Group-based SPTO

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Abstract

In this note we derive the classification of (bosonic) phases protected by (conventional) group-based symmetries. We find that phases are classified by the data $(G_u \leq G, [\alpha])$, where G_u is the unbroken symmetry group of the full symmetry G, and $[\alpha] \in \mathrm{H}^2(G_u, U(1))$ denotes an equivalence class of 2-cocycles [BGRS25][SPGC11] [LDOV23]. The broken symmetry group $G_b = G/G_u$ permutes between ground states, which in the MPS formalism translates to permuting normal blocks [BGRS25]. Each of these blocks has global G_u-symmetry, such that the more refined classification uses the well-known second-cohomology data labeling equivalence classes of different unitary projective representation which act on the virtual spaces of the MPS. This extends the Landua symmetry breaking theory of phases, which only uses the first piece of data included in $(G_u \leq G, [\alpha])$ [CGLW13].

1 Background

We begin by introducing Matrix Product States (MPS), which are constructed from three-valent tensors $A : \mathcal{V} \to \mathcal{H} \otimes \mathcal{V}$ of size $d \times D \times D$, and diagrammatically represented as a node with one physical leg and two virtual legs [CPGSV21]. d is the physical dimension, and D is the virtual/bond dimension. Without loss of generality, we assume the MPS to be translationally invariant [LDOV23]. We denote the virtual Hilbert spaces by $\mathcal{V} \cong \mathbb{C}^D$, and the physical Hilbert spaces by $\mathcal{H} \cong \mathbb{C}^d$. We call the MPS injective when it is injective as a map [BGRS25]:

$$A : \mathcal{V}_l^* \otimes \mathcal{V}_r \to H, \tag{1}$$

or equivalently that the matrices $\{A^i | i = 1, ..., d\} \in \mathcal{M}_d$ span $\mathcal{M}_D$ [BGRS25], where $\mathcal{M}_N$ is the space of $N \times N$-dimensional matrices. We call an MPS tensor A normal if $\exists k \in \mathbb{N}$, such that the blocked MPS $A^{\otimes k} \in \mathcal{V} \otimes \mathcal{H}^{\otimes k} \otimes \mathcal{V}$, is injective [BGRS25]. This process is referred to as blocking, and for $k < k'$ we have that also $A^{\otimes k'}$ is injective [BGRS25]. When putting an arbitrary tensor X on the boundary of the MPS state before contracting the two outermost legs, we say that the PMS state has arbitrary boundary contions (ABCs). When X is the trivial tensor $\mathbb{I}$, we say that the MPS state has periodic boundary conditions (PBCs) [LDOV23]. We consider MPS which are block-injective: $A^l = \bigoplus_a A_a^l$, where A_a^l is are $D_a \times D_a$ matrices and $a \in \{x, y, z, ...\}$ denotes the block labels. Henceforth, we assume that we are dealing with block-injective MPSs with orthogonal blocks, which can be achieved upon blocking a small number of sites [LDOV23]. Importantly, we can associate a gapped, local, and frustration-free Hamiltonian $\mathbb{H}_A$ to any block-injective MPS A [LDOV23] [SPGC11]. The ground space of the PBC Hamiltonian is spanned by the individual blocks $a = x, y, z, ...$ of the MPS: $|\psi_a^{A,\mathbb{I}}\rangle$, such that the ground state (GS) degeneracy equals the number of blocks [LDOV23]. We write an MPS with bulk tensors A and boundary X as :

$$|\psi_n^{A,X}\rangle = \sum_{\{l_i\}} \mathrm{Tr}[XA^{l_1}A^{l_2}\cdots A^{l_n}]\,|l_1, l_2, ..., l_n\rangle \tag{2}$$

Next, we introduce Matrix Product Operators (MPOs), which are constructed from four-valent tensors $T : \mathcal{V} \otimes \mathcal{H}_i \to \mathcal{H}_o \otimes \mathcal{V}$ of size $d \times d \times \chi \times \chi$, where $d = \dim(\mathcal{H})$ is the physical dimension and $\chi = \dim(\mathcal{W})$ is the virtual dimension [CPGSV21]. We write the MPO with arbitrary boundary condition B and bulk tensors T, as:

$$\mathcal{O}_n^{T,B} = \sum_{\{l_i,p_j\}} \mathrm{Tr}[BT^{l_1p_1}T^{l_2p_2}\cdots T^{l_np_n}]\,|l_1, l_2, ..., l_n\rangle\,\langle p_1, p_2, ..., p_n|\,. \tag{3}$$

We furthermore consider MPOs that form an algebra satisfying the *closedness condition*: $\forall X, Y \in \mathcal{M}_\chi : \exists Z \in \mathcal{M}_\chi s.t. \forall n \in \mathbb{N} : \mathcal{O}_n^{T,X} \cdot \mathcal{O}_n^{T,Y} = \mathcal{O}_n^{T,Z}$ [LDOV23]. The closedness condition implies the existence of fusion- and splitting tensors (denoted as $\phi_{ab}^{c,\mu}$ and $\bar{\phi}_{ab}^{c,\mu}$,respectively), determined up to a gauge transformation, and satisfying orthogonality criteria: $\phi_{ab}^{c,\mu}\bar{\phi}_{ab}^{d,\nu} = \delta_{c,d}\delta_{\mu,\nu}\mathbb{I}_{\chi_c}$ [LDOV23]. Associativity of the fusion of three objects is encoded in so-called F-symbols, which satisfy *pentagon equations*. Also the F-symbols are defined up to gauge transformations, stemming from the gauge transformations of the fusion- and splitting tensors, which lead to equivalent F-symbols [LDOV23].

Finally, we note on the action of MPOs on MPSs. Specifically, we consider MPSs symmetric under a set of MPOs. Acting with an MPO $\mathcal{O}^{T,B}$ on the MPS $|\psi^{A,X}\rangle$ yields $|\psi^{A,Y}\rangle$. This global condition implies the existence of three-valent tensors, called *action tensors* (denoted by ${}^A\phi_{\alpha a}^{b,i}$ where α labels the MPO blocks, a, b label MPS blocks, and $i = 1, ..., M_{\alpha a}^b$ is a multiplicity label), which enocde the action of the MPO on the MPS [LDOV23][LXY25]. We will emit the MPS label A from the action tensors when it is clear from the context. One can also regard the action tensors as the map $\phi_{\alpha a}^{b,i} : \mathcal{W}_a \otimes \mathcal{V}_x \to \mathcal{V}_y$ so that $\sum_l T_\alpha^{ml} A_x^l = \sum_{y,i} \bar{\phi}_{\alpha a}^{b,i} A_y^m \phi_{\alpha a}^{b,i}$. These should be thought of as objects which function similar to Clebsch-Gordan coefficients. Also the action tensors satisfy a orthogonality relations: $\phi_{\alpha a}^{b,i}\bar{\phi}_{\alpha a}^{c,i} = \delta_{b,c}\delta_{i,j}\mathbb{I}_{D_b}$. The action tensors are again defined up to a gauge transformation. Associotivity of applying two MPOs sequentially, or rather first fusing the MPOs first before apllying it to the MPS, defines so-called L-symbols [LDOV23][LXY25]. These are most easily understood using the graphical presentation, which we postpone until we can create these. Importantly, the L-symbols need to satisfy a *coupled pentagon equation*, involving both L- and F-symbols. Furthermore, the L-symbols are also equivalent up to a gauge transformation stemming from the gauge freedom of the action tensors, such that one must consider equilence classes of L-symbols. It is also shown in [LDOV23] that an injective MPS cannot be invariant undera non-trivial MPO, implying that a symmetric Hamiltonian cannot have a unique non-degenrate gapped ground state.

2 SSB and SPT for group-based symmetries

We proceed to classify boconic SPT phases symmetric under group-based symmetries. We say that two systems are in the same phase if the states, and by extension also the (parent) Hamiltonians, can be connected by a smooth path of local, gapped, and symmetric Hamiltonians [BGRS25][LDOV23][CPGSV21]. Equivalently, two systems are in the same phase if they can be connected by symmetry-preserving local unitary (LU) transformations, all the while the gap remains open[CGLW13]. The LU evolution can be realised by a finite depth quantum circuit [CGLW13]. For the one-dimensional case covered here, a complete classification of G-symmetric spin chains are completely classified by the data $(G_u \leq G, [\alpha] \in \mathrm{H}^2(G_u, U(1)))$ [CGW11][Oga21]. The *unbroken subgroup* G_u relates to SSB, yielding the degeneracy of the GS through the order of the *broken symmetry group* $G_b \coloneqq G/G_u$, that is: $|G_b|$. The second-cohomology class $[\alpha] \in \mathrm{H}^2(G_u, U(1))$ discriminates between models of equal GS-degeneracy, and it is precisely this data that represents the SPT character of the phase [BGRS25].

In the absence of symmeties, any twp MPS states with equal GS-degeneracy are in the same phase, the canonical example being product and GHZ-like states[CPGSV21]. Thus the only data classifying these cases are the GS-degeneracy. When symmetries are present, we will also have to specify extra data labelling the second cohomology class. Essentially, the latter will tell us about the reaization of the symmetry at the virtual level.

We start off by specifying the (PBC) MPOs realizing the group-based symmetry. The blocks of the MPO label group elements $g \in G$. Multiplication of MPOs follows $\mathcal{O}_g \cdot \mathcal{O}_h = \mathcal{O}_{gh}$, and the trivial group element $e \in G$ satisfies: $\mathcal{O}_e \cdot \mathcal{O}_g = \mathcal{O}_g \cdot \mathcal{O}_e = \mathcal{O}_g$ and $\mathcal{O}_{\bar{g}} \cdot \mathcal{O}_g = \mathcal{O}_g \cdot \mathcal{O}_{\bar{g}} = \mathcal{O}_e$. Note that, with the exception of the on-site symmetry case: $\mathcal{O}_g = (u_g)^{\otimes n} \quad \forall g \in G$ (such that blocks are one-diensional), $\mathcal{O}_e$ is not neccessarily the identity, but rather a projector onto the releveant PBC subspace [LDOV23]. The F-symbols have no multiplicity indices since group multiplication yields a unique product, and these symbols evaluate to 3-cocycles $\omega(g, h, k)$. The pentagon equations become the 3-cocycle condition:

$$\omega(g, h, k)\omega(g, hk, l)\omega(h, k, l) = \omega(gh, k, l)\omega(g, h, kl) \tag{4}$$

The fusion- and splitting tensors, again void of any multiplicity indices, read:

$$\phi^k_{gh} = \delta_{gh,k}; \quad \bar{\phi}^k_{gh} = \delta_{k,gh}. \tag{5}$$

As mentioned in section 1, the fusion- and splitting tensors have gauge freedom, labeled by a 3-coboundary β, such that we have an equivalence relation on 3-cocycles:

$$\omega'(g,h,k) \sim \omega(g,h,k)\frac{\beta_{h,k}\beta_{g,hk}}{\beta_{g,h}\beta_{gh,k}}. \tag{6}$$

The equivalence classes of 3-cocycles define the third-cohomology group $\mathrm{H}^3(G,U(1))$ [LDOV23]. Similarly, we write the L-symbols as L^a_{gh} instead of the full $(L^k_{gh})^{n,\mu}_{lm}$ since the group-elements are one-dimensional, we have no multiplicities, and k is uniquely determined: $k = gh$. The coupled pentagon equations reduce to:

$$L^a_{gh}\cdot L^a_{g,hk} = \omega(g,h,k)\cdot L^a_{g,h}. \tag{7}$$

As mentioned in section 1, the L-symbols also have gauge freedom stemming from the gauge freedom of the action tensors (labelled by γ) and fusion-/splitting tensors (labelled by β), such that we have the equivalence relation:

$$L^a_{gh} \sim \frac{\gamma_{g,ha}\cdot\gamma_{h,a}}{\gamma_{gh,a}\cdot\beta_{g,h}}\cdot L^a_{gh}. \tag{8}$$

We now proceed to answer the question: given a group G and a representative 3-cocycle, how do we classify solutions of eq.(7)? The answer will turn out to be that the different classes of solutions are encoded by the data $(G_u, [\alpha] \in \mathrm{H}^2(G_u, U(1)))$ [LDOV23]. The different solutions should be interpreted as defining different SPTOs. Let G_u denote the biggest subgroup of G that does not permute the blocks. We then have that the number of MPS blocks is $|G/G_u|$. In [LDOV23] it is shown that unbroken subgroup G_u trivializes the representative 3-cocyle $[\omega] \in \mathrm{H}^3(G_u, U(1))$. This fact imposes constraints on G_u, and thus also on the GS-degeneracy. When G has a non-trivial 3-cocyle, $G_u < G$ i.e. we cannot find a single-block MPS that is also invariant under the MPO defining the symmetry.

The fact that G_u trivializes the 3-cocycle $[\omega]$ implies that there exists a gauge for the fusion tensors such that for $u_1, u_2, u_3 \in G_u$ [LDOV23]:

$$\omega(u_1,u_2,u_3)\cdot\beta_{u_1,u_2}\cdot\beta_{u_1u_2,u_3}\cdot\bar{\beta}_{u_2,u_3}\bar{\beta}_{u_1,u_2u_3} = 1, \tag{9}$$

which in turn also modifies the L-symbols in eq.(7):

$$\frac{L^a_{u_1,u_2}}{\beta_{u_1,u_2}}\cdot\frac{L^a_{u_1u_2,u_3}}{\beta_{u_1u_2,u_3}} = \frac{L^a_{u_2,u_3}}{\beta_{u_2,u_3}}\cdot\frac{L^a_{u_1,u_2u_3}}{\beta_{u_1,u_2u_3}}, \tag{10}$$

which yields the 2-cocylce condition upon defining $\psi^a(u_i,u_j) = L_{u_i,u_j}/\beta_{u_i,u_j}$. Further fixing a choice of gauge on the action tensors modifies the restricted L-symbols from above, such that ψ^a belongs to the second cohomology class

$\mathrm{H}^2(G_u, U(1))$. In summary: given that the unbroken subgroup G_u trivializes ω, the different solutions are classified by $\mathrm{H}^2(G_u, U(1))$. Upon specification of G_u and ψ (given G and ω), all the L-symbols can be computed [LDOV23].

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