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OILTHIGH OBAR DHEATHAIN

Ultrarelativistic Charged Particles

Author: **David Sragli**

Supervisor: Dr. Charles Wang

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Declaration

This report is entirely my own composition. It has not been accepted in any previous application for a degree. It is a record of my own work and all verbatim extracts have been distinguished by quotation marks. I have specifically acknowledged my sources of information in the bibliography.

Signed:

David Sragli

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List of Symbols

$\mathbf{u}_a$	Co-variant vector
$\mathbf{u}^a$	Contra-variant vector
$\mathbf{d}f$	Infinitesimal change
δx	Variation of x (small change)
$\frac{\partial}{\partial x}$	Partial derivative
δ_{ij}	Kronecker delta
ϵ_{ijk}	Alternating tensor
$\delta(x)$	Dirac-delta
$\dot{x}$	x derivative with respect to time
$\ddot{x}$	x second derivative with respect to time
$\mathcal{L}$	Lagrangian
$\mathcal{L}_d$	Lagrangian density
$\mathcal{T}$	Kinetic energy
$\mathcal{V}$	Potential energy
$\mathcal{H}$	Hamiltonian
$F(k)$	Fourier transform
$\Box$	d'Alembert operator
$g_{\mu\nu}$	Metric or Minkowski Metric - diag(-1,1,1,1) is used in this thesis
$a_k, a_k^\dagger$	Particle annihilation and creation operators
$b_k, b_k^\dagger$	Anti-Particle annihilation and creation operators
$\dagger$	Adjoint symbol - conjugate diagonal of a matrix
$[A, B]$	Commutation relationship
$\langle A\|B\|C\rangle$	Bra-Ket notation
∇	Nabla operator
$\nabla \times \mathbf{f}$	Curl of **f**
$\nabla \cdot \mathbf{f}$	Divergence of **f**
∇f	Gradient of f

List of Acronyms

UCP	Ultrarelativistic charged particle
GR	General Relativity
SR	Special Relativity
QFT	Quantum Field Theory
ED	Electrodynamics
EM	Electromagnetic
A.-L.	Abraham-Lorentz
E.-L.	Euler-Lagrange
QM	Quantum Mechanics
MS	Minkowski space
RS	Rindler space
EP	Equivalence principle

Abstract

Ultrarelativistic charged particles are an exciting area of physics as they could be the key to the discovery of new physics. The Unruh radiation, which is a result of the Quantum Field theory, could potentially be realised with the right experiment - the acceleration of such particles to ultrarelativistic levels. This dissertation provides a critical review of the relevant topics required to understand the Unruh effect and ultrarelativistic charged particle physics. Throughout each chapter, mathematical formulation is introduced, with details of how they were developed. Starting from a classical view, the dissertation goes towards the quantum view, and discusses the Unruh effect and radiation as a final step, with the addition of the current research and the importance of the Unruh radiation. These could be the key to studying black hole dynamics in the laboratory since they are related through Einstein's equivalence principle, which would lead to the potential discovery of new physics.

Chapter 1

Introduction

1.1 Aims

This thesis aims to provide a critical review of the physics of Ultrarelativistic Charged Particles (UCP) from 1915, the founding of General Relativity (GR) to present day, where Unruh radiation (cousin to Hawking radiation) could potentially be described by such concepts. This thesis ultimately tries to discuss how Unruh radiation is encoded into the Abraham-Lorentz radiation of an accelerating charged particle.

From Chapter 2 to Chapter 5 the relevant background theory is discussed, where the reader should learn what fields of physics are required to understand UCP. This would consist of General Relativity, Special Relativity, Lagrangian Mechanics, Electrodynamics and Quantum Field Theory. In each of the background chapters, a general introduction to these topics is provided, as well as an outline of the relevant physics and equations to the field of interest. In the later chapters, using the background theory provided, the physics is built up to understand concepts like the Unruh effect and consequently the Unruh radiation. These chapters hence discuss the Unruh effect, its consequences and current research on the subject. In these chapters an introduction and a derivation will be presented to the Unruh effect and then a discussion on the predictions. Finally, some current analyses of research papers are shown on how the Unruh radiation could be observed.

1.2 Methodology

This dissertation is a critical review of the relevant literature, therefore books, research papers, and articles were studied from trusted sources. The main focus was on books that covered topics about GR, SR, ED, Lagrangians and a mix of them all, as these were the main sources for Chapters 2-6. For Chapter 6, research papers and articles were also used as this area is currently being researched. Furthermore, in cases of talking about the biography of relevant scientists, mostly biographies or reports on biographies were used.

1.3 Brief description of Ultrarelativistic charged particles

Ultrarelativistic means that the Lorentz factor γ is extremely high, reaching at least a 1000 - hence while describing UCP, the conditions are extreme. In nature, it is unlikely to observe such a phenomenon. An ultrarelativistic charged particle can be related to a particle that is under a huge gravitational field through Einstein's equivalence principle. In theory, if the acceleration is high enough it could be described by the same physics as the black hole horizon. Unruh radiation is theorized to be identical to Hawking radiation and arises in the case when there is a huge acceleration present rather than a gravitational field. Current research argues that the Unruh radiation is embedded into the Abraham-Lorentz (A.-L.) radiation and if a charged particle were accelerated to ultrarelativistic speeds one would be able to measure the Unruh radiation, which would be part of the A.-L. radiation.

1.4 Relevancy of the project

The relevancy of the project is to outline the relevant mathematics and physics to describe an ultrarelativistic charged particle and to understand the Unruh effect and radiation as we currently know it. With the understanding of such Physics, it could be speculated how the A.-L. radiation could be used to observe the Unruh radiation, as it has not been observed yet. Unruh radiation is quite an important concept as it would be the laboratory equivalent of the Hawking radiation of a black hole, hence it would help physicists understand Hawking radiation better. Furthermore, it could test the validity of the equivalence principle as Unruh and Hawking radiation arise from different situations - one that is due to a high acceleration and the other is due to a large gravitational field, respectively. If black hole dynamics are better understood it could mean that we could get another hint towards Quantum Gravity, Dark Matter or even Dark Energy - all of which are still a puzzle in current times.

Chapter 2

General relativity

After publishing the Special Theory of Relativity in 1905, Albert Einstein started working on his theory of General Relativity. The key component of the theory is that gravity is no longer regarded as a force - instead, it is the curvature of spacetime locally caused by mass and energy. This view further revolutionized the world of Physics and replaced Newton's law of gravitation. Einstein worked for numerous years on general relativity as the mathematics of it is extremely complex and required the development of new mathematical tools - hence Einstein worked with David Hilbert to solve the issues of mathematical complexity. The final form of the equations was published in 1915 - gravity is now described by field equations, which was a truly exceptional achievement as it was another field theory after the first, Maxwell's Unified field theory. These field equations were first proved by observing a solar eclipse in 1919. However, due to general relativity being incompatible with Quantum Theory [1, 2] - because of the difficulty of quantising gravity - the popularity and growth of QM and the earlier mentioned mathematical complexity of GR made it unattractive to many scientists of the era. Hence, general relativity was not a widely researched area of Physics until the 1950s.

General relativity is built on Einstein's Theory of Special Relativity, as GR assumes that every point in the universe is locally Lorentzian, meaning that the laws of SR hold locally. Furthermore, GR positions (coordinates) no longer have physical meaning - they are described by geometrical concepts bringing space and time coordinates under the same hat. This way of thinking made Physics independent of coordinate systems as geometrical concepts would mean the same thing in any frame of reference (coordinate system).

When talking about Ultrarelativistic Charged Particles due to the large speeds, and large acceleration; gravity gets mocked up. Hence while describing the physics of UCP, one only needs to consider flat spacetime and can ignore the effects of gravity. As gravity and acceleration are related through the equivalence principle, this method could describe curved space-time without the need to consider gravity. This approximation is quite useful as this way the incompatibility of GR with QFT can be ignored as there is no need for gravity and this phenomenon can be described on a quantum level as well. However, flat spacetime requires special relativity and it often uses the mathematics of GR - hence this section is dedicated to introducing the relevant mathematics that will be used for the Physics of UCP. For a more in-depth exploration of the fundamentals of general relativity refer to [3].

2.1 A geometrical viewpoint

The first concept to introduce is a manifold - which is a collection of points in space that can be identified using coordinates. The dimension of the coordinates is the number of coordinates for a point, a change from one coordinate system to another is called a coordinate transformation - geometrical objects are preserved under transformation. As geometrical objects are preserved the mathematics of flat spacetime (as well as curved spacetime) introduces coordinate system independence and in this area of Physics one uses only these geometrical objects, rather than positional coordinates as in Newtonian Physics. A point is now called an "event", an arrow linking two events is called a "vector", the generalisation of the vector is the "tangent vector" and the machine that gives the squared length of any vector is called a "metric". All of these concepts are geometric, there is no requirement for the notion of coordinates. There are two types of vectors worth mentioning: co-variant and contra-variant vectors.

2.1.1 Co-variant and contra-variant vectors

If a vector is expressed in terms of its components, those components are co-variant if and only if they change inversely with the transformation that changes the basis (or transform with

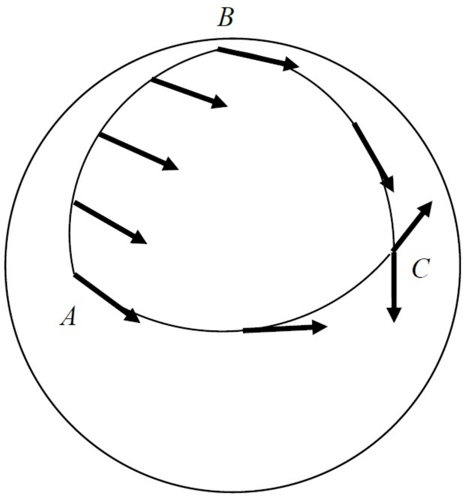

Figure 2.1: *Difference between going from A to C and B to C in curved spacetime. Curved spacetime is path-dependent, hence the need for introducing co-variant and contra-variant vectors. [4]*

the inverse Jacobian matrix). Furthermore, co-variant vectors are part of the co-tangent space, which represents the possible directional derivatives at this point. Mathematically a co-variant vector **u** can be represented by linearly combining the basis vectors u_a and infinitesimal change dx^a:

$$\mathbf{u} = u_a dx^a \tag{2.1}$$

as the component u_a represents the basis vector, to ensure that the co-variant vector changes the same way as the basis - the coordinate transformation of u_a can be written up as the following:

$$u'_a = \frac{\partial x^b}{\partial x'^a} u_b \tag{2.2}$$

If a vector is expressed in terms of its components, those components are contra-variant if and only if they change with the same transformation as the change of basis (or transform with the Jacobian matrix). Furthermore, contra-variant vectors are part of the tangent space, which represents the linear functions that map co-variant vectors to real numbers. Mathematically a contra-variant vector **U** can be represented by linearly combining the basis vectors U^a with $\frac{\partial}{\partial \mathbf{x}^a}$:

$$\mathbf{U} = U^a \frac{\partial}{\partial x^a} \tag{2.3}$$

as the component U^a represents the basis vector, to ensure that the contra-variant vector changes inversely as the basis does - the coordinate transformation for U^a can be written up as the following:

$$U'^a = \frac{\partial x'^a}{\partial x^b} U^b. \tag{2.4}$$

An important fact to note is a letter with an upper index represents a contra-variant component and a letter with a lower index represents a co-variant component.

2.1.2 Metric

The metric tensor is a machine (function) which can be used to calculate the squared length of a single vector or the scalar product of two different vectors. Hence the metric tensor can be defined as the following:

$$\mathbf{g}(\mathbf{a}\ ,\mathbf{b}) = \mathbf{u} \cdot \mathbf{v}. \tag{2.5}$$

The metric tensor has the following properties:

- Symmetric - $\mathbf{g}(\mathbf{a}, \mathbf{b}) = \mathbf{g}(\mathbf{b}, \mathbf{a})$
- Linear - $\mathbf{g}(\mathrm{n}\mathbf{a} + \mathrm{m}\mathbf{b}, \mathbf{c}) = \mathbf{g}(\mathbf{c}, \mathrm{n}\mathbf{a} + \mathrm{m}\mathbf{b}) = \mathrm{n}\mathbf{g}(\mathbf{a}, \mathbf{c}) + \mathrm{m}\mathbf{g}(\mathbf{b}, \mathbf{c})$
- A vector with itself gives back the squared length - $\mathbf{g}(\mathbf{a}, \mathbf{a}) = \mathbf{a}^2$

This machine proves extremely useful in general relativity and is crucial in special relativity as when looking at the Lorentz frame, the metric tensor gives information about whether directions and vectors are *spacelike* (positive squared length), *timelike* (negative squared length) or *lightlike* (null squared length).

Timelike separation means if one moves with a certain speed, one can see an event at A and after a certain time experience another event at B. Spacelike separation means that the two events are too far apart in space and there is no way of seeing them, as the observer watches event A then event B will already have happened by the time the observer arrives. Lightlike separation means that if the observer travels with the speed of light they can see the two events happening at the same time.

2.2 Basic Tensor algebra

The most general tensor **H** and its rank $\binom{n}{m}$ can be defined as a *linear* machine with n input slots for contra-variant vectors and m input slots for co-variant vectors that outputs a scalar **real** number.

$$\mathbf{H}(\mathbf{n}^0, \mathbf{n}^1, ..., \mathbf{n}^a, \mathbf{m}_0, \mathbf{m}_1, ..., \mathbf{m}_b) \tag{2.6}$$

Tensor algebra has many important properties which are essential for working with the geometrical view, even in special relativity, hence the following sections are dedicated to briefly summarising such properties. However, not all properties are labelled here, as flat spacetime does not require the description of curvature. For a more general introduction to tensor algebra refer to [7].

2.2.1 Tensor Types

Tensors can be classified based on their input and output. As a vector output can be turned into a scalar output using another tensor - there is no difference made here between them. However, based on their input it could be co-variant, contra-variant and mixed tensor.

Co-variant Tensor

Co-variant Tensor is such that their input is co-variant vectors and its output is a scalar real number. As co-variant vectors are labelled with lower indices, so are co-variant tensors. Hence a rank-a co-variant tensor is labelled as:

$$\mathbf{T}_{p_1,p_2..,p_a} \tag{2.7}$$

and its transformation from one coordinate system to another is

$$\mathbf{T'}_{q_1,q_2..,q_a} = \frac{\partial^{p_1}}{\partial'^{q_1}}...\frac{\partial^{p_a}}{\partial'^{q_a}}\mathbf{T}_{p_1,p_2..,p_a}. \tag{2.8}$$

Contra-variant Tensor

Contra-variant Tensor is such that their input is contra-variant vectors and its output is a scalar real number. As contra-variant vectors are labelled with upper indices, so are contra-variant tensors. Hence a rank-b contra-variant tensor is labelled as:

$$\mathbf{T}^{p_1,p_2..,p_b} \tag{2.9}$$

and its transformation from one coordinate system to another is

$$\mathbf{T'}^{q_1,q_2..,q_b} = \frac{\partial'^{q_1}}{\partial^{p_1}} \cdots \frac{\partial'^{q_b}}{\partial^{p_b}} \mathbf{T}^{p_1,p_2..,p_b}. \tag{2.10}$$

Mixed Tensor

Finally, a mixed tensor is a combination of co-variant and contra-variant tensors. Hence a rank $\binom{a}{b}$ tensor can be labelled as:

$$\mathbf{T}^{p_1,p_2..,p_a}_{q_1,q_2..,q_b} \tag{2.11}$$

and its transformation from one coordinate system to another is

$$\mathbf{T'}^{k_1,k_2..,k_a}_{l_1,l_2..,l_b} = \frac{\partial'^{k_1}}{\partial^{p_1}} \cdots \frac{\partial'^{k_a}}{\partial^{p_a}} \frac{\partial^{q_1}}{\partial'^{l_1}} \cdots \frac{\partial^{q_b}}{\partial'^{l_b}} \mathbf{T}^{p_1,p_2..,p_a}_{q_1,q_2..,q_b}. \tag{2.12}$$

2.2.2 Tensor Addition, Subtraction, Product and Contraction

New tensors can be obtained with $+, -, \otimes$, and with a process called contraction that provides a lower-rank tensor by removing a repeated index. Let **T** and **U** be tensors with rank $\binom{a}{b}$ and cd and f a constant. The operations look the following way:

- Addition and Subtraction - $\mathbf{S}^{p_1 \pm q_1, p_2 \pm q_2 .. p_a \pm q_c}_{p_1 \pm q_1, p_2 \pm q_2 .. p_b \pm q_d} = \mathbf{T}^{p_1,p_2..p_a}_{p_1,p_2..p_b} \pm \mathbf{U}^{q_1,q_2..,q_c}_{q_1,q_2..q_d}$
- Scalar multiplication - $\mathbf{S}^{p_1,p_2..p_a}_{p_1,p_2..p_b} = f\mathbf{T}^{p_1,p_2..p_a}_{p_1,p_2..p_b}$
- Product - $\mathbf{S} = \mathbf{T} \otimes \mathbf{U} = \mathbf{S}^{g_1,g_2..g_{a+c}}_{h_1,h_2,h_{c+d}} = \mathbf{T}^{p_1,p_2..p_a}_{p_1,p_2..p_b} \otimes \mathbf{U}^{q_1,q_2..,q_c}_{q_1,q_2..q_d}$
- Contraction - $\mathbf{S}^{p_1,p_3..p_a}_{p_1,p_3..p_b} = \mathbf{T}^{p_1,p_2..p_a}_{p_1,p_2..p_b}$

Hence addition and subtraction would produce a new tensor with the same rank, but different components and the operation is only valid if and only if a=c and b=d. Scalar multiplication just multiplies each component with a number, producing a same rank tensor. The tensor product makes a new tensor with $\binom{a+c}{c+d}$ rank and the operation is only valid if and only if b=c. Lastly, contraction produces a tensor with rank $\binom{a-1}{b-1}$, removing the repeated index if there is one.

2.2.3 p-forms (co-variant vector constructs)

p-forms are essentially co-variant vector constructs and serve as an essential tool in the description of fields, especially electromagnetic fields. A 1-form is a covariant vector, however, a 2-form would be a construct of two or more 1-forms - an example would be the Electromagnetic tensor, which is discussed in Chapter 4. As p-forms are covariant vector constructs, it makes sense to define the norm, which is as follows [3]:

$$||\mathbf{x}|| = x_{|i_1,i_2...,i_p|}x^{i_1,i_2,...,i_p} \tag{2.13}$$

The norm is used for the calculation of a squared length or electrodynamics and quantum field theory - all of which are discussed in later chapters. p-forms are technically geometrical constructs that entail a chosen orientation of space, visually speaking they look as follows [3]:

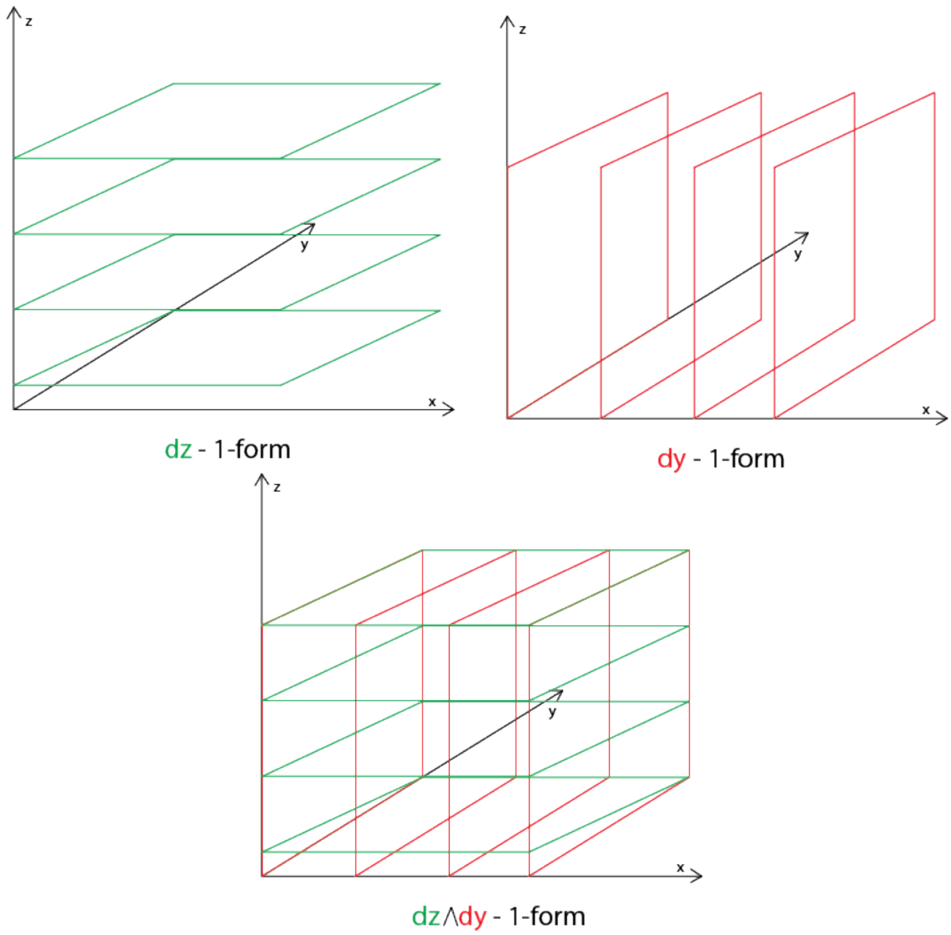

Figure 2.2: *Construction of a 2-form from two 1-forms. The 2-form* $\mathbf{d}y \wedge \mathbf{d}y$ *is covering the space constructed by y and z. It is an extremely useful concept for describing fields (see Chapters 4 and 5) [4]*

2.2.4 Symmetric and Antisymmetric Tensors

A tensor is symmetric if the following condition holds:

$$\mathbf{U}^{ab} = \mathbf{U}^{ba} \tag{2.14}$$

and antisymmetric if:

$$\mathbf{U}_{ab} = -\mathbf{U}_{ba} \tag{2.15}$$

holds. The previous properties are important as a symmetric tensor is associated with conservation laws (e.g. stress-energy tensor needs to be symmetric) and the electromagnetic field tensor is antisymmetric, a property that is used with UCP. It is a useful tool to know how to symmetrize and anti-symmetrize a tensor. One can achieve it by the following relations:

$$\mathbf{G}_{ab} = \frac{1}{2}(\mathbf{F}_{ab} + \mathbf{F}_{ba}) \tag{2.16}$$

$$\mathbf{G}_{ab} = \frac{1}{2}(\mathbf{F}_{ab} - \mathbf{F}_{ba}) \tag{2.17}$$

where (2.16) refers to the process of making **F** tensor symmetric by producing a new tensor **G**. Similarly (2.17) refers to the process of translating **F** tensor antisymmetric by constructing a new **G** tensor.

2.2.5 Kronecker delta and the Alternating Tensor

Kronecker delta and the Alternating tensors are important techniques when working with tensors. Kronecker delta helps filter the repeating indices, making the tensor calculations simpler - while the Alternating tensor helps in writing vector products more simply. The Kronecker delta is defined as:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.18}$$

and the Alternating tensor is defined as:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if i, j, k is a cyclic permutation of (1, 2, 3)} \\ -1, & \text{if i, j, k is an anti-cyclic permutation of (1, 2, 3)} \\ 0, & \text{if any of i, j, k are equal.} \end{cases} \tag{2.19}$$

2.3 Einstein's equation of Gravity

Einstein's equation of gravity treats gravity as a curvature of spacetime and uses the mathematical tool of *Riemann tensor* to describe it. It was found that energy-matter tells spacetime how to curve and in exchange spacetime tells energy-matter how to move. The relation was found to be the following:

$$\mathbf{G}_{\mu\nu} = \frac{8\pi G}{c^4}\mathbf{T}_{\mu\nu}, \tag{2.20}$$

where $\mathbf{G}_{\mu\nu}$ is the Einstein tensor, which is a special Riemann curvature tensor, G is the gravitational constant, c is the speed of light and $\mathbf{T}_{\mu\nu}$ is the stress-energy tensor. The stress-energy is used to describe energy and matter at a specific point, it contains conservation laws, Newton's law of motion and many more inside of it; all of which UCP Physics requires. A more general formula of Einstein's equation would be the following:

$$\mathbf{G}_{\mu\nu} + \Lambda\mathbf{g}_{\mu\nu} = \frac{8\pi G}{c^4}\mathbf{T}_{\mu\nu}, \tag{2.21}$$

where Λ is the cosmological constant that would only be relevant if the perspective of is the entirety of the Universe (in Cosmology), for other calculations, it can be ignored. It is important to note that GR usually talks in the language of geometry, which means that all the units are geometrized. In that perspective c^4 would be identical to G, only having 8π as a constant next to the stress-energy tensor.

In this critical review curved spacetime is not looked at more in-depth, as was mentioned before, since gravity can be ignored as the ultrarelativistic speeds *mock* it up, giving us *flat* spacetime. As in flat spacetime, the Einstein tensor vanishes - UCP would only require the stress-energy tensor combined with SR. If one's interested in the Mathematics of curved spacetime and the applications of general relativity in a more general sense then refer to [3, 4, 6]. Another important concept to mention is geometrized units; it is where the Boltzmann constant, the speed of light and the gravitational constant are taken to be unity ($k_B = c = G = 1$). Throughout this thesis, the geometrized units are used unless stated otherwise.

Chapter 3

Analytical Mechanics

Analytical Mechanics, or Lagrangian Mechanics, was formulated by J.L. Lagrange in the 18th century [7]. J. L. Lagrange was born in Turin, Sardinia - his father was of service to the King of Sardinia, however, his family was never wealthy. Lagrange's father wanted him to pursue a career in law, hence he never learnt from the masters of the fields of Maths, or Physics. After reading a copy of the work of Halley on optics using algebra, he was inspired to make a career in Mathematics. He was mostly self-taught, as he was on the path to becoming a lawyer first, however, this did not stop him from making a name for himself at an early stage. By the age of 19, he obtained important results on the tautochrone, which is a curve on which a body moves under gravity and will arrive at the lowest point at the same time regardless of the initial position. He sent these results to L. Euler, who was the director of Mathematics in Berlin at the time. Euler was impressed with the works and from that point, they worked together on numerous problems. In contrast to this critical review, the most important to mention is the *Euler - Lagrange Equation* (E.-L. equation), which is considered to be the pure equations of motion.

Lagrangian Mechanics is a universal method to extract the *pure equations of motion*, which are differential equations without any constraint forces (unknowns of the problem). Using these equations one can determine the system's motion - it can be applied on point particles, continuous rigid systems or on a system of point particles. In the case of a system of point particles, the E.-L. equation is equivalent to Newton's equation, however, they possess an important feature as they have the same form in every coordinate system. This independence of the coordinate system points out the underlying symmetry for a certain problem in different coordinate systems. This method is an incredibly successful application, which has survived the relativistic revolution and is one of the most important mathematics in Modern Physics, as using the E-L equation one can extract field equations, which is the language of GR or QFT.

3.1 Classical Lagrangian Mechanics

3.1.1 Brief derivation of the Euler-Lagrange Equation

The E-L equations can be derived from Newton's equations [7]. If we consider a system with N point particles, each particle (i) has a force $\mathbf{F}_i$ acting on it and due to the other particles reactive forces $\mathbf{R}_i$ acting on it. Hence, Newton's equation can be written up as:

$$m_i * \mathbf{a}_i = \mathbf{F}_i + \mathbf{R}_i, \tag{3.1}$$

which can be rewritten in the following way:

$$\frac{d^2\mathbf{x}_i}{dt^2} - \frac{1}{m_i}\mathbf{F}_i(t, \mathbf{x}_1(t), \mathbf{x}_2(t).., \mathbf{x}_n(t), \dot{x}_1(t), \dot{x}_2(t).., \dot{x}_n(t)) = \frac{1}{m_i}\mathbf{R}_i. \tag{3.2}$$

The reaction (constraint) forces are assumed to be ideal, in which case the tangent vector of the path at some time t should hold the following relation with the reaction forces:

$$\mathbf{R}_i * \delta\mathbf{x}_i = 0, \tag{3.3}$$

or for a system with N particles:

$$\sum_{i=1}^{n} \mathbf{R}_i \delta\mathbf{x}_i = 0. \tag{3.4}$$

$\delta\mathbf{x}_i$ can be rewritten in a different manner using the other tangent vectors:

$$\delta\mathbf{x}_i = \sum_{k=1}^{n} \frac{\partial\mathbf{x}_i}{\partial q^k}\delta\mathbf{q}_i = 0, for\ k = 1, 2, .., n. \tag{3.5}$$

Consequently putting (3.4) and (3.5) together a more subtle expression of how constraints behave can be obtained:

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\mathbf{R}_i\frac{\partial \mathbf{x}_i}{\partial q^k} * \delta\mathbf{q}_i = 0. \tag{3.6}$$

As $\delta\mathbf{q}_i$ is arbitrary (one can choose only q^1 to describe x_1, hence other sum parts would be 0 and so on for the others) it is possible to choose q variables in such a way that (3.6) simplifies to:

$$\sum_{k=1}^{n}\mathbf{R}_i\frac{\partial \mathbf{x}_i}{\partial q^k} = 0. \tag{3.7}$$

Now using (3.2) and (3.7) gives the following set of equations:

$$\sum_{k=1}^{n} m_i\mathbf{a}_i\frac{\partial \mathbf{x}_i}{\partial q^k} - \sum_{k=1}^{n}\mathbf{F}_i\frac{\partial \mathbf{x}_i}{\partial q^k} = 0 \tag{3.8}$$

However, the constraint forces arise from different potentials, which means F_i can be expressed as (however the direction is opposite, hence the negative sign arises):

$$\mathbf{F}_i(t, \mathbf{x}_1, \mathbf{x}_2.., \mathbf{x}_n) = -\nabla_{\mathbf{x}_i}\mathcal{V}(t, \mathbf{x}_2.., \mathbf{x}_n) \tag{3.9}$$

obtaining the expression:

$$\sum_{k=1}^{n}\mathbf{F}_i\frac{\partial \mathbf{x}_i}{\partial q^k} = \sum_{k=1}^{n} -\nabla_{\mathbf{x}_i}(\mathcal{V})\frac{\partial \mathbf{x}_i}{\partial q^k} = -\frac{\partial \mathcal{V}}{\partial q^k}. \tag{3.10}$$

Similarly, acceleration can be expressed using the kinetic energy as follows:

$$\sum_{k=1}^{n} m_i\mathbf{a}_i\frac{\partial \mathbf{x}_i}{\partial q^k} = \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}^k} - \frac{\partial \mathcal{T}}{\partial q^k}, \tag{3.11}$$

where a rather rigorous relation was used, which is proved in section 8.1. Consequently, the Lagrangian $\mathcal{L}$ can be defined as $\mathcal{L} = \mathcal{T} - \mathcal{V}$, as $\dot{q}^k$ does not depend on $\mathcal{V}$, it can be included as a dummy constant, and hence using (3.11), (3.10) and (3.8) the E.-L. equation can be found:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^k} - \frac{\partial \mathcal{L}}{\partial q^k} = 0. \tag{3.12}$$

Another way of deriving the E.-L. equation would be by using the principle of least action. Action (S) is defined to be a path between two points and is related to the Lagrangian ($\mathcal{L}$).

$$S = \int_{t_0}^{t_1} \mathcal{L}\, dt \tag{3.13}$$

As in nature, everything seems to go on the shortest past possible, or the most energy-efficient (least energy-consuming) path possible, by minimizing action (S) the E.-L. equations can also be found. However, the derivation from Newton's equations is more useful as it shows how the constraint forces are not present in the E.-L. equation, giving a coordinate-independent equation of motion. For this derivation refer to [8].

3.1.2 Using the Euler-Lagrange Equation

Consider a system with two masses, M_1 and M_2, that are connected with an inextensible string through a pulley hanging from the ceiling. M_1 is hanging x distance away from the ceiling and M_2 hanging $(l - x)$ distance away.

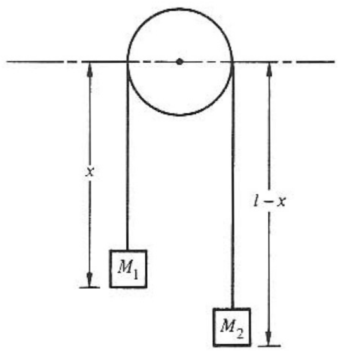

Figure 3.1: *Exemplary question proposed from [8]*

The potential energy and the kinetic energy, consequently the Lagrangian of the system is defined as the following:

$$\mathcal{V} = -M_1 g q - M_2 g(l - q) \tag{3.14}$$

$$\mathcal{T} = \frac{1}{2}(M_1 + M_2)\dot{q}^2 \tag{3.15}$$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{1}{2}(M_1 + M_2)\dot{q}^2 + M_1 g q + M_2 g(l - q), \tag{3.16}$$

where g is the gravitational field strength, q is the position, l is the length of the string and $\dot{q}$ is the velocity.

The partial derivatives of the Lagrangian with respect to q and $\dot{q}$ (velocity) are the following, as we are in 1 dimension only one E-L. equation is needed:

$$\frac{\partial \mathcal{L}}{\partial q} = (M_1 - M_2)g \tag{3.17}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = (M_1 - M_2)\dot{q} \tag{3.18}$$

Then using (3.12) the following is obtained:

$$(M_1 - M_2)\ddot{q} = (M_1 - M_2)g \tag{3.19}$$

It is obvious that instead of using the E.-L. equation, the problem could have been solved using Newton's equations. However, using Newton's equations the constraint forces would be present - here it would have been tension acting on the string. Using the E.-L. equation is not optimal in cases like this however when the problem is a lot more complicated, e.g. the constraint forces are not known and there are many of them present, then it is the best way to solve for the motion of the system. Hence, this exemplary problem shows why the E.-L. equation is such a power approach - no need for the consideration of the constraint forces.

3.1.3 Producing field equations using Lagrangian density

In modern Physics, especially when it is related to GR or ED, one is interested in the field equations of the system - as they describe the system in a way that is independent of the coordinate system. In other words, the point of interest is finding a set of equations that is **invariant** under transformations which include the Lorentz, Poincaré or Bogoliubov; all of which are areas of interest for UCP.

Lagrangian density ($\mathcal{L}_d$) is a function that can be used to determine the $\mathcal{L}$ of the system. The relation is the following:

$$\mathcal{L} = \int \mathcal{L}_d d\mathbf{x}, \tag{3.20}$$

where x could be a 1-,2-, 3- or even 4-D coordinate system - which is the case when considering problems in special relativity. A more in-depth introduction to the Lagrangian density can be found in [8].

Using the principle of least action the E.-L- equations can be derived for the Lagrangian density as well giving the equations of the following form:

$$\frac{d}{dx^{\nu}}\frac{\partial \mathcal{L}_d}{\partial \eta_{\rho,\nu}} - \frac{\partial \mathcal{L}_d}{\partial \eta_{\rho}} = 0, \tag{3.21}$$

where η refers to the field function, ρ refers to the number of different equations for the field and ν refers to the dimension - for spacetime ν would have (x, y, z, t) components. Hence for example if there is only one equation describing the field then ρ factor can be omitted and the equations would simplify down to:

$$\frac{d}{dx}\frac{\partial \mathcal{L}_d}{\frac{\partial \eta}{\partial x}} + \frac{d}{dy}\frac{\partial \mathcal{L}_d}{\frac{\partial \eta}{\partial y}} + \frac{d}{dz}\frac{\partial \mathcal{L}_d}{\frac{\partial \eta}{\partial z}} + \frac{d}{dt}\frac{\partial \mathcal{L}_d}{\frac{\partial \eta}{\partial t}} - \frac{\partial \mathcal{L}_d}{\partial \eta} = 0. \tag{3.22}$$

The derivation of the E.-L. equation for $\mathcal{L}_d$ is shown in Chapter 5 or for further reading refer to [7, 8]. Using the $\mathcal{L}_d$ and the E.-L. equation the field equations can be found - these field equations would then govern the motion of the system. Maxwell equations, and more importantly Quantum field equations describing the ultrarelativistic motion of an electron, can be found with the right $\mathcal{L}_d$.

3.2 Group Theory

Group theory is one of the most crucial elements of Physics. It outlines the importance of symmetry and choosing the right coordinate representation for a certain problem. Problems in classical physics can be solved in any coordinate system, however, some coordinate system representation makes the problem easier than others. As we enter the world of Quantum Physics or Relativity problems become increasingly difficult and finding the right coordinate system representation is **crucial** to being able to solve the problem. A **group** is defined to be a set of transformations that leave the problem invariant while transforming it from one coordinate sys-

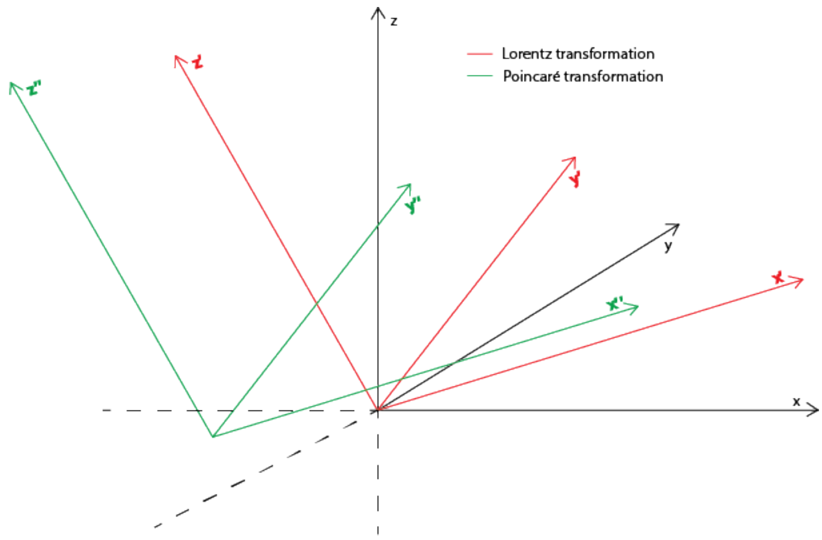

Figure 3.2: *Lorentz and Poincaré transformations*

tem to another. For example [3], if we take a plastic bottle and try to represent it in a cartesian coordinate system, there would be no issue with it as the bottle can be easily modelled in a 3-D space. However, if we assume that this bottle is squashed down to a volume of 0, then it would be impossible to represent its dynamics in a simple cartesian coordinate system - singularity would be reached. In this case, using a *group*, it would need to be transformed to another coordinate system that would have no singularity and would leave the problem *invariant*, meaning it would not change the base problem.

Particular groups of interest in this critical review are Lorentz, Poincaré [7] and Bogoliubov group [10]. In classical Physics, to transform one coordinate system to another, Galilean transformation (GT) is used, however, in Special relativity time is no longer absolute and the GT would not be suitable. Hence, to solve this problem the Lorentz group, or Lorentz transformation is used. Lorentz transformation could be looked at as a boost (scaling) and a rotation, which means if a problem involves a change of observer, or in other words a *translation*, a more general transformation called the **Poincaré** transformation need to be used. Poincaré transformation does solve the problem of translation, however, it still has a problem as it can not be used to transform between inertial and non-inertial frames. To fully generalise the transformations, the **Bogoliubov** transformation is used. The Lorentz and Poincaré are explored in this section as it is connected to relativistic lagrangian mechanics and the most general, Bogoliubov transformation, is explored in a later chapter.

3.2.1 Poincaré and Lorentz Group

Given a transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ with the following form[7]:

$$x'^{\mu} = a^{\mu} + \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\mu}, \mu = 0, 1, 2, 3, \tag{3.23}$$

where c^{μ} is a coordinate in $\mathbb{R}^4$, Λ^{μ}_{ν} is a set of maps that are part of the Minkowski space, x'^{μ} is the set of coordinates in the new frame and x^{μ} are the original coordinates. If $\Lambda^t \mathbf{g} \Lambda = \mathbf{g}$ satisfied, which means the metric tensor $\mathbf{g}$ (as it describes the problem) is invariant under a series of transformations described by Λ, then Equation 3.23 is called a Poincaré Group. If a^{μ} is not included then it is called a Lorentz group.

The Poincaré group is labelled as IO(1,3) and the Lorentz group is called O(1,3). After a multitude of rigorous mathematical processes (see [7] Ch. 10.4), the important subgroups of these groups can be obtained, which are used in the context of special relativity. These transformations are the following:

$$\textit{Temporal displacements:} \begin{cases} x'^{0}_{\nu} = x^0 + \nu \\ x'^{\mu}_{\nu} = x^{\mu}, \mu = 1, 2, 3 \end{cases} \quad \forall \nu \in \mathbb{R} \tag{3.24}$$

$$\textit{Spatial Displacements:} \begin{cases} \textit{along base vector } \mathbf{e}_1 \begin{cases} x'^{1}_{\nu} = x^1 + \nu \\ x'^{\mu}_{\nu} = x^{\mu}, \mu = 0, 2, 3 \end{cases} \\ \textit{along base vector } \mathbf{e}_2 \begin{cases} x'^{2}_{\nu} = x^2 + \nu \\ x'^{\mu}_{\nu} = x^{\mu}, \mu = 0, 1, 3 \end{cases} \\ \textit{along base vector } \mathbf{e}_3 \begin{cases} x'^{3}_{\nu} = x^3 + \nu \\ x'^{\mu}_{\nu} = x^{\mu}, \mu = 0, 1, 2 \end{cases} \end{cases} \quad \forall \nu \in \mathbb{R} \tag{3.25}$$

$$\text{Spatial Rotations: }\begin{cases}\text{around base vector } \mathbf{e}_1 \begin{cases} x'^2_\nu = x^2\cos(\nu) - x^3\sin(\nu) \\ x'^3_\nu = x^2\sin(\nu) + x^3\cos(\nu) \\ x'^\mu_\nu = x^\mu, \mu = 0, 1 \end{cases} \\ \text{around base vector } \mathbf{e}_2 \begin{cases} x'^1_\nu = x^1\cos(\nu) - x^3\sin(\nu) \\ x'^3_\nu = x^1\sin(\nu) + x^3\cos(\nu) \\ x'^\mu_\nu = x^\mu, \mu = 0, 2 \end{cases} \\ \text{around base vector } \mathbf{e}_3 \begin{cases} x'^1_\nu = x^1\cos(\nu) - x^2\sin(\nu) \\ x'^2_\nu = x^1\sin(\nu) + x^2\cos(\nu) \\ x'^\mu_\nu = x^\mu, \mu = 0, 3 \end{cases}\end{cases} \quad \forall \nu \in \mathbb{R} \tag{3.26}$$

$$\text{Lorentz Boosts: }\begin{cases}\text{along base vector } \mathbf{e}_1 \begin{cases} x'^0_\nu = x^0\cosh(\nu) + x^1\sinh(\nu) \\ x'^1_\nu = x^0\sinh(\nu) + x^1\cosh(\nu) \\ x'^\mu_\nu = x^\mu, \mu = 2, 3 \end{cases} \\ \text{along base vector } \mathbf{e}_2 \begin{cases} x'^0_\nu = x^0\cosh(\nu) + x^2\sinh(\nu) \\ x'^2_\nu = x^0\sinh(\nu) + x^2\cosh(\nu) \\ x'^\mu_\nu = x^\mu, \mu = 1, 3 \end{cases} \\ \text{along base vector } \mathbf{e}_3 \begin{cases} x'^0_\nu = x^0\cosh(\nu) + x^3\sinh(\nu) \\ x'^3_\nu = x^0\sinh(\nu) + x^3\cosh(\nu) \\ x'^\mu_\nu = x^\mu, \mu = 1, 2 \end{cases}\end{cases} \quad \forall \nu \in \mathbb{R} \tag{3.27}$$

If only (3.26) and (3.27) are applied during a coordinate transformation then it would be called a Lorentz transformation, if all of the above is applied then it would be called a Poincaré transformation.

3.3 Relativistic Lagrangian Mechanics

In relativistic mechanics the path of γ(t) is now called a world line and the time is no longer absolute, hence t becomes τ (proper time). Momentum is defined as P^μ and is defined by a four-velocity vector of spacetime called V_ν. The momentum conservation is still valid and at rest, it would be the following:

$$\sum_{i=1,2,..N} m_i V_i(\tau) = \sum_{j=1,2,..M} m'_j V'_j(\tau), \tag{3.28}$$

where N would represent N *histories* of point particles and after collision in an event it creates M *histories* of point particles.

Furthermore, there are some other results worthy of noting. If we take $\mathbf{g}(P, P)$ (where g is the metric tensor of the Minkowski frame) then it would yield the following:

$$\mathbf{g}(P, P) = -m^2c^2 \tag{3.29}$$

and if we would expand the metric tensor and assume that P^0 is positive, as in it is future-oriented:

$$P^0 = \sqrt{\sum_{\mu=1}^{3}(P^\mu)^2 + m^2c^2}. \tag{3.30}$$

If the velocity of the particle(s) approaches the c limit, that is when special relativity comes in. The momentum conservation becomes:

$$\sum_{i=1,2,..N} \frac{m_i v_i^\mu}{\sqrt{1-\frac{v_i^2}{c^2}}} = \sum_{j=1,2,..M} \frac{m'_j v'^\mu_j}{\sqrt{1-\frac{v_j^2}{c^2}}}, \tag{3.31}$$

similarly using $\mathbf{g}(P, \mathbb{R})$, P^0 can be obtained, furthermore if multiplied by c it gives back the *total mechanical energy*:

$$cP^0 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{3.32}$$

Using the total energy, as it is the sum of the energy due to the rest mass (using Einstein's

$E = mc^2$) and the kinetic energy, an expression for $\mathcal{T}$ can be obtained:

$$\mathcal{T} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2, \tag{3.33}$$

where $\mathcal{T}$ is called the **relativistic kinetic energy**. Using Taylor expansion around $\frac{v^2}{c^2}$ and assuming that v is much smaller than c, then it gives back the classical kinetic energy formula:

$$\mathcal{T} = \frac{1}{2}mv^2 + O(v^4/c^4). \tag{3.34}$$

Furthermore, if one wants to calculate the motion of a relativistic particle then the Impulse, Force, and Time relations are used, which look as follows:

$$\frac{dP}{d\tau} = F(\gamma(\tau), V(\tau)), \tag{3.35}$$

$$m\frac{d\dot{\gamma}}{ds} = F(\gamma(s), \dot{\gamma}). \tag{3.36}$$

Using further derivations (for full proofs refer to [7]), the relativistic kinetic energy theorem can be obtained from (3.36), which looks as follows:

$$\frac{d}{d\tau}\mathcal{T} = F.\mathbf{v}(\gamma). \tag{3.37}$$

After finding the $\mathcal{L}$ and for continuous systems (such as a field) finding $\mathcal{L}_d$ the equation of motion can be calculated using the above relativistic definitions after solving the E.-L. equation(s). It is clear that (3.35) and (3.36) could be used to find the relativistic motion, however, in many cases solving those equations is a lot more complex than the E.-L. equations.

One example of Lagrangian for a point-charged particle moving in an external electromagnetic field would be:

$$\mathcal{L} = \frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{v}.\mathbf{A}(\mathbf{x}, t) - q\Phi(\mathbf{r}, t), \tag{3.38}$$

where m is mass, v is the velocity, q is the charge, c is the speed of light, A is the electromagnetic potential and Φ is the electromagnetic field. For a relativistic charged particle it would be (without co-variant vectors):

$$\mathcal{L}(\mathbf{r}, \mathbf{v}, t) = -\frac{mc^2}{\gamma(\mathbf{v})} - q\Phi(\mathbf{r}, t) - q\mathbf{v}.\mathbf{A}(\mathbf{r}, t). \tag{3.39}$$

The relativistic Lagrangian is analyzed in the next chapter, after introducing relevant concepts in Electrodynamics, such as the Electromagnetic field tensor.

Chapter 4

Classical Theory of Electrodynamics

The main figure behind the first-ever field theory was James Clerk Maxwell. He described electricity and magnetism as one field and developed a geometrical theory, which is now used in modern physics. Maxwell was born in 1831 in Edinburgh and also attended the University of Edinburgh as an undergraduate from 1847-1850 [11]. Later he attended the University of Cambridge 1850-56, where he studied Maths, which was of great help to him when developing his equations of electromagnetism. After his studies finished as an undergraduate, he was a professor in Aberdeen and later in London, where he developed the so-called *Maxwell equations*. He published his equations in 1865, which connected electricity and magnetism, as in that time there was no discovered link between the two. This unification planted the seeds for the modern world, gave pathways to revolutionary ideas such as the internet and currently plays a crucial role in understanding our world using Physics. Further information on Maxwell's life can be found in [11].

In this chapter, the main focus is on introducing the Electromagnetic tensor as it is used when describing UCP, an introduction to the Abraham-Lorentz radiation and self-force, and finally, some calculations using previously described theory on the motion of a relativistic charged particle that is moving in an external electromagnetic field[12].

4.1 Electromagnetic equations

4.1.1 Maxwell Equations

Originally, Maxwell released 20 differential equations, however, those could be reduced to 4 equations, giving the following [12]:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \text{ ,}\textit{(Gauss' Law)} \tag{4.1}$$

$$\nabla \cdot \mathbf{B} = 0 \text{ ,}\textit{(No special name)} \tag{4.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ ,}\textit{(Faraday's Law)} \tag{4.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \text{ ,}\textit{(Ampere's law with Maxwell correction),} \tag{4.4}$$

where **B** is the magnetic field vector, **E** is the electric field vector, ρ is the charge density, **J** is the current density and μ_0, ϵ_0 are vacuum permeability and permittivity. This set of 4 equations describes electricity and magnetism in free space, however, if one would want to include *linear* medium these equations become as follows:

$$\nabla \cdot \mathbf{D} = \rho_f \text{ ,}\textit{(Gauss' Law)} \tag{4.5}$$

$$\nabla \cdot \mathbf{B} = 0 \text{ ,}\textit{(No special name)} \tag{4.6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ ,}\textit{(Faraday's Law)} \tag{4.7}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \text{ ,}\textit{(Ampere's law with Maxwell correction),} \tag{4.8}$$

where $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$ - a relation for magnetization and magnetic field, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ - a relation for electric field and polarization, ρ_f is the free charge density and $\mathbf{J}_b$ is the bounded current density. These equations would describe most of the problems concerning electricity, magnetism or both. However, if the medium would not be linear where the **E** or **B** acts, then (4.1)-(4.4) would not become (4.5)-(4.8). An extensive description and introduction to Maxwell's equations can be found in [12, 13].

4.1.2 Electromagnetic field tensors

Using the Lorentz transformations introduced in Chapter 3, one can derive how the Electric field and the Magnetic field transform into another frame. Assume we have three different frames $S_0, S, \bar{S}$ and the electric and magnetic field is transformed parallel along the x-axis. Then, **B** and **E** could be seen as a surface that stays invariant in the x direction, but changes in the y and z directions.

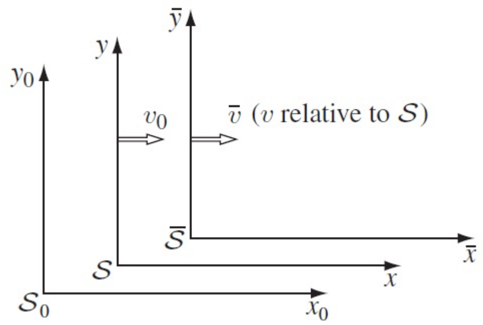

Figure 4.1: *$S_0, S, \bar{S}$ with x axes being parallel.*

After applying Lorentz transformations according to the conditions outlined it would yield:

$$\bar{E}_x = E_x \tag{4.9}$$

$$\bar{B}_x = B_x \tag{4.10}$$

$$\bar{E}_y = \gamma(E_y - vB_z) \tag{4.11}$$

$$\bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z) \tag{4.12}$$

$$\bar{E}_z = \gamma(E_z + vB_y) \tag{4.13}$$

$$\bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) \tag{4.14}$$

The full proof of these transformations can be found in section 8.2. Now using the more abstract, tensor notation for Lorentz transformation Λ the field tensor(s) can be obtained. (4.20) states that the Lorentz transformation between two frames is as follows:

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\mu}, \mu = 0, 1, 2, 3, \tag{4.15}$$

If the transformation is made for the particle moving in the x direction, the transformation tensor is as follows:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.16}$$

It is clear from (4.9)-(4.14) that the electromagnetic field tensor is going to be antisymmetric, hence the form using (2.15) [12]:

$$T^{\mu\nu} = \begin{pmatrix} 0 & T^{01} & T^{02} & T^{03} \\ -T^{01} & 0 & T^{12} & T^{13} \\ -T^{02} & -T^{12} & 0 & T^{23} \\ -T^{03} & -T^{13} & -T^{23} & 0 \end{pmatrix}. \tag{4.17}$$

The Λ here transforms a T tensor according to the following transformations (as it is a second-rank tensor):

$$\bar{T}^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} T^{\lambda\sigma}. \tag{4.18}$$

Hence for example the second item in the transformed T tensor is given by [12]:

$$\bar{T}^{02} = \Lambda^{0}_{\lambda} \Lambda^{2}_{\sigma} T^{\lambda\sigma} = \Lambda^{0}_{0} \Lambda^{2}_{2} T^{02} + \Lambda^{0}_{1} \Lambda^{2}_{2} T^{12} = \gamma(T^{02} - \beta T^{12}), \tag{4.19}$$

as other variations of σ, λ would yield 0, so for T^{02} only these two terms are used. Applying all this, the transformations can be found to be:

$$\bar{T}^{01} = T^{01} \tag{4.20}$$

$$\bar{T}^{23} = T^{23} \tag{4.21}$$

$$\bar{T}^{02} = \gamma(T^{02} - \beta T^{12}) \tag{4.22}$$

$$\bar{T}^{31} = \gamma(T^{31} + \beta T^{03}) \tag{4.23}$$

$$\bar{T}^{03} = \gamma(T^{03} + \beta T^{31}) \tag{4.24}$$

$$\bar{T}^{12} = \gamma(T^{12} - \beta T^{02}). \tag{4.25}$$

Pairing (4.9)-(4.14) with (4.20)-(4.25) the electromagnetic field tensor can be found. One way to do it is by pairing (4.22) with (4.11) and another way is by pairing (4.22) with (4.12) and following the rule that is set by those two. Using the first way the first field tensor is obtained:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix} \tag{4.26}$$

and using the second pairing method:

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{pmatrix}. \tag{4.27}$$

The tensor (4.26) is called the *Faraday* tensor and (4.27) is the *Maxwell* tensor, although it is rather an informal name and usually (4.26) is used as they describe the same thing and both can be referred to as electromagnetic tensor. Similarly, as co-variant and contra-variant vectors mean mostly the same thing in flat spacetime, so do these two. However, if curved spacetime is in question, then co-variant and contra-variant describe different things and it is said that they are dual to each other. Hence, the Maxwell and the Faraday field tensors are also dual to each other and the relevancy would only arise if one would not be working in flat spacetime.

Using these two tensors Maxwell's equations can be written in a lot simpler form and now can be applied to spacetime and relativity. Another important observation is that one can construct a polarization-magnetization (**P** and **M**) tensor and an electric displacement (**D** and **H**) tensor - for the derivation of these refer to [12, 13].

4.1.3 Maxwell Equations and Relativity

If speeds are close to the speed of light, then many things are affected by it in Electrodynamics. This includes charge density, volume and current density, henceforth it makes sense to talk about proper charge density and proper current density as due to length contraction, the volume also gets lower. Hence the components of current density are as follows:

$$J^{\mu} = (c\rho, J_x, J_y, J_z), \tag{4.28}$$

where ρ is the Lorentz-transformed charge density (not the proper charge density). Maxwell equations can now be described by the following two tensors:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu} \tag{4.29}$$

$$\frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0, \tag{4.30}$$

where if μ is summed over 0 to 3, all the Maxwell's equations can be retrieved. The Lorentz force becomes:

$$K^{\mu} = q\eta_{\nu} F^{\mu\nu}, \tag{4.31}$$

where K is the Lorentz force, η is the proper velocity and F is the Faraday tensor.

As both the **E** and **B** can be described in the classical sense by a potential **A**, so can be the Faraday tensor when considering relativity. The potential **A** becomes:

$$A^{\mu} = (V/c, A_x, A_y, A_z), \tag{4.32}$$

or

$$A_{\mu} = (-V/c, A_x, A_y, A_z), \tag{4.33}$$

where V is a scalar potential and **A** is a vector potential. The equation relating **A** with F looks as follows:

$$F_{\rho\sigma} = \frac{\partial A_\sigma}{\partial x^\rho} - \frac{\partial A_\rho}{\partial x^\sigma}. \tag{4.34}$$

To get back the contra-variant form of the EM tensor, the Minkowski metric ($g_{\mu\nu}$) can be used to lower or raise indices:

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}. \tag{4.35}$$

Furthermore, if we assume that **A** is in the *Lorentz gauge* and it is invariant in that gauge, then Maxwell's equations can be represented using the d'Alembertian, which looks as follows:

$$\Box^2 A^\mu = -\mu_0 J^\mu, \tag{4.36}$$

where $\Box^2$ is $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. Hence, when calculating the relativistic motion of a charged particle under an electromagnetic field, relations (4.28)-(4.36) are used. It is clear that there is no proof of these relations here, those can be found in [8, 12].

4.2 Abraham-Lorentz radiation

As a point charge, or continuous charge distribution, moves it radiates energy. This radiation can be calculated using the Poynting vector over the surface of the point charge - which would be a sphere [12]. Using the instantaneous acceleration, the power radiated can be obtained as follows:

$$P = \oint_{Surface} \mathbf{S}_{rad} d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int_{Surface} \frac{sin^2(\theta)}{r^2} r^2 sin(\theta) d\theta d\phi = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{4.37}$$

, where q is the charge, a is the instantaneous acceleration, c is the speed of light and S is the Poynting vector. If it approaches the speed of light it becomes as follows:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi\, c}\left(a^2 - \left|\frac{\mathbf{v}\times\mathbf{a}}{c}\right|^2\right), \tag{4.38}$$

where γ factor appeared and the acceleration is a bit more complicated than it was before. This equation shows that when a charge is moving at relativistic speeds, its radiation increases extremely quickly. One could imagine how enormous the radiation is if the particle would reach ultrarelativistic speeds ($\gamma >> 1000$). As there is radiation, there must be a force present - the force responsible for this radiation is the **Abraham-Lorentz** force or in other words the **self-force** of a particle. The reason for this self-force is that in a charge there are imbalances in the magnetic field and these small changes add up and create the **self-force** present. The following formula gives self-force [12]:

$$F_{rad} = \frac{\mu_0 q^2 \dot{a}}{6\pi c}, \tag{4.39}$$

and when the particle gets close to the speed of light, the relativistic formula is as follows:

$$F_{rad} = \frac{2kq^2}{3c^3}\left(\gamma^2 \dot{a} + \frac{\gamma^4 v(v \cdot \dot{a})}{c^2} + \frac{3\gamma^4 a(v \cdot a)}{c^2} + \frac{3\gamma^6 v(v \cdot a)^2}{c^4}\right), \tag{4.40}$$

where k is the Coulomb constant, a is the acceleration, $\dot{a}$ is the rate of change of acceleration, γ is the Lorentz factor, and v is the speed of the particle. It is theorized in current research that the Abraham-Lorentz radiation and self-force could be the key to observing Unruh radiation. However, for it to be observed the particle needs to reach ultrarelativistic speeds, which is currently limited by technology. Nevertheless, it is one of the most important concepts in modern Physics and understanding the concept can help us understand black holes better and consequently bring physicists closer to the *Holy Grail* that is Quantum Gravity. The derivation of Abraham-Lorentz self-force and radiation is quite rigorous; there are plenty of books written on the matter, hence for a more in-depth introduction to the A.-L. force refer to [12, 14, 15, 16, 17]

To understand the potential of A.-L. radiation it is necessary to study Quantum Field Theory (QFT), hence the next chapter focuses on the *quantum* side of the problem.

4.3 Relativsitic charged particle

However, before introducing or discussing QFT it is important to understand how a relativistic charged particle behaves under an electromagnetic field.

Consider the following Lagrangian (defined in Chapter 3), which describes a relativistic charged particle moving under an external electromagnetic field:

$$\mathcal{L}(\mathbf{r}, \mathbf{v}, t) = -\frac{mc^2}{\gamma(\mathbf{v})} - q\Phi(\mathbf{r}, t) + q\mathbf{v}.\mathbf{A}(\mathbf{r}, t) \tag{4.41}$$

To find the equation of motion for such particle the E.-L. equations are used. Hence using (3.12) the following can be obtained:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} + q\mathbf{A}\right), \tag{4.42}$$

and

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}} = \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{L}}{\partial y} + \frac{\partial \mathcal{L}}{\partial z} = -q\nabla\phi + q\nabla(\mathbf{v}\cdot\mathbf{A}). \tag{4.43}$$

To obtain the equation of motion the $q\mathbf{A}$ term is moved to the other side of the equation and $\nabla(\mathbf{v}\cdot\mathbf{A})$ is expanded:

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -q\frac{d\mathbf{A}}{dt} - q\nabla\phi + q(\nabla(\mathbf{A}\cdot\mathbf{v}) - (\mathbf{v}\cdot\nabla)\mathbf{A}). \tag{4.44}$$

Now using the Maxwell relations (4.1)-(4.4) and some basic vector calculus the following is given:

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{dP}{dt} = q\mathbf{E} + q\mathbf{v}\cdot\mathbf{B}, \tag{4.45}$$

which is the equation of motion using the definition of relativistic momentum. Furthermore, the Lagrangian (for a relativistic charged particle) using covariant notation can also be defined as follows:

$$\mathcal{L}(r, v) = -mc\sqrt{v_\mu v^\mu} - qA_\mu(r)v^\mu. \tag{4.46}$$

Similarly using the E.-L. equations the equation of motion can be obtained, however now with the electromagnetic tensor defined in (4.26) [18]:

$$\frac{dp_\mu}{dt} = qF_{\mu\nu}v^\nu. \tag{4.47}$$

It is clear how useful the electromagnetic tensor is, as it greatly simplifies the mathematics. In the next chapter QFT is explored, which is neccessary to understand the Unruh effect and to build up how ultrarelativistic charged particles can be useful to realise the Unruh radiation.

Chapter 5

Quantum Field Theory

Quantum Mechanics and Quantum Field Theory is not a field of Physics that is mostly developed by one or two scientists, but it is the contribution of many scientists over the years. In 1923 [21] Louis de Broglie was the first to suggest the possibility of describing particles like photons in terms of waves and with Erwin Schrödinger, he founded the so-called *wave mechanics*. Schrödinger first derived the relativistic wave equation [21], but became discouraged with it as it gave the wrong fine structure for hydrogen and then realized that the non-relativistic wave equation was of value, hence he published that first. Only later on (in his 6th paper) was the relativistic version introduced, however by that time Oskar Klein and Walter Gordon rediscovered the relativistic wave equation independently, hence it is called the *Klein-Gordon equation*.

The next stage in the development of quantum mechanics was the invention of matrix mechanics from 1925-26 by Werner Heisenberg, Max Born, Pascual Jordan and Wolfgang Pauli. The *Klein-Gordon equation* had an issue, which led to negative probabilities. Hence the next step in quantum theory was made by Paul Dirac, who replaced the wave equation with a different one solving the issue of negative probabilities, furthermore, this modification gave the right fine structure constant for hydrogen, which was predicted by Heisenberg and Jordan. Later on, in 1930-31, Dirac solved the issue that electrons interacting with radiation can produce a transition that moves the positive-energy electrons to a negative-energy state. This theory is regarded as the *Dirac hole* theory, where it is assumed that there is a *sea* of negative energy levels that are all filled - however, there are holes in this negative energy sea, which have opposite charge and mass to electrons.

The process of an electron meeting with a hole was called an annihilation event. This theory was the basis for the existence of anti-particles and the validity of the theory was proved by observing a cosmic ray in a Wilson cloud chamber in 1932. Nowadays it is known that every kind of particle has its anti-particle, except for bosons, as they can share the same energy level, and hence do not have the same problem as fermions. Later on in the 1930s with contributions from Dirac, Born, Jordan, Fermi, Rosenfeld, Pauli, Wigner, Heisenberg, Oppenheimer, Furry, Fock, Weisskopf, Yukawa and many more scientists contributed to the development of the quantum field theory, which unifies classical field theory, quantum mechanics and special relativity. Quantum field theory is one of the most important fields in modern physics and is of particular interest for this thesis as it predicts the Unruh effect and radiation. In this chapter the relevant mathematics of QFT is discussed, for introduction to Quantum Mechanics refer to [19, 20] and for a more in-depth introduction to QFT see [21, 22, 23, 24, 25, 26].

5.1 Lagrangians and Hamiltonians

In quantum field theory $\mathcal{L}$ and $\mathcal{L}_d$ are an extremely powerful tool. As it was discussed in Chapter 3, they are widely used because they satisfy the Lorentz invariance, hence they present the problem the same way in different coordinate systems. However, in some problems such as finding the S-matrix, which relates the initial state and the final state of a system undergoing scattering, the Hamiltonian needs to be found as it is used to calculate such matrix. Hamiltonians are related to Lagrangians and it is possible to find the equations of motion in terms of Hamiltonians rather than Lagrangians. In simple terms, the two are related through the following:

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = 2\mathcal{T} - \mathcal{H}, \tag{5.1}$$

where $\mathcal{H}$ is the Hamiltonian, $\mathcal{V}, \mathcal{T}$ are the potential energy and kinetic energy respectively and $\mathcal{L}$ is the Lagrangian. However, this representation is not general enough for fields and it is not clear how the Hamiltonian equation of motion can be found. In QFT consider the general

Lagrangian as $\mathcal{L}[\Psi(t), \dot{\Psi}(t)]$ which is a set of generic fields $\Psi^l(\mathbf{x}, t)$ and their time-derivatives $\dot{\Psi}^l(\mathbf{x}, t)$. Consider a conjugate field $\Pi_l(\mathbf{x}, t)$ that is defined as follows:

$$\Pi_l(\mathbf{x}, t) = \frac{\delta \mathcal{L}[\Psi(t), \dot{\Psi}(t)]}{\delta \dot{\Psi}^l(\mathbf{x}, t)}, \tag{5.2}$$

from which the equation of motion is defined as follows:

$$\dot{\Pi}_l(\mathbf{x}, t) = \frac{\delta \mathcal{L}[\Psi(t), \dot{\Psi}(t)]}{\delta \Psi^l(\mathbf{x}, t)}, \tag{5.3}$$

then using the action introduced in Chapter 3, the following is obtained:

$$S = \int_{-\infty}^{\infty} \mathcal{L}[\Psi(t), \dot{\Psi}(t)] dt, \tag{5.4}$$

using the fact that nature likes to go on the shortest path, minimizing this action and also using the Lagrangian density definition it is possible to obtain (3.12) and (3.21). In QFT mostly the Lagrangian density is used as one is interested in fields, hence the equation of interest is the E.-L. equation in the form of $\mathcal{L}_d$, which looks as follows:

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}_d}{\partial(\partial \Psi^l / \partial x^\mu)} = \frac{\partial \mathcal{L}_d}{\partial \Psi^l}. \tag{5.5}$$

In QFT when trying to identify fields it is easier to find the Lagrangian and then convert to a Hamiltonian as Hamiltonians are not so easy to find [21]. Hamiltonians can be found using *Legendre transform* from the Lagrangians:

$$\mathcal{H} = \sum_l \int_{-\infty}^{\infty} [\Pi_l(\mathbf{x}, t) \dot{\Psi}^l(\mathbf{x}, t) - \mathcal{L}[\Psi(t), \dot{\Psi}(t)]] d^3 x, \tag{5.6}$$

after a series of mathematical manipulations [21] on (5.6), the Hamiltonian equation of motion is obtained as:

$$-\dot{\Pi}_l(\mathbf{x}, t) = \frac{\delta \mathcal{H}}{\delta \Psi^l(\mathbf{x}, t)}, \tag{5.7}$$

which is almost identical to (5.3) except for a negative factor. Similarly, the E.-L. equations can be found, however now in terms of the Hamiltonian, by using the least action principle.

5.2 Early developments of QFT

5.2.1 Annihilation and Creation operators

In 1926, Born, Heisenberg and Jordan applied matrix mechanics to the free radiation field. They ignored the polarization of EM waves and only considered one space dimension. The Hamiltonian they took for this analysis is as follows:

$$\mathcal{H} = \frac{1}{2}\int_0^L \left(\left(\frac{\partial u}{\partial t}\right)^2 + c^2 \left(\frac{\partial u}{\partial x}\right)^2 \right) dx, \tag{5.8}$$

which was simplified using the fact that this field behaves like a sum of independent harmonic oscillators, hence using the following radiation field:

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin\left(\frac{\omega_k x}{c}\right), \tag{5.9}$$

where $\omega_k = k\pi c/L$ the following simplified Hamiltonian is obtained:

$$\mathcal{H} = \frac{L}{4}\sum_{k=1}^{\infty} \left(\dot{q}_k^2(t) + \omega_k^2 q_k^2(t) \right). \tag{5.10}$$

Using the Hamiltonian the equation of motion can be found [21], as momentum is conjugate to position the time derivative of position can be obtained as follows:

$$\dot{q}_k(t) = \frac{\partial}{\partial p_k(t)} \mathcal{H}(p(t), q(t)), \tag{5.11}$$

and the momentum is:

$$p_k(t) = \frac{L}{2}\dot{q}_k(t), \tag{5.12}$$

which gives rise to the commutation relations as follows:

$$[\dot{q}_k(t), q_j(t)] = \frac{2}{L}[p_k(t), q_j(t)] = \frac{-2i\hbar}{L}\delta_{kj} \tag{5.13}$$

$$[q_k(t), q_j(t)] = 0. \tag{5.14}$$

The Hamiltonian equation of motion hence then is as follows:

$$\ddot{q}_k(t) = \frac{2}{L}\dot{p}_k(t) = -\frac{2}{L}\frac{\partial \mathcal{H}}{\partial q_k(t)} = -\omega_k^2 q_k(t). \tag{5.15}$$

The matrix q_k was found by Born, Heisenberg and Jordan as the following:

$$q_k = \sqrt{\frac{\hbar}{L\omega_k}}[a_k exp(-i\omega_k t) + a_k^{\dagger} exp(i\omega_k t)], \tag{5.16}$$

where a_k is a time-independent matrix and $a_k^{\dagger}$ is its Hermitian adjoint (conjugate diagonal matrix) that satisfies the following commutation relations:

$$[a_k, a_j^{\dagger}] = \delta_{kj} \tag{5.17}$$

$$[a_k, a_j] = 0. \tag{5.18}$$

The matrices a_k and $a_k^{\dagger}$ can be generated from the following relations [21]:

$$(a_k)_{n'_1,n'_2,...,n_1,n_2,...} = \sqrt{n_k}\delta_{n'_k,n_k-1}\prod_{j\neq k}\delta_{n'_j,n_j} \tag{5.19}$$

$$(a_k^{\dagger})_{n'_1,n'_2,...,n_1,n_2,...} = \sqrt{n_k+1}\delta_{n'_k,n_k+1}\prod_{j\neq k}\delta_{n'_j,n_j}, \tag{5.20}$$

where $n'_1, n'_2..$ represent the final quantum states and $n_1, n_2, ..$ the initial quantum states. The a_k matrix is called the annihilation operator it removes one quantum state from the previous quantum state and the $a_k^{\dagger}$ is called the creation operator as it will create a quantum state in the kth mode. For example, vacuum would be just a 0 (as an element) that was annihilated by an a_k operator. This approach formed the basis of QFT as it explained quantum fields in terms of quantum states through the process of annihilation and creation. Using the previously introduced relations an n-diagonal representation of the Hamiltonian can also be obtained [21]:

$$(H)_{n'_1,n'_2,\ldots,n_1,n_2,\ldots} = \sum_k \hbar\omega_k(n_k + \frac{1}{2}) \prod_j \delta_{n'_j,n_j}. \tag{5.21}$$

This gives rise to an important property that is the energy level in vacuum is not 0, vacuum also has an energy and this property will later be important to the Unruh effect.

5.2.2 Problem with electrons

There were still issues with this representation as for electrons the Pauli exclusion principle prevents any normal mode k from having a value other than 0 or 1. Hence, the electron field could not be explained correctly with such relations, instead, they were found to be explained by the *anticommutation* relations:

$$a_k a_j^\dagger + a_j^\dagger a_k = \delta_{jk}, \tag{5.22}$$

$$a_k a_j + a_j a_k = 0, \tag{5.23}$$

as these relations would only allow the electrons to take values of 1 and 0. It was shown by Fierz and Pauli that whether a particle's quantum field is described by the commutation or the anticommutation relations is dependent on the particle's spin. Commutation relations are used for particles with spin integer and anticommutation relations are used for particles with half-integer spins.

For example, if we take the electron, it can only take 0 or 1 in a normal mode k, as electrons do not like each other. If there is a quantum mode k in a place, the annihilation operator a_k destroys that; and if there is no quantum mode in a place, then the creation operator $a_k^\dagger$ creates one. Similar reasoning is applied to particles like photons, only with the commutation relations and this way of thinking is ubiquitous in QFT.

As mentioned before the first relativistic quantum field equation was the Klein-Gordon equation, however, the theory of general quantum fields was laid out by Heisenberg and Pauli using Lagrangians. Pauli and Heisenberg took a complex scalar field $\Phi(x)$ and wrote its Lagrangian as:

$$\mathcal{L} = \int \left(\dot{\Phi}^\dagger \dot{\Phi} - c^2 (\nabla\Phi)^\dagger \cdot (\nabla\Phi) - \left(\frac{mc^2}{\hbar}\right)^2 \Phi^\dagger\Phi \right) d^3x, \tag{5.24}$$

after minimizing it the Klein-Gordon equation can be obtained [21] as follows:

$$\left(\Box - (\frac{mc}{\hbar})^2\right)\Phi = 0, \tag{5.25}$$

where $\Box$ is the d'Alembert operator. Here the Klein-Gordon equation was derived using a Lagrangian instead of modifying the wave equation, which it was derived first with. From this equation the *momenta* (π) and its adjoint can be defined as follows:

$$\pi = \frac{\delta L}{\delta \dot{\Phi}} = \dot{\Phi}^\dagger \tag{5.26}$$

$$\pi^\dagger = \frac{\delta L}{\delta \dot{\Phi}^\dagger} = \dot{\Phi}, \tag{5.27}$$

and their commutation relations are:

$$[\pi(\mathbf{x}, t), \Phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t)^\dagger, \Phi(\mathbf{y}, t)^\dagger] = -i\hbar\delta^3(\mathbf{x} - \mathbf{y})$$

$$[\pi(\mathbf{x}, t), \Phi(\mathbf{y}, t)^\dagger] = [\pi(\mathbf{x}, t)^\dagger, \Phi(\mathbf{y}, t)] = 0$$

$$[\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t)^\dagger, \pi(\mathbf{y}, t)^\dagger] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)^\dagger] = 0$$

$$[\Phi(\mathbf{x}, t), \Phi(\mathbf{y}, t)] = [\Phi(\mathbf{x}, t)^\dagger, \Phi(\mathbf{y}, t)^\dagger] = [\Phi(\mathbf{x}, t), \Phi(\mathbf{y}, t)^\dagger] = 0$$

Table 5.1: *Commutation relations between momenta and a complex scalar field.*

Furthermore, the Hamiltonian can be obtained using (5.6), which gives:

$$\mathcal{H} = \int \left(\pi\dot{\Phi} + \pi^\dagger\dot{\Phi}^\dagger\right) d^3x \tag{5.28}$$

5.2.3 Introduction of antiparticles

QFT was still not complete because of the negative probabilities the Klein-Gordon equation led to. The discovery of the positron in 1932 confirmed Dirac's idea of hole theory. In (5.21) the ω_k take up positive and negative values, half of it is negative and half of it is positive. Using Dirac's idea of *holes* another set of annihilation and creation operators were developed to account for the negative energy electrons, nowadays we would call them the antiparticle annihilation and creation operators. These operators are as follows:

$$b_k^\dagger = a_k \tag{5.29}$$

$$b_k = a_k^\dagger, \tag{5.30}$$

using these operators one can define another field, that would be called the Dirac field:

$$\Psi(x) = \sum_k^{(+)} a_k u_k(x) + \sum_k^{(-)} b_k^\dagger u_k(x), \tag{5.31}$$

and the Hamiltonian can be rewritten as well:

$$\mathcal{H} = \sum_k^{(+)} \hbar\omega_k a_k^\dagger a_k + \sum_k^{(-)} \hbar|\omega_k| b_k^\dagger b_k + E_0. \tag{5.32}$$

5.2.4 Challenge to the anticommutation relations

In 1934, Pauli and Weisskopf wrote a paper [21] where they resolved the problem of negative energy states. They managed to use creation and annihilation operators that satisfy the commutation relations rather than the anticommutation relations, which challenged Dirac's view. Pauli and Weisskopf considered a free-charged scalar field in plane waves in a cube and defined the field and the momenta as follows (V = L*L*L):

$$\Phi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} q(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.33}$$

$$\pi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} p(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{5.34}$$

and can obtain the momentum and position using the anti-fourier transform:

$$q(\mathbf{k}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \Phi(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{5.35}$$

$$p(\mathbf{k},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\pi(\mathbf{x},t)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.36}$$

p and q furthermore suffices the commutation relations:

$$[p(\mathbf{k},t), q(\mathbf{l},t)] = \frac{-i\hbar}{V}\int[e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{l}\cdot\mathbf{x}}]d^3x = -i\hbar\delta_{\mathbf{kl}} \tag{5.37}$$

$$[p(\mathbf{k},t), q^\dagger(\mathbf{l},t)] = [p(\mathbf{k},t), p(\mathbf{l},t)] = [p(\mathbf{k},t), p^\dagger(\mathbf{l},t)] = [q(\mathbf{k},t), q(\mathbf{l},t)] = [q(\mathbf{k},t), q^\dagger(\mathbf{l},t)] = 0. \tag{5.38}$$

The Hamiltonian can now be expressed as follows:

$$\mathcal{H} = \sum_{\mathbf{k}}[p^\dagger(\mathbf{k},t)p(\mathbf{k},t) + \omega_{\mathbf{k}}^2 q^\dagger(\mathbf{k},t)q(\mathbf{k},t)], \tag{5.39}$$

from which the Hamiltonian equation of motion can be obtained:

$$\dot{p}(\mathbf{k},t) = -\frac{\partial\mathcal{H}}{\partial q(\mathbf{k},t)} = -\omega_{\mathbf{k}}^2 q^\dagger(\mathbf{k},t), \tag{5.40}$$

which is equivalent to the Klein-Gordon wave equation [21]. Furthermore, q and p can be expressed in terms of particle and antiparticle annihilation and creation operators as follows:

$$q(\mathbf{k},t) = i\sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}}[a(\mathbf{k})e^{-i\omega_k t} - b^\dagger(\mathbf{k})e^{i\omega_k t}] \tag{5.41}$$

and for momentum:

$$p(\mathbf{k},t) = \sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2}}[b(\mathbf{k})e^{-i\omega_k t} + a^\dagger(\mathbf{k})e^{i\omega_k t}], \tag{5.42}$$

and the associated commutation relations are as follows:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = [b(\mathbf{k}), b^\dagger(\mathbf{l})] = \delta_{\mathbf{kl}} \tag{5.43}$$

$$[a(\mathbf{k}), a(\mathbf{l})] = [b(\mathbf{k}), b(\mathbf{l})] = 0 \tag{5.44}$$

$$[a(\mathbf{k}), b(\mathbf{l})] = [a(\mathbf{k}), b^\dagger(\mathbf{l})] = [a^\dagger(\mathbf{k}), b(\mathbf{l})] = [a^\dagger(\mathbf{k}), b^\dagger(\mathbf{l})] = 0, \tag{5.45}$$

and using the annihilation and creation operators and the commutation relations the most general form of the Hamiltonian can be found:

$$\mathcal{H} = \sum_{\mathbf{k}} \hbar\omega_k \left(b^\dagger(\mathbf{k}) b(\mathbf{k}) + a^\dagger(\mathbf{k}) a(\mathbf{k}) \right) + E_0. \tag{5.46}$$

Hence this type of QFT is a theory of two kinds of particles with the same mass, one the particle and one that is the antiparticle. In modern physics, all particles have their antiparticle pair and this theory helps us describe such systems. It is important to note that bosons do not have the same issue as fermions as they can share the same energy level, hence their antiparticle is not so obvious. One might say, for example, if we take photons we can take their antiparticles that would be called *virtual photons*, however, it would be of the same type as regular photons, with the same spin and same energy. Hence it would not make sense to overcomplicate the problem by introducing antiparticles for bosons when they do not require such treatment (although in some cases it is necessary to do so [26]). However, for fermions, it is required and without antiparticles, the theory would not be consistent. Another important consequence of such theory is that the wavefunctions or fields Ψ and Φ are no longer regarded as probability amplitudes, but they are operators that create and destroy particles - hence the term *negative* probabilities has a clear meaning.

QFT in the beginning faced many difficulties as it was inconsistent in certain areas - for example, it was assumed that only positrons, electrons and photons exist as particles - using only these three particles the theory was inconsistent. Later it was found out that there are many more particles, which now form the basis of the standard model as we currently know, and with such correction, QFT does explain systems well and is consistent with itself.

Another issue was the problem of inifinities, that is for example the electron's self-mass $e^2/6\pi ac^2$ in the centre blows up to infinity (if we integrate from a=0 to ∞) meaning it would have infinite self-mass, which is not possible. These issues were called ultraviolet and infrared divergences, in many cases the solutions were found (an example would be a process called renormalization to account for infinities), however, there are still cases where infinities are present and cause disturbance to the theory. These solutions usually cancel the infinities in a certain way (using some kind of mathematical tool), however in most cases, the infinities remain and the theory would only work for a specific cutoff wave-number. This cutoff wave-number is a limit that needs to be taken into account and if that is reached, the theory would be inconsistent with itself and QFT would not explain the physics there. In the next section, Electrodynamics is explored in QFT, however, there is a lot more to QFT than what is summarized here and for a more in-depth introduction consider reading the textbooks by Steven Weinberg (who unified the weak force and electromagnetic interaction between elementary particles), an expert in this area [21, 23, 24].

5.3 Electrodynamics in Quantum Field Theory

Most of Quantum electrodynamics is based on Fermi's 1934 paper as the scientists of the era learnt it from him [21]. Furthermore, after the introduction of the particles and antiparticles point of view, other theories have been also developed. One of such theories is the Yukawa's potential quantum field theory. He proposed that the nucleon-nucleon potential is given by:

$$V(r) \propto \frac{1}{r} e^{-\lambda r}, \tag{5.47}$$

where r is the separation between nucleons and λ is a parameter, this equation describes a particle of mass $\hbar\lambda/c$ and such particle is called a mesons, to be more specific π mesons. However, many other particle theories arose following Fermi's paper - but in this thesis, the focus is on electrons, hence in this section the focus is to introduce the quantized version of Maxwell's equations.

5.3.1 Gauge invariance

The electromagnetic force is mediated by photons. Photons are massless particles, meaning they have a helicity of 1 or -1 [21]. However, for massless particles, there is no way of constructing the field equations with annihilation and creation operators - this is one part of QFT where singularities arise. Instead, to treat this problem it is assumed that all electromagnetic interactions would require $\mathcal{F}_{\mu\nu}(x)$ and its derivatives [21]. By choosing a gauge (measurement) we set the electrostatic and magnetic potentials of some value that would then define the electromagnetic tensors. Gauge invariance means that by going from one gauge to another, the electrostatic and magnetic potentials remain invariant. For matter, it is demanded that the action must be invariant under a general gauge transformation that is:

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{\partial \epsilon(x)}{\partial x_\mu}. \tag{5.48}$$

One can define the change in the matter action under (5.48) as follows:

$$\delta S_M = \int \left(\frac{\delta I_M}{\delta A_\mu(x)} \frac{\partial \epsilon(x)}{\partial x_\mu} \right) d^4 x, \tag{5.49}$$

then after minimizing this action after a series of mathematical calculations, that can be found in [21] the following equation of motion can be found:

$$\frac{\delta S_M}{\delta A_\mu(x)} = J^\mu(x), \tag{5.50}$$

where J is the current density containing the electric charge and current densities, and A is the vector potential containing the electrostatic potential and the magnetic vector potential - both of them are 1-forms. Furthermore after considering the photons' interaction on themselves the following action can be considered [21]

$$S_\gamma = -\frac{1}{4} \int \left(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right) d^4 x, \tag{5.51}$$

after combining Equation 5.50 and 5.51 the inhomogeneous Maxwell equations can be found:

$$0 = \frac{\delta \left(S_M + S_\gamma \right)}{\delta A_\nu} = \frac{\partial \mathcal{F}^{\mu\nu}}{\partial x_\mu} + J^\nu. \tag{5.52}$$

Hence, in QFT electrodynamics can be described using the $\mathcal{F}^{\mu\nu}$ 2-form and the A, J 1-forms containing the electric and magnetic potentials and electric charge and current densities respectively.

Gauge Fields can be generalized with p-forms, meaning that, for example, for theories working in more than 4 spacetimes dimensions, such as string theory one would require more than 1- or 2-forms describing electrodynamics. In these cases p-forms and *exterior calculus* are used to describe such systems. However, for 4 spacetime dimensions, 1-forms and 2-forms are sufficient. Exterior calculus is another branch of mathematics that is quite useful for p-forms - for an introduction to the mathematical theory refer to [3].

However, with this chapter finished, all the relevant background theory is covered to understand the Unruh effect and radiation. Hence the next chapters focus on the Unruh effect, speculations on how the Unruh radiation can be first observed and why is it useful for Hawking radiation.

Chapter 6

Unruh effect

The Unruh effect argues that whether a particle is real or not is a purely observer-dependent concept. It predicts that when a particle detector (observer) accelerates it will perceive a bath of thermal radiation in a vacuum. This radiation is the so-called Unruh radiation and is connected to the Hawking radiation of a black hole through Einstein's equivalence principle. Hawking radiation is caused by the large gravitational field of the horizon and Unruh radiation is created by the large acceleration that would be equivalent to the black hole gravitational field as previously outlined by the equivalence principle.

The Unruh radiation has never been observed and current research speculated that it might be encoded into the A.-L. radiation (Larmor radiation) and an ultrarelativistic charged particle could help realise this radiation. However, there is much discussion on the matter and many scientists are in disagreement. For example, Unruh radiation is a purely quantum concept whereas Larmor radiation is purely classical. This would imply that the Larmor radiation formula would only need a quantum correction when entering the *quantum world*. However, some papers (which will be discussed in the next chapter) speculate that Larmor radiation becomes the Unruh radiation as it reaches ultrarelativistic speeds. Although, it would be more logical that the Unruh radiation would be just a quantum correction to the Larmor radiation as it does not have a heat profile (temperature). On the other hand, Unruh radiation must have a heat profile as it is equivalent to Hawking radiation (which also has a heat profile) - if it turns out that Unruh radiation does not have a heat profile it could challenge one of the fundamental principles of modern physics, that is the equivalence principle. Hence, some physicists would regard observing Unruh radiation as another test of the equivalence principle.

This chapter discusses the mathematics and meanings behind the Unruh effect, radiation and its connection to the Hawking effect and radiation. Furthermore, it would be a preparation to understand the last section of the chapter better - which focuses on how Unruh radiation can be realised using UCP.

6.1 Accelerating Observer

A passenger in a rocket feels *gravity* because he is under acceleration in flat spacetime. As mentioned before, due to the equivalence principle gravity and acceleration are identical and hence one can analyze the quantum field theory of curved spacetime, using, in fact, flat spacetime that is under large acceleration. However, such a system would only work if the vicinity, an infinitesimal version of spacetime, is supplied by a moving frame or the world line is not bigger than the *inverse acceleration* of the accelerating particle. This solution was introduced as the *Fermi-Walker transported orthonormal tetrad* [3, 27, 28, 29].

This solution yields the motion of the particle as a hyperbolic motion, which could be defined as follows considering a two-dimensional Minkowski space (using the Minkowski metric):

$$g(d\bar{u}, d\bar{v}) = ds^2 = d\bar{u}d\bar{v} = dt^2 - dx^2, \tag{6.1}$$

and t and x are defined as follows:

$$t = a^{-1}e^{a\xi}\sinh(a\eta), \tag{6.2}$$

$$x = a^{-1}e^{a\xi}\cosh(a\eta), \tag{6.3}$$

with a being a constant bigger than 0 and $-\infty < \eta$ and labels the time coordinates and is a straight line from the origin, $\xi < \infty$ and labels the worldline of a uniformly accelerated observer. Using the hyperbolic identities and defining $u = \eta - \xi$ and $v = \eta + \xi$ the following

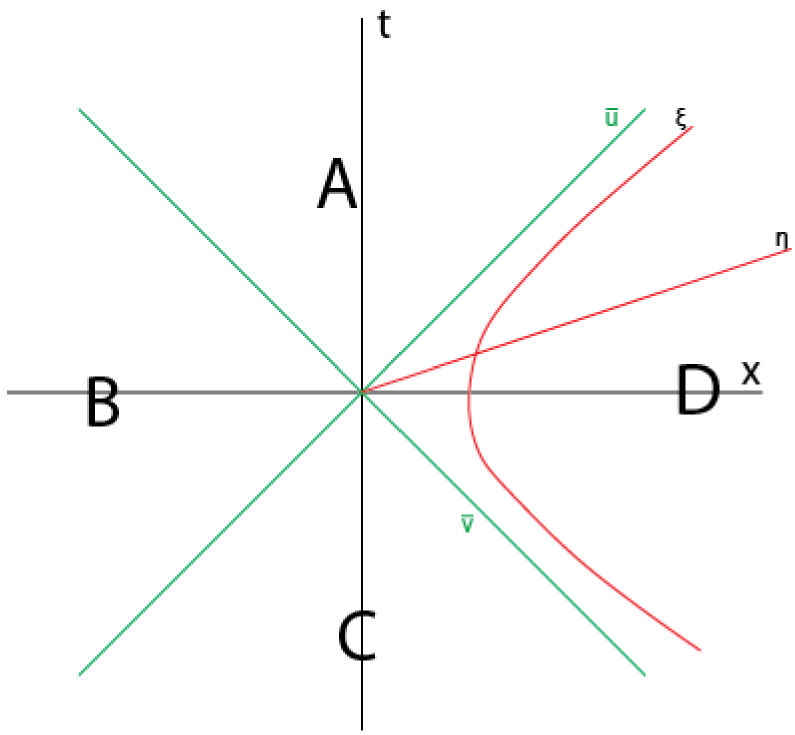

Figure 6.1: *Rindler coordinates of Minkowski space. There could be another region plotted in the B region, where $x < |t|$ and the corresponding coordinates can be obtained by taking negative signs of* (6.2)-(6.7). *Hence B region would describe a timelike separation and the D region would describe a spacelike separation.*

can be obtained:

$$\bar{u} = -a^{-1}e^{-au}, \tag{6.4}$$

$$\bar{v} = a^{-1}e^{av}. \tag{6.5}$$

The inverse acceleration can be defined as $\alpha = \sqrt{x^2 - t^2} = \sqrt{a^{-2}e^{2a\xi}}$ and since the accelerated observer approaches the speed of light as η approaches $\pm\infty$, the equation gives back the proper time as $\tau = e^{a\xi}\eta$. Using these relations t and x can be rewritten to be:

$$t = \alpha \sinh\left(\frac{\tau}{\alpha}\right), \tag{6.6}$$

$$x = \alpha \cosh\left(\frac{\tau}{\alpha}\right), \tag{6.7}$$

which are called the Rindler coordinates of Minkowski space for an accelerating observer.

6.2 Bogoliubov transformation

As mentioned in Chapter 3, Bogoliubov's transformation is a more general transformation than the Poincaré transformation. One would need to apply the Bogoliubov transformation when describing physics in *curved* spacetime. Previously it was mentioned that the Unruh effect can be derived in flat spacetime, which is still the case - however, by the equivalence principle, the accelerating observer would see a curved spacetime, or at least simulate it. The Poincaré group would no longer suitable descriptor for describing the conservations laws, hence the Bogoliubov transformation needs to be used to give the correct result back.

In (5.21) it was shown how the Hamiltonian of a field is computed without using the particle, antiparticle annihilation and creation operators. If there are no particles present in the Minkowski space then the energy would be $\sum_k \frac{1}{2}\hbar w_k$. This would imply that a vacuum has an infinite amount of energy density, which is not true. This issue can be solved by introducing annihilation and creation operators as was done in (5.46). This result hence means that the vacuum still has energy and consists of modes, however it is the lowest possible energy level it can be in. Hence, the Minkowski observer (that is not accelerating) would be in this lowest energy state, this would be called the Fock space and is labelled as $a_i|0\rangle = 0$ (as there are no particles present).

Consider the $\mathcal{L}_d$ of a curved scalar field $\phi(x)$ as follows [26]:

$$\mathcal{L}_d = \frac{1}{2}[-g(x)]^{\frac{1}{2}} \left(g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x)\right), \tag{6.8}$$

where the only new variable, that was also introduced in Chapter 2 is the R(x) variable, which is a scalar curvature. Using the E.-L. equation (5.5) the relations for the scalar field $\phi(x)$ can be found [26]. Furthermore, the steps can be found in section 8.3. However, there is not one general solution that can be found, since $\phi(x)$ can be expanded as a set of modes, but also by a set of orthonormal modes to the previous modes. Hence the solutions can take the following forms:

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)], \tag{6.9}$$

where u(x) and its conjugate u*(x) are the modes and the solution using the set of orthonormal modes $\bar{u}_j(x)$ is:

$$\phi(x) = \sum_i [\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x)], \tag{6.10}$$

which gives rise to another vacuum state that is:

$$\bar{a}_j|\bar{0}\rangle = 0, \forall j. \tag{6.11}$$

The original modes and the orthonormal modes are related through the process of Bogoliubov transformations, which is as follows:

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*) \textbf{ or } \int d\mathbf{k}(\alpha_{kp} u_p + \beta_{kp} u_p^*), \tag{6.12}$$

and its inverse is:

$$u_i = \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*) \textbf{ or } \int d\mathbf{k}(\alpha_{kp}^* \bar{u}_k - \beta_{kp} \bar{u}_k^*). \tag{6.13}$$

The α and β coefficients are called the Bogoliubov coefficients and are computed as follows:

$$\alpha_{ji} = \alpha_{kp} = (\bar{u}_i, u_j) \tag{6.14}$$

$$\beta_{ji} = \beta_{kp} = -(\bar{u}_i, u_j^*) \tag{6.15}$$

Furthermore, the annihilation operator can be calculated using the original mode solution and the new mode solution using the Bogoliubov coefficients as follows:

$$a_i = \sum_j [\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger], \tag{6.16}$$

and

$$\bar{a}_j = \sum_j [\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger]. \tag{6.17}$$

Furthermore, they have the following properties:

$$\sum_k (\alpha_{ij} \alpha_{ik}^* - \beta_{ij} \beta_{ik}^*) = \delta_{jk} = \delta(j-k), \tag{6.18}$$

$$\sum_k (\alpha_{ij}\beta_{ik} - \beta_{ij}\alpha_{ik}) = 0. \tag{6.19}$$

This is quite an interesting result as it means that the new vacuum defined by the orthonormal modes will not be annihilated by a_i, meaning $a_i|\bar{0}\rangle = \sum_j \beta^*_{ji}|\bar{1}_j\rangle \neq 0$. Furthermore, it means that the vacuum of $\bar{u}_j$ mode contains $\langle\bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta^2_{ji}|$ number of particles that are in u_i mode.

This result is the basis of the Unruh effect, which outlines that whether a particle is detected or not is purely observer-dependent. If an observer is viewing from the orthonormal mode point of view they would detect particles in **vacuum**, however, the observer in the original mode point of view would detect particles. Hence, the Bogoliubov transformation is a quite useful tool, as it predicts particle creation, which is speculated to have happened by gravitational fields during the Big Bang [30, 10] and it serves as a basis for the Unruh effect and the Hawking radiation as well.

6.3 Maths behind the Unruh effect

To derive the Unruh effect, we would consider an accelerating particle detector in Rindler spacetime and using the Bogoliubov transformation one can relate the Minkowski frame to the Rindler frame. Here I will be using the method that was proposed in [31] to rewrite the Rindler coordinates in lightcone coordinates, which goes as follows:

$$\begin{array}{ll} \bar{u} = t - x & u = \eta - \xi \\ \bar{v} = t + x & v = \eta + \xi \\ ds^2 = d\bar{u}d\bar{v} & ds^2 = e^{a(v-u)}dudv, \end{array}$$

Table 6.1: $(\bar{u}, \bar{v})$ *are the coordinates of the Minkowski space(MS) and (u,v) are the coordinates of the Rindler space(RS).*

Using (6.9) and (6.10) the mode expansions of a scalar field ϕ can be found and as $u = \frac{1}{(4\pi\omega)^{\frac{1}{2}}}e^{i(kx-\omega t)}$ satisfies the scalar field equation [26, 31]. Furthermore, as both the Minkowski and the Rindler frame scalar field satisfy the Klein-Gordon equation (5.25), ϕ can be expanded as a linear combination of the lightcone coordinates in MS and RS. Hence $\phi(\bar{u}, \bar{v}) = A(\bar{u}) + B(\bar{v})$ and $\phi(u, v) = C(u) + D(v)$. The idea is to find the relation between the A and B functions, which would be described by the Bogoloiubov transformations and hence the number of particles in the accelerated Rindler frame can be found.

For the mode expansions a continuous system is used now as it is easier to handle that, hence ϕ is:

$$\phi(t,x) = \int_{-\infty}^{0} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} \left(e^{ik(x+t)} a_k + e^{-ik(t+x)} a_k^\dagger \right) + \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2k}} \left(e^{-ik(t-x)} a_k + e^{ik(t-x)} a_k^\dagger \right), \tag{6.20}$$

where $w = |k|$ and $-\infty \to 0$ integral represent when $|k| < 0$ and $0 \to \infty$ integral represents $|k| > 0$ with the annihilation and creation operators from (6.9). Similarly, the Rindler frame is:

$$\phi(\eta,\xi) = \int_{-\infty}^{0} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} \left(e^{ik(\xi+\eta)} b_k + e^{-ik(\eta+\xi)} b_k^\dagger \right) + \int_0^\infty \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2k}} \left(e^{-ik(\eta-\xi)} b_k + e^{ik(\eta-\xi)} b_k^\dagger \right), \tag{6.21}$$

where $b, b^\dagger$ labels similarly the annihilation and creation operators, not to be confused with the antiparticle annihilation and creation operators. Hence using the lightcone coordinates for MS and RS, furthermore using the fact that $\omega = |k|$ these integrals can be rewritten as one integral from $0 \to \infty$.

$$\phi(\bar{u},\bar{v}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega(\bar{u})} a_\omega + e^{i\omega(\bar{u})} a_\omega^\dagger + e^{-i\omega(\bar{v})} a_{-\omega} + e^{i\omega(\bar{v})} a_{-\omega}^\dagger \right), \tag{6.22}$$

and for the RS:

$$\phi(u,v) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega(u)} b_\Omega + e^{i\Omega(u)} b_\Omega^\dagger + e^{-i\Omega(v)} b_{-\Omega} + e^{i\Omega(v)} b_{-\Omega}^\dagger \right), \tag{6.23}$$

where Ω is the momenta, but in the Rindler frame. As mentioned before Φ can be expanded as a linear combination of the lightcone coordinates, hence using those relations it is evident that:

$$C(u) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega(u)} b_\Omega + e^{i\Omega(u)} b_\Omega^\dagger \right) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega(\bar{u})} a_\omega + e^{i\omega(\bar{u})} a_\omega^\dagger \right) = A(\bar{u}), \tag{6.24}$$

and

$$D(v) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left(e^{-i\Omega(v)} b_{-\Omega} + e^{i\Omega(v)} b^\dagger_{-\Omega} \right) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left(e^{-i\omega(\bar{v})} a_{-\omega} + e^{i\omega(\bar{v})} a^\dagger_{-\omega} \right) = B(\bar{v}). \tag{6.25}$$

Hence now the Bogoliubov transformation needs to be found, more importantly, the Bogoliubov coefficients. This can be done by either Fourier transforming (6.24) or (6.25). Furthermore, it is important to choose F(Ω) as the Fourier transform function, so the relation between RS's and MS's annihilation and creation operator can be found, hence the Bogoliubov coefficients. The left-hand side of the equation is:

$$\int_{-\infty}^{\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} C(u) = \frac{1}{\sqrt{2\Omega}} b_\Omega, \tag{6.26}$$

where Mathematica was used to perform the calculations (the error function was used here) and the annihilation operator in the result would change to the creation operator if the $\Omega < 0$, however C(u) is defined as positive momenta. Similarly the right-hand side is:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{du}{\sqrt{2\pi}} e^{i\Omega u} A(\bar{u}) = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \int_{\infty}^{\infty} \frac{du}{2\pi} \left(e^{i\Omega u - i\omega(\bar{u})} a_\omega + e^{i\Omega u + i\omega(\bar{u})} a^\dagger_\omega \right) = \\ = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \left(F(\omega, \Omega) a_\omega + F(-\omega, \Omega) a^\dagger_\omega \right), \end{aligned} \tag{6.27}$$

where $F(\omega, \Omega)$ is defined as follows:

$$F(\omega, \Omega) = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{i\Omega u - i\omega(\bar{u})} = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{i\Omega u - i\frac{\omega}{a} e^{-au}}, \tag{6.28}$$

where a stands for acceleration as defined in Table 6.1. Then finally the annihilation operator in the RS is as follows (using (6.27) and (6.26)):

$$b_\Omega = \int_0^\infty d\omega \frac{\sqrt{\Omega}}{\sqrt{\omega}} \left(F(\omega, \Omega) a_\omega + F(-\omega, \Omega) a^\dagger_\omega \right). \tag{6.29}$$

Then using (6.16) the Bogoliubov coefficients can be found (by pairing the relevant terms up) , which are as follows:

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega, \Omega), \tag{6.30}$$

and

$$\beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega, \Omega). \tag{6.31}$$

Then using (6.18) and the result that $F(\omega, \Omega) = F(-\omega, \Omega)e^{\frac{\pi\Omega}{a}}$, which can be achieved by a contour integration over the complex plane [31]. (6.18) yields the following:

$$\delta(\Omega - \Omega') = \int_0^\infty d\omega \sqrt{\frac{\Omega}{\omega}} \sqrt{\frac{\Omega'}{\omega}} F(\omega, \Omega) F^*(\omega, \Omega') - \sqrt{\frac{\Omega}{\omega}} \sqrt{\frac{\Omega'}{\omega}} F(-\omega, \Omega) F^*(-\omega, \Omega'), \tag{6.32}$$

then using the previously mentioned relation between $F(\omega, \Omega)$ and $F(-\omega, \Omega)$:

$$\delta(\Omega - \Omega') = \int_0^\infty \frac{d\omega \sqrt{\Omega\Omega'}}{\omega} \left(e^{\frac{\pi\Omega}{a}} F(-\omega, \Omega) e^{\frac{\pi\Omega'}{a}} F^*(-\omega, \Omega') - F(-\omega, \Omega) F^*(-\omega, \Omega') \right), \tag{6.33}$$

which simplifies down to:

$$\delta(\Omega - \Omega') = \left(e^{\frac{\pi(\Omega + \Omega')}{a}} - 1 \right) \int_0^\infty \frac{d\omega \sqrt{\Omega\Omega'}}{\omega} F(-\omega, \Omega) F^*(-\omega, \Omega'), \tag{6.34}$$

which can be rearranged and when setting $\Omega = \Omega'$ then:

$$\int_0^\infty \frac{d\omega\Omega}{\omega} |F(-\omega, \Omega)|^2 = \left(e^{\frac{2\pi(\Omega)}{a}} - 1 \right)^{-1} \delta(0) \tag{6.35}$$

With this result in mind, the calculation of the number of particles the Rindler particle detector detects in the vacuum (MS) can be calculated using the result from Section 6.2:

$$\langle 0_M | N_\Omega | 0_M \rangle = \langle 0_M | b_\Omega^\dagger b_\omega | 0_M \rangle = \int d\omega |\beta_{\omega\Omega}|^2, \tag{6.36}$$

Hence using (6.31) for the $\beta_{\omega\Omega}$ it yields (6.35) back. It means that:

$$\langle N_\Omega \rangle = \int d\omega |\beta_{\omega\Omega}|^2 = \int_0^\infty \frac{d\omega\Omega}{\omega} |F(-\omega, \Omega)|^2 = \left(e^{\frac{2\pi(\Omega)}{a}} - 1 \right)^{-1} \delta(0). \tag{6.37}$$

However, this result would cause a divergence as it is all the particles in the entire space. The delta function represents the part where the equation blows up to infinity. This happens because the integration took place over all spaces, meaning the N_Ω represents the number of particles over all spaces, which would indeed cause a divergence. Hence the delta function represents the **volume** of the entire space and the remaining term is just the number of particles density [31]. $\langle n_\Omega \rangle$ is as follows:

$$\langle n_\Omega \rangle = \left(e^{\frac{2\pi(\Omega)}{a}} - 1 \right)^{-1}, \tag{6.38}$$

which is nothing else, but the **Planck** distribution, it means that the Rindler particle detector will detect particles with (depending on the space it would cover) temperature, that is the **thermal bath** as mentioned before. Using the Bose-Einstein distribution for a massless particle (with $\Omega = E$:

$$n(E) = \frac{1}{e^{\frac{E}{T}} - 1}, \tag{6.39}$$

the temperature is as follows:

$$T = \frac{a}{2\pi}, \tag{6.40}$$

where a is the acceleration of the Rindler observer. However, after restoring the units a more elegant expression of T can be obtained, which can be done by considering the Planck distribution [32]:

$$u(f)df = \frac{8\pi h f^3}{c^3} \frac{1}{(e^{hf/k_B T} - 1)} df, \tag{6.41}$$

where f is the frequency, h is Planck's constant, c is the speed of light in vacuum, k_B is the Boltzmann constant and T is the temperature. After making the exponential term dimensionless with the speed of light in (6.38) (restoring the units from geometrized ones), the exponential terms in (6.38) and (6.41) can be paired up as both describe the Planck distribution:

$$\frac{hf}{k_B T} = \frac{2\pi\Omega}{a} = \frac{2\pi 2\pi f c}{a}, \tag{6.42}$$

rearranging for T gives the following:

$$T = \frac{\hbar a}{2\pi c k_B}. \tag{6.43}$$

Hence the thermal bath temperature the Rindler particle detector observes can be calculated by (6.43). It is worth noting that the Hawking effect (the temperature that is outside of the *black hole* horizon observer sees) is given by [26, 31]:

$$T = \frac{\hbar g}{2\pi c k_B}, \tag{6.44}$$

where g is the gravitational field strength. Hence the only difference between the two is the acceleration and the gravitational strength term. This is an extraordinary similarity, which gives another theoretical hint at the validity of the equivalence principle (EP). Hence, if the Unruh effect would be observed in a laboratory then the EP can have another validity test and it could potentially help us understand Hawking radiation through Unruh radiation, although there is much debate about whether Unruh radiation is a real phenomenon or not.

6.4 Unruh and Hawking radiation connection

Hawking effect states the same thing as the Unruh effect, however, there is another concept that arises from the Hawking effect, which is Hawking radiation. The observer outside of the black hole would see particles when there should be no particles present as the huge gravitational pull does not let anything escape and pulls everything to its epicentre. However, due to the Hawking effect, there are particles present and the temperature they have is given by (6.44). Previously it was thought that nothing can escape a black hole, however, this counter-intuitive result predicts Hawking radiation, which is that light can escape the horizon of the black hole and it has to get the energy from somewhere, which has to be the black hole. Hence black holes slowly evaporate due to them experiencing Hawking radiation. This phenomenon was observed recently [33] and hence the Hawking radiation is considered to be a real and observable result of mathematics.

However, Unruh radiation is yet to be observed and some of the scientists even claim that it is not a real thing, unlike Hawking radiation. As in the black hole's scenario, there is an emission of photons (because it gets it from the black hole), but whether an accelerated observer radiates the Unruh radiation is debatable. Furthermore, another difficulty would be observing

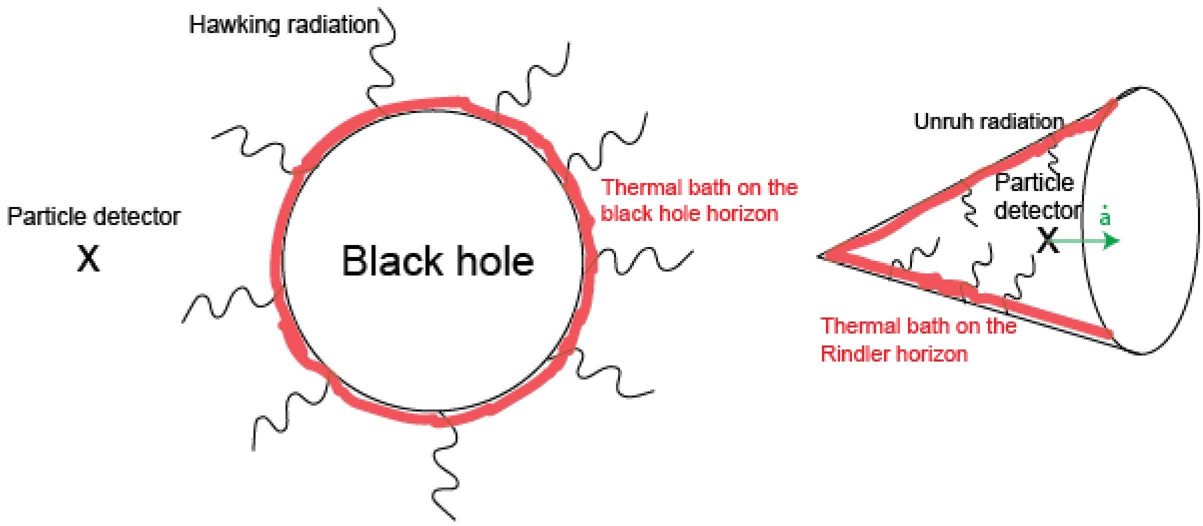

Figure 6.2: *Picture showing why the Hawking radiation is "easily" observable and the Unruh radiation is not. Unruh radiation would be observed in the accelerating particle detector's frame rather than the laboratory frame. Hawking radiation can be seen from the outside of the black hole, hence it was possible to observe it. [33]*

the particles as it is the accelerating observer that detects the particles and not the observer outside the accelerating frame. Hence, it is not so clear if the laboratory observer would see [34, 35, 36] the Unruh radiation or what exactly to look for when it arises for an accelerating particle detector. Unruh radiation is indeed an exciting research area nowadays as physicists are trying to construct the right experiment to measure it (observe it for the first time). One of the methods is by using charged particles, as there is an argument that the A.-L. radiation contains the Unruh radiation, therefore it could potentially be detected when it reaches ultrarelativistic speeds. The next section hence discusses the recent research on the Unruh effect and radiation and tries to summarize the relevant results and methods used.

6.5 Current research on the Unruh effect

The main interests of this section are relativistic quantum computing [43, 44], quantum heat engines [40, 41, 42], hydrodynamics as a way to understand highly theoretical concepts [39] and the possible realisation of the Unruh radiation using ultrarelativistic charged particles [37, 38]. There are myriads of research papers on the subject, however, only a couple were chosen - hence if one's interested in a certain area more deeply then refer to Physics Journals and websites containing the relevant research papers (such as https://arxiv.org/).

6.5.1 Measuring the Unruh radiation

Several of the recent papers [37, 38] argue that with the availability of highly powerful lasers charged particles such as electrons could be accelerated up to ultrarelativistic speeds. These ultrarelativistic particles under high acceleration could potentially be the key to realise the Unruh radiation because, as it was mentioned in Chapter 4, an accelerating charge radiates. Hence if a charge reaches certain speeds, it detects the Unruh radiation, which must get its energy from somewhere - this must be coming from the charge itself. This means that the A.-L. self-force formula requires a modification as there will be a *recoil* the charge experiences as it is giving power to the Unruh radiation. [37] argues that the A.-L. force in the laboratory's frame would look as follows:

$$F = \frac{2}{3}\frac{\hbar}{c^2}\frac{g^2}{4\pi}\left(1 - \eta\frac{k_B T}{mc^2}\right)\dot{a}. \tag{6.45}$$

Hence, given a powerful enough laser, the A.-L. force should be lower than is predicted by the relativistic formula in Chapter 4. However, such powerful enough lasers do not yet exist, hence the Unruh radiation is still something that remains a mystery.

Another interesting approach [38] argues that the *Sokolov-Ternov effect* could be used to measure the Unruh radiation. The Ternov effect is the self-polarization of electrons at relativistic speeds due to an emission of spin-flip radiation. However, it is still a theoretical work, which requires experimental proof.

6.5.2 Exploring curved spacetime using Hydrodynamics

Hawking's work on black hole radiation was thought to give rise to a new area of physics, however as it is not possible to replicate black hole dynamics in laboratory, scientists looked at alternatives to understand such phenomena. It was found that hydrodynamics uses the same physics that describes many classical and quantum phenomena [39]. Hence, many scientists try to replicate experiments using a certain variation of the Naiver-Stokes equation and trying to observe how it behaves under the *same* mathematics that would explain quantum effects in spacetime (for example, black hole dynamics on a quantum level). While this research does not bring our understanding towards the Unruh effect, blackhole dynamics or quantum gravity closer; they give useful insight into how mathematics works in such areas. As the same mathematics can describe a black hole as a certain case in hydrodynamics, it would mean that it could potentially give a solution that would give new information about how black holes work. However, this research area of physics could also hinder the proper research into such matters, as these usually do not come up with new mathematics or new ways of looking at the theory, rather they apply the already known theory to experiments with liquids.

6.5.3 Quantum Otto Engines

Recently it was theorized that the Unruh effect can be utilized to make a Quantum Otto Engine [40] to extract work. The system consists of an Unruh-DeWitt particle detector, which would generate a thermal bath of hot or cold heat based on the varying acceleration. This cold heat is also called the anti-Unruh effect [41]. Even though the theory works and it could be used as a heat engine to generate work, it is not practical as energy is required to accelerate the detector where it would generate heat. However, perhaps if new physics is found and achieving such acceleration would not be a problem, such a system seems a quite interesting application of the Unruh effect.

6.5.4 Relativistic Quantum Information Theory

Perhaps the most interesting application of the Unruh effect would be a relativistic quantum information theory and its application [43, 44]. Relativistic quantum information studies how quantum information is affected by relativistic motion or gravitational fields. Quantum computing is still in its early stages however, once complete, it could make computers even faster, enabling humans to build even more powerful frameworks. For example, ChatGPT is impressive, yet imagine how impressive it would be with quantum computers, where the speeds of the computer are a lot faster. However, quantum computing is still a long way to go, as we are still trying to figure out how quantum entanglement can be used to successfully transport information in computing. It is important to note that quantum entanglement is an observed phenomenon and it states that if two particles are entangled then a direct change on one of the particles would **instantly** affect the other particle. This would mean that information can travel faster than the speed of light, but it would not violate special relativity as quantum entanglement is a consequence of a *non-local* theory and, locally speaking, speed of light is still a hard limit. There is some success in sending information through quantum entanglement. It is still an area that is in development, hence it is clear how particle creation with relativistic quantum information would be an even harder concept to apply - especially if a more known concept's application has not yet been developed fully. Nevertheless, relativistic quantum computing would be another revolutionising breakthrough in science.

Chapter 7

Conclusion

7.1 Summary of content

Starting from Chapter 1 of this thesis, the relevance of the project and the methodology was introduced. It is clear now why the Unruh effect is one of the most important concepts in modern physics: it is one of the results of QFT in curved spacetime and properly understanding such theory could lead to the *Holy Grail* that is quantum gravity (as it is still a puzzle these days). From Chapters 2-5 the relevant theory was introduced, the language of modern physics; what is exactly required for the Unruh effect.

Chapter 2 focused on the introduction of GR concepts and mainly the mathematics behind it. It gave a brief introduction to tensor calculus and Einstein's equation of gravity, furthermore, it become clear how gravity does not need to be used due to the relativistic speeds *mocking* gravity up. Chapter 3 starts with one of the most important theories in Physics, which is Analytical Mechanics. By the end of the chapter, it was clear how the Lagrangian formulation of motion is essential in modern physics and also a brief introduction to special relativity with Lagrangian mechanics was discussed. The discussion on group theory was started here and it was outlined how in special relativity the Lorentz and the Poincaré groups were used to describe the transformation from one coordinate system to another. It was briefly mentioned how such transformation laws collapse under curved spacetime, where Bogoliubov transformations are used. Chapter 4 discussed classical electrodynamics and the formulation of the Electromagnetic tensor, but more importantly a discussion on the A.-L. force was shown. A.-L. radiation was of particular interest for ultrarelativistic charged particles as they could be used to realise Unruh radiation as it was discussed in a later chapter. Lastly, in Chapter 4 it was shown how an electron would move under an external electromagnetic field using the Lagrangian formulation. Hence, Chapters 2-4 discussed the classical theory of Physics, laying down the basics to make the necessary steps in the quantum world.

Chapter 5 started the discussion on Quantum field theory, mainly focusing on the development of the theory starting from the relativistic wave equation to the particle, anti-particle annihilation and creation operators. In this chapter, a brief introduction to mathematics and history was discussed, all of which are needed to understand the physics behind the Unruh effect. Furthermore, the formulation of the Hamiltonian equation of motion was introduced, which is labelled as almost identical to the Lagrangian equation of motion; however, sometimes it is easier to use the Hamiltonian formulation. As a last section in Chapter 5, the electrodynamics in QFT was briefly introduced as it is necessary to include such a theory in a thesis about ultra-relativistic charged particles. Even though the exact usage of such a theory was not discussed here, it goes without saying that the research papers discussed in Chapter 6 used such physics.

Finally, the last chapter of the thesis was about the Unruh effect. In this chapter, the derivation of the effect was shown in flat spacetime by relating the Minkowski and the Rindler observer through the Bogoliobuv transformation. It was also discussed how in curved spacetime the Poincaré group no longer works and it was mentioned how the Bogoliubov transformations lead to particle creation and entanglement. Furthermore, the discussion was started on the Hawking effect and radiation and how it is related to the Unruh effect and radiation. In the last section, the current research was summarized on the Unruh effect and it was discussed how the Unruh radiation could be realised and why is it hard to observe such phenomena.

7.2 Future outlook

This thesis hence summarized the important physics and development of the classical and quantum side of ultrarelativistic charged particles and the Unruh effect. Throughout the thesis, it was intended to introduce the concepts through mathematical proofs and consistency; hence even someone not educated on the topic can understand how the Unruh effect arises and why is it such an important concept.

Once Unruh radiation is realised in a laboratory, it is clear that it could give rise to new physics as it allows scientists to study black hole dynamics through the equivalence principle. As it was mentioned before, black holes could be the key to understanding how gravity behaves on a quantum level, or perhaps with more knowledge, experiments could be constructed to test theories such as string theory or closed-loop gravity. It is nevertheless one of the most exciting research areas in physics as the only component left to reach the *Theory of Everything* is to make gravity work on a quantum level. Furthermore, such theories could give rise to revolutionary applications such as in quantum computing, where perhaps an even smarter AI could be reached - perhaps it would be possible to model the brain of a human. The human brain is theorized to make 10^{18} calculations per second, which no computer is capable of in current times, hence it means if relativistic quantum computing would be a thing, then such a rate of calculations can be achieved and perhaps sentient AIs would come to existence.

This project could be expanded by further studying the quantum field theory of curved spacetime in detail and perhaps with the help of hydrodynamics it would be possible to better understand the maths and universal laws guiding such concepts. However, it would not lead to new physics, rather one could find the replica of mathematical equations in hydrodynamics for Unruh radiation. Such a system then could be understood more deeply, gaining a new perspective that would allow us to develop new physics.

Furthermore, it is clear that curvature was not deeply discussed and certainly, there are many other theories one can explore to gain a better insight into the matter of the Unruh effect or radiation. Black hole dynamics and the theory of Hawking radiation could be one direction that needs more exploring, but a deeper dive into GR would also be extremely important. In present times there are theories such as the *string theory*, which would also be an important field to explore as it could also give more insight into the concepts behind Unruh radiation.

Nevertheless, the most important goal is to find a way to realise the Unruh radiation. Such discovery would be a breakthrough in quantum field theory and it would bring humanity closer to finding a working quantum theory of general relativity. Once the *Holy Grail* is reached, it is then a matter of time until we will be able to answer life's biggest question: how we came into existence, and how the universe was created.

Chapter 8

Appendix

8.1 Proof of relation (3.11)

Recall relation (3.11):

$$\sum_{k=1}^{n} m_i \mathbf{a}_i \frac{\partial \mathbf{x}_i}{\partial q^k} = \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}^k} - \frac{\partial \mathcal{T}}{\partial q^k}. \tag{8.1}$$

Kinetic energy can be defined as follows:

$$\mathcal{T} = \sum_{i=1}^{n} \frac{1}{2} m_i \mathbf{v}_i^2. \tag{8.2}$$

Using the right-hand side of (8.1) the left-hand side can be reached [7]:

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{d}{dt}\frac{\partial}{\partial \dot{q}^k}\left(\frac{1}{2} m_i \mathbf{v}_i^2\right) - \sum_{i=1}^{n} \frac{\partial}{\partial q^k}\left(\frac{1}{2} m_i \mathbf{v}_i^2\right) = \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial \dot{q}^k}\right) - \sum_{i=0}^{n} m_i \mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial q^k} = \\
&= \sum_{i=1}^{n} m_i \frac{d\mathbf{v}_i}{dt}\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} + \sum_{i=1}^{n} m_i \mathbf{v}_i \left(\frac{d}{dt}\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k}\right) - \sum_{i=0}^{n} m_i \mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial q^k} = \sum_{i=1}^{n} m_i \mathbf{a}_i \frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} + \sum_{i=1}^{n} m_i \mathbf{v}_i \left(\frac{d}{dt}\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} - \frac{\partial \mathbf{v}_i}{\partial q^k}\right).
\end{aligned} \tag{8.3}$$

However, we know the following:

$$\mathbf{v}_i = \frac{\mathbf{x}_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q^j} \dot{q}^j, \tag{8.4}$$

which means that:

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} = \frac{\partial \mathbf{x}_i}{\partial q^k}. \tag{8.5}$$

Consequently, the proof is almost finished:

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}^k} - \frac{\partial \mathcal{T}}{\partial q^k} = \sum_{i=1}^{n} m_i \mathbf{a}_i \frac{\partial \mathbf{x}_i}{\partial q^k} + \sum_{i=1}^{n} m_i \mathbf{v}_i [\frac{d}{dt}\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} - \frac{\partial \mathbf{v}_i}{\partial q^k}]. \tag{8.6}$$

Finally, to finish the proof $\frac{d}{dt}\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} - \frac{\partial \mathbf{v}_i}{\partial q^k} = 0$ needs to be proved. Note that $\dot{q}^k = \frac{dq^k}{dt}$, hence:

$$\frac{d}{dt}\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} - \frac{\partial \mathbf{v}_i}{\partial q^k} = \frac{\partial \mathbf{v}_i}{\partial q^k} - \frac{\partial \mathbf{v}_i}{\partial q^k} = 0. \tag{8.7}$$

Thus, the proof is complete as we have reached the left-hand side from the right-hand side in (8.1):

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}^k} - \frac{\partial \mathcal{T}}{\partial q^k} = \sum_{k=1}^{n} m_i \mathbf{a}_i \frac{\partial \mathbf{x}_i}{\partial q^k}. \tag{8.8}$$

8.2 Derivation of (4.9)-(4.14)

To derive (4.9)-(4.14) the Lorentz transformation is applied to the electric and magnetic fields along the x direction. To find out the Lorentz transformed fields the surface charge density is used. The surface charge density transformed from a rest frame S_0 to a frame S is given by:

$$\sigma = \gamma_0 \sigma_0, \tag{8.9}$$

and if we are to transform to a frame $\bar{S}$, moving with high speeds, from S_0, then the surface charge density is given by:

$$\bar{\sigma} = \bar{\gamma}\bar{\sigma}_0 = \frac{\bar{\gamma}}{\gamma_0}\sigma. \tag{8.10}$$

The Lorentz factors can be further simplified as follows:

$$
\begin{aligned}
\frac{\bar{\gamma}}{\gamma_0} &= \frac{\sqrt{1-\frac{v_0^2}{c^2}}}{\sqrt{1-\frac{\bar{v}^2}{c^2}}} = \sqrt{\frac{1-\frac{v_0^2}{c^2}}{1-\frac{(v+v_0)^2}{\left(c+\frac{vv_0}{c}\right)^2}}} = \sqrt{\left(1-\frac{v_0^2}{c^2}\right)\frac{\left(c^2+2vv_0+\frac{v^2v_0^2}{c^2}-v^2-2vv_0-v_0^2\right)}{c^2+2vv_0+\frac{v^2v_0^2}{c^2}}} = \\
&= \sqrt{\left(1-\frac{v_0^2}{c^2}\right)\frac{\left(1+\frac{2vv_0}{c^2}+\frac{v^2v_0^2}{c^4}\right)}{\left(1-\frac{v^2}{c^2}-\frac{v_0^2}{c^2}+\frac{v^2v_0^2}{c^4}\right)}} = \sqrt{\left(1-\frac{v_0^2}{c^2}\right)\frac{\left(1+\frac{vv_0}{c^2}\right)^2}{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{v_0^2}{c^2}\right)}} = \gamma\left(1+\frac{vv_0}{c^2}\right), \qquad (8.11)
\end{aligned}
$$

where we used $\gamma = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$ and $\bar{v} = \frac{v+v_0}{1+\frac{vv+0}{c^2}}$.

Hence the transformed surface charge density can be expressed as:

$$\bar{\sigma} = \gamma\left(1+\frac{vv_0}{c^2}\right)\sigma. \qquad (8.12)$$

To define how the electric field and the magnetic field transform, first, it has to be noted that because of the right-hand rule $\mathbf{E}_z$ direction of the electric field is influenced by the $\mathbf{B}_y$ direction of the magnetic field and vice versa. Similarly the $\mathbf{E}_y$ influences $\mathbf{B}_z$ and so does $\mathbf{B}_z$ affect $\mathbf{E}_y$. Thus the transformations of the fields look as follows:

$$
\begin{array}{lll}
\bar{E}_x = E_x & \bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0} & \bar{E}_z = \frac{\bar{\sigma}}{\epsilon_0} \\
\bar{B}_x = B_x & \bar{B}_y = \mu_0\bar{\sigma}\bar{v} & \bar{B}_z = -\mu_0\bar{\sigma}\bar{v}
\end{array}
$$

Table 8.1: *Electric field and Magnetic field transformations under Lorentz transformation in x-direction.*

Furthermore, the **E** and the **B** in the rest frame are:

$$
\begin{array}{lll}
E_x = \frac{\sigma_x}{\epsilon_0} & E_y = \frac{\sigma_y}{\epsilon_0} & E_z = \frac{\sigma_z}{\epsilon_0} \\
B_x = \mu_0\sigma_x v_x & B_y = \mu_0\sigma_z v_z & B_z = -\mu_0\sigma_y v_y
\end{array}
$$

Table 8.2: *Electric field and Magnetic field in the rest frame.*

Finally, (3.26) and Table 8.2 can be used to simplify the values in Table 8.1 to get back the equations (4.9)-(4.14):

$$\bar{E}_y = \gamma(1+\frac{vv_0}{c^2})\frac{\sigma}{\epsilon_0} = \gamma(E_y - \frac{v}{c^2\epsilon_0\mu_0}B_z) = \gamma(E_y - vB_z), \qquad (8.13)$$

$$\bar{E}_z = \gamma(1+\frac{vv_0}{c^2})\frac{\sigma}{\epsilon_0} = \gamma(E_z + \frac{v}{c^2\epsilon_0\mu_0}B_y) = \gamma(E_z + vB_y), \qquad (8.14)$$

$$\bar{B}_y = \bar{\sigma} = \gamma(1 + \frac{vv_0}{c^2})\sigma\mu_0(\frac{(v + v_0)}{(1 + \frac{vv_0}{c^2})}) = \gamma(B_y + \frac{v}{c^2}E_z), \tag{8.15}$$

$$\bar{B}_z = \bar{\sigma} = \gamma(1 + \frac{vv_0}{c^2})\sigma\mu_0(\frac{(v + v_0)}{(1 + \frac{vv_0}{c^2})}) = \gamma(B_z - \frac{v}{c^2}E_y), \tag{8.16}$$

where (8.13) and (8.15) assumes that $\sigma = \sigma_y$, $v = v_y$, $\bar{v} = \bar{v}_y$, $v_0 = v_{0y}$ and (8.14) and (8.16) assumes that $\sigma = \sigma_z$, $v = v_z$, $\bar{v} = \bar{v}_z$, $v_0 = v_{0z}$. Hence (8.13)-(8.16) are the Lorentz-transformed electromagnetic field equations; equivalent to (4.9)-(4.14).

8.3 Solving $\mathcal{L}_d$ of (6.8)

Consider the $\mathcal{L}_d$ introduced in (6.8) [26]:

$$\mathcal{L}_d = \frac{1}{2}[-g(x)]^{\frac{1}{2}}\left(g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} - [m^2 + \xi R(x)]\phi^2(x)\right). \tag{8.17}$$

To find the solutions of the scalar field $\phi(x)$ one can write the action up and then minimize it:

$$S = \int d^n x \mathcal{L}_d, \tag{8.18}$$

where n is the dimension of space(time). Hence by setting the variation with respect to ϕ the solution can be found:

$$\frac{\delta\mathcal{L}_d}{\delta\phi} = \frac{\delta\left(\frac{1}{2}(-g(x))^{\frac{1}{2}}[g^{\mu\nu}(x)\phi_{,\mu}(x)\phi_{,\nu}(x)]\right)}{\delta\phi(x)} - \frac{\delta[\frac{1}{2}(-g(x))^{\frac{1}{2}}[[m^2 + \xi R(x)]]\phi^2(x)]}{\delta\phi(x)} = 0. \tag{8.19}$$

For the second term, the variation is quite simple as it is in terms of $\phi(x)$, for the first term however it will be different specifically the first term's variation is:

$$\frac{\delta[\frac{1}{2}(-g(x))^{\frac{1}{2}}[g^{\mu\nu}(x)\phi_{,\mu}(x)\phi_{,\nu}(x)]]}{\delta\phi(x)} = \left(\frac{1}{2}(-g(x))^{\frac{1}{2}}\partial_\mu[g^{\mu\nu}(x)\partial_\nu\phi(x)]\right) = -(-g(x))^{\frac{1}{2}}\Box\phi(x), \tag{8.20}$$

where the $\Box$ operator was used that is defined as $-(-g(x))^{\frac{-1}{2}}\frac{1}{2}(-g(x))^{\frac{1}{2}}[\partial_\mu g^{\mu\nu}(x)\partial_\nu]$ from [26]. The second term in (8.19) becomes as follows:

$$\frac{\delta[\frac{1}{2}(-g(x))^{\frac{1}{2}}[[m^2+\xi R(x)]]\phi^2(x)]}{\delta\phi(x)} = (-g(x))^{\frac{1}{2}}[[m^2+\xi R(x)]]\phi(x)]. \tag{8.21}$$

Consequently, putting (8.19), (8.20), and (8.21) together the following equation can be found:

$$[\Box + m^2 + \xi R(x)]\phi(x) = 0. \tag{8.22}$$

Using the fact that a scalar field $\phi(x)$ can be expanded in terms of modes $u_k(x)$ with the annihilation and creation operators (as introduced in Chapter 5), it must mean that the solution to $\phi(x)$ takes the following form:

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)], \tag{8.23}$$

which is exactly what was used as a result of (6.8) in Chapter 6.

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