Herstein — Groups — Section 2.5 Problems

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Problem 1. If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G.

Proof. If H and K are disjoint, $H \cap K = \{e\}$ is a subgroup trivially. Then assume $H \cap K \neq \{e\}$. Let $h \in H \cap K$, so $h \in H$ and $h \in K$. Then $h^{-1} \in H$ and $h^{-1} \in K$. Thus $h^{-1} \in H \cap K$.

Let $a, b \in H \cap K$, so $a, b \in H$ and $a, b \in K$. Because H and K are subgroups, then $ab \in H$ and $ab \in K$. Thus $ab \in H \cap K$. So $H \cap K$ is a subgroup of G. □

Problem 2. Let G be a group such that the intersection of all its subgroups which are different from (e) is a subgroup different from (e). Prove that every element in G has finite order.

Proof. By contrapositive, assume there is an element $g \in G$ such that (g) has infinite order. There will always be such a g that forms a proper subgroup of G, since if $(g) = G$, (g^2) is also a subgroup of G with infinite order and does not contain every element of (g). So assume (g) is a proper subgroup of G.

Then there must be at least two distinct right cosets of (g) in G, since $e \in (g)$ means every element in G is contained in at least one right coset of (g), and if they are all identical then $(g) = G$. Let $(g)x$ and $(g)y$ for $x, y \in G$ be these distinct right cosets. $(g)x \neq (g)y \neq (e)$, but $(g)x \cap (g)y = (e)$. □

Problem 3. If G has no nontrivial subgroups, show that G must be finite of prime order.

Proof. Assume G is infinite. Then there exists $g \in G$ where (g) is a proper subgroup of G, because if $(g) = G$ then (g^2) is also a subgroup of G but doesn't contain every element of (g), so it is a nontrivial subgroup of G.

Now assume G has finite composite order, so $o(G)$ can be written mn for some $m, n \in \mathbb{Z}$. If there is no element $g \in G$ such that $(g) = G$, then (g) is a nontrivial subgroup of G for any $g \in G$. Assume $g \in G$ exists such that $(g) = G$. Then $(g^m) \neq G$ and has order n, so it is a nontrivial subgroup of G. □