Extensions to the Multi-Product Allocation Problem

Incorporating Uncertainty, Dynamics, Capacity, and Environmental Regulation

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Abstract

This paper extends the multi-product allocation framework presented in [1], where commodities A and B are allocated across products P, Q, and R with waste generation from joint production. We develop four comprehensive extensions: (1) stochastic production functions incorporating yield and price uncertainty with risk-averse formulations; (2) dynamic optimization with inventory considerations across multiple periods; (3) capacity constraints with endogenous expansion decisions; and (4) environmental regulation through carbon taxes and cap-and-trade systems. Each extension preserves the fundamental trade-off between specialized single-input production and joint production with waste externalities while adding realistic operational and policy dimensions.

The paper ends with “The End”

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1 Introduction

The baseline multi-product allocation problem [1] establishes a firm's optimization over two commodities (A and B) allocated across three products (P, Q, and R), where Products P and Q utilize single inputs while Product R emerges from joint production technology that simultaneously generates waste. The profit maximization problem is:

$$\max_{a,b,m,n} \pi = p_P \cdot P + p_Q \cdot Q + p_R \cdot R - c_A \cdot a - c_B \cdot b - c_w \cdot w \tag{1}$$

subject to production functions $P = f_1(m)$, $Q = f_2(n)$, $R = f_3(a-m, b-n)$, and $w = g(a-m, b-n)$.

This paper develops four extensions suggested for future research, transforming the static, deterministic model into a comprehensive framework suitable for realistic production planning.

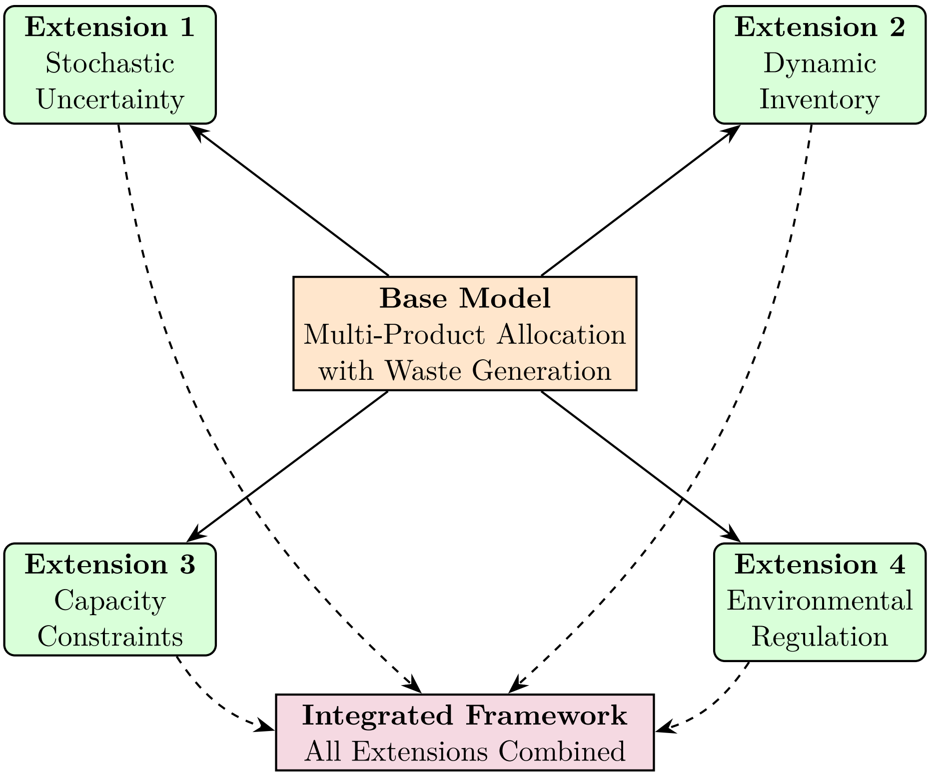

Figure 1: Overview of extensions to the base multi-product allocation model.

2 Extension 1: Stochastic Production Functions with Uncertainty

2.1 Sources of Uncertainty

We introduce stochastic elements into the production system following approaches in stochastic programming literature [2,3].

2.1.1 Production Yield Uncertainty

Production yields are subject to random shocks:

$$\tilde{P} = f_1(m) \cdot \tilde{\epsilon}_P \quad (2)$$
$$\tilde{Q} = f_2(n) \cdot \tilde{\epsilon}_Q \quad (3)$$
$$\tilde{R} = f_3(a - m, b - n) \cdot \tilde{\epsilon}_R \quad (4)$$

where $\tilde{\epsilon}_i$ are random yield factors with $\mathbb{E}[\tilde{\epsilon}_i] = 1$ and $\mathrm{Var}(\tilde{\epsilon}_i) = \sigma_i^2$.

2.1.2 Demand and Price Uncertainty

Output prices follow stochastic processes:

$$\tilde{p}_P, \tilde{p}_Q, \tilde{p}_R \sim F_p(\cdot) \quad (5)$$

2.1.3 Waste Generation Uncertainty

Waste generation is also uncertain:

$$\tilde{w} = g(a - m, b - n) \cdot \tilde{\epsilon}_w \quad (6)$$

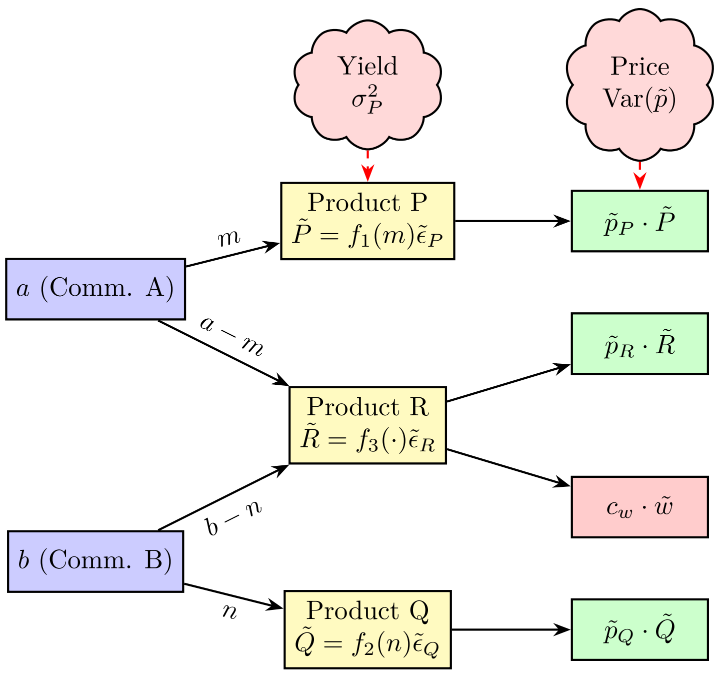

Figure 2: Production flow with stochastic elements.

Uncertainty affects yields (production shocks ϵ_i) and prices (market volatility).

2.2 Expected Profit Maximization

The firm's stochastic optimization problem becomes:

$$\max_{a,b,m,n} \mathbb{E}[\tilde{\pi}] = \mathbb{E}[\tilde{p}_P \cdot \tilde{P} + \tilde{p}_Q \cdot \tilde{Q} + \tilde{p}_R \cdot \tilde{R}] - c_A \cdot a - c_B \cdot b - c_w \cdot \mathbb{E}[\tilde{w}] \quad (7)$$

Under independence assumptions:

$$\mathbb{E}[\tilde{\pi}] = \bar{p}_P \cdot f_1(m) + \bar{p}_Q \cdot f_2(n) + \bar{p}_R \cdot f_3(a - m, b - n) - c_A a - c_B b - c_w \cdot g(a - m, b - n) \quad (8)$$

where $\bar{p}_i = \mathbb{E}[\tilde{p}_i]$ denotes expected prices.

2.3 Risk-Averse Formulation

For a risk-averse firm, we incorporate variance through a mean-variance objective:

$$\max_{a,b,m,n} \mathbb{E}[\tilde{\pi}] - \lambda \cdot \mathrm{Var}(\tilde{\pi}) \tag{9}$$

where $\lambda > 0$ is the risk aversion parameter.

Definition 2.1 (Certainty Equivalent)**.** *The certainty equivalent profit π^{CE} is the deterministic profit level that yields the same utility as the risky profit:*

$$U(\pi^{CE}) = \mathbb{E}[U(\tilde{\pi})] \tag{10}$$

2.3.1 Chance-Constrained Programming

Alternatively, following robust optimization approaches [4], we impose probabilistic constraints:

$$\Pr(\tilde{\pi} \geq \pi_{\min}) \geq 1 - \alpha \tag{11}$$

where α is the acceptable probability of failing to meet the minimum profit threshold.

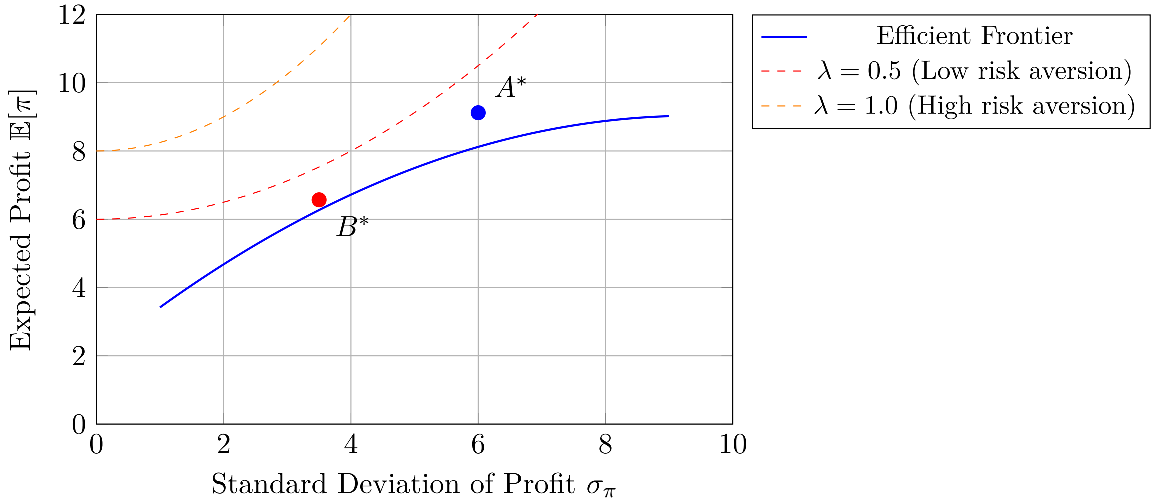

Figure 3: Risk-return tradeoff in stochastic production.

The efficient frontier shows maximum expected profit for each risk level. Optimal allocation depends on risk aversion λ.

2.4 Two-Stage Stochastic Model

Adopting a two-stage framework:

Stage 1 (Here-and-Now): Decide input purchases (a, b) before uncertainty resolves.

Stage 2 (Wait-and-See): Decide allocations (m, n) after observing realizations.

$$\max_{a,b} \left\{ -c_A a - c_B b + \mathbb{E}_\xi \left[\max_{m,n} \pi_2(m, n, \xi \mid a, b) \right] \right\} \tag{12}$$

where ξ represents the realized uncertainty.

2.5 Modified First-Order Conditions Under Uncertainty

For the expected profit case, the FOCs become:

$$\bar{p}_P \cdot \mathbb{E}\left[\frac{\partial f_1}{\partial m}\right] = \bar{p}_R \cdot \mathbb{E}\left[\frac{\partial f_3}{\partial m}\right] + c_w \cdot \mathbb{E}\left[\frac{\partial g}{\partial m}\right] \tag{13}$$

Proposition 2.2 (Uncertainty Premium)**.** *Under risk aversion, the firm allocates more resources to less volatile production processes. Formally, if $\sigma_P^2 > \sigma_R^2$, then $m^*_{\textit{risk-averse}} < m^*_{\textit{risk-neutral}}$.*

Proof. From the mean-variance objective (9), the first-order condition with respect to m includes:

$$\frac{\partial}{\partial m}\left[\mathbb{E}[\tilde{\pi}] - \lambda \mathrm{Var}(\tilde{\pi})\right] = 0 \tag{14}$$

The variance term $\mathrm{Var}(\tilde{\pi})$ increases in m when $\sigma_P^2 > \sigma_R^2$, creating an additional marginal cost for P-production. This shifts optimal allocation away from the more volatile process. □

3 Extension 2: Dynamic Optimization with Inventory Considerations

3.1 Multi-Period Framework with Inventories

We extend to a discrete-time horizon $t = 1, 2, \ldots, T$, introducing inventory dynamics for all commodities and products [5, 6].

3.1.1 State Variables

- I_A^t, I_B^t: Input inventories at time t
- I_P^t, I_Q^t, I_R^t: Output inventories at time t
- I_w^t: Accumulated waste at time t

3.2 Inventory Dynamics

Input Inventories:

$$I_A^{t+1} = I_A^t + a^t - x_A^t \tag{15}$$
$$I_B^{t+1} = I_B^t + b^t - x_B^t \tag{16}$$

where x_A^t, x_B^t are total inputs used in period t.

Output Inventories:

$$I_P^{t+1} = I_P^t + P^t - D_P^t \tag{17}$$
$$I_Q^{t+1} = I_Q^t + Q^t - D_Q^t \tag{18}$$
$$I_R^{t+1} = I_R^t + R^t - D_R^t \tag{19}$$

where D_i^t is sales/demand for product i in period t.

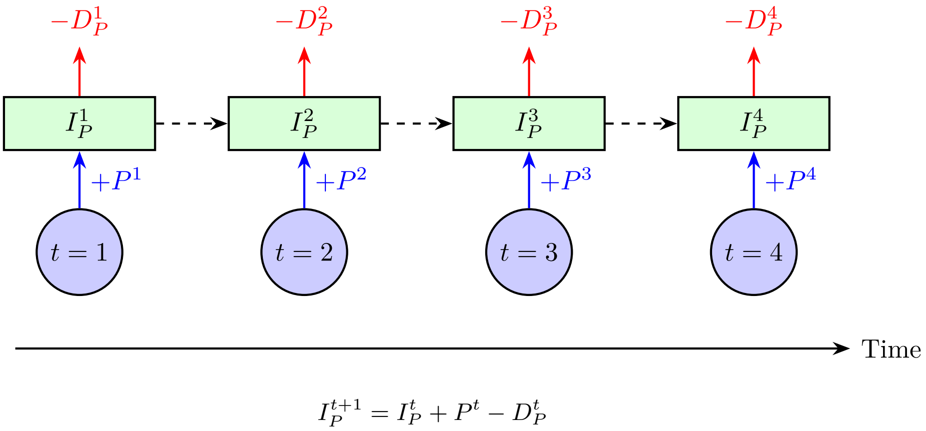

Figure 4: Inventory dynamics for Product P across multiple time periods. Production adds to inventory while demand depletes it.

3.3 Dynamic Profit Maximization

$$\max \sum_{t=1}^{T} \delta^{t-1} \left[\sum_{i \in \{P,Q,R\}} p_i^t D_i^t - c_A a^t - c_B b^t - c_w w^t - \sum_j h_j I_j^t \right] \tag{20}$$

where:

- $\delta \in (0, 1]$ is the discount factor
- h_j are holding costs for inventory type j

Subject to:

- Inventory dynamics (above)
- Non-negativity: $I_j^t \geq 0$ for all j, t
- Sales constraints: $D_i^t \leq I_i^t + \text{Production}_i^t$

3.4 Bellman Equation Formulation

The dynamic programming formulation follows [7]:

$$V^t(\mathbf{I}^t) = \max_{a^t, b^t, m^t, n^t, \mathbf{D}^t} \left\{ \pi^t + \delta \cdot V^{t+1}(\mathbf{I}^{t+1}) \right\} \tag{21}$$

where $\mathbf{I}^t = (I_A^t, I_B^t, I_P^t, I_Q^t, I_R^t, I_w^t)$ is the state vector.

3.5 Optimal Inventory Policy

Proposition 3.1 (Inventory Smoothing). *Under convex holding costs and concave production functions, the optimal policy smooths production across periods rather than matching production to demand in each period.*

Modified FOC for Allocation:

$$p_P^t \cdot MP_m^P - h_P = p_R^t \cdot MP_m^R - h_R + c_w \cdot MW_m + h_w \tag{22}$$

The holding cost differentials now influence allocation decisions.

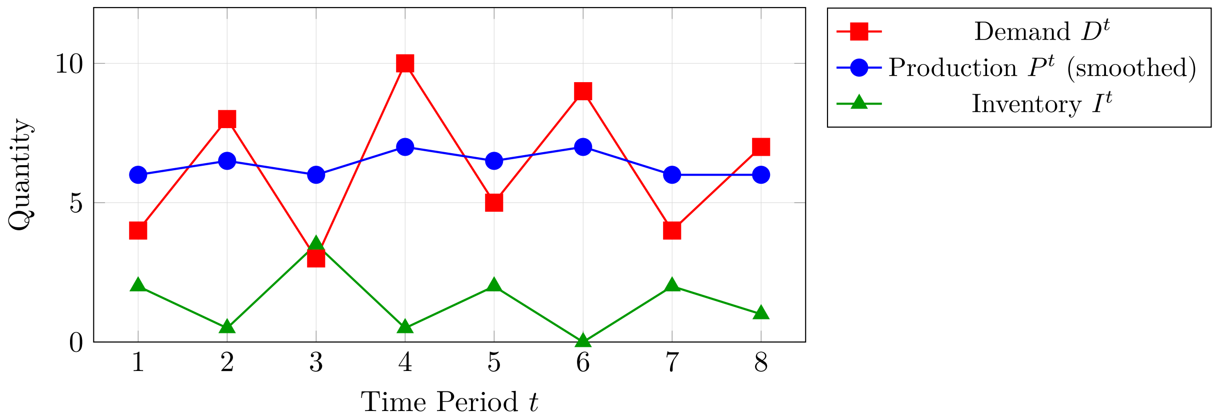

Figure 5: Production smoothing under inventory management.

Optimal policy maintains relatively stable production despite demand fluctuations, using inventory as a buffer.

3.6 Economic Order Quantity Integration

For input procurement, we incorporate setup costs K_A, K_B:

$$\text{EOQ}_A = \sqrt{\frac{2K_A \cdot \bar{x}_A}{h_A}} \tag{23}$$

where $\bar{x}_A$ is average input usage per period.

4 Extension 3: Multiple Time Periods with Capacity Constraints

4.1 Capacity-Constrained Production

Building on the multi-period framework, we add production capacity constraints [8, 9].

Production Capacity:

$$P^t \leq \bar{K}_P^t, \quad Q^t \leq \bar{K}_Q^t, \quad R^t \leq \bar{K}_R^t \tag{24}$$

Input Processing Capacity:

$$m^t + (a^t - m^t) \leq \bar{C}_A^t \tag{25}$$
$$n^t + (b^t - n^t) \leq \bar{C}_B^t \tag{26}$$

Waste Handling Capacity:

$$w^t \leq \bar{W}^t \tag{27}$$

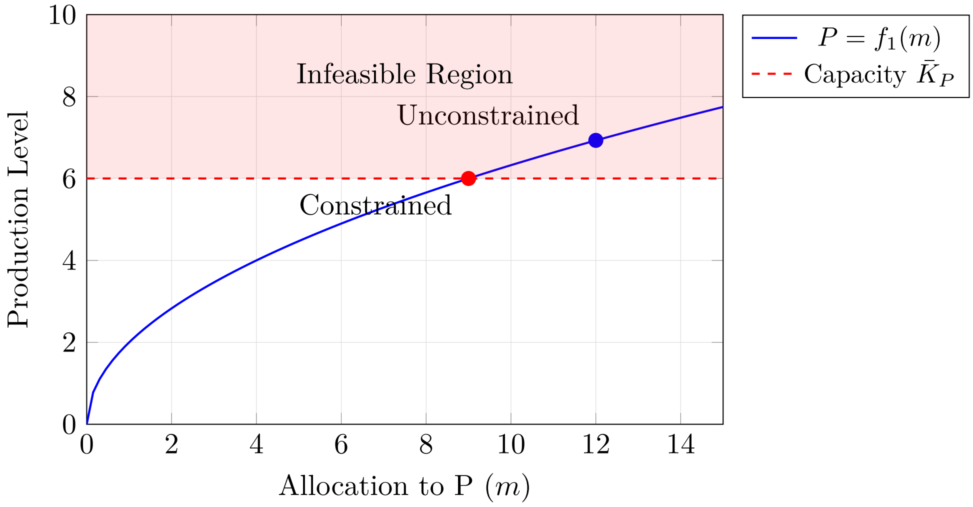

Figure 6: Capacity constraint on Product P production.

When unconstrained optimum exceeds capacity, production is capped at $\bar{K}_P$.

4.2 Capacity Expansion Decisions

The firm can invest in capacity expansion:

$$\bar{K}_i^{t+1} = \bar{K}_i^t + \Delta K_i^t - \text{depreciation} \tag{28}$$

with investment cost $\gamma_i(\Delta K_i^t)$, typically convex.

4.3 Extended Optimization Problem

$$\max \sum_{t=1}^{T} \delta^{t-1} \left[\pi^t - \sum_i \gamma_i(\Delta K_i^t) \right] \tag{29}$$

Subject to all constraints from the base model plus capacity constraints.

4.4 Lagrangian Analysis

The Lagrangian incorporating capacity constraints:

$$\begin{aligned}\mathcal{L} = \sum_t \delta^{t-1}\pi^t - \sum_t \mu_P^t(P^t - \bar{K}_P^t) - \sum_t \mu_Q^t(Q^t - \bar{K}_Q^t) \\ - \sum_t \mu_R^t(R^t - \bar{K}_R^t) - \sum_t \mu_w^t(w^t - \bar{W}^t)\end{aligned} \tag{30}$$

Modified FOCs:

$$(p_P - \mu_P) \cdot MP_m^P = (p_R - \mu_R) \cdot MP_m^R + (c_w + \mu_w) \cdot MW_m \tag{31}$$

4.5 Shadow Price Interpretation

Proposition 4.1 (Capacity Shadow Prices)**.** *The multiplier μ_i^t represents the marginal value of relaxing capacity constraint i in period t. When $\mu_i^t > 0$, capacity is binding and expansion is valuable.*

Capacity Investment Rule:

$$\gamma_i'(\Delta K_i^t) = \sum_{\tau=t+1}^{T} \delta^{\tau-t-1} \mu_i^\tau \tag{32}$$

Invest until marginal investment cost equals discounted future shadow prices.

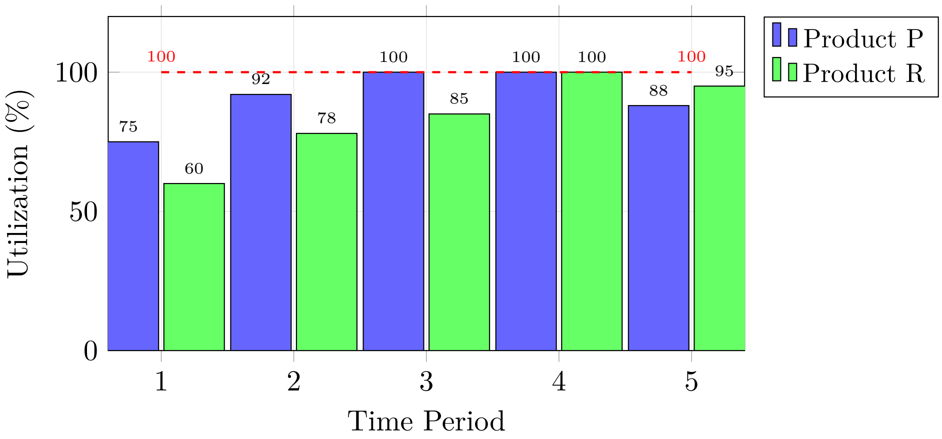

Figure 7: Capacity utilization over time.

Periods 3–4 show binding capacity constraints for Product P (100% utilization).

Table 1: Capacity Constraint Summary by Period

Period	P Utilization	R Utilization	Shadow Price μ_P	Binding?
1	75%	60%	0	No
2	92%	78%	0	No
3	100%	85%	> 0	Yes (P)
4	100%	100%	> 0	Yes (P, R)
5	88%	95%	0	No

5 Extension 4: Environmental Regulation and Emission Permits

5.1 Regulatory Framework

We incorporate environmental constraints following approaches in environmental economics [10, 11].

Types of Regulation:

1. **Command-and-Control:** Hard emission limits
2. **Carbon Tax:** Per-unit emission charge
3. **Cap-and-Trade:** Tradeable emission permits

5.2 Emission Function

Total emissions are linked to waste and production:

$$E = \alpha_w \cdot w + \alpha_P \cdot P + \alpha_Q \cdot Q + \alpha_R \cdot R \tag{33}$$

where α_i are emission coefficients (tons CO_2 per unit).

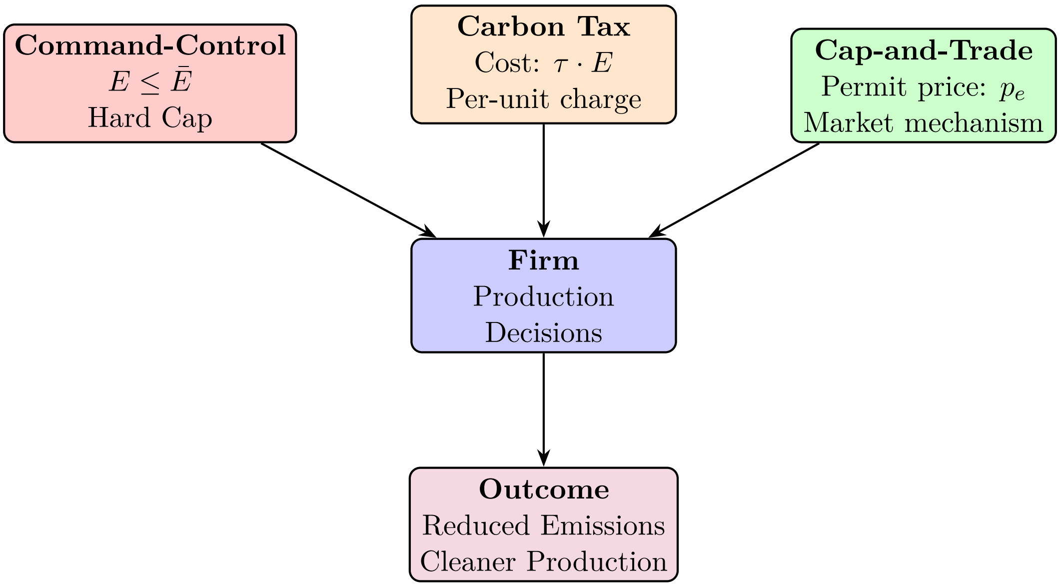

Figure 8: Three environmental regulatory mechanisms affecting firm decisions.

5.3 Command-and-Control Regulation

Hard Emission Cap:

$$E \leq \bar{E} \tag{34}$$

The firm's problem becomes:

$$\max_{a,b,m,n} \pi \quad \text{s.t.} \quad \alpha_w g(a-m, b-n) + \sum_i \alpha_i \cdot \text{Output}_i \leq \bar{E} \tag{35}$$

Proposition 5.1 (Emission Constraint Effect)**.** *When the emission constraint binds, the firm shifts production toward cleaner products. If $\alpha_R > \alpha_P$ and $\alpha_w > 0$, optimal m^* increases relative to the unconstrained case.*

5.4 Carbon Tax Framework

With carbon tax τ per unit emission:

$$\pi = p_P P + p_Q Q + p_R R - c_A a - c_B b - c_w w - \tau \cdot E \tag{36}$$

Rearranging:

$$\pi = p_P P + p_Q Q + p_R R - c_A a - c_B b - (c_w + \tau\alpha_w) w - \tau(\alpha_P P + \alpha_Q Q + \alpha_R R) \tag{37}$$

Effective prices become:

$$\tilde{p}_i = p_i - \tau\alpha_i \tag{38}$$

Modified FOC:

$$(p_P - \tau\alpha_P) \cdot MP_m^P = (p_R - \tau\alpha_R) \cdot MP_m^R + (c_w + \tau\alpha_w) \cdot MW_m \tag{39}$$

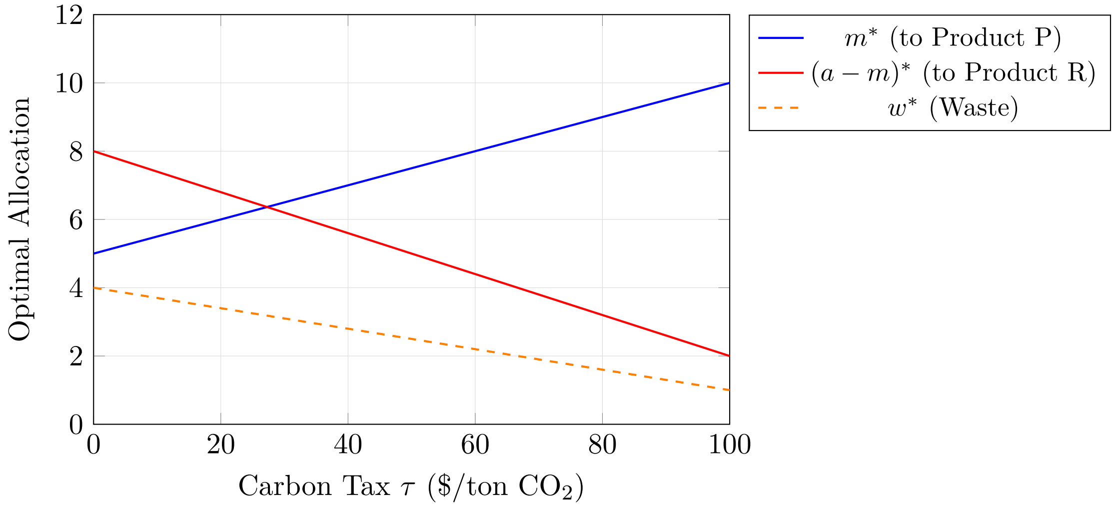

Figure 9: Comparative statics: Effect of carbon tax on optimal allocation.

Higher τ shifts resources from joint production (R) to cleaner single-input production (P).

5.5 Cap-and-Trade System

The firm can buy/sell emission permits at price p_e:

$$\max_{a,b,m,n,e} \pi - p_e \cdot (E - e_0 + e_{\text{sold}} - e_{\text{bought}}) \tag{40}$$

where e_0 is the initial permit allocation.

Market Equilibrium Condition: At equilibrium, $p_e = \tau^*$ (the optimal carbon tax that achieves the same emission level as the cap).

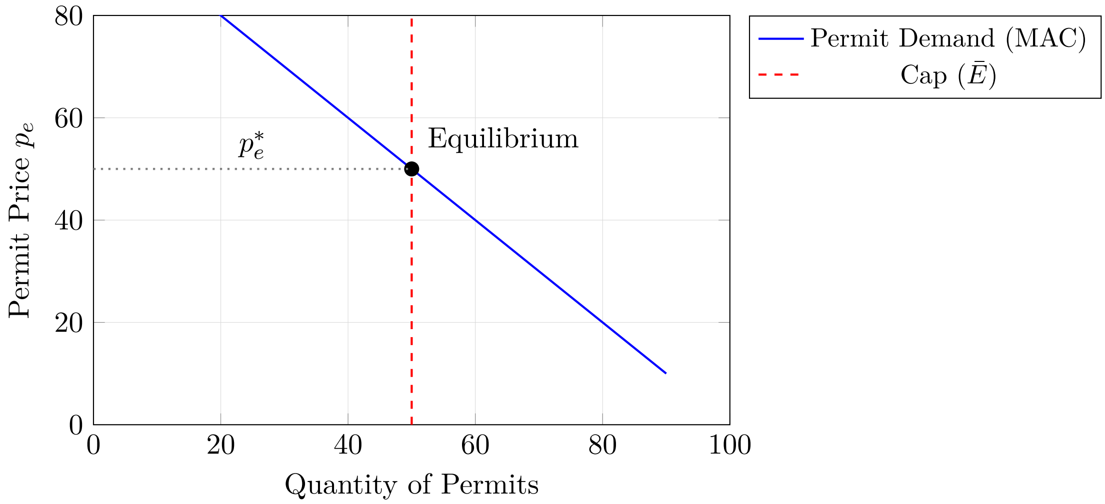

Figure 10: Emission permit market equilibrium.

The permit price equals the marginal abatement cost (MAC) at the cap level $\bar{E}$.

5.6 Dynamic Environmental Policy

Following stochastic dynamic programming approaches for environmental policy [12]:

$$V^t(K^t, E^t) = \max \left\{ \pi^t - \tau^t E^t + \delta V^{t+1}(K^{t+1}, E^{t+1}) \right\} \tag{41}$$

where K^t is capital stock, E^t is cumulative emissions, and τ^t may evolve according to policy trajectory.

Proposition 5.2 (Optimal Abatement Path)**.** *With increasing carbon taxes ($\tau^{t+1} > \tau^t$), the firm optimally front-loads dirty production and transitions toward cleaner technologies over time.*

5.7 Comparative Statics: Carbon Tax Effects

Table 2: Effects of Increasing Carbon Tax τ

Variable	**Effect of** $\uparrow \tau$	**Intuition**
m^*	$\uparrow$ (if $\alpha_P < \alpha_R$)	Shift to cleaner product
n^*	$\uparrow$ (if $\alpha_Q < \alpha_R$)	Shift to cleaner product
R^*	$\downarrow$	Joint production more costly
w^*	$\downarrow$	Waste generates emissions
Total profit	$\downarrow$	Higher compliance costs

6 Integrated Framework

Combining all extensions yields a comprehensive model:

$$\max \sum_{t=1}^{T} \delta^{t-1} \left[\mathbb{E}[\tilde{\pi}^t] - \lambda \mathrm{Var}(\tilde{\pi}^t) - \tau^t E^t - \sum_j h_j I_j^t - \gamma(\Delta K^t) \right] \quad (42)$$

Subject to:

- Production functions with uncertainty (Extension 1)
- Inventory dynamics (Extension 2)
- Capacity constraints: $\mathrm{Output}_i^t \leq \bar{K}_i^t$ (Extension 3)
- Environmental constraints: $E^t \leq \bar{E}^t$ or tax τ^t (Extension 4)

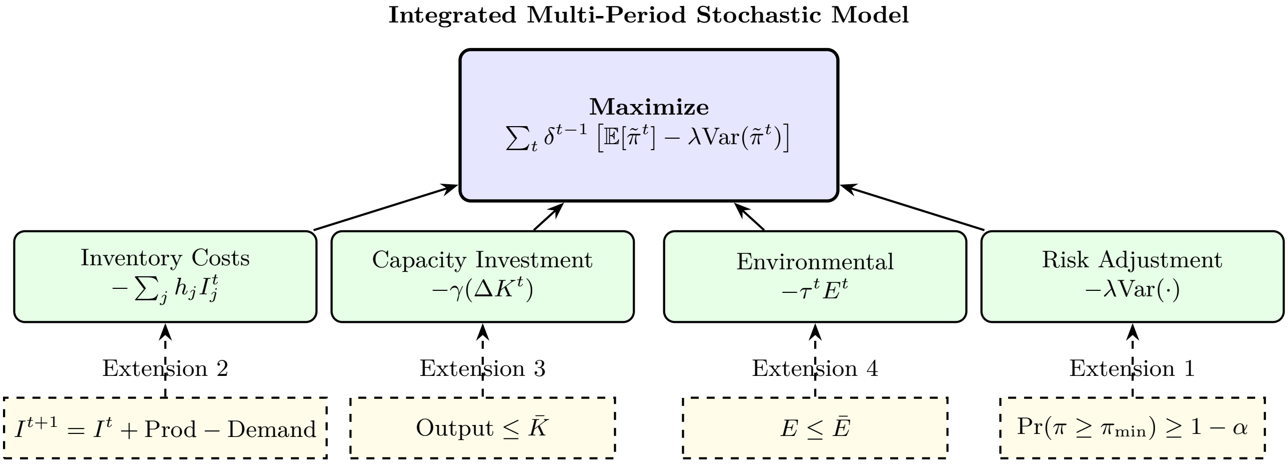

Figure 11: Structure of the integrated optimization framework combining all four extensions.

Table 3: Summary of Extensions to the Base Model

Extension	Key Addition	New Variables	New Constraints
1. Stochastic	Uncertainty in yields, prices	Risk parameter λ	Chance constraints
2. Dynamic/ Inventory	Time periods, storage	I_j^t, D_i^t	Inventory dynamics
3. Capacity	Production limits	ΔK_i^t	Output $\leq \bar{K}$
4. Environmental	Emissions, permits	$e_{\text{bought/sold}}$	$E \leq \bar{E}$ or tax τ

7 Conclusion

This paper has developed four comprehensive extensions to the multi-product allocation problem with joint production externalities [1]. The extensions address:

1. **Uncertainty:** Risk-averse decision-making under stochastic yields and prices, with two-stage recourse formulations.

2. **Dynamics:** Multi-period optimization with inventory management, production smoothing, and intertemporal trade-offs.

3. **Capacity:** Production constraints with endogenous expansion decisions guided by shadow prices.

4. **Environment:** Regulatory frameworks including carbon taxes and cap-and-trade systems that internalize emission externalities.

The integrated framework provides a realistic basis for production planning that accounts for operational constraints, market uncertainties, and environmental responsibilities. Future work could explore computational methods for solving large-scale instances and empirical applications to specific industries.

Glossary

Bellman Equation
A recursive equation expressing the value function at time t as the immediate payoff plus the discounted continuation value: $V^t = \max\{\pi^t + \delta V^{t+1}\}$.

Cap-and-Trade
An environmental policy mechanism that sets a total emission cap and allows firms to trade emission permits, creating a market price for emissions.

Carbon Tax A per-unit charge τ on emissions, which increases production costs proportionally to emission intensity.

Certainty Equivalent
The guaranteed profit level that yields the same expected utility as a risky profit distribution for a risk-averse decision-maker.

Chance Constraint
A probabilistic constraint requiring that a condition holds with at least a specified probability: $\Pr(X \geq x_0) \geq 1 - \alpha$.

Discount Factor
The parameter $\delta \in (0, 1]$ representing the present value of one unit received in the next period; reflects time preference and opportunity cost of capital.

Economic Order Quantity (EOQ)
The optimal order quantity that minimizes total inventory costs, balancing ordering costs against holding costs.

Emission Coefficient
The parameter α_i representing emissions per unit of output i (e.g., tons CO_2 per unit product).

Expected Profit
The probability-weighted average of profit over all possible realizations of uncertainty: $\mathbb{E}[\tilde{\pi}]$.

Holding Cost The per-unit cost h_j of maintaining inventory, including storage, insurance, and opportunity costs.

Inventory Dynamics
The state transition equations governing how inventory levels evolve: $I^{t+1} = I^t + \text{Production} - \text{Demand}$.

Mean-Variance Objective
An objective function $\mathbb{E}[\pi] - \lambda \text{Var}(\pi)$ that trades off expected return against risk, parameterized by risk aversion λ.

Production Smoothing
The optimal policy of maintaining stable production levels across periods, using inventory to buffer demand fluctuations.

Risk Aversion The preference to avoid uncertainty; formally, a concave utility function where $U''(\cdot) < 0$.

Risk Aversion Parameter
The coefficient $\lambda > 0$ in mean-variance optimization representing the penalty per unit of variance.

Shadow Price The marginal value of relaxing a constraint by one unit; the Lagrange multiplier at optimum (e.g., μ_i for capacity constraint i).

Stochastic Programming
Optimization under uncertainty where some parameters are random variables with known distributions.

Two-Stage Model
A decision framework where Stage 1 decisions are made before uncertainty resolves (here-and-now) and Stage 2 decisions are made after observing realizations (wait-and-see).

Uncertainty Premium
The additional return required by risk-averse agents to compensate for bearing risk.

Value Function

$V^t(\mathbf{I}^t)$: the maximum expected discounted profit achievable from state $\mathbf{I}^t$ at time t under optimal policy.

Yield Uncertainty

Randomness in production output given inputs, modeled as $\tilde{P} = f(m) \cdot \tilde{\epsilon}$ where $\tilde{\epsilon}$ is a random shock.

References

[1] Ghosh, S. (2025). Solving a Multi-Product Allocation Problem with Joint Production Externalities through Waste Generation. *Working Paper.*

[2] Birge, J. R., & Louveaux, F. (2011). *Introduction to Stochastic Programming* (2nd ed.). Springer.

[3] Shapiro, A., Dentcheva, D., & Ruszczyński, A. (2014). *Lectures on Stochastic Programming: Modeling and Theory* (2nd ed.). SIAM.

[4] Ben-Tal, A., El Ghaoui, L., & Nemirovski, A. (2009). *Robust Optimization.* Princeton University Press.

[5] Arrow, K. J., Karlin, S., & Scarf, H. (1958). *Studies in the Mathematical Theory of Inventory and Production.* Stanford University Press.

[6] Porteus, E. L. (2002). *Foundations of Stochastic Inventory Theory.* Stanford University Press.

[7] Bellman, R. (1957). *Dynamic Programming.* Princeton University Press.

[8] Manne, A. S. (1961). Capacity expansion and probabilistic growth. *Econometrica*, 29(4), 632–649.

[9] Luss, H. (1982). Operations research and capacity expansion problems: A survey. *Operations Research*, 30(5), 907–947.

[10] Nordhaus, W. D. (2013). *The Climate Casino: Risk, Uncertainty, and Economics for a Warming World.* Yale University Press.

[11] Weitzman, M. L. (1974). Prices vs. quantities. *Review of Economic Studies*, 41(4), 477–491.

[12] Pindyck, R. S. (2013). Climate change policy: What do the models tell us? *Journal of Economic Literature*, 51(3), 860–872.

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