(Small) Non-Abelian Groups

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Abstract

In this note, we explicitly work out some properties and representation theory of the smallest non-Abelian groups that will be used for future work. Specifically, we will write down matrix presentations of group elements for all the irreducible representations (irreps), use character theory to determine the fusion rules of irreps, and determine the Clebsch-Gordan coëfficients for the groups.

Definitions and Conventions

A representation of a finite group G is a homomorphism (i.e. a structure presserving map) $\Gamma : G \to GL_{\mathbb{C}}(V)$, where $GL_{\mathbb{C}}(V)$ is the group of invertible $\mathbb{C}$-linear transformations of the complex vector space V.We denote the dimension of the representation by d_Γ. If a representation can not be written as a direct sum of other representations, we call them irreducible representations (irreps). Equivalently, a representation is irreducible if the only invariant subspaces of V are $\{0\}$ and V itself. The most important theorem regarding irreps is Schur's Lemma, stating that any matrix which commutes with all representation matrices of a particular irrep must be a scalar multiple of the identity.
The character of a representation Γ, which we henceforth denote as $\chi_\Gamma(g)$ for $g \in G$, is defined by the tracing operation: $\chi_\Gamma(g) = \mathrm{Tr}(\Gamma(g)$. From the properties of the trace, it is obvious that the characters remain the same over a conjugacy class C_i, such that we shall also write $\chi(C_i)$.

We can decompose any reducible representation Γ_{red} into a direct sum of irreps Γ_a with multiplicity $N_a \in \mathbb{Z}$:

$$\Gamma_{red} = \bigoplus_a N_a \Gamma_a \tag{1}$$

The multiplicity N_a can be determined using the character orthogonality relation:

$$N_a = \frac{1}{|G|} \sum_{g \in G} \chi_{red}(g) \chi_a^*(g) = \frac{1}{|G|} \sum_i n_i \chi_{red}(C_i) \chi_a^*(C_i) \tag{2}$$

wherein $|G|$ is the order of the group, n_i is the order of conjugacy class C_i, and $(-)^*$ denotes the complex conjugate. The prevous two formulas allow us to consider so-called 'fusion rules', which essentially seeks to multiply two irreps, and since the product is not necessarily also irreducible, we can decompose the product into a direct sum of irreps:

$$\Gamma_\alpha \otimes \Gamma_\beta = \bigoplus_k N^\gamma_{\alpha\beta}\Gamma_k \tag{3}$$

Using again the properties of the trace operation we have that $\chi_{\alpha\otimes\beta}(g) = \chi_\alpha(g)\chi_\beta(g)$, which implies that we can determine the multiplicities $N^\gamma_{\alpha\beta}$ using:

$$N^\gamma_{\alpha\beta} = \frac{1}{|G|}\sum_i n_i[\chi_\alpha(C_i)\chi_\beta(C_i)]\chi^*_\gamma(C_i) \tag{4}$$

Whilst the fusion rules determine which irreps and the multiplicity of the irreps in the decompostion of the product, we still need matrices relating the basis states of the uncoupled/unfused tensor product to the basis of the coupled/fused states. These are the generalized CLebsch-Gordan (CG) coëfficients. Choosing bases $|a\rangle$ for $/Gamma_\alpha$ and $|b\rangle$ for Γ_β, the basis states for $\Gamma_\alpha \otimes \Gamma_\beta$ reads $|a, b\rangle$. The coupled basis states for Γ_γ reads $|c\rangle$. We then have the unitary matrix of CG coëfficients:

$$|\Gamma_\gamma c\rangle = \sum_{a,b}\begin{pmatrix}\Gamma_\alpha & \Gamma_\beta & | & \Gamma_\gamma & \mu \\ a & b & & c & \end{pmatrix}|\Gamma_\alpha a, \Gamma_\beta b\rangle = \sum_{a,b} C^{\alpha\beta\gamma}_{ab,c}\,|a, b\rangle \tag{5}$$

For simply reducible groups, i.e groups that have multiplicities of either zero or one in the irrep product decompositions, we abandon the multiplicity index μ. By leveraging the great orthogonality theorem, we get an expression for the CG coëfficients:

$$\begin{pmatrix}\Gamma_\alpha & \Gamma_\beta & | & \Gamma \\ a & b & & c\end{pmatrix} = \begin{pmatrix}\Gamma_\alpha & \Gamma_\beta & | & \Gamma \\ a_0 & b_0 & & c_0\end{pmatrix}^{-1}\frac{d_\Gamma}{|G|}\sum_{g\in G}[\Gamma_\alpha(g)]_{aa_0}[\Gamma_\beta(g)]_{bb_0}[\Gamma(g)]_{cc_0} \tag{6}$$

wherein the $\begin{pmatrix}\Gamma_\alpha & \Gamma_\beta & | & \Gamma \\ a_0 & b_0 & & c_0\end{pmatrix}$-factor amounts to a phase convention.

1 THE SYMETRIC GROUP $\mathcal{S}_3$

The symmetric group $\mathcal{S}_3$ has order 6, contains 3 conjugacy classes (labelled by C_1, C_2, and C_3), and thus also 3 irreps (labelled by A_1, A_2, and E). A_1 is the trivial irrep, A_2 is the sign irrep, and E is the two-dimensional irrep which seeks to permute the coördinates of an equilateral triangle in the plane. Table 1 summarizes the characters. The first two irreps A_1 and A_2 are one-dimensional and their matrices are given by:

$$\Gamma_{A_1}(E) = 1, \quad \Gamma_{A_1}(R) = 1, \quad \Gamma_{A_1}(W) = 1, \quad \Gamma_{A_1}(\sigma) = 1, \quad \Gamma_{A_1}(\sigma') = 1, \quad \Gamma_{A_1}(\sigma'') = 1.$$

$$\Gamma_{A_2}(E)=1,\quad \Gamma_{A_2}(R)=1,\quad \Gamma_{A_2}(W)=1,\quad \Gamma_{A_2}(\sigma)=-1,\quad \Gamma_{A_2}(\sigma')=-1,\quad \Gamma_{A_2}(\sigma'')=-1.$$

The third representation, called E, is two-dimensional, with:

$$\Gamma_E(E)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \Gamma_E(R)=\frac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix},\quad \Gamma_E(W)=\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix},$$
$$\Gamma_E(\sigma)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad \Gamma_E(\sigma')=\frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix},\qquad \Gamma_E(\sigma'')=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}.$$

One can readily check that these matrices are compatible with the charachters in table 1. For the fusion rules, which we summarize in table 2, we use eq.4. Taking as an example the fusion of $E\otimes E$:

$$N_{EE}^{A_1}=\frac{1}{6}[1(2)(2)(1)+3(0)(0)(1)+2(-1)(-1)(1)]=1$$

$$N_{EE}^{A_2}=\frac{1}{6}[1(2)(2)(1)+3(0)(0)(-1)+2(-1)(-1)(1)]=1$$

$$N_{EE}^{E}=\frac{1}{6}[1(2)(2)(2)+3(0)(0)(0)+2(-1)(-1)(-1)]=1$$

One should also confirm that the dimensions agree: $\underline{2}\otimes\underline{2}=\underline{1}\oplus\underline{1}\oplus\underline{2}$.

Table 1: Character table for the symmetric group S_3.

$\mathcal{C}\rightarrow$	C_1	$3C_2$	$2C_3$
$\Gamma\downarrow$	$\{E\}$	$\{\sigma,\sigma',\sigma''\}$	$\{R,W\}$
A_1	1	1	1
A_2	1	-1	1
E	2	0	-1

Table 2: Fusion Rules for $\mathcal{S}_3$

$\otimes$	A_1	A_2	E
A_1	A_1	A_2	E
A_2	A_2	A_1	E
E	E	E	$A_1\oplus A_2\oplus E$

We now move on to calcuate the (generalized) Clebsch)Gordan coëfficients for $\mathcal{S}_3$. We remind the reader that the CG coëfficients $C_{\alpha\beta}^{\gamma}$ are symmetric in the arguments $\alpha \leftrightarrow \beta$. Products of one-dimensional irreps are simply read off from the character table:

$$\begin{pmatrix} A_1 & A_1 & \vert & A_1 \\ 1 & 1 & & 1 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & \vert & A_2 \\ 1 & 1 & & 1 \end{pmatrix} = \begin{pmatrix} A_2 & A_2 & \vert & A_1 \\ 1 & 1 & & 1 \end{pmatrix} = 1$$

By leveraging eq.6 we can also write down:

$$\begin{pmatrix} A_1 & E & \vert & E \\ 1 & \mu & & \nu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu}, \quad \begin{pmatrix} A_2 & E & \vert & E \\ 1 & \mu & & \nu \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\mu\nu}$$

$$\left(\begin{array}{cc|c} E & E & A_1 \\ \mu & \nu & 1 \end{array} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu}, \quad \left(\begin{array}{cc|c} E & E & A_2 \\ \mu & \nu & 1 \end{array} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\mu\nu}$$

$$\left(\begin{array}{cc|c} E & E & E \\ \mu & \nu & 1 \end{array} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mu\nu}, \quad \left(\begin{array}{cc|c} E & E & E \\ \mu & \nu & 2 \end{array} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mu\nu}$$

As an example, we explicitly work out the CG coëfficients for the fusion $E \otimes E$ into the channels A_2 and E:

First we need to choose a triple (a_0, b_0, c_0) which we will use a normalization in the $\begin{pmatrix} E & E & \vert & A_2 \\ a_0 & b_0 & & c_0 \end{pmatrix}$-factor. We need to make sure this factor is nonzero. We choose $(a_0, b_0, c_0) = (1, 2, 1)$:

$$\left(\begin{array}{cc|c} E & E & A_2 \\ 1 & 2 & 1 \end{array} \right) = \frac{d_{A_2}}{|G|} \sum_{g \in S_3} [\Gamma_E(g)]_{11} [\Gamma_E(g)]_{22} [\Gamma_{A_2}(g)]_{11} = \sqrt{\frac{1}{6}[1 + \frac{1}{4} + \frac{1}{4} + 1 + \frac{1}{4} + \frac{1}{4}]} = \frac{1}{\sqrt{2}} = \mathcal{N}^{-1}$$

$$\left(\begin{array}{cc|c} E & E & A_2 \\ 1 & 1 & 1 \end{array} \right) = \frac{\mathcal{N}}{6} \left[(1)(0)(1) + 2 \cdot \left(-\frac{1}{2}\right) \left(-\frac{\sqrt{3}}{2}\right) (1) + 1 \cdot (-1)(0)(-1) + 2 \cdot \left(\frac{1}{2}\right) \left(\frac{\sqrt{3}}{2}\right) (-1) \right]$$

$$= \frac{\mathcal{N}}{6} \left[0 + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} \right] = 0$$

$$\left(\begin{array}{cc|c} E & E & A_2 \\ 1 & 2 & 1 \end{array} \right) = \frac{1}{6} \left[(1)(1)(1) + 2 \cdot \left(-\frac{\mathcal{N}}{2}\right) \left(-\frac{1}{2}\right) (1) + 1 \cdot (-1)(1)(-1) + 2 \cdot \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) (-1) \right]$$

$$= \frac{\mathcal{N}}{6} \left[1 + \frac{1}{2} + 1 + \frac{1}{2} \right] = \frac{1}{6} \cdot 3 = \frac{\mathcal{N}}{2} = \frac{1}{\sqrt{2}}$$

$$\left(\begin{array}{cc|c} E & E & A_2 \\ 2 & 1 & 1 \end{array} \right) = \frac{\mathcal{N}}{6} \left[(0)(0)(1) + 2 \cdot \left(\frac{\sqrt{3}}{2}\right) \left(-\frac{\sqrt{3}}{2}\right) (1) + 1 \cdot (0)(0)(-1) + 2 \cdot \left(\frac{\sqrt{3}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) (-1) \right]$$

$$= \frac{\mathcal{N}}{6}\left[0 - \frac{3}{2} + 0 - \frac{3}{2}\right] = \frac{1}{6}\cdot(-3) = -\frac{\mathcal{N}}{2} = -\frac{1}{\sqrt{2}}$$

$$\left(\begin{array}{cc|c} E & E & A_2 \\ 2 & 2 & 1 \end{array}\right) = \frac{\mathcal{N}}{6}\left[(0)(1)(1) + 2\cdot\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)(1) + 1\cdot(0)(1)(-1) + 2\cdot\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)(-1)\right]$$

$$= \frac{\mathcal{N}}{6}\left[0 - \frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2}\right] = 0$$

So for the CG coëfficient $\begin{pmatrix} E & E & | & A_2 \\ \mu & \nu & & 1 \end{pmatrix}$ we have indeed that $\begin{pmatrix} E & E & | & A_2 \\ \mu & \nu & & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\mu\nu}$. This could also have been derived by noting that A_2 is anti-symmetric, which implies that $(\mu,\nu) = (1,1)$ or $(2,2)$ must vanish and the coëfficient for $(1,2)$ must be negative of that of $(2,1)$. Then it only remains to normalize, which yields the desired result. TODO: repeat for $\begin{pmatrix} E & E & | & E \\ \mu & \nu & & 1 \end{pmatrix}$ and $\begin{pmatrix} E & E & | & E \\ \mu & \nu & & 2 \end{pmatrix}$

2 THE DIHEDRAL GROUP $\mathcal{D}_4$

The dihedral group $\mathcal{D}_4$ has order 8, consists of 5 conjugacy classes (labelled C_1 to C_5), and thus also 5 irreps (labelled by A_1, A_2, B_1, B_2, and E).The group is generated by the symmetries of the square, which amounts to combinations of rotations R by $\pi/2$ and flips/reflections F.Table 3 summarizes the characters of each conjugacy class for a given irrep.

Table 3: Character Table for the Dihedral Group D_4. The group is generated by a 90° rotation (R) and a reflection/flip (F). The A_i and B_i irreps are one-dimensional, whilst the E irrep is two-dimensional.

$C \rightarrow$	C_1	$2C_2$	C_3	$2C_4$	$2C_5$
$\Gamma \downarrow$	$\{E\}$	$\{R, R^3\}$	$\{R^2\}$	$\{F, R^2F\}$	$\{RF, R^3F\}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	-1	1	1	-1
B_2	1	-1	1	-1	1
E	2	0	-2	0	0

Matrix representations for $\mathcal{D}_4$ are:

$$\Gamma_{A_1}: \quad \Gamma_{A_1}(E)=1, \quad \Gamma_{A_1}(R)=1, \quad \Gamma_{A_1}(R^2)=1, \quad \Gamma_{A_1}(R^3)=1,$$
$$\Gamma_{A_1}(F)=1, \quad \Gamma_{A_1}(RF)=1, \quad \Gamma_{A_1}(R^2F)=1, \quad \Gamma_{A_1}(R^3F)=1.$$

$$\Gamma_{A_2}: \quad \Gamma_{A_2}(E)=1, \quad \Gamma_{A_2}(R)=1, \quad \Gamma_{A_2}(R^2)=1, \quad \Gamma_{A_2}(R^3)=1,$$
$$\Gamma_{A_2}(F)=-1, \quad \Gamma_{A_2}(RF)=-1, \quad \Gamma_{A_2}(R^2F)=-1, \quad \Gamma_{A_2}(R^3F)=-1.$$

$$\Gamma_{B_1}: \quad \Gamma_{B_1}(E)=1, \quad \Gamma_{B_1}(R)=-1, \quad \Gamma_{B_1}(R^2)=1, \quad \Gamma_{B_1}(R^3)=-1,$$
$$\Gamma_{B_1}(F)=1, \quad \Gamma_{B_1}(RF)=-1, \quad \Gamma_{B_1}(R^2F)=1, \quad \Gamma_{B_1}(R^3F)=-1.$$

$$\Gamma_{B_2}: \quad \Gamma_{B_2}(E)=1, \quad \Gamma_{B_2}(R)=-1, \quad \Gamma_{B_2}(R^2)=1, \quad \Gamma_{B_2}(R^3)=-1,$$
$$\Gamma_{B_2}(F)=-1, \quad \Gamma_{B_2}(RF)=1, \quad \Gamma_{B_2}(R^2F)=-1, \quad \Gamma_{B_2}(R^3F)=1.$$

$$\Gamma_E: \quad \Gamma_E(E)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \Gamma_E(R)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \Gamma_E(R^2)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad \Gamma_E(R^3)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

$$\Gamma_E(F)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \Gamma_E(RF)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \Gamma_E(R^2F)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \Gamma_E(R^3F)=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}.$$

Fusion rules for the dihedral group $\mathcal{D}_4$ are obtained by using eq.4. We explicitly calculate the decomposition of $E \otimes E$:

$$N_{EE}^{A_1}=\frac{1}{8}[1(2)(2)(1)+2(0)(0)(1)+1(-2)(-2)(1)+2(0)(0)(1)+2(0)(0)(1)]=1 \tag{7}$$

$$N_{EE}^{A_2}=\frac{1}{8}[1(2)(2)(1)+2(0)(0)(1)+1(-2)(-2)(1)+2(0)(0)(-1)+2(0)(0)(-1)]=1 \tag{8}$$

$$N_{EE}^{B_1}=\frac{1}{8}[1(2)(2)(1)+2(0)(0)(-1)+1(-2)(-2)(1)+2(0)(0)(1)+2(0)(0)(-1)]=1 \tag{9}$$

$$N_{EE}^{B_2}=\frac{1}{8}[1(2)(2)(1)+2(0)(0)(-1)+1(-2)(-2)(1)+2(0)(0)(-1)+2(0)(0)(1)] \tag{10}$$

$$N_{EE}^{E} = \frac{1}{8}[1(2)(2)(2)+2(0)(0)(0)+1(-2)(-2)(-2)+2(0)(0)01)+2(0)(0)(0)] = 0 \tag{11}$$

We thus have $E \otimes E = A_1 \oplus A_2 \oplus B_1 \oplus B_2$. The dimensions check out: $\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{1} \oplus \underline{1} \oplus \underline{1}$.

Table 4: Fusion Rules for the Dihedral Group D_4

$\otimes$	A_1	A_2	B_1	B_2	E
A_1	A_1	A_2	B_1	B_2	E
A_2	A_2	A_1	B_2	B_1	E
B_1	B_1	B_2	A_1	A_2	E
B_2	B_2	B_1	A_2	A_1	E
E	E	E	E	E	$A_1 \oplus A_2 \oplus B_1 \oplus B_2$

3 THE ALTERNATING GROUP A_4

The alternating group A_4 has order 12, consists of 4 conjugacy classes (labelled by C_1 to C_4), and thus has 4 irreps: three of which are one-dimensional (labelled A_1, A_2, and A_3), and a three-dimensional irrep (labelled by T). Denote $\omega = \exp(2\pi i/3)$.

Table 5: Character Table for the Alternating Group A_4. The A_i irreps are one-dimensional, whilst the T irrep is three-dimensional. The C_2-class corresponds to double transpositions e.g. (12)(34), the C_3-class to 3-cycles e.g. (123), and the C_4-class corresponds to inverse 3-cycles e.g. (132). $\omega = \exp(2\pi i/3)$

$C \rightarrow$ $\Gamma \downarrow$	C_1 $\{E\}$	$3C_2$ $\{(22)\}$	$4C_3$ $\{(3)\}$	$4C_4$ $\{(3)^{-1}\}$
A_1	1	1	1	1
A_2	1	1	ω	ω^2
A_3	1	1	ω^2	ω
T	3	-1	0	0

We provide the matrix representations for the generators of the group, which we label R (a 3-cycle $\Rightarrow R^3 = E$ and S (a 2-cycle/double transposition $\Rightarrow S^2 = E$):

$$\Gamma_{A_1}: \quad \Gamma_{A_1}(E) = 1, \quad \Gamma_{A_1}(R) = 1, \quad \Gamma_{A_1}(R^2) = 1, \quad \Gamma_{A_1}(S) = 1.$$

$$\Gamma_{A_2}: \quad \Gamma_{A_2}(E) = 1, \quad \Gamma_{A_2}(R) = \omega, \quad \Gamma_{A_2}(R^2) = \omega^2, \quad \Gamma_{A_2}(S) = 1.$$

$$\Gamma_{A_3}: \quad \Gamma_{A_3}(E) = 1, \quad \Gamma_{A_3}(R) = \omega^2, \quad \Gamma_{A_3}(R^2) = \omega, \quad \Gamma_{A_3}(S) = 1.$$

$$\Gamma_T: \quad \Gamma_T(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_T(R) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Gamma_T(R^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\Gamma_T(S) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Gamma_T(RSR^2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Gamma_T(R^2SR) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The fusion rules are again worked out, and summarized in table 5. For this we again used eq.4. As an example we explicitly work out $T \otimes T$:

$$N_{TT}^{A_1} = \frac{1}{12}[1(3)(3)(1) + 3(-1)(-1)(1) + 4(0)(0)(1) + 4(0)(0)(1)] = 1 \tag{12}$$

$$N_{TT}^{A_2} = \frac{1}{12}[1(3)(3)(1) + 3(-1)(-1)(1) + 4(0)(0)(\omega) + 4(0)(0)(\omega^2)] = 1 \tag{13}$$

$$N_{TT}^{A_3} = \frac{1}{12}[1(3)(3)(1) + 3(-1)(-1)(1) + 4(0)(0)(\omega^2) + 4(0)(0)(\omega)] = 1 \tag{14}$$

$$N_{TT}^{T} = \frac{1}{12}[1(3)(3)(3) + 3(-1)(-1)(-1) + 4(0)(0)(0) + 4(0)(0)(0)] = 2 \tag{15}$$

We thus have that $T \otimes T = A_1 \oplus A_2 \oplus A_3 \oplus 2T$, which also checks out dimensionally: $\underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{1} \oplus \underline{1} \oplus \underline{3} \oplus \underline{3}$.

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Table 6: Fusion Rules for the alternating group A_4. Note that the multiplcity of T in the decomposition of $T \otimes T$ is greater than 1, and thus the alternating group A_4 is not simply reducible.

$\otimes$	A_1	A_2	A_3	T
A_1	A_1	A_2	A_3	T
A_2	A_2	A_3	A_1	T
A_3	A_3	A_1	A_2	T
T	T	T	T	$A_1 \oplus A_2 \oplus A_3 \oplus 2T$