Chapter 1

Seminar — 3 Oct. 2023, Rev. 1

1.1 Studiaţi monotonia şirului

$x_n = \frac{3n+5}{4n+6}, n \in \mathbb{N}$

$x_{n+1} = \frac{3(n+1)+5}{4(n+1)+6} = \frac{3n+8}{4n+1}$

$$\begin{aligned} x_{n+1} - x_n &= \frac{3n+8}{4n+1} - \frac{3n+5}{4n+7} = \\ &= \frac{(4n+7)(4n+8)}{(4n+1)(4n+7)} = \\ &= \frac{(3n+8)(4n+7)-(3n+5)(4n+1)}{(4n+1)(4n+7)} = \\ &= \frac{(12n+56)-(12n+5)}{16n+7} = \\ &= \frac{51}{16n+7} \end{aligned}$$

1.2 Calculaţi $\lim\limits_{n\to\infty} n \cdot \ln{(1 - \frac{1}{n})}$

$\lim\limits_{n\to\infty} n \cdot \ln{(1 - \frac{1}{n})} = 0 \cdot \infty$

$$C.N.: \frac{0}{0}; \frac{\infty}{\infty}; 1^{\infty}; 0 \cdot \infty; 0^0; \infty - \infty; 0^{\infty}$$

1.2.1 Metoda I

Criteriul log.: $\lim\limits_{n\to\infty} \frac{\ln(1+x_n)}{x_n} \overset{\frac{0}{0}}{=} 1$, pentru $\lim\limits_{n\to\infty} x_n = 0$

$l = \lim\limits_{n\to\infty} \frac{n \cdot \ln(1-\frac{1}{n})}{-\frac{1}{n}} \cdot (-\frac{1}{n}) = \lim\limits_{n\to\infty} \cancel{n}(-\frac{1}{\cancel{n}}) = -1$

1.2.2 Metoda II

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

Criteriul lui Euler: $\lim_{n\to\infty} (1+x_n)^{\frac{1}{x_n}} = l$, pentru $\lim_{n\to\infty} x_n = 0$

$$l = \lim_{n\to\infty} \ln{(1-\tfrac{1}{n})}^n = \ln(\lim_{n\to\infty} \underbrace{(1-\frac{1}{n})^n}_{1^\infty}) = \ln[\lim_{n\to\infty}(1-\tfrac{1}{n})]$$

$$\lim n \to \infty (x_n)^{y^n} = (\lim_{n\to\infty}, x_n)$$

1.3 $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-\sqrt{1+x^2}}{x}$

$$\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-\sqrt{1+x^2}}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{1+x+x^2-(1+x^2)}{x(\sqrt{1+x+x^2}+\sqrt{1+x^2})} =$$
$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x+x^2}+\sqrt{1+x^2})} =$$
$$= \frac{1}{2}$$

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$$
$$(x-y)(x+y) = x^2-y^2$$

1.4 Studiaţi continuitatea funcţiei

$$f(x) = \begin{cases} \frac{\sqrt{1+x}-x}{x}, \; x< 0 \\ \frac{\lim{(1+2x)}}{x}, \; x\geq 0 \end{cases}$$

I. $x < 0 \Rightarrow f(x) = \frac{\sqrt{1+x}-1}{x}$ — elementară, deci continuă pe $x \in (-\infty, 0)$

II. $x > 0 \Rightarrow f(x) = \frac{\ln{(1+2x)}}{x}$ — elementară, deci continuă pe $x \in (0, \infty)$

III. $x = 0 :$ f — continuă $\Leftrightarrow ls(0) = ld(0) = f(0)$

$$ls(0) = \lim_{\substack{x\to 0\\ x<0}} \frac{x(\sqrt{1+x}-1)}{x} =$$
$$= \lim_{\substack{x\to 0\\ x<0}} \frac{1+x-1}{x(\sqrt{1+x}+1)} =$$
$$= \lim_{\substack{x\to 0\\ x<0}} \frac{x}{x(\sqrt{1+x}+1)} =$$
$$= \frac{1}{2}$$

$\lim\limits_{\substack{x\to 0\\ x>0}} \ln \frac{(1+2x)}{2x} \cdot 2 = 2$

$ls(0) \neq ld(0) \Rightarrow f$ nu este continuă

1.5

a. $f(x) = \arctan \frac{1-x^2}{1+x^2}$, $f'(x) = ?$

$$\begin{aligned} f'(x) &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \left(\frac{1-x^2}{1+x^2}\right)' = \\ &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \frac{(-2x)(xx^2)-(1-x^2)\cdot 2x}{(1+x^2)^2} = \\ &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \frac{-2x-\cancel{2x^3}-2x+\cancel{2x^2}}{(1+x^2)^2} = \\ &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot -\left(\frac{4x}{(1+x^2)^2}\right) \end{aligned}$$

b. $f'(x) = x^x$

$$a^b = 2^{b\ln a}$$

$$f'(x) = (e^x \cdot \ln x)' = (e^{u(x)})' = e^{u(x)} \cdot u'(x)$$

$$(e^x)' = e^x$$

$$(e^{u(x)})' = e^{u(x)} \cdot u'(x)$$

$$\begin{aligned} &e^x \cdot \ln^x \cdot [x' \cdot \ln x + x \cdot (\ln x)'] = \\ &= e^x \cdot \ln^x \cdot (\ln x + x \cdot \tfrac{1}{x}) = \\ &= e^x \cdot \ln^x \cdot \ln x \end{aligned}$$

c. $f'(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0')}{x-x_0}$ $f'(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0')}{x-x_0} = \lim\limits_{x\to x_0} \frac{e^x-e^{x_0}}{x-x_0}$

Criteriul exponenţial: $\lim\limits_{x\to a} \frac{e^{f(x)}-1}{f(x)} = 1$, pentru $\lim\limits_{x\to a} f(x) = 0$

$$\lim_{x\to x_0} \frac{e^x-e^{x_0}}{x-x_0} \overset{\frac{0}{0}}{=} \lim_{x\to x_0} \frac{e^{x-x_0}-1}{x-x_0} = e^{x_0}$$

1.6

$$\int \frac{\cos x - \sin x}{\cos x + \sin x} dx = \int \frac{1}{t} dt =$$
$$= \ln t + C =$$
$$= \ln(-\sin x + \cos x) + C$$

$\cos x + \sin x = t \mid d \Leftrightarrow (-\sin x + \cos x) \cdot dx = dt$

$$x_1 \neq \; x_2 \Rightarrow f(x_1) \neq f(x_2) \text{ — funcţie injectivă}$$
$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ — funcţie injectivă}$$

1.7

$\int \frac{x^3}{(1+x^2)^2} dx = \int \frac{x \cdot x^2}{(1+x^2)^2} dx = \int \frac{2x \cdot x^2}{(1+x^2)^2} \cdot \frac{1}{2} dx = \frac{1}{2} \int \frac{x^2}{1+x^2} 2x \; dx$

$(1+x^2)^2 = t \mid d \Leftrightarrow 2 \cdot (1+x^2)(1+x^2)' dx = dt$

$1 + x^2 = t \mid d \Leftrightarrow 2x \; dx = dt =$

$$= \int \frac{1}{x} - \int t^2 =$$
$$= \ln(t) - t \frac{-2+1}{2-2+1} + C$$