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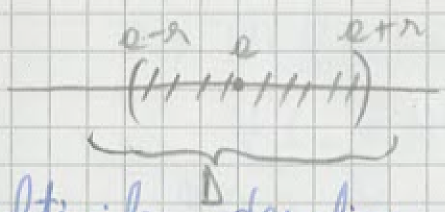
Borelianul mulțimii $\mathbb{R}$
(Emile Borel)

$\mathcal{T}_{\mathbb{R}}$ – topologia lui $\mathbb{R}$ = mulțimea (familia) mulțimilor **deschise** din $\mathbb{R}$

$\Delta \subset \mathbb{R}$ – deschisă dacă $\Delta = \emptyset$ sau
$\forall a \in \Delta, \exists r > 0 : (a-r, a+r) \subset \Delta$ $(a \in \mathring{\Delta})$



Mulțimile deschise din $\mathbb{R}$:
1) intervale deschise $(a, b), (-\infty, a), (a, \infty), (-\infty, \infty) = \mathbb{R}$
2) Reuniunile finite sau numărabile de intervale deschise disjuncte

Def: $\mathcal{B}_{\mathbb{R}} = \mathcal{F}(\mathcal{T}_{\mathbb{R}}) \subset \mathcal{P}(\mathbb{R})$ borelianul mulțimii $\mathbb{R}$

TEOREMA:
1) $\mathcal{B}_{\mathbb{R}}$ – Conține toate tipurile de intervale și reuniunile sau intersecțiile cel mult numărabile de intervale.
2) $\mathcal{B}_{\mathbb{R}}$ este generat de familia tuturor intervalelor de un anumit tip

Dem.: 2) — EXEMPLU:

$$\mathcal{M} = \{(-\infty, a] \mid a \in \mathbb{R}\}$$

Arătăm că $\boxed{\mathcal{B}_{\mathbb{R}} = \mathcal{F}(\mathcal{M})}$

Dem

I. $\mathcal{B}_{\mathbb{R}} \subset \mathcal{F}(\mathcal{M})$

$$(-\infty, a) = \bigcup_{n=1}^{\infty} \underbrace{(-\infty, a-\tfrac{1}{n}]}_{\in \mathcal{M}} \in \mathcal{F}(\mathcal{M}), \forall a \in \mathbb{R}$$

$$(a, \infty) = \underbrace{(-\infty, a]}_{\in \mathcal{M}}{}^{c} \in \mathcal{F}(\mathcal{M}), \forall a \in \mathbb{R}$$

$$(a, b) = \underbrace{(-\infty, b)}_{\in \mathcal{F}(\mathcal{M})} \setminus \underbrace{(-\infty, a]}_{\in \mathcal{M}} \in \mathcal{F}(\mathcal{M}), \forall a, b \in \mathbb{R}, a<b$$

Deducem $\mathcal{T}_{\mathbb{R}} \subset \mathcal{F}(\mathcal{M}) \Rightarrow \mathcal{B}_{\mathbb{R}} \subset \mathcal{F}(\mathcal{M})$

II. $\mathcal{F}(\mathcal{M}) \subset \mathcal{B}_{\mathbb{R}}$

$(-\infty, a] = (a, \infty)^{c} \in \mathcal{B}_{\mathbb{R}}, \forall a \in \mathbb{R} \Rightarrow \mathcal{M} \subset \underbrace{\mathcal{B}_{\mathbb{R}}}_{\sigma\text{-alg}} \Rightarrow$

$\Rightarrow \mathcal{F}(\mathcal{M}) \subset \mathcal{B}_{\mathbb{R}}$

Deci $\mathcal{B}_{\mathbb{R}} = \mathcal{F}(\mathcal{M})$

PROBABILITATEA

$(\Omega, \mathcal{F}, \mathbb{P})$ – spațiu de probabilitate

unde

$\Omega \neq \emptyset$

$\mathcal{F}$ – σ-corp peste Ω (mulțimea evenimentelor)

$\mathbb{P} : \mathcal{F} \to \mathbb{R}$

$\mathbb{P}$ – probabilitate (   ) dacă:

$P_1)$ $\mathbb{P}(A) \geq 0, \ \forall A \in \mathcal{F}$

$P_2)$ $\mathbb{P}(\Omega) = 1$

$P_3)$ $\mathbb{P}\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mathbb{P}(A_n), \ \forall \{A_n\}_{n \geq 1} \subset \mathcal{F}$ ai $A_n \cap A_k = \emptyset$,
$\forall k, n \in \mathbb{N}^*, k \neq n$

TEOREMĂ Funcția de probabilitate $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ are urm. proprietăți:

P_4 $\mathbb{P}(\emptyset) = 0$

Dem: $\emptyset \in \mathcal{F}$. Presupunem prin absurd $\mathbb{P}(\emptyset) \neq 0$.
Atunci $\mathbb{P}(\emptyset) = p > 0$.

Definim: $A_n = \emptyset, \ n \in \mathbb{N}^*$

$$p = \mathbb{P}(\emptyset) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \overset{P_3}{=} \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \overbrace{\mathbb{P}(\emptyset)}^{p} = \infty$$

Contradicție: Deci $\mathbb{P}(\emptyset) = 0$

P_5 $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{P}(A_k)$, $\forall \{A_1, A_2, A_3 \dots A_n\} \subset \mathcal{F}$,

$A_i \cap A_j = \emptyset$ pentru $i \neq j$

Dem.: Fie $A_1, \dots, A_n \in \mathcal{F}$ $(n \geq 2)$, a.î. $A_i \cap A_j = \emptyset$

Definim: $A_k = \emptyset$, $k \in \mathbb{N}^*$, $k \geq n+1$

$\{A_k\}_{k \geq 1}$ — șir de evenimente punctual incompatibile

Atunci $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) \overset{P_3}{=} \sum_{k=1}^{n} \mathbb{P}(A_k) = \sum_{k=1}^{}) P(A_k) + \sum_{k=}^{\infty} \underbrace{P(A_k)}_{0 \, (P_4)} = \sum_{k=1}^{n} \mathbb{P}(A_k)$

P_6 $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$, $\forall A, B \in \mathcal{F}$ a.î. $A \subset B$

Dem.: Fie $A, B \in \mathcal{F}$ a.î. $A \subset B$

$B = A \cup (B \setminus A)$

$A \cap (B \setminus A) = \emptyset$

$\mathbb{P}(B) = \mathbb{P}[A \cup (B \setminus A)] \overset{P_5}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A)$

$\Rightarrow \mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$

P_7 Funcție $\mathbb{P}$ este monoton crescătoare

$\forall A, B \in \mathcal{F}: \quad A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

DEM.: Fie $A, B \in \mathcal{F}$, $A \subset B$

Cf P_6: $\mathbb{P}(B) - \mathbb{P}(A) = \mathbb{P}(B \setminus A) \geq 0 \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

P_8 $\mathbb{P}(A) \in [0,1], \forall A \in \mathcal{F}$

Dem.: Fie $A \in \mathcal{F}$

$$\left.\begin{array}{l} \mathbb{P}(A) \geq 0 \text{ (cf } P_1) \\ A \subset \Omega \overset{P_4}{\Rightarrow} \mathbb{P}(A) \leq \mathbb{P}(\Omega) \overset{P_2}{=} 1 \end{array}\right\} \Rightarrow \mathbb{P}(A) \in [0,1]$$

P_9 $\mathbb{P}(A^c) = 1 - \mathbb{P}(A), \forall A \in \mathcal{F}$

Dem.: Fie $A \in \mathcal{F}$

$$\begin{cases} A \cup A^c = \Omega \\ A \cap A^c = \emptyset \end{cases}$$

Atunci $1 \overset{P_2}{=} \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) \overset{P_5}{=} \mathbb{P}(A) + \mathbb{P}(A^c)$

deci $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

P_{10} Formula lui Poincaré

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{1 \leq k_1 < k_2 \leq n} \mathbb{P}(A_{k_1} \cap A_{k_2}) +$$

$$+ \sum_{1 \leq k_1 < k_2 < k_3 \leq n} \mathbb{P}(A_{k_1} \cap A_{k_2} \cap A_{k_3}) - \dots +$$

$$+ (-1)^{n-1} \mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq k_1 < k_2 < \dots < k_i \leq n} \mathbb{P}(A_{k_1} \cap A_{k_2} \cap \dots A_{k_i})$$

$\forall A_1, A_2, \dots, A_n \in \mathcal{F}, n \geq 2$

Cazul $n=2$

$$\mathbb{P}(A\cup B)=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B),\ \forall A,B\in\mathcal{F}$$

Dem.: Fie $A,B\in\mathcal{F}$

$$A\cup B=[A\setminus(A\cap B)]\cup[B\setminus(A\cap B)]\cup(A\cap B)$$

$$\mathbb{P}(A\cup B)=\mathbb{P}(A\setminus(A\cap B))+\mathbb{P}(B\setminus(A\cap B))+\mathbb{P}(A\cap B)\overset{P_6}{=}$$

$$\overset{P_6}{=}[\mathbb{P}(A)-P(A\cap B)]+[\mathbb{P}(B)-\mathbb{P}(A\cup B)]+\mathbb{P}(A\cap B)=$$

$$=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B)$$

$\boxed{\text{Temă: Formula lui ~~Pinteroi ptr~~ Poincaré pt 3 mulțimi}}$

P_{11} Subaditivitatea

a) $\mathbb{P}\left(\bigcup\limits_{k=1}^{n} A_k\right)\leqslant\sum\limits_{k=1}^{n}\mathbb{P}(A_k),\ \forall\{A_k,\ k=\overline{1,n}\}\subset\mathcal{F}$

b) $\mathbb{P}\left(\bigcup\limits_{n=1}^{\infty} A_n\right)\leqslant\sum\limits_{n=1}^{\infty}\mathbb{P}(A_n),\ \forall\{A_n\}_{n\geqslant 1}\subset\mathcal{F}$

Dem.:

a) Fie $A_1,A_2,\dots,A_n\in\mathcal{F}$

Definim evenimentele: $B_k=\begin{cases}A_1, & k=1\\ A_k\setminus\left(\bigcup\limits_{i=1}^{k-1}A_i\right), & k=2,\dots,n\end{cases}$

Avem:
1) $B_k\subset A_k,\ k=\overline{1,n}$
2) $\bigcup\limits_{k=1}^{n}B_k=\bigcup\limits_{k=1}^{n}A_k$
3) $B_j\cap B_k=\emptyset,\ \forall j<k$

REZULTĂ

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) \overset{2)}{=} \mathbb{P}\left(\bigcup_{k=1}^{n} B_k\right) \underset{P_5}{\overset{3)}{=}} \sum_{k=1}^{n} \mathbb{P}(B_k) \underset{P_7}{\overset{1)}{\leq}} \sum_{k=1}^{n} \mathbb{P}(A_k)$$

P12 Ineg. Boole

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) \geq 1 - \sum_{k=1}^{n} \mathbb{P}(A_k^c)$$

$$\forall A_1, \dots, A_n \in \mathcal{F}$$

Dem.:

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) \overset{P_9}{=} 1 - \mathbb{P}\left(\left(\bigcap_{k=1}^{n} A_k\right)^c\right) \overset{\text{de Morgan}}{=\!=\!=} 1 - \sum_{k=1}^{n} \mathbb{P}(A_k^c)$$