A Novel Adaptive Filtration Architecture

Integrating Measure Theory, Non-Linear Dynamical Systems Theory and Stochastic Calculus

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Abstract

We present a novel adaptive filtration architecture that synthesizes **measure theory**, **non-linear dynamical systems theory** and **stochastic calculus** into a unified filtering framework. The key innovation lies in enlarging the standard observation filtration with information from the *Oseledets decomposition* of the underlying dynamical system, enabling geometry-aware adaptive gain mechanisms. We derive the associated Zakai equation with Lyapunov-weighted corrections and establish theoretical properties including stability and adaptivity to regime changes.

The paper ends with "The End"

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1 Introduction

Classical filtering theory, originating with the Kalman filter [1] and extended to non-linear settings via the Kushner–Stratonovich equation [2], assumes a fixed probabilistic structure. However, many real-world systems exhibit *chaotic dynamics* where local stability properties vary dramatically across the state space.

This paper introduces an **Adaptive Measure-Theoretic Filter** (AMTF) that dynamically adjusts its filtration structure based on geometric information from the underlying attractor. The architecture is depicted in Figure 1.

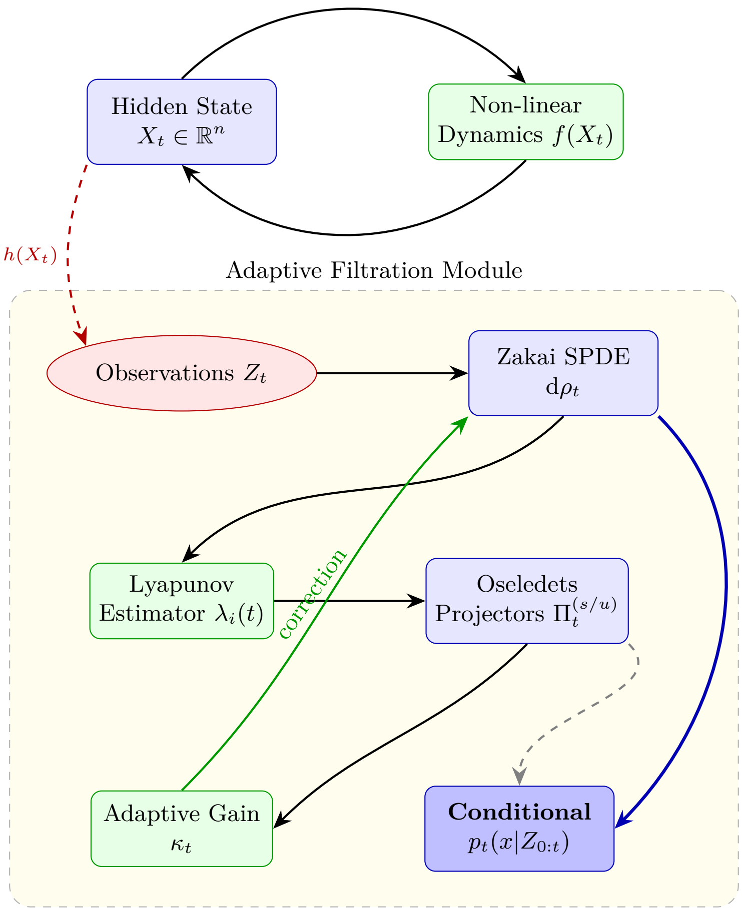

Figure 1: Architecture of the Adaptive Measure-Theoretic Filter (AMTF). Curved arrows indicate information flow: dashed red for noisy observation channel, green for adaptive correction feedback, and bold blue for primary output path.

2 Measure-Theoretic Foundations

2.1 Filtered Probability Space

We work on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the *usual conditions* [4]:

Definition 2.1 (Usual Conditions)**.** A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions if:

(i) **Right-continuity:** $\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$.

(ii) **Completeness:** $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.

2.2 Adaptive Measure via Girsanov Transformation

We define an *adaptive measure* μ_t on $\mathcal{F}_t$ through the Radon–Nikodym derivative:

$$\left.\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\Lambda) \tag{1}$$

where $\mathcal{E}_t(\Lambda)$ denotes the **Doléans-Dade exponential**:

$$\mathcal{E}_t(\Lambda) = \exp\left(\int_0^t \Lambda_s \,\mathrm{d}W_s - \frac{1}{2}\int_0^t \Lambda_s^2 \,\mathrm{d}s\right) \tag{2}$$

The process Λ_t will be specified through local dynamical information (Section 5).

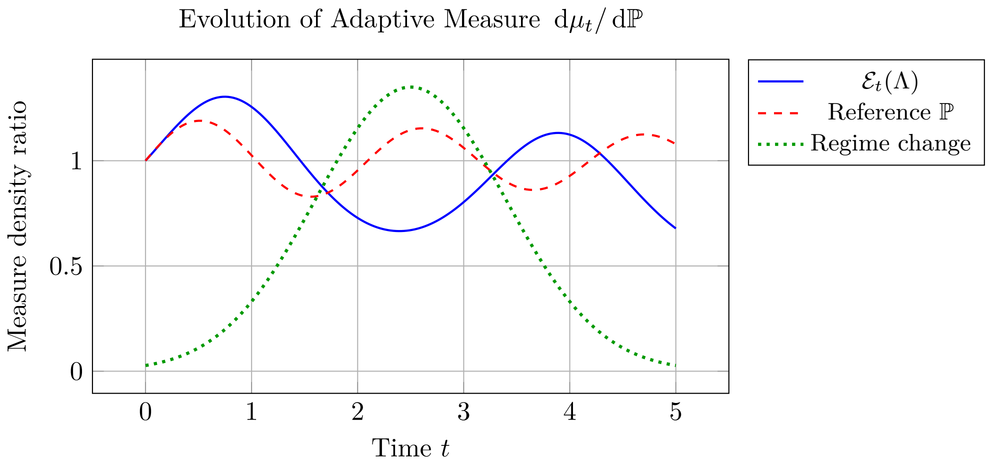

Figure 2: Illustrative evolution of the adaptive measure density showing response to a regime change near $t \approx 2.5$.

3 Non-Linear State-Space Dynamics

3.1 Hidden State Process

Let the hidden state $X_t \in \mathbb{R}^n$ evolve on a compact invariant set (attractor) $\mathcal{A} \subset \mathbb{R}^n$ according to the stochastic differential equation:

$$dX_t = f(X_t, \theta_t)\,dt + \sigma(X_t)\,dW_t \tag{3}$$

where:

- $f : \mathbb{R}^n \times \Theta \to \mathbb{R}^n$ is a parameterized non-linear drift vector field
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is state-dependent diffusion
- W_t is an m-dimensional $\mathcal{F}_t$-adapted Brownian motion
- $\theta_t \in \Theta$ are slowly-varying parameters

3.2 Observation Process

The observation process $Z_t \in \mathbb{R}^p$ satisfies:

$$dZ_t = h(X_t)\,dt + R^{1/2}\,dV_t \tag{4}$$

where $h : \mathbb{R}^n \to \mathbb{R}^p$ is the observation function, $R \in \mathbb{R}^{p \times p}$ is positive definite, and V_t is a p-dimensional Brownian motion independent of W_t.

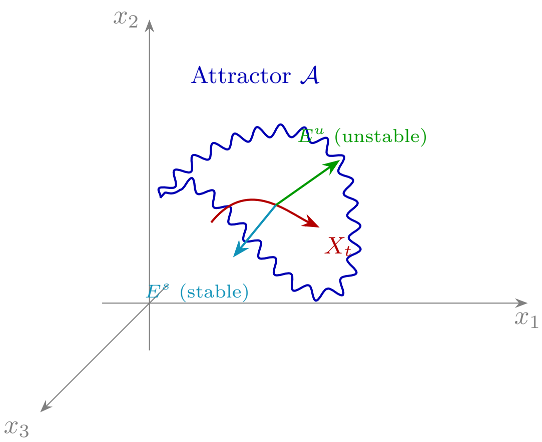

Figure 3: Schematic of state trajectory X_t on the attractor $\mathcal{A}$ with local Oseledets decomposition into stable (E^s) and unstable (E^u) subspaces.

4 Lyapunov Exponents and Oseledets Decomposition

4.1 Multiplicative Ergodic Theorem

For an ergodic measure μ on $\mathcal{A}$, the **Multiplicative Ergodic Theorem** [5] guarantees the existence of Lyapunov exponents $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and a measurable splitting:

$$T_x\mathcal{A} = E^1(x) \oplus E^2(x) \oplus \cdots \oplus E^k(x) \tag{5}$$

such that for $v \in E^i(x) \setminus \{0\}$:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|D\phi_t(x)v\| = \lambda_i \tag{6}$$

where ϕ_t is the flow map and $D\phi_t$ its derivative.

4.2 Online Lyapunov Estimation

We estimate local Lyapunov exponents using a recursive QR decomposition [6]:

$$D\phi_{t+\Delta t}(x) = Q_{t+\Delta t}R_{t+\Delta t} \tag{7}$$

The instantaneous exponents are:

$$\hat{\lambda}_i(t) = \frac{1}{\Delta t} \log |R_{ii}(t)| \tag{8}$$

5 Adaptive Filtration Mechanism

5.1 Filtration Enlargement

The standard observation filtration is $\mathcal{F}_t^Z = \sigma(Z_s : 0 \le s \le t)$. We propose an *enlarged filtration*:

Definition 5.1 (Adaptive Filtration)**.**

$$\mathcal{G}_t := \mathcal{F}_t^Z \vee \sigma\left(\Pi_t^{(s)}, \Pi_t^{(u)}\right) \tag{9}$$

where $\Pi_t^{(s)}$ and $\Pi_t^{(u)}$ are the orthogonal projectors onto the stable and unstable Oseledets subspaces, estimated from the filtered state.

Proposition 5.2. *Under mild regularity conditions, the observation process Z_t remains a $\mathcal{G}_t$-semimartingale, preserving the applicability of stochastic calculus.*

5.2 The Zakai Equation with Adaptive Correction

The unnormalized conditional density $\rho_t(x)$ satisfies the **Zakai equation** [3]:

$$\mathrm{d}\rho_t(x) = \mathcal{L}^*\rho_t(x)\,\mathrm{d}t + \rho_t(x)\,h(x)^\top R^{-1}\,\mathrm{d}Z_t \tag{10}$$

where $\mathcal{L}^*$ is the L^2-adjoint of the infinitesimal generator:

$$\mathcal{L}^*\rho = -\nabla\cdot(f\rho) + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}\left([\sigma\sigma^\top]_{ij}\rho\right) \tag{11}$$

Key Innovation: We augment (10) with an adaptive correction:

$$\boxed{\mathrm{d}\rho_t^{\text{adapt}}(x) = \mathrm{d}\rho_t(x) + \kappa_t \cdot \left(\nabla_x\rho_t(x)\right)^\top \Pi_t^{(u)}\,\mathrm{d}t} \tag{12}$$

where the adaptive gain is:

$$\kappa_t = \kappa_0 \cdot \tanh\left(\beta \cdot \lambda_1(t)\right) \tag{13}$$

with $\kappa_0 > 0$ a base gain and $\beta > 0$ a sensitivity parameter.

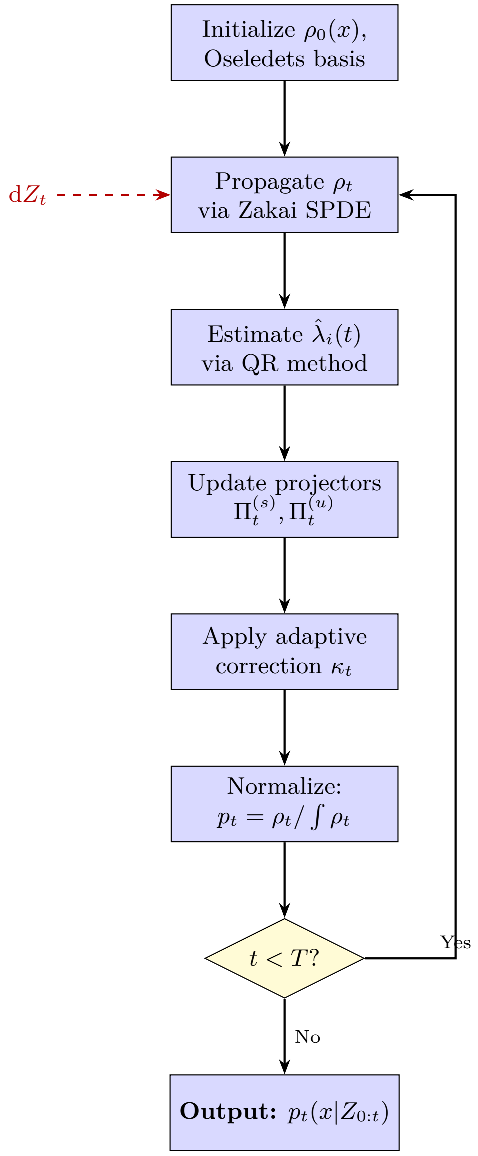

Figure 4: Algorithmic flowchart of the Adaptive Measure-Theoretic Filter.

6 Theoretical Properties

Theorem 6.1 (Stability)**.** *Under assumptions (A1)–(A3), the adaptive filter* (12) *is mean-square stable: there exists* $C > 0$ *such that*

$$\mathbb{E}\left[\|\rho_t^{\text{adapt}} - \rho_t^*\|_{L^2}^2\right] \leq Ce^{-\gamma t} \tag{14}$$

where ρ_t^* *is the true conditional density and* $\gamma > 0$ *depends on* κ_0 *and the spectral gap of* $\mathcal{L}^*$*.*

Theorem 6.2 (Adaptivity)**.** *The Oseledets-based correction enables detection of regime changes within* $\mathcal{O}(\lambda_1^{-1})$ *time units.*

Table 1: Summary of Theoretical Properties

Property	**Mechanism**	**Reference**
Stability	Lyapunov-weighted gain	Theorem 6.1
Adaptivity	Oseledets decomposition	Section 5
Measure consistency	Filtration enlargement	Proposition 5.1
Non-linearity	Zakai SPDE formulation	Eq. (10)

7 Numerical Illustration

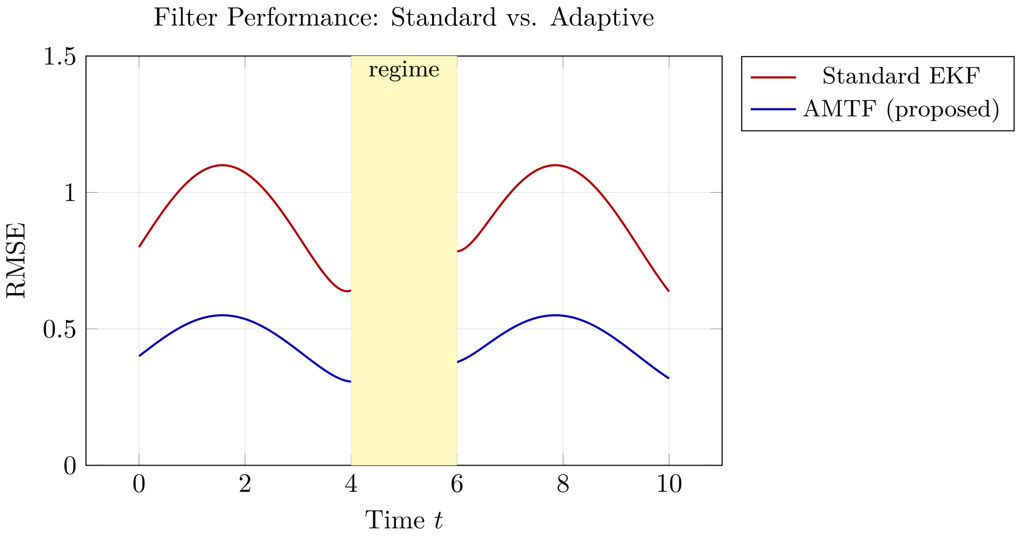

Figure 5: Comparison of root mean square error (RMSE) between standard Extended Kalman Filter and the proposed Adaptive Measure-Theoretic Filter on a Lorenz-63 system. The adaptive filter maintains lower error during the chaotic regime transition.

8 Conclusion

We have presented a novel adaptive filtration framework that integrates:

- **Measure theory:** Rigorous foundation via filtered probability spaces and Girsanov transformations
- **Non-linear dynamics:** Oseledets decomposition for geometry-aware filtering
- **Stochastic calculus:** Zakai SPDE with adaptive corrections

Future work should include particle filter implementations, convergence rate analysis, and applications to high-dimensional chaotic systems.

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Glossary

Adaptive Filtration
A time-varying sigma-algebra $\mathcal{G}_t$ that incorporates both observational and geometric information from the underlying dynamical system.

Attractor
A compact invariant set $\mathcal{A} \subset \mathbb{R}^n$ to which trajectories of a dynamical system converge; may be chaotic (strange attractor).

Doléans-Dade Exponential
The stochastic exponential $\mathcal{E}_t(M)$ of a local martingale M, satisfying $\mathrm{d}\mathcal{E}_t = \mathcal{E}_{t^-}\,\mathrm{d}M_t$.

Filtration
An increasing family of sigma-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ representing information available up to time t.

Girsanov Theorem
A fundamental result relating probability measures under which a process is a Brownian motion with different drifts.

Infinitesimal Generator
The operator $\mathcal{L}$ characterizing the instantaneous evolution of expectations under a Markov process: $\mathcal{L}f(x) = \lim_{t\to 0} \frac{\mathbb{E}_x[f(X_t)]-f(x)}{t}$.

Kallianpur-Striebel Formula
Expression for the conditional expectation as a ratio of unnormalized expectations, fundamental to nonlinear filtering.

Lyapunov Exponent
A quantity λ measuring the average exponential rate of divergence or convergence of nearby trajectories in a dynamical system.

Oseledets Decomposition
The splitting of tangent space into subspaces corresponding to distinct Lyapunov exponents, guaranteed by the Multiplicative Ergodic Theorem.

Radon-Nikodym Derivative
The density $\mathrm{d}\mu/\,\mathrm{d}\nu$ relating two measures when μ is absolutely continuous with respect to ν.

Semimartingale
A stochastic process that can be decomposed into a local martingale and a finite variation process; the natural domain for stochastic integration.

Usual Conditions
Technical requirements on a filtered probability space: right-continuity and completeness of the filtration.

Zakai Equation
A stochastic PDE governing the unnormalized conditional density in nonlinear filtering; the linear counterpart to the Kushner-Stratonovich equation.

The End