The Complete Treatise on Gambling
A Mathematical, Historical, and Strategic Compendium

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Contents

1 Introduction

Gambling—the wagering of something of value on an event with an uncertain outcome—has accompanied human civilisation for millennia. Archaeological evidence dates dice games to at least 3000 BCE in Mesopotamia [1], while formalised probability theory did not emerge until the correspondence between Pascal and Fermat in 1654 [2].

This treatise presents a rigorous yet accessible treatment of the mathematics, strategy, and history of gambling. Every major result is stated as a theorem and proved; key concepts are illustrated with PGF/TikZ vector graphics; and a glossary of terms is appended.

2 Foundations of Probability

2.1 Sample Spaces and Events

Definition 2.1 (Sample Space)**.** A **sample space** Ω is the set of all possible outcomes of a random experiment. An **event** $A \subseteq \Omega$ is any subset of the sample space.

Example 2.2 (Single Die)**.** Rolling a fair six-sided die gives $\Omega = \{1, 2, 3, 4, 5, 6\}$ and, for instance, the event "rolling an even number" is $A = \{2, 4, 6\}$.

2.2 Axioms of Probability

Definition 2.3 (Kolmogorov Axioms)**.** A probability measure P on $(\Omega, \mathcal{F})$ satisfies:

(i) $P(A) \geq 0$ for every $A \in \mathcal{F}$.

(ii) $P(\Omega) = 1$.

(iii) For mutually exclusive events $A_1, A_2, \ldots,$ $P\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} P(A_i)$.

Theorem 2.4 (Complement Rule)**.** *For any event A,* $P(A^c) = 1 - P(A)$.

Proof. Since A and A^c are mutually exclusive and $A \cup A^c = \Omega$, axiom (iii) gives $P(A) + P(A^c) = P(\Omega) = 1$, whence $P(A^c) = 1 - P(A)$. □

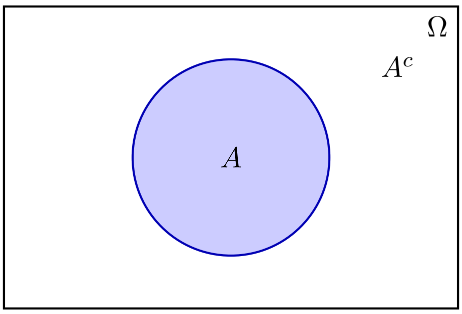

Figure 1: Venn diagram showing event A and its complement A^c within Ω.

2.3 Conditional Probability and Bayes' Theorem

Definition 2.5 (Conditional Probability)**.** If $P(B) > 0$, the conditional probability of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Theorem 2.6 (Bayes' Theorem)**.** *Let $\{B_1, \ldots, B_n\}$ be a partition of Ω with $P(B_j) > 0$ for all j. Then*

$$P(B_k \mid A) = \frac{P(A \mid B_k)\, P(B_k)}{\sum_{j=1}^{n} P(A \mid B_j)\, P(B_j)}.$$

Proof. By definition, $P(B_k \mid A) = P(A \cap B_k)/P(A)$. Since the B_j partition Ω, $P(A) = \sum_j P(A \cap B_j) = \sum_j P(A \mid B_j)P(B_j)$, and $P(A \cap B_k) = P(A \mid B_k)P(B_k)$. Substituting completes the proof. □

3 Expected Value and the House Edge

Definition 3.1 (Expected Value)**.** For a discrete random variable X taking values $x_1, x_2, \ldots$ with probabilities $p_1, p_2, \ldots$,

$$E[X] = \sum_i x_i\, p_i.$$

Theorem 3.2 (Linearity of Expectation)**.** *For any random variables X, Y and constants $a, b \in \mathbb{R}$,*

$$E[aX + bY] = aE[X] + bE[Y].$$

Proof.

$$\begin{aligned} E[aX + bY] &= \sum_{\omega \in \Omega} \big(aX(\omega) + bY(\omega)\big) P(\omega) \\ &= a \sum_{\omega} X(\omega)P(\omega) + b \sum_{\omega} Y(\omega)P(\omega) \\ &= aE[X] + bE[Y]. \end{aligned}$$

□

Definition 3.3 (House Edge)**.** The **house edge** (or house advantage) H of a wager is

$$H = -\frac{E[X]}{W},$$

where X is the player's net gain and W is the amount wagered. A positive house edge means the casino profits on average.

Example 3.4 (American Roulette — Single-Number Bet)**.** A single-number bet pays 35:1. With 38 pockets (numbers 1–36, 0, 00):

$$E[X] = \frac{1}{38}(+35) + \frac{37}{38}(-1) = \frac{35 - 37}{38} = -\frac{2}{38} \approx -0.0526.$$

Hence $H \approx 5.26\%$.

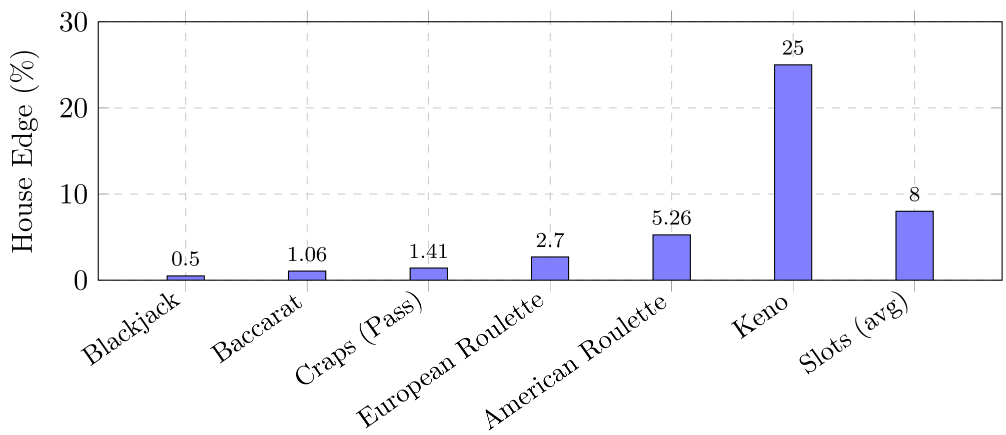

Figure 2: Approximate house edges for popular casino games.

4 Dice Games and Combinatorics

4.1 Sum of Two Dice

Proposition 4.1. *When two fair six-sided dice are rolled, the probability of the sum $S = k$ for $k \in \{2, \ldots, 12\}$ is*

$$P(S = k) = \frac{6 - |k - 7|}{36}.$$

Proof. There are 36 equally likely outcomes. For a target sum k, the first die can show any value $d_1 \in \{1, \ldots, 6\}$ such that $d_2 = k - d_1 \in \{1, \ldots, 6\}$. One verifies directly that the number of valid pairs is $6 - |k - 7|$ for each k, giving the result. □

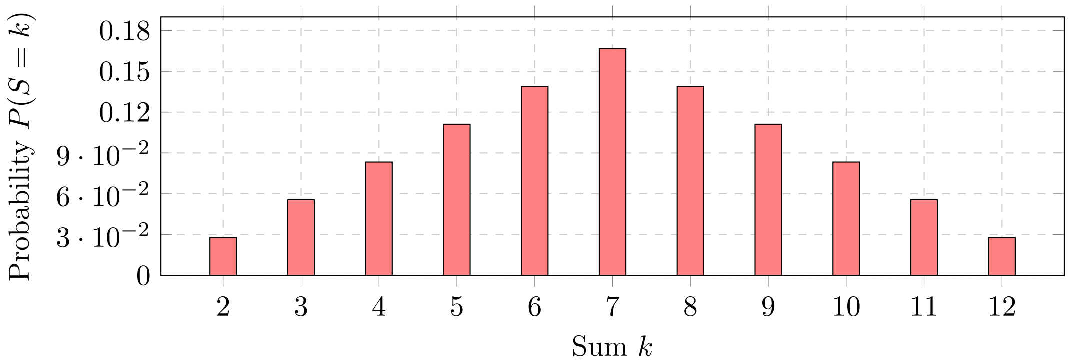

Figure 3: Probability distribution of the sum of two fair dice.

4.2 Craps

In the game of Craps, the *Pass Line* bet proceeds as follows:

1. **Come-out roll:** if the sum is 7 or 11 the bettor wins; if it is 2, 3, or 12 the bettor loses.
2. Otherwise the sum becomes the **point** p. The dice are rolled repeatedly until the sum is p (win) or 7 (lose).

Theorem 4.2 (Pass-Line Probability)**.** *The probability of winning a Pass-Line bet in Craps is*

$$P(win) = \frac{244}{495} \approx 0.4929.$$

Proof. On the come-out roll, $P(\text{win immediately}) = P(7) + P(11) = 6/36 + 2/36 = 8/36$. $P(\text{lose immediately}) = P(2) + P(3) + P(12) = 1/36 + 2/36 + 1/36 = 4/36$.

For each possible point p, let n_p denote the number of ways to roll sum p. Given point p, the probability of making the point before a 7 is $n_p/(n_p + 6)$. The contribution of each point to the overall win probability is $(n_p/36) \cdot n_p/(n_p + 6)$. Points come in symmetric pairs:

Point p	n_p	$n_p^2/\big(36(n_p + 6)\big)$
4 or 10	3	$2 \times 9/(36 \times 9) = 2/36$
5 or 9	4	$2 \times 16/(36 \times 10) = 32/360$
6 or 8	5	$2 \times 25/(36 \times 11) = 50/396$

Summing all contributions over a common denominator of 1980:

$$P(\text{win}) = \frac{8}{36} + \frac{2}{36} + \frac{32}{360} + \frac{50}{396} = \frac{440 + 110 + 176 + 250}{1980} = \frac{976}{1980} = \frac{244}{495}.$$

The house edge on the Pass Line is therefore $1 - 2 \cdot \frac{244}{495} = \frac{7}{495} \approx 1.41\%$. □

5 Card Games

5.1 Blackjack: Basic Strategy

Blackjack is one of the few casino games where skilled play can reduce the house edge to below 1% [6]. The player and dealer each receive cards; the goal is to approach a hand value of 21 without exceeding it.

Theorem 5.1 (Blackjack Insurance Is Unfavourable)**.** *In a freshly shuffled single-deck game of Blackjack, the insurance side bet (paying* 2:1 *when the dealer's hole card is a ten-value card) has a negative expected value for the player.*

Proof. After seeing the dealer's Ace and the player's two cards (neither of which is a ten-value card for the worst case), 49 cards remain of which 16 are ten-value. The probability the dealer has Blackjack is 16/49. The expected value of a unit insurance bet is

$$E = \frac{16}{49} \cdot (+2) + \frac{33}{49} \cdot (-1) = \frac{32 - 33}{49} = -\frac{1}{49} < 0.$$

If one or both player cards are ten-value, even fewer tens remain, making the bet strictly worse. $\square$

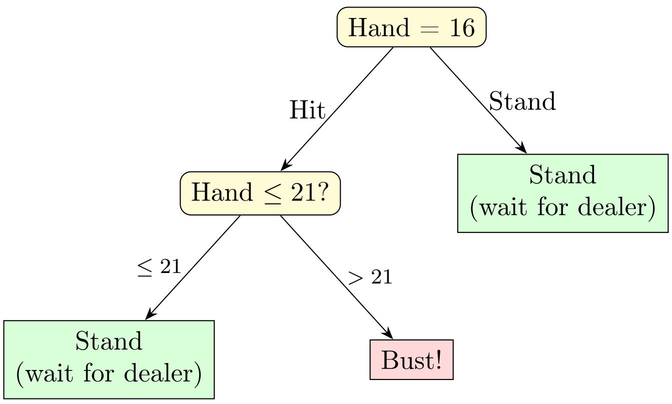

Figure 4: Simplified decision tree for a Blackjack hand totalling 16.

5.2 Poker: Hand Rankings and Combinatorics

A standard five-card poker hand is drawn from a 52-card deck.

Proposition 5.2 (Number of Flush Hands)**.** *The number of flush hands (five cards of the same suit, excluding straight flushes) is* 5108*.*

Proof. There are $\binom{4}{1}$ ways to choose a suit and $\binom{13}{5} = 1287$ ways to choose 5 cards from that suit. Subtract the 10 straight flushes per suit:

$$4\left(\binom{13}{5} - 10\right) = 4(1287 - 10) = 4 \times 1277 = 5108. \qquad \square$$

Royal Flush (4)

Straight Flush (36)

Four of a Kind (624)

Full House (3 744)

Flush (5 108)

Straight (10 200)

Three of a Kind (54 912)

Two Pair (123 552)

One Pair (1 098 240)

Figure 5: Poker hand rankings with number of distinct five-card combinations (out of $\binom{52}{5} =$ 2,598,960).

6 The Gambler's Ruin

One of the oldest problems in probability theory concerns a gambler who repeatedly makes even-money bets.

Theorem 6.1 (Gambler's Ruin)**.** *A gambler starts with i dollars and bets \$1 each round against an infinitely wealthy casino, winning each bet with probability p and losing with $q = 1 - p$. Let r_i be the probability of eventual ruin (reaching \$0).*

(a) If $p \neq q$:

$$r_i = \left(\frac{q}{p}\right)^i, \qquad i \geq 0.$$

(b) If $p = q = \frac{1}{2}$:

$$r_i = 1 \quad \textit{for all } i \geq 0.$$

Proof. Let W_n denote the gambler's wealth after n rounds. The ruin probability satisfies the recurrence

$$r_i = p\, r_{i+1} + q\, r_{i-1}, \qquad i \geq 1,$$

with boundary condition $r_0 = 1$ (already ruined).

Case $p \neq q$**.** We seek solutions of the form $r_i = \lambda^i$. Substituting: $\lambda^i = p\lambda^{i+1} + q\lambda^{i-1}$, i.e., $p\lambda^2 - \lambda + q = 0$. The roots are $\lambda = 1$ and $\lambda = q/p$. The general solution is $r_i = A + B(q/p)^i$. Since $r_0 = 1$, we have $A + B = 1$.

If the casino has finite wealth N, $r_i = \frac{(q/p)^i - (q/p)^N}{1-(q/p)^N}$, and sending $N \to \infty$:

- If $p < q$: $(q/p)^N \to \infty$, so $r_i \to 1$.
- If $p > q$: $(q/p)^N \to 0$, so $r_i = (q/p)^i$.

Combining: $r_i = \min\bigl(1, (q/p)^i\bigr)$. Since $q/p < 1$ when $p > q$, we have $r_i = (q/p)^i < 1$. When $p < q$, ruin is certain.

Case $p = q = \frac{1}{2}$**.** The characteristic equation has a repeated root $\lambda = 1$. The general solution is $r_i = A + Bi$. With the finite-casino boundary $r_0 = 1$, $r_N = 0$, $r_i = 1 - i/N$. Letting $N \to \infty$, $r_i \to 1$ for every finite i. $\square$

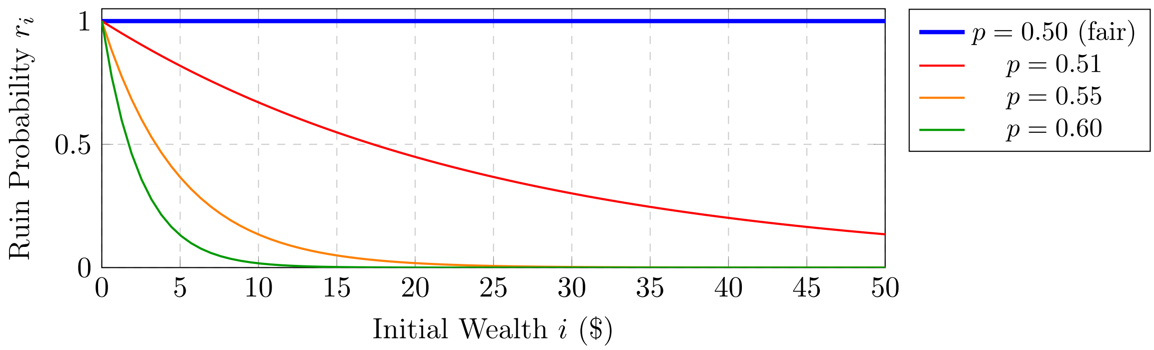

Figure 6: Gambler's ruin probability $r_i = (q/p)^i$ for various win probabilities p, playing against an infinitely wealthy casino.

7 The Law of Large Numbers and Casino Profitability

Theorem 7.1 (Weak Law of Large Numbers). *Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean μ and finite variance σ^2. Then for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right) = 0.$$

Proof. By Chebyshev's inequality,

$$P\big(|\bar{X}_n - \mu| \geq \varepsilon\big) \leq \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \xrightarrow{n\to\infty} 0. \qquad \square$$

Corollary 7.2. *For a casino offering a game with house edge $H > 0$, the casino's average profit per unit wagered converges in probability to H as the number of bets grows.*

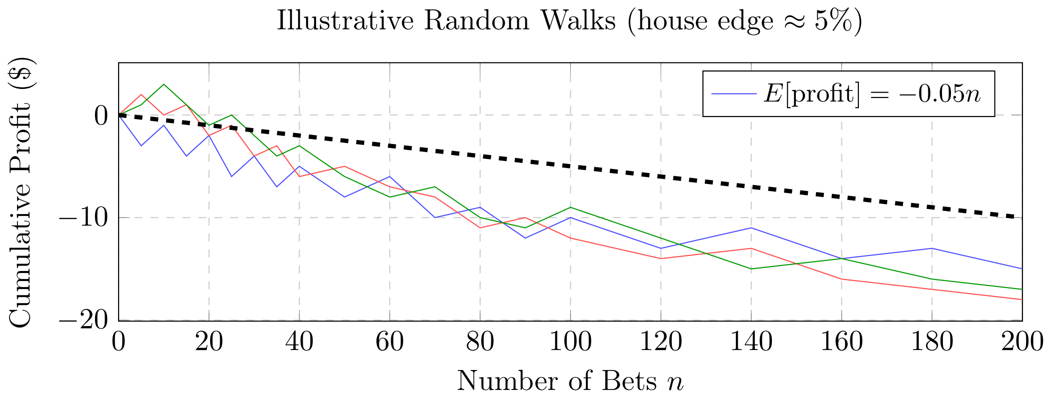

Figure 7: Three illustrative player profit trajectories converging toward the expected loss line (dashed).

8 Optimal Betting: The Kelly Criterion

Theorem 8.1 (Kelly Criterion [5]). *Consider a bet that wins with probability p and loses with probability $q = 1 - p$, paying even money. The fraction f^* of one's bankroll that maximises the expected logarithmic growth rate is*

$$f^* = p - q = 2p - 1.$$

More generally, if the bet pays b:1,

$$f^* = \frac{bp - q}{b} = p - \frac{q}{b}.$$

Proof. After one round, the bankroll is multiplied by $1 + f$ with probability p and $1 - f$ with probability q. The expected log-growth per round is

$$g(f) = p \ln(1 + f) + q \ln(1 - f).$$

Setting $g'(f) = 0$:

$$\frac{p}{1+f} - \frac{q}{1-f} = 0 \implies p(1-f) = q(1+f) \implies f^* = p - q.$$

Since $g''(f) = -p/(1+f)^2 - q/(1-f)^2 < 0$, this is a maximum.

For the general b:1 payout, the bankroll multiplier on a win is $1 + bf$, so $g(f) = p \ln(1 + bf) + q \ln(1 - f)$. Setting $g'(f) = 0$ yields $f^* = (bp - q)/b$. □

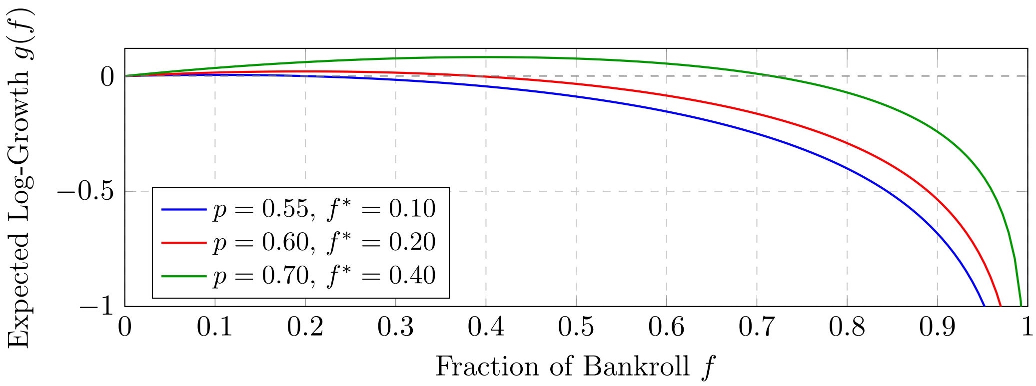

Figure 8: Expected log-growth $g(f)$ for different win probabilities. The peak of each curve indicates the Kelly-optimal fraction f^*.

9 Roulette: A Detailed Analysis

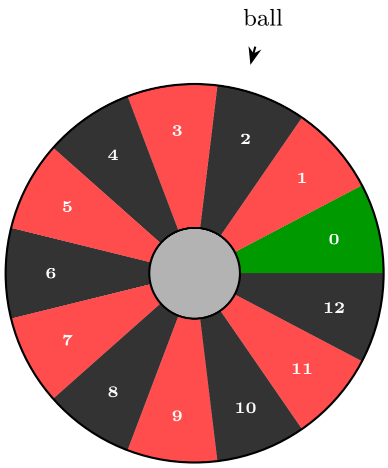

Figure 9: Simplified roulette wheel (12 numbered pockets plus a zero pocket, for illustrative purposes).

Theorem 9.1 (European vs. American Roulette)**.** *The house edge for any standard bet is:*

- ***European (single zero):*** $H = \dfrac{1}{37} \approx 2.70\%$.
- ***American (double zero):*** $H = \dfrac{2}{38} \approx 5.26\%$.

Proof. Standard roulette payouts are computed as though there were only 36 pockets. A bet on k numbers pays $(36 - k)/k$ to 1. With N total pockets ($N = 37$ for European, $N = 38$ for American), the expected value of a unit bet is

$$E = \frac{k}{N} \cdot \frac{36-k}{k} - \frac{N-k}{N} = \frac{36-k}{N} - \frac{N-k}{N} = \frac{36-N}{N}.$$

For European roulette: $E = (36 - 37)/37 = -1/37$; house edge $= 1/37 \approx 2.70\%$.
For American roulette: $E = (36 - 38)/38 = -2/38 = -1/19$; house edge $= 1/19 \approx 5.26\%$. □

10 Martingale and Other Betting Systems

Definition 10.1 (Martingale System)**.** After each loss on an even-money bet, the gambler doubles the wager. After any win, the gambler returns to the base stake.

Theorem 10.2 (Futility of the Martingale)**.** *Under the Martingale system with an initial wager of \$1, after n consecutive losses the required wager is 2^n dollars and the cumulative loss is $2^n - 1$ dollars. The expected profit of each completed "cycle" is non-positive whenever $p \le \frac{1}{2}$.*

Proof. After k consecutive losses ($k = 0, 1, \ldots, n-1$) followed by a win on round $k+1$, the net profit is

$$\text{profit} = 2^k - \sum_{j=0}^{k-1} 2^j = 2^k - (2^k - 1) = +1.$$

The probability of this event is $q^k p$. The probability of losing all n rounds is q^n, incurring a loss of $2^n - 1$. Hence the expected profit per cycle is

$$E = \sum_{k=0}^{n-1} q^k p \cdot 1 - q^n(2^n - 1) = p\,\frac{1-q^n}{1-q} - q^n(2^n - 1).$$

Since $1 - q = p$, the first term simplifies to $1 - q^n$. Thus

$$E = 1 - q^n - q^n(2^n - 1) = 1 - q^n \cdot 2^n = 1 - (2q)^n.$$

When $p = q = \frac{1}{2}$: $E = 1 - 1 = 0$. When $p < \frac{1}{2}$: $2q > 1$, so $(2q)^n > 1$ and $E < 0$. □

Remark. Even in the fair case ($p = \frac{1}{2}$), the Martingale is impractical because table limits and finite bankrolls cap n.

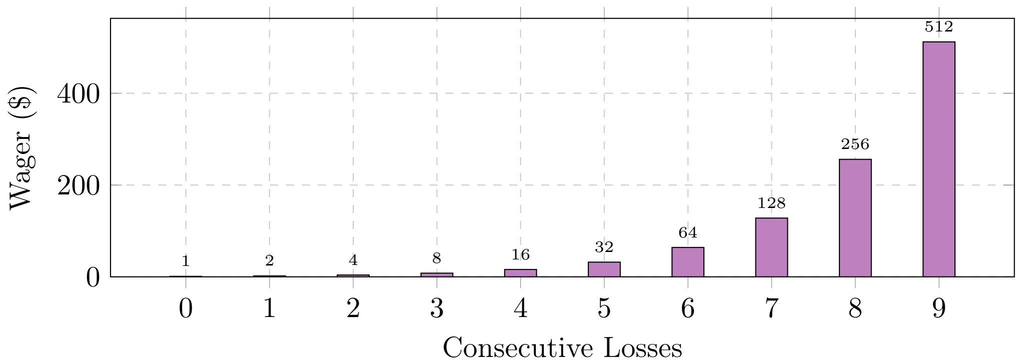

Figure 10: Exponential growth of the Martingale wager after consecutive losses.

11 A Brief History of Gambling

c. 3000 BCE Earliest known dice found in Mesopotamia [1].

c. 500 BCE Greeks and Romans gamble on chariot races and gladiatorial combat.

1564 Gerolamo Cardano writes *Liber de Ludo Aleae*, the first systematic treatment of probability in gambling [3].

1654 Pascal–Fermat correspondence formalises the "Problem of Points" [2].

1713 Jacob Bernoulli's *Ars Conjectandi* published posthumously, containing the first proof of the Law of Large Numbers [4].

1866 The Monte Carlo Casino opens in Monaco.

1956 John Kelly publishes the Kelly Criterion at Bell Labs [5].

1962 Edward Thorp's *Beat the Dealer* introduces card counting to the public [6].

1990s–present Online gambling emerges and grows into a multi-billion dollar industry.

12 Responsible Gambling

No mathematical treatise on gambling is complete without an ethical note. Because the house edge guarantees long-run losses for the player (Theorem 7.1), gambling should be viewed strictly as entertainment with a known cost, never as a source of income. Readers experiencing difficulty controlling gambling behaviour are encouraged to contact the **National Council on Problem Gambling** (USA: 1-800-522-4700) or equivalent organisations worldwide.

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Glossary

Bankroll The total amount of money a gambler has set aside for wagering.

Bet (Wager) An amount of money risked on the outcome of a random event.

Bust In Blackjack, exceeding a hand total of 21.

Come-out Roll The first roll of the dice in a round of Craps.

Expected Value (EV) The long-run average outcome of a random variable; $E[X] = \sum x_i p_i$.

House Edge The average profit the casino makes per unit wagered, expressed as a percentage.

Kelly Criterion An optimal betting strategy that maximises the expected logarithmic growth rate of a bankroll.

Martingale A betting system in which the stake is doubled after every loss.

Odds A ratio representing the likelihood of a particular outcome, or the payout ratio of a bet.

Point In Craps, the number established on the come-out roll (4, 5, 6, 8, 9, or 10) that the shooter must repeat before rolling a 7.

Probability A measure between 0 and 1 indicating the likelihood of an event.

Random Variable A numerical quantity whose value is determined by a random experiment.

Sample Space (Ω) The set of all possible outcomes of a random experiment.

Variance A measure of the spread of a distribution; $\mathrm{Var}(X) = E[(X - \mu)^2]$.

The End