Chapter 1

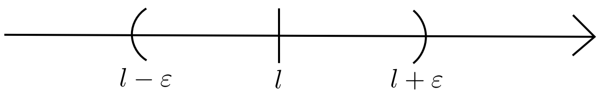
Seminar — 17 Oct. 2023, Rev. 1

1.1 Limite de şiruri

1.1.1 Teorie

Definiţie

Fie $(x_n)_n \in \mathbb{N}$, şir de numere reale. Spunem că şirul $x_n, n \in \mathbb{N}$ admite limita $l \in \mathbb{R}$ dacă de la un anumit rang $n_\varepsilon(n \geq n \geq n_\varepsilon)$ doar un număr finit de termeni ai şirului nu se află într-o vecinătate al lui l.



Definiţie

$x_n, n \in \mathbb{N}$ admite limita $l \in \mathbb{R}$ dacă:

$\forall \varepsilon > 0, (\exists)\ n_\varepsilon \in \mathbb{N}$ cu $n \geq n_\varepsilon$ a.î. $|x_n - l| < \varepsilon$

$|x_n - l| < \varepsilon \Leftrightarrow -\varepsilon < x_n - l < \varepsilon \mid +l \Leftrightarrow l - \varepsilon < x_n = l + \varepsilon$

Dacă $l =$ finit $\Rightarrow (x_n)_n \geq 1$ şir convergent

1.1.2 Exerciţii

1. Arătaţi că:

   a. $\lim\limits_{n\to\infty} \frac{2n+5}{3n+7} = \frac{2}{3}$

$|\frac{2n+5}{3n+7} - \frac{2}{3}| < \varepsilon \Leftrightarrow |\frac{6n+15-6n-14}{9n+21}| < \varepsilon \Leftrightarrow |\frac{1}{9n+21}| < \varepsilon \Leftrightarrow \frac{1}{9n+21} < \varepsilon \Leftrightarrow$
$9n + 21 > \frac{1}{\varepsilon} \Leftrightarrow 9n > \frac{1}{\varepsilon} - 21$

$9n > \frac{1-21\varepsilon}{\varepsilon}$

?

$n > \frac{1-21\varepsilon}{9\varepsilon} \Rightarrow n_\varepsilon = \left[\frac{1-21\varepsilon}{9\varepsilon}\right] + 1 \Rightarrow \lim\limits_{n\to\infty} \frac{2n+5}{3n+7} = \frac{2}{3}$

b. $\lim\limits_{n\to\infty} \frac{n^2}{2n^2+2} = \frac{1}{2}$

$|\frac{n^2}{2n^2+2} - \frac{1}{2}| \Leftrightarrow |\frac{2n^2}{4n^2+4} - \frac{2n^2+2}{4n^2+4}| \Leftrightarrow |\frac{2n^2}{4n^2+4} - \frac{2n^2+2}{4n^2+4}| \Leftrightarrow |\frac{\cancel{2n^2}-\cancel{2n^2}-2}{4n^2+4}| \Leftrightarrow$
$|\frac{-2}{4n^2+4}| \Leftrightarrow |\frac{-2}{4(2n^2+2)}| \Rightarrow |\frac{-1}{2n^2+2}| < \varepsilon \Rightarrow 2n^2 + 2 < \frac{1}{\varepsilon}$

$2n^2 > \frac{1}{\varepsilon} - 2 \Rightarrow 2n^2 > \frac{1-2\varepsilon}{\varepsilon} \Rightarrow n^2 > \frac{1-2\varepsilon}{2\varepsilon} \Rightarrow n > \sqrt{\frac{1-2\varepsilon}{2\varepsilon}}$

$n_\varepsilon = \left[\sqrt{\frac{1-2\varepsilon}{2\varepsilon}}\right] + 1$

1.1.3

Şirul $(x_n)_n \in \mathbb{N}$; (x_{k_n}) — subşir al lui $(x_n)_n \geq 1$

$k_n = 2n$
$k_n = 2n + 1$
$x_n = n$

Lemă

Dacă $\lim\limits_{n\to\infty} x_n = l$, atunci $(\forall) k_n \in \mathbb{N} \quad \lim\limits_{n\to\infty} x_{k_n} = l$

Observaţie

Dacă $(\exists)$ k_n şi $l_n \in \mathbb{N}$ a.î. $\lim\limits_{n\to\infty} x_{k_n} \neq \lim\limits_{n\to\infty} x_{l_n}$, atunci:

$$(\nexists) \lim_{n\to\infty} x_n = l$$

Exemple

1. $x_n = \cos(\pi \cdot n), n \in \mathbb{N}$

   $\lim\limits_{n\to\infty} x_n = \lim_{n\to\infty} \cos(\pi \cdot n)$

   $\cos(\pi \cdot n) = \lim\limits_{n\to\infty} (-1)^n$

$n = 2k \Rightarrow \lim\limits_{n\to\infty} x_{2k} = \lim\limits_{n\to\infty} (-1)^{2k} = 1$

$n = 2k+1 \Rightarrow \lim\limits_{n\to\infty} x_{2k+1} = \lim\limits_{n\to\infty} (-1)^{2k+1} = -1$

2. $x_n = \{\frac{n}{2}\}$

   $n = 3k$

   $n \in \{k|3 \nmid d\} \Rightarrow \lim\limits_{n\to\infty} x_n \in \mathbb{Q},\ \lim\limits_{n\to\infty} x_n = 0 \Rightarrow (\nexists) \lim\limits_{n\to\infty} \{\frac{n}{3}\}$

   $\sqrt{2} = \lim\limits_{n\to\infty} x_n$

Criteriul lui d'Alébert $(\frac{\infty}{\infty})$

$\lim\limits_{n\to\infty} \frac{a_p \cdot n^p + a_{p-1}\cdot n^{p-1} + \cdots + a_{in} + a_0}{b_q \cdot n^q + b_{q-1}\cdot n^{q-1} + \cdots + b_{in} + b_0} \overset{\frac{\infty}{\infty}}{=}$

$$= \begin{cases} 0, & p < q \\ \frac{a_p}{b_q}, & p = q \\ \infty, & p > q \end{cases}$$

Exemple

1. $\lim\limits_{n\to\infty} \frac{4n^5+8n^2+31}{9n^2-16n+3} \overset{\frac{\infty}{\infty}}{=} \frac{4}{9}$

2. $\lim\limits_{n\to\infty} \sqrt{n^2+2n+3} - \sqrt{n^2+n+1} =$

   $= \lim\limits_{n\to\infty} \frac{(\sqrt{n^2+2n+3}-\sqrt{n^2+n+1})(\sqrt{n^2+2n+3}+\sqrt{n^2+n+1})}{\sqrt{n^2+2n+3}+\sqrt{n^2+n+1}} =$

   $= \lim\limits_{n\to\infty} \frac{\cancel{n^2}+2n+3-\cancel{n^2}-n-1}{\sqrt{n^2(1+\frac{2}{n}+\frac{3}{n^2})}+\sqrt{n^2(1+\frac{1}{n}+\frac{1}{n^2})}} =$

   $= \lim\limits_{n\to\infty} \frac{n+2}{\sqrt{n^2}+\sqrt{n^2}} =$

   $= \lim\limits_{n\to\infty} \frac{n+2}{2n} = \frac{1}{2}$

3. $\lim\limits_{n\to\infty} \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} =$

   $= \lim\limits_{n\to\infty} \sqrt{n+2} - \sqrt{n+1} + \sqrt{n} - \sqrt{n+1} =$

   $= \lim\limits_{n\to\infty} \frac{n+2-n-1}{\sqrt{n+2}+\sqrt{n+1}} + \sqrt{n-n-1}\sqrt{n} - \sqrt{n+1} - \frac{1}{\sqrt{n}-\sqrt{n+1}} = 0$

4. $\lim\limits_{n\to\infty} \frac{4^n+8^n}{9^n+11^n} =$

$$= \lim_{n\to\infty} \frac{8^n\left(\overbrace{\left(\frac{4}{8}\right)^n}^{0}+1\right)}{11^n\left(\underbrace{\left(\frac{4}{8}\right)^n}_{0}+1\right)} =$$

$$= \lim_{n\to\infty} \frac{8^n}{11^n} = 0$$

5. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{k^2+k} =$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} =$$

$$= \lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}\right) =$$

$$= \lim_{n\to\infty} \left(\frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1}\right) =$$

$$= \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

6. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \cdots + \frac{1}{\sqrt{n^2+n}}$

$\frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+n}}$

$\frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$

$\vdots$

$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}$ (-1)

$\Rightarrow x_n \leq \frac{n}{\sqrt{n^2+1}}$

$\frac{n}{\sqrt{n^2+1}} > \frac{n}{\sqrt{n^2+n}}$

$\frac{n}{\sqrt{n^2+n}} \geq \frac{n}{\sqrt{n^2+n}}$ (+)

$x_n \geq \frac{n}{\sqrt{n^2+2}}$

$\frac{n}{\sqrt{n^2+n}} \leq x_n \leq \frac{n}{\sqrt{n^2+1}}$

**Săgeată de la toate inegalităţile către un 1*

Lema cleştelui

$a_n \leq x_n \leq b_n$şi $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = l \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} x_{n=l}$

Exemple

1. $\lim\limits_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} =$

   (progresie geometrică)

   $$= \lim n \to \infty 1 \cdot \frac{1 - \overbrace{\left(\frac{1}{2}\right)^{n+1}}^{0}}{1-\frac{1}{2}} = 2$$

2. $\lim\limits_{n\to\infty} \left(\frac{3n+2}{3n+1}\right)^n \overset{1^\infty}{=}$

   $\lim\limits_{n\to\infty} \left(1 + \frac{3n+2}{3n+1} - 1\right)^n =$

   $\lim\limits_{n\to\infty} \left(1 + \frac{\cancel{3n}+2-\cancel{3n}-1}{3n+1} - 1\right)^n =$

   $\lim\limits_{n\to\infty} \left[\left(1 + \frac{1}{(3n+1)}\right)^{3n+1}\right]^{\frac{n}{3n+1}} =$

   $\lim\limits_{n\to\infty} e \lim\limits_{n\to\infty} = \frac{n}{3n+1} = e^{\frac{1}{3}}$

Criteriul lui Euler (1^∞)

$\lim\limits_{n\to\infty} (1 + x_n)^{\frac{1}{-en}} = e$, dacă $\lim\limits_{n\to\infty} x_n = 0$

Exemple

1. $\lim\limits_{n\to\infty} n \cdot \ln\left(\frac{n^2+2n+2}{n^2+n+1}\right) =$

   $\lim\limits_{n\to\infty} \ln\left(\frac{n^2+2n+2}{n^2+n+1}\right)^n =$

   $\ln \lim\limits_{n\to\infty} \left(\frac{n^2+2n+2}{n^2+n+1}\right)^n =$

   $\ln \lim\limits_{n\to\infty} \left[\left(1 + \frac{n+1}{n^2+n+1}\right) \frac{n^2+n+1}{n+1}\right] \frac{n+1}{n^2+n+1} \cdot n =$

$\ln e \lim\limits_{n\to\infty} \frac{n^2+n}{n^2+n+1} = \ln e = 1$