Generalized (2+1)d Cluster State

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Abstract

In this note we study the generalized cluster state in (2+1)d, which displays a generalized symmetry in both the categorical- and in the higher-form sense. We explicitely construct the state in three ways: (1) a linear depth unitary circuit, (2) the ground state of a stabilizer Hamiltonian, and (3) a PEPS representation. We distinguish the G-cluster state from the trivial product state by means of a duality/gauging argument.

1 Notations and Conventions

1.1 Qubits $\rightarrow$ G-qudits

We generalize the qubits $\{|0\rangle, |1\rangle\}$, the $\mathbb{Z}_2$-case, to G-valued qudits $\{|g\rangle \,|g \in G\}$, such that local Hilbert spaces is the group algebra $\mathbb{C}[G]$, labelled by elements of the finite group G. The dual basis is labeled by elements of the irreps, defined as:

$$|\Gamma_{\alpha\beta}\rangle := \sqrt{\frac{d_\Gamma}{|G|}} \sum_{g \in G} [\Gamma(g)]_{\alpha\beta} |g\rangle. \tag{1}$$

Specializing to the trivial irrep yiels:

$$|\mathbb{I}\rangle_{\alpha\beta} = \sqrt{\frac{1}{|G|}} \sum_{g \in G} \delta_{\alpha,\beta} |g\rangle. \tag{2}$$

1.2 Generalized Pauli Operators

As a reminder we note that the left- and right regular multiplication, which generalizes the Pauli X-operator (i.e. the Pauli σ^x-matrix) is given by:

$$\overrightarrow{X}_g \equiv L_g = \sum_{h \in G} |gh\rangle \langle h| \iff L_g |h\rangle = |gh\rangle. \tag{3}$$

$$\overleftarrow{X}_g \equiv R_g = \sum_{h \in G} |h\rangle \langle hg| \iff R_g |h\rangle = |h\bar{g}\rangle \tag{4}$$

The generalized Pauli Z-operators Z_Γ are given by:

$$Z_\Gamma = \sum_{g \in G} \Gamma(g) \otimes |g\rangle \langle g| \iff Z_\Gamma |g\rangle = \Gamma(g) \otimes |g\rangle \tag{5}$$

$$Z_\Gamma^\dagger = \sum_{g \in G} \Gamma(\bar{g}) \otimes |g\rangle \langle g| \iff Z_\Gamma^\dagger |g\rangle = \Gamma(\bar{g}) |g\rangle \tag{6}$$

2 G-Cluster State

We begin by constructing the generalization of the cluster state by replacing the group $\mathbb{Z}_2$ by an arbitrary finite (possibly non-Abelian) group G. We present complementary perspectives on this state: first we present an explicit construction of the state through a linear-depth local unitary circuit (2.1), then we present the state as the unique ground state of the G-cluster state stabilizer Hamiltonian (2.2), and finally we present the Projected Entangled-Pair State (PEPS) construction of this state (2.3). In subsection (2.4) we discuss the symmetries of the G-cluster state, and find that it has a $G^{(0)} \times \text{Rep}(G)^{(1)}$-symmetry. Occasionaly, we shall consider the case where $G = \mathbb{Z}_2$.

2.1 Circuit Construction

Our circuit construction closely follows that of Brell [Bre15], inspired by the generalization of the toric code to Kitaev's quantum double models. The first obstacle when considering the generalization is the observation that the $\mathbb{Z}_2$-cluster state is not a CSS stabilizer code (the stabilizers have mixed X- and Z-operators). To this end, one requires applying a Hadamard gate to one of the sublattices of the bipartite graph Λ on which the model is defined. Furthermore, it is not immediately clear if one should replace the Pauli X-operators by either left- or right regular representations of the group G. This problem is mended by requiring that the lattice upon which the model is defined, also be directed $\rightarrow$. Finally, since we also mean to generalize to non-Abelian groups, we also need to specify an ordering # on the the vertices of the lattice.

$$\tag{7}$$

Explicitely, consider a directed bipartite planar graph $(\Lambda, \rightarrow, \#)$, depicted in eq.(7), and an arbitrary finite group G. We further distinguish between edges $e \in E \subset \Lambda$ and vertices $v \in V \subset \Lambda$. Put a qudit on each vertex $v \in V$ and each edge $e \in E$ with local Hilbert space isomprphic to the group algebra of

G: $\mathcal{H}_{v/e} \cong \mathbb{C}[G]$. The data $(\Lambda, \rightarrow, \#)$ admits a full specification of the CSS G-cluster state $|\mathcal{C}\rangle$ through the circuit construction:

$$|\mathcal{C}\rangle \sim \prod_{v \in V} \overrightarrow{\prod_{v \sim e}} CX_{(v,e)} |\mathbb{I}\rangle^{\otimes V} |e\rangle^{\otimes E} =: \mathcal{U}_\mathcal{C} |\mathbb{I}\rangle^{\otimes V} |e\rangle^{\otimes E} \tag{8}$$

That is, start with the product state $|\mathbb{I}\rangle^{\otimes V} |e\rangle^{\otimes E}$, where $\mathbb{I} = \frac{1}{|G|} \sum_{g \in G} |g\rangle$, $e \in G$ is the trivial group element, and the tensor product is over all vertices $v \in V$ and edges $e \in E$. Subsequently, apply controlled multiplication operators $CX_{(v,e)}$ with as control the vertex $v \in V$ and as target the edges $e \in E$ neighbouring v. Here, we make the choice of applying $C\overrightarrow{X}_{(v,e)} \equiv CL_{(v,e)}$ when $v = s(\mathsf{e})$ i.e. when the edge e connecting the vertex qudit v and the edge qudit e is directed from the vertex v as source $s(\cdot)$ and as target $t(\cdot)$ the edge qudit e. Conversely, we apply $C\overleftarrow{X}_{(v,e)} \equiv CR_{(v,e)}$ when the vertex $v = t(\mathsf{e})$, i.e. when the edge connecting the vertex qudit and the edge qudit is directed from the edge qudit $e = s(\mathsf{e})$ to the vertex qudit $v = t(\mathsf{e})$[1].

In this note, we shall consider a square lattice $\Lambda_\square$, with all horizontal edges directed from left to right, and all vertical edges directed from the bottom of the page towards the top of the page. We use the same convention for multiplication specified in the previous paragraph. We use clockwise ordering for all multiplication. We note that we are free to choose a start- and endpoint when multiplying operators around a given cycle, so long as the relative order stays the same [Bre15]. The G-cluster state for this geometry is given by:

$$|\mathcal{C}\rangle \sim \prod_v CR_{(v,l)} CR_{(v,b)} CL_{(v,r)} CL_{(v,t)} |\mathbb{I}\rangle^{\otimes V} |e\rangle^{\otimes E}, \tag{9}$$

where $l, b, r, t \in E$ are the left-, bottom-, right-, and top edge qudits neigbouring vertex qudit $v \in V$. Thus we can construct the G-cluster state $|\mathcal{C}\rangle$ by means of a linear-depth local unitary circuit from the product state $|\mathbb{I}\rangle^{\otimes V} |e\rangle^{\otimes E}$.

2.2 Ground State of a Stabilizer Hamiltonian

We generalize the $(2+1)$d $\mathbb{Z}_2$-cluster state Hamiltonian [CSSZ25][VBV+24][Yos16] by replacing Pauli operators with generalized Pauli operators following Fechisin [FTA25]. This requires introducing directionality and ordering on the lattice, for which we use the same conventions as specified in subsection (2.1).
The $(2+1)$d (CSS) G-cluster state Hamiltonian then reads:

$$\mathbb{H}_\mathcal{C}^G = -\frac{1}{|G|} \sum_{<v,w>} \sum_{\Gamma \in \text{Rep}(G)} d_\Gamma \text{Tr}[Z_\Gamma^{(v)\dagger} \cdot Z_\Gamma^{(v,w)} \cdot Z_\Gamma^{(w)}] - \frac{1}{|G|} \sum_v \sum_{g \in G} L_g^{(v)} \overrightarrow{\prod_{e \subset v}} X_g^{(e)}. \tag{10}$$

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[1] We shall differentiate between the edge qudits $e \in E$ and the edges e of the graph Λ.

The first summation runs over al neigbouring vertices $v, w \in V$. The second summation puts a *star operator* on each vertex $v \in V$, with the $\overrightarrow{\prod}$ product indicating a product depending on the directionality of the lattice, with choice of convention for left- or right-regular multiplication as specified in subsection (2.1). We note that for the square lattice $\Lambda_\square$ with arrows pointing left-to-right and bottom-to-top, the star operators are of the form:

The individual terms are interpreted as stabilizers of the G-cluster state. One can show that the G-cluster state is the unique ground state (GS) of the Hamiltonian $\mathbb{H}_\mathcal{C}^G$ by counting the degrees of freedom on the lattice and the number of stabilizers contained in $\mathbb{H}_\mathcal{C}^G$.

For the case where $G = \mathbb{Z}_2$, the generalized Pauli Z- and X-operators (Z_Γ, and R_g or L_g) reduce to the usual Pauli matrices Z and X. The Hamiltonian $\mathbb{H}_\mathcal{C}^G$ becomes:

$$\mathbb{H}_\mathcal{C}^{\mathbb{Z}_2} = -\sum_{<v,w>} Z^{(v)} Z^{(v,w)} Z^{(w)} - \sum_v X^{(v)} \prod_{e \subset v} X^{(e)}, \qquad (11)$$

wherein we ommited the trivial stabilizers, and $\overrightarrow{\prod}$ is replaced by $\prod$.

2.3 G-Cluster State PEPS

$$(12)$$

We now present the PEPS representation [CPGSV21] of the G-cluster state, which we depict in eq.(**??**). The horizontal- and vertical legs of each tensor indicate virtual legs (coloured blue), whereas diagonal legs of each tensor indicate physical legs (coloured black). The individual tensors[2] are given by:

$$(13)$$

In order to establish that eq.(**??**) is the correct PEPS representation, it suffices to check that it stabilizes, i.e. it is in the $+1$ eigenspace, of all stabilizers contained in $\mathbb{H}_\mathcal{C}^G$. To this end, we give the symmetries of the vertex- and edge tensors:

$$(14)$$

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[2]We proceed to write first the virtual legs of the tensor, followed by the physical leg of the tensor.

$$\tag{15}$$

$$\tag{16}$$

$$=$$

$$=$$

$$=$$

It should be clear from the above symmetries that the PEPS representation (eq.**??**) is indeed the GS of $\mathbb{H}_{\mathcal{C}}^{G}$. Indeed, pulling the stabilizers through to the virtual legs, one get the identity operators with +1 eigenvalue. Take for example a ZZZ-operator: on the vertices one pulls the first $Z_{\Gamma}^{\dagger}$ throught toward the left virtual leg of the edge tensor and the last Z_{Γ} throught towards the right virtual leg of the edge tensors, meanwhile one pulls the middle Z_{Γ} through which yields a Z_{Γ} and a $Z_{\Gamma}^{\dagger}$ on the left and right virtual legs off the edge tensor, respectively. What remains is the identity, with trace equal to d_{Γ}, such that the full term yields eigenvalue +1. A similar argument can be made for the terms in the second sum.

2.4   Symmetries of the G-cluster State

In this subsection, we analyse the symmetries of the G-cluster state. Specifically, we find that the state has the following independent families of global symmetries:

$$G_R^{(0)} : \overleftarrow{A}_g := \prod_{v \in V} R_g^{(v)} \tag{17}$$

$$\mathrm{Rep}(G)^{(1)} : \mathcal{U}_\Gamma(\gamma) := \mathrm{Tr}[\overrightarrow{\prod_{e \subset \gamma}} Z_\Gamma^{(e)}], \tag{18}$$

where γ is some closed oriented loop defined on the edges of the lattice Λ. The $\overrightarrow{\prod}$-product indicates a directed product, wherein we takes Z_Γ ($Z_\Gamma^\dagger$) when γ runs parallel (anti-parallel) to Λ. Importantly, we note that the $G_R^{(0)}$- and the $\mathrm{Rep}(G)^{(1)}$-symmetries act on different sublattices, such that they commute with one another. The superscripts (0) and (1) indicate that we are dealing with a (conventional) 0-form, and a 1-form symmetry, respectively. Furthermore, since multiplication of irreps do not generically multiply according to the rules of a group, we have that the $\mathrm{Rep}(G)^{(1)}$-symmetry is also a generalized symmetry, multiplying according to the rules of the fusion category $\mathrm{Rep}(G)$. We thus have that the $\mathrm{Rep}(G)^{(1)}$-symmetry generalizes the concept of symmetries in two ways: both in the categorical sense, and in the higher-form sense [FTA25].

3   Duality: $G \times G$ SSB $\rightarrow G \times \mathrm{Rep}(G)$ SPT

In this section we give a duality argument, similar to the (1+1)d case in [FTA25]. We will show that the G-cluster state is distinct from the trivial product state by showing that upon gauging/dualising models with different spontaneous symmetry breaking (SSB) patterns, we obtain the aforementioned states as unique GSs of the dual models. Consider the following two models defined on the directed bipartite lattice Λ' consisiting of even and odd sites:

$$\mathbb{H}_{\mathrm{SSB1}} = -\frac{1}{|G|} \sum_{\substack{w \text{ even} \\ w' \subset\subset w}} \sum_{\Gamma \in \mathrm{Rep}(G)} \mathrm{Tr}[Z_\Gamma^{(w)} \cdot Z_\Gamma^{(w')\dagger}] d_\Gamma - \frac{1}{|G|} \sum_{v \text{ odd}} \sum_{g \in G} L_g^{(v)}, \tag{19}$$

$$\mathbb{H}_{\mathrm{SSB2}} = -\frac{1}{|G|} \sum_{\substack{w \text{ even}, \, w \subset\subset w' \\ v \subset w, v' \subset w', v \neq v'}} \sum_{\Gamma \in \mathrm{Rep}(G)} \mathrm{Tr}[Z_\Gamma^{(v)\dagger} \cdot Z_\Gamma^{(w)} \cdot Z_\Gamma^{(w')\dagger} \cdot Z_\Gamma^{(v')}] d_\Gamma \tag{20}$$

$$-\frac{1}{|G|} \sum_{\substack{w \text{ even} \\ v \subset w}} \sum_{g \in G} L_g^{(v)} L_g^{(w)}, \tag{21}$$

wherein we denote $w \in \Lambda'_e$ and $v \in \Lambda'_o$ as sites contained in the even- and odd sublattices of Λ', respectively. We also denote by $\subset\subset$ the nearest neighbour on the same sublattice, whilst the meaning of $\subset$ remains the same as in the previous section. Note that both of the model Hamiltonians $\mathbb{H}_{\mathrm{SSB1}}$ and $\mathbb{H}_{\mathrm{SSB2}}$ have the following families of symmetries on each sublattice:

$$G_R^{\text{o}} : \prod_{v \in \Lambda_o'} R_g^{(v)} \tag{22}$$

$$G_R^{\text{e}} : \prod_{w \in \Lambda_e'} R_g^{(w)} \tag{23}$$

Since these symmetries act indepently the total symmetry is $G_R^{\text{odd}} \times G_R^{\text{even}}$. In both models, the GS spontaneously breaks symmetry. For SSB1 the GS is given by $|\psi_0^{\text{SSB1}}\rangle := |\mathbb{I}\rangle^{\otimes \text{ODD}} |g\rangle^{\otimes \text{EVEN}}$ for $g \in G$, with unbroken symmetry G_R^{odd}. For SSB2 the GS is given by $|\psi_0^{\text{SSB2}}\rangle := \prod_{o \in \Lambda_o'} \overrightarrow{\prod}_{e \subset o} CX_{(o,e)} |\mathbb{I}\rangle^{\otimes \text{ODD}} |g\rangle^{\otimes \text{EVEN}}$ for $g \in G$, which has unbroken symmetry[3] $R_h^{(o)} \prod_{e \subset o} R_{\bar{g}hg}^{(e)}$ for the state labeled by g, where $o \in \Lambda_o'$ and $e \in \Lambda_e'$. Comparing the GSs, we see that they are related by the circuit:

$$U_{\text{SSB}} := \prod_{o \in \Lambda_o'} \overrightarrow{\prod}_{e \subset o} CX_{(o,e)}, \tag{24}$$

such that $|\psi_0^{\text{SSB2}}\rangle = U_{\text{SSB}} |\psi_0^{\text{SSB1}}\rangle$.

We now apply the gauging procedure outlined in [HVAS$^+$15] to the above models. Specifically, we gauge the G_R^{e}-symmetry. This is done by first introducing edge degrees of freedom (d.o.f.) i.e. G-qudits on the edges between the even sites $w \in \Lambda_e'$. We refer to the original even sites as the matter d.o.f., and the newly introduced edge d.o.f. as gauge d.o.f.. Then we apply a group-averaged projector $\mathcal{P} = \frac{1}{|G|} \sum_{g \in G} \mathcal{P}_g$, projecting onto the locally gauge-invariant subspace. $\mathcal{P}_g$ is interpreted as a (local) Gauss law. We supplement the gauging projector by a *disentangler* $\mathcal{D}$. The disentangler, due to totally gauging the G_R^e-symmetry, decouples the (even) matter d.o.f. from the gauge d.o.f.. We defer details of the procedure to appendix A. For our purpose, we only mention the following results:

$$Z_\Gamma^{(e)} Z_\Gamma^{(e')\dagger} \mapsto Z_\Gamma^{(e,e')} \tag{25}$$

$$L_g^{(e)} \mapsto R_g^{(l)} R_g^{(b)} L_g^{(r)} L_g^{(t)}, \tag{26}$$

where (e, e') and l, b, r, t denote the edges connecting even sites e and e', and the left- (l), bottom-(b), right-(r), and top-(t) edges of the even site e, respectively. Upon applying a shift of the odd sites by one site, we obtain the following mapping:

$$\mathbb{H}_{\text{SSB1}} \mapsto -\frac{1}{|G|} \sum_{<v,w>} \sum_{\Gamma \in \text{Rep}(G)} \text{Tr}[Z_\Gamma^{(v,w)\dagger}] d_\Gamma - \frac{1}{|G|} \sum_v \sum_{g \in G} L_g^{(v)} \tag{27}$$

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[3]This is easy to check using a quantum circuit diagram.

$$\mathbb{H}_{\text{SSB2}} \mapsto -\frac{1}{|G|} \sum_{<v,w>} \sum_{\Gamma \in \text{Rep}(G)} d_\Gamma \text{Tr}[Z_\Gamma^{(v)\dagger} \cdot Z_\Gamma^{(v,w)} \cdot Z_\Gamma^{(w)}] - \frac{1}{|G|} \sum_v \sum_{g \in G} L_g^{(v)} \overrightarrow{\prod}_{e \subset v} X_g^{(e)},$$

(28)

wherin we have relabeled the odd sites $o \in \Lambda'_o$ on the original lattice to vertices $v \in V$, and the gauge d.o.f. to edge sites $E \ni (v,w) \sim e \in E$. Hence, we find that SSB1 is mapped to a model with unique (product) GS $|\Psi_0^{SSB1}\rangle :=$ $|\mathbb{I}\rangle^{\otimes E} |e\rangle^{\otimes V}$. Meanwhile, the SSB2 Hamiltonian $\mathbb{H}_{\text{SSB2}}$ is mapped to the G-cluster state Hamiltonian $\mathbb{H}_\mathcal{C}^G$, with unique ground state the (2+1)d G-cluster state $|\mathcal{C}\rangle$, which is a SPTO state. Finally, we note the similarity between the circuit $\mathcal{U}_\mathcal{C}$ used to prepare the G-cluster state in subsection 2.1 and U_{SSB}, which have equivalent (in the sense of the gauging map above) controls, and targets (even sites in the SSB model, which are mapped to edge sites in the SPT model). In particular, the gauged model inherits the directionality and ordering of the SSB model lattice, but differ in that the SSB model lattice have only sites, whereas the SPT model lattice have both edge- and vertex sites.

Appendix A: Gauging a G-symmetry

Appendices:
- deforming the 1-form symmetry using the Gausβ law
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