String Order Parameters, and Interface Algebras

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Abstract

We consider a general method for constructing String Order Parameters (SOPs) which can be be used to identify SPT phases. For both of these constructions, we will first regard the interface (symmetry) algebra, which encodes the action of symmetry operators on domain walls/boundaries. We explicitely work out the action tensors, L-symbols, interface algebra, and SOPs for: $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathrm{Rep}(D_3)$. We provide the general results for $\mathrm{Rep}(G)$, where G denotes some finite group.

1 Review of MPSs and MPOs

1.1 Non-invertible symmetries as MPOs

We use Matrix Product Operator (MPO) representations of the non-invertible symmetries, described by the fusion category $\mathcal{C}$, as explicit lattice realizations of these symmetries. We label the simple objects of the $\mathcal{C}$ by $x, y, ... \in \mathrm{Simp}(\mathcal{C})$. The symmetry operators $\mathcal{O}_x$ should be thought of as topological lines. The fusion of the toplogical lines $\mathcal{O}_x$ with $\mathcal{O}_y$ should also satisfy:

$$\mathcal{O}_x \otimes \mathcal{O}_y = \sum_{z \in \mathrm{Simp}(\mathcal{C})} \sum_{\mu} (\phi_{xy}^{z,\mu}) \mathcal{O}_z (\bar{\phi}_{xy}^{z,\mu}) \tag{1}$$

where $\mu = 1, ..., N_{xy}^z$. For finite groups G, the $\phi_{xy}^{z,\mu}$ and $\bar{\phi}_{xy}^{z,\mu}$ are Generalized Clebsch-Gordan Coefficients (GCGCs). We call these $\phi_{xy}^{z,\mu}$ and $\bar{\phi}_{xy}^{z,\mu}$ the fusion- and splitting tensors, respectively. We also draw arrows on the legs of these tensors, such that a fusion tensors have two incoming- and one outgoing leg, and vice versa for the splitting tensors. Importantly, the tensors DO NOT always satisfy the strong orthogonality condition:

$$\sum_{x,y} \phi_{xy}^{z,\mu} \bar{\phi}_{xy}^{z',\nu} = \delta_{\mu,\nu} \delta_{z,z'} \mathbb{I}_z, \tag{2}$$

rather, they satisfy the weak orthogonality condition, which we come back to in the next subsection when discussing so-called 'action tensors'. We do note however that the stong orthogonality condition is satisfied if the bond dimensions

χ_x of the MPO $\mathcal{O}_x$ agree with the quantum dimensions of x for all $x \in \mathrm{Simp}(\mathcal{C})$, i.e. eq.(2) $\iff \chi_x = \dim(x) \quad \forall x \in \mathrm{Simp}(\mathcal{C})$.

The multiplication of the MPOs follow the same algebra of the simple objects of $\mathcal{C}$, namely:

$$\mathcal{O}_x \times \mathcal{O}_y = \sum_z N^z_{xy} \mathcal{O}_z \tag{3}$$

Finally, the multiplication of three simple objects $(x \otimes y) \otimes z$ and $x \otimes (y \otimes z)$ is encoded in the F-symbols, which satisfy a pentagon equation- obtained via the five ways in which one can comboine four objects. We postpone discussion of these symbols for a future note.

1.2 MPSs as symmetric gapped ground states

The action of an MPO on an MPS is encoded by so-called action tensors:

$$\mathcal{O}_x |A\rangle = \sum_i ({}^A\phi_x)_i \otimes |A\rangle \otimes ({}^A\phi_x)_i \tag{4}$$

where $i = 1, ..., d_x$. We will drop the left superscript A when it is clear from the context, but it is important to note that the action tensors are specific to the MPSs and hence also the phase which one considers. Importantly the action tensors in general DO NOT satisfy the (strong) orthogonality condition:

$$(\bar{\phi}_x)_i \otimes (\phi_{x'})_j = \delta_{i,j}\delta_{x,x'}\mathbb{I}, \tag{5}$$

rather, they satisfy the weak orthogonality condition:

$$\sum_i (\phi_x)_j \otimes [\psi] \otimes (\bar{\psi}_x)_i (\phi_x) i = (\phi_x)_j \otimes [\psi], \tag{6}$$

where $\psi \in \mathrm{Aut}(V)$ (V is the virtual/bond Hilbert space with bond dimension χ i.e $V \cong \mathbb{C}^\chi$). The strong orthogonality relation holds when the bond dimension χ_x of the MPO $\mathcal{O}_x$ is equal to $\dim(x)$ $\forall x \in \mathrm{Simp}(\mathcal{C})$. The weak orthogonality condition containing fusion- and splitting tensors reads:

$$\sum_{z,\mu} (\phi_x)_i (\phi_y)_j [(\phi^z_{xy})_\mu \otimes (\bar{\phi}^z_{xy})_\mu] = (\phi_x)_i (\phi_y)_j \tag{7}$$

Finally, the product of two action tensors on the MPS ,$|A\rangle$ also define so-calles L-symbols, defined by:

$$({}^A\phi_y)_j ({}^A\phi_y)_j |A\rangle = \sum_{z,\mu} \sum_k (L^z_{xy})^{k,\mu}_{ij} (\phi^z_{xy})_\mu ({}^A\phi_z)_k |A\rangle \tag{8}$$

The LHS of the above equation separately applies the action tensors $({}^A\phi_y)_j$ and $({}^A\phi_x)_i$, whereas the RHS first employs the fusion tensor $(\phi^z_{xy})_\mu$ and subsequently applies the action tensor $({}^A\phi_z)_k$. The L-symbols have gauge ambiguities due to the choice of action tensors $(\phi_x)i$. A (unitary) gauge transformation of the action tensors is given by:

$$(\phi_x)_i \to {}^U(\phi_x)_i = \sum_j (U_x)_{ij}(\phi_x)_j \tag{9}$$

$$(\bar{\phi}_x)_i \to {}^U(\phi_x)_i = \sum_j (\bar{\phi}_x)_j(U_x^\dagger)_{ji} \tag{10}$$

The corresponding gauge transformation o the L-symbols follows:

$$(L^z_{xy})^{k,\mu}_{ij} \to {}^U(L^z_{xy})^{z,\mu}_{ij} = \sum_{i'j'k'} (U_x)_{ii'}(U_x)_{jj'}(U_x^\dagger)_{k'k}(L^z_{xy})^{k',\mu}_{i'j'}. \tag{11}$$

Equivalence classes of L-symbols classify the $\mathcal{C}$-symmetric (SPTO) phases, whose ground states are efficiently represented by injective MPSs. By identifying the L-symbols of the associated $\mathcal{C}$-symmetric injective MPS with the data of a fiber functor $(F(x), J_{xy})$ [1], we find that the classification of (SPT) phases via $\mathcal{C}$-module categories is equivalent to the classification via fiber functors. Indeed, the constistency conditions of the isomorphism J_{xy} of the fiber functor agree with the consistency conditions on the L-symbols of the $\mathcal{C}$-module category. Furthermore, the gauge transformations of the L-symbols can be identified with the isomorphisms of the corresponding fiber functor. In conclusion: equivalence classes of L-symbols correspond to isomorphism classes of fiber functors

1.3 Example i: L-ssymbols for $\mathbb{Z}_2 \times \mathbb{Z}_2$ Cluster state

1.4 Example ii: L-symbols for $G \times$ Rep(G)-symmetric G-Cluster State

We introduced the G-cluster state in the note "G-Cluster State". There we identified the G-symmetry as the topolical lines labeled by $g \in G$ which acts with the right regular representation on odd sites. Conversely, we identified the Rep(G)-symmetry as the topological lines labelled by irreps $\rho \in \text{Rep}(G)$ which acts on even sites. We proceed to calcuate the L-symbols associated with the Rep(G)-symmetry only. Specifically, we use the following symmetry:

$$\mathcal{O}_\rho \left|\mathcal{C}_i\right\rangle = Z^{\text{phys}}_\rho \left|\mathcal{C}_i\right\rangle = (Z^{\text{virt},L}_\rho)^\dagger (Z^{\text{virt},R}_\rho) \left|\mathcal{C}_i\right\rangle, \tag{12}$$

where $|\mathcal{C}_i\rangle$ is a two-site unit cell. The bond/virtual dimension of the MPO and hence also the action tensors is the dimension of the irrep. Let $\{v^\rho_i | i = 1, 2, ..., \dim(\rho)\}$ be a basis for the virtual space V_ρ, and do the same for the dual basis V^*_ρ. This allows us to calculate the L-symbols as:

[1] A fiber functor of a fusion category $\mathcal{C}$ is defined as the tensor functor from $\mathcal{C}$ to Vec, i.e F is a functor mapping objects $x \in \mathcal{C}$ to vector space $F(x)$, whilst preserving the composition law. Furthermore, the natural transformations J_{xy} satsisfy the commutative diagram, called the hexagon equations.

$$(L^{\rho_3}_{\rho_1\rho_2})^{k,\mu}_{ij} = \frac{1}{|G|}\sum_{g\in G}\sum_{i'j'k'} \rho_1(g)^\dagger_{ii'}\rho_2(g)^\dagger_{jj'}((\phi^{\rho_3}_{\rho_1\rho_2})^{k,\mu}_{ij}\rho_3(g)_{k'k} \tag{13}$$

$$(\bar{L}^{\rho_3}_{\rho_1\rho_2})^{k,\mu}_{ij} = \frac{1}{|G|}\sum_{g\in G}\sum_{i'j'k'} \rho_3(g)^\dagger_{kk'}((\phi^{\rho_3}_{\rho_1\rho_2})^{k'}_{i'j'}\rho_1(g)_{i'i}\rho_2(g)_{j'j}, \tag{14}$$

where the fusion- and splitting tensors now evaluate to Generalized CLebsch-Gordan coefficients (GCGCs). Leveraging the Great orthogonality Theorem (GOT), we can rewrite the L-symbols as:

$$(L^{\rho_3}_{\rho_1\rho 2})^{k,\mu}_{ij} = (\phi^{\rho_3}_{\rho 1\rho 2})^{k,\mu}_{ij} \quad (\bar{L}^{\rho_3}_{\rho_1\rho 2})^{k,\mu}_{ij} = (\bar{\phi}^{\rho_3}_{\rho 1\rho 2})^{k,\mu}_{ij} \tag{15}$$

1.5 Example iii: L-symbols for Rep($\mathcal{D}_3$) -symmetric cluster state

2 Interface algebras

Consider an infinite MPS, where the left-half is built from tensors A, whilst the right-half is built from tensors B. We denote this product of half-infinte chains by $|A,B\rangle = |A\rangle_L |B\rangle_R$. Both A and B are symmetric with respect to the symmetry operators $\mathcal{O}_x$ where $x \in \mathrm{Simp}(\mathcal{C})$. When acting with one of the symmetry operators $\mathcal{O}_x$ on this state, we obtain:

$$\mathcal{O}_x |A,B\rangle = \sum_{i,i'} |A\rangle_L ({}^A\bar{\phi}_x)_i \otimes ({}^B\phi_x)_{i'} |B\rangle_R, \tag{16}$$

where $({}^A\bar{\phi}_x)_i$ acts on the rightmost edge of $|A\rangle_L$ and $({}^B\phi_x)_{i'}$ acts on the leftmost edge of $|B\rangle_R$. We hence define an element of the interface (symmetry) algebra $\mathcal{A}^{A|B}$ as:

$$\mathcal{I}^{A|B}_{x,(i,i')} = ({}^A\bar{\phi}_x)_i({}^B\phi_x)_{i'} \tag{17}$$

Multiplication in the interface algebra $\mathcal{A}^{A|B}$ follows:

$$\mathcal{I}^{A|B}_{x,(i,i')} \times \mathcal{I}^{A|B}_{y,(j,j')} = ({}^A\bar{\phi}_x)_i({}^A\bar{\phi}_y)_j \otimes \psi^{A|B} \otimes ({}^B\bar{\phi}_y)_{j'}({}^A\bar{\phi}_x)_{i'} \tag{18}$$

$$= \sum_{z,\mu}({}^A\phi_x)_i(\phi^z_{xy})_\mu({}^A\phi_y)_j \otimes \psi^{A|B} \otimes ({}^B\phi_y)_{j'}(\bar{\phi}^z_{xy})_\mu({}^B\phi_x)_{i'} \tag{19}$$

$$= \sum_{z,\mu}\sum_{k,k'}({}^A\phi_x)_i(\phi^z_{xy})_\mu({}^A\phi_y)_j({}^A\bar{\phi}_z)_k({}^A\phi_z)_k \otimes \psi^{A|B} \otimes ({}^B\bar{\phi}_z)_{k'}({}^B\phi_z)_{k'}({}^B\phi_y)_{j'}(\bar{\phi}^z_{xy})_\mu({}^B\phi_x)_{i'} \tag{20}$$

$$= \sum_{z,\mu}\sum_{k,k'}({}^A\bar{L}^z_{xy})^{k,\mu}_{ij}({}^BL^z_{xy})^{k',\mu}_{i'j'}\mathcal{I}^{A|B}_{z,(k,k')}, \tag{21}$$

wherein we have used the weak orthogonality condition in the second- and third lines, and used the definition of the L-symbols in the fourth line.

When A and B describe the same phase, we call the interface algebra a self-interface algebra, and the algebra reduces to that of the action tensors of the phase A.

Next, we show that if the interface algebra $\mathcal{A}^{A|B}$ has a one-dimensional representation (describing non-degenerate interface modes), then A and B are in the same SPT phase[2]. Conversely, the interface algebra $\mathcal{A}^{A|B}$ does not have one-dimensional representations if A and B are in different SPT phases, thus the boundary of different adjacent SPT phases is anomalous.

Statement: The interface algebra $\mathcal{A}^{A|B}$ has a one-dimensional representation iff A and B are in the same SPT phase.

Proof: $\Longleftarrow$: As noted above, when A and B are in the same SPT phase, the associated L-symbols are equivalent, up to a set of unitary matrices $\{U_x | x \in \mathrm{Simp}(\mathcal{C})\}$:

$$({}^A L^z_{xy})^{k,\mu}_{ij} = \sum_{i'j'k'} (U_x)_{ii'} (U_x)_{jj'} (U^\dagger_x)_{k'k} ({}^B L^z_{xy})^{k',\mu}_{i'j'} \tag{22}$$

Using the invertibility of the unitary matrices U_x, U_y, and U_z, we have:

$$(U_x)_{ii'} (U_y)_{jj'} = \sum_{z,\mu} \sum_{k,k'} ({}^A L^z_{xy})^{k,\mu}_{ij} ({}^B \bar{L}^z_{xy})^{k',\mu}_{i'j'} (U_z)_{kk'} \tag{23}$$

Hence, we find that the matrix elements of the unitary transformations U_x defines a one-dimensional representation of the interface algebra $\mathcal{A}^{A|B}$.

$\Longrightarrow$: Given that the interface algebra $\mathcal{A}^{A|B}$ has a one-dimensional representation, we note that also $\mathcal{A}^{B|A}$ has a one-dimensional representation. This is provided by the unitary property of the fusion category $\mathcal{C}$, which is assumed throughout this note. This in turn implies that the L-symbols can be written as unitary matrices. Given the one-dimensional representation $(U^{A|B}_x)_{ij}$ of $\mathcal{A}^{A|B}$, we obtain the corresponding one-dimensional representation of $\mathcal{A}^{B|A}$ by taking the Hermitian conjugate such that $(U^{B|A}_x)_{ij} \coloneqq (U^{A|B}_x)^\dagger_{ij} = (U^{A|B}_x)^*_{ji}$. These $(U^{B|A}_x)_{ij}$ satisfy a similar equation as eq.(23) when $(U^{B|A}_x)_{ij}$ is one-dimensional. By taking the product $(U^{A|B}_x U^{B|A}_x)_{ij} = \sum_k (U^{A|B}_x)_{ik} (U^{B|A}_x)_{kj}$, we have a one-dimensional representation of the self-interface algebra $\mathcal{A}^{A|A} = \mathcal{A}^A$. Similarly, $(U^{B|A}_x U^{A|B}_x)_{ij} = \sum_k (U^{B|A}_x)_{ik} (U^{A|B}_x)_{kj}$, defines a one-dimensional representation of the self-interface algebra $\mathcal{A}^B$. Taking powers of thes elements is still one-dimensional, and because the self-interface algebras $\mathcal{A}^A$ and $\mathcal{A}^B$ have finitely many one-dimensional representations, there exists a power n such that $(\mathcal{O}^A_x)_{ij} \coloneqq (U^{A|B}_x U^{B|A}_x)^n_{ij}$ is isomorphic to the trivial one-dimensional representation. The product $(U^{A|B}_x U^{B|A}_x)_{ij}$ thus has non-zero determinant, implying inveribility, such that eq.(23) gives an equivalence of the L-symbols ${}^A L$ and ${}^B L$, which means that A and B are in the same SPT phase.

[2] We remark that the one-dimensional representations are in one-to-one correspondence with the automorphisms of a fiber functor F which form a group $\mathrm{Aut}(F)$. This might be an important fact we'll come back to later...

2.1 Example i: $\mathbb{Z}_2 \times \mathbb{Z}_2$

2.2 Example ii: Rep(G)

2.3 Example iii: Rep($\mathcal{D}_3$)

3 String Order Parameters

We construct string order parameters (SOPs) by taking a symmetry operator, truncating it so that it acts on some finite set of sites, and subsequently decorating the loose ends by charges which we here call charge decoration operators $\mathcal{E}_x$. The charge decoration operators $\mathcal{E}_x$ essentially act on the physical leg in such a way to convert the symmetry operation to only one of the virtual legs. Explicitely, the SOPs are given by:

$$({}^{A}\mathcal{O}_x^{\text{string}})_{i;I,J} = {}^{A}(\mathcal{E}_x)_{i,I}\mathcal{O}_{x,I+1}\cdots\mathcal{O}_{x,J-1}{}^{A}(\mathcal{E}_x)_{i,J} \tag{24}$$

The indices on the charge decoration operators $\mathcal{E}_x$ label the components of the action tensors. Measurement of the SOP yields:

$$\langle\Psi|\,({}^{A}\mathcal{O}_x^{\text{string}})_{i;I,J}\,|\Psi\rangle \sim \begin{cases}(O(1) & \Psi \equiv A\\ \exp -L & \Psi \not\equiv A\end{cases}, \tag{25}$$

where the conditional equivalences denote equivalence of phases. We can intuitively interpret the SOPs as acting with the untruncated topological line on the state Ψ, which upon truncation leads to the loose ends having unused action tensors $({}^{A}\phi_x)$. These unused action tensors are then absorbed by the charge decoration operators, by popping the bubble (=using the orthogonality condition). If Ψ had been in a different phase, say phase B, the absorption can not take place, yielding a vanishing two-point correlation function.

3.1 Example i: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

The unit cell of the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetric cluster state contains two qubits $\mathbb{C}^2\otimes\mathbb{C}^2$. The symmetry is defined on the two sublattices A and B, as:

$$U_A = \Pi_{i\in A} X_i \tag{26}$$

$$U_B = \Pi_{j\in B} X_j \tag{27}$$

We locally implement these symmetries via the MPO:

$$\mathcal{O}_{(a,b)} = X(a)\otimes X(b) = X^a \otimes X^b \tag{28}$$

wherein $a, b = \{0, 1\}$. The MPO has bond dimension one, and thus no virtual legs. Applying the MPO to a specific state, yields the action tensors specific to that state: acting with $\mathcal{O}_{(a,b)}$ on the trivial product state $|\psi_0\rangle = \bigotimes_i |0\rangle_i$ gives trivial action tensors, in contrast to the cluster state (CS) $|\mathcal{C}\rangle = \bigotimes_i |0+\rangle$, which are:

$$({}^{CS}\phi_{(a,b)}) = X^a \otimes Z^b \qquad ({}^{CS}\bar{\phi}_{(a,b)}) = Z^b \otimes X^a \tag{29}$$

Pulling these action tensors back out of the MPS unit cell, yields the charge decoration operatoprs:

$$({}^{CS}\bar{\mathcal{E}}_{(a,b)}) = Z^b \otimes [Z^b X^a] \tag{30}$$

$$ {}^{CS}\mathcal{E}_{(a,b)}) = [Z^b X^a] \otimes Z^a \tag{31}$$

Gluing the charge decoration operators to the trunctated symmetry operator $U_{\text{trunc}} = \Pi_{i=I}^{J} X_{2i}^{a} X_{2i+1}^{b}$ yields the CS SOP:

$${}^{CS}\mathcal{O}_{I,J}^{\text{string}} = Z_{2I-2}^{b} \otimes [Z^b X^a]_{2I-1} \Pi_{i=I}^{J} X_{2i}^{a} X_{2i+1}^{b} [Z^b X^a]_{2J+2} \otimes Z_{2J+3}^{a} \tag{32}$$

3.2 Example ii: Rep(G)

We take as example the G-cluster state (GCS), as discussed in the note 'G-Cluster State', which has $G \times \text{Rep}(G)$ symetry. Here we only consider the Rep(G) part, defined by:

$$\hat{B}_\Gamma := \text{Tr}[\Pi_{i \text{ even}} Z_\Gamma^{(i)}] \tag{33}$$

We again implement this symmetry using MPOs. We note that acting with a Z_Γ on the physical leg of an even site is equivalent to acting with Z_Γ and $Z_\Gamma^\dagger$ on its left- and right virtual legs, respectively. By considering the previous fact, we have that the action tensors read:

$${}^{GCS}\phi_\Gamma = Z_\Gamma \qquad {}^{GCS}\bar{\phi}_\Gamma = Z_\Gamma^\dagger \tag{34}$$

Next, we find the charge decoration operators. For this, we note that for an odd site we can freely move a Z_Γ around on any of its indices. We thus have that the charge decoration operators for the GCS are:

$${}^{GCS}\mathcal{E}_\Gamma = Z_\Gamma \qquad {}^{GCS}\mathcal{E}_\Gamma = Z_\Gamma^\dagger \tag{35}$$

Finally, we can construct the SOP for the GCS by gluing the charge decoration operators at the ends of the truncated symmetry operator $\hat{B}_{\text{trunc}} = \text{Tr}[\Pi_{i=I}^{J} Z_\Gamma^{(i)}]$:

$${}^{GCS}\mathcal{O}_{I,J}^{\text{string}} = \sum_{\Gamma \neq \mathbf{1}} \frac{d_\Gamma}{|G|-1} \text{Tr}\Big[(Z_\Gamma^\dagger)^{I-1} (\Pi_{i=I}^{J} Z_\Gamma^{(i)}) Z_\Gamma^{J+1}\Big], \tag{36}$$

where I, J label even sites. Here we have summed over all the irreps, and used a suitable normalization factor as to match with the SOP of Fechisin et. al..

3.3 Example iii: Rep($\mathcal{D}_3$)

4 Outlook