 17.11.2023

Teorema de medie

Fie $f:[a,b]\to\mathbb{R}$ continuă

Atunci există $c\in[a,b]$ astfel încât $\int_a^b f(x)\,dx = f(c)(b-a)$

Demonstrație:

Deoarece $f:[a,b]\to\mathbb{R}$ este continuă, conform teoremei lui Weierstrass, există $x_*, x^*\in[a,b)$ a. î.

$$f(x_*) \le f(x) \le f(x^*) \quad (\forall)\, x\in[a,b]$$

$$\longrightarrow \int_a^b f(x_*)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(x^*)\,dx$$

$$f(x_*)\int_a^b 1\,dx \le \int_a^b f(x)\,dx \le f(x^*)\int_a^b 1\,dx$$

$$(b-a)f(x^*) \le \int_a^b f(x)\,dx \le (b-a)f(x^*)$$

$$f(x_*) \le \frac{\int_a^b f(x)\,dx}{b-a} \le f(x^*)$$

$\Downarrow$ f are proprietatea lui Darboux

există $c\in[a,b]$ a. î. $\dfrac{\int_a^b f(x)\,dx}{b-a} = f(c)$

TEOREMĂ

Fie $f:[a,b]\to\mathbb{R}$ - continuă

Atunci $F:[a,b]\to\mathbb{R}$, dată de $F(x)=\int_a^x f(t)\,dt$ pentru orice $x\in[a,b]$, este derivabilă și

$F'(x)=f(x)$ pentru orice $x\in[a,b]$

i.e. F este o primitivă a lui f

$f:[a,b]\to\mathbb{R}$ continuă

$\int_a^b f(x)\,dx = F(b)-F(a)$

$F'=f$

$\to \boxed{F(b)=F(a)+\int_a^b F'(x)\,dx}$

$F(x)=\int_a^x f(t)\,dt$

$F'(x)=f(x) \quad (\forall)\, x\in[a,b]$

$\to \boxed{\left(\int_a^x f\right)'=f(x)}$

$f\to\int f\to\left(\int f\right)'\to f$

Demonstrație

Fie $x_0\in[a,b]$

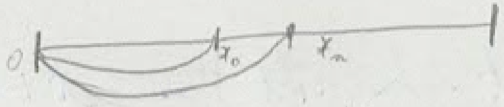
Dorim să arătăm că există $\lim\limits_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0)$

Este suficient să alegem un șir arbitrar $(x_n)_n\subseteq[a,b]-\{x_0\}$ cu $\lim\limits_{n\to\infty}x_n=x_0$

și să arătăm că $\lim\limits_{n\to\infty}\frac{F(x_n)-F(x_0)}{x_n-x_0}=f(x_0)$

$$\frac{F(x_n)-F(x_0)}{x_n-x_0} = \frac{\int_0^{x_n} f(t)\,dt - \int_0^{x_0} f(t)\,dt}{x_n-x_0} = \frac{\int_{x_0}^{x_n} f(t)\,dt}{x_n-x_0} =$$

$$= f(c_n) \xrightarrow{n\to\infty} f(x_0)$$



conform teoremei de medie, există c_n între x_0 și x_n a.î.

$$\int_{x_0}^{x_n} f(t)\,dt = (x_n - x_0)\cdot f(c_n)$$

EXEMPLU

$$\lim_{x\to 0} \frac{\int_0^x \ln(1+t^2)\,dt}{\sin^3 x} = ?$$

$$\int_0^x \ln(1+t^2)\,dt = x\ln(1+c_x^2) \to 0 \ln 1 = 0\cdot 0$$

c_x între 0 și x

$x \to 0$

$c_x \to 0$

$\xrightarrow{V_1}$ l'Hospital

$\xrightarrow{V_2}$ $\int_0^x \ln(1+t^2)\,dt = \ldots$

V_2 $\int_0^x \ln(1+t^2)\,dt =$

$$\int \ln(1+t^2)\,dt = \int t' \cdot \ln(1+t^2)\,dt = t\cdot\ln(1+t^2) - \int t \cdot \ldots$$

$$\ldots \frac{1}{1+t^2}\cdot 2t\,dt =$$

$$= t \cdot \ln(1+t^2) - 2\int \frac{t^2}{1+t^2}\,dt =$$

$$= t \cdot \ln(1+t^2) - 2\left[\int 1\,dt - \int \frac{1}{t^2+1}\,dt\right]$$

$$t \cdot \ln(1+t^2) - 2t + 2\,\mathrm{arctg}\, t$$

$$\lim_{x \to 0} \frac{x \cdot \ln(1+x^2) - 2x + 2\,\mathrm{arctg}\, x}{\sin x^3} \overset{0}{\underset{0}{=}} \text{etc} \ldots$$

$$V_1 \quad \lim_{x \to 0} \frac{\left(\int_0^x \ln(1+t^2)\,dt\right)'}{(\sin^3 x)'} = \lim_{x \to 0} \frac{\ln(1+x^2)}{3\sin^2 x \cdot \cos x} =$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\ln(1+x^2)}{x^2} \cdot \frac{x^2}{\sin^2 x} = \frac{1}{3}$$

Teorema de schimbare de variabilă

Fie $\varphi : [a, b] \to [\alpha ; \beta] \to \mathbb{R}$ a. î.:

i) φ derivabilă

ii) φ' continuă

iii) f continuă

φ' nu este continuă în 0 $\qquad \varphi : \mathbb{R} \to \mathbb{R}$

$$\varphi'(x) \begin{cases} 2\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad \varphi(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Atunci $\int_a^b f(\varphi(x)) \cdot \varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$

<u>Demonstrație</u>

f continuă $\Rightarrow$ există F derivabilă cu $F' = f$

Leibniz-Newton $\Rightarrow \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt = F\Big|_{\varphi(a)}^{\varphi(b)} = F(\varphi(b)) - F(\varphi(a))$

$(f \circ \varphi) \cdot \varphi' = (F' \circ \varphi) \cdot \varphi' =$
$= (F \circ \varphi)'$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$(f \circ g)' = (f' \circ g) \cdot g'$

$\int_a^b h(x)\,dx = \int_a^b f(\underbrace{\varphi(x)}_{\text{expresie în } \varphi}) \cdot \underbrace{\varphi'(x)}_{\varphi'}\,dx = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$

$x = a \rightarrow t = \varphi(a) \qquad \varphi(x) = t$

$x = b \rightarrow t = \varphi(b) \qquad \varphi'(x)\,dx = dt$

EXEMPLU

$$\int_0^1 x^2 \cdot e^{x^3} dx = \frac{1}{3}\int_0^1 e^t dt = \frac{1}{3} e^t \Big|_0^1 (e^1 - e^0) = \frac{e-1}{3}$$

$x^3 = t$

$(x^3)' dx = dt$

$3x^2 dx = dt$

$x^2 dx = \frac{1}{3} dt$

$x = 0 \to t = 0^3 = 0$

$x = 1 \to t = 1^3 = 1$

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2}\int_1^2 \frac{1}{t} dt = \frac{1}{2} \ln t \Big|_1^2 = \frac{\ln 2 - \ln 1}{2} = \frac{\ln 2}{2}$$

$1 + x^2 = t$

$(1+x^2)' dx = dt$

$2x\, dx = dt$

$x\, dx = \frac{1}{2} dt$

$x = 0 \to t = 1$

$x = 1 \to t = 2$

$$\int_0^1 \frac{x}{1+x^4} dx = \int_0^1 \frac{1}{1+(x^2)^2} \cdot x\, dx = \int_0^1 \frac{1}{1+t^2} dt =$$

$1 + x^4 = t$

$4x^3 dx = dt$

$x^2 \cdot \cancel{x}\, dx = \cancel{dt}$

$x^2 = t$

$2x\, dx = dt$

$x\, dx = \frac{1}{2} dt$

$x = 0 \to t = 0$

$x = 1 \to t = 1$

$$= \frac{1}{2} \operatorname{arctg} t \Big|_0^1 = \frac{1}{2}(\operatorname{arctg} 1 - \operatorname{arctg} 0) =$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$$

$$\int_0^{\frac{\pi}{4}} tg\,x + tg^3 x \, dx =$$

$$(tg\,x)' = \frac{1}{\cos^2 x} =$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} =$$

$$= 1 + tg^2 x$$

$$= \int_0^{\frac{\pi}{4}} tg\,x(1 + tg^2 x)\, dx = \int_0^1 t\, dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$tg\,x = t$

$(tg\,x)'\,dx = dt$

$(1 + tg^2 x)\,dx = dt$

Teorema de integrare prin părți

Fie $f, g : [a, b] \to \mathbb{R}$ derivabile a.î. f' și g' continue

Atunci $\int_a^b f'(x) \cdot g(x)\, dx = \boxed{f(b)\, g(b) - f(a)\, g(a)} - \int_a^b f(x) \cdot \ldots$

$\ldots \cdot g'(x)\, dx$

$$\Updownarrow$$

$$\boxed{\int_a^b f' g = f \cdot g\Big|_a^b - \int_a^b f \cdot g'}$$

$(f \cdot g)' = f' \cdot g + f \cdot g'$

$\int_a^b (f \cdot g)' = \int_a^b f' \cdot g + f \cdot g' =$

$f \cdot g \Big|_a^b = \int_a^b f' \cdot g + \int_a^b f \cdot g'$

Exemplu:

$\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx =$

$\int_0^{\frac{\pi}{2}} \left(\frac{x^2}{2}\right)' \sin x \, dx = \frac{x^2}{2} \sin x \Big|_0^{\frac{\pi}{2}} -$

$- \int_0^{\frac{\pi}{2}} \frac{x^2}{2} (\sin x)' dx =$

$= \frac{\pi}{8}^2 - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cdot \cos x \, dx =$

$= ???$

$\int_a^{\frac{\pi}{2}} (-\cos x)' \cdot x \, dx =$

$= (-\cos x) \cdot x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) \cdot x' \cdot dx =$

$= \int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1$

$$\int_0^1 x \cdot \operatorname{arctg} x \, dx = \int_0^1 \left(\frac{x^2}{2}\right)' \operatorname{arctg} x \, dx =$$

$$= \frac{x^2 \cdot \operatorname{arctg} x}{x} \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\, dx =$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}\, dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 1 \, dx + \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\, dx =$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{2} \operatorname{arctg} x \Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}$$

$$\int_1^e \ln x \, dx = \int_1^e x' \cdot \ln x \, dx =$$

$$= x \ln x \Big|_1^e - \int_1^e x \cdot \frac{1}{x}\, dx =$$

$$= e - (e-1) = 1$$

$$\underline{\underline{I}} = \int_0^\pi e^x \cdot \sin x \, dx = \int_0^\pi (e^x)' \sin x \, dx = e^x \sin x \Big|_0^\pi - \int_0^\pi e^x \cdot (\sin x)' dx =$$

$$= -\int_0^\pi e^x \cos x \, dx =$$

$$= -\int_0^\pi (e^x)' \cos x \, dx =$$

$$= -\left[e^x \cdot \cos x \Big|_0^\pi - \int_0^\pi e^x \cdot (\cos x)' \, dx\right] =$$

$$= -\left[-e^{-\pi} - 1 + \int_0^\pi e^x \sin x \, dx\right] =$$

$$= -\int_0^\pi e^x \sin x + 1 + \frac{1}{e^\pi} = 1 + \frac{1}{e^\pi} - I$$

$2I = 1 + \frac{1}{e^{\pi}}$

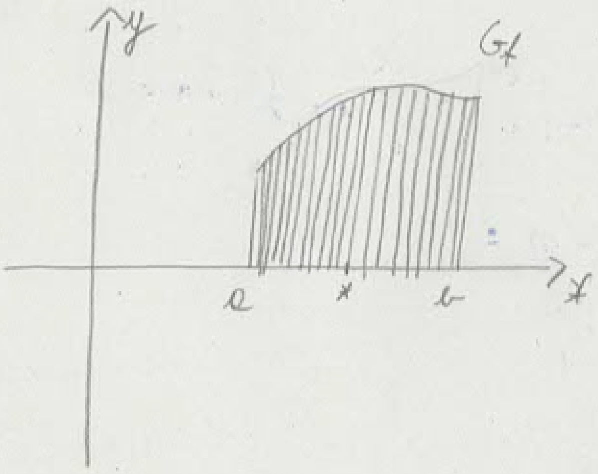
$I = \frac{1}{2}\left(1 + \frac{1}{e^{\pi}}\right)$

Teoremă

$f : [a, b] \to \mathbb{R}_+$

integrabilă

$S = \{(x, y) \mid x \in [a, b],\ y \in [0, f(x)]\}$



$\Rightarrow S$ are arie și aria lui $S = \int_a^b f(x)\,dx$

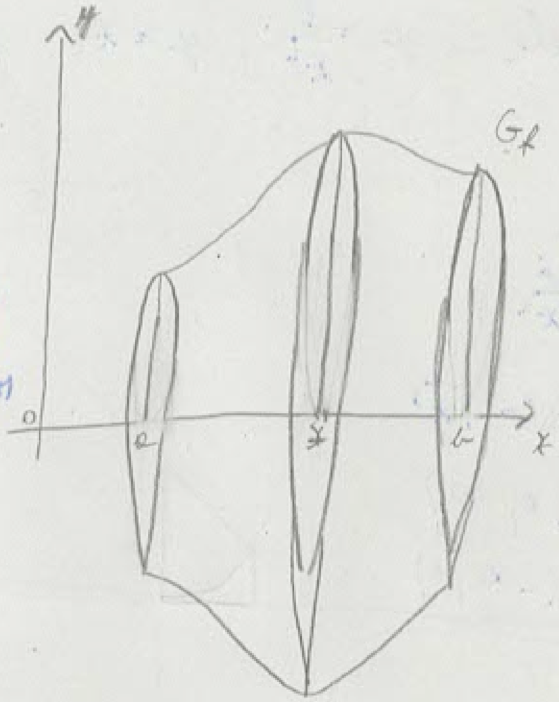
TEOREMĂ

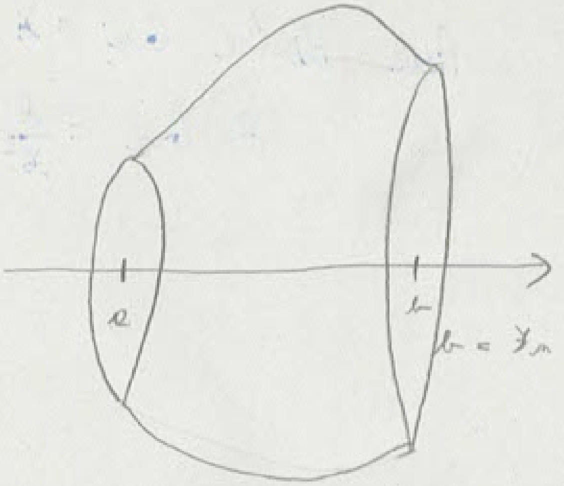
volumul unui corp

$f : [a, b] \to \mathbb{R}_+$
integrabilă

$V = \{(x, y, z) \in \mathbb{R}^3 \,/\, x \in [a, b],\ y^2 + z^2 \le f^2(x)\}$
are volum

și volumul lui V este





EXEMPLU

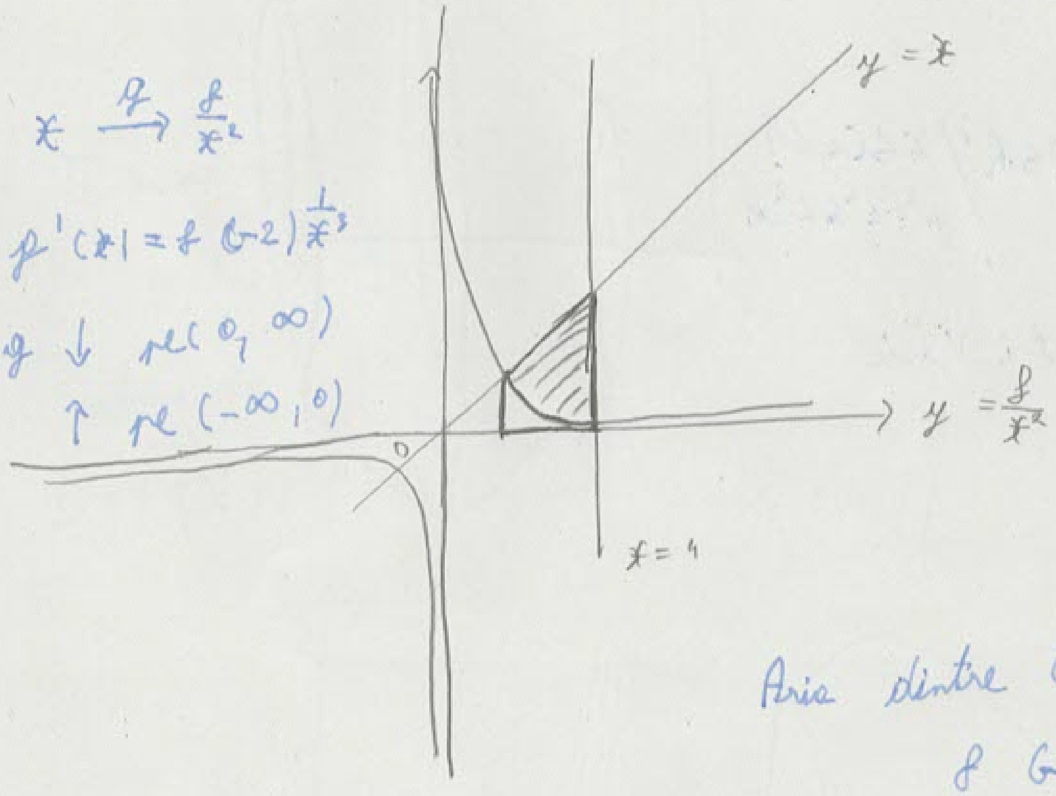
Să se calculeze aria suprafeței plane determinată de curbele $y = \frac{8}{x^2}$, $y = x$ și $x = 4$

$x \xrightarrow{f} \frac{8}{x^2}$

$f'(x) = 8(-2)\frac{1}{x^3}$

$f \downarrow$ pe $(0, \infty)$

$\uparrow$ pe $(-\infty, 0)$



Aria dintre $G_{y=x}$

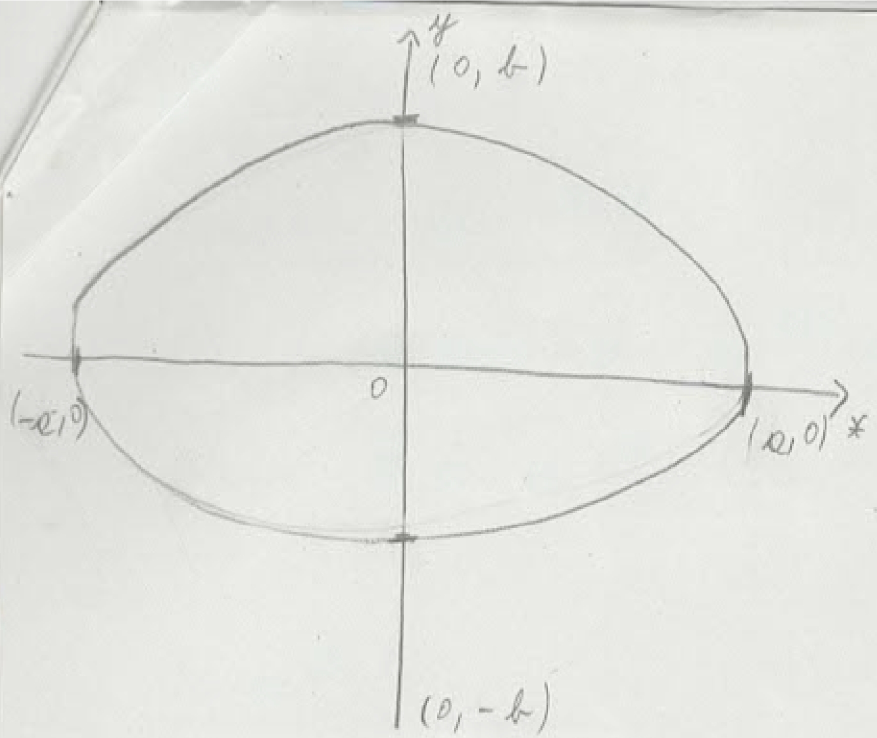
& $G_{y=\frac{8}{x^2}}$

$y = x$

$y = \frac{8}{x^2}$

$\Rightarrow x^3 = 8 \Rightarrow x = 2$

$$A = \int_2^4 x\,dx - \int_2^4 \frac{8}{x^2}\,dx = \frac{x^2}{2}\Big|_2^4 - 8\cdot\frac{(-2)}{x^3}\Big|_2^4 = \dots$$



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

$$f : [0, b] \to \mathbb{R}$$

$$f(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

Exemplu

$$V = \pi \int_0^a f^2(x)\,dx =$$

$$= \pi b^2 \left(x\Big|_0^a - \frac{1}{a^2} \cdot \frac{x^3}{3}\Big|_0^a \right) =$$

$$= \pi b^2 \left(a - \frac{1}{\cancel{a^2}} \cdot \frac{a^{\cancel{3}}}{3} \right) =$$

$$= \pi b^2 \cdot \frac{2a}{3} = \frac{2a}{3} b^2 \cdot \pi$$

Volumul elipsoidului de axe a & b este $\frac{4}{2} \pi a b^2$

Volumul sferei $\frac{4}{3} \pi R^3$

Aria cercului πR^2 derivare

Arie sferei $4 \pi R^2$ derivare

Lungimea cercului $2 \pi R$