Gauging & Duality

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Abstract

1 INTRODUCTION

The process of promoting a global symmetry to a local symmetry, formally referred to as 'gauging', is introduced in the tensor-network framework. Succintly, this is done by introducing so-called gauge degrees of freedom, enlarging the Hilbert space, after which we then project onto the locally gauge-invariant subspace. This process is encapsulated in what we call the 'gauging map', and when supplemented with a 'disentangler', we retrieve Kramers-Wannier-like dualities.

2 THE GAUGING PROCEDURE

We gauge at the level of states, which should be interpreted as the map $\mathcal{G}$: $\mathcal{H}_m \to \mathcal{H}_{m+g}$ such that the global symmetry $U(g)\,|\psi\rangle = \bigotimes_i u_i(g)\,|\psi\rangle = |\psi\rangle$ becomes a local symmetry $\hat{u}_i(g)\,|\hat{\psi}\rangle = |\hat{\psi}\rangle \;\; \forall i$. Herein, we have introduced gauge degrees of freedom (d.o.f) on the edges between the sites- denoted by $i+1/2$- such that the local Hilbert spaces become: $\hat{\mathcal{H}}_i \coloneqq \mathcal{H}_{i-1/2} \otimes \mathcal{H}_i \otimes \mathcal{H}_{i+1/2}$. For all practical purposes we shall also initiate the gauge d.o.f in the trivial state $|e\rangle_{i+\frac{1}{2}}$. For the case we will be considering, we use the group algebra $\mathbb{C}[G]$ to represent the gauge d.o.f labelled by the group elements: $\{|g\rangle \mid \; g \in G\}$. We will also be using the left- and right regular representations:

$$L_g^i \equiv L(g)^i \equiv \overrightarrow{X}_i(g) \coloneqq \sum_{h\in G} |gh\rangle\,\langle h| \tag{1}$$

$$R_g^i \equiv R^i(g) \equiv \overleftarrow{X}_i(g) \coloneqq \sum_{h\in G} |h\bar{g}\rangle\,\langle h| \tag{2}$$

The local Gauss law reads $\hat{u}_g^i = R_g^{i-\frac{1}{2}} \otimes u_g^i \otimes L_g^{i+\frac{1}{2}}$. Upon confirming that the left- and right regular representations acting on the same site commute:

$[L_g, R_h] = 0$, we can construct a projector onto the locally gauge-invariant subspace using the group-averaged type projector:

$$\mathcal{P}_i \coloneqq \frac{1}{|G|}\sum_{g\in G} u_g^i \tag{3}$$

Naturally, the global projector is taken over all sites: $\mathcal{P} = \Pi_i \mathcal{P}_i$. Summarizing: the enlarging of the local Hilbert spaces and subsequent projection onto the states wich are locally gauge-invariant is encapsulated in the gauging map $\mathcal{G}\,|\psi\rangle = \mathcal{P}(|\psi\rangle \bigotimes_i |e\rangle_{i+\frac{1}{2}})$.

Gauging of operators O is defined by $\Gamma : \mathcal{L}(\mathcal{H}_m) \to \mathcal{L}(\mathcal{H}_{m+g})$, and satisfy:

$$\text{(i)}\;\; u_g^i \mathcal{G} = \mathcal{G} \tag{4}$$

$$\text{(ii)}\;\; [u_g^i, \Gamma[O]] = 0 \tag{5}$$

$$\text{(iii)}\;\; \mathcal{G}\cdot U_g = \mathcal{G} \tag{6}$$

$$\text{(iv)}\;\; \Gamma[O]\mathcal{G}\,|\psi\rangle = \mathcal{G}O\,|\psi\rangle \tag{7}$$

Note: remember to add the fundamental theorem of MPS somewhere!

Identifying the possible symmetries of the tensors which constitute the gauging map greatly aids in the diagrammatic derivations which will be used. For example [list some examples here with diagrams]. Similarly, we do this for the left- and right legs of the projectors: [insert symmetries of R and L here, wtih diagrams]

The gauging map, complemented with a so-called 'disentangler' $\mathcal{U}$ is what constitutes a 'duality'[explain what is meant by the term duality]. By pulling the disentangler through the gauging map, we possibly get a simpler action which reduces the complexity of identifying operators on each side of the duality.

3 Gauging $\mathbb{Z}_2$

In this section, we illustrate the gauging procedure via the simplest example of $\mathbb{Z}_2$-symmetry. Supplementing the gauging by an appropriate disentangler will yield the Kramers-Wannier duality [ref KW].

4 Gauging of G-symmetry for G-Cluster State SPTO

In this section, we gauge the G-symmetry of SSB states and use an appropriate disentangler so as to obtain the trivial- and G-cluster states.

REFERENCES

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