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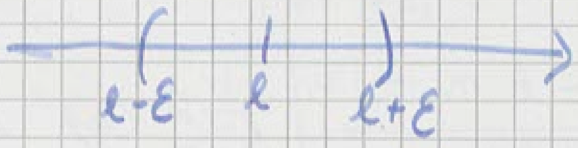
Limite de şiruri

I Def: Fie $(x_n)_{n\in\mathbb{N}}$, şir de numere reale. Spunem că şirul x_n, $n\in\mathbb{N}$ admite limita $l\in\mathbb{R}$ dacă de la un anumit ~~rag~~ rang n_ε $(n \geq n \geq n_\varepsilon)$ doar un număr finit de termeni ai şirului nu se află într-o vecinătate a lui l.



Def: $x_n, n\in\mathbb{N}$ admite limita $l\in\mathbb{R}$ dacă:

$\forall \varepsilon>0, (\exists)\, n_\varepsilon\in\mathbb{N}$ cu $n\geq n_\varepsilon$ a.î. $|x_n - l|<\varepsilon$

$|x_n-l|<\varepsilon \Leftrightarrow -\varepsilon < x_n - l < \varepsilon \;\; |+l \Leftrightarrow l-\varepsilon < \dots$

$\dots < x_n = l+\varepsilon$

Dacă l = finit $\Rightarrow (x_n)_{n\geq 1}$ şir convergent

II. Exerciții

1. Arătați că:

a) $\lim\limits_{n \to \infty} \dfrac{2n+5}{3n+7} = \dfrac{2}{3}$

$$\left| \frac{2n+5}{3n+7} - \frac{2}{3} \right| < \varepsilon \Leftrightarrow \ldots$$

$$\ldots \Leftrightarrow \left| \frac{6n+15-6n-14}{9n+21} \right| < \varepsilon \Leftrightarrow \ldots$$

$$\ldots \Leftrightarrow \left| \frac{1}{9n+21} \right| < \varepsilon \Leftrightarrow \frac{1}{9n+21} < \varepsilon \Leftrightarrow \ldots$$

$$\ldots \Leftrightarrow 9n+21 > \frac{1}{\varepsilon} \Leftrightarrow 9n > \frac{1}{\varepsilon} - 21$$

$$9n > \frac{1-21\varepsilon}{\varepsilon}$$

$$n > \frac{1-21\varepsilon}{9\varepsilon} \Rightarrow n_\varepsilon = \left[\frac{1-21\varepsilon}{9\varepsilon} \right] + 1 \Rightarrow \ldots$$

$$\ldots \Rightarrow \lim_{n \to \infty} \frac{2n+5}{3n+7} = \frac{2}{3}$$

b) $\lim_{n\to\infty} \frac{n^2}{2n^2+2} = \frac{1}{2}$

$$\left|\frac{n^2}{2n^2+2} - \frac{1}{2}\right| \Leftrightarrow \left|\frac{2n^2}{4n^2+4} - \frac{2n^2+2}{4n^2+4}\right| \Leftrightarrow \dots$$

$$\dots \Leftrightarrow \left|\frac{\cancel{2n^2} - \cancel{2n^2} - 2}{4n^2+4}\right| \Leftrightarrow \left|\frac{-2}{4n^2+4}\right| \Leftrightarrow \left|\frac{-2}{4(2n^2+2)}\right| \Leftrightarrow \dots$$

$$\dots \Leftrightarrow \left|\frac{-2}{4(2n^2+2)}\right| = \left|\frac{-1}{2n^2+2}\right| < \varepsilon \Rightarrow \dots$$

$$\dots \Rightarrow 2n^2+2 < \frac{1}{\varepsilon}$$

$$2n^2 > \frac{1}{\varepsilon} - 2 \Rightarrow 2n^2 > \frac{1-2\varepsilon}{\varepsilon} \Rightarrow n^2 > \frac{1-2\varepsilon}{2\varepsilon} \Rightarrow$$

$$\Rightarrow n > \sqrt{\frac{1-2\varepsilon}{2\varepsilon}}$$

$$n_\varepsilon = \left[\sqrt{\frac{1-2\varepsilon}{2\varepsilon}}\right] + 1$$

Șirul $(x_n)_{n \in \mathbb{N}}$; (x_{k_n}) - subșir al lui $(x_n)_{n \geq 1}$

$k_n = 2n$

$k_n = 2n+1$

$x_n = n$

Lemă: Dacă $\lim_{n \to \infty} x_n = l$, atunci $(\forall)\ k_n \in \mathbb{N}$ ...

... $\lim_{n \to \infty} x_{k_n} = l$

Obs: Dacă $(\exists)\ k_n$ și $l_n \in \mathbb{N}$ a.î. ...

... $\lim_{n \to \infty} x_{k_n} \neq \lim_{n \to \infty} x_{l_n}$, atunci:

$(\nexists)\ \lim_{n \to \infty} x_n = l$

Exemple:

1) $x_n = \cos(\pi n)$, $n \in \mathbb{N}$

$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \cos(\pi \cdot n)$

$\cos(\pi \cdot n) = \lim_{n \to \infty} (-1)^n =$

$n = 2k \Rightarrow \lim_{n \to \infty} x_{2k} = \lim_{n \to \infty} (-1)^{2k} = 1$

$n = 2k+1 \Rightarrow \lim_{n \to \infty} x_{2k+1} =$

$\ldots \lim_{x \to \infty} (-1)^{2k+1} = -1$

Exerciții:

2) $x_n = \left\{ \frac{n}{3} \right\}$

$n = 3k$

$n \in \{ k \mid 3 \nmid k \} \Rightarrow \lim_{n\to\infty} x_n \in \mathbb{Q}^*$

(nu divide)

$\Rightarrow \lim_{n\to\infty} x_n = 0$

$\Rightarrow \ldots$

$\ldots \Rightarrow (\nexists) \lim_{n\to\infty} \left\{ \frac{n}{3} \right\}$

$$\boxed{\sqrt{2} = \lim_{n\to\infty} x_n}$$

Criteriul lui d'Alembert $\left(\frac{\infty}{\infty}\right)$

$$\lim_{n\to\infty} \frac{a_p n^p + a_{p-1} n^{p-1} + \ldots + a_1 n + a_0}{b_q n^q + b_{q-1} \cdot n^{q-1} + \ldots b_1 n + b_0} \overset{\frac{\infty}{\infty}}{=} \ldots$$

$$\ldots \begin{cases} 0 & , p < q \\ \frac{a_p}{b_q} & , p = q \\ \infty & , p > q \end{cases}$$

Ex.: 1.) $\lim\limits_{x \to \infty} \dfrac{4x^5 + 8x^2 + 31}{9x^2 - 16x + 3} \overset{\frac{\infty}{\infty}}{=} \dfrac{4}{9}$

$\sqrt{x^2+2x+3} + \sqrt{x^2+x+1}$ (conjugate)

2) $\lim\limits_{x \to \infty} \sqrt{x^2+2x+3} - \sqrt{x^2+x+1} \overset{\infty - \infty}{=}$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{x^2+2x+3} - \sqrt{x^2+x+1}\right)\left(\sqrt{x^2+2x+3} + \sqrt{x^2+x+1}\right)}{\sqrt{x^2+2x+3} + \sqrt{x^2+x+1}} =$$

$$= \lim_{x \to \infty} \frac{\cancel{x^2} + 2x + 3 - \cancel{x^2} - x - 1}{\sqrt{x^2\left(1 + \underset{\downarrow\atop 0}{\frac{2}{x}} + \underset{\downarrow\atop 0}{\frac{3}{x^2}}\right)} + \sqrt{x^2\left(1 + \frac{1}{x} - \frac{1}{x^2}\right)}} =$$

$$= \lim_{x \to \infty} \frac{x+2}{\sqrt{x^2} + \sqrt{x^2}} =$$

$$= \lim_{x \to \infty} \frac{x+2}{2x} = \frac{1}{2}$$

3) $\lim\limits_{x \to \infty} \sqrt{x+2} - 2\sqrt{x+1} + \sqrt{x} =$

$$= \lim_{x \to \infty} \sqrt{x+2} - \sqrt{x+1} + \sqrt{x} - \sqrt{x+1} =$$

$$= \frac{x+2-x-1}{\sqrt{x+2} + \sqrt{x+1}} + \frac{x-x-1}{\sqrt{x} + \sqrt{x+1}} - \frac{1}{\sqrt{x} - \sqrt{x+1}} = 0$$

4) $\lim\limits_{n\to\infty} \dfrac{4^n + 8^n}{9^n + 11^n} =$

$= \lim\limits_{n\to\infty} \dfrac{8^n\left(\left(\frac{4}{8}\right)^{n} + 1\right)}{11^n\left(\left(\frac{9}{11}\right)^n + 1\right)} = \lim\limits_{n\to\infty} \dfrac{8^n}{11^n} = 0$

$\left(\frac{4}{8}\right)^n \to 0$, $\left(\frac{9}{11}\right)^n \to 0$

5) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{k^2+k} = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \dfrac{1}{k(k+1)} =$

$= \lim\limits_{n\to\infty} \left(\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dots + \dfrac{1}{n(n+1)} \right) =$

$= \lim\limits_{n\to\infty} \left(\dfrac{1}{1} - \cancel{\dfrac{1}{2}} + \cancel{\dfrac{1}{2}} - \cancel{\dfrac{1}{3}} + \dots + \cancel{\dfrac{1}{n}} - \dfrac{1}{n+1} \right) =$

$= \lim\limits_{n\to\infty} 1 - \dfrac{1}{n+1} = 1$

6) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \dots + \dfrac{1}{\sqrt{n^2+n}}$

$\dfrac{1}{\sqrt{n^2+1}} \leq \dfrac{1}{\sqrt{n^2+n}}$

$\dfrac{1}{\sqrt{n^2+2}} < \dfrac{1}{\sqrt{n^2+1}}$

$\vdots$

$\dfrac{1}{\sqrt{n^2+n}} < \dfrac{1}{\sqrt{n^2+1}}$ (-1)

$$\Rightarrow x_n \le \frac{n}{\sqrt{n^2+1}}$$

$$\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{n^2+n}}$$

$$\frac{1}{\sqrt{n^2+n}} \ge \frac{1}{\sqrt{n^2+n}} \quad (+)$$

$$x_n \ge \frac{n}{\sqrt{n^2+n}}$$

$$\frac{n}{\sqrt{n^2+n}} \le x_n \le \frac{n}{\sqrt{n^2+1}}$$

$$\to 1$$

$\boxed{\text{Lema cleştelui:}}$ $a_n \le x_n \le b_n$ pt ..

şi $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = l \Rightarrow \lim\limits_{n\to\infty} x_n = l$

7) $\lim_{n \to \infty} 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} =$

(progresie geometrică)

$= \lim_{n \to \infty} 1 \cdot \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$, unde $\left(\frac{1}{2}\right)^{n+1} \to 0$

8) $\lim_{n \to \infty} \left(\frac{3n+2}{3n+1}\right)^n \overset{1^\infty}{=}$

$= \lim_{n \to \infty} \left(1 + \frac{3n+2}{3n+1} - \frac{3n+1}{1}\right)^n =$

$= \lim_{n \to \infty} \left(1 + \frac{\cancel{3n} + 2 - \cancel{3n} - 1}{3n+1}\right)^n =$

$= \lim_{n \to \infty} \left[\left(1 + \frac{1}{(3n+1)}\right)^{3n+1}\right]^{\frac{n}{3n+1}} =$

$= e^{\lim_{n \to \infty} \frac{n}{3n+1}} = e^{\frac{1}{3}}$

Criteriul lui Euler (1^∞)

$\lim_{n \to \infty} (1 + x_n)^{\frac{1}{x_n}} = e$, dacă $\lim_{n \to \infty} x_n = 0$

9) $\lim_{n \to \infty} n \cdot \ln\left(\frac{n^2 + 2n + 2}{n^2 + n + 1}\right) =$

$= \lim_{n \to \infty} \ln\left(\frac{n^2 + 2n + 2}{n^2 + n + 1}\right)^n =$

$= \ln \lim_{n \to \infty} \left(\frac{n^2 + 2n + 2}{n^2 + n + 1}\right)^n$

$$= \ln \lim_{n \to \infty} \left(1 + \frac{n^2 + 2n + 2 - n^2 - n - 1}{n^2 + n + 1}\right)^n =$$

$$= \ln \lim_{n \to \infty} \left[\left(1 + \frac{n+1}{n^2+n+1}\right)^{\frac{n^2+n+1}{n+1}}\right]^{\frac{n+1}{n^2+n+1} \cdot n}$$

$$= \ln e^{\lim_{n \to \infty} \frac{n^2+n}{n^2+n+1}} = \ln e = 1$$