ALGAD_8_S_2023_11_01 01.11.2023

!!! ÎNTRĂ LA EXAMEN !!!

* Continuare ALGAD_6_S_2023_10_25

$e_1'' = e_{11} e_1' + e_{21} e_2' + e_{31} e_3'$

$(1,0,0) = e_{11}(1,1,0) + e_{21}(1,0,1) + e_{31}(1,0,-1)$

$(1,0,1) = (e_{11}, e_{11}, 0) + (e_{21}, 0, e_{21}) + (e_{31}, 0, -e_{31})$

$$\begin{cases} e_{11} + e_{21} + e_{31} = 1 \\ e_{21} + 0 + 0 = 0 \\ 0 + e_{32} + e_{33} = 0 \end{cases} \Rightarrow \boxed{e_{11} = 0} \;\Big|\; \Rightarrow \begin{cases} e_{21} + e_{31} = 1 \\ e_{21} - e_{31} = 0 \ (+) \end{cases}$$

$$2e_{21} = 1$$

$$\boxed{e_{21} = \frac{1}{2}}$$

$$\boxed{e_{31} = \frac{1}{2}}$$

$e_2'' = e_{12} e_1' + e_{22} e_2' + e_{32} e_3'$

$(1,1,0) = e_{12}(1,1,0) + e_{22}(1,0,1) + e_{32}(1,0,-1)$

$$\begin{cases} e_{12} + e_{22} + e_{32} = 1 \\ \boxed{e_{12} = 1} \\ e_{22} - e_{32} = 0 \end{cases} \;\Big|\; \Rightarrow e_{22} = e_{32} = 0$$

$e_3'' = e_{13} e_1' + e_{23} e_2' + e_{33} e_3'$

$(1,1,1) = e_{13}(1,1,0) + e_{23}(1,0,1) + e_{33}(1,0,-1)$

$$(1,1,1) = e_{13}(1,1,0) + e_{23}(1,0,1) + e_{33}(1,0,-1)$$

$$\begin{cases} e_{13} + e_{23} + e_{33} = 1 \\ \boxed{e_{13} = 1} \\ e_{23} - e_{33} = 1 \end{cases} \Rightarrow \begin{cases} e_{23} + e_{33} = 0 \\ e_{23} - e_{33} = 1 \quad (+) \end{cases}$$

$$2e_{23} = 1$$

$$e_{23} = \frac{1}{2}$$

$$e_{33} = \frac{1}{2}$$

$$\text{col}_1 : \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad \text{col}_2 : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$M_{B'B''} = \begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

c) $x = \alpha_1 \cdot e_1' + \alpha_2 e_2' + \alpha_3 e_3'$

$$(2,-1,1) = \alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(1,0,-1)$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \boxed{\alpha_1 = -1} \quad \Rightarrow \quad * \\ \alpha_2 - \alpha_3 = 1 \end{cases}$$

$* \Rightarrow \alpha_2 + \alpha_3 = 3$

$$\begin{cases} \alpha_2 + \alpha_3 = 3 \\ \alpha_2 - \alpha_3 = 1 \quad (+) \end{cases}$$

$$2\alpha_2 = 4 \Rightarrow \boxed{\alpha_2 = 2}$$

$$2 - \alpha_3 = 1 \Rightarrow \boxed{\alpha_3 = 1}$$

Coord. lui x în B':

$x' = (\alpha_1, \alpha_2, \alpha_3)$

$\Rightarrow x' = (-1, 2, 1)$

$$X = \beta_1 e_1'' + \beta_2 e_2'' + \beta_3 e_3''$$

$$(2,1,1) = \beta_1(1,0,0) + \beta_2(1,1,0) + \beta_3(1,1,1)$$

$\beta_1 + \beta_2 + \beta_3 = 2 \Rightarrow \beta_1 = 3$

$\beta_2 + \beta_3 = -1 \Rightarrow \beta_2 = -2$

$\boxed{\beta_3 = 1}$

coord lui X în B''

$$X'' = \begin{array}{c} \beta_1 \quad \beta_2 \quad \beta_3 \\ (3, -2, 1) \end{array}$$

d) $e_1' = e_{11} e_1 + e_{21} e_2 + e_{31} e_3$

$$(1,1,0) = e_{11}(1,0,0) + e_{21}(0,1,0) + e_{31}(0,0,1)$$

$$\begin{aligned} e_{11} &= 1 \\ e_{21} &= 1 \\ e_{31} &= 0 \end{aligned} \quad \Rightarrow M_{BB'} \begin{array}{c} \begin{array}{ccc} e_1' & e_2' & e_3' \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{array}$$

Analog $M_{BB''} = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} e_1'' & e_2'' & e_3'' \end{array} \end{array}$

$$M_{BB''} = M_{BB'} \cdot M_{B'B''}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{2}+\frac{1}{2} \end{pmatrix} \quad A$$

Ex.: 2 $B' = \{ e_1' = (1, 0, 1),\ e_2' = (0, 1, -1),\ e_3' = (-1, 1, 0) \}$

Temă $B'' = \{ e_1'' = (1, -1, 0),\ e_2'' = (1, 2, 1),\ e_3'' = (0, 0, 1) \}$

$X = (1, 2, 3)$

Transformări liniare. Endomorfisme

Def.: 1) $T : V \to W$, $(V, +, \cdot_K)$, $(W, \oplus, \odot_K)$ sp. vec. peste K

T este transformare liniară $\Leftrightarrow$

$C_1 : T(v_1 + v_2) = T(v_1) \oplus T(v_2)$

$C_2 : T(\alpha v) = \alpha \odot T(v)$

2) Dacă $W = V$; $T : V \to V$ transf. liniară $\Rightarrow$

$\Rightarrow T =$ endomorfism

Ex. 1) Stabiliți care dintre următoarele aplicații sunt transformări liniare:

a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_1(x_1, x_2) = (x_1 + x_2, x_1 - 3x_2, 2x_1)$

Temă b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_2\ (x_2 - x_3, 2x_1 + x_2, x_2 - 1)$

c) $T_3 : \mathbb{R}^3 \to \mathbb{R}^2$, $T_3(x_1, x_2, x_3) = (x_1 x_2, 2x_3)$

Temă d) $T_4 : \mathbb{R}^3 \to \mathbb{R}^2$, $T_4(x_1, x_2, x_3) = (x_1 + x_3, 2x_2 - x_3)$

a) $C_1: T_1(v_1+v_2) = T_1(v_1) + T_1(v_2) \quad \forall v_1, v_2 \in \mathbb{R}^2 \Rightarrow$

$\Rightarrow v_1 = (x_1, x_2)$

$v_2 = (y_1, y_2)$

$T_1((x_1, x_2) + (y_1, y_2)) = T_1(x_1, x_2) + T_1(y_1, y_2)$

$T_1(x_1 + y_1, x_2 + y_2) = (x_1 + x_2, x_1 - 3x_2, 2x_1) +$
$+ (y_1 + y_2; y_1 - 3y_2, 2y_1)$

$(x_1 + y_1 + x_2 + y_2; x_1 + y_1 - 3(x_2 + y_2); 2(x_1 + y_1)) =$
$= (x_1 + x_2 + y_1 + y_2, x_1 + y_1 - 3x_2 - 3y_2, 2x_1 + 2y_1)$ ADV.

$C_2: T(\alpha v) = \alpha T(v) \quad \forall v \in \mathbb{R}^2 \Rightarrow v(x_1, x_2) \; \forall \alpha \in \mathbb{R}$

$T(\alpha(x_1; x_2)) = \alpha T(x_1; x_2) \Rightarrow T(\alpha x_1; \alpha x_2) =$

$= \alpha(x_1 + x_2; x_1 - 3x_2; 2x_1)$

$(\alpha x_1 + \alpha x_2; \alpha x - 3\alpha x_2; 2\alpha x_1) = (\alpha x_1 + \alpha x_2; \alpha x_1 - 3\alpha x_2,$
$, 2\alpha x_1) \to$ ADV.

$C_1; C_2 \Rightarrow T_1 \to$ trans. liniară

c) $T_3(v_1+v_2) = T_{(v_1)} + T_{(v_2)}$, $\forall v_1, v_2 \in \mathbb{R}^3$

$\Rightarrow v_1 = (x_1, x_2, x_3)$

$v_2 = (y_1, y_2, y_3)$

$T_3[(x_1, x_2, x_3) + (y_1, x_2, y_3)] = T_3(x_1, x_2, x_3) + T_3(y_1, y_2, y_3)$

$T_3(x_1+y_1, x_2+y_2, x_3+y_3) = (x_1x_2, 2x_3) + (x_1y_2, 2y_3)$

$((x_1+y_2)(x_2+y_2), 2(x_3+y_3)) = (x_1x_2 + y_1x_2; 2x_3 + 2y_3)$

$(x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2, 2x_3 + 2y_3) = (x_1x_2 + y_1y_2, 2x_3 + 2y_3)$

$\Rightarrow x_1y_2 + y_1x_2 = 0$, $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$ FALS $\Rightarrow$

$\Rightarrow T_3$ nu este trans. liniară
(aplicație)

Ex ② a) Arătați că aplicația $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x_1, x_2) = (3x_1 + x_2, x_1 - x_2)$ este endomorfism

b) Det. matricea lui T în baza canonică din $\mathbb{R}^2 \Rightarrow$

$B = \{e_1 = (1,0), e_2(0,1)\}$

$\hookrightarrow T(e_i) = \sum_{j=1}^{n} a_{ji} e_j$

c) Det. $\ker T$, $\operatorname{Im} T$, și verificați Th. dimensiunii
(teorema)

$\operatorname{Ker} T = \{ v \in V \mid T(v) = 0_W \} \to$ nucleul

$\operatorname{Ker} T = \{0_V\} \Leftrightarrow T$ - inj.

$\operatorname{Im} T = \{ w \in W \mid \exists\, v \in V \text{ a.î. } T(v) = w \}$

$\operatorname{Im} T = W \Leftrightarrow T$ - surjectiv

Th. dim: $\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$

Tema a) Verif. $c_1 + c_2 \Rightarrow$ endomorfism.

b) $T(e_1) = a_{11} e_1 + a_{21} e_2$

$T(e_2) = a_{12} e_1 + a_{22} e_2$

$\Rightarrow$

$T(1,0) = a_{11}(1,0) + a_{21}(0,1)$

$(3,1) = (a_{11}, 0) + (0, a_{21})$

$(3,1) = (a_{11}, a_{21}) \Rightarrow \boxed{\begin{matrix} a_{11} = 3 \\ a_{21} = 1 \end{matrix}}$

$T(e_2) = a_{21} e_1 + a_{22} e_2$

$T(0,1) = a_{12}(1,0) + a_{22}(0,1)$

$(1,-1) = (a_{12}, a_{22}) \Rightarrow \boxed{\begin{matrix} a_{12} = 1 \\ a_{22} = -1 \end{matrix}}$

c) $Ker\,T = \{v \in \mathbb{R}^2 \mid T(v) = 0_{\mathbb{R}^2}\}$

$v = (x_1, x_2) \in \mathbb{R}^2$ a.î.

$T(x_1, x_2) = (0, 0)$

$(3x_1 + x_2,\ 2x_1 - x_2) = (0, 0)$

$$\begin{cases} 3x_1 + x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}$$

$5x_1 = 0$

$x_1 = 0 \qquad x_2 = 0$

$Ker\,T = \{(0,0)\} \Rightarrow T$ inj.

$\dim Ker\,T = 0$

$Im\,T = \{w \in \mathbb{R}^2 \mid \exists\, v \in \mathbb{R}^2$ a.î. $T(v) = w\}$

Fie $w = (w_1, w_2) \in \mathbb{R}^2$

$v = (x_1, x_2) \in \mathbb{R}^2$

$T(x_1, x_2) = (w_1, w_2)$

$(3x_1 + x_2;\ 2x_1 - x_2) = (w_1, w_2)$

$$\begin{cases} 3x_1 + x_2 = w_1 \\ 2x_1 - x_2 = w_2 \quad (+) \end{cases}$$

$5x_1 = w_1 + w_2$

$x_1 = \dfrac{w_1 + w_2}{5}$

$2 \cdot \frac{w_1 + w_2}{5} - X_2 = w_2$

$\Rightarrow 2w_1 + 2w_2 - 5X_2 = 5w_2$

$2w_1 - 3w_2 = 5X_2$

$X_2 = \frac{2w_1 - 3w_2}{5}$

$\mathrm{Im}\,T = \{ (w_1, w_2) \mid \exists v = \left(\frac{w_1 + w_2}{5} ; \frac{2w_1 - 3w_2}{5} \right)$

a.î. $T(v) = w_2 \}$

$\mathrm{Im}\,T = \mathbb{R}^2 \Rightarrow \dim \mathrm{Im}\,T = \dim \mathbb{R}^2 = 2 \Rightarrow T$ surj.

$\dim \mathrm{Ker}\,T + \dim \mathrm{Im}\,T = \dim \mathbb{R}^2$

$0 + 2 = 2$

3. Fie endomorfismul $T : \mathbb{R}^3 \to \mathbb{R}^3$ definit prin matricea sa în baza canonică B:

$$M_B(T) = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Găsiți $T(1, 0, 1)$ și det. expr. lui $T(x, y, z)$

$$\boxed{T(v)^t = M_B(T)\, v^t}$$

$$T(1,0,1) = M_B(T)\cdot(1,0,1)^t = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$T(1,0,1) = (-1,0,1)$

$$T(x,y,z)^t = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 0 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y-3 \\ y \\ x \end{pmatrix}$$

$T(x,y,z) = (2y\cdot 3, y, x)$