G-TFIM

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Abstract

In this note, we generalize the Transverse Field Ising Model (TFIM) from $\mathbb{Z}_2$ to G. The corresponding Hamiltonians will be $\text{Rep}(G)$ symmetric. We analyse string order parameters, edge modes, and entanglement spectrum degeneracies- features associated with Symmetry-Protected Topological Order (SPTO).

1 From Qubits to G-Qudits

1.1 $\mathbb{Z}_2$-TFIM

The standard TFIM Hamiltonian reads:

$$H_{TFIM} = -J \sum_i Z_i Z_{i+1}^\dagger - h \sum_i X_i + h.c. \tag{1}$$

wherein X and Z are the usual Pauli operators, $J > 0$ is the ferromagnetic coupling, and h is the transverse field strength. The model has a $\mathbb{Z}_2$ symmetry generated by $U = \Pi_i X_i$. At large field, we have the unique symmetric ground state- called the paramagnetic state $\bigotimes_i |+\rangle_i$. At zero field $h = 0$, we have spontaneous symmetry breaking SSB, and two gapped ground states- called the ferromagnetic ground states:$|\mathbf{0}\rangle = \bigotimes_i |0\rangle_i$ and $|\mathbf{1}\rangle = \bigotimes_i |1\rangle_i$. Furthermore, the symmetry breaking can be detected by acting with the symmtry operator U on the local order parameter Z_i, which in turn is charged under U.

1.2 $\mathbb{Z}_N$-TFIM

Next we extend to the Abelian group $\mathbb{Z}_N$, called the 'clock' model. For this we need to replace the qubits with N-state qudits, labbeled as $|n\rangle$. We also need to generalize the Pauli operators to 'clock operators' $Z^k |n\rangle = \exp i2\pi kn |n\rangle$, and 'shift operators' $X^k |n\rangle = |n + m\rangle$. (Reminder that addition is modulo N). The commutation relations read:

$$Z^K X^m = \exp i2\pi \frac{km}{N} X^m Z^k \tag{2}$$

Similarly to eq.1, the Hamiltonian reads:

$$H_{clock} = -J \sum_i Z_i Z_{i+1}^{\dagger} - h \sum_i X_i + h.c., \tag{3}$$

which has $\mathbb{Z}_N$ symmetry generated by $U = \Pi_i X_i$. The trivial state at large field is the symmetric superposition $\bigotimes_i (\sum_n |n\rangle)_i$, and the symmerty broken states at small field are $|\boldsymbol{n}\rangle = \bigotimes_i |n\rangle_i$- which are intercahnged upon acting with U^l. The clock operators serve as local order parameters for the SSB phase, since they are charged under U.

1.3 G-TFIM

Consider a group G, which may be non-Abalian. Local Hilbert spaces are spanned by $\{|g\rangle \, |g \in G\}$, referred to as a G-qudit. Furthermore, the Pauli operators are generalized by the left- and right regular representations:

$$\overrightarrow{X}_g |h\rangle = |gh\rangle \quad \overleftarrow{X}_g |h\rangle = |h\bar{g}\rangle \quad Z_{\Gamma,\alpha\beta} |g\rangle = \Gamma_{\alpha\beta}(g) |g\rangle \,, \tag{4}$$

which satisfy the commutation relations:

$$Z_{\Gamma,\alpha\beta} \overrightarrow{X}_g = \sum_\kappa \Gamma_{\alpha\kappa}(g) \overrightarrow{X}_g Z_{\Gamma,\kappa\beta} \tag{5}$$

The simplest G-symmetric TFIM Hamiltonian reads:

$$H_G = -J \sum_i \sum_\Gamma d_\Gamma Tr[Z_\Gamma^i \cdot Z_{\bar{\Gamma}}^{i+1}] - h \sum_i \sum_g \overleftarrow{X^i}_g + h.c. \tag{6}$$

This Hamiltonian has G-symmetry generated by $U_g = \Pi_i \overrightarrow{X}_g^i$.The first term is a projector forcing neighboring sites to have the same g-label. the second term disorders this allignment. The small field GS is given by $|\boldsymbol{g}\rangle = \bigotimes_i |g\rangle_i$, spontaneously breaking the G-symmetry. Th elarge field GS is given by the symmetric superposition on each site: $\bigotimes_i (\sum_g |g\rangle)_i$.

Under a duality, or equivalently gauging the G-symmetry, as outlined in the 'Gauging' note, we obtain the dual Hamiltonian:

$$H_{\tilde{G}} = -J \sum_i \sum_\Gamma d_\Gamma Tr[Z_\Gamma^i] - h \sum_i \sum_g \overleftarrow{X^i}_g \overrightarrow{X^{i+1}}_g = -\tilde{h} \sum_i \sum_\Gamma d_\Gamma Tr[Z_\Gamma^i] - \tilde{J} \sum_i \sum_g \overleftarrow{X^i}_g \overrightarrow{X^{i+1}}_g \tag{7}$$

In the last line, we have replaced the coupling constants as is usually done when considering the Kramers-Wannier duality of the $\mathbb{Z}_2$-TFIM. We note that this Hamiltonian has Rep(G) symmetry, which has a simple objects the irreducible representations of the group G, and which in general is non-invertible. This can be seen by acting with one of the generators R_Γ on an arbitrary state: since if $d_\Gamma > 1$ for some Γ, a conjugacy class will have character necessarily zero, thus annhilating certain states- we thus have that R_Γ has a non-trivial kernel which implies non-invertibility. Details of the Rep(G) symmetry and its algebra can also be found in toe note regarding the G-cluster state.

2 Example: Rep($\mathcal{D}_3$) SPT

TODO: comment on

- (Robust) Entanglement spectrum degenracy
- String Order Parameter (SOP)
- Domain walls, edge modes