

A useful class of symmetric, positive semi-definite and rank one matrices with special properties

Soumadeep Ghosh

Kolkata, India

Abstract

We construct a general class of 9×9 matrices of the form $M_{ij} = \sqrt{x_i x_j}$ with $0 < x_i < 1$ for all i , show that each such matrix can be written as a rank-one outer product, and characterize its algebraic, spectral, and geometric structure. A fully explicit numerical instance is provided as a worked example.

The paper ends with “The End”

1 General construction

Let

$$x_1, \dots, x_9 \in (0, 1) \tag{1}$$

be arbitrary positive scalars strictly smaller than one and define

$$a_i = \sqrt{x_i}, \quad i = 1, \dots, 9. \tag{2}$$

Introduce the vector

$$X = (a_1, \dots, a_9)^\top \in \mathbb{R}^9. \tag{3}$$

We define the matrix

$$M = XX^\top. \tag{4}$$

Componentwise,

$$M_{ij} = a_i a_j = \sqrt{x_i x_j}, \quad i, j = 1, \dots, 9. \tag{5}$$

2 Algebraic and spectral properties

2.1 Symmetry and positive semi-definiteness

By construction,

$$M = XX^\top = (XX^\top)^\top, \tag{6}$$

so M is symmetric. Moreover, for any $v \in \mathbb{R}^9$,

$$v^\top M v = v^\top X X^\top v = (X^\top v)^2 \geq 0, \tag{7}$$

which proves that M is positive semi-definite.

2.2 Rank

Since $X \neq 0$ and M is the outer product of a single vector with itself,

$$\text{rank}(M) = 1. \tag{8}$$

2.3 Eigenvalues and eigenvectors

The spectral structure is completely determined by X . Indeed,

$$MX = X(X^\top X). \quad (9)$$

Hence X is an eigenvector and the unique nonzero eigenvalue is

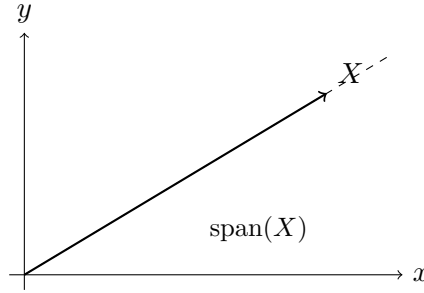
$$\lambda_1 = X^\top X = \sum_{i=1}^9 a_i^2 = \sum_{i=1}^9 x_i. \quad (10)$$

All remaining eight eigenvalues are equal to zero. Any vector orthogonal to X spans the null eigenspace.

3 Geometric interpretation

The matrix M is the Gram matrix of the single vector X . Every column of M is a scalar multiple of X , and therefore the column space is the one-dimensional subspace

$$\text{Col}(M) = \text{span}\{X\}. \quad (11)$$



This schematic emphasizes that the entire matrix is generated by a single direction in \mathbb{R}^9 .

4 Special structure

The separable representation

$$M_{ij} = a_i a_j \quad (12)$$

induces an exact one-factor structure. In particular, if one rescales rows and columns to unit variance, all pairwise correlations are equal to one. In econometrics and mathematical finance this coincides with the limiting case of a single common latent factor with no idiosyncratic component.

5 Worked numerical example

We now specialize the general construction to a concrete choice satisfying $0 < x_i < 1$. Let

$$x_i = \left(\frac{i}{10}\right)^2, \quad i = 1, \dots, 9. \quad (13)$$

Then

$$a_i = \sqrt{x_i} = \frac{i}{10}, \quad (14)$$

so that

$$X = \frac{1}{10}(1, 2, 3, 4, 5, 6, 7, 8, 9)^\top. \quad (15)$$

The resulting matrix is

$$M = \frac{1}{100} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{pmatrix}. \quad (16)$$

The unique nonzero eigenvalue is

$$\lambda_1 = \sum_{i=1}^9 x_i = \frac{1}{100} \sum_{i=1}^9 i^2 = \frac{285}{100} = 2.85, \quad (17)$$

with eigenvector X .

References

- [1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.
- [2] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed., Johns Hopkins University Press, 2013.
- [3] H. Lütkepohl, *New Introduction to Multiple Time Series Analysis*, Springer, 2005.

Glossary

Gram matrix A matrix of inner products of a collection of vectors. In the present case, M is the Gram matrix generated by the single vector X .

Positive semi-definite matrix A symmetric matrix A such that $v^\top A v \geq 0$ for all vectors v .

Rank-one matrix A matrix that can be written as an outer product uv^\top of two nonzero vectors.

Outer product For vectors u and v , the matrix uv^\top .

Eigenvalue A scalar λ such that $Au = \lambda u$ for some nonzero vector u .

One-factor structure A representation in which all comovement is driven by a single latent factor.

The End