Matrice

1. Demonstrați că $\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ are loc identitatea

** $A^2 - (a+d)A + \det(A) I_2 = O_2$ → Cayley - Hamilton

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\boxed{O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$

$tr(A) = a + d$ → urma matricei A

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix}$$

$$(a+d)A = (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

$$= \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\det(A) \cdot I_2 = (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

** $\begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & bc + a^2 \end{pmatrix} = \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} +$

$+ \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = O_2$

$$= \begin{pmatrix} \cancel{a^2} + \cancel{bc} - \cancel{a^2} - \cancel{ad} + \cancel{ad} - \cancel{bc} & \cancel{ab} + \cancel{bd} - \cancel{ba} - \cancel{bd} \\ -\cancel{ac} + \cancel{dc} - \cancel{ca} - \cancel{cd} & \cancel{bc} + \cancel{a^2} - \cancel{dc} - \cancel{d^2} + \cancel{ad} - \cancel{bc} \end{pmatrix}$$

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2. Rez. ec. $X A = B$ pt.

a) $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ -1 & 2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 7 & -13 \\ 5 & 0 & 5 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix}$

Rez.

a. $\det A = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1 \cdot 3 \cdot 2 + 0 + 0 - 3 - 2 - 0 =$

$\begin{matrix} 1 & 0 & -1 \\ 0 & 3 & 1 \end{matrix}$

$= 6 - 5 = 1 \neq 1 \Rightarrow$

$\Rightarrow \exists A^{-1} = \frac{1}{\det A} \cdot A^*$

$$^tA = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 2 \\ -1 & 1 & 2 \end{pmatrix}$$

$$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 1 \cdot 4 = 4$$

$$C_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = -1 \cdot (+2) = -2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} = -1 \cdot (+3) = +3$$

$$C_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = -1 \cdot (+1) = -1$$

$$C_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 \cdot 1 = 1$$

$$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = -1 \cdot 1 = -1$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 1 \cdot (+3) = 3$$

$$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -1 \cdot 2 = -2$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 1 \cdot 3 = 3$$

$$A^{-1} = \frac{1}{1} \cdot \begin{pmatrix} 4 & -2 & -3 \\ -1 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix} =$$

$$X \cdot A = B \quad | A^{-1} \text{ ld}$$

$$X \cdot \underbrace{A \cdot A^{-1}}_{} = B \cdot A^{-1}$$

$$X \cdot I_3 = B \cdot A^{-1}$$

$$X = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 4 & -2 & 3 \\ -1 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix}$$

$$X = \begin{pmatrix} 10 & -6 & 9 \\ -2 & 1 & -1 \\ -10 & 7 & -10 \end{pmatrix}$$

b) $XA = B$

tip: $(m, n) \cdot (2, 3) = (2, 3) \Rightarrow X \in M_2(\mathbb{R})$

fie $X \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \Rightarrow XA = B \Leftrightarrow$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 & -13 \\ 5 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} a + 2b & 2a - b & -3a + 4b \\ c + 2d & 2c - d & -3c + 4d \end{pmatrix}$$

$$\begin{cases} a + 2b = 1 \\ 2a - b = 7 \\ -3a + 4b = 13 \quad (+) \end{cases}$$

$$/ \;+ 5b = -5 \Rightarrow \boxed{b = -1}$$

$a + 2b = 1$

$a - 2 = 1$

$\boxed{a = 3}$

$$\begin{cases} c + 2d = 5 \\ 2c - d = 0 \\ -3c + 4d = 5 \quad (+) \end{cases}$$

$$/ \;+ 5d = 10 \Rightarrow \boxed{d = 2}$$

$c + 2d = 5$

$c + 4 = 5$

$\boxed{c = 1}$

$$X = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$$

c) $tip = (m, n) \cdot (3, 3) = (2, 3)$

$$X = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$X \cdot A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} =$$

$$\Rightarrow \begin{pmatrix} a+2c & 2a+b & 2b+c \\ d+2f & 2d+e & 2e+f \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix}$$

$$\begin{cases} a+2c = 2 \\ 2a+b = 1 \\ 2b+c = 0 \Rightarrow c = -2b \end{cases}$$

$$\begin{cases} a - 4b = 2 \quad | \cdot (-2) \\ 2a+b = 1 \end{cases}$$

$$\begin{cases} -2a + 8b = -4 \\ 2a + b = 1 \end{cases} +$$

$9b = -3 \Rightarrow b = -\frac{1}{3} \Rightarrow c = -2 \cdot -\frac{1}{3} = \frac{2}{3}$

$a + \frac{4}{3} = 2 \Rightarrow a = 2 - \frac{4}{3} = \frac{2}{3}$

$$\begin{cases} d + 2f = -1 \quad | \cdot (-2) \\ 2d + e = 3 \\ 2e + f = 2 \end{cases}$$

$$\begin{cases} 2d + 4f = 2 \\ 2d + e = 3 \\ 2e + f = 2 \end{cases}$$

$$\begin{cases} 2d + 4f = 2 \\ 2d + e = 3 \end{cases} \oplus$$

$e - 4f = 5$

$2e + f = 2 \quad | \cdot 4 \Rightarrow \begin{cases} e - 4f = 5 \\ 8e + 4f = 8 \end{cases} \oplus$

$9e = 13 \Rightarrow e = \frac{13}{9}$

$$\frac{26}{9}+f=2 \Rightarrow f=2-\frac{26}{9}=\frac{-8}{9}$$

$$d-\frac{16}{9}=-1 \Rightarrow d=-1+\frac{16}{9}=\frac{7}{9}$$

$$X=\begin{pmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\\ \frac{7}{9} & \frac{13}{9} & -\frac{8}{9}\end{pmatrix}=\frac{1}{9}\begin{pmatrix}6 & -3 & 6\\ 7 & 13 & 8\end{pmatrix}$$

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3. Rez. sistemul

$$\begin{cases}2X-5Y=A\\ -X+3Y=B\end{cases}$$

unde $A=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}$

$B=\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}$

$$\begin{cases}2X-5Y=A\\ -X+3Y=B \mid \cdot 2\end{cases}$$

$$\begin{cases}2X-5Y=A\\ -2X+6Y=2B \quad +\end{cases}$$

$Y=A+2B \Leftrightarrow Y=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}+2\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}$ *

* $Y=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}+\begin{pmatrix}4 & 2\\ 6 & 0\end{pmatrix} \Rightarrow Y=\begin{pmatrix}5 & 0\\ 6 & 1\end{pmatrix}$

$2X-5Y=A$

$2X=A+5Y$

$2X=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}+5\begin{pmatrix}5 & 0\\ 6 & 1\end{pmatrix}$

$2X=\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}+\begin{pmatrix}25 & 0\\ 30 & 5\end{pmatrix} \Rightarrow 2X=\begin{pmatrix}26 & -2\\ 30 & 6\end{pmatrix} \Rightarrow X=\begin{pmatrix}13 & -1\\ 15 & 3\end{pmatrix}$

4. Aflați toate matricele inversabile $A \in M_2(\mathbb{N})$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{N}$

A este inversabilă $\Leftrightarrow$ $\det A \neq 0 \Leftrightarrow ad - bc \neq 0$

$\exists A^{-1} = \frac{1}{\Delta A} \cdot A^* \in M_2(\mathbb{N})$

$\det A = ad - bc$

$A^{t} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Rightarrow A^* = \begin{pmatrix} +d & -b \\ -c & +a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \in M_2(\mathbb{N}) \Rightarrow ad - bc = \pm 1$

I $ad - bc = 1$

$\left.\begin{array}{l} A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in M_2(\mathbb{N}) \\ a, b, c, d \in \mathbb{N} \end{array}\right\} \Rightarrow \left.\begin{array}{r} b = 0, c = 0 \Rightarrow ad = 1 \\ a, d \in \mathbb{N} \end{array}\right| \Rightarrow$

$\Rightarrow a = d = 1$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

II $ad - bc = -1$

$\left.\begin{array}{l} A^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{N}) \\ a, b, c, d \in \mathbb{N} \end{array}\right| \Rightarrow d = a = 0 \Rightarrow \left.\begin{array}{r} -bc = -1 \\ bc = 1 \\ b, c \in \mathbb{N} \end{array}\right| \Rightarrow b = c = 1$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

5. Dem. că ∀ matrice $A \in M_n(\mathbb{R})$ poate fi scrisă ca suma dintre:

- o matrice simetrică B și ($B^t = B$)
- o matrice anti-simetrică C ($C^t = -C$)

$$A = B + C$$

$$A^t = (B+C)^t = B^t + C^t = B - C$$

$$\begin{cases} B + C = A \\ B - C = A^t \end{cases} \quad +$$

$$2B = A + A^t \Rightarrow B = \frac{A + A^t}{2}$$

$$C = A - \frac{A + A^t}{2} \Rightarrow C = \frac{2A - A - A^t}{2} \Rightarrow C = \frac{A - A^t}{2}$$

6. Dem. că dacă matricea A are proprietatea $A^2 = A$, atunci matricea $B = 2A - I_n$ este involutivă ($B^2 = I_n$)

Ind. $A \cdot I_n = I_n \cdot A = A$

$\hookrightarrow B^2 = \dots$ formula de calcul

7. a) $A = \begin{pmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Ind.: $A = I_3 + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{B}$, $B^3 = O_3$

7. a). $A^2 = \begin{pmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{pmatrix} \begin{pmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{pmatrix} =$

$$= \begin{pmatrix} -\cos^2 - \sin^2 x & -2\sin x \cdot \cos x \\ 2\sin x \cdot \cos x & -\sin^2 x + \cos^2 x \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2 x & -\sin^2 x \\ \sin 2x & \cos 2x \end{pmatrix}$$

$A^n = \begin{pmatrix} -\cos nx & -\sin nx \\ \sin nx & \cos nx \end{pmatrix}$

Pr. $P(n)$ adev. Dem: $P(n+1)$ adev.:

$A^{n+1} = A^n \cdot A =$