

$$f'''(x) = \frac{(-4)^3 [(1+2x)^4] - (-4) \cdot 4[(1+2x)^3]^1}{(1+2x)^9} = \frac{-16}{2(1+2x)^3} \Rightarrow$$

$$\Rightarrow f'(0) = \frac{2}{2} = 2 \quad \Rightarrow f'''(0) = 16$$

Teme: De rezolvat din nou:

Calculati ca putemul lui Taylor de 3 ord.

$$\left(\lim_{x \rightarrow 0} \frac{\ln(1+2x) - \sin 2x + 2x^2}{x^3} \right)$$

$$T_3(f, x) \text{ la } x_0 = 0 \quad \text{pt } f(x) = \ln(1+2x) - \sin 2x$$

* Verzi +T-ANMARE-S-13-2023-II-14

Teoreme de medie

Eile $f: [a, b] \rightarrow \mathbb{R}$ continuă

Atunci există $c \in [a, b]$ astfel încât $\int_a^b f(x) dx = f(c)(b-a)$

Demonstratie:

Deoarece $f: [a, b] \rightarrow \mathbb{R}$ este continuă și conform teoremei lui Weierstrass, există $x_*, x^* \in [a, b]$ s.t.

$$f(x_*) \leq f(x) \leq f(x^*) \quad (\forall) x \in [a, b]$$

$$\rightarrow \int_a^b f(x_*) dx \leq \int_a^b f(x) dx \leq \int_a^b f(x^*) dx$$

$$f(x_*) \int_a^b 1 dx \leq \int_a^b f(x) dx \leq f(x^*) \int_a^b 1 dx$$

$$(b-a)f(x^*) \leq \int_a^b f(x) dx \leq (b-a)f(x^*)$$

$$f(x_*) \leq \frac{\int_a^b f(x) dx}{b-a} \leq f(x^*)$$



f are proprietatea lui Darboux

există $c \in [a, b]$ s.t. $\frac{\int_a^b f(x) dx}{b-a} = f(c)$

TEOREMĂ

Fie $f: [a, b] \rightarrow \mathbb{R}$ - continuă

Așadar $F: [a, b] \rightarrow \mathbb{R}$, dată de $F(x) = \int_a^x f(t) dt$
pentru orice $x \in [a, b]$, este derivabilă și
 $F'(x) = f(x)$ pentru orice $x \in [a, b]$
I.e. F este o primitivă a lui f

$f: [a, b] \rightarrow \mathbb{R}$
continuă

$$\begin{aligned}
 & \int_a^b f(x) dx = F(b) - F(a) \\
 & F' = f \quad \rightarrow F(b) = F(a) + \int_a^b F'(x) dx \\
 \\
 & F(x) = \int_a^x f(t) dt \\
 & F'(x) = f(x) \quad (\forall) x \in [a, b] \quad \rightarrow \left(\int_a^x f(t) dt \right)' = f(x) \\
 \\
 & f \rightarrow Sf \rightarrow \left(\int_a^x f(t) dt \right)' \rightarrow f
 \end{aligned}$$

Demonstratie

Fie $x_0 \in [a, b]$

Dăm să arătăm că există $\lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$

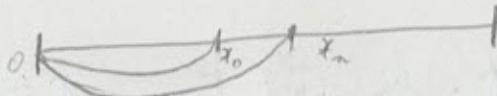
Este suficient să demonstreăm un sir arbitrar

$(x_n)_n \subseteq [a, b] - \{x_0\}$ cu $\lim_{n \rightarrow \infty} x_n = x_0$.

și să arătăm că $\lim_{n \rightarrow \infty} \frac{F(x_n) - F(x_0)}{x_n - x_0} = f(x_0)$

$$\frac{F(x_n) - F(x_0)}{x_n - x_0} = \frac{\int_{x_0}^{x_n} f(t) dt - \int_{x_0}^{x_0} f(t) dt}{x_n - x_0} = \frac{\int_{x_0}^{x_n} f(t) dt}{x_n - x_0} =$$

$$= f(c_n) \xrightarrow{n \rightarrow \infty} f(x_0).$$



Conform teoremei de medie există un între x_0 și x_n s.t.

$$\int_{x_0}^{x_n} f(t) dt = (x_n - x_0) \cdot f(c_n)$$

EXEMPLU

$$\lim_{x \rightarrow 0} \frac{\int_0^x \ln(1+t^2) dt}{\sin^3 x} = ?$$

$$\int_0^x \ln(1+t^2) dt = x \ln(1+x^2) \rightarrow 0 \ln 1 = 0 \cdot 0$$

c_x între 0 și x

$$\begin{cases} x \rightarrow 0 \\ c_x \rightarrow 0 \end{cases}$$

$\xrightarrow{V_1}$ l'Hopital

$\xrightarrow{V_2}$ $\int_0^x \ln(1+t^2) dt = \dots$

V_2 $\int_0^x \ln(1+t^2) dt =$

$$\int \ln(1+t^2) dt = \int t' \cdot \ln(1+t^2) dt = t \cdot \ln(1+t^2) - \int t \cdot \dots$$

$$\dots \frac{1}{1+t^2} \cdot 2t dt =$$

$$= t \cdot \ln(1+t^2) - 2 \int \frac{t^2+1}{t^2+1} dt =$$

$$= t \cdot \ln(1+t^2) - 2 \left[\int 1 dt - \int \frac{1}{t^2+1} dt \right]$$

$$t \cdot \ln(1+t^2) - 2t + 2 \arctan t$$

$$\lim_{x \rightarrow 0} \frac{x \cdot \ln(1+x^2) - 2x + 2 \arctan x}{\sin x^3} \stackrel{0}{=} \text{etc}$$

$$\text{V}_1 \lim_{x \rightarrow 0} \frac{\left(\int_0^x \ln(1+t^2) dt \right)'}{\sin^3 x} = \lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{3 \sin^2 x - \cos x} =$$

$$= \frac{1}{3} \lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{x^2} \cdot \frac{x^2}{\sin^2 x} = \dots \frac{1}{3}$$

Teoreme de schimbare de variabile

Eie $\mathcal{C} : [a, b] \rightarrow [\alpha, \beta] \rightarrow \mathbb{R}$ s. r.:

i) \mathcal{C} derivabilă

ii) \mathcal{C}' continuă

iii) t continuă

\mathcal{C}' nu este continuă în 0

$\mathcal{C} : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathcal{C}'(x) \begin{cases} 2 \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\mathcal{C}(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{Atunci } \int_a^b f(\varphi(x)) \cdot \varphi'(x) dx = \int_{\varphi(a)}^{\varphi(b)} f(t) dt$$

Demonstratie

f continuă \Rightarrow există F derivabilă cu $F' = f$

$$\text{Leibniz - Newton} \Rightarrow \int_{\varphi(a)}^{\varphi(b)} f(t) dt = F \Big|_{\varphi(a)}^{\varphi(b)} = F(\varphi(b)) - F(\varphi(a))$$

$$(f \circ \varphi)' \cdot \varphi' = (F' \circ \varphi) \cdot \varphi' = \\ = (F \circ \varphi)'$$

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$(f \circ g)' = (f' \circ g) \cdot g'$$

$$\int_a^b h(x) dx = \int_a^b f(\varphi(x)) \cdot \varphi'(x) dx = \int_{\varphi(a)}^{\varphi(b)} f(t) dt$$

exprimare în φ

$$x = a \rightarrow t = \varphi(a) \quad \varphi(x) \leftarrow t$$

$$x = b \rightarrow t = \varphi(b) \quad \varphi'(x) dx = dt$$

EXEMPLO

$$\int_0^1 x^2 \cdot e^{x^3} dx = \frac{1}{3} \int_0^1 e^t dt = \frac{1}{3} e^t \Big|_0^1 (e^1 - e^0) = \frac{e-1}{3}$$

$$x^3 = t \\ (x^3)' dx = dt$$

$$3x^2 dx = dt \\ x^2 dx = \frac{1}{3} dt$$

$$x=0 \rightarrow t = 0^3 = 0$$

$$x=1 \rightarrow t = 1^3 = 1$$

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{1}{2} \int_1^2 \frac{1}{t} dt = \frac{1}{2} \ln t \Big|_1^2 = \frac{\ln 2 - \ln 1}{2} = \frac{\ln 2}{2}$$

$$1+x^2 = t \\ (1+x^2)' dx = dt \\ 2x dx = dt \\ x dx = \frac{1}{2} dt$$

$$x=0 \rightarrow t=1$$

$$x=1 \rightarrow t=2$$

$$\int_0^1 \frac{x}{1+x^2} dx = \int_0^1 \frac{1}{1+(x^2)^2} \cdot x dx = \int_0^1 \frac{1}{1+t^2} dt =$$

$$1+x^2 = t \\ 2x dx = dt \\ x^2 dx = dt$$

$$x^2 = t \\ 2x dx = dt \\ x dx = \frac{1}{2} dt$$

$$x=0 \rightarrow t=0 \\ x=1 \rightarrow t=1$$

$$= \frac{1}{2} \arctan t \Big|_0^1 = \frac{1}{2} (\arctan 1 - \arctan 0) =$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$$

$$\int_0^{\frac{\pi}{4}} \tan x + \tan^2 x \, dx =$$

$$(\tan x)' = \frac{1}{\cos^2 x} =$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} =$$

$$= 1 + \tan^2 x$$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \tan x (1 + \tan^2 x) \, dx = \int_0^1 t \, dt = \frac{t^2}{2} \Big|_0^1 = \frac{1}{2}$$

$$\tan x = t$$

$$(\tan x)' dx = dt$$

$$(1 + \tan^2 x) dx = dt$$

Teorema de integrare prin parti

Este $f, g : [a, b] \rightarrow \mathbb{R}$ derivabile s.t. f' și g' contin

Atunci $\int_a^b f'(x) \cdot g(x) \, dx = \boxed{f(b)g(b) - f(a)g(a)} - \int_a^b f(x) \cdot g'(x) \, dx$.

$$\therefore \int_a^b f'(x) \, dx$$

$$\boxed{\int_a^b f'g = f \cdot g \Big|_a^b - \int_a^b f \cdot g'}$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$\int_a^b (f \cdot g)' = \int_a^b f' \cdot g + f \cdot g' =$$

$$f \cdot g \Big|_a^b = \int_a^b f' \cdot g + \int_a^b f \cdot g'$$

Exemple :

$$\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx =$$

$$\begin{aligned} & \rightarrow \int_0^{\frac{\pi}{2}} \left(\frac{x^2}{2}\right)' \sin x \, dx = \frac{x^2}{2} \sin x \Big|_0^{\frac{\pi}{2}} - \\ & - \int_0^{\frac{\pi}{2}} \frac{x^2}{2} (\sin x)' \, dx = \\ & = \frac{\pi^2}{8} - \frac{1}{2} \int_0^{\frac{\pi}{2}} -x \sin x \, dx = \\ & = ?? \end{aligned}$$

$$\downarrow \int_0^{\frac{\pi}{2}} (-\cos x)' \cdot x \, dx =$$

$$= (-\cos x) \cdot x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) \cdot x' - dx =$$

$$= \int_0^{\frac{\pi}{2}} -\cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1$$

$$\begin{aligned}
 \int_0^1 x \cdot \operatorname{arctg} x \, dx &= \int_0^1 \left(\frac{x^2}{2} \right)' \operatorname{arctg} x \, dx = \\
 &= \frac{x^2 \cdot \operatorname{arctg} x}{2} \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx = \\
 &= \frac{\pi}{8} - \frac{1}{2} \int_0^1 \frac{x^2 + 1 - 1}{1+x^2} \, dx = \frac{\pi}{8} - \frac{1}{2} \int_0^1 1 \, dx + \frac{1}{2} \int_0^1 \frac{1}{1+x^2} \, dx = \\
 &= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{2} \operatorname{arctg} x \Big|_0^1 = \frac{\pi}{8} - \frac{1}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \int_1^e \ln x \, dx &= \int_1^e x^1 \cdot \ln x \, dx = \\
 &= x \ln x \Big|_1^e - \int_1^e x \cdot \frac{1}{x} \, dx = \\
 &= e - (e - 1) = 1
 \end{aligned}$$

$$\begin{aligned}
 I &= \int_0^\pi e^x \cdot \sin x \, dx = \int_0^\pi (e^x)' \sin x \, dx = e^x \sin x \Big|_0^\pi - \int_0^\pi e^x \cdot (\sin x)' \, dx = \\
 &= - \int_0^\pi e^x \cdot \cos x \, dx = \mu \\
 &= - \underbrace{\int_0^\pi (e^x)' \cdot \cos x \, dx}_{= - \left[e^x \cdot \cos x \Big|_0^\pi - \int_0^\pi e^x \cdot (\cos x)' \, dx \right]} = \\
 &= - \left[e^x \cdot \cos x \Big|_0^\pi - \int_0^\pi e^x \cdot (\cos x)' \, dx \right] = \\
 &= - \left[-e^{-\pi} - 1 + \int_0^\pi e^x \cdot \sin x \, dx \right] = \\
 &= - \int_0^\pi e^x \sin x + 1 + \frac{1}{e^\pi} = 1 + \frac{1}{e^\pi} - I
 \end{aligned}$$

$$2I = 1 + \frac{1}{e^\pi}$$

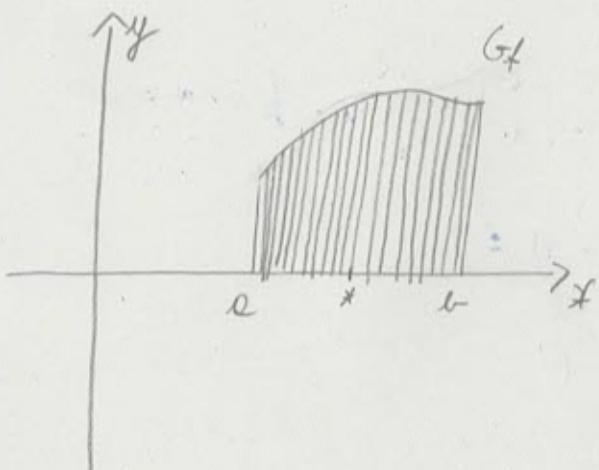
$$I = \frac{1}{2} \left(1 + \frac{1}{e^\pi} \right)$$

Teoreme

$$f: [a, b] \rightarrow \mathbb{R}_+$$

integrabilă

$$S = \{(x, y) \mid x \in [a, b], y \in [0, f(x)]\}$$



$$\Rightarrow S \text{ are arie} \quad \text{și aria lui } S = \int_a^b f(x) dx$$

TEOREMĂ

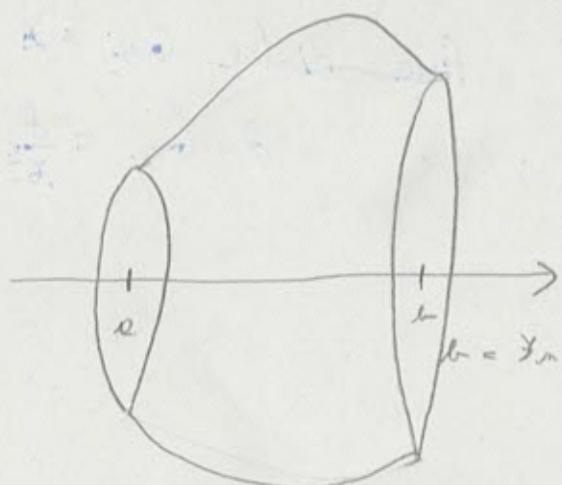
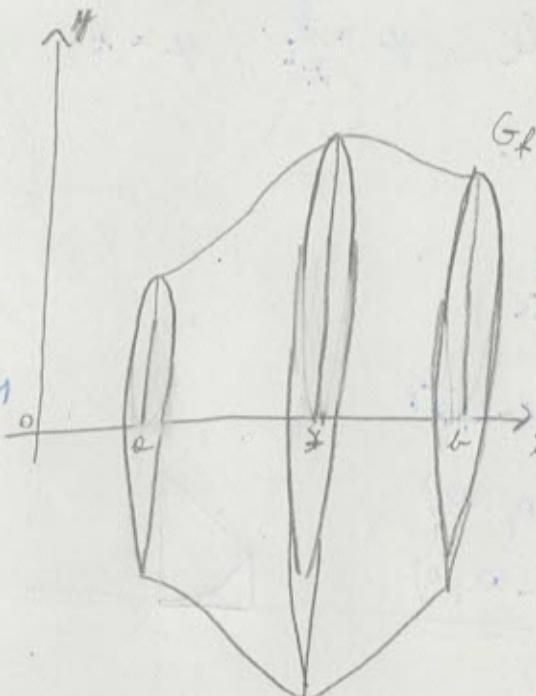
volumul unui corp

$f: [a, b] \rightarrow \mathbb{R}$
intregabilă

$$V = \left\{ (x, y, z) \in \mathbb{R}^3 / x \in [a, b], y^2 + z^2 \leq f(x)^2 \right\}$$

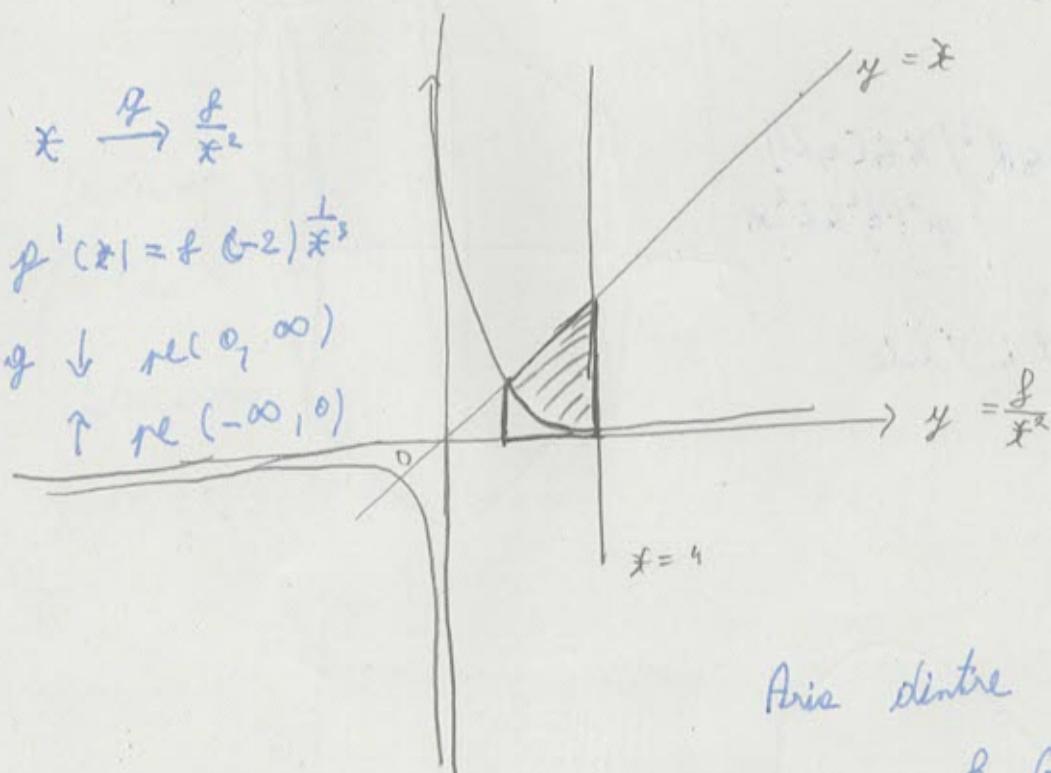
are volum

și volumul lui V este



Exemplu

Să se calculeze aria suprafeței plane determinată de
curbele $y = \frac{f}{x^2}$, $y = x$ și $x = 4$



Aria dintre $y = x$

$$\text{de } y = \frac{f}{x^2}$$

$$y = x$$

$$y = \frac{f}{x^2}$$

$$\Rightarrow x^3 = f \Rightarrow x = 2$$

$$A = \int_2^4 x \, dx - \int_2^4 \frac{f}{x^2} \, dx = \frac{x^2}{2} \Big|_2^4 - f \cdot \frac{(-2)}{x^3} \Big|_2^4 = \dots$$