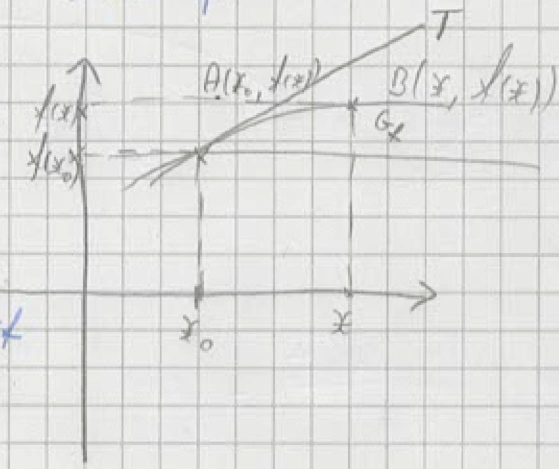
Funcții derivabile

$f'(x_0)$ – panta tangentei la G_f –

$m_T = tg\,\alpha = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$

(T): $y - y_0 = f'(x_0)\cdot(x-x_0)$ –

– ec. tangentei în x_0 la G_f



Ex: Ec. tang. în $x_0 = -1$ la G_f, $f:\mathbb{R}\to\mathbb{R}$,

$f(x) = x^3 - x^2 + x + 5$

$f(-1) = (-1) - 1 - 1 + 5 = 2 = 9$

$f'(-1) = 3 - 2 + 1 = 2$

$y - 2 = 6(x+1)$

$y - 2 = 6x + 6$

$y = 6x + 8$

$-6x + y - 8 = 0$

$f:\mathbb{R}\to\mathbb{R}$ $f(x) = ax^2 + bx + c$, $a, b, c = ?$

a.î. $A(1, 2) \in G_f$

tg la G_f în $B(2,1)$ este paralelă cu Ox

$f'(x) = 2ax + b$

$A \in G_f \Rightarrow f(1) = 2 \Leftrightarrow a + b + c = 2$

$tg \parallel Ox \Rightarrow m_{Ag} = 0 \Leftrightarrow f'(2) = 0 \Leftrightarrow 4a + b = 0$

$B \in G_f \Rightarrow f(2) = 1 \Rightarrow 4a + 2b + c = 1$

$$\begin{cases} a + b + c = 2 \\ 4a + b = 0 \\ 4a + 2b + c = 1 \end{cases}$$

Teoremele de medie

Teorema lui Lagrange:

$f: \mathbb{R} \to \mathbb{R}$ derivabilă și continuă pe $\mathbb{R}$, atunci $(\exists)$ un punct $c \in (a, b) \subset \mathbb{R}$ a.î. $f'(c) = \frac{f(b) - f(a)}{b - a}$

1) Arătați că: $\frac{1}{\sqrt{n}} - \sqrt{\ln\left(\frac{n+1}{n}\right)} > 0 \ \forall\ n \in \mathbb{N}^*$

$f: (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$ - cont. și derivabilă pe $(0, \infty) \Rightarrow (\exists)\ c \in (n, n+1)$ a.î. $f'(c) = \frac{f(n+1) - f(n)}{n+1-n} \Leftrightarrow$

$f'(x) = \frac{1}{x} \Rightarrow \boxed{f'(c) = \frac{1}{c}}$ $\Leftrightarrow f'(c) = \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right)$

$c \in (n, n+1) \Leftrightarrow n < c < n+1 \Leftrightarrow \frac{1}{n+1} < \frac{1}{c} < \frac{1}{n} \Leftrightarrow$

$\Leftrightarrow \frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n} \quad \Big| \sqrt{} \Leftrightarrow \frac{1}{\sqrt{n+1}} < \sqrt{\ln\left(\frac{n+1}{n}\right)} <$

$< \frac{1}{\sqrt{n}}$

$\frac{1}{\sqrt{n}} - \sqrt{\ln\left(\frac{n+1}{n}\right)} > 0$

Consecințe 1 : i) f - monoton - crescătoare dacă $f'(x) > 0, \forall x \in \mathbb{R}$

ii) f — ıı — descrescătoare dacă $f'(x) < 0, \forall x \in \mathbb{R}$

$a < b$, atunci f crescătoare dacă: $\underbrace{\frac{f(b) - f(a)}{b - a}}_{\vee} > 0 \Leftrightarrow$
$\Leftrightarrow f(b) - f(a) > 0 \Leftrightarrow \boxed{f(b) > f(a)}$

$f'(c) \geq 0 \Rightarrow f$ - crescătoare pe (a, b)

Ex.: $(1 + x)^n \geq 1 + a \cdot x, \forall x > -1, a \in \mathbb{R}, n \in \mathbb{N}$

Are loc inegalitatea $\forall a \in \mathbb{R}$?

$f : (-\cancel{1}^{0}, \infty) \to \mathbb{R}, f(t) = (1 + t)^n$ - cont. și derivabilă pe $(-1, \infty) \Rightarrow$

$\left.\begin{array}{l} \Rightarrow (\exists)\ c \in (0, x) \text{ a.î. } f'(c) = \frac{f(x) - f(0)}{x - 0} \\ f'(x) = n(1 + x)^{n-1} \Leftrightarrow f'(c) = n(1 + c)^{n-1} \end{array}\right\} \Rightarrow$

$\Rightarrow n(1 + c)^{n-1} = \frac{(1 + x)^n - 1}{x}$

$c \in (0, x) \Leftrightarrow 0 < c < x \quad | +1 \quad \Leftrightarrow$

$\Leftrightarrow 1 < 1 + c < 1 + x \quad |^{n-1} \quad \Leftrightarrow 1 < (1 + c)^{n-1} < (1 + x)^{n-1} \quad | \cdot n \Leftrightarrow$

$\Leftrightarrow n < n(1 + c)^{n-1} < (1 + x)^{n-1} \cdot n \Leftrightarrow n < \frac{(1 + x)^n - 1}{x} <$

$< n(1 + x)^{n-1} \Leftrightarrow n \cdot x < (1 + x)^n - 1 \Leftrightarrow (1 + x)^n > 1 + n \cdot x$

ARE LOC DOAR PENTRU $a \in \mathbb{N}$

3) $\sqrt[4]{260} = ?$

$f:(0,\infty) \to \mathbb{R}$, $f(x) = \sqrt[4]{x}$ — cont. și derivabilă $\Rightarrow$

$4^4 = 256$

$\Rightarrow \exists\, c \in [256, 260]$

a. î. $f'(c) = \dfrac{f(260) - f(256)}{4} = \dfrac{\sqrt[4]{260} - 4}{4}$

$f'(x) = \dfrac{1}{4\sqrt[4]{x^3}} \Rightarrow f'(c) = \dfrac{1}{4\sqrt[4]{c^3}}$

$c \in (256, 260) \Leftrightarrow 256 < c < 260 \Leftrightarrow \dfrac{1}{260} < \dfrac{1}{c} < \dfrac{1}{256} \Big|^3 \Leftrightarrow$

$\Leftrightarrow \dfrac{1}{4\sqrt[4]{c^3}} < \dfrac{1}{4\sqrt[4]{(256)^3}} \Leftrightarrow \dfrac{\sqrt[4]{260} - 4}{\cancel{4}} < \dfrac{1}{\cancel{4}\sqrt[4]{(256)^3}} \Leftrightarrow$

$\Leftrightarrow \sqrt[4]{260} - 4 < \dfrac{1}{4^3} \Leftrightarrow \sqrt[4]{260} < 4 + 0,001\ldots < 4,0001$

4) $\arcsin\sqrt{1-x^2} + \arccos x = \pi, \forall x \in (-1, 0)$

Consecință 2: $f'(x) = 0, \forall x \in \mathbb{R} \Rightarrow f(x) =$ constantă

$f:(-1,0) \to \mathbb{R}$, $f(x) = \arcsin\sqrt{1-x^2} + \arccos x$

$f'(x) = \dfrac{1}{1-\sqrt{(1-x^2)}^2} \cdot \dfrac{1}{\cancel{2}\sqrt{1-x^2}} \cdot (-\cancel{2}x) - \dfrac{1}{1-x^2}$

~~$\Rightarrow f'(x) = \dfrac{-x}{x\sqrt{1-x^2}} + \dfrac{1}{\quad} \Rightarrow f'(x) = 0$~~

$\Rightarrow f'(x) = \dfrac{1}{\sqrt{1-(1-x^2)}} \cdot \dfrac{1}{\sqrt{1-x^2}} \cdot (-x) - \dfrac{1}{\sqrt{1-x^2}} =$

$= \dfrac{1}{\sqrt{x^2}} \cdot \dfrac{1}{\sqrt{1-x^2}} \cdot (-x) - \dfrac{1}{\sqrt{1-x^2}} =$

$= \dfrac{1}{|x|} \cdot \dfrac{1}{\sqrt{1-x^2}} \cdot (-x) - \dfrac{1}{\sqrt{1-x^2}} = \dfrac{-\cancel{x}}{-\cancel{x} \cdot \sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} =$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$\Rightarrow f(x) = k \Leftrightarrow \arcsin\sqrt{1-x^2} + \arccos x = k$

$x = -\frac{1}{2} \Rightarrow \arcsin\sqrt{1-\frac{1}{4}} + \arccos\left(-\frac{1}{2}\right) = k \Rightarrow$

$\Rightarrow \arcsin\sqrt{\frac{3}{4}} + \arccos\left(-\frac{1}{2}\right) = k$

$\Rightarrow \arcsin\frac{\sqrt{3}}{2} + \arccos\left(\frac{1}{2}\right) = k \Rightarrow \frac{\pi}{3} + \frac{2\pi}{2} = \pi = k$

<u>Teorema lui Fermat:</u>

$f: I \subset \mathbb{R} \to \mathbb{R}$, I - interval, f - derivabilă pe I atunci dacă $x_0 \in I$ — pct. de extrem pt. f, avem că $f(x_0) = 0$. Reciproc, nu e mereu adevărată ($f'(x_0) = 0 \not\Rightarrow x_0$ — pct. de extrem).

<u>Ex.:</u> $a^x + b^x + c^x > 3$, $\forall x \in \mathbb{R}$, $a, b, c \in \mathbb{R}$. Calculaţi $a \cdot b \cdot c = ?$

$f: \mathbb{R} \to (0, \infty)$ a.î. $f(x) = a^x + b^x + c^x$

$x = 0 \Rightarrow f(0) = 3$

$f(x) \geq f(0), \forall x \in \mathbb{R} \Rightarrow x = 0$ — pct. de minim $\Rightarrow$

$\overset{T.F.}{\Rightarrow} f'(0) = 0$

$f'(x) = a^x \ln a + b^x \ln b + c^x \cdot \ln c \Rightarrow f'(0) = \ln a + \ln b + \ln c = \ln(a \cdot b \cdot c)$

$\ln(a \cdot b \cdot c) = 0 \Leftrightarrow a \cdot b \cdot c = 1$