

Chapter 1

Seminar — 3 Oct. 2023, Rev. 1

1.1 Studiați monotonia șirului

$$x_n = \frac{3n+5}{4n+6}, n \in \mathbb{N}$$

$$x_{n+1} = \frac{3(n+1)+5}{4(n+1)+6} = \frac{3n+8}{4n+1}$$

$$\begin{aligned} x_{n+1} - x_n &= \frac{3n+8}{4n+1} - \frac{3n+5}{4n+7} = \\ &= \frac{(4n+7)(4n+8)}{(4n+1)(4n+7)} = \\ &= \frac{(3n+8)(4n+7) - (3n+5)(4n+1)}{(4n+1)(4n+7)} = \\ &= \frac{(12n+56) - (12n+5)}{16n+7} = \\ &= \frac{51}{16n+7} \end{aligned}$$

1.2 Calculați $\lim_{n \rightarrow \infty} n \cdot \ln(1 - \frac{1}{n})$

$$\lim_{n \rightarrow \infty} n \cdot \ln(1 - \frac{1}{n}) = 0 \cdot \infty$$

$$C.N. : \frac{0}{0}; \frac{\infty}{\infty}; 1^\infty; 0 \cdot \infty; 0^0; \infty - \infty; 0^\infty$$

1.2.1 Metoda I

$$\text{Criteriul log.: } \lim_{n \rightarrow \infty} \frac{\ln(1+x_n)}{x_n} \stackrel{\frac{0}{0}}{=} 1, \text{ pentru } \lim_{n \rightarrow \infty} x_n = 0$$

$$l = \lim_{n \rightarrow \infty} \frac{n \cdot \ln(1 - \frac{1}{n})}{-\frac{1}{n}} \cdot (-\frac{1}{n}) = \lim_{n \rightarrow \infty} \mathcal{H}(-\frac{1}{n}) = -1$$

1.2.2 Metoda II

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right)$$

Criteriul lui Euler: $\lim_{n \rightarrow \infty} (1 + x_n)^{\frac{1}{x_n}} = l$, pentru $\lim_{n \rightarrow \infty} x_n = 0$

$$l = \lim_{n \rightarrow \infty} \ln\left(1 - \frac{1}{n}\right)^n = \ln\left(\lim_{n \rightarrow \infty} \underbrace{\left(1 - \frac{1}{n}\right)^n}_{1^\infty}\right) = \ln\left[\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)\right]$$

$$\lim_{n \rightarrow \infty} n \rightarrow \infty (x_n)^{y^n} = \left(\lim_{n \rightarrow \infty} x_n\right)^{\lim_{n \rightarrow \infty} y^n}$$

1.3 $\lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{1+x^2}}{x}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+x+x^2} - \sqrt{1+x^2}}{x} &\stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{1+x+x^2 - (1+x^2)}{x(\sqrt{1+x+x^2} + \sqrt{1+x^2})} = \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x+x^2} + \sqrt{1+x^2})} = \\ &= \frac{1}{2} \end{aligned}$$

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$$

$$(x - y)(x + y) = x^2 - y^2$$

1.4 Studiați continuitatea funcției

$$f(x) = \begin{cases} \frac{\sqrt{1+x}-x}{x}, & x < 0 \\ \frac{\ln(1+2x)}{x}, & x \geq 0 \end{cases}$$

I. $x < 0 \Rightarrow f(x) = \frac{\sqrt{1+x}-1}{x}$ — elementară, deci continuă pe $x \in (-\infty, 0)$

II. $x > 0 \Rightarrow f(x) = \frac{\ln(1+2x)}{x}$ — elementară, deci continuă pe $x \in (0, \infty)$

III. $x = 0 : f$ — continuă $\Leftrightarrow ls(0) = ld(0) = f(0)$

$$\begin{aligned} ls(0) &= \lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{x(\sqrt{1+x}-1)}{x} = \\ &= \lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \\ &= \lim_{\substack{x \rightarrow 0 \\ x < 0}} \frac{x}{x(\sqrt{1+x}+1)} = \\ &= \frac{1}{2} \end{aligned}$$

$$\lim_{\substack{x \rightarrow 0 \\ x > 0}} \ln \frac{(1+2x)}{2x} \cdot 2 = 2$$

$ls(0) \neq ld(0) \Rightarrow f$ nu este continuă

1.5

a. $f(x) = \arctan \frac{1-x^2}{1+x^2}$, $f'(x) = ?$

$$\begin{aligned} f'(x) &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \left(\frac{1-x^2}{1+x^2}\right)' = \\ &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \frac{(-2x)(xx^2) - (1-x^2) \cdot 2x}{(1+x^2)^2} = \\ &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2} = \\ &= \frac{1}{1+\left(\frac{1-x^2}{1+x^2}\right)^2} \cdot -\left(\frac{4x}{(1+x^2)^2}\right) \end{aligned}$$

b. $f'(x) = x^x$

$$a^b = 2^{b \ln a}$$

$$f'(x) = (e^x \cdot \ln x)' = (e^{u(x)})' = e^{u(x)} \cdot u'(x)$$

$$(e^x)' = e^x$$

$$(e^{u(x)})' = e^{u(x)} \cdot u'(x)$$

$$\begin{aligned} e^x \cdot \ln^x \cdot [x' \cdot \ln x + x \cdot (\ln x)'] &= \\ = e^x \cdot \ln^x \cdot (\ln x + x \cdot \frac{1}{x}) &= \\ = e^x \cdot \ln^x \cdot \ln x \end{aligned}$$

c. $f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{e^x - e^{x_0}}{x - x_0}$

Criteriul exponențial: $\lim_{x \rightarrow a} \frac{e^{f(x)} - 1}{f(x)} = 1$, pentru $\lim_{x \rightarrow a} f(x) = 0$

$$\lim_{x \rightarrow x_0} \frac{e^x - e^{x_0}}{x - x_0} \stackrel{0}{=} \lim_{x \rightarrow x_0} \frac{e^{x-x_0} - 1}{x - x_0} = e^{x_0}$$

1.6

$$\begin{aligned}\int \frac{\cos x - \sin x}{\cos x + \sin x} dx &= \int \frac{1}{t} dt = \\ &= \ln t + C = \\ &= \ln(-\sin x + \cos x) + C\end{aligned}$$

$$\cos x + \sin x = t \mid d \Leftrightarrow (-\sin x + \cos x) \cdot dx = dt$$

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \text{ — funcție injectivă}$$

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ — funcție injectivă}$$

1.7

$$\begin{aligned}\int \frac{x^3}{(1+x^2)^2} dx &= \int \frac{x \cdot x^2}{(1+x^2)^2} dx = \int \frac{2x \cdot x^2}{(1+x^2)^2} \cdot \frac{1}{2} dx = \frac{1}{2} \int \frac{x^2}{1+x^2} 2x \, dx \\ (1+x^2)^2 = t \mid d &\Leftrightarrow 2 \cdot (1+x^2)(1+x^2)' dx = dt\end{aligned}$$

$$\begin{aligned}1+x^2 = t \mid d &\Leftrightarrow 2x \, dx = dt = \\ &= \int \frac{1}{x} - \int t^2 = \\ &= \ln(t) - t^{\frac{-2+1}{2-2+1}} + C\end{aligned}$$