TPSM – 04 – C – 2024 – 03 – 04 04.03.2024

Probabilități limită

Def. Fie $(\Omega, \mathcal{F}, \mathbb{P})$ un spațiu de probabilitate
$(A_n)_{n \geq 1}$ – șir de evenimente $(A_n \in \mathcal{F}, \forall n \in \mathbb{N}^*)$

1) $(A_n)_{n \geq 1}$ – șir monoton crescător dacă $A_n \subseteq A_{n+1}$, $\forall n \in \mathbb{N}^*$

$$\lim_{n \to \infty} A_n \overset{def}{=} \bigcup_{n=1}^{\infty} A_n \overset{not.}{=} A \in \mathcal{F}$$

Notație $\boxed{A_n \uparrow A}$

2) $(A_n)_{n \geq 1}$ – șir monoton descrescătoare dacă

$$A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}^*$$

$$\lim_{n \to \infty} A_n \overset{def.}{=} \bigcap_{n=1}^{\infty} A_n \overset{not}{=} A \in \mathcal{F}$$

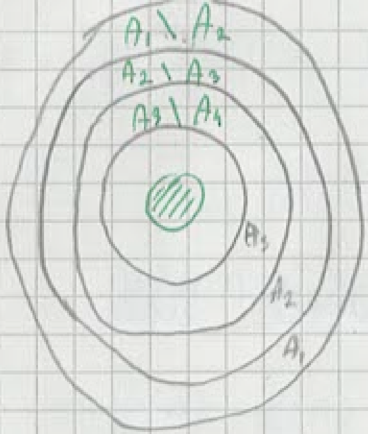
$\boxed{\text{Notație } A_n \downarrow A}$

TEOREMĂ $(\Omega, \mathcal{F}, \mathbb{P})$ – sp. prob.

$(A_n)_{n \geq 1}$ – șir monoton de $\Theta \odot$. Atunci

$$\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n \to \infty} A_n)$$

DEM. I. $A_n \downarrow \phi$, deci $A_n \supseteq A_{n+1}, \forall n \in \mathbb{N}^*$ și $\bigcap_{n=1}^{\infty} A_n = \phi$



$$A_1 = \bigcup_{n=1}^{\infty} (A_n \setminus A_{n+1})$$

$$\mathbb{P}(A_1) = \left(\bigcup_{n=1}^{\infty} (A_n \setminus A_{n+1})\right) \overset{P_3}{=}$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(A_n \setminus A_{n+1})$$

Rezult. $\lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k \setminus A_{k+1}) = 0$

Dar $\sum_{k=n}^{\infty} \mathbb{P}(A_k \setminus A_{k+1}) = \mathbb{P}\left(\bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1})\right) = \mathbb{P}(A_n)$ $\Bigg\} \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(A_n) = 0 = \mathbb{P}(\emptyset) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right)$$

II. $A_n \downarrow A \Rightarrow A_n \setminus A \downarrow \emptyset \overset{I}{\Rightarrow} \lim_{n\to\infty} \mathbb{P}(A_n \setminus A) = 0$

Dar $A \subset A_n$, $\forall n \in \mathbb{N}^*$.

Atunci $\mathbb{P}(A_n \setminus A) = \mathbb{P}(A_n) - \mathbb{P}(A)$, $\forall n \in \mathbb{N}^*$

III. $A_n \uparrow A$ $\left(A = \bigcup_{n=1}^{\infty} A_n\right)$

$A_n \uparrow A \Rightarrow A_n^c \downarrow A^c \overset{II}{\Longrightarrow} \lim_{n\to\infty} \mathbb{P}(A_n^c) = \mathbb{P}(A^c) \Rightarrow$

$\Rightarrow \lim_{n\to\infty} \left[1 - \mathbb{P}(A_n)\right] = 1 - \mathbb{P}(A) \Rightarrow \lim_{n\to\infty} \mathbb{P}(A) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right) =$

$= \mathbb{P}(\emptyset) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right)$

EVENIMENTE INDEPENDENTE

Def.: $(\Omega, \mathcal{F}, \mathbb{P})$ sp. de prob.

1) $A, B \in \mathcal{F}$ s.n. evenimente independente dacă
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

2) $\{A_n, n \geq 1\} \subset \mathcal{F}$ - șir de evenimente distincte independente dacă $\mathbb{P}(A_{n_1} \cap A_{n_2} \cap \dots \cap A_{n_p}) = \mathbb{P}(A_{n_1}) \cdot P(A_{n_2}) \cdot \dots \cdot \mathbb{P}(A_{n_p})$
$$\forall \underbrace{n_1, n_2, n_3, \dots, n_p}_{\text{distincte}} \in \mathbb{N}^* \ (p \geq 2)$$

Consecințe

I.1) ϕ, A - ev. independente, $\forall A \in \mathcal{F}$

Ω, A - ev. indep., $\forall A \in \mathcal{F}$

I.2) $A, B \in \mathcal{F}$ - ev. indep. $\Rightarrow \begin{cases} A, B^c \text{ - ev. indep.} \\ A^c, B^c \text{ - ev. indep.} \end{cases}$

I.3) $A_1, \dots, A_n \in \mathcal{F}$ - ev. distincte, independente

Atunci
$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 1 - \prod_{k=1}^{n} \mathbb{P}(A_k^c)$$

DEM.

I_1 Fie $A \in \mathcal{F}$.
$$\mathbb{P}(A \cap \phi) = \mathbb{P}(\phi) = 0 = \mathbb{P}(A) \cdot \underbrace{\mathbb{P}(\phi)}_{\overset{\shortparallel}{0}} \Rightarrow$$
$\Rightarrow A, \phi$ - independ.

$\mathbb{P}(A \cap \Omega) = \mathbb{P}(A) \cdot \underbrace{\mathbb{P}(\Omega)}_{\overset{\shortparallel}{1}} \Rightarrow A, \Omega$ - independente

I_2. Fie $A, B \in \mathcal{F}$ ev. indep.

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A \setminus (A \cap B)) = \mathbb{P}(A) - \mathbb{P}(A \cap B) =$$

$$= \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A) \cdot \mathbb{P}(B^c)$$

A, B^c - ev. independente

A, B^c - independente $\Rightarrow A^c, B^c$ - indep.

I_3. $\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 1 - \mathbb{P}\left(\left(\bigcup_{k=1}^{n} A_k\right)^c\right) \overset{\text{de Morgan}}{=\!=} 1 - \mathbb{P}\left(\bigcap_{k=1}^{n} A_k^c\right) \underset{I_2}{\overset{\text{ind}}{=\!=}}$

$$= 1 - \prod_{k=1}^{n} \mathbb{P}(A_k^c)$$

EVENIMENTE CONDIȚIONATE

Def.: $(\Omega, \mathcal{F}, \mathbb{P})$ - sp. de prob.

1. $A \in \mathcal{F}$ a.î. $\mathbb{P}(A) > 0$ - A eveniment nenegljabil (neglijabil)

2. $A, B \in \mathcal{F}$, $\mathbb{P}(A) > 0$

$$\underbrace{\mathbb{P}(B|A)}_{B \text{ condiționat de } A} \overset{\text{def.}}{=} \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \in [0, 1]$$

Proprietăți:

C_1 $A, B \in \mathcal{F}$, $\mathbb{P}(A) > 0 \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A)$

C_2 $A, B \in \mathcal{F}$, $\mathbb{P}(A), \mathbb{P}(B) > 0 \Rightarrow \mathbb{P}(B|A) \cdot \mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B)$

C_3 $\left.\begin{array}{l} A, B \in \mathcal{F} \text{ - ev. indep.} \\ \mathbb{P}(A) > 0 \end{array}\right\} \Rightarrow \mathbb{P}(B|A) = \mathbb{P}(B)$

C_4. $A \in \mathcal{F}$, $\mathbb{P}(A) > 0$

$\mathbb{P}_A : \mathcal{F} \to \mathbb{R}$, $\mathbb{P}_A(X) \stackrel{def}{=} \mathbb{P}(X|A)$, $X \in \mathcal{F}$

$\mathbb{P}_A$ — funcție de probabilitate

Dem.:

C_1.: $\mathbb{P}(B|A) \stackrel{def.}{=} \dfrac{\mathbb{P}(A \cap B)}{\underbrace{\mathbb{P}(A)}_{>0}} \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(B|A) \cdot \mathbb{P}(A)$

C_2.: Din C_1

C_3.: $\mathbb{P}(B|A) \stackrel{def.}{=} \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} \stackrel{indep.}{=} \dfrac{\cancel{\mathbb{P}(A)} \cdot \mathbb{P}(B)}{\cancel{\mathbb{P}(A)}} = \mathbb{P}(B)$

Formule de calculul probabilităților care implică evenimentele condiționate

Def.: $(\Omega, \mathcal{F}, \mathbb{P})$ — sp. de prob.

$S = \{A_i, i \in I\} \subset \mathcal{F}$, I — cel mult numărabilă

S — sistem complet de evenimente dacă:

1) $A_i \cap A_j = \emptyset$, $\forall i, j \in I$, $i \neq j$

2) $\bigcup_{i \in I} A_i = \Omega$

3) $\mathbb{P}(A_i) > 0$, $\forall i \in I$

Formula probabilității totale (FPT)

Fie $S = \{A_i, i \in I\}$ – sistem complet de evenimente

$$\mathbb{P}(A) = \sum_{i \in I} \mathbb{P}(A \mid A_i) \cdot \mathbb{P}(A_i), \forall A \in \mathcal{F}$$

Dem.: Fie $A \in \mathcal{F}$

$$\mathbb{P}(A) = \mathbb{P}(A \cap \Omega) \overset{2)}{=} \mathbb{P}\Big(A \cap \big(\bigcup_{i \in I} A_i\big)\Big) =$$

$$= \mathbb{P}\Big(\bigcup_{i \in I} (A \cap A_i)\Big) = \sum_{i \in I} \mathbb{P}(A \cap A_i) \overset{c_2}{=} \sum_{i \in I} \mathbb{P}(A \mid A_i) \cdot \mathbb{P}(A_i)$$

$$(A \cap A_i) \cap (A \cap A_j) = \emptyset, \ i \neq j$$

Formula lui Bayes

$S = \{A_i, i \in I\}$ – sistem complet de evenimente

$A \in \mathcal{F}$, $\mathbb{P}(A) > 0$

$$\mathbb{P}(A_k \mid A) = \frac{\mathbb{P}(A \mid A_k) \cdot \mathbb{P}(A_k)}{\sum_{i \in I} \mathbb{P}(A \mid A_i) \cdot \mathbb{P}(A_i)}, \forall k \in I$$

Dem.

Fie $k \in I$

$$\mathbb{P}(A_k \mid A) \overset{def}{=} \frac{\mathbb{P}(A_k \cap A)}{\mathbb{P}(A)} \underset{(FPT)}{\overset{c_1}{=}} \frac{\mathbb{P}(A \mid A_k) \cdot \mathbb{P}(A_k)}{\sum_{i \in I} \mathbb{P}(A \mid A_k)\, \mathbb{P}(A)}$$