Produsul scalar se notează cu "·"

Def.: $\bar{a}\cdot\bar{b} = \|\bar{a}\|\cdot\|\bar{b}\|\cdot\cos(\widehat{\bar{a},\bar{b}}) \in \mathbb{R}$

Expresie analitică

$$\left.\begin{array}{l}\bar{a}\,(a_1, a_2, a_3)\\ \bar{b}\,(b_1, b_2, b_3)\end{array}\right\} \Rightarrow \bar{a}\cdot\bar{b} = a_1\cdot b_1 + a_2 b_2 + a_3 b_3$$

Proprietăți:

1) Comutativ: $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$

2) $\bar{a}\cdot\bar{a} = \|\bar{a}\|^2\cdot\underbrace{\cos 0}_{1} \Rightarrow \|\bar{a}\| = \sqrt{\bar{a}\cdot\bar{a}}$

3) $\bar{a}\perp\bar{b} \Rightarrow \cos 90^\circ = 0 \Leftrightarrow \bar{a}\cdot\bar{b} = 0$

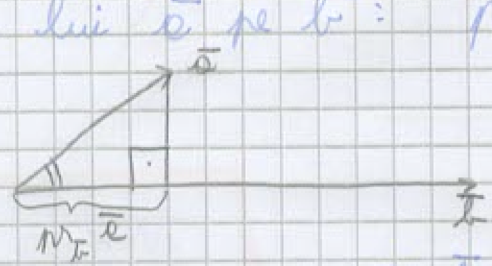
4) $\cos(\widehat{\bar{a},\bar{b}}) = \dfrac{\bar{a}\cdot\bar{b}}{\|\bar{a}\|\cdot\|\bar{b}\|}$ $> 0 \Rightarrow (\widehat{\bar{a},\bar{b}})$ - ascuțit

$< 0 \Rightarrow (\widehat{\bar{a},\bar{b}})$ - obtuz

5) Versorul unui vector $\bar{a} \cdot u(\bar{a}) = \dfrac{\bar{a}}{\|\bar{a}\|}$

(vector de lungime 1)

6) Proiecția lui $\bar{a}$ pe $\bar{b}$: $pr_{\bar{b}}\,\bar{a}$



$$\cos(\bar{a},\bar{b}) = \frac{\bar{a}\cdot\bar{b}}{\|\bar{a}\|\cdot\|\bar{b}\|}$$

$$\cos = \frac{\text{cat. al}}{\text{ip}} = \frac{pr_{\bar{b}}\,\bar{a}}{\|\bar{a}\|}$$

$$\Rightarrow pr_{\bar{b}}\,\bar{a} = \frac{\bar{a}\cdot\bar{b}}{\|\bar{b}\|}$$

Ex① $A(2,4,5)$, $B(3,0,4)$, $C(1,2,3)$

a) coord vectorilor $\overline{AB}$, $\overline{AC}$, $\overline{BC}$

b) produsele scalare: $\overline{AB}\cdot\overline{AC}$, $\overline{AB}\cdot\overline{BC}$, $\overline{AC}\cdot\overline{BC}$

c) lungimile lat. Δ-ului ABC

d) ∡ Δ-lui ABC

$\overline{AB} = (x_B - x_A)\bar{i} + (y_B - y_A)\bar{j} + (z_B - z_A)\bar{k} =$

$= \bar{i} - 4\bar{j} - \bar{k} \Leftrightarrow \overline{AB}(1, -4, -1)$

$\overline{AC}(-1, -2, -2)$

$\overline{BC}(-2, 2, -1)$

$= 1\cdot(-1) + (-4)\cdot(-2) + (-1)\cdot(-2)$

b) $\overline{AB}\cdot\overline{AC} = (1, -4, -1)\cdot(-1, -2, -2) = -1 + 8 + 2 = 9$

$\overline{AB}\cdot\overline{BC} = 1\cdot(-2) + (-4)\cdot 2 + (-1)(-1) = -2 - 8 + 1 = -9$

$\overline{AC}\cdot\overline{BC} = (-1)\cdot(-2) + (-2)\cdot 2 + (-2)\cdot(-1) = 2 - 4 + 2 = 0 \Rightarrow$
$\Rightarrow \overline{AC} \perp \overline{BC} \Rightarrow \Delta ABC$ dr. în C

c) $AB = \|\overline{AB}\| = \sqrt{(1)^2 + (-4)^2 + (-1)^2} = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

$AC = \|\overline{AC}\| = \sqrt{1+4+4} = \sqrt{9} = 3$

$BC = \|\overline{BC}\| = \sqrt{4+4+1}$ $\Rightarrow \Delta ABC$ - isoscel

d) ΔABC - dr. isoscel $\rightarrow \hat{C} = 90^\circ$, $\hat{A} = \hat{B} = 45^\circ$

2) $A(1,2,1)$, $B(3,0,1)$, $C(1,2,0)$

Calculați: a) $\overline{AB} \cdot \overline{AC}$

b) $\|\overline{BC}\|$

c) $\cos(\overline{AB}, \overline{AC})$

d) $pr_{\overline{AC}} \overline{BC}$

e) versorul lui $\overline{AB}$

* COPIAZĂ POZĂ

Vezi foaie $\boxed{\#03}$

#08 ALGAD - S - 12 - 2023 - 11 - 15
(continuare ex. 2)

a) $A(1,2,1)$
$B(3,0,1)$
$C(1,2,0)$

$\overline{AB} = (x_B - x_A)\bar{i} + (y_B - y_A)\bar{j} + (z_B - z_A)\bar{k}$
$\overline{AB} = (3-1)\bar{i} + (0-2)\bar{j} + (1-1)\bar{k}$
$\overline{AB} = 2\bar{i} - 2\bar{j} + 0k$
$\overline{AB}\ (2,-2,0)$

$\overline{AC} = (x_C - x_A)\bar{i} + (y_C - y_A)\bar{j} + (z_C - z_A)\bar{k}$
$\overline{AC} = 0\bar{i} + 0\bar{j} - \bar{k}$
$\overline{AC}\ (0,0,-1)$

$\overline{AB}\ \overline{AC} = (2,-2,0)\ (0,0,-1) =$
$= 0+0+0 =$
$= 0$

c) $\cos(\overline{AB}, \overline{AC}) = \dfrac{\overline{AB} \cdot \overline{AC}}{\|\overline{AB}\| \cdot \|\overline{AC}\|} = 0 \Rightarrow \overline{AB} \perp \overline{AC}$

$\left.\begin{array}{l}\overline{AB}\ (2,-2,0) \\ \overline{AC}\ (0,0,-1)\end{array}\right| \Rightarrow \overline{AB} \cdot \overline{AC} = 2 \cdot 0 + (-2) \cdot 0 + 0 \cdot (-1) =$
$= 0+0+0 = 0$

$\|\overline{AB}\| = \sqrt{4+4+0} = \sqrt{8} = 2\sqrt{2}$
$\|\overline{AC}\| = \sqrt{0+0+1} = \sqrt{1} = 1$

b) $\|\overline{BC}\| = ?$

$B(3,0,1)$
$C(1,2,0)$
$\overline{BC}\ (x_C - x_B,\ y_C - y_B,\ z_C - z_B)$
$\overline{BC}\ (1-3,\ 2,\ -1)$
$\overline{BC}\ (-2,\ 2,\ -1)$

$\|\overrightarrow{BC}\| = \sqrt{(2)^2+2^2+(-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

d) $pr_{\overrightarrow{AC}} \overrightarrow{BC} = \dfrac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{\|\overrightarrow{AC}\|} = \dfrac{1}{1} = 1$

$\overrightarrow{AC}\,(0,0,-1)$

$\|\overrightarrow{AC}\| = \sqrt{0^2+0^2+(1)^2} = 1$

$\overrightarrow{AC}\cdot\overrightarrow{BC} = (0,0,-1)\cdot(-2,2,-1) = 0+0+1 = 1$

e) versorul lui $\overrightarrow{AB}$

$u(\overrightarrow{AB}) = \dfrac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \dfrac{2\bar{i}-2\bar{j}}{2\sqrt{2}} = \dfrac{\bar{i}-\bar{j}}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\,\bar{i} - \dfrac{1}{\sqrt{2}}\,\bar{j}$

$\overrightarrow{AB}\,(2,-2,0)\ ,\ \|\overrightarrow{AB}\| = 2\sqrt{2}$

$\bar{u}(\overrightarrow{AB})\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}, 0\right)$

Ex.3 $\overline{OA} = 12\bar{i} - 4\bar{j} + 3\bar{k}$

$\overline{OB} = 3\bar{i} + 12\bar{j} - 4\bar{k}$

$\overline{OC} = 2\bar{i} + 3\bar{j} - 4\bar{k}$

a) ΔAOB isoscel

b) ΔAOC dreptunghic

c) $P_{\Delta ABC}$

Metoda 1

$\overline{AB} = \overline{AO} + \overline{OB}$

$\overline{AC} = \overline{AO} + \overline{OC}$

$\overline{BC} = \overline{BO} + \overline{OC}$

Metoda 2

$\overline{OA}, \overline{OB}, \overline{OC}$ — vectori de poziție

$A(12, -4, 3)$

$B($

$C($

a)

$A(12, -4, 3)$

$B(3, 12, -4)$

$C(2, 3, -4)$

$\overline{AB}(-9, 16, -7)$

$\|\overline{AO}\| = \sqrt{12^2 + (-4)^2 + 3^2} = \sqrt{144 + 16 + 9} = \sqrt{169} = 13$

$\|\overline{BO}\| = \sqrt{9 + 144 + 16} = 13$

$\|\overline{AB}\| = \sqrt{81 + 256 + 49} = \sqrt{130 + 256} = \sqrt{386}$

$\Rightarrow \Delta AOB$ isoscel

#04

ALG AB - S - 12 - 2023 - 11 - 15
(continuare ex. 3, b))

b) $AO = \|\overline{OA}\| = \sqrt{12^2 + (-4)^2 + 3^2} = \sqrt{144 + 16 + 9} = \sqrt{160 + 9} = \sqrt{169} = 13$

$OC = \|\overline{OC}\| = \sqrt{2^2 + 3^2 + (-4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$

$\overline{AC} = \overline{AO} + \overline{OC} = -\overline{OA} + \overline{OC}$

$= -12\bar{i} + 4\bar{j} - 3\bar{k} + 2\bar{i} + 3\bar{j} - 4\bar{k} =$

$= -10\bar{i} + 7\bar{j} - 7\bar{k}$

$AC = \|\overline{AC}\| = \sqrt{100 + 49 + 49} = \sqrt{198}$

$\left.\begin{array}{l} AO^2 = 169 \\ OC^2 = 29 \\ AC^2 = 198 \end{array}\right| \Rightarrow AO^2 + OC^2 = AC^2$

$\xRightarrow{R.T.P.\perp} \Delta AOC$ dr, $\hat{O} = 90°$

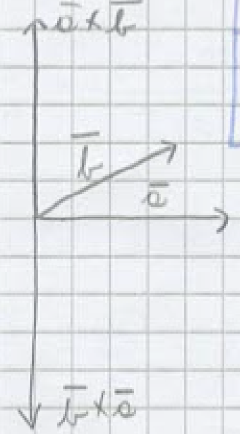
c) $\overline{BC}(-1, -9, 0)$

$\Rightarrow \|BC\| = \sqrt{1+81+0} = \sqrt{82}$

$\Rightarrow P_{ABC} = AB + BC + AC = \sqrt{386} + \sqrt{82} + \sqrt{198}$

Produsul vectorial se notează cu „x"

$\bar{a} \times \bar{b} =$
- → direcția: $\perp (\bar{a}, \bar{b})$
- → sensul: regula burghiului (sens trig ↑)
- → lungimea: $\|\bar{a} \times \bar{b}\| = \underset{+}{\|\bar{a}\|} \cdot \underset{+}{\|\bar{b}\|} \cdot \sin\underset{+}{(\bar{a}, \bar{b})} \geq 0$



Proprietăți: 1) $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$ (anticomutativ)

2) $A_{\square(\bar{a}, \bar{b})} = \|\bar{a} \times \bar{b}\|$

$A_{\Delta}(\bar{a}, \bar{b}) = \frac{1}{2}\|\bar{a} \times \bar{b}\|$

3) $\bar{a}, \bar{b}$ − coliniari ($\exists\, \alpha \in \mathbb{R}$ a.î. $\bar{b} = \alpha \bar{a}$)
sau $\bar{a} \parallel \bar{b} \Rightarrow (\bar{a}, \bar{b}) \in \{0, 180\}$

$\Leftrightarrow \bar{a} \times \bar{b} = 0$

Expresia analitică:

$\bar{a}(a_1, a_2, a_3)$
$\bar{b}(b_1, b_2, b_3)$
$\Rightarrow \bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$

$= \bar{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \bar{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \bar{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

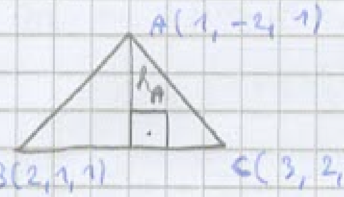
Ex. 4 Calculați $A_{\Delta ABC}$ și h_A pt.:

a) $A(1,-2,1)$, $B(2,1,1)$, $C(3,2,-6)$

Temă b) $A(1,2,1)$, $B(3,0,1)$, $C(1,2,0)$

Temă c) $A(1,0,0)$, $B(1,0,1)$, $C(2,3,1)$

a)



$A_{\Delta ABC} = \frac{1}{2} \| \overline{AB} \times \overline{BC} \| =$

$\overline{AB} = (1, 3, 0)$

$\overline{BC} = (1, 1, -7)$

$$\overline{AB} \times \overline{BC} = \begin{vmatrix} i & j & k \\ 1 & 3 & 0 \\ 1 & 1 & -7 \end{vmatrix} = \bar{i}\begin{vmatrix} 3 & 0 \\ 1 & -7 \end{vmatrix} - \bar{j}\begin{vmatrix} 1 & 0 \\ 1 & -7 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} =$$

$$= -21\bar{i} + 7\bar{j} - 2\bar{k}$$

$$= \frac{1}{2}\sqrt{(-21)^2+(7)^2+(-2)^2} = \frac{1}{2}\sqrt{441+49+4} = \frac{1}{2}\sqrt{494}$$

$$A_{\Delta ABC} = \frac{BC \cdot h_A}{2} \Rightarrow h_A = \frac{2\cdot\left(\frac{1}{2}\sqrt{494}\right)}{\sqrt{51}} = \frac{\sqrt{494}}{\sqrt{51}}$$

Ex. 5 $\bar{a}(2,1,2)$

$\bar{b}(3,1,4)$

$\bar{c}(2,2,5)$

Det.: a) Suprafața de arie maximă a fețelor paralelipipedului $(\bar{a}, \bar{b}, \bar{c})$

b) Găsiți un vector $\perp \bar{a}$ și $\perp b$ de lungime 2.

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{vmatrix} = \bar{i} \underbrace{\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}}_{2} - \bar{j} \underbrace{\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix}}_{2} + \bar{k} \underbrace{\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}}_{-1} =$$

$$= 2\bar{i} - 2\bar{j} - \bar{k}$$

b) $\bar{v} \perp \bar{a} \Rightarrow$ coliniar cu $\bar{a} \times \bar{b}$

$\bar{v} \perp \bar{b} \Rightarrow \bar{v} = \alpha(\bar{a} \times \bar{b})$

$\|\bar{v}\| = 2$

$\|\alpha(\bar{a} \times \bar{b})\| = 2$

$|\alpha| \cdot \underbrace{\|\bar{a} \times \bar{b}\|}_{3} = 2$

$|\alpha| = \frac{2}{3}$ <
- $\alpha = \frac{2}{3} \Rightarrow \bar{v}_1 = \frac{2}{3}(2\bar{i} - 2\bar{j} - \bar{k})$
- $\alpha = -\frac{2}{3} \Rightarrow \bar{v}_2 = -\frac{2}{3}(2\bar{i} - 2\bar{j} - \bar{k})$

Ex 6 $\bar{a} = \bar{m} + 2\bar{n}$

$\bar{b} = \bar{m} - 3\bar{n}$

$\|\bar{m}\| = 1, \ \|\bar{n}\| = 2$

$\sphericalangle(m, n) = \frac{\pi}{3}$

a) lungimile lat. ▭$(\bar{a}, \bar{b})$

Temă b) lungimile diag. ▭$(\bar{a}, \bar{b})$ și $\sphericalangle$ dintre ele

c) $A_{▭}(\bar{a}, \bar{b})$

b) $\bar{d}_1 = \bar{a} + \bar{b} = 2\bar{m} - \bar{n}$

$\bar{d}_2 = \bar{a} - \bar{b} = 5\bar{n}$

$\|\bar{d}_2\| = \|5\bar{n}\| = |5| \cdot \|\bar{n}\| = 5\|\bar{n}\| = 5 \cdot 2 = 10$

b) $\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{(\vec{m}+2\vec{n})\cdot(\vec{m}+2\vec{n})} =$

$= \sqrt{\vec{m}\cdot\vec{m} + 2\cdot\vec{m}\cdot\vec{n} + 2\vec{n}\cdot\vec{m} + 4\vec{n}\cdot\vec{m}}$

$\|\vec{m}\|^2 = 1^2$

$\vec{m}\cdot\vec{n} = \|\vec{m}\|\cdot\|\vec{n}\|\cdot\cos(\widehat{m,n}) = 1\cdot2\cdot\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$

$= \sqrt{1+2+2+16} = \sqrt{21}$

$\|\vec{b}\| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{(\vec{m}-3\vec{n})\cdot(\vec{m}-3\vec{n})} =$

$= \sqrt{\underbrace{\vec{m}\cdot\vec{m}}_{\|\vec{m}\|^2=1^2} - \underbrace{3\vec{m}\cdot\vec{n}}_{1} - \underbrace{3\vec{n}\cdot\vec{m}}_{1} + \underbrace{9\vec{n}\cdot\vec{n}}_{\|\vec{n}\|^2=4}}$

c) $A_{\square(\vec{a},\vec{b})} = \|\vec{a}\times\vec{b}\| = \|(\vec{m}, 2\vec{n}) \times (\vec{m}-3\vec{n})\| =$

$= \|\underbrace{\vec{m}\times\vec{m}}_{0} - 3\vec{m}\times\vec{n} + \underbrace{2\vec{n}\times\vec{m}}_{-\vec{m}\times\vec{n}} - \underbrace{6\vec{n}\times\vec{n}}_{0}\| =$

$= \|-5\vec{m}\times\vec{n}\| = |-5|\cdot\|\vec{m}\times\vec{n}\| = 5\cdot\|\vec{m}\|\,\|\vec{n}\|\cdot\sin(\vec{m},\vec{n}) =$

$= 5\cdot1\cdot2\cdot\sin\frac{\pi}{3} = 10\cdot\frac{\sqrt{3}}{\cancel{2}} = 5\sqrt{3}$