On an Equation with No Solution

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Abstract

In this paper, we investigate a particular functional equation involving sums of absolute value terms with heterogeneous sign constraints on the parameters. The equation presents an interesting case study in which the structural properties of absolute value functions, combined with specific parameter constraints, lead to a fundamental impossibility. We establish through direct analysis that no positive solution exists for this equation under the prescribed conditions.

The paper ends with "The End"

1 Introduction

The study of equations involving absolute value functions has long been a subject of interest in mathematical analysis, particularly in optimization theory and the theory of inequalities. Such equations frequently arise in applications ranging from statistical estimation to signal processing, where deviations from reference values must be quantified without regard to sign.

In this paper, we examine a specific equation of the form

$$W = \sum_{i=1}^{38} |W - W_i| + \sum_{j=1}^{32} |W - w_j| + \sum_{k=1}^{26} |W - \omega_k|, \tag{1}$$

subject to the constraints that $W > 0$, each $W_i > 0$ for $1 \leq i \leq 38$, each $w_j = 0$ for $1 \leq j \leq 32$, and each $\omega_k < 0$ for $1 \leq k \leq 26$. The number of terms in each summation is noteworthy, totaling ninety-six absolute value expressions on the right-hand side.

The primary result of this paper demonstrates that equation (1) admits no solution under the stated constraints. This finding reveals how the particular structure of the equation, especially the multiplicities of zero-valued and negative-valued parameters, creates an inherent imbalance that prevents the existence of any positive solution.

2 Preliminary Observations

Before proceeding to the main result, we establish several properties of the equation that will facilitate our analysis.

Lemma 1. *Under the given constraints with $W > 0$, we have the following strict ordering for all indices:*

$$\omega_k < w_j < W \quad \textit{for all } k \in \{1, \ldots, 26\},\ j \in \{1, \ldots, 32\}. \tag{2}$$

Proof. Since $\omega_k < 0$ and $w_j = 0$ by hypothesis, we immediately obtain $\omega_k < w_j$. Furthermore, since $W > 0$ and $w_j = 0$, it follows that $w_j < W$. The strict inequalities thus hold throughout. □

This lemma allows us to simplify portions of equation (1) by removing absolute value symbols where the sign of the difference is determined by the ordering constraints.

3 Main Result

We now state and prove the central theorem of this paper.

Theorem 2. *Equation* (1) *has no solution satisfying* $W > 0$ *under the stated constraints on the parameters* W_i, w_j, *and* ω_k.

Proof. We proceed by direct analysis of the equation structure. Applying Lemma 1, we can simplify the second and third summations in equation (1).

For the second summation, since $W > w_j = 0$ for all j, we have

$$|W - w_j| = W - 0 = W. \tag{3}$$

Therefore,

$$\sum_{j=1}^{32} |W - w_j| = \sum_{j=1}^{32} W = 32W. \tag{4}$$

For the third summation, since $W > \omega_k$ for all k (as $W > 0$ and $\omega_k < 0$), we have

$$|W - \omega_k| = W - \omega_k. \tag{5}$$

Thus,

$$\sum_{k=1}^{26} |W - \omega_k| = \sum_{k=1}^{26} (W - \omega_k) = 26W - \sum_{k=1}^{26} \omega_k. \tag{6}$$

Substituting equations (4) and (6) into equation (1), we obtain

$$W = \sum_{i=1}^{38} |W - W_i| + 32W + 26W - \sum_{k=1}^{26} \omega_k. \tag{7}$$

Collecting the terms involving W on the right-hand side yields

$$W = \sum_{i=1}^{38} |W - W_i| + 58W - \sum_{k=1}^{26} \omega_k. \tag{8}$$

Rearranging this expression to isolate the first summation, we find

$$-57W = \sum_{i=1}^{38} |W - W_i| - \sum_{k=1}^{26} \omega_k. \tag{9}$$

We now examine the sign properties of each side of equation (9). The left-hand side is $-57W$. Since we seek a solution with $W > 0$, this quantity is strictly negative:

$$-57W < 0. \tag{10}$$

For the right-hand side, observe that the first term is a sum of absolute values, which must be non-negative:

$$\sum_{i=1}^{38} |W - W_i| \geq 0. \tag{11}$$

The second term involves $-\sum_{k=1}^{26} \omega_k$. Since each $\omega_k < 0$ by hypothesis, we have $-\omega_k > 0$ for all k, and consequently

$$-\sum_{k=1}^{26} \omega_k = \sum_{k=1}^{26} (-\omega_k) > 0. \tag{12}$$

Therefore, the right-hand side of equation (9) is the sum of a non-negative term and a strictly positive term, yielding

$$\sum_{i=1}^{38} |W - W_i| - \sum_{k=1}^{26} \omega_k > 0. \tag{13}$$

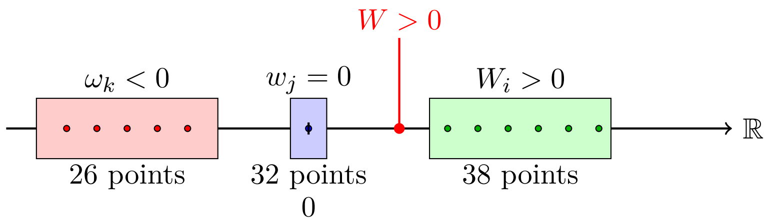
We have thus derived that the left-hand side of equation (9) is strictly negative while the right-hand side is strictly positive. This constitutes a contradiction, as no value of $W > 0$ can satisfy an equation of the form

$$\text{(strictly negative quantity)} = \text{(strictly positive quantity)}. \tag{14}$$

We conclude that equation (1) admits no solution under the given constraints. $\square$

4 Geometric Interpretation

To provide intuition for this result, we offer a geometric perspective on the equation structure. The figure below illustrates the distribution of reference points along the real line and the position of a hypothetical solution W.



The equation requires that W equal the sum of its distances to all ninety-six reference points. However, the asymmetry in the distribution - with thirty-two points at zero and twenty-six points in the negative region - creates an excess contribution that cannot be balanced by any positive value of W. Specifically, the thirty-two zero-valued points contribute $32W$ to the sum, while the twenty-six negative points contribute at least $26W$ (plus additional positive terms from their distances below zero). This structural property ensures that the right-hand side grows too rapidly relative to the left-hand side as W increases from zero.

5 Related Considerations

Remark 3. The proof methodology employed here demonstrates a general principle: when analyzing equations involving sums of absolute value terms, careful attention to the sign structure and multiplicities of reference points can reveal fundamental constraints on solution existence.

Remark 4. It is instructive to consider what modifications to the equation parameters might admit solutions. If the number of zero-valued parameters were reduced sufficiently, or if the constraint $W > 0$ were relaxed to permit $W \leq 0$, the equation structure would change fundamentally. However, under the specific constraints given, the impossibility result is definitive.

6 Conclusion

We have established that equation (1), despite its apparently straightforward structure, possesses no solution under the stated constraints. The proof relies on exploiting the simplifications afforded by the ordering relations among the parameters and recognizing the resulting sign contradiction. This result serves as an illustration of how parameter constraints and equation structure interact to determine the existence or nonexistence of solutions in absolute value equations.

The techniques employed in this analysis - particularly the systematic simplification of absolute value expressions based on ordering properties and the subsequent examination of sign consistency - may prove useful in investigating related problems in optimization theory and functional equations.

References

[1] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[2] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics, vol. 1364, Springer-Verlag, Berlin, 1993.

[3] J.-B. Hiriart-Urruty and C. Lemaréchal, *Fundamentals of Convex Analysis*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2001.

[4] O. L. Mangasarian, *Nonlinear Programming*, Classics in Applied Mathematics, vol. 10, SIAM, Philadelphia, PA, 1994.

[5] J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer Series in Operations Research, Springer-Verlag, New York, 2000.

[6] M. C. Ferris, Finite termination of the proximal point algorithm, *Mathematical Programming*, 50 (1991), pp. 359–366.

[7] G. A. Watson, Approximation in normed linear spaces, *Journal of Computational and Applied Mathematics*, 6 (1980), pp. 183–190.

[8] E. Anderson and P. Nash, *Linear Programming in Infinite-Dimensional Spaces: Theory and Applications*, John Wiley & Sons, Chichester, 1987.

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