AnMate_17_S_2023_11_28 28.11.2023

Integrale prin părți

$$\int_a^b f\cdot g' = f\cdot g\Big|_a^b - \int_a^b f'\cdot g$$

1) $\int_1^e x^2\cdot \ln^2 x\, dx = I$

$$\left.\begin{array}{l} f(x) = (\ln x)^2 \Rightarrow f'(x) = \dfrac{2\ln x}{x} \\ g'(x) = x^2 \Rightarrow g(x) = \dfrac{x^3}{3} \end{array}\right| \Rightarrow I = \frac{x^3}{3}\cdot \ln^2 x\Big|_1^e -$$

$$-\int_1^e \frac{2\ln x}{\cancel{x}}\cdot\frac{x^{\cancel{3}\,2}}{3}\,dx =$$

$$= \frac{e^3}{3} - \frac{2}{3}\underbrace{\int_1^e x^2\cdot \ln x\, dx}_{I_1}$$

$$\left.\begin{array}{l} f(x) = \ln x \\ g'(x) = x^2 \end{array} \Rightarrow \begin{array}{l} f'(x) = \dfrac{1}{x} \\ g(x) = \dfrac{x^3}{3} \end{array}\right| \Rightarrow I_1 = \frac{x^3}{3}\cdot \ln x\Big|_1^e -$$

$$-\int_1^e \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{3}\,2}}{3}\,dx = \frac{e^3}{3} - \frac{x^3}{9}\Big|_1^e = \frac{1^3}{3} - \frac{e^3}{9} + \frac{1}{9}$$

$$I = \frac{e^3}{3} - \frac{2e^3}{9} + \frac{2(e^3-1)}{27}$$

2) $\int_1^e \ln x\,dx = \int_1^e 1\cdot \ln x\,dx = \int_1^e \underbrace{(x)'}_{g}\cdot \underbrace{\ln x}_{f} =$

$= x\cdot\ln x\big|_1^e - \int_1^e x\cdot\frac{1}{x}\,dx = e - x\big|_1^e = 1$

3) $J=\int_0^1 \overset{\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{1-x^2}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} - \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \arcsin x\big|_0^1 + \int_0^1 x\cdot\frac{x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{4} + \int_0^1 \underbrace{x}_{f}\underbrace{(\sqrt{1-x^2})'}_{g} =$

$= \frac{\pi}{4} + x\cdot\sqrt{1-x^2}\big|_0^1 - \int_0^1\sqrt{1-x^2}\,dx \Leftrightarrow J = \frac{\pi}{4} - J \Rightarrow$

$\Rightarrow \overset{?}{2J = \frac{\pi}{4}} \Rightarrow \boxed{J = \frac{\pi}{5}}$ corect

4) $J=\int_0^{\pi/2} e^{2x}\cdot\cos x\,dx = \int_0^{\pi/2}\underbrace{e^{2x}}_{f}\cdot\underbrace{(\sin x)'}_{g}\,dx =$

$= e^{2x}\sin x\big|^{\pi/2} - 2\int_0^{\pi/2} e^{2x}\sin x\,dx =$

$= e^{\pi} - 2\int_0^{\pi/2}\underbrace{e^{2x}}_{f}\underbrace{(-\cos x)'}_{g}\,dx = e^{\pi} - 2\left(-e^{2x}\cos x\big|_0^{\pi/2} +\right.$

$+\underbrace{\int_0^{\pi/2} 2e^{2x}\cdot\cos x\,dx}_{2J}$

$J = e^{\pi} - 2 - 4J \Rightarrow 5J = e^{\pi}-2 \Rightarrow J = \frac{e^{\pi}-2}{5}$

5) $\int_1^3 \operatorname{arctg}\sqrt{x}\,dx = \int_1^3 1\cdot\operatorname{arctg}\sqrt{x}\,dx =$

$= \int_1^3 \underbrace{x'}_{g}\cdot\underbrace{\operatorname{arctg}\sqrt{x}}_{f}\,dx = x\operatorname{arctg}\sqrt{x}\Big|_1^3 - \int_1^3 \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\cdot x\,dx =$

$$\boxed{\left(\operatorname{arctg}\sqrt{x}\right)' = \frac{1}{1+(\sqrt{x})^2}\cdot(\sqrt{x})' = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}}$$

$= 3\operatorname{arctg}\sqrt{3} - \operatorname{arctg}1 - \frac{1}{2}\int_1^3 \frac{\sqrt{x}}{(1+x)\cdot\cancel{x}}\cancel{x}\,dx \quad (\operatorname{arctg}\sqrt{x}) =$

$= \frac{1}{1+(\sqrt{x})^2}\cdot(\sqrt{x})' = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}$

$\sqrt{x} = t \Big|_{\sqrt{3}}^{2} \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

$I_1 = \int_1^{\sqrt{3}} \frac{t}{1+t^2}\,2t\,dt = 2\int_1^{\sqrt{3}} \frac{t^2}{1+t^2}\,dt =$

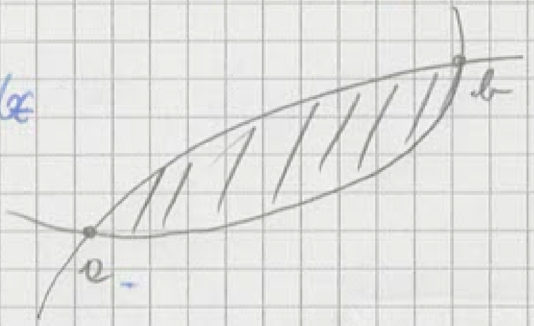
$= 2\int_1^{\sqrt{3}} \frac{1-t^2-1}{1+t^2}\,dt = 2\left(\int_1^{\sqrt{3}} dt - \underbrace{\int_1^{\sqrt{3}} \frac{1}{1+t^2}\,dt}_{\operatorname{arctg}}\right)$

Arii și volume

1) Calculați aria suprafeței cuprinse între curbele:

$$y = x^2 \quad \text{și} \quad y = \sqrt{x}$$

$$A = \int_a^b |x^2 - \sqrt{x}|\, dx$$



Rezolvăm sistemul: $\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases} \Rightarrow x^2 = \sqrt{x} \,\Big|^2 \Rightarrow x^4 = x \Leftrightarrow$

$\Leftrightarrow x^4 - x = 0 \Leftrightarrow x(x^3 - 1) = 0$

$x = 0$

$x = 1$

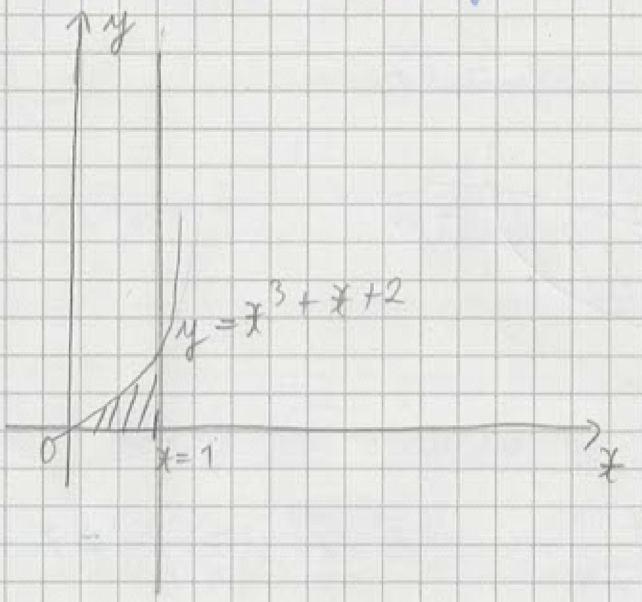
$\left.\begin{array}{l} x = 0 \Rightarrow y = 0 \\ x = 1 \Rightarrow y = 1 \end{array}\right| \Rightarrow A = \int_0^1 |x^2 - \sqrt{x}|\, dx = \int_0^1 (\sqrt{x} - x^2)\, dx =$

$= \left.\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 - \left.\dfrac{x^3}{3}\right|_0^1 =$

2) Calculați aria suprafeței

$$y = x^3 + x + 2$$

cuprinsă între $y = 0$ și $x = 1$



$$A = \int_0^1 (x^3 + x + 2)\,dx$$

Rezolvăm sistemul: $\begin{cases} y = x^3 + x + 2 \\ y = 0 \end{cases} \Rightarrow x^3 + x + 2 = 0$

$$x^3 + 1 + x + 1 \Leftrightarrow (x+1)(x^2 - x + 1) + (x+1) = 0 \Leftrightarrow$$

$$\Leftrightarrow (x+1)(x^2 - x + 2) = 0$$

$$x + 1 = 0 \Rightarrow x = -1$$

$$x^2 - x + 2 = 0$$

$$\Delta = 1 - 8 = -7 < 0 \Rightarrow x_{2,3} \notin \mathbb{R}$$

$$A = \int_{-1}^{1} (x^3 + x + 2)\,dx = \underbrace{\frac{x^4}{4}\Big|_{-1}^{1}}_{=0} + \underbrace{\frac{x^2}{2}\Big|_{-1}^{1}}_{=0} + 2x\Big|_{-1}^{1} = 2 + 2 = 4$$

3) Volumul unui corp obtinut prin rotirea lui G_f în jurul axei de sistem este:

$$V = \pi \int_a^b f^2(x)\, dx \quad : \quad f: [a, b] \to \mathbb{R}$$

$f : [0, 1] \to \mathbb{R}$, $f(x) = x \cdot e^{x}$

$V_c = ?$

$$V_c = \pi \cdot \int_0^1 f^2(x)\, dx = \pi \cdot \int_0^1 (x \cdot e^x)^2 dx = \pi \cdot \int_0^1 x^2 \cdot e^{2x} dx =$$

$$= \pi \cdot \int_0^1 \underbrace{\left(\frac{e^{2x}}{2}\right)'}_{g} \cdot \underbrace{x^2}_{f} dx =$$

$$= \pi \cdot \left(x^2 \cdot \frac{e^{2x}}{2} \Big|_0^1 - \int_0^1 \cancel{2}x \cdot \frac{e^{2x}}{\cancel{2}} dx \right) =$$

$$= \pi \cdot \left(\frac{e^2}{2} - \int_0^1 \underbrace{x}_{f} \underbrace{\left(\frac{e^{2x}}{2}\right)'}_{g} dx \right) =$$

$$= \pi \cdot \left(\frac{e^2}{2} - x \cdot \frac{e^{2x}}{2} \Big|_0^1 + \int_0^1 \frac{e^{2x}}{2} dx \right) =$$

$$= \pi \left(\cancel{\frac{e^2}{2}} - \cancel{\frac{e^2}{2}} + \frac{e^{2x}}{4} \Big|_0^1 \right) = \pi \left(\frac{e^2}{4} - \frac{1}{4} \right)$$

Serii numerice

Def.: Fie $(x_n)_{n\in\mathbb{N}}$ șir de nr. naturale.

Numim serie suma: $\sum_{n=0}^{\infty} x_n$

$$S_n = \sum_{k=0}^{n} x_k = x_0 + x_1 + x_2 + \dots + x_n$$ – șirul sumelor parțiale asociat seriei S, $n\in\mathbb{N}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} x_k = S$$

↳ suma seriei

Ex.: 1) e $\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots =$

$$= 2,5 + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots =$$

$$= 2.71 \quad \dots$$

2) $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$, α $(-1, 1)$

3) $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots > 1 + \frac{1}{2} + \frac{1}{4} + \dots$

$\dots = 1 + \frac{1}{2} + \frac{1}{2} + \underset{\infty \text{ ori}}{\dots} + \frac{1}{2} = \infty \Rightarrow \sum \frac{1}{n} = \infty = -D$