ALGAD _ S _ 14 _ 2023 _ 11 - 22 22.11.2023

* PI 5 - 08:00 miercuri

Produsul mixt (box product)

Def: $(\bar{a}, \bar{b}, \bar{c}) = \bar{a} \cdot \underbrace{(\bar{b} \times \bar{c})}_{\bar{v}} \in \mathbb{R}$

Expresia analitică

$$\left.\begin{array}{l} \bar{a}(a_1, a_2, a_3) \\ \bar{b}(b_1, b_2, b_3) \\ \bar{c}(c_1, c_2, c_3) \end{array}\right. \Rightarrow (\bar{a}, \bar{b}, \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= |(\bar{a}, \bar{b}, \bar{c})|$$

Proprietăți

a) $V_{\square}$ $(\bar{a}, \bar{b}, \bar{c})$

$V_{tetraedru} = \frac{1}{6} |(\bar{a}, \bar{b}, \bar{c})|$

b) $(\bar{a}, \bar{b}, \bar{c}) = 0 \Leftrightarrow \bar{a}, \bar{b}, \bar{c}$ — coplanari

Ex. (1) $V_{ABCD} = ?$, $h_A - ?$

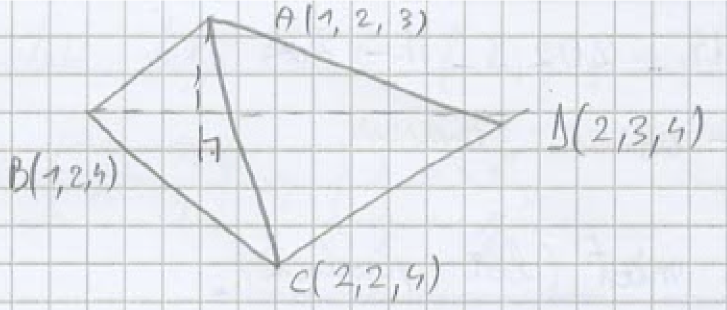
a) $A(1,2,3)$, $B(1,2,4)$, $C(2,2,4)$, $D(2,3,4)$

TEMĂ b) $A(1,0,0)$, $B(1,1,0)$, $C(1,1,1)$, $D(3,1,2)$

c) $A(1,-5,4)$, $B(0,-3,1)$, $C(-2,-4,3)$, $D(4,4,-2)$

REZ.:

a)



$$V_{ABCD} = \frac{1}{6} \left| (\overline{AB}, \overline{BC}, \overline{BD}) \right|$$

$$\overline{AB} = (X_B - X_A, Y_B - Y_A, Z_B - Z_A) = (0, 0, 1)$$

$$\overline{BC} = (1, 0, 0)$$

$$\overline{BD} = (1, 1, 0)$$

$$(\overline{AB}, \overline{BC}, \overline{BD}) = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 1 \cdot (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} =$$

$$= 1 \Rightarrow V_{ABCD} = \frac{1}{6}$$

$$V_{ABCD} = \frac{h_A \cdot A_{BCD}}{3} \Rightarrow h_A = \frac{3 \cdot V_{ABCD}}{A_{BCD}}$$

$$A_{BCD} = \frac{1}{2} \| \overline{BC} \times \overline{BD} \|$$

$$\overline{BC} \times \overline{BD} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \bar{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \bar{j} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \bar{k} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \| \bar{k} \| = \frac{1}{2}$$

$$V_{ABCD} = \frac{1}{6}\left|(\overline{AB}, \overline{BC}, \overline{BD})\right| =$$

$\overline{AB}(-1, 2, -3)$.

$\overline{BC}(-2, -1, 2)$

$\overline{BD}(4, 7, -3)$

$$(\overline{AB}, \overline{BC}, \overline{BD}) = \begin{vmatrix} -1 & 2 & -3 \\ -2 & -1 & 2 \\ 4 & 7 & -3 \end{vmatrix} = -3 + (14 \cdot 3) + 16 - 12 - 12 + 14 =$$

$$= -3 + 42 + 16 - 12 - 12 + 14 =$$

$$= 58 + 14 - 27 = 72 - 27 = \underline{45}$$

$$= \frac{1}{6} \cdot |45| = \frac{45}{6} = \frac{15}{2}$$

$$h_A = \frac{3\, V_{ABCD}}{A_{BCD}} = \frac{3 \cdot \frac{15}{2}}{\left(\frac{15}{2}\right)} = 3$$

$$\overline{BC} \times \overline{BD} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & -1 & 2 \\ 4 & 7 & -3 \end{vmatrix} = -11\bar{i} + 2\bar{j} - 10\bar{k}$$

$$A_{BCD} = \frac{1}{2} \cdot \|\overline{BC} \times \overline{BD}\| = \frac{1}{2}\sqrt{(-11)^2 + 2^2 + (-10)^2} = \frac{1}{2}\sqrt{121 + 4 + 100} =$$

$$= \frac{1}{2}\sqrt{225} = \frac{1}{2} 15 = \left(\frac{15}{2}\right)$$

Ex. 2:

$A(1, 0, 0)$

$B(1, 1, 0)$

$C(2, 1, \alpha)$, $\alpha \in \mathbb{R}$

$D(2, 2, 3)$

a) $dist(A, (BCD)) = h_A$ în ABCD

b) $\alpha = ?$ a.î. A, B, C, D - coplanare!

REŠ.:

$$h_A = \frac{3 \cdot V_{ABCD}}{A_{BCD}}$$

$$V_{ABCD} = \frac{1}{6} \left|(\overline{AB}, \overline{BC}, \overline{BD})\right| =$$

$\overline{AB} = (0, 1, 0)$

$\overline{BC} = (1, 0, \alpha)$

$\overline{BD} = (1, 1, 3)$

$$(\overline{AB}, \overline{BC}, \overline{BD}) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha \\ 1 & 1 & 3 \end{vmatrix} = \alpha - 3$$

$$= \frac{1}{6} \cdot |\alpha - 3|$$

$$A_{BCD} = \frac{1}{2} \| \overline{BC} \times \overline{BD} \|$$

$$\overline{BC} \times \overline{BD} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & \alpha \\ 1 & 1 & 3 \end{vmatrix} = \bar{i} \begin{vmatrix} 0 & \alpha \\ 1 & 3 \end{vmatrix} - \bar{j} \begin{vmatrix} 1 & \alpha \\ 1 & 3 \end{vmatrix} + \bar{k} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} =$$

$$= -\alpha \bar{i} - (3 - \alpha) \bar{j} + \bar{k}$$

$$A_{BCD} = \frac{1}{2} \| \overline{BC} \times \overline{BD} \| = \frac{\sqrt{\alpha^2 + 9 - 6\alpha + \alpha^2 + 1}}{2} = \frac{\sqrt{2\alpha^2 - 6\alpha + 10}}{2} =$$

$$= \frac{\sqrt{2\alpha^2 - 6\alpha + 10}}{2}$$

$$h_A = \frac{3 \cdot V_{ABCD}}{A_{BCD}} = \frac{\frac{1}{2}|\alpha - 3|}{\frac{\sqrt{2\alpha^2 - 6\alpha + 10}}{2}}$$

b) A, B, C, D - copl. $\Rightarrow (\overline{AB}, \overline{BC}, \overline{BD}) = 0$

$$\alpha - 3 = 0$$
$$\alpha = 3$$

Ex.: 3 $\quad \bar{a} = (1, 2, 3)$

$\bar{b} = (2, 0, 2)$

$\bar{c} = (4, 4, \alpha)$

$\Rightarrow \alpha = ?$ ptr a) $\bar{a} \perp \bar{c}$ ($\bar{a}, \bar{c}$ ortogonali)

b) $\bar{b} \parallel \bar{c}$ ($\bar{b}, \bar{c}$ coliniari)

c) $\bar{a}, \bar{b}, \bar{c}$ coplanari

$\bar{a} \cdot \bar{c} = 0$

$\bar{b} \times \bar{c} = \bar{0}$

$(\bar{a}, \bar{b}, \bar{c}) = 0$

a) $\bar{a} \perp \bar{c} = \bar{a} \cdot \bar{c} = 0$

$1 \cdot 4 + 2 \cdot 4 + 3\alpha = 0$

$12 + 3\alpha = 0$

$\alpha = \frac{-12}{3}$

$\alpha = -4$

b) $\bar{b} \times \bar{c} = \bar{0}$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 0 & 2 \\ 4 & 4 & \alpha \end{vmatrix} = \bar{i} \begin{vmatrix} 0 & 2 \\ 4 & \alpha \end{vmatrix} - \bar{j} \begin{vmatrix} 2 & 2 \\ 4 & \alpha \end{vmatrix} + \bar{k} \begin{vmatrix} 2 & 0 \\ 4 & 4 \end{vmatrix} =$$

$$= -8\bar{i} - (2\alpha - 8)\bar{j} + 8\bar{k} =$$

$$= 8\bar{i} - (2\alpha - 8)\bar{j} + 8\bar{k} \neq \bar{0}, \forall \alpha \in \mathbb{R}$$

$\bar{b} \nparallel \bar{c}, \forall \alpha \in \mathbb{R}, S = \emptyset$

c) $\bar{a}, \bar{b}, \bar{c}$ - coplanari $\Leftrightarrow (\bar{a}, \bar{b}, \bar{c}) = 0 \Rightarrow$

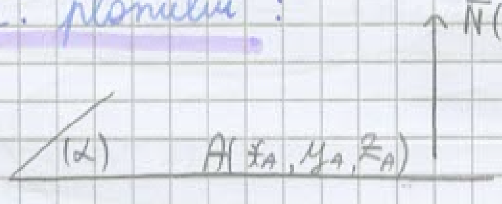
$$\Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 2 \\ 4 & 4 & \alpha \end{vmatrix} = 0 \Rightarrow 0 + 24 + 16 - 0 - 4\alpha - 8 = 0 \Rightarrow$$

$$\Rightarrow 40 - 4\alpha - 8 = 0 \Rightarrow 32 - 4\alpha = 0 \Rightarrow 4(8 - \alpha) = 0 \Rightarrow$$

$$\Rightarrow \alpha = 8$$

Ecuația planului și dreptei în spațiu

Ec. planului:

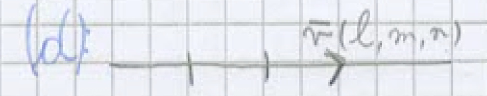


$\Rightarrow (\alpha): A(x - x_A) + B(y - y_A) + C(z - z_A) = \underline{0}$!

$(\alpha): Ax + By + Cz + D = 0 = I$

N

Ec. dreptei:



$$\Rightarrow (d): \frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} \; (= t)$$

ec. parametrice

$$\begin{cases} x = lt + x_A \\ y = mt + y_A \\ z = nt + z_A \end{cases}$$

Ex 1. Scrieți ecuația planului α pt. fiecare caz:

a) $A(1,2,-1) \in (\alpha)$ și are $\bar{N} = 2\bar{i} + 3\bar{j} + 4\bar{k}$

b) $A(2,3,5) \in (\alpha)$ și $(\alpha) \parallel (P): x + y + 2z - 3 = 0$

c) $A(1,2,3) \in (\alpha)$ și $(\alpha) \parallel xOz$

d) $A(1,-1,0) \in (\alpha)$ și $\bar{v_1}(1,2,8) \subset (\alpha)$, $\bar{v_2}(0,1,2) \subset (\alpha)$,

e) $A(1,0,1)$, $B(2,0,1)$, $C(2,1,2) \in (\alpha)$

f) $O_z \subset (\alpha)$ și $A(1,3,2) \in (\alpha)$

g) $(\alpha) \perp (P_1): x + y + z = 0$

$(\alpha) \perp (P_2): x - 2y + 3z = 0$

și trece prin origine

h) mij $[BC] \in (\alpha)$ și $\bar{AB} \perp (\alpha)$,

$A(3,-1,3)$, $B(5,1,-1)$, $C(0,5,-3)$

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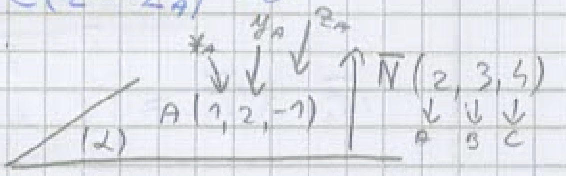
a) $(\alpha): A(x - x_A) + B(y \cdot y_A) + C(z - z_A) = 0$



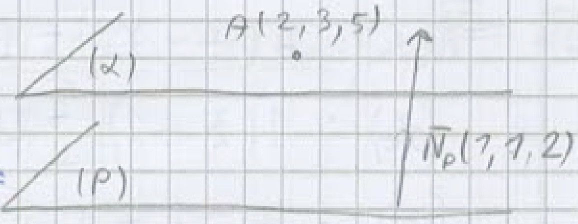
$(\alpha): 2(x-1) + 3(y-2) + 4(z+1) = 0$

$2x - 2 + 3y - 6 + 4z + 4 = 0$

$(\alpha) \quad 2x + 3y + 4z - 4 = 0$

b)

$(\alpha) \parallel (P) \Rightarrow \bar{N}_\alpha = \bar{N}_P(1,1,2)$



$(\alpha): A(x - x_A) + B(y - y_A) + C(z - z_A) =$

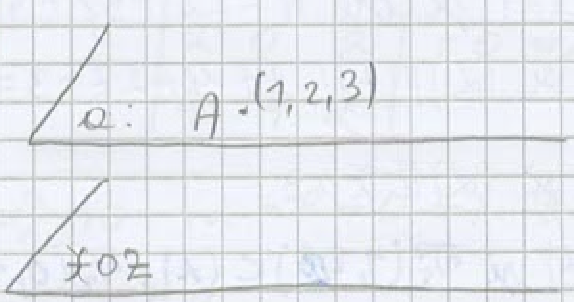
$= 0$

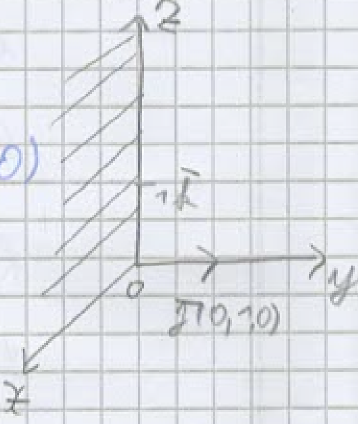
$(\alpha): 1(x-2) + 1(y-3) + 2(z-5) = 0$

$(\alpha): x - 2 + y - 3 + 2z - 10 = 0$

$x + y + 2z - 15 = 0$

c)





$(\alpha) \parallel (xOz) \Rightarrow \overline{N_\alpha} = \overline{N}_{xOz} = \overline{j}(0,1,0)$

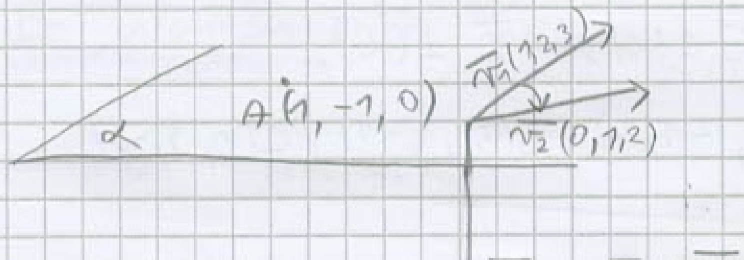
$\alpha: A(x - x_A) + B(y - y_A) + C(z - z_A) = 0$

$0 + 1(y-2) + 0 = 0$

$y - 2 = 0$

$y = 2$

d)



$\overline{N} = \overline{v_1} \times \overline{v_2}$

$$\overline{v_1} \times \overline{v_2} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = \overline{i}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - \overline{j}\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + \overline{k}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= \overline{i} - 2\overline{j} + \overline{k}$$

$\Rightarrow \overline{N}(1, -2, 1)$

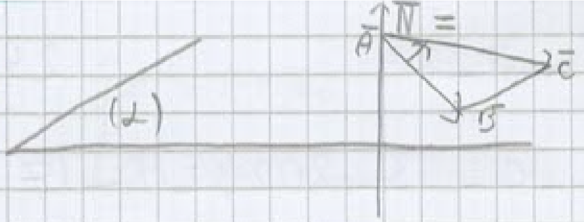
$\Rightarrow (\alpha): A(x - x_A) + B(y - y_A) + C(z - z_A) = 0$

$(\alpha): 1(x-1) - 2(y+1) + 1(z-0) = 0$

$(\alpha): x - 1 - 2y - 2 + z = 0$

$(\alpha): x - 2y + z = 3$

e)



$$\overline{AB} \times \overline{AC} = \longrightarrow = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \bar{i}\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} - \bar{j}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} =$$

$\overline{AB} = (1, 0, 0)$

$\overline{AC} = (1, 1, 1)$

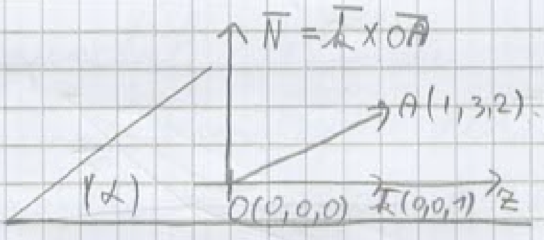
$$= -\bar{j} + \vec{k}$$

$\overline{N}(0, -1, 1) \Rightarrow (\alpha): \quad 0(x-1) - 1(y-0) + 1(z-1) = 0$

$(\alpha): \quad 0 - y + z - 1 = 0$

$(\alpha): \quad -y + z - 1 = 0$

f)



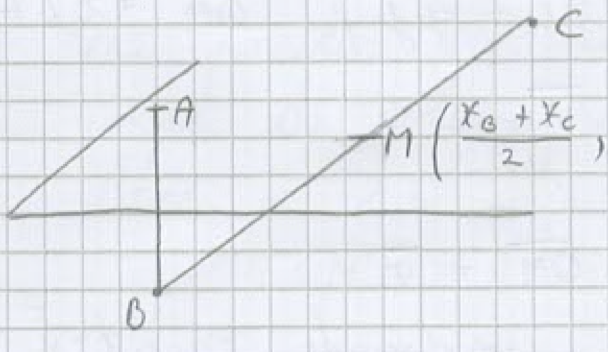
Temă

g) $(\alpha) \perp (P_1) \Rightarrow \overline{N_\alpha} \perp \overline{N_{P_1}}(1, 1, 1)$

$(\alpha) \perp (P_2) \Rightarrow \overline{N_\alpha} \perp \overline{N_{P_2}}(1, -2, 3)$

$\Longrightarrow \overline{N_\alpha} = \overline{N_{P_1}} \times \overline{N_{P_2}}$

h)



$M\left(\frac{x_B + x_C}{2}, \frac{y_B + y_C}{2}, \frac{z_B + z_C}{2}\right)$

$\overline{N} = \overline{AB}$