27. 11. 2023

AnMate_15_S_2023-11-27

Integrale definite

1.

a) $\int_1^4 \left(2\sqrt{x} - x\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right) dx = 2\int_1^4 \sqrt{x} - \int_1^4 x\sqrt{x} + \int_1^4 \frac{1}{\sqrt[3]{x}}\, dx =$

$= 2\int_1^4 x^{\frac{1}{2}}\, dx - \int_1^4 x \cdot x^{\frac{1}{2}} + \int_1^4 x^{-\frac{1}{3}}\, dx =$

$= \left(2 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right)\Bigg|_1^4 =$

$= 2 \cdot \frac{4^{\frac{3}{2}}}{\frac{3}{2}} - 2 \cdot \frac{1}{\frac{3}{2}} - \frac{4^{\frac{5}{2}}}{\frac{5}{2}} + \frac{1}{\frac{5}{2}} + \frac{4^{\frac{2}{3}}}{\frac{2}{3}} - \frac{1}{\frac{2}{3}}$

b) $\int_1^2 \frac{(x-1)^2}{x^3}\, dx = \int_1^2 \frac{x^2 - 2x + 1}{x^3}\, dx =$

$= \int_1^2 \left(\frac{x^2}{x^3} - \frac{2x}{x^3} + \frac{1}{x^3}\right) dx =$

$= \int_1^2 \left(\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}\right) dx = \int_1^2 \frac{1}{x}\, dx - \int_1^2 \frac{2}{x^2}\, dx + \int_1^2 \frac{1}{x^3}\, dx =$

$= \ln|x|\Big|_1^2 - 2 \cdot \int_1^2 x^{-2}\, dx + \int_1^2 x^{-3} =$

$= \ln 2 - 2 \cdot \frac{x^{-1}}{-1}\Big|_1^2 + \frac{x^{-2}}{-2}\Big|_1^2$

c) $\int_0^1 \left(2^x + e^{2x} - \frac{1}{e^x}\right) dx = \int_0^1 2^x + \int_0^1 e^{2x} - \int_0^1 e^{-x}\, dx = \frac{2^x}{\ln 2}\Big|_0^1 + \frac{e^{2x}}{2}\Big|_0^1 +$

$+ e^{-x}\Big|_0^1$

d) $\int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{1}{\sqrt{4x^2-8}} + \frac{1}{\sqrt{4-x^2}} + \frac{x^2}{4-x^2} \right) dx =$

$= \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{4(x^2-2)}} + \arcsin\frac{x}{2}\Big|_{-\sqrt{2}}^{\sqrt{2}} + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^2+4-4}{4-x^2} =$

$= \underbrace{\frac{1}{2}\ln\left|x+\sqrt{x^2-2}\right|\Big|_{-\sqrt{2}}^{\sqrt{2}} + \arcsin\frac{\sqrt{2}}{2} - \arcsin\left(\frac{-\sqrt{2}}{2}\right)}_{\dots} - \int_{-\sqrt{2}}^{\sqrt{2}} \frac{\cancel{x^2-4}}{\cancel{x^2-4}} + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{4}{4-x^2} =$

$= \dots - \int_{-\sqrt{2}}^{\sqrt{2}} 1\, dx + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{4}{(2-x)(2+x)} =$

$= \dots - x\Big|_{-\sqrt{2}}^{\sqrt{2}} + \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2+x+(2-x)}{(2+x)(2-x)} \Rightarrow I_1$

$I_1 = \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{\cancel{2+x}}{\cancel{(2+x)}(2-x)} + \frac{\cancel{2-x}}{(2+x)\cancel{(2-x)}} \right] dx =$

...

e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cdot \cos^2 x}\, dx = \int_{\pi/6}^{\pi/3} \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x}\, dx =$

$= \int_{\pi/6}^{\pi/3} \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x} + \int_{\pi/6}^{\pi/3} \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x}\, dx =$

$= \int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x}\, dx + \int_{\pi/6}^{\pi/3} \frac{1}{\sin^2 x}\, dx = \int_{\pi/6}^{\pi/3} (tg\, x)'\, dx - \int_{\pi/6}^{\pi/3} (ctg\, x)'\, dx =$

$= tg\, x \Big|_{\pi/6}^{\pi/3} - ctg\, x \Big|_{\pi/6}^{\pi/3}$

! f) $\int_{-2}^{2} |x^2 - 1|\, dx =$

$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$x^2 - 1 = 0 \iff x = \pm 1 \Rightarrow$

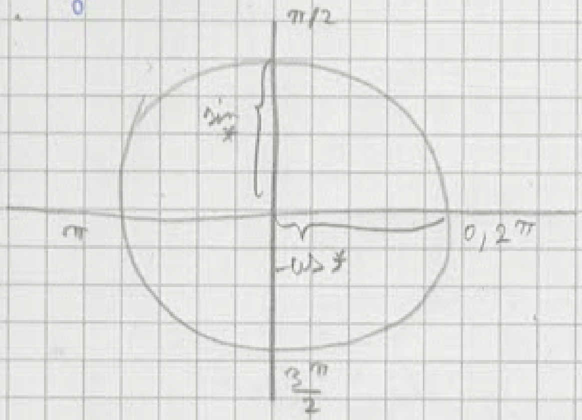
x	-2			-1			1		2
x^2-1	$+$	$+$	$+$	0	$-$	$-$	0	$+$	$+$ $+$

$\Rightarrow = \int_{-2}^{-1} x^2 - 1 + \int_{-1}^{1} 1 - x^2 + \int_{1}^{2} x^2 - 1 =$

g) $\int_0^2 \min\{x, x^2\}\, dx = \int_0^1 \min\{x, x^2\}\, dx + \int_1^2 \min\{x, x^2\} =$

$= \int_0^1 x^2\, dx + \int_0^1 x\, dx$

$$\int_0^{\pi} |\cos 2x|\, dx = \int_0^{\pi/4} \cos 2x + \int_{\pi/4}^{\pi/2} (-\cos 2x)\, dx - \int_{\pi/2}^{3\pi/4} \cos 2x + \int_{3\pi/4}^{\pi} \cos 2x\, dx$$



$$\int \cos 2x = \frac{\sin 2x}{2} + C$$

2) Calculați: a) $\lim\limits_{n \to \infty} \int_0^1 x^n \cdot \sin x\, dx$

$$f \le g \Rightarrow \int f \le \int g$$

$$|x^n \cdot \sin x| = |x^n| \cdot |\sin x| = x^n \cdot \underbrace{|\sin x|}_{\le 1} \Leftrightarrow -x^n \le x^n \cdot \sin x \le x^n$$

$$\int_0 \quad -1 \le \sin x \;\; 1 \;\Big|\cdot x^2$$

$$-\int_0^1 x^n\, dx \le I_n \le \int_0^1 x^n\, dx$$

$$-\frac{x^{n+1}}{n+1}\Big|_0^1 \le \underset{\downarrow}{I_n} \le \frac{x^{n+1}}{n+1}\Big|_0^1$$

$$0$$

b) $\lim\limits_{x \to \infty} (1-x)^n \cdot e^x\, dx$

$$0 \le (1-x)^n \cdot e^x \le (1-x)^n \;\Big|\int_0^1 \Leftrightarrow \int_0^1 0 \le I_n \le \int_0^1 e(1-x)^n$$

$$\int_0^1 e(1-x)^n\, dx = e \int_0^1 (1-x)^n\, dx =$$

$$= e \cdot \frac{(1-x)^{n+1}}{n+1} \Big|_0^1 = \frac{1}{n+1}$$

c) $\int f(x) \cdot g(x)\, dx = \underline{\int f(x)\, dx \cdot \int g(x)\, dx}$ NU AȘA

$\int f(x) \cdot f'(x)\, dx$ sau $\int \frac{f(x)}{f'(x)}\, dx$ —— Schimbare de variabilă

$f(x) = t$

$dt = f'(x)\, dx$

sau

$dt = \frac{1}{f'(x)}\, dx$

$$\underbrace{\int f(x) \cdot g'(x)\, dx = f(x)\, g(x) - \int f'(x) \cdot g(x)\, dx}$$

↳ integrare prin părți

3) e) $\int_e^{e^2} \frac{x}{\ln(x^{x^2})}\, dx = \int_e^{e^2} \frac{x}{x^x \cdot \ln x}\, dx = \int_e^{e^2} \frac{1}{x \ln x}\, dx$

$\ln x = t \Rightarrow dt = \frac{1}{x}\, dx$

$x = e \Rightarrow t = 1$

$x = e^2 \Rightarrow t = 2$

$\Rightarrow I = \int_1^2 \frac{1}{t}\, dt = \ln|t| \Big|_1^2 = \ln 2$

b) $\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx =$

$\sqrt{x} = t \Rightarrow dt = \frac{1}{2\sqrt{x}}\, dx$

$x = 1 \Rightarrow t = 1$

$x = 2 \Rightarrow t = \sqrt{2}$

$= 2\int_1^{\sqrt{2}} e^t dt = 2 \cdot e^t \Big|_1^{\sqrt{2}}$

c) $\int_{-\pi/2}^{\pi/4} \frac{1}{\sin^2 x \sqrt[n]{-ctg\, x}}\, dx$

$\Big|_{-\pi/2}^{\pi/4} -ctg\, x = t \Big|_0^1 \Rightarrow dt = -\frac{1}{\sin^2 x}\, dx$

$J_1 = -\int_0^1 \frac{1}{\sqrt[n]{t}}\, dt = \frac{t^{-\frac{1}{n}+1}}{-\frac{0}{n}+1}\Big|_0^1$

d) $\int_0^{-\pi/2} \frac{x}{x^2 + x + 1}\, dx =$

$x^2 + x + 1 = t$

$\hookrightarrow (2x+1)\, dx = dt$

$= \frac{1}{2}\int_0^{-\pi/2} \frac{2x+1-1}{x^2+x+1}\, dx = \frac{1}{2}\int_0^{-\pi/2} \frac{2x+1}{x^2+x+1} - \frac{1}{2}\int_0^{-\pi/2} \frac{1}{x^2+x+1}\, dx =$

$= \frac{1}{2} \ln (x^2 + x + 1) \Big|_0^{-\frac{\pi}{2}} - \frac{1}{2}\int_0^{-\pi/2} \frac{1}{a(x-x_v)^2 + y_v} =$

$= \dots - \frac{1}{2}\int_0^{-\pi/2} \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}\, dx$

$$I_1 = \int_{-\frac{1}{2}}^{\left(\frac{-\pi-1}{2}\right)} \frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \cdot \operatorname{arctg} \frac{t}{\frac{\sqrt{3}}{2}} \Bigg|_{-\frac{1}{2}}^{-\frac{(\pi-1)}{2}}$$

$$\Big|_{0}^{-\pi/2} \quad x - \frac{1}{2} = t$$

$$dx = dt$$