Matrices, determinants and linear systems I

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The scoring system

- 7 points for solving the exercises in the written exam
- 2 points for the seminar activity
- 1 point ex officio

The way to ask questions

- at the end of every lecture we will allocate 10 minutes for questions
- the questions that arise after the individual study can be addressed during the seminar

Academic advices

- it is very important that you study the lectures and the seminaries weekly
- the worst strategy is to start studying during the exam session
- studying Linear Algebra and Analytic Geometry must be made using a pen or pencil; simply reading the lecture notes is not enough, it should be completed with notes and discussions (during the seminar and between you)
- we will study elements of linear algebra and analytic geometry with a much higher rigour compared to what is done in high school, emphasizing the understanding of concepts
- the topics are mostly new, but a good knowledge of high school algebra and geometry is useful

What topics will we study?

- Matrices, determinants, systems of equations
- Trigonometry
- Vector spaces
- Euclidean spaces
- Linear transformations
- The space of free vectors, vector products
- Planes, lines, angles, distances
- Conic sections
- Quadrics
- Vector fields

What are linear algebra and analytic geometry used for?

They represent essential tools for solving mathematical modelling problems that arise in:

- physics
- chemistry
- biology
- informatics
- engineering
- economy
- medicine
- sports

References

Theory and applications

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- E. Stoica, N. Minculete, Algebră Liniară, Geometrie Analitică, Editura Fair Parteners, București, 2016.
- G. Toader, S. Toader, T. Lazăr, Algebră Liniară, Geometrie Analitică și Geometrie Diferențială, U.T. Press, Cluj-Napoca, 2014.

References

Applications

- S. Chiriță, Probleme de Matematici Superioare, Editura Didactică și Pedagogică, București, 1989.
- P. Georgescu, G. Popa, Structuri fundamentale în algebra liniară, geometria vectorială și geometria analitică, Editura Matrix Rom, București, 2003.
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The concept of matrix

Definition. *Let us consider* $m, n \in \mathbb{N}^*$. *A function*

$$a : \{1, ..., m\} \times \{1, ..., n\} \to K$$

is called $m \times n$ *matrix over* K.

Remark. *Here* $K = \mathbb{R}$ *or* $K = \mathbb{C}$.

The classical notation

Remark. *If we use the notation*

$$a(i,j) \overset{not}{=} a_{ij},$$

for every $(i,j) \in \{1,...,m\} \times \{1,...,n\}$, *we obtain a representation of the matrix as a rectangular array having:*

- *m rows*

and

- *n columns,*

as follows:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \overset{not}{=} (a_{ij}).$$

The entries a_{ij}

Definition. *a_{ij} are called the entries of the matrix A.*

- a_{ij} is the entry which appears in the i th row and j th column

- i is the row index

- j is the column index

$\mathcal{M}_{m,n}$ and $\mathcal{M}_n$

Definition. *The set of all $m \times n$ matrices will be denoted by $\mathcal{M}_{m,n}$.*

Remark. *For $m = n$, we shall use the following notation:*

$$\mathcal{M}_{m,n} \overset{not}{=} \mathcal{M}_n.$$

Equal matrices

Definition. *The matrices* $A = (a_{ij}) \in \mathcal{M}_{m,n}$ *and* $B = (b_{kl}) \in \mathcal{M}_{p,q}$ *are equal if:*

i)

$$m = p \text{ and } n = q;$$

ii)

$$a_{ij} = b_{ij},$$

for each $i \in \{1, ..., m\}$ *and each* $j \in \{1, ..., n\}$.

The sum of two matrices

Definition. *Let us consider* $A = (a_{ij}) \in \mathcal{M}_{m,n}$ *and* $B = (b_{ij}) \in \mathcal{M}_{m,n}$. *The sum* $A + B$ *is an element of* $\mathcal{M}_{m,n}$ *whose entries are*

$$c_{ij} = a_{ij} + b_{ij}.$$

So

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}).$$

The product of a scalar with a matrix

Definition. *Let us consider* $A = (a_{ij}) \in \mathcal{M}_{m,n}$ *and a scalar* λ. *The product* λA *is an element of* $\mathcal{M}_{m,n}$ *whose entries are*

$$c_{ij} = \lambda a_{ij}.$$

So

$$\lambda(a_{ij}) = (\lambda a_{ij}).$$

The product of two matrices

Definition. *Let us consider* $A = (a_{ij}) \in \mathcal{M}_{m,n}$ and $B = (b_{jk}) \in \mathcal{M}_{n,p}$. *The product* AB *is an element of* $\mathcal{M}_{m,p}$ *whose entries are*

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} =$$

$$= a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk}.$$

So

$$(a_{ij})(b_{jk}) = (\sum_{j=1}^{n} a_{ij} b_{jk}).$$

Note. Frequently we shall write products such as AB without explicitly mentioning the sizes of the factors and in such cases it will be understood that the product is defined.

Examples I

1.

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 & -1 & 2 \\ 15 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 5 & -1 & 2 \\ 0 & 7 & 2 \end{pmatrix}.$$

2.

$$\begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{pmatrix}.$$

Examples II

3.

$$\begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 29 \end{pmatrix}.$$

4.

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 6 & 12 \end{pmatrix}.$$

5.

$$\begin{pmatrix} 2 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = (10).$$

The zero matrix

Definition. *The matrix* $(0) \in \mathcal{M}_{m,n}$ *is denoted by* 0 *and it is called the zero matrix.*

The unit matrix

Definition. *The matrix* $(\delta_{ij}) \in \mathcal{M}_n$, *where*

$$\delta_{ij} = \left\{ \begin{array}{ll} 1, & i = j \\ 0, & i \neq j \end{array} \right. ,$$

is denoted by I_n *and it is called the unit matrix of order* n.

Diagonal matrices

Definition. *A matrix* $A = (a_{ij}) \in \mathcal{M}_n$ *is called diagonal if:*
i) there exists $k \in \{1, ..., n\}$ *such that*

$$a_{kk} \neq 0;$$

ii)

$$a_{ij} = 0,$$

for every $i \neq j$, $i, j \in \{1, ..., n\}$.

The transpose of a matrix

Definition. *For* $A = (a_{ij}) \in \mathcal{M}_{m,n}$, *the transpose of* A *is the matrix* ${}^tA = (b_{ij}) \in \mathcal{M}_{n,m}$, *where*

$$b_{ij} = a_{ij},$$

for every $i \in \{1, ..., m\}$, $j \in \{1, ..., n\}$.

So ${}^tA$ *is obtained by transforming rows into columns for* A.

Examples

1.

$$
{}^t\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.
$$

2.

$$
{}^t\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.
$$

3.

$$
{}^t\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.
$$

Symmetric and skew symmetric matrices I

Definition. *A matrix* $A \in \mathcal{M}_n$ *is called symmetric if*

$$A = {}^tA.$$

Definition. *A matrix* $A \in \mathcal{M}_n$ *is called skew symmetric if*

$$A = -{}^tA.$$

Symmetric and skew symmetric matrices II

Remark. *For a matrix $A \in \mathcal{M}_n$, we have*

$$A = \frac{A \ + \ {}^tA}{2} + \frac{A \ - \ {}^tA}{2}.$$

Note that $\frac{A + {}^tA}{2}$ *is symmetric and* $\frac{A - {}^tA}{2}$ *is skew symmetric.*

Triangular matrices

Definition. *A matrix $A \in \mathcal{M}_n$ is called triangular if it has all its non-zero elements under or above the main diagonal.*

Orthogonal matrices I

Definition. *A matrix $A \in \mathcal{M}_n$ is called orthogonal if*

$$A\,{}^tA = {}^tA A = I_n.$$

Orthogonal matrices II

The matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathcal{M}_2$ is orthogonal since

$$A^t A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{pmatrix} = I_2$$

and

$${}^t AA = I_2.$$

The powers of a matrix

Definition. *Given a matrix* $A \in \mathcal{M}_m$ *we can define the powers of* A, *by using the mathematical induction, in the following way:*

$$A^0 = I_m$$

$$A^{n+1} = AA^n,$$

for every $n \in \mathbb{N}$.

The matrix polynomial

Definition. *Given a matrix* $A \in \mathcal{M}_m$ *and a polynomial function*

$$f = a_0 + a_1 x + ... + a_p x^p,$$

the matrix

$$a_0 I_m + a_1 A + ... + a_p A^p$$

is called a matrix polynomial which is denoted by $f(A)$*.*

Example

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2$ *and the polynomial function*

$$f = ad - bc - (a+d)x + x^2,$$

we have

$$f(A) = 0.$$

The properties of matrix operations I

Proposition. *For every $A, B \in \mathcal{M}_n$ we have:*

i)

$$A + B = B + A;$$

ii)

$$(A + B) + C = A + (B + C);$$

iii)

$$A + 0 = A;$$

iv)

$$A + (-A) = (-A) + A = 0;$$

v)

$$^{t}(A + B) = {}^{t}A + {}^{t}B.$$

The properties of matrix operations II

Proposition. *For every* $A, B \in \mathcal{M}_n$ *and every scalars* λ, μ, *we have:*

i)

$$1A = A;$$

ii)

$$(\lambda + \mu)A = \lambda A + \mu A;$$

iii)

$$(\lambda\mu)A = \lambda(\mu A);$$

iv)

$$\lambda(A + B) = \lambda A + \lambda B.$$

The properties of matrix operations III

Proposition. *For every* $A, B, C, D \in \mathcal{M}_n$, *we have:*

i)

$$A(BC) = (AB)C;$$

ii)

$$A(B + C) = AB + AC;$$

iii)

$$(B + C)D = BD + CD;$$

iv)

$$AI_n = I_nA = A;$$

v)

$$^t(AB) = {}^tB\ {}^tA.$$

A word of warning I

Matrix multiplication is not commutative.

Just consider

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}$$

A word of warning II

It is not true that

$$AB = 0 \Rightarrow A = 0 \textit{ or } B = 0.$$

Just consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Homework I

1. Prove that for $A, B \in \mathcal{M}_m$ such that $AB = BA$, we have

$$(A+B)^n = \sum_{k=1}^{n} C_n^k A^{n-k} B^k, \qquad (*)$$

for every $n \in \mathbb{N}$.
Prove that the condition $AB = BA$ is crucial.

Hint.
For $(*)$ one can use the mathematical induction method.
For the second part just consider

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}.$$

Homework II

2. Prove that

$$\begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix},$$

for every $n \in \mathbb{N}$.

3. Find all $X \in \mathcal{M}_2$ such that

$$X^2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The notion of n-permutation

Definition. *A function $\sigma : \{1, ..., n\} \rightarrow \{1, ..., n\}$, where $n \in \mathbb{N}^*$, which is one-to-one is called n-permutation.*

Notation.

$$\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ \sigma(1) & \sigma(2) & ... & \sigma(n) \end{pmatrix}.$$

The set of all n-permutations

Notation.

$$S_n = \{\sigma : \{1, ..., n\} \to \{1, ..., n\} \mid \sigma \textit{ is a n-permutation}\}.$$

Remark.

$$\mathrm{Card}(S_n) = n!.$$

The sign of a permutation

Definition. *The sign of a n-permutation σ, denoted by $\varepsilon(\sigma)$, is*

$$\varepsilon(\sigma) = (-1)^{\mathrm{Card}\{(i,j)\in\{1,\dots,n\}\times\{1,\dots,n\} \mid 1\le i<j\le n \text{ and } \sigma(i)>\sigma(j)\}} \in \{-1,1\}.$$

Definition. *A permutation σ is called even if $\varepsilon(\sigma) = 1$.*

Definition. *A permutation σ is called odd if $\varepsilon(\sigma) = -1$.*

Remark.

$$\mathrm{Card}\{\sigma \in S_n \mid \varepsilon(\sigma) = 1\} = \mathrm{Card}\{\sigma \in S_n \mid \varepsilon(\sigma) = -1\} = \frac{n!}{2}.$$

The notion of determinant

Definition. *The determinant of the matrix*

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (a_{ij}) \in \mathcal{M}_n,$$ *denoted by* $\det(A)$, *is*

$$\det(A) = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}.$$

Remark. *n is called the order of the determinant.*

Remark. *We shall also use the following notation:*

$$\det(A) \overset{not}{=} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

Determinants of order 2

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Determinants of order 3

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =
$$

$$
a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} -
$$

$$
-\left(a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}\right).
$$