chaos started when he entered a contest that was sponsored by the king of Sweden (Henri Poincaré, n.d.). One of the questions in this contest was a generalization of the famous three body problem and was to show that the solar system as modeled by Newton’s equations is dynamically stable. Poincaré could not get a complete solution because difficulty arose in showing that solutions in terms of invariants converged, but nonetheless his work was so impressive that he won the prize and Weierstrass said that the publication of his work would usher in a new era in the history of celestial mechanics.

Poincaré described the concept of chaos as “…small differences in the initial conditions produce very great ones in the final phenomena” (Henri Poincaré, n.d.).

## 2.2. Edward Lorenz

Edward Lorenz grew up fascinated with the weather and mathematics, and so upon graduating from Dartmouth College in 1938 he planned to go into the field of math (This Month in Physics History, 2003). His plans were changed, however, when he got drafted into World War II where he was a weather forecaster in the Army Air Corps. At the conclusion of his service, Lorenz decided to pursue meteorology and published on many topics such as the general circulation of the atmosphere.

### 2.2.1. Lorenz’s Discovery of Chaos

With the advent of computers, Lorenz realized he could combine meteorology with math and thus began trying to construct a model of the weather using differential equations. His original model was a system of 12 differential equations (This Month in Physics History, 2003) and he ran a continuous simulation on it to observe data. At one point, Lorenz wanted to examine a specific sequence in greater detail and so took a shortcut by restarting his simulation using data form printouts for initial conditions. After a short break he returned to his data output to observe that his system had drastically changed from what he had previously observed and thought that his computer was broken; upon further analysis he realized that the computer stored data up to six decimal places but only printed out three decimal places—and thus he discovered that his system had a sensitivity to initial conditions, the trait of chaos.

### 2.2.2. The New System

To further study this phenomenon, Lorenz simplified his system into one of three differential equations that modeled rolling fluid convection (Harris):

where σ represents the ratio of fluid viscosity to thermal conductivity (Prandtl number), ρ represents the difference in temperature between the top and bottom of the system (Rayleigh number), and β represents the ratio of the width and height of the container of the system. Lorenz chose his parameter values as σ=10, ρ=28, and β=8/3.

This system created what is now commonly known as the Lorenz Attractor—known for its double spiral shape that resembles butterfly wings.

**Definition 2.2.2.1.** (Alam & Ahmed, 2017)

**A closed set *A* is an attractor if:**

1. ***A* is invariant.**
2. **There is an open set *U* containing *A* such that if , then the distance from *X* to *A* tends to zero as .**
3. ***A* is minimal.**

**Note that an attractor is said to be a strange attractor if it has a fractal structure—the Lorenz Attractor is a strange attractor.**

## 2.3. Analysis of the Three-Dimensional System

### 2.3.1. Equilibrium Solutions

Solving the system yields three equilibrium solutions:

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### 2.3.2. Local Stability Analysis

To analyze the stability, one must first linearize the system at each equilibrium solution; this is done using the Jacobian matrix of the system and then plugging in the respective equilibrium point:

The result for this system is two different characteristic equations (the non-origin solutions are symmetric):

The Routh-Hurwitz conditions for third-degree polynomials are as follows (Thanoon & F.Al-Azzawi, 2008):

where is the coefficient of the squared term, is the coefficient of the first-degree term, and is the constant. Thus for stability at the origin, assuming , is always positive, is positive when , and the third condition is met when with .

### 2.3.3. Global Stability Analysis

For global stability, a Lyapunov function is necessary. There are three conditions imposed on a Lyapunov function to guarantee global stability, and they are:

For the purposes of this document, the Lyapunov function chosen is . Taking the derivative and substituting in the system yields

Since a Lyapunov function can be defined that satisfies all three conditions, the system has global stability about the origin.

# 3. The Four-Dimensional Lorenz System

## 3.1. Difference between Chaos and Hyperchaos

Chaotic behavior is primarily determined by a system’s Lyapunov exponents—quantities that characterize the rate of separation of infinitesimally close trajectories—and is exhibited with at least one positive exponent; another way to word this is that chaos exists when a system has a dependence on initial conditions. When the sum of a system’s Lyapunov exponents is less than zero, the system has a dissipative structure; this means that being far from an equilibrium point allows for the creation of large-scale patterns. It is also true that “any (bounded) infinite trajectory that does not converge towards a fixed point is characterized by at least one zero LE” (Politi, 2013)—meaning a system that exhibits chaos has at least one zero Lyapunov exponent.

When a system has only one positive Lyapunov exponent, it can exhibit basic chaotic behavior. When the number of positive Lyapunov exponents increases to two or more, however, the behavior becomes hyperchaotic which has no applied physical interpretation. Hyperchaos has several conditions that need met (Alam & Ahmed, 2017), and they are:

1. “The system should have a dissipative structure.”
2. “The smallest dimension of the system should be at least four.”
3. “The number of terms giving rise to instability should be more than one.” (More than one positive Lyapunov exponent)

### 3.1.1. Lyapunov Exponents

It was already stated that Lyapunov exponents characterize the rate of separation of infinitesimally close trajectories, but what does that look like?

Let . If then is said to be the Lyapunov exponent (Guan). This prompts the solving of , which happens to need simple algebraic skills. Dividing both sides of the equation by , taking the natural log of both sides, and then dividing both sides by *t* yields . The new form leads us to the definition of the maximum Lyapunov exponent (MLE):

“The limit ensures the validity of the linear approximation at any time” (Guan).

The MLE is generally the most sought-after Lyapunov exponent because it dictates the asymptotic behavior of trajectories by being larger than the other exponents.

Another concept pertaining to the Lyapunov exponents is the Lyapunov spectrum which uses localized Jacobian matrices to shed light on the fractal dimension of an attractor (Schreiber, 2000).

## 3.2. The Researched System

The system under analysis was formed by first learning about hyperchaos using the system found in (Gang-Quan, Hui, & Yan-Bin, 2010):

This system meets the necessary conditions for hyperchaos and the article states that the Lyapunov exponents are .

## 3.3. Analysis of the Modified System

To analyze the effects of the powers in , changing the powers to 2, 3, 5, and 6 were examined. The odd powers appear to result in unexplained behavior and without a thorough understanding of Lyapunov exponents the reason is unknown; however the even powers appeared to have similar behavior so the power of 2 was chosen for the modified system, that is:

### 3.3.1. Equilibrium Solutions

Solving the system by setting each equation equal to zero yields the only equilibrium point, which is the origin.

### 3.3.2. Local Stability Analysis

To analyze the stability, one must first linearize the system at the origin; this is done using the Jacobian matrix of the system and then plugging in the origin:

The result for this system is the characteristic equation

= 0

which, when is factored out, becomes

Since we are able to factor out and get a third-degree polynomial, we can use the same Routh-Hurwitz conditions as were used on the original Lorenz system on **pg. 4** of this paper. Let , , and . Thus the Routh-Hurwitz stability conditions are:

Since σ represents a physical measurement, assume . Using the conditions, it follows then that and . So for the system to be stable about the origin, it must satisfy the conditions:

Note that upon observing that changing the powers in does not affect the characteristic equation, it is conjectured that even though behavior may be different, every power has the same stability conditions.

### 3.3.3. Local Stability Graph

To obtain all graphs used in this document, Maple was used with the parameters along with the initial condition of . Note that was left out of the list because it is the parameter being changed to analyze chaos and stability.

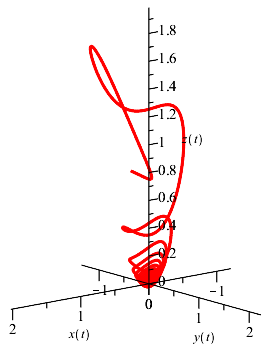
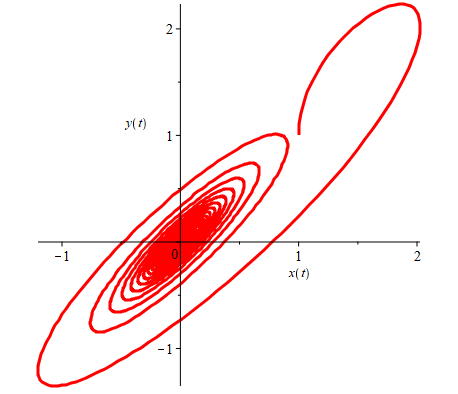
 

Figure 1.

Figure 2.

**Figure 1** is a space curve of , and **Figure 2** is a phase portrait of . Both were created using so that the stability conditions are met, and it is shown that solutions converge to the origin as expected.

### 3.3.4. Hyperchaotic Behavior

As learned earlier in this paper, achieving chaotic behavior can be done by altering ; but in order for hyperchaotic behavior to appear there must be at least a second Lyapunov exponent that becomes positive. Since no packages or methods to find Lyapunov exponents were found with the time allotted for this project, hyperchaotic behavior in the modified system cannot be proven. However, altering graphs eventually led to the same kind of hyperchaotic behavior that was shown in (Gang-Quan, Hui, & Yan-Bin, 2010) and so it is assumed that the modified system satisfies all conditions for hyperchaos.

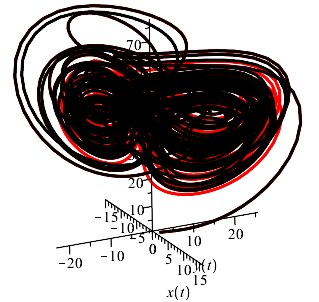
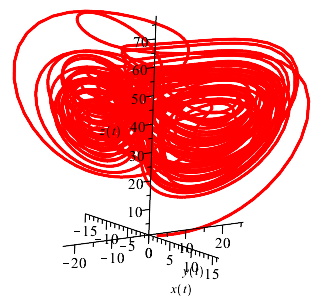


Figure 3.

Figure 4.

**Figure 3** and **Figure 4** were obtained with and the time interval . **Figure 3** is a space curve of and shows the hyperchaos obtained when breaking the stability conditions. **Figure 4** is a similar graph, only it contains two different initial conditions to show that the system is indeed sensitive to initial conditions; the initial condition for the red is and the initial condition for the black is and yet after only 50 units of time they are diverging from each other.

## 3.3 Adaptive Controller

Adaptive controllers became popular when aircraft began needing better ways to implement automatic control. The advantage that an adaptive controller has over a conventional controller is it uses real-time input to adjust control parameters in order to keep a good control of the system over time (Adaptive Control and its Applications in the Industry, 2018).

(Gang-Quan, Hui, & Yan-Bin, 2010) implemented an adaptive controller to obtain global stability of their system about the origin, and a similar system was applied to this document’s modified system. The controller will be defined as

with the adaptive parameters

The controlled system will now look like

where

and adaptive laws of the adaptive parameters are then defined as

where is a constant that controls the speed of convergence of the adaptive laws. With the adaptive controller in place, it would now make sense to graph the system and analyze the affects that way. For the purposes of this document a stability analysis will be conducted, using a Lyapunov function to prove that the system has global stability about the origin and therefore the chaos of the system is controlled.

Let

where is a constant. Taking the derivative yields

Substituting in the system, the adaptive laws, and the adaptive parameters and then simplifying down to where terms cancel out yields . Observe that can be written as

Let and . Then . Thus *B* is a positive definite matrix, and therefore . Since a Lyapunov function was found for the adaptive controlled system that satisfies the necessary conditions for stability, it can be said that the controlled system has global stability about the origin and so the controller is conjectured to work as intended.

# 4. Conclusion

This document began with a history of both chaos theory and Edward Lorenz, a notable contributor to the field of Chaos Theory and the creator of the Lorenz System. Following the introduction was an analysis of the three-dimensional Lorenz System to explain the concepts of chaos (having sensitivity to initial conditions), stability analysis using Routh-Hurwitz conditions and a Lyapunov function, and chaotic attractors. Note that all stability analysis and graphs were done by the author and fact checked by comparing to various research articles that performed similar analyses (except for the modified system).

The main portion of this document was an analysis of a modified four-dimensional Lorenz System where the formal analysis in which has fourth degree variables was done by (Gang-Quan, Hui, & Yan-Bin, 2010), which was applied to variables of second degree () and of sixth degree (). With regards to odd degrees, no conclusions could be obtained so it is conjectured that such systems either are not stable or exhibit no formal chaos—no confirmation or information could be found in any literature and so it needs to be further explored. Important concepts to take away are that hyperchaos exists only when: the system has a dimension of at least four, the system has a dissipative structure, the number of positive Lyapunov exponents should be more than one, and the sum of the Lyapunov exponents should be less than zero; Routh-Hurwitz criterion can be used to prove local stability for third- and fourth-degree characteristic polynomials; Lyapunov functions can be used to prove global stability; and a chaotic attractor is a set of bounded points in which trajectories always approach but never touch.

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