AnMate - 18 - S - 2023 - 12 - 05 05. 12. 2023

## Serii numerice (continuare)

$\sum_{n=0}^{\infty} x_n$ - serie numerică, $S_n = \sum_{k=0}^{n} x_k$ - şirul sumelor parţiale

1) Studiaţi natura seriilor

a) $\sum_{n=1}^{\infty} = \frac{1}{16n^2+8n-3}$

$S_n = \sum_{k=1}^{n} \frac{1}{16k^2+8k-3} =$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_{k+1} - b_k = \cancel{b_2} - b_1 + \cancel{b_3} - \cancel{b_2} + \cancel{b_4} - \cancel{b_3} + \ldots + b_{n+1} - \cancel{b_n}$$

$a_k = b_{k+1} - b_k$

$$= b_{n+1} - b_1$$

$16k^2 + 8k - 3 = 0$

$\Delta = 64 + 12 \cdot 16 = 256$

$k_{1,2} = \frac{-8 \pm 16}{32}$ ⟨ $\frac{1}{4}$, $-\frac{3}{4}$ $\Rightarrow$ $16k^2 + 8k - 3 = 16(k - \frac{1}{4})(k + \frac{3}{4}) =$

$= (4k-1)(4k+3)$

$$= \sum_{k=1}^{n} \frac{1}{(4k-1)(4k+3)} = \frac{1}{4} \sum_{k=1}^{n} \frac{4k+3-(4k-1)}{(4k+3)(4k-1)} =$$

$$= \frac{1}{4}\left(\sum_{k=1}^{n} \frac{\cancel{4k+3}}{\cancel{(4k+3)}(4k-1)} - \frac{\cancel{4k-1}}{(4k+3)\cancel{(4k-1)}}\right) =$$

$$= \frac{1}{4}\sum_{k=1}^{n} \frac{1}{4k-1} - \frac{1}{4k+3} = \frac{1}{4}\left(\frac{1}{3} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \frac{1}{11} + \dots + \cancel{\frac{1}{4n-1}} - \frac{1}{4n+3}\right) =$$

$$= \frac{1}{4}\left(\frac{1}{3} + \frac{1}{4n+3}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} = \frac{1}{12} - \frac{1}{16n+12} = \frac{1}{12} \Rightarrow$$

$$\Rightarrow \sum \frac{1}{16n^2+8n-3} - C \qquad = \sim \frac{1}{12}$$

b) $$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

$$S_n = \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n} \ln(k+1) - \ln k =$$

$$= \cancel{\ln 2} - \ln 1 + \cancel{\ln 3} + \cancel{\ln 2} + \dots + \ln(n+1) - \cancel{\ln n} =$$

$$= \ln(n+1) \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = \infty \Rightarrow$$

$$\Rightarrow \ln\left(\frac{n+1}{n}\right) - D$$

c) $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

$$S_n = \sum_{k=1}^{n} \frac{\overset{\sqrt{k+1}-\sqrt{k}}{1}}{\sqrt{k} + \sqrt{k+1}} = \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{k+1-k} =$$

$$= \sum_{k=1}^{n} \sqrt{k+1} - \sqrt{k} =$$

$$= \cancel{\sqrt{2}} - \sqrt{1} + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \dots + \sqrt{n+1} - \cancel{\sqrt{n}} =$$

$$= \sqrt{n+1} - 1 \Rightarrow \lim_{n \to \infty} S_n = \infty$$

d) $$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k\sqrt{k+1} + (k+1)\sqrt{k}} =$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{k^2(k+1)} + \sqrt{k(k+1)^2}} =$$

$$= \sum_{k=1}^{n} \frac{\frac{\sqrt{k+1} - \sqrt{k}}{1}}{\sqrt{k(k+1)}\left(\sqrt{k} + \sqrt{k+1}\right)} =$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k(k+1)}} =$$

$$= \sum_{k=1}^{n} \frac{\cancel{\sqrt{k+1}}}{\sqrt{k} \cdot \cancel{\sqrt{k+1}}} - \frac{\cancel{\sqrt{k}}}{\cancel{\sqrt{k}} \cdot \sqrt{k+1}} =$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} = \frac{1}{1} - \frac{1}{\sqrt{2}} \dots + \cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}} =$$

$$= 1 - \frac{1}{\sqrt{n+1}} \dots \lim$$

e) $\sum_{n=0}^{\infty} \frac{n}{(n+1)!}$ — Convergent

$$S_n = \sum_{k=0}^{n} \frac{k}{(k+1)!} = \sum_{k=0}^{n} \frac{k}{k!(k+1)} =$$

$$= \sum_{k=0}^{n} \frac{k+1-1}{k!(k+1)} = \sum_{k=0}^{n} \frac{\cancel{k+1}}{k!\cancel{(k+1)}} - \frac{1}{k!(k+1)} =$$

$$= \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{(k+1)!} = \frac{1}{1!} - \cancel{\frac{1}{2!}} + \cancel{\frac{1}{2!}} \cdots + \cancel{\frac{1}{n!}} - \frac{1}{(n+1)!} =$$

$$= 1 - \frac{1}{(n+1)!} \Rightarrow \lim_{n \to \infty} S_n = 1$$

f) $\sum_{n=1}^{\infty} \frac{n}{n!} \overset{?}{=} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{m=0}^{\infty} \frac{1}{m!} = e$

$n-1 = m$

$n = 1 \Rightarrow m = 0$

g) $\sum_{n=0}^{\infty} \frac{2^n + 3^n + (-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n + \left(-\frac{1}{5}\right)^n =$

$$= \sum \left(\frac{2}{5}\right)^n + \sum \left(\frac{3}{5}\right)^n + \sum \left(-\frac{1}{5}\right)^n =$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}, \text{ dacă } \alpha \in (-1, 1)$$

$$= \frac{1}{1-\frac{2}{5}} + \frac{1}{1-\frac{3}{5}} + \frac{1}{1+\frac{1}{5}} = \frac{5}{3} + \frac{5}{2} + \frac{5}{6} \dots$$

$$\sum_{k=0}^{n} \alpha^k = 1 + \alpha + \alpha^2 + \dots + \alpha^n = 1 \cdot \frac{1-\alpha^{n+1}}{1-\alpha}$$

$\alpha^{n+1} \to 0$ dacă $\alpha \in (-1, 1)$

$$\lim_{n\to\infty} \Rightarrow \sum_{k=0}^{\infty} \alpha^k = \lim_{n\to\infty} \frac{1-\alpha^{n+1}}{1-\alpha} = \frac{1}{1-\alpha}$$

h) $$\sum_{n=0}^{\infty} \frac{4n^4+9n^2+17n+1}{5n^4+8n^2+9n^3-2n-1}$$

$$\lim_{n\to\infty} \frac{4n^4+9n^2+17n+1}{5n^4+8n^2+9n^3-2n-1} = \frac{4}{5} \neq 0 \Rightarrow \sum(\dots) - D$$

$\sum x_n - C \Rightarrow \lim_{n\to\infty} x_n = 0$ Reciproca nu e adevărată

i) $$\sum_{n=0}^{\infty} \frac{n+1}{n^4+1} \simeq \sum \frac{n}{n^4} = \sum \frac{1}{n^3}$$

$\sum \frac{1}{n^3} - C$. (Serie Riemann pt $\alpha = 3$

$$\boxed{\sum \frac{1}{n^\alpha} - \begin{cases} C, & \alpha > 1 \\ D, & \alpha \leq 1 \end{cases}}$$

j) $\sum_{n=0}^{\infty} \frac{3n\sqrt{n}+5}{n^2+n+1}$

Criteriul comparației cu inegalități:

$\sum x_n$, $\sum y_n$, $x_n, y_n \geq 0$, $\forall n \in \mathbb{N}$ și

și $l = \lim_{n\to\infty} \frac{x_n}{y_n} \in (0, \infty)$. Atunci

$$\sum x_n \simeq \sum y_n$$

→ au aceeași natură

De obicei: $\sum x_n$ - seria a cărei natură vrem să o aflăm,

$$\sum y_n = \sum \frac{1}{n^\alpha}$$

$x_n = \frac{3n\sqrt{n}+5}{n^2+n+1}$ și $y_n = \frac{1}{n^\alpha}$,

$$l = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\frac{3n\sqrt{n}+5}{n^2+n+1}}{\frac{1}{n^\alpha}} = \lim_{n\to\infty} \frac{3n \cdot n^{\frac{1}{2}}+5}{n^2+n+1} \cdot n^\alpha$$

$$= \lim_{n\to\infty} \frac{3n^{3/2+\alpha}+5n^\alpha}{n^2+n+1} \overset{\alpha=\frac{1}{2}}{=\!=} \lim_{n\to\infty} \frac{3n^2+5n^{1/2}}{n^2+n+1} =$$

$$= 3 \in (0,\infty) \Rightarrow \sum x_n \simeq \sum \frac{1}{n^{1/2}}$$

$$\sum \frac{1}{\sqrt{n}} \text{ — D}$$

$$\sum_{n=0}^{\infty} \frac{3n\sqrt{n}+5}{n^2+n+1} \simeq \sum \frac{n\sqrt{n}}{n^2} = \sum \frac{\sqrt{n}}{n} = \sum \frac{1}{\sqrt{n}}$$

k) $\sum_{n=1}^{\infty} n \cdot \sin \frac{1}{n^2}$

$x_n = n \cdot \sin\frac{1}{n^2}$ ; $y_n = \frac{1}{n^2}$ ; $l = \lim_{n\to\infty} \frac{n \cdot \sin \frac{1}{n^2}}{\frac{1}{n^2}} =$

$$= \lim_{n\to\infty} \frac{n \cdot \overbrace{\sin \frac{1}{n^2}}^{\to 1}}{\frac{1}{n^2}} \cdot \frac{1}{n^2} \cdot n^2$$

$\lim_{n\to\infty} \frac{n^{\alpha+1}}{n^2} \overset{\alpha=1}{=} 1 \Rightarrow \sum x_n \simeq \sum \frac{1}{n}$

$\sum \frac{1}{n}$ — D.

l) $\sum_{n=1}^{\infty} \frac{1}{n\cdot\sqrt[3]{n}+\sqrt{n}}$

$$\left.\frac{1}{n\cdot\sqrt[3]{n}+\sqrt{n}} < \frac{1}{n\sqrt[3]{n}} \Rightarrow \sum \frac{1}{n\sqrt[3]{n}+\sqrt{n}} < \sum \frac{1}{n\sqrt[3]{n}} = \sum \frac{1}{n^{\frac{4}{3}}} \right\} \Rightarrow$$

$\sum \frac{1}{n^{4/3}}$ — C.

$\Rightarrow \sum x_n$ — C.

$$\left.\begin{array}{l} \frac{1}{n\cdot\sqrt[3]{n}+\sqrt{n}} < \frac{1}{\sqrt{n}} \\ \sum \frac{1}{\sqrt{n}} = \infty \end{array}\right\} \Rightarrow \sum x_n < \infty$$

$$\sum x_n < S$$

$$\left.\sum y_n = \infty \;;\quad \sum x_n > \sum y_n \right\} \Rightarrow$$

$$\Rightarrow \sum x_n = \infty$$

m) $\sum_{n=1}^{\infty} \left(\sqrt{(n+1)(n+a)} - n\right)^n, \; a > 0$

Aplicăm crt. rădăcinii, adică calculăm:

$$l = \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\left(\sqrt{(n+1)(n+a)} - n\right)^{\cancel{n}}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{(n+1)(n+a)} + n}{\sqrt{(n+1)(n+a)} + n}\left(\sqrt{(n+1)(n+a)} - n\right) =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} + n + an + a - \cancel{n^2}}{\sqrt{n^2 + (a+1)n + a} + n} = \lim_{n\to\infty} \frac{(a+1)\cdot n + a}{\sqrt{n^2\left(1 + \frac{a+1}{n} + \frac{a}{n^2}\right)} + n} = \frac{a+1}{2}$$

I Dacă $l > 1 \Rightarrow \sum x_n$ — D

$l > 1 \iff \frac{a+1}{2} > 1 \iff a > 1$

II Dacă $l < 1$ $(a < 1)$ $\Rightarrow \sum x_n$ — C

III. $l = 1 \Rightarrow \sum \left(\sqrt{(n+1)^2} - n\right)^n =$

$\Downarrow$

$a = 1$

$$= \sum (\cancel{n} + 1 - \cancel{n})^n =$$

$$= \sum 1 = \infty$$