Generalized (2+1)d Cluster State

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December 2025

Abstract

In this note we study the generalized cluster state in (2+1)d, which displays a generalized symmetry in both the categorical- and in the higher-form sense. We explicitely construct the state in three ways: (1) a linear depth unitary circuit, (2) the ground state of a stabilizer Hamiltonian, and (3) a PEPS representation. We distinguish the G-cluster state from the trivial product state by means of a duality/gauging argument.

1 Introduction

2 Notations and Conventions

2.1 Qubits $\to$ G-qudits

We generalize the qubits $\{|0\rangle, |1\rangle\}$, the $\mathbb{Z}_2$-case, to G-valued qudits $\{|g\rangle \, | g \in G\}$, such that local Hilbert spaces is the group algebra $\mathbb{C}[G]$, labelled by elements of the finite group G. The dual basis is labeled by elements of the irreps, defined as:

$$|\Gamma_{\alpha\beta}\rangle \coloneqq \sqrt{\frac{d_\Gamma}{|G|}} \sum_{g\in G} [\Gamma(g)]_{\alpha\beta} |g\rangle . \quad (1)$$

Specializing to the trivial irrep yiels:

$$|\mathbb{I}\rangle_{\alpha\beta} = \sqrt{\frac{1}{|G|}} \sum_{g\in G} \delta_{\alpha,\beta} |g\rangle . \quad (2)$$

2.2 Generalized Pauli Operators

As a reminder we note that the left- and right regular multiplication, which generalizes the Pauli X-operator (i.e. the Pauli σ^x-matrix) is given by:

$$\overrightarrow{X}_g \equiv L_g = \sum_{h\in G} |gh\rangle \langle h| \iff L_g |h\rangle = |gh\rangle . \tag{3}$$

$$\overleftarrow{X}_g \equiv R_g = \sum_{h\in G} |h\rangle \langle hg| \iff R_g |h\rangle = |h\bar{g}\rangle \tag{4}$$

The generalized Pauli Z-operators Z_Γ are given by:

$$Z_\Gamma = \sum_{g\in G} \Gamma(g) \otimes |g\rangle \langle g| \iff Z_\Gamma |g\rangle = \Gamma(g) \otimes |g\rangle \tag{5}$$

$$Z_\Gamma^\dagger = \sum_{g\in G} \Gamma(\bar{g}) \otimes |g\rangle \langle g| \iff Z_\Gamma^\dagger |g\rangle = \Gamma(\bar{g}) |g\rangle \tag{6}$$

-Commutation relations -graphical presentation -delta function of the group -

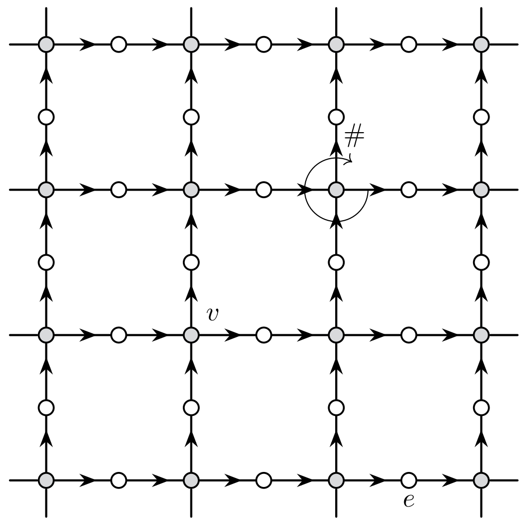
2.3 MPSs and MPOs

3 G-Cluster State

We begin by constructing the generalization of the cluster state by replacing the group $\mathbb{Z}_2$ by an arbitrary finite (possibly non-Abelian) group G. We present complementary perspectives on this state: first we present an explicit construction of the state through a constant-depth local unitary quantum circuit (3.1), then we present the state as the unique ground state of the G-cluster state stabilizer Hamiltonian (3.2), and finally we present the Projected Entangled-Pair State (PEPS) construction of this state (3.3). In subsection (3.4) we discuss the symmetries of the G-cluster state, and find that it has a $G^{(0)} \times \mathrm{Rep}(G)^{(1)}$-symmetry. Occasionaly, we shall consider the case where $G = \mathbb{Z}_2$.

3.1 Circuit Construction

Our circuit construction closely follows that of Brell [Bre15], inspired by the generalization of the toric code to Kitaev's quantum double models. The first obstacle when considering the generalization is the observation that the $\mathbb{Z}_2$-cluster state is not a CSS stabilizer code (the stabilizers have mixed X- and Z-operators). To this end, one requires applying a Hadamard gate to one of the sublattices of the bipartite graph Λ on which the model is defined. Furthermore, it is not immediately clear if one should replace the Pauli X-operators by either left- or right regular representations of the group G. This problem is mended by requiring that the lattice upon which the model is defined, also be directed $\rightarrow$.



(7)

Explicitely, consider a directed bipartite planar graph $(\Lambda, \rightarrow)$, depicted in eq.(7), and an arbitrary finite group G. We further distinguish between edges $\mathsf{e} \in E \subset \Lambda$ and vertices $\mathsf{v} \in V \subset \Lambda$. Put a qudit v or e on each vertex $\mathsf{v} \in V$ and each edge $e \in E$, respectively, with local Hilbert space isomprphic to the group algebra of G: $\mathcal{H}_{v/e} \cong \mathbb{C}[G]$. The data $(\Lambda, \rightarrow)$ admits a full specification of the CSS G-cluster state $|\mathcal{C}\rangle$ through the circuit construction:

$$|\mathcal{C}\rangle \sim \prod_{\mathsf{e}\in E} \overrightarrow{\prod_{\mathsf{v}\sim\mathsf{e}}} CX_{(v,e)} \, |\mathbb{I}\rangle^{\otimes V} \, |e\rangle^{\otimes E} =: \mathcal{U}_{\mathcal{C}} \, |\mathbb{I}\rangle^{\otimes V} \, |e\rangle^{\otimes E} \tag{8}$$

That is, start with the product state $|\mathbb{I}\rangle^{\otimes V} \, |e\rangle^{\otimes E}$, where $\mathbb{I} = \frac{1}{|G|} \sum_{g \in G} |g\rangle$, $e \in G$ is the trivial group element, and the tensor product is over all vertices $\mathsf{v} \in V$ and edges $\mathsf{e} \in E$. Subsequently, apply controlled multiplication operators $CX_{(v,e)}$ with as control the vertex qudit v and as target the edge qudit e neighbouring v. Here, we make the choice of applying $C\overrightarrow{X}_{(v,e)} \equiv CL_{(v,e)}$ when $\mathsf{v} = s(\mathsf{e})$. Conversely, we apply $C\overleftarrow{X}_{(v,e)} \equiv CR_{(v,e)}$ when the vertex $\mathsf{v} = t(\mathsf{e})$[1].

In this note, we shall consider a square lattice $\Lambda_{\square}$, with all horizontal edges directed from left to right, and all vertical edges directed from the bottom of the page towards the top of the page. We use the same convention for multiplication specified in the previous paragraph. The G-cluster state in our convention is given by:

$$|\mathcal{C}\rangle \sim \prod_{\mathsf{e}} CL_{(s(\mathsf{e}),e)} CR_{(t(\mathsf{e}),e)} \, |\mathbb{I}\rangle^{\otimes V} \, |e\rangle^{\otimes E} \,, \tag{9}$$

[1]Note that we differentiate between the vertices v and edges e of the lattice, and the vertex-v and edge qudits e.

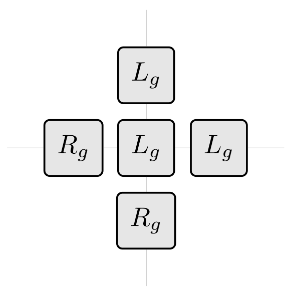
Thus we can construct the G-cluster state $|\mathcal{C}\rangle$ by means of a constant-depth local unitary quantum circuit from the product state $|\mathbb{I}\rangle^{\otimes V}\, |e\rangle^{\otimes E}$.

3.2 Ground State of a Stabilizer Hamiltonian

We generalize the (2+1)d $\mathbb{Z}_2$-cluster state Hamiltonian [CSSZ25][VBV$^+$24][Yos16] by replacing Pauli operators with generalized Pauli operators following Fechisin [FTA25]. We again require a directed bipartite lattice as in section (3.1).
The (2+1)d (CSS) G-cluster state Hamiltonian then reads:

$$\mathbb{H}_{\mathcal{C}}^{G} = -\frac{1}{|G|}\sum_{\mathsf{e}}\sum_{\Gamma\in\mathrm{Rep}(G)} d_\Gamma \mathrm{Tr}[Z_\Gamma^{s(\mathsf{e})}\cdot Z_\Gamma^{(\mathsf{e})}\cdot Z_\Gamma^{t(\mathsf{e})\dagger}] - \frac{1}{|G|}\sum_{\mathsf{v}}\sum_{g\in G}\mathsf{Star}_g^{\mathsf{v}} \tag{10}$$

The first summation runs over all edges $\mathsf{e} \in E$. The second summation puts a *star operator* Star_g (see eq. 11) on each vertex $\mathsf{v} \in V$, which depends on the directionality of the lattice, with choice of convention for left- or right-regular multiplication as specified in subsection (3.1). We note that for the square lattice $\Lambda_\square$ with arrows pointing left-to-right and bottom-to-top, the star operators centered on vertex $\mathsf{v} \in V$ connected to edges $\mathsf{e} \in E$ are of the form:



$$:= \mathsf{Star}_g^{\mathsf{v}} \tag{11}$$

The individual terms are interpreted as stabilizers of the G-cluster state. One can show that the G-cluster state is the unique ground state (GS) of the Hamiltonian $\mathbb{H}_{\mathcal{C}}^{G}$ by counting the degrees of freedom on the lattice and the number of stabilizers contained in $\mathbb{H}_{\mathcal{C}}^{G}$. We also provide a Hamiltonian $\mathbb{H}_{\mathrm{TRIV}}^{G}$, to which one can compare $\mathbb{H}_{\mathcal{C}}^{G}$ to:

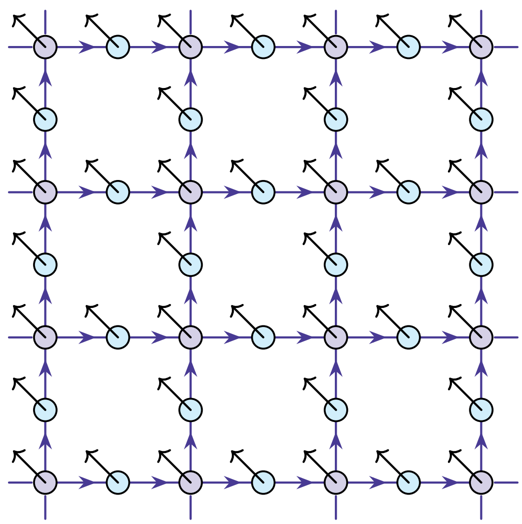
$$\mathbb{H}_{\mathrm{TRIV}}^{G} := -\frac{1}{|G|}\sum_{\mathsf{e}}\sum_{\Gamma\in\mathrm{Rep}(G)} \mathrm{Tr}[Z_\Gamma^{(\mathsf{e})}]d_\Gamma - \frac{1}{|G|}\sum_{v}\sum_{g\in G} L_g^{(v)}, \tag{12}$$

which has as GS the product state $|\psi_0^{\mathrm{TRIV}}\rangle = |\mathbb{I}\rangle^{\otimes E}\, |e\rangle^{\otimes V}$.
For the case where $G = \mathbb{Z}_2$, the generalized Pauli Z- and X-operators (Z_Γ, and R_g or L_g) reduce to the usual Pauli matrices Z and X. The Hamiltonian $\mathbb{H}_{\mathcal{C}}^{G}$ becomes:

$$\mathbb{H}_{\mathcal{C}}^{\mathbb{Z}_2} = -\sum_{\mathsf{e}} Z^{s(\mathsf{e})} Z^{(\mathsf{e})} Z^{t(\mathsf{e})} - \sum_{\mathsf{v}} \mathsf{Star}, \tag{13}$$

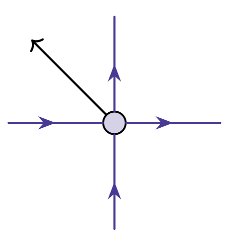
wherein we ommited the trivial stabilizers. Here Star is of the same form as eq. 11, but with Pauli X-operators only.

3.3 G-Cluster State PEPS

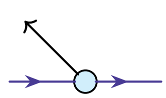


(14)

We now present the PEPS representation [CPGSV21] of the G-cluster state, which we depict in eq.(14). The horizontal- and vertical legs of each tensor indicate virtual legs (coloured blue), whereas diagonal legs of each tensor indicate physical legs (coloured black). The individual tensors[2] are given by:



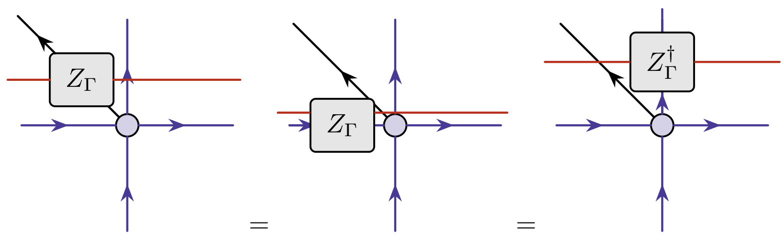
$$= \sum_{g \in G} |g\rangle_l \, |g\rangle_b \, \langle g|_r \, \langle g|_t \otimes |g\rangle \tag{15}$$



$$= \sum_{g \in G} L_g \otimes |g\rangle \tag{16}$$

[2] We proceed to write first the virtual legs of the tensor, followed by the physical leg of the tensor.

In order to establish that eq.(14) is the correct PEPS representation, it suffices to check that it stabilizes, i.e. it is in the +1 eigenspace, of all stabilizers contained in $\mathbb{H}_{\mathcal{C}}^{G}$. To this end, we give the symmetries of the vertex- and edge tensors:



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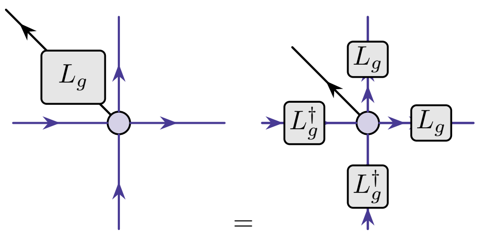

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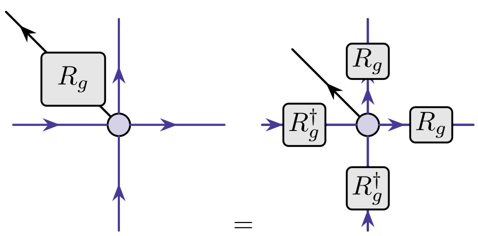

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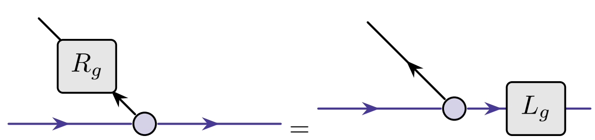

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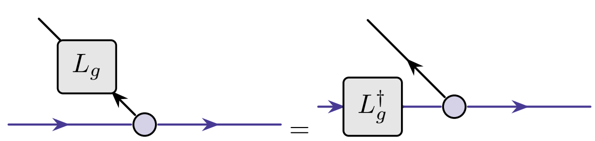

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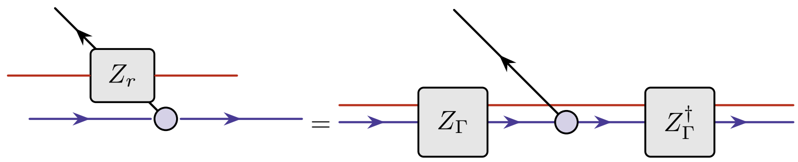


(22)

It should be clear from the above symmetries that the PEPS representation (eq.14) is indeed the GS of $\mathbb{H}_{\mathcal{C}}^{G}$. Indeed, pulling the stabilizers through to the virtual legs, one get the identity operators with $+1$ eigenvalue. Take for example a ZZZ-operator: on the vertices one pulls the first $Z_\Gamma^\dagger$ throught toward the left virtual leg of the edge tensor and the last Z_Γ throught towards the right virtual leg of the edge tensors, meanwhile one pulls the middle Z_Γ through which yields a Z_Γ and a $Z_\Gamma^\dagger$ on the left and right virtual legs off the edge tensor, respectively. What remains is the identity, with trace equal to d_Γ, such that the full term yields eigenvalue $+1$. A similar argument can be made for the terms in the second sum.

3.4 Symmetries of the G-cluster State

In this subsection, we analyse the symmetries of the G-cluster state. Specifically, we find that the state has the following independent families of global symmetries:

$$G_R^{(0)} : \overleftarrow{A}_g \coloneqq \prod_{\mathsf{v}\in V} R_g^{\mathsf{v}} \tag{23}$$

$$\mathrm{Rep}(G)^{(1)} : \mathcal{U}_\Gamma(\gamma) \coloneqq \mathrm{Tr}[\overrightarrow{\prod_{\mathsf{e}\subset\gamma}} Z_\Gamma^{\mathsf{e}}], \tag{24}$$

where γ is some closed oriented loop defined on the edges of the lattice Λ. The $\overrightarrow{\prod}$-product indicates a directed product, wherein we takes Z_Γ ($Z_\Gamma^\dagger$) when γ runs parallel (anti-parallel) to Λ. Importantly, we note that the $G_R^{(0)}$- and the $\mathrm{Rep}(G)^{(1)}$-symmetries act on different sublattices, such that they commute with one another. The superscripts (0) and (1) indicate that we are dealing with a (conventional) 0-form, and a 1-form symmetry, respectively. Furthermore, since multiplication of irreps do not generically multiply according to the rules of a group, we have that the $\mathrm{Rep}(G)^{(1)}$-symmetry is also a generalized symmetry, multiplying according to the rules of the fusion category $\mathrm{Rep}(G)$. We thus have that the $\mathrm{Rep}(G)^{(1)}$-symmetry generalizes the concept of symmetries in two ways: both in the categorical sense, and in the higher-form sense [FTA25].

4 Duality: $G \times G$ SSB $\to$ $G \times \mathrm{Rep}(G)$ SPT

In this section we give a duality argument, similar to the (1+1)d case in [FTA25]. We will show that the G-cluster state is distinct from the trivial product state by showing that upon gauging/ dualising models with different spontaneous symmetry breaking (SSB) patterns, we obtain the aforementioned states as unique GSs of the dual models. Consider the following two models defined on the directed lattice Λ' consisiting of two-component vertices labeled $V \ni \mathsf{v} \equiv (\mathsf{v}_1, \mathsf{v}_2)$:

$$\mathbb{H}_{\text{SSB1}} = -\frac{1}{|G|}\sum_{\mathsf{e}}\sum_{\Gamma\in\text{Rep}(G)} \text{Tr}[Z_\Gamma^{s(\mathsf{e})_2} \cdot Z_\Gamma^{t(\mathsf{e})_2\dagger}]d_\Gamma - \frac{1}{|G|}\sum_{\mathsf{v}}\sum_{g\in G} L_g^{\mathsf{v}_1}, \tag{25}$$

$$\mathbb{H}_{\text{SSB2}} = -\frac{1}{|G|}\sum_{\mathsf{e}}\sum_{\Gamma\in\text{Rep}(G)} \text{Tr}[Z_\Gamma^{s(\mathsf{e})_1} \cdot Z_\Gamma^{s(\mathsf{e})_2} \cdot Z_\Gamma^{t(\mathsf{e})_2\dagger} \cdot Z_\Gamma^{t(\mathsf{e})_1\dagger}]d_\Gamma \tag{26}$$

$$-\frac{1}{|G|}\sum_{\mathsf{v}}\sum_{g\in G} L_g^{\mathsf{v}}, \tag{27}$$

Note that $L_g^{\mathsf{v}} \equiv L_g^{\mathsf{v}_1} L_g^{\mathsf{v}_2}$. Both of the model Hamiltonians $\mathbb{H}_{\text{SSB1}}$ and $\mathbb{H}_{\text{SSB2}}$ have the following families of symmetries which act on the first- and second components of each vertex $\mathsf{v} \in \Lambda'$, respectively:

$$G_I : \prod_{\mathsf{v}} R_g^{\mathsf{v}_1} \tag{28}$$

$$G_{II} : \prod_{\mathsf{v}} R_g^{\mathsf{v}_2} \tag{29}$$

Since these symmetries act indepentently, the total symmetry is $G_I \times G_{II}$. In both models, the GS spontaneously breaks symmetry. For SSB1 the GS is given[3] by $|\psi_0^{\text{SSB1}}\rangle := |\mathbb{I}\rangle^{\otimes \text{V}1} |g\rangle^{\otimes V2}$ for $g \in G$, with unbroken symmetry G_I. For SSB2 the GS is given by $|\psi_0^{\text{SSB2}}\rangle := \prod_{\mathsf{e}} CL_{(s(\mathsf{e})_1, t(\mathsf{e})_2)} CR_{(t(\mathsf{e})_1, s(\mathsf{e})_2)} |\mathbb{I}\rangle^{\otimes V1} |g\rangle^{\otimes V2}$ for $g \in G$, which has unbroken symmetry[4] $\{\prod_{\mathsf{e}} R_h^{s(\mathsf{e})_1} R_{\bar{g}hg}^{t(\mathsf{e})_2} L_h^{t(\mathsf{e})_1} L_{g\bar{h}\bar{g}}^{s(\mathsf{e})_2} | h \in G\}$ for the state labeled by g. Comparing the GSs, we see that they are related by the circuit:

$$U_{\text{SSB}} := \prod_{\mathsf{e}} CL_{(s(\mathsf{e})_1, t(\mathsf{e})_2)} CR_{(t(\mathsf{e})_1, s(\mathsf{e})_2)} \tag{30}$$

such that $|\psi_0^{\text{SSB2}}\rangle = U_{\text{SSB}} |\psi_0^{\text{SSB1}}\rangle$ and $U_{\text{SSB}} \mathbb{H}_{\text{SSB1}} U_{\text{SSB}}^\dagger = \mathbb{H}_{\text{SSB2}}$.

We now apply the gauging procedure outlined in [HVAS$^+$15] to the above models. Specifically, we gauge the G_{II}-symmetry. This is done by first introducing edge degrees of freedom (d.o.f.) i.e. G-qudits on the edges between the

[3] We denote by $V1$ all the first components v_1 for each vertex $\mathsf{v} \in V \subset \Lambda'$. Similarly, we denote by $V2$ all the second components.

[4] Ambiguity of the ordering is resolved by fixing the ordering here to be clockwise around any given vertex $\mathsf{v} \in V$.

sites $\mathsf{v} \in \Lambda'$. We refer to the original sites as the matter d.o.f., and the newly introduced edge d.o.f. as gauge d.o.f.. Then we apply a group-averaged projector $\mathcal{P} = \frac{1}{|G|}\sum_{g\in G}\mathcal{P}_g$, projecting onto the locally gauge-invariant subspace. $\mathcal{P}_g$ is interpreted as a (local) Gauss law. We supplement the gauging projector by a *disentangler* $\mathcal{D}$. The disentangler, due to totally gauging the G_{II}-symmetry, decouples the (second component of the) matter d.o.f. from the gauge d.o.f.. We defer details of the procedure to appendix A. For our purpose, we only mention the following results:

$$Z_\Gamma^{s(\mathsf{e})} \cdot Z_\Gamma^{t(\mathsf{e})\dagger} \;\mapsto\; Z_\Gamma^{\mathsf{e}}, \tag{31}$$

$$L_g^{\mathsf{v}} \;\mapsto\; \prod_{s(\mathsf{e})=\mathsf{v}} L_g^{\mathsf{e}} \prod_{t(\tilde{\mathsf{e}})=\mathsf{v}} R_g^{\tilde{\mathsf{e}}}. \tag{32}$$

Applying the above results to the SSB Hamiltonians (upon a relabelling of $\mathsf{v}_1 \mapsto \mathsf{v}$) yields:

$$\mathbb{H}_{\mathrm{SSB1}} \mapsto -\frac{1}{|G|}\sum_{\mathsf{e}}\sum_{\Gamma\in\mathrm{Rep}(G)} \mathrm{Tr}[Z_\Gamma^{(\mathsf{e})}]d_\Gamma - \frac{1}{|G|}\sum_{v}\sum_{g\in G} L_g^{(v)} \tag{33}$$

$$= \mathbb{H}_{\mathrm{TRIV}}^G \tag{34}$$

[Define $\mathbb{H}_{\mathrm{TRIV}}^G$ in GS of Stabilizer Hamiltonian section, compare GS with G-cluster state]

$$\mathbb{H}_{\mathrm{SSB2}} \mapsto -\frac{1}{|G|}\sum_{\mathsf{e}}\sum_{\Gamma\in\mathrm{Rep}(G)} d_\Gamma \mathrm{Tr}[Z_\Gamma^{s(\mathsf{e})} \cdot Z_\Gamma^{(\mathsf{e})} \cdot Z_\Gamma^{t(\mathsf{e})\dagger}] - \frac{1}{|G|}\sum_{\mathsf{v}}\sum_{g\in G} \mathsf{Star}_g^{\mathsf{v}} \tag{35}$$

$$= \mathbb{H}_{\mathcal{C}}^G \tag{36}$$

$$\tag{37}$$

Hence, we find that SSB1 is mapped to a model with unique (product) GS $|\Psi_0^{SSB1}\rangle := |\mathbb{I}\rangle^{\otimes E}|e\rangle^{\otimes V}$. Meanwhile, the SSB2 Hamiltonian $\mathbb{H}_{\mathrm{SSB2}}$ is mapped to the G-cluster state Hamiltonian $\mathbb{H}_{\mathcal{C}}^G$, with unique ground state the (2+1)d G-cluster state $|\mathcal{C}\rangle$, which is a SPTO state. Finally, we note the similarity between the circuit $\mathcal{U}_{\mathcal{C}}$ used to prepare the G-cluster state in subsection 3.1 and U_{SSB}, which have equivalent (in the sense of the gauging map above) controls, and targets (even sites in the SSB model, which are mapped to edge sites in the SPT model).

-elaborate on duality argument -recreate fig 2 of Fechisin in own style

5 SPTO Signatures

5.1 Interface Modes and GS Degeneracy

5.2 String Order Parameters

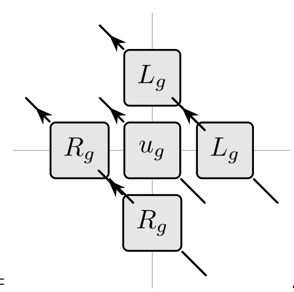
5.3 Topological response and Thouless Pump

Appendix A: Gauging a G-symmetry

We follow the gauging procedure at the level of states outlined in [HVAS+15]. We illustrate the gauging of G_{II} defined in section 4. Since G_{II} only acts upon the second components of each vertex $\mathsf{v} \equiv (\mathsf{v}_1, \mathsf{v}_2)$, we temporarily 'forget' the first components $\mathsf{v}_1 \quad \forall \mathsf{v} \in V$, and add these back in at the end of the dualitisation procedure. We use the term *dualisation* as the result of *gauging* appended by *disentangling*. The disentangler, to be introduced below, essentially seeks to decouple the gauge d.o.f. from the matter d.o.f..

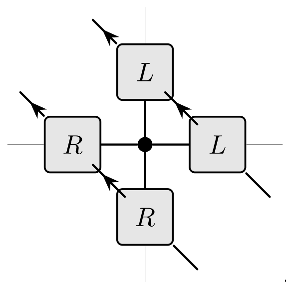
Given the directed lattice Λ', denote by E and V the sets consisting of all the edges e and vertices v, respectively. For a specific edge e, the arrow on this edge defines the source $s(\mathsf{e})$ and target $t(\mathsf{e})$. Given the finite (possibly non-Abelian) group G , introduce on each edge $\mathsf{e} \in E$ a qudit e with local Hilbert space the group algebra $\mathbb{C}[G]$ such that an arbitrary single-site state may be written as $|\phi\rangle = \sum_{g \in G} c_g |g\rangle$ where $c_g \in \mathbb{C}$. We refer to the newly introduced qudits on the edges as *gauge* d.o.f..

Next, we set up the group-averaged projector $\mathcal{P} = \frac{1}{|G|} \sum_{g \in G} \mathcal{P}_g$, with $\mathcal{P}_g := \bigotimes_{\mathsf{v} \in V} \mathcal{P}_g^{\mathsf{v}}$. For the case where Λ' is a square lattice with all edges directed left-to-right and bottom-to-top, $\mathcal{P}_g^{\mathsf{v}}$ reads:

$\mathcal{P}_g^{\mathsf{v}} :=$  , (38)

with $u_g = R_g$ [HVAS+15]. The $\mathcal{P}_g^{\mathsf{v}}$ locally enforce a *Gauss law*. We are free to choose any $g \in G$ to initialize the gauge d.o.f., and choose the trivial group element $e \in G$ for simplicity. The distinction between edges and the trivial group element should always be clear from the context. If a $g \neq e$ is chosen, one simply appends an extra layer to the quantum circuit implementing the duality to correct this lapse in judgement.

We now introduce the disentangler $\mathcal{D} := \bigotimes_{\mathsf{v} \in V} \mathcal{D}^{\mathsf{v}}$. There is no clear-cut formula to determine the form of the disentangler. Inspired by the duality procedure used for the duality argument for the (1+1)d G-cluster state in [FTA25], we find the following disentangler for the square lattice Λ':

$\mathcal{D}^{\mathsf{v}} :=$  , (39)

centered on v and consisting of generalized CX gates. The gates have as common control the vertex v and as targets the qudits e on the adjacent edges e. Importantly, we make the following observation:

$$\mathcal{D}^{\mathsf{v}}\mathcal{P}^{\mathsf{v}}_g = R^{\mathsf{v}}_g\mathcal{D}^{\mathsf{v}} \implies \mathcal{D}^{\mathsf{v}}\mathcal{P}^{\mathsf{v}} = \frac{1}{|G|}\sum_{g\in G} R^{\mathsf{v}}_g\mathcal{D}^{\mathsf{v}} := \mathcal{T}^{\mathsf{v}}\mathcal{D}^{\mathsf{v}}, \tag{40}$$

wherein we have defined the *trivializer* $\mathcal{T}^{\mathsf{v}}$ which maps the vertex state to $|\mathbb{I}\rangle$. In summary, we first applied the gauging projector $\mathcal{P}$ and subsequently applied the disentangler $\mathcal{D}$. The observation in eq. 40 significantly simplifies the quantum circuit implementing the duality, by trivializing the vertex state.
Eqs. 31 can now be rewritten in terms of the *duality* operator $\mathbb{D}$ [SS24]:

$$\mathbb{D}\Big[Z_\Gamma^{s(\mathsf{e})}\, Z_\Gamma^{t(\mathsf{e})\dagger}\Big] = Z_\Gamma^{\mathsf{e}}\,\mathbb{D} \tag{41}$$

$$\mathbb{D}\big[L_g^{\mathsf{v}}\big] = \left[\prod_{s(\mathsf{e})=\mathsf{v}} L_g^{\mathsf{e}} \prod_{t(\tilde{\mathsf{e}})=\mathsf{v}} R_g^{\tilde{\mathsf{e}}}\right]\mathbb{D}, \tag{42}$$

which are readily verified by utilizing quantum circuit diagrams.

Appendix B: Deforming the Rep$(G)^{(1)}$-symmetry

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