G-Cluster States, and SPT Order

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Abstract

We study explicitly a generalization of the cluster state for some finite group G. Using an MPS representation, we study its symmetries and a stabilizing Hamiltonian. We show that the G-cluster state, which displays SPT order, is distinct from the (trivial) symmetric product state through a duality/gauging argument.

1 INTRODUCTION

We study a generalization of symmetry operators $\{O_a\}_a$ which no longer multiply according to the structure of a group, but which multiply with respect to the structure of a fusion category [FTA25]:

$$O_a O_b = \sum_c N^c_{a,b} O_c \tag{1}$$

We have replaced the conventional presentation of a symmetry operator U by O to emphasize the possibility of non-unitarity of the operators O. Note that the fusion of two objects equal a sum of objects, which ultimately leads to the existence of non-invertible objects- being symmetries in our case [FTA25].

Here, we construct an explicit microscopic model with fusion category SPTO in 1D, generalizing the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Abelian SPTO. Specifically, we replace the qubits by G-valued basis states. When G is non-Abelian the symmetry becomes $G \times \mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the fusion category of finite-dimensional representations of the group G [Bre15].

2 NOTATIONS AND CONVENTIONS

We follow the notations and conventions of Fechisin et. al. [FTA25]. Denote group elements as $g \in G$. and their inverses as $\bar{g} \in G$. Irreducible representations (irreps) are defined via the map $\Gamma : G \to GL_{d_\Gamma}(\mathbb{C})$, and respect the group structure of G. Denote the trivial irrep as $\mathbf{1}$. The tensor product of irreps satisfy:

$$\Gamma_i \otimes \Gamma_j \cong \bigoplus_k N^{\Gamma_k}_{\Gamma_i,\Gamma_j}\Gamma_k \tag{2}$$

We represent our states as a MPS [PGVWC07], written as:

$$|\psi_A\rangle = \sum_{\{g_i\}} Tr[BA^{(1)}_{g_1}\cdots A^{(N}_{g_N}]\,|g_1,\cdots,g_N\rangle \tag{3}$$

wherein A^i_g is the matrix evaluated at site i when $g_i = g$, and B encodes the choice of boundary conditions. Similarly, we write MPOs as:

$$O_a = \sum_{\{g_i,g'_i\}} Tr[BA^{(1)}_{g_1,g'_1}\cdots A^{(N}_{g_N,g'_N}]\,|g_1,\cdots,g_N\rangle\langle g'_1,\cdots,g'_N| \tag{4}$$

Both the MPSs and the MPOs can be represented diagrammatically using Penrose notation as 3-valent tensors and 4-valent tensors, respectively. Joining the ends of the string of either of these objects amounts to taking the trace. Gluing a bra or a ket to the left or the right, respectively, one can select a particular matrix element.

We label local Hilbert space basis kets by elements of the finite group G, and which follow the group algebra $\mathbb{C}[G]$. An arbitrary single-site state is written as a superposition of these basis kets as $|\psi\rangle = \sum_{g\in G} c_g\,|g\rangle$ with $c_g \in \mathbb{C}$. The generalization of qubit Pauli operators to group-based Pauli operators are used extensively. The group-based X-type operators are labeled by group elements $g \in G$:

$$\overrightarrow{X_g} = \sum_{h\in G} |gh\rangle\langle h| \iff \overrightarrow{X_g}\,|h\rangle = |gh\rangle \tag{5}$$

$$\overleftarrow{X_g} = \sum_{h\in G} |h\bar{g}\rangle\langle h| \iff \overleftarrow{X_g}\,|h\rangle = |h\bar{g}\rangle \tag{6}$$

The left-X-type and the right-X type can be combined into a conjugation operator $\overleftrightarrow{X}_g$. The group-based Z-type operators are labeled by irreps Γ:

$$Z_\Gamma = \sum_{g\in G} \Gamma(g)\otimes|g\rangle\langle g| \iff Z_\Gamma\,|g\rangle = \Gamma(g)\otimes|g\rangle \tag{7}$$

$$Z^\dagger_\Gamma = \sum_{g\in G} \Gamma(\bar{g})\otimes|g\rangle\langle g| \iff Z^\dagger_\Gamma\,|g\rangle = \Gamma(\bar{g})\otimes|g\rangle \tag{8}$$

The delta function on the group is given by:

$$\delta^G_{g,h} = \sum_{\Gamma\in Rep(G)} \frac{d_\Gamma}{|G|} Tr[\Gamma(\bar{g}h)] \tag{9}$$

Since in general G is non-Abelian, $\Gamma(g_i)$ and $\Gamma(g_j)$ do not necessarily commute which means we must be careful in the ordering of operators. The commutation relations follow:

$$[\overrightarrow{X_g^{(i)}}, \overleftarrow{X_h^{(j)}}] = 0 \tag{10}$$

$$[\overrightarrow{X}_g^{(i)}, \overrightarrow{X}_h^{(j)}] \propto [\overleftarrow{X}_g^{(i)}, \overleftarrow{X}_h^{(j)}] \propto \delta_{i,j} \tag{11}$$

$$\overrightarrow{X}_g Z_\Gamma = \Gamma(\bar{g}).Z_\Gamma \overrightarrow{X}_g \tag{12}$$

$$\overleftarrow{X}_g Z_\Gamma = Z_\Gamma.\Gamma(g)\overleftarrow{X}_g \tag{13}$$

Swapping of X- and Z-type operators on the same site yields a matrix $\Gamma(h)$ which acts on the virtual space. Similar to the case where we generalized the computational basis to G-valued basis states, we generalize the qubit basis $|+\rangle$ and $|-\rangle$ via basis states labeled by matrix elements of irreps:

$$|\Gamma_{\alpha\beta}\rangle \coloneqq \sqrt{\frac{d_\Gamma}{|G|}} \sum_{g\in G} [\Gamma(g)]_{\alpha\beta} |g\rangle \tag{14}$$

3 THE GENERALIZED CLUSTER STATE

For a finite group G, there does not exist a natural isomorphism between group elements $g \in G$ and irreps Γ of G [FTA25]. This complicates matters when generalizing the stabilizers, as the ZXZ-cluster state is not a CSS stabilizer code [FTA25]. This caveat is circumvented by using a Hadamard gate on even sites, such that we obtain a state called the CSS-cluster state.This can be seen from the fact that ZXZ-operators are either transformed into ZZZ- or XXX- operators. Alternatively, the CSS-cluster state may aslo be prepared by acting with a circuit of CX-gates on the product state $|+\rangle |0\rangle |+\rangle \cdots$. This method readily generalizes for the state preparation of the G-cluster state, where one should instead use $|\mathbf{1}\rangle |e\rangle |\mathbf{1}\cdots\rangle$, where $e \in G$ denotes the identity and $\mathbf{1} \in Rep(G)$ is the trivial irrep. In contrast to the qubit-based cluster state presented as a simple graph, the G-cluster state neccetates a bipartite and directed graph. The bipartiteness stems from the fact that CX -gates (and subsequent generalizations thereof) are not symmetric in their controls and targets, such that we need to assign a particular subset of the lattice as controls [FTA25]. The requirement for being directed is due to the fact that there exist different group-based CX-operators, being $C\overrightarrow{X}^{(i,j)}$ and $C\overleftarrow{X}^{(i,j)}$ in our case:

$$C\overrightarrow{X}^{(i,j)} |g_i, g_j\rangle \coloneqq |g_i, g_i g_j\rangle \quad C\overleftarrow{X}^{(i,j)} |g_i, g_j\rangle \coloneqq |g_i, g_j \bar{g}_i\rangle \tag{15}$$

Here, we study a particular group-based cluster state specified by choosing odd-sites as controls, with all edges oriented to the left, henceforth referred to as the G-cluster state. Defining a 1D chain with n sites, each of which have a local Hilbert space $\mathbb{C}[G]$, the G-cluster state reads:

$$|\mathcal{C}\rangle = \mathcal{N} \sum_{\{g_i\}\in G} |g_1\rangle |g_1\bar{g}_2\rangle |g_2\rangle \cdots |g_{N-1}\bar{g}_N\rangle |g_N\rangle \tag{16}$$

Alternatively, G-cluster state preparation may be implemented by acting on the product state $|\psi_0\rangle \coloneqq |\mathbf{1}\rangle\,|e\rangle\,|\mathbf{1}\rangle\cdots$ with the finite depth circuit $U_{\mathcal{C}}$ [FTA25]:

$$U_{\mathcal{C}} = \prod_{i \text{ odd}} C\overrightarrow{X}^{(i,i+1)} C\overleftarrow{X}^{(i,i-1)} \tag{17}$$

Regrouping of indices of the tensors defining the G-cluster state allows a graphical representation of the state. Furthermore, these single-site tensors respect a set of "pulling-through" identities, which allow transferring operators on the physical indices to the virtual indices [BGRS25].
A Hamiltonian with ground state $|\mathcal{C}\rangle$ can be constructed as a sum of commuting stabilizers:

$$H_{\mathcal{C}} = -\frac{1}{|G|}\sum_{i \text{ odd}}\Big(\sum_{\Gamma\in Rep(G)} Tr[Z_\Gamma^{\dagger(i)}.Z_\Gamma^{(i+1)}Z_\Gamma^{(i+2)}]d_\Gamma + \sum_{g\in G}\overleftarrow{X}_g^{(i+1)}\overrightarrow{X}_g^{(i+2)}\overrightarrow{X}_g^{(i+3)}\Big) \tag{18}$$

wherein the sum over $\Gamma \in Rep(G)$ denotes a sum over irreps Γ of G (i.e. simple objects in $Rep(G)$). The group-based Pauli terms all commute with one another, and the ground-state $|\mathcal{C}\rangle$ is the joint eigenspace of all the operators with maximum eigenvalue ($1/|G|$ for X-type terms, and $d_\Gamma^2/|G|$ for the Z-type terms. The cluster-state Hamiltonian has four independent families of global symmetries, which in turn are respected by the ground state $|\mathcal{C}\rangle$:

$$G_R : \overrightarrow{A}_g \coloneqq \prod_{i \text{ odd}} \overleftarrow{X}_g^{(i)}, \tag{19}$$

$$G_L : \overrightarrow{A}_g \coloneqq \prod_{i \text{ odd}} \overrightarrow{X}_g^{(i)}\overleftrightarrow{X}_g^{(i+1)}, \tag{20}$$

$$\text{Rep}(G) : \hat{B}_\Gamma \coloneqq Tr\Big[\prod_{i \text{ even}} Z_\Gamma^{(i)}\Big] \tag{21}$$

$$\text{Inn}(G) : \hat{C}_g \coloneqq \prod_{i \text{ even}}\Big(\overleftarrow{X}_{g'g\bar{g}'}^{(i)}\delta_{g'}^{(i+1)}\overrightarrow{X}_{g'g\bar{g}'}^{(i+2)}\Big) \tag{22}$$

We thus have: two G symmetries, a Rep(G) symmetry, and an Inn(G) symmetry corresponding to the inner automorphism group of G. For Abelian groups, the two G symmetries are identified and the Inn(G) trivializes. What remains are the two symmetries, $G \times$ Rep(G), that protect the SPTO. For the non-Abelian case, it is analytically and numerically proved that $G_R \times$ Rep(G) is the minimal subgroup also protecting the SPTO [FTA25]. Since $\overleftarrow{A}_g$ and $\hat{B}_\Gamma$ are supported on different sublattices, they commute. Whilst the $\overleftarrow{A}_g$ operators form a representation of a group, the $\hat{B}_\Gamma$ operators multiply according to the fusion rules of Rep(G):

$$\hat{B}_{\Gamma_i}\hat{B}_{\Gamma_j} = \sum_k N_{\Gamma_i,\Gamma_j}^{\Gamma_k}\hat{B}_{\Gamma_k} \tag{23}$$

We refer to this symmetry as a fusion category symmetry, whilst high-energy nomenclature refers to these as non-invertible symmetries. This stems from the fact that the Z_Γ operators in general do not have an inverse. This is again rooted in the fact that an object and its dual, yields a sum of objects including the trivial irrep, implying that the fusion to the trivial object is not the only fusion channel.
If G is Abelian, we have that $\mathrm{Rep}(G) \cong G$ such that the character group is isomorphic to G.
The fusion category symmetry $G \times \mathrm{Rep}(G)$ is labelled by pair (g, Γ) and multiply as:

$$(g_i, \Gamma_i) \ \otimes \ (g_j, \Gamma_j) = \bigoplus_k N^{\Gamma_k}_{\Gamma_i, \Gamma_j} \ (gh, \Gamma_k) \tag{24}$$

From which the $\overleftarrow{A}_g$ and $\hat{B}_\Gamma$ symmetry operators follow the same multiplication. Notice that the multiplication is non-commutative since $gh \neq hg$, regardless of the fact that $N^{\Gamma_k}_{\Gamma_i, \Gamma_j}$ is nonzero for multiple triples (i, j, k). This implies that $G \times \mathrm{Rep}(G)$ is not the fusion algebra of irreps of some other finite group G': $G \times \mathrm{Rep}(G) \neq \mathrm{Rep}(G')$. However, as long as the fusion category is nonanomolous, we can relate $G \times \mathrm{Rep}(G)$ to $\mathrm{Rep}(H)$ for some semisimple Hopf algebra H, such that: $G \times \mathrm{Rep}(G) \cong \mathrm{Rep}(H)$. This is satisfied for $H = \mathbb{C}[G]^* \otimes \mathbb{C}[G]$.

4 DISTINCTNESS FROM THE SYMMETRIC PRODUCT STATE: DUALITY ARGUMENT

We show that the G-cluster state is distinct from the trivial product state by mapping two distinct spontaneous symmetry breaking (SSB) states to the aforementioned states [FTA25]. In particular, the mapping is a duality, which is obtained by first gauging the G-symmetry present in the SSB models, and subsequently supplementing the gauging map with a disentangler so as to completely decouple the gauge d.o.f from the matter d.o.f.

4.1 Gauging G-symmetry

Starting from two models that are $G \times G$ symmetric:

$$H_{\mathrm{SSB1}} = -\frac{1}{|G|} \sum_{i \text{ odd}} \Big(\sum_{\Gamma \in \mathrm{Rep}(G)} d_\Gamma \mathrm{Tr}[Z_\Gamma^{(i+1)} \cdot Z_\Gamma^{\dagger(i+3)}] + \sum_{g \in G} \overrightarrow{X}_g^{(i)} \Big) = - \sum_{i \text{ odd}} \Big(\delta^G_{g_i+1, g_i+3} + \frac{1}{|G|} \sum_{g \in G} \overrightarrow{X}_g^{(i)} \Big) \tag{25}$$

$$H_{\mathrm{SSB2}} = -\frac{1}{|G|} \sum_{i \text{ odd}} \Big(\sum_{\Gamma \in \mathrm{Rep}(G)} d_\Gamma \mathrm{Tr}[Z_\Gamma^{\dagger(i)} \cdot Z_\Gamma^{(i+1)} \cdot Z_\Gamma^{\dagger(i+3)} \cdot Z_\Gamma^{(i+2)}] + \sum_{g \in G} \overrightarrow{X}_g^{(i)} \overrightarrow{X}_g^{(i+1)} \Big), \tag{26}$$

We notice that both of these models obey the symmetries:

$$G_R^{\text{odd}} : \Pi_{i \text{ odd}} \overleftarrow{X}_g^{(i)}, \quad G_R^{\text{even}} : \Pi_{i \text{ even}} \overleftarrow{X}_g^{(i)} \tag{27}$$

Since these symmetries act on different sublattices, they act independently such that the total symmetry is $G_R^{\text{odd}} \times G_R^{\text{even}}$. For both these Hamiltonians, the symmetry is spontaneously broken, such that the GS subspaces are given by:

$$H_{\text{SSB1}} : \{|\mathbf{1}, g, \mathbf{1}, g, ...\rangle \, |g \in G\} \tag{28}$$

$$H_{\text{SSB2}} : \{ \sum_{\{g_i\} \in G} |g_1, g_1 g, g_2, ...\rangle \, |g \in G\} \tag{29}$$

For each we have the SSB pattern $G \times G \to G$, although with different unbroken subgroups. Thus, the two models realize ground states of distinct gapped phases under $G_R^{\text{odd}} \times G_R^{\text{even}}$.

We now gauge the G_R^{even} symmetry following the prescriptions in [HVAS+15]. For further reference, we recommend reading the note on 'Gauging & Duality'. For this, we introduce 'gauge' d.o.f.s on the edges between sites, which we henceforth call 'matter'. Locally we enforce a Gauss law, which essentialy is a projector of the enlarged Hilbert space (conssiting of both matter and gauge), onto the locally gage invariant subspace [BGRS25]. This is of course done for all sites. Specifically, the projector reads:

$$P^{(i)} = \frac{1}{|G|} \sum_{g \in G} \overleftarrow{X}_g^{(i-1/2)} \otimes \overleftarrow{X}_g^{(i)} \otimes \overleftarrow{X}_g^{(i+1/2)} = \frac{1}{|G|} \sum_{g \in G} R_g^{(i-1/2)} \otimes R_g^{(i)} \otimes L_g^{(i+1/2)}, \tag{30}$$

such that the total projector reads $P = \bigotimes_{i \text{ even}} P^{(i)}$. Denote the left- and right regular representations of G by L_g and R_g, respectively. Notice that we have only applied the projector to the even sites, as else we would have also gauged G_R^{odd}.

4.2 Disentangling Matter from Gauge

Consider the disentangler:

$$\mathcal{U}^{(i)} := (cR)_{i,i-1/2} (cL)_{(i,i+1/2)}, \tag{31}$$

such that $\mathcal{U} = \bigotimes_{i \text{ even}} \mathcal{U}^{(i)}$.

One can check that the following identity holds:

$$\mathcal{U}^{(i)} \mathcal{G}_g^{(i)} = R_g^{(i)} \mathcal{U}^{(i)} \tag{32}$$

$$\implies \mathcal{U}^{(i)} \mathcal{G}^{(i)} = \sum_g R_g^{(i)} \mathcal{U}^{(i),} \tag{33}$$

wherein $\mathcal{G}_g^{(i)}$ is the gauging map on site i using only $g \in G$, and $\mathcal{G}^{(i)}$ applies the full projector operator eq. (30). The previous identity allows decoupling matter from gauge, since the matter is transformed into the trivial state $|\mathbf{1}\rangle$. One can now pull operators through this quantum circuit so as to derive the duality transformation. Practically, this is done by first identifying symmetries of the involved tensors. For example, we find that putting three Z_Γ operators on the legs of a R_g MPO, (one of which has a dagger), is the same as having no Z_Γ's at all. Similarly, one must check what identities one can derive when putting L_g's or R_g's on the legs. Finally, one also checks how one can pull these operatos through the controlled multiplication tensors $(cR)_{(i,i-1/2)}$ and $(cL)_{(i,i+1/2)}$. Ultimately, this leads to the duality transformation:

$$\overrightarrow{X}_g^{(i)} \to \overleftarrow{X}_g^{(i-1/2)} \overrightarrow{X}_g^{(i+1/2)}; \quad Z_\Gamma^{(i)} \cdot Z_\Gamma^{\dagger(i+1)} \to Z_\Gamma^{(i+1/2)} \tag{34}$$

Applying this to the SSB1 and SSB 2 Hamiltonians, we get:

$$H_{\mathrm{SSB1}} \to -\sum_{i \text{ odd}} \left(\sum_{\Gamma \in \mathrm{Rep}(G)} d_\Gamma \operatorname{Tr}\big[(Z_\Gamma^{(i+\frac{1}{2})})^\dagger\big] + \sum_{g \in G} \overrightarrow{X}_g^{(i)} \right), \tag{35}$$

$$H_{\mathrm{SSB2}} \to -\sum_{i \text{ odd}} \left(\sum_{\Gamma \in \mathrm{Rep}(G)} d_\Gamma \operatorname{Tr}\big[(Z_\Gamma^{(i-\frac{1}{2})})^\dagger Z_\Gamma^{(i+\frac{1}{2})} Z_\Gamma^{(i+\frac{3}{2})}\big] + \sum_{g \in G} \overleftarrow{X}_g^{(i-\frac{1}{2})} \overrightarrow{X}_g^{(i+\frac{1}{2})} \overrightarrow{X}_g^{(i+\frac{3}{2})} \right), \tag{36}$$

which we idenify with a trivial- and the G-cluster state Hamiltonian, respectively. (Here we mean trivial in the sense that it has a ground state the trivial product state $|\psi_0\rangle = |\mathbf{1}, e, \mathbf{1}, e, ...\rangle$, which was also our reference $G \times \mathrm{Rep}(G)$-symetric product state.)

4.3 DUALITY ARGUMENT

We have shown that under the duality transformation, consisting of gauging and disentangling, we get different resulting Hamiltonians with corresponding different ground states. Thus, due to the fact that the SSB1-phase and SSB2-phase have different symmetry breaking patterns, also the trivial product state and the G-cluster state belong to different phases. We say that two systems are in the same phase if their Hamiltonains can be connected via a smooth, gapped path of local, symmetric Hamiltonians [GRLM23]. We can furthermore argue that the G-cluster state is robust against weak symmetric perturbations, $\delta H_{G \times \mathrm{Rep}(fi)}$:

$$H'_{\mathcal{C}} = H_{\mathcal{C}} + \delta H_{G \times \mathrm{Rep}(G)}. \tag{37}$$

Since local symmetric operators are mapped to local symmetric operators [LDOV23](e.g a G-symmetric local operator for the SSB phases get mapped to $\mathrm{Rep}(G)$-symmetric local operators under the above duality transformation), applying the reverse of the duality defined above, we map $H'_{\mathcal{C}}$ to:

$$H'_{\mathrm{SSB2}} = H_{\mathrm{SSB2}} + \delta H_{G\times G} \tag{38}$$

Since we know that the SSB order H_{SSB2} is robust to local symmetric perturbations, ground states of H'_{SSB2} and H_{SSB2} lie in the same phase, such that alsi the ground states of $H'_{\mathcal{C}}$ and $H_{\mathcal{C}}$ lie in the same phase. In conclusion, the SPTO of $|\mathcal{C}\rangle$ is robust to local, symmetric perturbations, a fact that can also be confirmed numerically [FTA25].

5 SIGNATURES OF SPT ORDERS

5.1 Edge Modes and Ground State Degeneracy

Open SPT chains support edge modes that are robust against local symmetric perturbations. These edge modes are responsible for the ground state degeneracy which can be seen in the entanglement spectrum. Using the tensor network framework, this becomes explicit when pulling the symmetry MPOs from the physical- to the virtual legs of the MPS tensors. To illustrate this, we again consider the G-cluster state.

Consider the topological line $\overleftarrow{A}_g$ acting on $|\mathcal{C}\rangle$:

$$\overleftarrow{A}_g \left|\mathcal{C}\right\rangle = \left(\Pi_{i\ \mathrm{odd}} \overleftarrow{X}_g)^{(i)}\right)^{\mathrm{phys}} \left|\mathcal{C}\right\rangle \tag{39}$$

$$= (\overleftarrow{X_g^{\dagger}})_L^{\mathrm{virt}} (\overleftarrow{X_g})_R^{\mathrm{virt}} \left|\mathcal{C}\right\rangle, \tag{40}$$

where in the second line we have used the symetry of the tensors from which the G-cluster state is built to transfer the $\overleftarrow{X}_g$ from the physical- to its virtual legs. We see that in the bulk of the chain, we get the cancellation $\overleftarrow{X_g^{\dagger}}\overleftarrow{X_g} = \mathbb{I}_V$, where V is the virtual/bond Hilbert space. Importantly, at the left- and right ends of the open chain, there are the remaining operators $\overleftarrow{X_g^{\dagger}}$ and $\overleftarrow{X_g}$ acting on the virtual Hilbert spaces of the left- and right edges, respectively. Applying the same reasoning for the topological line $\hat{B}_\Gamma$, we get:

$$\hat{B}_\Gamma \left|\mathcal{C}\right\rangle = \mathrm{Tr}[\Pi_{i\ \mathrm{even}} Z_\Gamma^{(i)}]^{\mathrm{phys}} \left|\mathcal{C}\right\rangle \tag{41}$$

$$= (Z_\Gamma)_L^{\mathrm{virt}} (Z_\Gamma^{\dagger})_R^{\mathrm{virt}} \left|\mathcal{C}\right\rangle. \tag{42}$$

Diagrammatically, this is particularly enlightening. We can now read off the edge modes, which are $\overleftarrow{X_g}^{(L)}\overleftarrow{X_g}^{(R)}$ for the G-symmetry and $[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{ab}$ for the Rep(G)-symmetry, respectively. Furthermore, the edge modes do not commute:

$$\overleftarrow{X_g}^{(L)}[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{ab} = [Z_\Gamma^{(L)} \cdot \Gamma(g) \cdot Z_\Gamma^{\dagger(R)}]_{ab}\overleftarrow{X_g}^{(L)} \tag{43}$$

$$\overleftarrow{X_g}^{(R)}[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{ab} = [Z_\Gamma^{(L)} \cdot \Gamma(\bar{g}) \cdot Z_\Gamma^{\dagger(R)}]_{ab}\overleftarrow{X_g}^{(R)}, \tag{44}$$

that is, the modes fail to commute up to a matrix $\Gamma(g)$ or $\Gamma(\bar{g})$. Notice that the product $\overleftarrow{X_g}^{(L)}\overleftarrow{X_g}^{(R)}$ still commutes with $[Z_\Gamma^{(L)} \cdot Z_\Gamma^{\dagger(R)}]_{ab}$.

[TODO: derive the existence of edge modes from symmetry topological line and stabilizer Hamiltonian]

5.2 String Order Parameter

String Order Parameters (SOPs) can detect SPTOs [PGWS$^+$08]. We construct the SOP for the G-cluster state following the note 'SOPs and Interface Algebras' [LXY25]. We only consider the Rep(G)-symmetry of the model, which is given by the following topological line:

$$\hat{B}_\Gamma := \mathrm{Tr}[\Pi_{i \text{ even}} Z_\Gamma^{(i)}] \tag{45}$$

Consider the action of a single MPO tensor $Z_\Gamma^{(i)}$ acting on the physical leg of the even site i. We can pull the MPO through to the virtual legs of site i, using the symmetry of the G-cluster state MPS. We find that this translates to a Z_Γ and a $Z_\Gamma^\dagger$ acting on the left- and right virtual Hilbert spaces. We thus have as action tensors [IO24]:

$$\phi_\Gamma = Z_\Gamma \quad \bar{\phi}_\Gamma = Z_\Gamma^\dagger \tag{46}$$

Truncating the infinite topological line $\hat{B}_\Gamma$, and dressing it with the appropriate charge decoration operators, simply the action tensors in this case, yields:

$$(\mathcal{O}_\Gamma^{\text{string}})_{I,J} = \mathrm{Tr}[(Z_\Gamma)^{(I-1)}(\Pi_I^J Z_\Gamma^{(i)})(Z_\Gamma^\dagger)^{(J+1)}], \tag{47}$$

where I,J label even sites. Multiplying by an appropriate normalization factor and summing over all non-trivial irreps yields:

$$(\mathcal{O}^{\text{string}})_{I,J} = \sum_{\Gamma \neq \mathbf{1}} \frac{d_\Gamma}{|G|-1} \mathrm{Tr}[(Z_\Gamma)^{(I-1)}(\Pi_I^J Z_\Gamma^{(i)})(Z_\Gamma^\dagger)^{(J+1)}] \tag{48}$$

This SOP agrees with the SOP used in [FTA25]. There the SOP is derived by pushing the order parameter of the SSB2 phase through the duality circuit. To this end, we start by providing the order parameter for the SSB2 phase:

$${}^{SSB2}\mathcal{O} = \sum_{\Gamma \neq \mathbf{1}} \mathrm{Tr}([Z_\Gamma^\dagger)^{(I-1)} \cdot (Z_\Gamma^{(I)} \cdot (Z_\Gamma)^{(I)} \cdot (Z_\Gamma^\dagger)^{(J)} \cdot (Z_\Gamma)^{(J+1)}] \tag{49}$$

In the previous section, and in the note on 'Gauging and Duality', we described the duality circuit which maps $\overrightarrow{X}_g \to \overleftarrow{X}_g \overrightarrow{X}_g$ and $Z_\Gamma Z_\Gamma^\dagger \to Z_\Gamma$. Note that we can put $Z_\Gamma \cdot Z_\Gamma^\dagger$ on all the even sites. For the middel part, we use this last fact to pull a Z_Γ onto the gauge d.o.f. For the ends of the string, we similarly use the duality map to obtain a $Z_\Gamma^\dagger$ on the odd site on the left side of the string, and a Z_Γ on the odd site on the right end of the string. We then have the same SOP as given in eq.(48).

5.3 Topological Response

A hallmark of topological- and SPT orders is the ability to pump quantized symmetry charge upon insertion of symmetry flux. This is a generalization of the Thouless pump, which adiabatically evolves a a state so as to insert a symmetry flux, such that the state responds by condensing a symmetry charge. A notable example of topological response is the quantum Hall effect, which pumps $U(1)$ symmetry charge (in the form of electric charge) in response to insertion of $U(1)$ symmetry flux (in the form of magnetic field) -the topological response being the quantized Hall conductivity.

We present the topological response of the G-cluster state from two complementing viewpoints. First we show that by threading the G-cluster state with a g-flux ($g \in G$), which can be seen as inserting a twisted boundary condition, we get a non-trivial response when acting with the $\mathrm{Rep}(G)$-symmetry. Similarly, we can thread a Γ_{ab}-flux ($\Gamma \in \mathrm{Rep}(G); a, b = 1, ..., \dim(\Gamma)$), and work out the associated charge attachment. Next, we will discuss the generalized Thouless pump, and see how to formulate this for the G-cluster state.

We start off with the open G-cluster state and put it on a ring. Trivially identifying the ends of the open chain is the same as imposing periodic boundary conditions. Threading a flux at a loose end before joing the ends of the open chain, is equivalent to imposing twisted boundary conditions. One can think about this twist as inserting a domain wall between the first and the last sites. Let us start of by threading a g flux through $|\mathcal{C}\rangle$, resulting in:

$$|\mathcal{C}_g\rangle \coloneqq \mathcal{N} \sum_{\{g_i\}} |g_1\rangle\, |g_1\bar{g}_2\rangle\, |g_2\rangle \cdots |g_{N-1} g \bar{g}_1\rangle\,, \tag{50}$$

which is obtained by inserting $\overleftarrow{X}_g$ on the right virtual leg of the N'th site of $|\mathcal{C}\rangle$. We denote $|\mathcal{C}_g\rangle$ is the g-twisted G-cluster state, and specifically $|\mathcal{C}\rangle$ is the G-cluster state with an e-twist, i.e. the trivial twist, equivalent to imposing periodic boundary conditions. We now want to know what happens when we apply the $\mathrm{Rep}(G)$-symmetry to $|\mathcal{C}_g\rangle$:

$$\hat{B}_\Gamma\, |\mathcal{C}_g\rangle = \mathrm{Tr}[\Pi_{i \text{ even}} Z_\Gamma^{(i)}]\, |\mathcal{C}_g\rangle = \mathrm{Tr}[\Gamma(g)]\, |\mathcal{C}_g\rangle\,. \tag{51}$$

We observe that $|\mathcal{C}_g\rangle$ has a non-trivial eigenvalue (when $g \neq e$), when acting on it with $\hat{B}_\Gamma$: the response to insertion of g-flux is given by the character of g in Γ. We note that the character is an invariant of the conjugacy class, implying that group elements g belonging to the same conjugacy class will induce the same topological response. We say that $|\mathcal{C}_g\rangle$ is non-trivially charged under $\mathrm{Rep}(G)$ when $|\mathcal{C}_g\rangle$ has a eigenvalue different than 1 when acting with $\hat{B}_\Gamma$ on $|\mathcal{C}_g\rangle$. As a final note, we observe that $|\mathcal{C}_g\rangle$ is an eigenstate of $\hat{B}_\Gamma$, i.e. it transforms as a one-dimensional irrep of the fusion category $\mathrm{Rep}(G)$. In general, states which are charged under the symmetry will transform as some irrep of the symmetry (REMARK: what about extensions/cases beyound groups and their representations?). This is a fact intimately related to Tannaka-Krein duality, which we hope to explore more about later.

What hapens when threading a Γ_{ab}-flux through $|\mathcal{C}\rangle$? We get the state:

$$|\mathcal{C}_{\Gamma_{ab}}\rangle \coloneqq \sum_{\{g_i\}} [\Gamma(g_1)]_{ab} |g_1\rangle |g_1\bar{g}_2\rangle \cdots |g_{N-1}\bar{g}_1\rangle , \tag{52}$$

which corresponds to adding the Z_Γ MPO on the virtual leg joining the first and last sites when gluing the loose ends together, and subsequently terminating the remaining free legs with bra and ket $\langle a|$ and $|b\rangle$, respectively. We now act with the G-symmetry on $|\mathcal{C}_{\Gamma_{ab}}\rangle$:

$$\overleftarrow{A}_g |\mathcal{C}_{\Gamma_{ab}}\rangle = [\Gamma(g_1 g)]_{ab} |g_1\rangle |g_1\bar{g}_2\rangle \cdots |g_{N-1}\bar{g}_1\rangle \mathcal{N} \sum_{\{g_i\}} \tag{53}$$

$$= \sum_c [\Gamma(g)]_{cb} |\mathcal{C}_{\Gamma_{ac}}\rangle , \tag{54}$$

such that for a given a, the states $|\mathcal{C}_{\Gamma_{ab}}\rangle$ (with $b = 1, ..., \dim(\Gamma)$), transform under the irrp Γ when acting upon the states with $\overleftarrow{A}_g$. Since also $a = 1, ..., \dim(\Gamma)$, the states $|\mathcal{C}_{\Gamma_{ab}}\rangle$ transform as the direct sum of $\dim(\Gamma)$ copies of Γ, which we denote as $\dim(\Gamma) \cdot \Gamma$. This is precisely the nontrivial topological response when threading the G-cluster state with $\mathrm{Rep}(G)$-flux.

We note that the trivial product state $|\psi_0\rangle$ is unaltered upon insertion of symetry flux. This is a direct consequence of the fact that it is the ground state of a Hamiltonian consisiting of only single-site stabilizers. This fact means that there are no terms to modify upon symmetry flux insertions between sites. Put differently, because it is a product state, adding a symmetry twist between to sites, one can simply pull the twist to one of the adjacent sites, where it acts trivially due to the fact that it is symmetric with respect to the state. Obviously, the product state can be seen to have no symmetry charge response to symmetry flux insertions, such that we can henceforth distinguish SPTs from trivial product states.

We now move on to the discussion regarding the generalization of the Thouless pump. PUT THIS ON HOLD FOR NOW

A MULTIPLICATION OF MPOs

Here we explicitly multiply the MPOs of the $\mathrm{Rep}(G)$ symmetry. First we introduce the notation for the fusion of two irreps of the group G, which we refer to as generalized Clebsch-Gordan coëfficients. This is presented by a tuning fork type diagram with two incoming legs and one outgoing leg [insert a diagram], which corresponds to the generalized Clebsch-Gordan coefficiënt:

$$\begin{pmatrix} \Gamma_i & \Gamma_j & | & \Gamma_k & \mu \\ \alpha & \beta & & \gamma & \end{pmatrix} \tag{55}$$

As a reminder we also write out the explicit form of the Z_Γ MPO [insert Z

MPO here]:

$$alpha - MPO(Z_\Gamma) - beta = \sum_{g \in G} [\Gamma(g)]_{\alpha\beta} \left| g \right\rangle \left\langle g \right| \tag{56}$$

This allows the multiplication of two Z_Γ's, which is represented by so called 'pulling-through'- or 'zipper' diagrams [insert pulling-through/zipper diagram here]:

Explicitly, by choosing basis vectors we can write:

$$\sum_{\epsilon,\delta} [Z_{\Gamma_i}]_{\epsilon\alpha} [Z_{\Gamma_j}]_{\delta\beta} \begin{pmatrix} \Gamma_i & \Gamma_j & | & \Gamma_k & \mu \\ \epsilon & \delta & & \gamma & \end{pmatrix} = \sum_{\kappa} \sum_{\mu} [Z_{\Gamma_k}]_{\gamma\kappa} \begin{pmatrix} \Gamma_i & \Gamma_j & | & \Gamma_k & \mu \\ \alpha & \beta & & \kappa & \end{pmatrix} \tag{57}$$

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