# G-Cluster States, and SPT Order

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**Abstract**

We study explicitly a generalization of the cluster state for some finite group $G$. Using an MPS representation, we study its symmetries and a stabilizing Hamiltonian. We show that the G-cluster state, which displays SPT order, is distinct from the (trivial) symmetric product state through a duality/gauging argument.

## 1 INTRODUCTION

We study a generalization of symmetry operators $\{O_a\}_a$ which no longer multiply according to the structure of a group, but which multiply with respect to the structure of a fusion category:

$$O_a O_b = \sum_c N_{a,b}^c O_c \tag{1}$$

We have replaced the conventional presentation of a symmetry operator $U$ by $O$ to emphasize the possibility of non-unitarity of the operators $O$. Note that the fusion of two objects equal a sum of objects, which ultimately leads to the existence of non-invertible objects- being symmetries in our case.

Here, we construct an explicit microscopic model with fusion category SPTO in 1D, generalizing the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Abelian SPTO. Specifically, we replace the qubits by $G$-valued basis states. When G is non-Abelian the symmetry becomes $G \times \mathrm{Rep}(G)$, where $\mathrm{Rep}(G)$ is the fusion category of finite-dimensional representations of the group $G$.

## 2 NOTATIONS AND CONVENTIONS

We follow the notations and conventions of Fechisin et. al.. Denote group elements as $g \in G$. and their inverses as $\bar{g} \in G$. Irreducible representations (irreps) are defined via the map $\Gamma : G \to GL_{d_\Gamma}(\mathbb{C})$, and respect the group structure of $G$. Denote the trivial irrep as $\mathbf{1}$. The tensor product of irreps satisfy:

$$\Gamma_i \otimes \Gamma_j \cong \bigoplus_k N^{\Gamma_k}_{\Gamma_i,\Gamma_j} \Gamma_k \tag{2}$$

We represent our states as a MPS, written as:

$$|\psi_A\rangle = \sum_{\{g_i\}} Tr[BA^{(1)}_{g_1} \cdots A^{(N)}_{g_N}] \, |g_1, \cdots, g_N\rangle \tag{3}$$

wherein $A^i_g$ is the matrix evaluated at site $i$ when $g_i = g$, and $B$ encodes the choice of boundary conditions. Similarly, we write MPOs as:

$$O_a = \sum_{\{g_i, g_i'\}} Tr[BA^{(1)}_{g_1,g_1'} \cdots A^{(N}_{g_N,g_N'}] \, |g_1, \cdots, g_N\rangle \langle g_1', \cdots, g_N'| \tag{4}$$

Both the MPSs and the MPOs can be represented diagrammatically using Penrose notation as 3-valent tensors and 4-valent tensors, respectively. Joining the ends of the string of either of these objects amounts to taking the trace. Gluing a bra or a ket to the left or the right, respectively, one can select a particular matrix element.

We label local Hilbert space basis kets by elements of the finite group $G$, and which follow the group algebra $\mathbb{C}[G]$. An arbitrary single-site state is written as a superposition of these basis kets as $|\psi\rangle = \sum_{g\in G} c_g |g\rangle$ with $c_g \in \mathbb{C}$. The generalization of qubit Pauli operators to group-based Pauli operators are used extensively. The group-based $X$-type operators are labeled by group elements $g \in G$:

$$\overrightarrow{X_g} = \sum_{h\in G} |gh\rangle \langle h| \iff \overrightarrow{X_g} |h\rangle = |gh\rangle \tag{5}$$

$$\overleftarrow{X_g} = \sum_{h\in G} |h\bar{g}\rangle \langle h| \iff \overleftarrow{X_g} |h\rangle = |h\bar{g}\rangle \tag{6}$$

The left-X-type and the right-X type can be combined into a conjugation operator $\overleftrightarrow{X}_g$. The group-based $Z$-type operators are labeled by irreps $\Gamma$:

$$Z_\Gamma = \sum_{g\in G} \Gamma(g) \otimes |g\rangle \langle g| \iff Z_\Gamma |g\rangle = \Gamma(g) \otimes |g\rangle \tag{7}$$

$$Z^\dagger_\Gamma = \sum_{g\in G} \Gamma(\bar{g}) \otimes |g\rangle \langle g| \iff Z^\dagger_\Gamma |g\rangle = \Gamma(\bar{g}) \otimes |g\rangle \tag{8}$$

The delta function on the group is given by:

$$\delta^G_{g,h} = \sum_{\Gamma \in Rep(G)} \frac{d_\Gamma}{|G|} Tr[\Gamma(\bar{g}h)] \tag{9}$$

Since in general $G$ is non-Abelian, $\Gamma(g_i)$ and $\Gamma(g_j)$ do not necessarily commute which means we must be careful in the ordering of operators. The commutation relations follow:

$$[\overrightarrow{X_g^{(i)}}, \overleftarrow{X_h^{(j)}}] = 0 \tag{10}$$

$$[\overrightarrow{X}_g^{(i)}, \overrightarrow{X}_h^{(j)}] \propto [\overleftarrow{X}_g^{(i)}, \overleftarrow{X}_h^{(j)}] \propto \delta_{i,j} \tag{11}$$

$$\overrightarrow{X}_g Z_\Gamma = \Gamma(\bar{g}).Z_\Gamma \overrightarrow{X}_g \tag{12}$$

$$\overleftarrow{X}_g Z_\Gamma = Z_\Gamma.\Gamma(g)\overleftarrow{X}_g \tag{13}$$

Swapping of $X$- and $Z$-type operators on the same site yields a matrix $\Gamma(h)$ which acts on the virtual space. Similar to the case where we generalized the computational basis to $G$-valued basis states, we generalize the qubit basis $|+\rangle$ and $|-\rangle$ via basis states labeled by matrix elements of irreps:

$$|\Gamma_{\alpha\beta}\rangle := \sqrt{\frac{d_\Gamma}{|G|}} \sum_{g \in G} [\Gamma(g)]_{\alpha\beta} |g\rangle \tag{14}$$

# 3 THE GENERALIZED CLUSTER STATE

For a finite group $G$, there does not exist a natural isomorphism between group elements $g \in G$ and irreps $\Gamma$ of $G$. This complicates matters when generalizing the stabilizers, as the ZXZ-cluster state is not a CSS stabilizer code. This caveat is circumvented by using a Hadamard gate on even sites, such that we obtain a state called the CSS-cluster state.This can be seen from the fact that $ZXZ$-operators are either transformed into $ZZZ$- or $XXX$- operators. Alternatively, the CSS-cluster state may aslo be prepared by acting with a circuit of $CX$-gates on the product state $|+\rangle|0\rangle|+\rangle\cdots$. This method readily generalizes for the state preparation of the G-cluster state, where one should instead use $|\mathbf{1}\rangle|e\rangle|\mathbf{1}\cdots\rangle$, where $e \in G$ denotes the identity and $\mathbf{1} \in Rep(G)$ is the trivial irrep. In contrast to the qubit-based cluster state presented as a simple graph, the G-cluster state neccetates a bipartite and directed graph. The bipartiteness stems from the fact that $CX$ -gates (and subsequent generalizations thereof) are not symmetric in their controls and targets, such that we need to assign a particular subset of the lattice as controls. The requirement for being directed is due to the fact that there exist different group-based $CX$-operators, being $C\overrightarrow{X}^{(i,j)}$ and $C\overleftarrow{X}^{(i,j)}$ in our case:

$$C\overrightarrow{X}^{(i,j)} |g_i, g_j\rangle := |g_i, g_i g_j\rangle \quad C\overleftarrow{X}^{(i,j)} |g_i, g_j\rangle := |g_i, g_j \bar{g}_i\rangle \tag{15}$$

Here, we study a particular group-based cluster state specified by choosing odd-sites as controls, with all edges oriented to the left, henceforth referred to as the $G$-cluster state. Defining a 1D chain with $n$ sites, each of which have a local Hilbert space $\mathbb{C}[G]$, the $G$-cluster state reads:

$$|\mathcal{C}\rangle = \mathcal{N} \sum_{\{g_i\} \in G} |g_1\rangle |g_1\bar{g}_2\rangle |g_2\rangle \cdots |g_{N-1}\bar{g}_N\rangle |g_N\rangle \tag{16}$$

Alternatively, $G$-cluster state preparation may be implemented by acting on the product state $|\psi_0\rangle := |\mathbf{1}\rangle\,|e\rangle\,|\mathbf{1}\rangle \cdots$ with the finite depth circuit $U_{\mathcal{C}}$:

$$U_{\mathcal{C}} = \prod_{i \text{ odd}} C\overrightarrow{X}^{(i,i+1)} C\overleftarrow{X}^{(i,i-1)} \tag{17}$$

Regrouping of indices of the tensors defining the $G$-clusetr state allows a graphical representation of the state. Furthermore, these single-site tensors respect a set of "pulling-through" identities, which allow transferring operators on the physical indices to the virtual indices.
A Hamiltonian with ground state $|\mathcal{C}\rangle$ can be constructed as a sum of commuting stabilizers:

$$H_{\mathcal{C}} = -\frac{1}{|G|}\sum_{i \text{ odd}}\Big(\sum_{\Gamma\in Rep(G)} Tr[Z_{\Gamma}^{\dagger(i)}.Z_{\Gamma}^{(i+1)}Z_{\Gamma}^{(i+2)}]d_{\Gamma}+\sum_{g\in G}\overleftarrow{X}_g^{(i+1)}\overrightarrow{X}_g^{(i+2)}\overrightarrow{X}_g^{(i+3)}\Big) \tag{18}$$

wherein the sum over $\Gamma \in Rep(G)$ denotes a sum over irreps $\Gamma$ of $G$ (i.e. simple objects in $Rep(G)$). The group-based Pauli terms all commute with one another, and the ground-state $|\mathcal{C}\rangle$ is the joint eigenspace of all the operators with maximum eigenvalue ($1/|G|$ for $X$-type terms, and $d_{\Gamma}^2/|G|$ for the $Z$-type terms. The cluster-state Hamiltonian has four independent families of global symmetries, which in turn are respected by the ground state $|\mathcal{C}\rangle$:

$$G_R : \overrightarrow{A}_g := \prod_{i \text{ odd}} \overleftarrow{X}_g^{(i)}, \tag{19}$$

$$G_L : \overrightarrow{A}_g := \prod_{i \text{ odd}} \overrightarrow{X}_g^{(i)}\overleftrightarrow{X}_g^{(i+1)}, \tag{20}$$

$$\mathrm{Rep}(G) : \hat{B}_{\Gamma} := Tr\Big[\prod_{i \text{ even}} Z_{\Gamma}^{(i)}\Big] \tag{21}$$

$$\mathrm{Inn}(G) : \hat{C}_g := \prod_{i \text{ even}}\Big(\overleftarrow{X}_{g'g\bar{g}'}^{(i)}\delta_{g'}^{(i+1)}\overrightarrow{X}_{g'g\bar{g}'}^{(i+2)}\Big) \tag{22}$$

We thus have: two $G$ symmetries, a $\mathrm{Rep}(G)$ symmetry, and an $\mathrm{Inn}(G)$ symmetry corresponding to the inner automorphism group of $G$. For Abelian groups, the two $G$ symmetries are identified and the $\mathrm{Inn}(G)$ trivializes. What remains are the two symmetries, $G \times \mathrm{Rep}(G)$, that protect the SPTO. For the non-Abelian case, it is analytically and numerically proved that $G_R \times \mathrm{Rep}(G)$ is the minimal subgroup also protecting the SPTO. Since $\overleftarrow{A}_g$ and $\hat{B}_{\Gamma}$ are supported on different sublattices, they commute. Whilst the $\overleftarrow{A}_g$ operators form a representation of a group, the $\hat{B}_{\Gamma}$ operators multiply according to the fusion rules of $\mathrm{Rep}(G)$:

$$\hat{B}_{\Gamma_i}\hat{B}_{\Gamma_j} = \sum_k N_{\Gamma_i,\Gamma_j}^{\Gamma_k}\hat{B}_{\Gamma_k} \tag{23}$$

We refer to this symmetry as a fusion category symmetry, whilst high-energy nomenclature refers to these as non-invertible symmetries. See Appendix **??** This stems from the fact that the $Z_\Gamma$ operators in general do not have an inverse. This is again rooted in the fact that an object and its dual, yields a sum of objects including the trivial irrep, implying that the fusion to the trivial object is not the only fusion channel.
If $G$ is Abelian, we have that $\mathrm{Rep}(G) \cong G$ such that the character group is isomorphic to $G$.
The fusion category symmetry $G \times \mathrm{Rep}(G)$ is labelled by pair $(g, \Gamma)$ and multiply as:

$$(g_i, \Gamma_i) \ \otimes \ (g_j, \Gamma_j) = \bigoplus_k N^{\Gamma_k}_{\Gamma_i, \Gamma_j} \ (gh, \Gamma_k) \tag{24}$$

From which the $\overleftarrow{A}_g$ and $\hat{B}_\Gamma$ symmetry operators follow the same multiplication. Notice that the multiplication is non-commutative since $gh \neq hg$, regardless of the fact that $N^{\Gamma_k}_{\Gamma_i, \Gamma_j}$ is nonzero for multiple triples $(i, j, k)$. This implies that $G \times \mathrm{Rep}(G)$ is not the fusion algebra of irreps of some other finite group $G'$: $G \times \mathrm{Rep}(G) \neq \mathrm{Rep}(G')$. However, as long as the fusion category is nonanomolous, we can relate $G \times \mathrm{Rep}(G)$ to $\mathrm{Rep}(H)$ for some semisimple Hopf algebra $H$, such that: $G \times \mathrm{Rep}(G) \cong \mathrm{Rep}(H)$. This is satisfied for $H = \mathbb{C}[G]^* \otimes \mathbb{C}[G]$.

# 4 DISTINCTNESS FROM THE SYMMETRIC PRODUCT STATE: STACKING VS DUALITY

# 5 SIGNATURES OF SPT ORDERS: EDGES AND STRINGS

## References


- Noninvertible Symmetry-Protected Topological Order in a Group-Based Cluster State; C. Fechisin, N. Tantivasadakarn, V. Albert; 2025; DOI: https://doi.org/10.1103/PhysRevX.15.011058


# A MULTIPLICATION OF MPOs

Here we explicitly multiply the MPOs of the $\mathrm{Rep}(G)$ symmetry. First we introduce the notation for the fusion of two irreps of the group $G$, which we refer to as generalized Clebsch-Gordan coëfficients. This is presented by a tuning fork type diagram with two incoming legs and one outgoing leg [insert a diagram], which corresponds to the generalized Clebsch-Gordan coëfficiënt:

$$\begin{pmatrix} \Gamma_i & \Gamma_j & | & \Gamma_k & \mu \\ \alpha & \beta & & \gamma & \end{pmatrix} \tag{25}$$

As a reminder we also write out the explicit form of the $Z_\Gamma$ MPO [insert Z MPO here]:

$$alpha - MPO(Z_\Gamma) - beta = \sum_{g \in G} [\Gamma(g)]_{\alpha\beta} \left| g \right\rangle \left\langle g \right| \tag{26}$$

This allows the multiplication of two $Z_\Gamma$'s, which is represented by so called 'pulling-through'- or 'zipper' diagrams [insert pulling-through/zipper diagram here]:

Explicitly, by choosing basis vectors we can write:

$$\sum_{\epsilon,\delta} [Z_{\Gamma_i}]_{\epsilon\alpha} [Z_{\Gamma_j}]_{\delta\beta} \begin{pmatrix} \Gamma_i & \Gamma_j & | & \Gamma_k & \mu \\ \epsilon & \delta & & \gamma & \end{pmatrix} = \sum_{\kappa} \sum_{\mu} [Z_{\Gamma_k}]_{\gamma\kappa} \begin{pmatrix} \Gamma_i & \Gamma_j & | & \Gamma_k & \mu \\ \alpha & \beta & & \kappa & \end{pmatrix} \tag{27}$$