31.10.2023

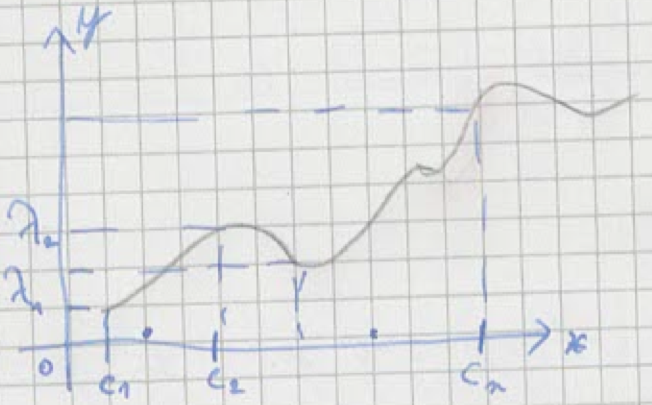
AnMate _ 8 _ S _ 2023 10 - 31

Proprietatea lui Darboux

$f: D \to \mathbb{R}$ şi $a, b \in D$ a.î. $a < b$. Spunem că f are proprietatea lui Darboux dacă $\forall$, ..

.., $\lambda \in [f(a), f(b)]$ a.î. $(\exists)$ $c \in (a, b)$ a.î.

$f(c) \in \lambda$



f - continuă $\Rightarrow$ f - are propr. lui Darboux

f - are proprietatea lui Darboux $\not\Rightarrow$ f - continuă

$f(I) = J$

Temă (5 pct): găsiţi o fct. care să aibă propr. lui Darboux, dar să != continuă

Ex.: Care din următoarele funcţii are propr. lui Darboux?

a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$x = 0$ $\cancel{\in}$ Im f

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{1}{x} = \infty$

Alegem $\lambda = 0 \Rightarrow (\exists)$

(Vezi poză)!

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{3^{x-1}-1}{x-1}, & x<1 \\ \ln(x+2), & x \geq 1 \end{cases}$

$ls(1) = ld(1) = f(1)$

$\underset{\substack{x \to 1 \\ x<1}}{ls} f(x) = \underset{x\to 1}{ls} \frac{3^{x-1}-1}{x-1} = \frac{3^0-1}{1-1} = \frac{1-1}{0} \overset{C.N}{=} \frac{0}{0} = \ln 3$

$\boxed{\lim_{x\to b} \frac{a^{f(x)}-1}{f(x)} = \ln a \text{ dacă } \lim_{x\to b} f(x) = 0}$

~~$\underset{\substack{x\to 1 \\ x>1}}{ld(1)} f(x) = ld(1)$~~

$\underset{\substack{x\to 1 \\ x>1}}{ld(1)} f(x) = \underset{x\to 1}{ld(1)} \ln(x+2) = \ln 1+2 = \ln 3$

$f(x) = \underbrace{\ln(x+2) = \ln 3}$ f continuă $\Rightarrow$ f are propr. lui Darboux

$ls = ld = f(x)$

$\ln 3 = \ln 3 = \ln 3$

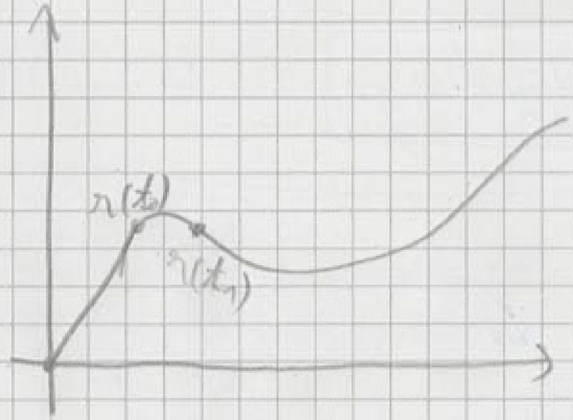
~~Funcții~~

Funcții derivabile

Def: Spunem că o fct. $f: \Delta \to \mathbb{R}$ este derivabilă în $x_0 \in \Delta$ dacă există și e finită limita:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Interpretare geometrică: $f'(x_0)$ = panta tangentei în x_0 la G_f.



1) $f(x) = k \Rightarrow f'(x) = 0$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{k - k}{x - x_0} = 0 \Rightarrow f'(x_0) = 0$$

2) $f(x) = x^n \Rightarrow f'(x) = n x^{n-1}$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} \stackrel{0/0}{=}$$

$$= \lim_{x \to x_0} \frac{(x - x_0)(x^{n-1} + x^{n-2} \cdot x_0 + \ldots + x \cdot x_0^{n-2} + x_0^{n-1})}{x - x_0} =$$

$$= \lim_{x \to x_0} x^{n-1} + \ldots\ldots$$

3) $f(x) = e^x \Rightarrow f'(x) = e^x \cdot \ln e$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} \stackrel{0/0}{=} \lim_{x \to x_0} \frac{e^{x_0}\left(\frac{e^x}{e^{x_0}} - 1\right)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{e^{x_0}\left(e^{x - x_0} - 1\right)}{x - x_0} = \lim_{x \to x_0} e^{x_0} \ln e = e^{x_0} \ln e$$

4) $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

$$f'(x_0) = \ln_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\ln x - \ln x_0}{x - x_0} \stackrel{0/0}{=}$$

$$= \ln_{x \to x_0} \frac{\ln \frac{x}{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{\ln\left(1 + \frac{x}{x_0} - 1\right)}{x - x_0} = \lim_{x \to x_0} \underbrace{\frac{\ln\left(1 + \frac{x - x_0}{x_0}\right)}{\frac{x - x_0}{x_0}}}_{\to 1} \cdot x_0 = \frac{1}{x_0}$$

5) $f(x) = \sin x \Rightarrow f'(x) = \cos x$

6) $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

7) $f(x) = \operatorname{arctg} x \Rightarrow f'(x) = \frac{1}{1+x^2}$

8) $f(x) = \arcsin x \Rightarrow f'(x) = \underline{\qquad\qquad}$

Operații cu derivate:

1. $(f \pm g)' = f' \pm g'$
2. $(f \cdot g)' = f' \cdot g + f \cdot g'$
3. $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

Ex: 1 Calculați derivatele următoarelor funcții:

a) $f(x) = tg\, x$;

b) $f(x) = 3 \cos^2 x^2$

a) $tg\,(x) = \frac{\sin x}{\cos x} \Rightarrow f'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cdot \cos x - \sin x \cdot \cos' x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

b) $3\cos^2 x^2 \Rightarrow f'(x) = (3\cos^2 x^2)' = 3(\cos^2 x^2)' =$

$= 3\cdot(\cos x^2 \cdot \cos x^2)' =$ ~~$3(\cos x^2 \cdot \cos x^2 ...)$~~

$= 6\cdot \cos' x^2 \cdot \cos x^2 = 6 - \sin x^2 \cdot 2x \cdot \cos x^2$

c) $f(x) = \sqrt[n]{x}$; d) $f(x) = \sqrt{\arcsin(x)^2}$

$f(x) = x^{\frac{1}{n}} \Rightarrow f'(x) = \frac{1}{n} x^{\frac{1}{n} - \frac{n}{1}} = \frac{1}{n} x^{\frac{1-n}{n}} =$

$= \frac{1}{n}(x^{n-1})^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{x^{n-1}}}$

d) $f(x) = \sqrt{\arcsin(x)^2} \Rightarrow$ ~~$f' =$~~

$\Rightarrow \sqrt{u(x)} \rightarrow f'(x) = \frac{1}{2\sqrt{u(x)}} \cdot u'(x) =$

$= \frac{1}{2\sqrt{\arcsin(x)^2}} \cdot (\arcsin(x)^2)' =$

$= \frac{1}{2\sqrt{\arcsin(x^2)}} \cdot \frac{1}{\sqrt{1-x^4}} \cdot 2x$

e) $f(x) = (x^x)' = (e^{x \cdot \ln x})' = (e^{u(x)})' =$

$= e^{u(x)} \cdot u'(x) = e^{x \cdot \ln x} \cdot (x \cdot \ln x)' =$

$= e^{x \cdot \ln x}(x' \cdot \ln x + x \cdot \ln' x) = e^{x \cdot \ln x}(\ln x + 1)$

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