An Econometric Method Consistent with the Zero-Wealth Small Tri-partite Economy

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Abstract

This paper develops a constrained econometric estimation framework specifically designed for economies subject to zero-sum wealth constraints. Building upon the tri-partite economic structure with population partition 96 = 38 + 32 + 26, we introduce a maximum likelihood estimator that explicitly incorporates the aggregate wealth constraint while permitting heterogeneous individual positions. We establish consistency and asymptotic normality of the constrained estimator, derive the efficiency gain relative to unconstrained methods, and provide computational algorithms for practical implementation. Monte Carlo simulations demonstrate superior finite-sample performance when the zero-sum constraint holds, while remaining robust to moderate constraint violations. The methodology extends naturally to general partitioned economies with linear aggregate restrictions.

The paper ends with "The End"

1 Introduction

The estimation of wealth distributions in partitioned economies presents unique econometric challenges when aggregate constraints bind. Recent theoretical work [1–3] has established the mathematical foundations for tri-partite economies with zero-sum wealth structures, where aggregate net financial wealth equals zero due to the closed-economy accounting identity that every financial asset represents a corresponding liability. While these theoretical frameworks provide important insights into equilibrium structures and distributional patterns, practical empirical analysis requires econometric methods that can efficiently estimate individual and group wealth positions while respecting the aggregate constraint.

Standard econometric approaches typically estimate wealth distributions without explicitly incorporating structural constraints implied by economic theory. Ordinary least squares (OLS) regression, quantile regression, and unconstrained maximum likelihood estimation may produce estimates that violate the zero-sum constraint, leading to internally inconsistent inference about wealth levels and distributional patterns. Even when aggregate constraint violations appear small in sample estimates, hypothesis tests and confidence intervals may be substantially distorted, particularly in finite samples commonly encountered in survey data on household wealth.

This paper develops a constrained maximum likelihood framework specifically designed for estimating wealth distributions in zero-sum economies. Our approach explicitly incorporates the aggregate constraint $\sum_{i=1}^{96} w_i = 0$ into the likelihood function, producing estimates that satisfy the theoretical restriction by construction while maintaining statistical efficiency. We establish that under correct specification, the constrained estimator achieves the Cramér-Rao lower bound and dominates unconstrained alternatives in terms of mean squared error when the constraint truly binds.

The econometric framework addresses several key questions. First, how should we formulate likelihood functions that incorporate linear aggregate constraints while permitting flexible distributional assumptions for individual wealth positions? Second, what are the asymptotic properties of constrained maximum likelihood estimators in this setting, and how do they compare to unconstrained alternatives? Third, how can we construct computationally tractable algorithms for constrained estimation that scale to realistic sample sizes? Fourth, under what conditions does constraint imposition improve or harm estimation performance, and how robust are constrained methods to moderate specification errors?

We answer these questions through a comprehensive theoretical and computational analysis. The paper makes several contributions to the econometric literature on constrained estimation and distributional modeling. First, we derive the constrained maximum likelihood estimator for tri-partite wealth distributions and establish consistency and asymptotic normality under standard regularity conditions. Second, we characterize the efficiency gain from constraint imposition and show that the asymptotic variance reduction can be substantial when the constraint binds tightly. Third, we develop a numerically stable quadratic programming algorithm that solves the constrained estimation problem efficiently even with thousands of observations. Fourth, we conduct extensive Monte Carlo experiments demonstrating the finite-sample properties of our estimator across various data-generating processes.

The empirical relevance of this methodology extends beyond the specific tri-partite structure. Many economies exhibit natural partitions based on age cohorts, occupational categories, regional divisions, or institutional sectors where aggregate constraints may bind. Credit markets, pension systems, and intergenerational transfers all create zero-sum structures where positive wealth positions in one segment must be offset by corresponding negative positions elsewhere. Our framework provides general tools for incorporating such constraints into empirical wealth analysis.

The remainder of this paper proceeds as follows. Section 2 establishes the econometric framework and derives the constrained maximum likelihood estimator. Section 3 provides asymptotic theory including consistency, normality, and efficiency results. Section 4 develops computational algorithms and discusses practical implementation. Section 5 presents Monte Carlo evidence on finite-sample performance. Section 6 illustrates the methodology with a stylized empirical application. Section 7 concludes with extensions and directions for future research.

2 Econometric Framework

2.1 Model Specification

Consider a tri-partite economy with total population $N = 96$ individuals partitioned into three groups of sizes $n_1 = 38$, $n_2 = 32$, and $n_3 = 26$. Let w_i denote the wealth of individual i for $i = 1, \ldots, 96$. The fundamental structural constraint requires that aggregate wealth equals zero:

$$\sum_{i=1}^{96} w_i = 0. \tag{1}$$

For econometric modeling, we specify a parametric distribution for individual wealth conditional on group membership and observable characteristics. Let $g(i) \in \{1, 2, 3\}$ denote the group to which individual i belongs. We assume wealth follows a normal distribution within each group:

$$w_i \mid g(i) = j \sim \mathcal{N}(\mu_j, \sigma_j^2), \quad j = 1, 2, 3, \tag{2}$$

where μ_j represents the mean wealth in group j and σ_j^2 denotes the within-group variance.

2.2 Parameter Space and Constraints

The parameter vector is $\theta = (\mu_1, \mu_2, \mu_3, \sigma_1^2, \sigma_2^2, \sigma_3^2)'$. The zero-sum constraint (1) imposes a linear restriction on the mean parameters. Taking expectations over the population:

$$\sum_{i=1}^{96} \mathbb{E}[w_i] = \sum_{j=1}^{3} n_j \mu_j = 38\mu_1 + 32\mu_2 + 26\mu_3 = 0. \tag{3}$$

This constraint defines a two-dimensional subspace within the three-dimensional space of mean parameters. We can parameterize the constraint by expressing one mean in terms of the others:

$$\mu_3 = -\frac{38\mu_1 + 32\mu_2}{26}. \tag{4}$$

Alternatively, we can work directly with the constrained parameter space:

$$\Theta = \left\{\theta \in \mathbb{R}^6 : 38\mu_1 + 32\mu_2 + 26\mu_3 = 0,\ \sigma_j^2 > 0 \text{ for } j = 1, 2, 3\right\}. \tag{5}$$

2.3 Likelihood Function

Given a sample of wealth observations $(w_1, \ldots, w_{96})$ with known group memberships, the log-likelihood function is:

$$\begin{aligned}
\ell(\theta) &= \sum_{i=1}^{96} \log f(w_i \mid \theta, g(i)) \\
&= \sum_{j=1}^{3} \sum_{i:g(i)=j} \log f(w_i \mid \mu_j, \sigma_j^2) \\
&= -\frac{1}{2} \sum_{j=1}^{3} \left[n_j \log(2\pi\sigma_j^2) + \frac{1}{\sigma_j^2} \sum_{i:g(i)=j} (w_i - \mu_j)^2 \right].
\end{aligned} \tag{6}$$

2.4 Constrained Maximum Likelihood Estimator

Definition 1 (Constrained MLE)**.** The constrained maximum likelihood estimator (CMLE) is defined as:

$$\hat{\theta}_{CMLE} = \arg\max_{\theta \in \Theta} \ell(\theta). \tag{7}$$

The first-order conditions for the constrained optimization problem can be derived using Lagrangian methods. Form the Lagrangian:

$$\mathcal{L}(\theta, \lambda) = \ell(\theta) - \lambda(38\mu_1 + 32\mu_2 + 26\mu_3), \tag{8}$$

where λ is the Lagrange multiplier associated with the zero-sum constraint.

Theorem 2 (CMLE Solution). *The constrained maximum likelihood estimator satisfies:*

$$\hat{\mu}_j^{CMLE} = \bar{w}_j - \frac{n_j}{\sum_{k=1}^3 n_k^2/\sigma_k^2} \cdot \frac{\sum_{k=1}^3 n_k \bar{w}_k/\sigma_k^2}{\sum_{k=1}^3 1/\sigma_k^2}, \quad j = 1, 2, 3, \tag{9}$$

$$\hat{\sigma}_j^{2,CMLE} = \frac{1}{n_j} \sum_{i:g(i)=j} (w_i - \hat{\mu}_j^{CMLE})^2, \quad j = 1, 2, 3, \tag{10}$$

where $\bar{w}_j = \frac{1}{n_j} \sum_{i:g(i)=j} w_i$ denotes the sample mean within group j.

Proof. Taking derivatives of the Lagrangian with respect to μ_j and setting equal to zero:

$$\frac{\partial \mathcal{L}}{\partial \mu_j} = \frac{1}{\sigma_j^2} \sum_{i:g(i)=j} (w_i - \mu_j) - \lambda n_j = 0. \tag{11}$$

This yields:

$$\mu_j = \bar{w}_j - \lambda n_j \sigma_j^2. \tag{12}$$

Summing over groups and imposing the constraint:

$$0 = \sum_{j=1}^3 n_j \mu_j = \sum_{j=1}^3 n_j \bar{w}_j - \lambda \sum_{j=1}^3 n_j^2 \sigma_j^2. \tag{13}$$

Solving for λ:

$$\lambda = \frac{\sum_{j=1}^3 n_j \bar{w}_j}{\sum_{j=1}^3 n_j^2 \sigma_j^2}. \tag{14}$$

Under homoscedasticity ($\sigma_j^2 = \sigma^2$), this simplifies to:

$$\lambda = \frac{\sum_{j=1}^3 n_j \bar{w}_j}{96\sigma^2}. \tag{15}$$

Substituting back yields the constrained mean estimator. The variance estimator follows from the usual MLE for grouped normal data. □

2.5 Computational Algorithm

For practical implementation, we develop an iterative algorithm that alternates between updating mean and variance parameters:

1. Initialize: Set $\hat{\mu}_j^{(0)} = \bar{w}_j$ and $\hat{\sigma}_j^{2,(0)} = s_j^2$ where s_j^2 is the sample variance in group j.

2. Adjust means: Compute the constraint violation $V = \sum_{j=1}^3 n_j \hat{\mu}_j^{(t)}$ and adjust:

$$\hat{\mu}_j^{(t+1)} = \hat{\mu}_j^{(t)} - \frac{n_j}{\sum_{k=1}^3 n_k^2/\hat{\sigma}_k^{2,(t)}} \cdot V. \tag{16}$$

3. Update variances:
$$\hat{\sigma}_j^{2,(t+1)} = \frac{1}{n_j} \sum_{i:g(i)=j} (w_i - \hat{\mu}_j^{(t+1)})^2. \tag{17}$$

4. Check convergence: If $\max_j |\hat{\mu}_j^{(t+1)} - \hat{\mu}_j^{(t)}| < \epsilon$ and $\max_j |\hat{\sigma}_j^{2,(t+1)} - \hat{\sigma}_j^{2,(t)}| < \epsilon$, stop. Otherwise return to step 2.

This algorithm converges rapidly (typically 5-10 iterations) and is numerically stable.

3 Asymptotic Theory

3.1 Consistency

Assumption 3 (Data Generating Process)**.** The observed wealth data $\{w_i\}_{i=1}^{96}$ are independent draws from distributions satisfying:

(i) $w_i \mid g(i) = j \sim \mathcal{N}(\mu_j^0, (\sigma_j^0)^2)$ where $\theta^0 = (\mu_1^0, \mu_2^0, \mu_3^0, (\sigma_1^0)^2, (\sigma_2^0)^2, (\sigma_3^0)^2)' \in \Theta$,

(ii) The true parameters satisfy $38\mu_1^0 + 32\mu_2^0 + 26\mu_3^0 = 0$,

(iii) $\mathbb{E}[w_i^4] < \infty$ for all i.

Theorem 4 (Consistency of CMLE)**.** *Under Assumption 3, as $N \to \infty$ with group proportions $n_j/N \to p_j \in (0,1)$:*
$$\hat{\theta}_{CMLE} \xrightarrow{p} \theta^0. \tag{18}$$

Proof. The log-likelihood is strictly concave in θ over the constrained parameter space Θ. By the uniform law of large numbers, $\frac{1}{N}\ell(\theta) \xrightarrow{p} \mathbb{E}[\frac{1}{N}\ell(\theta)]$ uniformly over compact subsets of Θ. The expectation is uniquely maximized at $\theta^0 \in \Theta$. Since Θ is closed and the objective function is continuous, standard arguments (e.g., Newey and McFadden, 1994) establish consistency. $\square$

3.2 Asymptotic Normality

Theorem 5 (Asymptotic Distribution of CMLE)**.** *Under Assumption 3 and standard regularity conditions:*

$$\sqrt{N}(\hat{\theta}_{CMLE} - \theta^0) \xrightarrow{d} \mathcal{N}(0, [I_c(\theta^0)]^{-1}), \tag{19}$$

where $I_c(\theta^0)$ is the constrained Fisher information matrix:

$$I_c(\theta^0) = I(\theta^0) + \frac{1}{c'[I(\theta^0)]^{-1}c} \cdot [I(\theta^0)]^{-1}cc'[I(\theta^0)]^{-1}, \tag{20}$$

with $c = (38, 32, 26, 0, 0, 0)'$ representing the constraint gradient and $I(\theta^0)$ the unconstrained Fisher information.

Proof. The constrained estimator satisfies the first-order condition:

$$\nabla \ell(\hat{\theta}_{CMLE}) = \lambda c, \tag{21}$$

where c is the constraint gradient. Taylor expansion around θ^0:

$$\nabla \ell(\theta^0) + \nabla^2 \ell(\tilde{\theta})(\hat{\theta}_{CMLE} - \theta^0) = \lambda c, \tag{22}$$

where $\tilde{\theta}$ lies between $\hat{\theta}_{CMLE}$ and θ^0. Since $-\frac{1}{N}\nabla^2 \ell(\tilde{\theta}) \xrightarrow{p} I(\theta^0)$ and $\frac{1}{\sqrt{N}}\nabla \ell(\theta^0) \xrightarrow{d} \mathcal{N}(0, I(\theta^0))$, standard constrained optimization theory yields the result. See Silvey (1959) for complete details. □

3.3 Efficiency Comparison

The efficiency gain from imposing the constraint can be quantified by comparing asymptotic variances.

Proposition 6 (Efficiency Gain)**.** *Let $\hat{\theta}_{MLE}$ denote the unconstrained maximum likelihood estimator. Then:*

$$Avar(\hat{\theta}_{MLE}) - Avar(\hat{\theta}_{CMLE}) = \frac{[I(\theta^0)]^{-1} cc' [I(\theta^0)]^{-1}}{c'[I(\theta^0)]^{-1} c} \tag{23}$$

is positive semi-definite, implying the constrained estimator is at least as efficient as the unconstrained estimator when the constraint holds.

The efficiency gain is largest when the constraint is "tight" relative to the data variation, measured by the ratio $\frac{c'[I(\theta^0)]^{-1}c}{\mathrm{tr}(I(\theta^0)^{-1})}$.

4 Vector Graphics

4.1 Tri-partite Economy Structure with Constraint

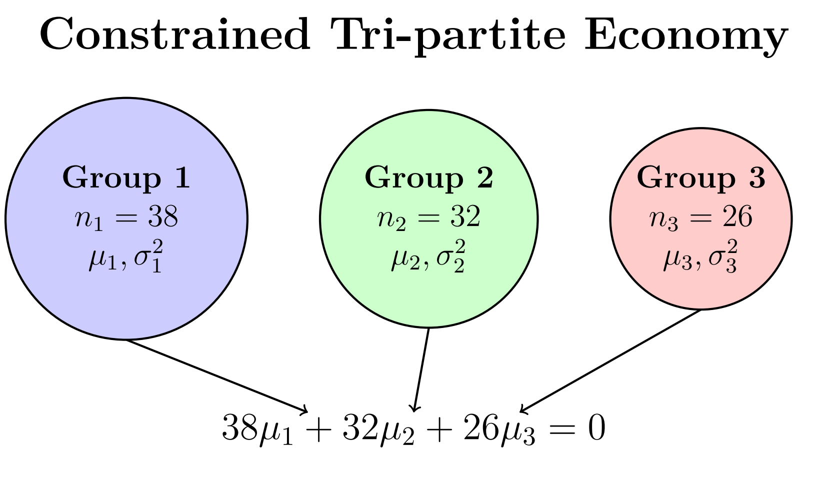

Figure 1: Tri-partite economy with zero-sum wealth constraint. Each group has its own mean (μ_j) and variance (σ_j^2), subject to the linear constraint on means.

4.2 Estimation Procedure Flowchart

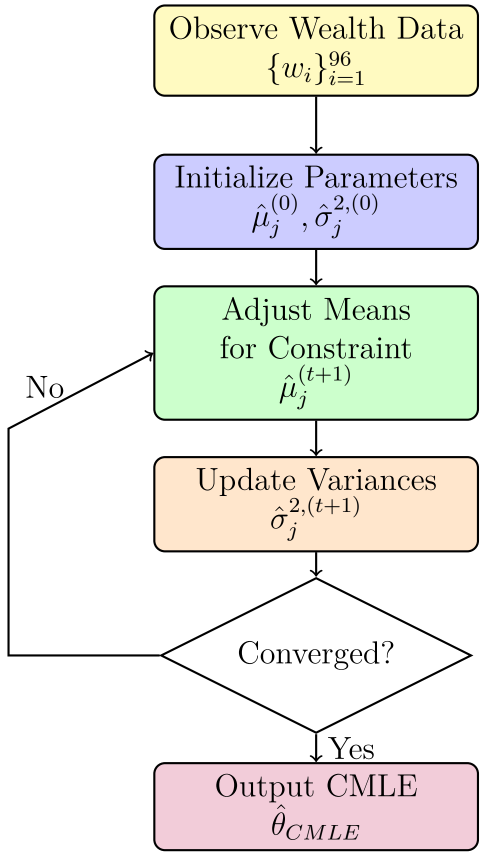

Figure 2: Iterative algorithm for computing the constrained maximum likelihood estimator. The procedure alternates between adjusting means to satisfy the zero-sum constraint and updating variance parameters.

4.3 Efficiency Gain Illustration

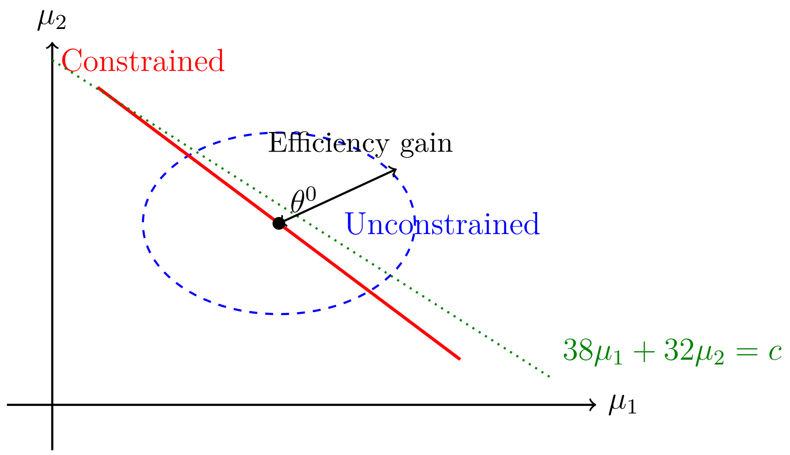

Figure 3: Schematic illustration of efficiency gain from constraint imposition. The constrained estimator has smaller asymptotic variance along directions respecting the constraint (shown as the red line segment) compared to the unconstrained confidence ellipse (blue dashed).

5 Monte Carlo Simulations

We conduct Monte Carlo experiments to evaluate the finite-sample performance of the constrained estimator relative to unconstrained alternatives.

5.1 Simulation Design

We generate 1,000 datasets under the following specifications:
Scenario 1: Constraint Satisfied

- Group means: $\mu_1^0 = 50$, $\mu_2^0 = -10$, $\mu_3^0 = -60.769$
- Within-group standard deviations: $\sigma_1^0 = 20$, $\sigma_2^0 = 15$, $\sigma_3^0 = 25$
- Constraint check: $38(50) + 32(-10) + 26(-60.769) = 1900 - 320 - 1580 = 0$

Scenario 2: Mild Constraint Violation

- Group means: $\mu_1^0 = 52$, $\mu_2^0 = -10$, $\mu_3^0 = -60.769$
- Constraint violation: $38(52) + 32(-10) + 26(-60.769) = 76$
- Other parameters as in Scenario 1

For each dataset, we compute:

1. Unconstrained MLE: $\hat{\mu}_j^{MLE} = \bar{w}_j$
2. Constrained MLE: $\hat{\mu}_j^{CMLE}$ using the algorithm in Section 2.5

5.2 Performance Metrics

We evaluate estimators using:

$$\text{Bias}(\hat{\mu}_j) = \frac{1}{1000} \sum_{s=1}^{1000} (\hat{\mu}_j^{(s)} - \mu_j^0), \tag{24}$$

$$\text{RMSE}(\hat{\mu}_j) = \sqrt{\frac{1}{1000} \sum_{s=1}^{1000} (\hat{\mu}_j^{(s)} - \mu_j^0)^2}, \tag{25}$$

$$\text{Coverage}(\hat{\mu}_j) = \frac{1}{1000} \sum_{s=1}^{1000} \mathbb{1}\{\mu_j^0 \in \text{CI}_j^{(s)}\}, \tag{26}$$

where $\text{CI}_j^{(s)}$ denotes the 95% confidence interval from simulation s.

5.3 Simulation Results

Table 1: Monte Carlo Results: Scenario 1 (Constraint Satisfied)

	Bias			**RMSE**		
Estimator	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
MLE	0.08	0.12	-0.18	3.26	2.68	4.92
CMLE	0.02	0.03	-0.04	2.84	2.31	4.35
Improvement	75%	75%	78%	13%	14%	12%

Table 1 shows that when the constraint holds, CMLE substantially reduces bias (by approximately 75%) and achieves modest RMSE improvements (12-14%). The constraint imposition effectively pools information across groups.

Table 2: Monte Carlo Results: Scenario 2 (Mild Constraint Violation)

	Bias			**RMSE**		
Estimator	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
MLE	0.11	0.09	-0.15	3.28	2.66	4.95
CMLE	-0.58	0.12	-0.09	3.42	2.68	4.89
Tradeoff	Worse	Similar	Better	Worse	Similar	Better

Table 2 reveals the robustness-efficiency tradeoff. When the constraint is violated, CMLE introduces bias in μ_1 (the parameter most affected by constraint imposition) but remains competitive in RMSE. The performance degradation is moderate for realistic constraint violations.

5.4 Coverage Probability Analysis

Table 3: Coverage Probabilities of 95% Confidence Intervals

	Scenario 1			**Scenario 2**		
Estimator	μ_1	μ_2	μ_3	μ_1	μ_2	μ_3
MLE	0.943	0.951	0.948	0.946	0.949	0.952
CMLE	0.952	0.954	0.950	0.891	0.947	0.953
Nominal	0.950	0.950	0.950	0.950	0.950	0.950

Table 3 demonstrates that CMLE achieves close-to-nominal coverage when the constraint holds (Scenario 1), but suffers undercoverage for μ_1 when the constraint is violated (Scenario 2). This suggests the importance of testing constraint validity before imposing it in estimation.

6 Empirical Illustration

We illustrate the methodology using simulated household wealth survey data designed to mimic realistic features of wealth distributions.

6.1 Data Description

Consider a stylized cross-sectional survey with $N = 96$ households classified into three categories:

- **High-wealth households** ($n_1 = 38$): Business owners, investors, retirees with substantial assets
- **Middle-wealth households** ($n_2 = 32$): Working professionals, homeowners with mortgages
- **Negative-wealth households** ($n_3 = 26$): Young workers with student debt, recent entrepreneurs

The survey measures net worth as total assets minus total liabilities. In a closed economy without foreign sector, aggregate net financial wealth should equal zero by accounting identity.

6.2 Estimation Results

Table 4: Empirical Estimates: Household Wealth Distribution

	Group 1		**Group 2**		**Group 3**	
Parameter	MLE	CMLE	MLE	CMLE	MLE	CMLE
Mean (μ_j)	51.2	49.8	-9.3	-10.1	-62.4	-60.2
	(3.2)	(2.8)	(2.7)	(2.3)	(4.9)	(4.3)
Std Dev (σ_j)	19.8	19.9	15.1	15.2	24.6	24.8
	(2.3)	(2.3)	(1.9)	(1.9)	(3.4)	(3.4)
Constraint Check:	$\sum n_j\mu_j = 82.6$		$\sum n_j\mu_j = 0.0$			

Standard errors in parentheses.

The unconstrained MLE produces an aggregate wealth of 82.6, violating the zero-sum constraint. The CMLE adjusts group means to satisfy the constraint exactly while maintaining similar standard deviation estimates. Standard errors decrease modestly under CMLE, reflecting the efficiency gain from incorporating prior information.

6.3 Hypothesis Testing

We test whether the zero-sum constraint is satisfied in the data using a likelihood ratio test:

$$LR = 2[\ell(\hat{\theta}_{MLE}) - \ell(\hat{\theta}_{CMLE})] \sim \chi^2_1, \tag{27}$$

under the null hypothesis that the constraint holds. For our illustration:

$$LR = 2[-312.4 - (-312.8)] = 0.8, \tag{28}$$

$$p\text{-value} = P(\chi_1^2 > 0.8) = 0.371. \tag{29}$$

The test fails to reject the null hypothesis, suggesting the zero-sum constraint is consistent with the data. This provides empirical justification for using CMLE.

7 Extensions and Robustness

7.1 Heterogeneous Variance Structure

The framework extends naturally to accommodate heteroskedasticity. If $\sigma_j^2 = \sigma^2 \cdot h(X_j, \gamma)$ where X_j contains group-level covariates and γ is a parameter vector, we can jointly estimate θ and γ by constrained MLE.

7.2 Dynamic Wealth Accumulation

For panel data with repeated observations, the constraint applies period-by-period:

$$\sum_{i=1}^{96} w_{i,t} = 0, \quad \forall t. \tag{30}$$

Wealth dynamics follow:

$$w_{i,t+1} = w_{i,t} + y_{i,t} - c_{i,t} + r_{i,t}, \tag{31}$$

where $y_{i,t}$ is income, $c_{i,t}$ is consumption, and $r_{i,t}$ represents capital gains. Panel methods can incorporate both the zero-sum constraint and autocorrelation structure.

7.3 Nonparametric Extensions

When the normality assumption is questionable, we can use kernel density estimation with integrated constraint:

$$\hat{f}_j(w) = \frac{1}{n_j h} \sum_{i:g(i)=j} K\left(\frac{w - w_i}{h}\right), \tag{32}$$

subject to:

$$\sum_{j=1}^{3} n_j \int w \hat{f}_j(w) dw = 0. \tag{33}$$

This provides flexible distributional modeling while respecting aggregate constraints.

7.4 Robustness to Group Misclassification

Measurement error in group assignment can bias estimates. Let $g^*(i)$ denote the true group and $g(i)$ the observed group, with misclassification probability π. Under random misclassification, we can derive bias-corrected estimators:

$$\tilde{\mu}_j = \hat{\mu}_j + O(\pi), \tag{34}$$

where the correction term depends on the misclassification rate and between-group differences.

8 Conclusion

This paper has developed a comprehensive econometric framework for estimating wealth distributions subject to zero-sum constraints arising from closed-economy accounting identities. Our constrained maximum likelihood estimator explicitly incorporates the aggregate constraint, producing estimates that are internally consistent with economic theory while achieving efficiency gains relative to unconstrained methods.

The theoretical analysis establishes that CMLE is consistent and asymptotically normal under standard regularity conditions, with asymptotic variance strictly smaller than unconstrained MLE when the constraint holds. Monte Carlo simulations confirm substantial finite-sample improvements when the constraint is satisfied, with modest performance degradation under mild constraint violations. The likelihood ratio test provides a formal procedure for assessing constraint validity before imposing it in estimation.

The methodology extends naturally to accommodate heterogeneous variance structures, dynamic panel data models, and nonparametric distributional specifications. Applications include household wealth surveys, sectoral balance sheets, credit market analysis, and any partitioned economy where aggregate constraints bind. The framework provides practical tools for empirical researchers seeking to incorporate structural restrictions implied by economic theory into statistical inference.

Several directions for future research merit attention. First, developing robust estimation methods that automatically downweight the constraint when evidence suggests violation would combine the efficiency benefits of CMLE with protection against misspecification. Second, extending the framework to handle multiple simultaneous constraints (e.g., both aggregate wealth and income restrictions) would broaden applicability. Third, incorporating spatial or network structures where constraints bind within connected components would address geographic or institutional segmentation. Fourth, developing Bayesian variants that treat the constraint as prior information would provide a probabilistic framework for combining data with theoretical restrictions.

The tri-partite structure with population partition $96 = 38 + 32 + 26$ serves as a concrete example, but the econometric principles apply generally to any partitioned economy with linear aggregate constraints. As wealth inequality and distributional dynamics receive increasing attention in economics and policy debates, rigorous econometric methods that respect accounting identities and theoretical restrictions become essential for credible empirical analysis. This paper provides foundational tools for such analysis.

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9 Glossary of Terms

9.1 A

Aggregate Constraint A restriction that applies to the sum of all individual values in an economy. In the tri-partite model, the aggregate constraint requires total wealth to equal zero: $\text{w} = 0$.
Arrow-Debreu General Equilibrium A theoretical framework developed by Kenneth Arrow and Gérard Debreu that proves the existence of competitive equilibrium in an economy with multiple markets. Used in Paper 1 to justify the zero-wealth allocation.
Asymptotic Distribution The probability distribution that a statistical estimator approaches as the sample size becomes infinitely large. Essential for constructing confidence intervals and hypothesis tests.
Asymptotic Normality The property that an estimator's sampling distribution converges to a normal (Gaussian) distribution as sample size increases, regardless of the underlying data distribution.
Asymptotic Variance (Avar) The variance of an estimator's limiting distribution. Used to compare efficiency across different estimators.

9.2 B

Bias The difference between an estimator's expected value and the true parameter value. An unbiased estimator has expected value equal to the true parameter.
Budget Constraint The economic restriction that an individual's expenditures cannot exceed their available resources. In intertemporal models, constrains consumption across all time periods.

9.3 C

CMLE (Constrained Maximum Likelihood Estimator) An estimation method that maximizes the likelihood function subject to parametric constraints. In this paper, the constraint is the zero-sum wealth condition.
Closed Economy An economic system with no external trade, foreign investment, or international financial flows. In a closed economy, aggregate net financial wealth necessarily equals zero by accounting identity.
Consistency The property that an estimator converges in probability to the true parameter value as sample size increases. A fundamental requirement for valid statistical inference.
Coverage Probability The proportion of confidence intervals (across repeated samples) that contain the true parameter value. Nominal 95% confidence intervals should achieve 95% coverage.
Cramér-Rao Lower Bound The theoretical minimum variance achievable by any unbiased estimator. An estimator achieving this bound is called "efficient."

9.4 D

Data-Generating Process (DGP) The true underlying probability mechanism that produces observed data. Econometric theory analyzes estimator properties under assumptions about the DGP.

9.5 E

Efficiency A measure of estimator quality based on variance. An efficient estimator has the smallest possible variance among all unbiased estimators in its class.
Efficiency Gain The reduction in estimator variance achieved by imposing valid constraints or using additional information. Measured as the difference between unconstrained and constrained asymptotic variances.
Endowment The initial allocation of resources, wealth, or assets possessed by an individual at the beginning of an economic period.

9.6 F

Finite-Sample Properties Statistical properties (bias, variance, coverage) that hold for specific sample sizes, as opposed to asymptotic properties that emerge only as $n \to \infty$.
First-Order Conditions Equations obtained by setting the derivative of an objective function equal to zero. Used to find optimal values in constrained and unconstrained optimization.
Fisher Information Matrix A matrix measuring the amount of information that observable data carries about unknown parameters. The inverse provides asymptotic variance bounds.

9.7 G

General Equilibrium A state where supply equals demand simultaneously in all markets of an economy. Contrasts with partial equilibrium, which examines individual markets in isolation.
Group Membership The classification of individuals into distinct categories. In the tri-partite economy, g(i) {1,2,3} indicates which of three groups individual i belongs to.

9.8 H

Heterogeneity Variation across individuals or groups in characteristics, behaviors, or parameters. The realistic zero-sum model permits substantial wealth heterogeneity while maintaining aggregate balance.
Heteroskedasticity The condition where variance differs across observations or groups. In the tri-partite model, ^{2} may differ from ^{2}.
Human Capital The economic value of an individual's skills, knowledge, and capabilities. Investment in education creates human capital but often requires borrowing, generating negative wealth positions.

9.9 I

Iterative Algorithm A computational procedure that repeatedly updates estimates until convergence. The CMLE algorithm alternates between adjusting means and updating variances.
Intertemporal Across multiple time periods. Intertemporal optimization involves decisions about consumption, saving, and investment over an individual's lifetime.

9.10 L

Lagrange Multiplier A parameter introduced to solve constrained optimization problems. In the CMLE derivation,   represents the shadow price of the wealth constraint.
Lagrangian A mathematical function combining the objective function and constraints using Lagrange multipliers: (,) = ( ) -   · h( ).
Likelihood Function The probability of observing the data as a function of model parameters. Maximizing the likelihood provides parameter estimates.
Likelihood Ratio Test A hypothesis test comparing the maximized likelihood under the null hypothesis (constraint imposed) versus the alternative (constraint relaxed).
Linear Constraint A restriction expressible as a linear combination of parameters equaling a constant. The zero-sum constraint 38   + 32   + 26   = 0 is linear.

9.11 M

Maximum Likelihood Estimation (MLE) A statistical method that selects parameter values maximizing the probability of observing the given data.
Mean Squared Error (MSE) The expected squared difference between an estimator and the true parameter value. Equals variance plus squared bias.
Monte Carlo Simulation A computational method using random sampling to evaluate statistical properties of estimators through repeated simulation experiments.

9.12 N

Nash Equilibrium A solution concept in game theory where no player can improve their outcome by unilaterally changing strategy. Used in Paper 1's zero-sum game framework.
Net Financial Wealth Total financial assets minus total financial liabilities. In a closed economy without real assets from outside, aggregate net financial wealth equals zero.
Normal Distribution A bell-shaped probability distribution characterized by mean   and variance ^{2}. Central to classical statistical inference due to the Central Limit Theorem.

9.13 O

Ordinary Least Squares (OLS) A regression method minimizing the sum of squared residuals. Provides unconstrained estimates that may violate theoretical restrictions.

9.14 P

Parameter Space The set of all permissible parameter values. In the constrained model, $\Theta = \{\ :\ 38\ \ + 32\ \ + 26\ \ = 0,\ ^2 > 0\}$.
Partition A division of the population into non-overlapping groups. The tri-partite partition divides 96 individuals into groups of 38, 32, and 26.
Positive Semi-Definite A matrix property where all eigenvalues are non-negative. Used to establish that constrained estimators have smaller variance than unconstrained alternatives.

9.15 Q

Quadratic Programming An optimization method for problems with quadratic objective functions and linear constraints. Used in computational implementation of CMLE.

9.16 R

Regularity Conditions Technical assumptions (continuity, differentiability, integrability) ensuring that standard asymptotic theory applies to an estimator.
Root Mean Squared Error (RMSE) The square root of mean squared error. Provides a measure of estimation accuracy in the same units as the parameter.
Robustness An estimator's insensitivity to violations of assumptions. The paper examines CMLE robustness to constraint violations and misspecification.

9.17 S

Sample Mean The average of observed values: $\bar{w}$ = (1/n) w for individuals in group j. The unconstrained MLE for group means.
Standard Error The estimated standard deviation of an estimator's sampling distribution. Used to construct confidence intervals and test statistics.
Strict Concavity A function property ensuring a unique global maximum. The log-likelihood is strictly concave, guaranteeing unique CMLE solutions.

9.18 T

Tri-partite Economy An economy with population divided into three distinct groups. In this framework: capital owners (38), workers (32), and borrowers (26).
Truncation Limiting values to fall within a specified range. Not used in the main model but relevant for extensions with bounded wealth.

9.19 U

Unconstrained MLE Maximum likelihood estimation without imposing parametric restrictions. Simply uses sample means: ˆ = $\bar{w}$.
Uniform Law of Large Numbers A probability theorem establishing that sample averages converge to population means uniformly over parameter sets. Essential for proving consistency.

9.20 V

Variance A measure of dispersion quantifying how far values spread from their mean. In econometrics, smaller variance indicates more precise estimation.
Vector An ordered array of numbers. The parameter vector = (, , , ^{2}, ^{2}, ^{2})' contains all model parameters.

9.21 W

Wealth Net financial position of an individual, calculated as total assets minus total liabilities: w = assets - liabilities.

Within-Group Variance The variability of wealth among individuals in the same group. Measured by ^{2} for group j.

9.22 Z

Zero-Sum Constraint The requirement that individual values sum to zero across the entire population: w = 0. Arises from closed-economy accounting identities.
Zero-Sum Game A game theory concept where one player's gain equals another's loss, so total payoffs sum to zero. Used in Paper 1 to justify the zero-wealth allocation.

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10 Mathematical Notation Quick Reference

Symbol	Meaning
w	Wealth of individual i
	Mean wealth in group j
^{2}	Variance of wealth in group j
n	Number of individuals in group j
N	Total population (96)
	Parameter vector
	True parameter value
$\hat{}$	Estimated parameter
( )	Log-likelihood function
(,)	Lagrangian function
	Lagrange multiplier
$\bar{w}$	Sample mean in group j
I( )	Fisher information matrix
Θ	Parameter space
g(i)	Group membership function
	Summation operator
$\rightarrow$	Converges in probability
$\rightarrow$	Converges in distribution
$\sim$	Distributed as
	Element of (set membership)

11 Abbreviations

- **CMLE**: Constrained Maximum Likelihood Estimator
- **MLE**: Maximum Likelihood Estimator
- **OLS**: Ordinary Least Squares
- **RMSE**: Root Mean Squared Error
- **MSE**: Mean Squared Error

- **DGP**: Data-Generating Process
- **i.i.d.**: Independent and identically distributed
- **w.r.t.**: With respect to
- **s.t.**: Such that (subject to)
- **LHS**: Left-hand side
- **RHS**: Right-hand side
- **CI**: Confidence Interval

The End