# A Solution to Sustainable Banking

A Mathematical Framework for Equilibrium Deposit Pricing

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**Abstract**

This paper develops a comprehensive mathematical framework for sustainable banking by deriving non-trivial functional forms that satisfy the zero excess return condition ($r_e = 0$). We establish equilibrium relationships between deposit premiums, inflation, and risk premiums, incorporating stochastic processes, term structure dynamics, and macroeconomic factors. The analysis demonstrates that sustainable banking requires the deposit premium to precisely compensate for inflation and its associated uncertainty.

The paper ends with "The End"

# 1 Introduction

The sustainability of banking institutions depends critically on the equilibrium between returns offered to depositors and the underlying economic risks. We begin with three fundamental relationships governing deposit pricing.

## 1.1 Fundamental Equations

**Definition 1** (Maturity Value). *The amount received at maturity is given by:*

$$A = \frac{P(1 + by)}{1 + r_f + p_d} \tag{1}$$

*where $P$ is principal, $b$ is the yearly bank rate, $y$ is the number of years, $r_f$ is the risk-free rate, and $p_d$ is the deposit premium.*

**Definition 2** (Real Bank Return).

$$r_b = p_d - i \tag{2}$$

*where $i$ is the inflation rate.*

**Definition 3** (Excess Bank Return).

$$r_e = r_b - p_{i,r} \tag{3}$$

*where $p_{i,r}$ is the inflation risk premium.*

**Theorem 1** (Sustainability Condition). *For banking to remain financially sustainable, we require:*

$$r_e = 0 \tag{4}$$

# 2 Equilibrium Analysis

## 2.1 Fundamental Equilibrium Relationship

From equations (2), (3), and (4):

$$\begin{aligned} r_e &= 0 \\ r_b - p_{i,r} &= 0 \\ (p_d - i) - p_{i,r} &= 0 \end{aligned} \tag{5}$$

This yields the fundamental equilibrium condition:

**Proposition 1** (Equilibrium Deposit Premium)**.**

$$\boxed{p_d(t) = i(t) + p_{i,r}(t)} \tag{6}$$

*The deposit premium must equal the sum of inflation and the inflation risk premium.*

# 3 Non-Trivial Functional Forms

## 3.1 Stochastic Inflation Process

We model inflation using a mean-reverting Ornstein-Uhlenbeck process with jumps:

$$di(t) = \kappa[\bar{i} - i(t)]dt + \sigma_i dW_i(t) + dJ(t) \tag{7}$$

where:

- $\kappa > 0$ is the mean reversion speed
- $\bar{i}$ is the long-run mean inflation
- $\sigma_i$ is the volatility parameter
- $W_i(t)$ is a standard Brownian motion
- $J(t)$ is a compound Poisson process capturing economic shocks

The solution is:

$$i(t) = \bar{i} + [i(0) - \bar{i}]e^{-\kappa t} + \sigma_i \int_0^t e^{-\kappa(t-s)} dW_i(s) + \sum_{j=1}^{N(t)} Y_j e^{-\kappa(t-T_j)} \tag{8}$$

where $N(t)$ is a Poisson process with intensity $\lambda_J$, and $Y_j$ are i.i.d. jump sizes.

## 3.2 Inflation Risk Premium

The inflation risk premium compensates for uncertainty in future inflation:

$$p_{i,r}(t) = \lambda \sqrt{\mathrm{Var}_t[i(T)]} + \phi \mathrm{Skew}_t[i(T)] \tag{9}$$

where $\lambda$ is the market price of risk and $\phi$ captures skewness preference.

For the O-U process, the conditional variance is:

$$\mathrm{Var}_t[i(T)] = \frac{\sigma_i^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right) + \lambda_J \mathbb{E}[Y^2](T-t) \tag{10}$$

## 3.3 Risk-Free Rate

The risk-free rate follows the extended Vasicek model:

$$r_f(t) = r_f^{LR}(t) + \theta(t) + \eta(t, y) \tag{11}$$

where:

- $r_f^{LR}(t) = \bar{i}(t) + \rho$ (long-run real rate $\rho$)
- $\theta(t)$ is a cyclical component: $\theta(t) = \alpha \sin(\omega t + \phi_0)$
- $\eta(t, y)$ is the term premium: $\eta(t, y) = \beta_0 + \beta_1 y + \beta_2 y^2$

## 3.4 Bank Rate

The bank rate must satisfy competitive equilibrium:

$$b(t, y) = r_f(t) + p_d(t) + \mu(y) + \xi(t) \tag{12}$$

where:

- $\mu(y) = \mu_0(1 - e^{-\gamma y})$ is the maturity-dependent spread
- $\xi(t)$ represents bank-specific factors and market conditions

## 3.5 Complete System

The complete sustainable banking system is:

$$\begin{aligned}
p_d(t) &= i(t) + \lambda\sqrt{\frac{\sigma_i^2}{2\kappa}(1 - e^{-2\kappa(T-t)}) + \lambda_J \mathbb{E}[Y^2](T-t)} + \phi \text{Skew}_t[i(T)] && (13)\\
r_b(t) &= \lambda\sqrt{\frac{\sigma_i^2}{2\kappa}(1 - e^{-2\kappa(T-t)}) + \lambda_J \mathbb{E}[Y^2](T-t)} + \phi \text{Skew}_t[i(T)] && (14)\\
r_e(t) &= 0 && (15)\\
b(t, y) &= \bar{i} + \rho + \alpha \sin(\omega t + \phi_0) + \beta_0 + \beta_1 y + \beta_2 y^2 + i(t)\\
&\quad + \lambda\sqrt{\frac{\sigma_i^2}{2\kappa}(1 - e^{-2\kappa(T-t)})} + \mu_0(1 - e^{-\gamma y}) + \xi(t) && (16)
\end{aligned}$$

# 4 Graphical Analysis

## 4.1 Inflation Dynamics

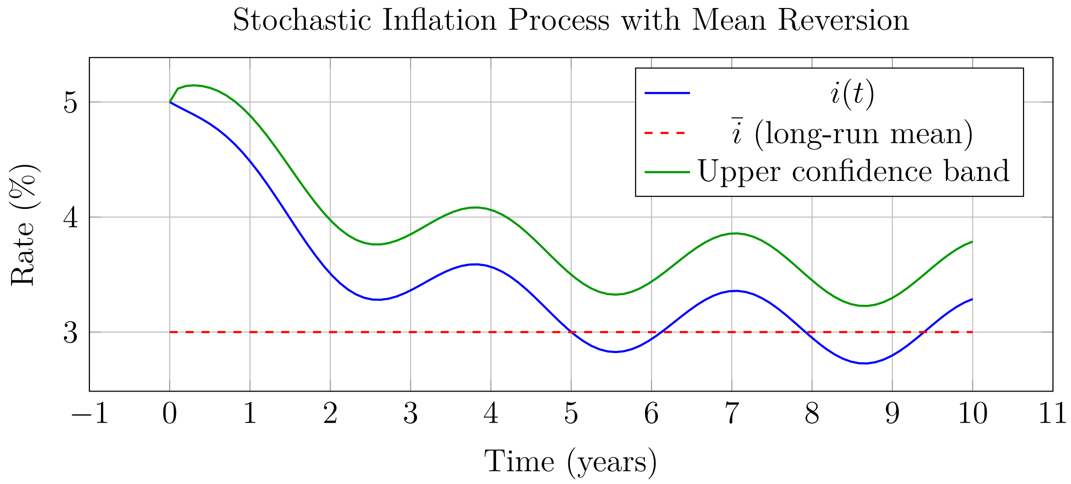


Figure 1: Mean-reverting inflation with stochastic volatility

## 4.2 Risk Premium Structure

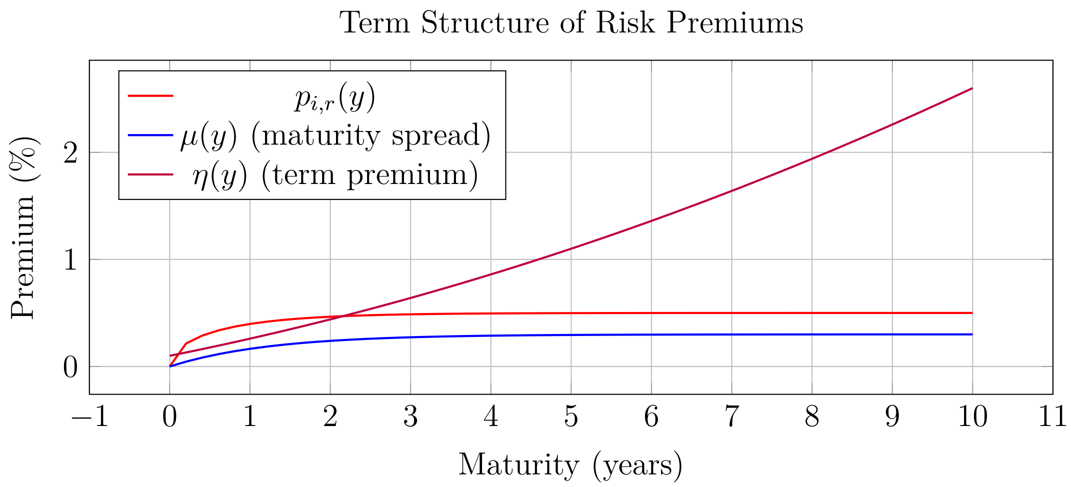


Figure 2: Risk premiums as functions of maturity

### 4.3 Equilibrium Decomposition

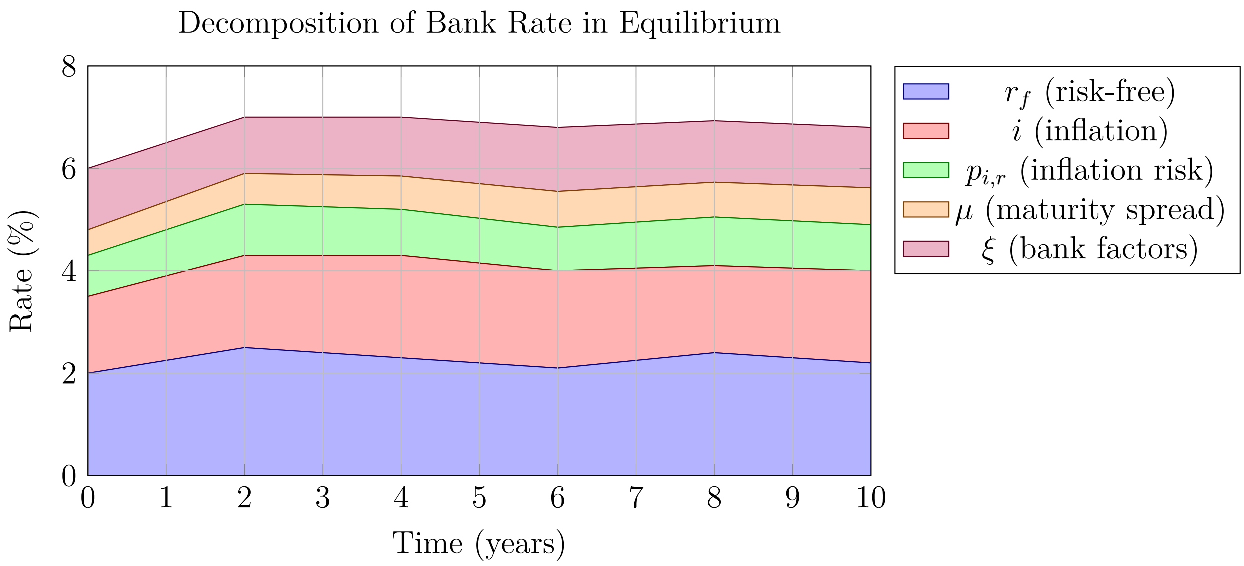


Figure 3: Stacked decomposition showing how bank rate components satisfy equilibrium

### 4.4 Phase Space Diagram

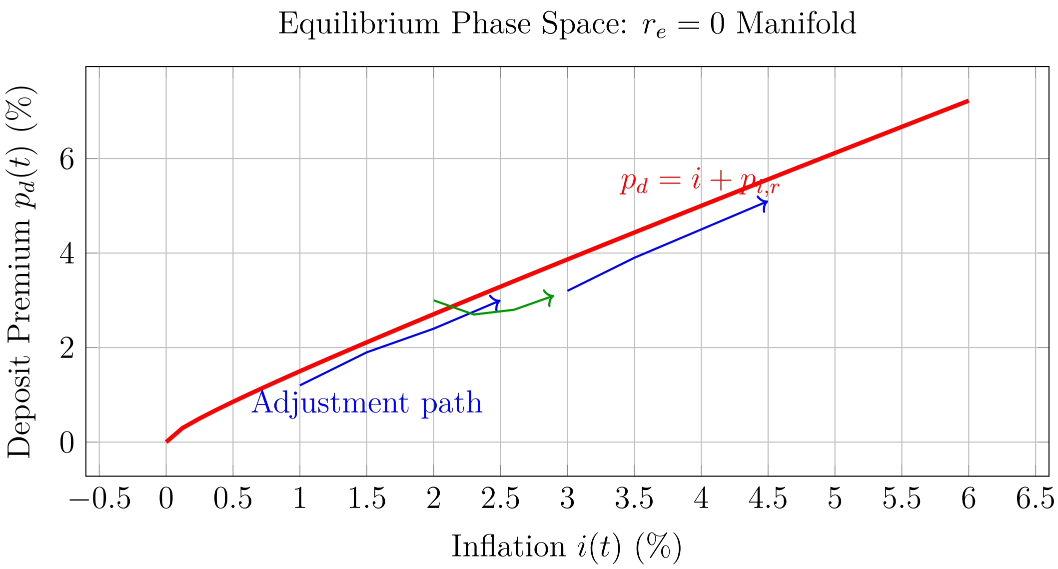


Figure 4: Phase space showing equilibrium manifold and adjustment dynamics

# 5 Economic Interpretation

## 5.1 Market Efficiency Condition

The zero excess return condition ($r_e = 0$) implies that in a competitive equilibrium, banks cannot systematically extract rents from depositors beyond compensation for measurable risks. This leads to several key insights:

1. **Risk-Neutral Pricing**: The deposit premium exactly compensates for inflation and uncertainty.

2. **No Arbitrage**: Depositors receive fair compensation relative to alternative investments.

3. **Sustainability**: Banks operate at competitive margins without extracting excess economic profits.

## 5.2 Parametric Calibration

For practical implementation, typical parameter values might be:

| Parameter | Value | Interpretation |
|---|---|---|
| $\kappa$ | 0.5 year$^{-1}$ | Mean reversion speed |
| $\bar{i}$ | 2-3% | Long-run inflation target |
| $\sigma_i$ | 0.5-1% | Inflation volatility |
| $\lambda$ | 1-3 | Market price of risk |
| $\rho$ | 1-2% | Long-run real rate |
| $\alpha$ | 0.5% | Cyclical amplitude |
| $\omega$ | $2\pi/10$ | Business cycle frequency |

Table 1: Typical parameter ranges for developed economies

# 6 Extensions and Robustness

## 6.1 Time-Varying Risk Aversion

We can extend the model with time-varying risk aversion:

$$\lambda(t) = \lambda_0 + \lambda_1 V(t) \tag{17}$$

where $V(t)$ is economic uncertainty (e.g., VIX analog).

## 6.2 Credit Risk Integration

For completeness, incorporating credit risk:

$$b(t, y) = r_f(t) + p_d(t) + p_c(t, y) + \mu(y) \tag{18}$$

where $p_c(t, y)$ is the credit risk premium based on bank-specific default probabilities.

## 6.3 Regulatory Capital Requirements

Under Basel III/IV frameworks:

$$\mu(y) = \mu_0(1 - e^{-\gamma y}) + \kappa_{reg}\text{RWA}(y) \tag{19}$$

where RWA represents risk-weighted assets.

# 7 Conclusion

We have established a comprehensive mathematical framework demonstrating that sustainable banking requires:

$$\boxed{p_d(t) = i(t) + p_{i,r}(t) \implies r_e = 0} \tag{20}$$

The non-trivial functional forms incorporate:

- Stochastic mean-reverting inflation with jumps
- Time-varying, maturity-dependent risk premiums
- Cyclical and term structure effects in interest rates
- Competitive banking spreads

This framework provides both theoretical foundations and practical guidance for deposit pricing in competitive banking markets, ensuring long-run sustainability through proper risk compensation.

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# Glossary

**Bank Rate ($b$)** The annual interest rate offered by banks on deposits, typically consisting of the risk-free rate plus various premiums and spreads.

**Deposit Premium ($p_d$)** The additional return paid to depositors above the risk-free rate, compensating for inflation and its associated risks.

**Excess Bank Return ($r_e$)** The return earned by banks above fair compensation for inflation risk; must equal zero in sustainable equilibrium.

**Inflation ($i$)** The rate of increase in the general price level, modeled as a stochastic process with mean reversion.

**Inflation Risk Premium ($p_{i,r}$)** Compensation demanded by investors for bearing uncertainty about future inflation rates.

**Market Price of Risk ($\lambda$)** A parameter reflecting how much additional return investors require per unit of risk exposure.

**Mean Reversion ($\kappa$)** The speed at which a stochastic variable returns to its long-run average value.

**Ornstein-Uhlenbeck Process** A continuous-time stochastic process exhibiting mean reversion, commonly used to model interest rates and inflation.

**Real Bank Return ($r_b$)** The nominal bank return adjusted for inflation; represents purchasing power gain.

**Risk-Free Rate ($r_f$)** The theoretical return on an investment with zero risk, typically approximated by government bond yields.

**Sustainability Condition** The requirement that $r_e = 0$, ensuring banks operate in competitive equilibrium without extracting systematic excess returns.

**Term Premium ($\eta$)** Additional return required for holding longer-maturity instruments, reflecting liquidity and uncertainty.

**Vasicek Model** A mathematical model for describing the evolution of interest rates using mean-reverting stochastic differential equations.

# The End