# Matrices, determinants and linear systems -II

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## Row (column) expansion of a determinant
(i.e. recursive definition of the determinant)

**Proposition.**

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + \dots + a_{in}A_{in},$$

*for each* $i \in \mathbb{N}^*$, *where*

$$A_{ij} = (-1)^{i+j} D_{ij}$$

*and* $D_{ij}$ *is the determinant of order* $n-1$ *obtained by deleting the* $i$*th row and the* $j$*th column of* $\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$ *for every* $i, j \in \{1, \dots, n\}$.

*The equality*

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = a_{i1}A_{i1} + \dots + a_{in}A_{in}$$

*is called the expansion of* $\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$ *on the* $i$*th row.*

*In a similar way we can consider the expansion of* $\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$ *on a column.*

## Properties of the determinants I

**1**.

$$\det(A) = \det({}^{t}A),$$

*for every* $A \in \mathcal{M}_n$.

*Therefore every property which is stated for rows is also true for columns.*

**2**.

$$\det(A) = 0,$$

*for every* $A \in \mathcal{M}_n$ *whose entries of a row are* 0.

## Properties of the determinants II

**3**.

$$\det(A) = -\det(A^{'}),$$

*for every* $A \in \mathcal{M}_n$, *where* $A^{'}$ *is obtained by interchanging two rows of* $A$.

**4**.

$$\det(A) = 0,$$

*for every* $A \in \mathcal{M}_n$ *having two identical rows.*

## Properties of the determinants III

**5**.

$$\det(A^{'}) = \lambda \det(A),$$

*for every* $A \in \mathcal{M}_n$*, where* $A^{'}$ *is obtained by multiplying by* $\lambda$ *a row of* $A$*.*

**6**.

$$\det(A) = 0$$

*for every* $A \in \mathcal{M}_n$ *having two proportional rows (i.e. one row is obtained by multiplication with some scalar the entries of another row).*

## Properties of the determinants IV

**7**.

$$\det(A) = \det(A^{'}) + \det(A^{''}),$$

*for every* $A, A^{'}, A^{''} \in \mathcal{M}_n$ *such that they are identical except for the* $i$th *row and the* $i$th *row of* $A$ *is the sum of* $i$th *rows of* $A^{'}$ *and* $A^{''}$.

**8**.

$$\det(A) = \det(A^{'}),$$

*for every* $A \in \mathcal{M}_n$, *where* $A^{'}$ *is obtained by adding a row of* $A$ *multiplied by a scalar to another row of* $A$.

## Properties of the determinants V

**9**.

$$\det(AB) = \det(A)\det(B),$$

*for every* $A, B \in \mathcal{M}_n$.

### A word of warning

*In general, the equality*

$$\det(A+B) = \det(A) + \det(B)$$

*is not valid.*

## Homework I

**1**. Prove that

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \prod_{i,j\in\{1,\dots,n\}, i>j} (a_i - a_j),$$

for every $n \in \mathbb{N}$, $n \geq 2$ and every $a_1$, ..., $a_n \in \mathbb{C}$.

**2**. Prove that

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = 8(b-a)(c-a)(b-c).$$

## Homework II

**3**. Compute

$$\begin{vmatrix} \sin a & \cos a & \sin(a+x) \\ \sin b & \cos b & \sin(b+x) \\ \sin c & \cos c & \sin(c+x) \end{vmatrix}.$$

**4**. Compute $\det(A)$, where $A \in \mathcal{M}_n$ is such that $A = -\,{}^tA$.

## The concept of invertible matrix

**Definition**. *A matrix* $A \in \mathcal{M}_n$ *is called invertible if there exists* $A^{-1} \in \mathcal{M}_n$ *such that*

$$AA^{-1} = A^{-1}A = I_n.$$

$A^{-1}$ *is called the inverse of* $A$.

## Some properties of invertible matrices

**Proposition**. *Let $A, B \in \mathcal{M}_n$ be invertible matrices. Then:*

*i) $A^{-1}$ is invertible and*

$$(A^{-1})^{-1} = A.$$

*ii) $AB$ is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

## How to compute the inverse of an invertible matrix I

**Definition**. *Let us consider* $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = (a_{ij}) \in \mathcal{M}_n$ *and*

$$A_{ij} = (-1)^{i+j} \cdot \Gamma_{ij},$$

*where* $\Gamma_{ij}$ *is the determinant of the matrix obtained by deleting the* $i$*th row and the* $j$*th column of* $A$.
*The matrix*

$$A^* = \begin{pmatrix} A_{11} & \dots & A_{n1} \\ \vdots & \cdots & \vdots \\ A_{1n} & \dots & A_{nn} \end{pmatrix} \in \mathcal{M}_n$$

*is called the adjoint of* $A$.

## How to compute the inverse of an invertible matrix II

**Proposition**. *For $A \in \mathcal{M}_n$, we have*

$$AA^* = A^*A = \det(A)I_n.$$

**Theorem**. *For $A \in \mathcal{M}_n$ the following statements are equivalent:*
*i) $A$ is invertible*
*ii)*

$$\det(A) \neq 0.$$

*Moreover, when $A$ is invertible, we have*

$$A^{-1} = \frac{1}{\det(A)}A^*.$$

## An example

Let us consider

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Since

$$\det(A) = 9 \neq 0,$$

the matrix $A$ is invertible.
We have

$$A^* = \begin{pmatrix} 1 & -2 & 4 \\ 4 & 1 & -2 \\ -2 & 4 & 1 \end{pmatrix},$$

so

$$A^{-1} = \begin{pmatrix} \frac{1}{9} & -\frac{2}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{1}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{pmatrix}.$$

## Homework

**1**. *Prove that the matrix* $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *is invertible and find its inverse.*

**2**. *Find* $m$ *such that* $\begin{pmatrix} 1 & -1 & m \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ *is invertible.*

**3**. *Let us consider* $A \in \mathcal{M}_n$ *such that* $A^2 = 0$. *Prove that* $A + I_n$ *and* $A - I_n$ *are invertible.*

## The concept of a minor of a matrix

**Definition**. *Let us consider* $A \in \mathcal{M}_{m,n}$ *and* $k \in \mathbb{N}^*$, $k \leq \min\{m, n\}$. *The determinant of a matrix from* $\mathcal{M}_k$ *obtained by crossing out* $m - k$ *rows and* $n - k$ *columns of* $A$ *is called a minor of* $A$ *of order* $k$.

## The rank of a matrix I

**Definition**. *The maximum order of a non-zero minor of a matrix $A$ is called the rank of $A$ and it is denoted by $rank(A)$.*

So $rank(A) = r$ if:

$\alpha$) there exists a non-zero minor of $A$ of order $r$

and

$\beta$) all minors of $A$ of order greater or equal to $r+1$ (if there exist) are zero.

## The rank of a matrix II

**Remark**. *For* $A \in \mathcal{M}_{m,n}$ *and* $k \in \mathbb{N}^*$, $k \leq \min\{m, n\}$, *the following statements are equivalent:*

*i)* $rank(A) = r$;

*ii)*

$\alpha$) *there exists a non-zero minor of* $A$ *of order* $r$

*and*

$\beta$) *all the minors of order* $r + 1$ *(if there exist) obtained by adding to the non-zero minor of order* $r$ *a row and a column of* $A$ *are zero.*

## Example

Let

$$A = \begin{pmatrix} 1 & 2 & 5 & -3 & 4 \\ 0 & 4 & 2 & 6 & 1 \\ 2 & 4 & 10 & -6 & 8 \end{pmatrix}.$$

Since $\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4 \neq 0$ and all minors of order 3 are zero (since their first and third rows are proportional), we conclude that $rank(A) = 2$.

## Homework

**1**. *Find the rank of the matrix* $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$.

**2**. *Find a and b such the the matrices* $\begin{pmatrix} 1 & -2 & -2 \\ 3 & 1 & a \\ 3 & -1 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & -2 & -2 & 4 \\ 3 & 1 & a & 4 \\ 3 & -1 & 1 & b \end{pmatrix}$ *have the same rank.*

**3**. *Let us consider* $A \in \mathcal{M}_2$ *such that* $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Find* $rank(A)$.

## The notion of system of linear equations

We shall consider systems of linear equations, i.e. systems having the following form:

$$\left\{ \begin{array}{c} a_{11}x_1 + ... + a_{1n}x_n = b_1 \\ ... \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{array} \right. . \qquad (S)$$

Here the unknowns are $x_1$, ..., $x_n$.

The matrix $A = (a_{ij}) \in \mathcal{M}_{m,n}$ is called the matrix of the system.

The matrix $B = \left( \begin{array}{c} b_1 \\ \vdots \\ b_m \end{array} \right) \in \mathcal{M}_{m,1}$ is called the column of the free terms of the system.

## Cramer's rule

Let us consider a system of linear equations

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \dots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases}$$

such that

$$\Delta = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \neq 0.$$

So:

- the number of equations is the same with the number of the unknowns

- the determinant of the matrix of the system is non-zero.

## Cramer' s rule

Then

$$x_j = \frac{\Delta_j}{\Delta},$$

for every $j \in \{1, ..., n\}$, where

$$\Delta_j = \begin{vmatrix} a_{11} & \dots & a_{1j-1} & b_1 & a_{1j+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nj-1} & b_n & a_{nj+1} & \dots & a_{nn} \end{vmatrix}$$

is obtained by replacing the $j$th column of $\Delta$ with the column of the free terms of the system.

## An example

Solve the system

$$\left\{ \begin{array}{c} x - y + z = 2 \\ 2x - y + z = 2 \\ x + z = 1 \end{array} \right. .$$

We have

$$\Delta = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1$$

and

$$\Delta_x = \begin{vmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0,$$

$$\Delta_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1$$

and

$$\Delta_z = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 1.$$

Hence

$$x = 0,\ y = -1 \text{ and } z = 1.$$

## The notion of a principal minor of a system of linear equations

**Definition**. *A principal minor of the system $S$ is a minor having the order equal with the rank of the matrix of the system.*

**Remark**. *The principal minor of the system $S$ is not unique.*

## The notion of main equations and side equations of a system of linear equations

Let us suppose that we fixed a principal minor of the system $S$.

**Definition**. *The equations of* $S$ *corresponding to the rows of the principal minor are called the main equations. The other ones are called the side equations of* $S$.

**Remark**. *The main equations and side equations of* $S$ *depend on the chosen principal minor.*

## The notion of characteristic determinant of a system of linear equations

Let us suppose that we fixed a principal minor of the system $S$.

**Definition**. *A characteristic determinant of the system $S$ is a determinant obtained by adding to the principal minor a row with corresponding coefficients from a side equation and a column consisting of the free terms situated on the chosen rows.*

**Remark**. *A characteristic determinant of the system $S$ depends on the chosen principal minor.*

**Remark**. *If there are no side equations, then the system has no characteristic determinants.*

## How to solve a system of linear equations I

Let us suppose that we fixed a principal minor of the system $S$.

*If there exists a non-zero characteristic determinant of the system* $S$*, then the system has no solutions.*

*In the opposite case* (i.e. there are no characteristic determinants or all of them are zero)*, the system has at least one solution.*

## How to solve a system of linear equations II

In the latter case, we get a Cramer system in the following way:

- cross out the side equations

- move the side unknowns to the column of free terms.

By solving this Cramer system (note that its number of equations is the same with the number of unknowns and that its deteminant is non-zero - being the principal minor of $S$ -) we obtain the solutions of $S$ (the principal unknowns being presented as functions of the side unknowns).

## Terminology

**Remark**. *A system of linear equations may have zero, one or infinite solutions.*

**Definition**. *A system of linear equations which has at least one solution is called consistent.*

**Definition**. *The number of side unknowns is called the degree of freedom of the system of linear equations.*

## Homogenous systems of linear equations

**Definition**. *The system of linear equations $S$ is called homogenous if* $b_1 = ... = b_m = 0$.

**Remark**. *All homogenous systems of linear equations are consistent* (one solution being $x_1 = ... = x_n = 0$).

**Remark**. *A homogenous system of linear equations has other solutions besides the trivial one if and only if* $rank(A) < n$.

## Examples

**1**. Solve the system

$$\begin{cases} x + y + z = 1 \\ 2x + 3y + 5z = 3 \end{cases}.$$

We can chose as principal minor $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$.

As we do not have characteristic determinants, the system is consistent.

The main equations are the first two.

The main unknowns are $x$ and $y$, and $z$ is the side unknown.

Using Cramer's rule to solve the system

$$\begin{cases} x + y = 1 - z \\ 2x + 3y = 3 - 5z \end{cases}$$

we get the solutions $x = 2\alpha$, $y = 1 - 3\alpha$ and $z = \alpha$, where $\alpha \in \mathbb{R}$.

The degree of freedom of this system is 1.

**2**. Solve the system

$$\left\{ \begin{array}{c} 2x - y + 3z + t = 0 \\ x + y + 2z - 2t = 2 \\ 7x + y + 12z - 4t = 6 \end{array} \right. .$$

We can chose as principal minor $\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$.

We have just one characteristic determinant, namely $\begin{vmatrix} 2 & -1 & 0 \\ 1 & 1 & 2 \\ 7 & 1 & 6 \end{vmatrix} = 0$, so the system is consistent.

The main equations are the first two.

The main unknowns are $x$ and $y$; $z$ and $t$ are the side unknowns.

Using Cramer's rule to solve the system

$$\begin{cases} 2x - y = -3z - t \\ x + y = 2 - 2z + 2t \end{cases}$$

we get the solutions $x = \frac{-5\alpha+\beta+2}{3}$, $y = \frac{-\alpha+5\beta+4}{3}$ and $z = \alpha$, $t = \beta$ where $\alpha, \beta \in \mathbb{R}$.

The degree of freedom of this system is 2.

**3**. Solve the system

$$\left\{\begin{array}{c} x+y+z=1 \\ 2x-y+3z=2 \\ x+5y-6z=4 \\ 4x+5y-2z=5 \end{array}\right. .$$

We can chose as principal minor $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 5 & -6 \end{vmatrix}$.

We have just one characteristic determinant, namely
$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 1 & 5 & -6 & 4 \\ 4 & 5 & -2 & 5 \end{vmatrix} = -34 \neq 0$, so the system is not consistent.

## Homework

Solve the systems:

$$\begin{cases} 2x + y = 2 \\ 3x - y = 3 \\ x + y = 1 \end{cases},$$

$$\begin{cases} 2x - y + z + t = 1 \\ 3x + y + 5t = 3 \\ x + 7y - 4z + 11t = 1 \end{cases}$$

and

$$\begin{cases} x - 2y + 4z - 7t = 0 \\ 2x + 3y - z + 5t = 0 \\ 3x - y + 2z - 7t = 0 \\ 4x + y - 3z + 6t = 0 \end{cases}.$$