A Formal Theory of Multi-Party Coalition Elections under Proportional Representation

Dominance, Subordination, and Parliamentary Arithmetic

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Abstract

We develop a rigorous mathematical framework for multi-party elections organised into pre-electoral coalitions, each comprising a single *dominant* party and a finite set of *subordinate* parties. Seats are allocated by the D'Hondt divisor method after parties have individually crossed an electoral threshold. We prove that the method satisfies upper quota and inter-party monotonicity, while lower quota may fail—systematically favouring larger parties. We characterise the conditions under which a single coalition attains an outright parliamentary majority, and derive a stochastic monotonicity result showing that higher thresholds increase the probability of a hung parliament. Intra-coalition power is quantified through the Shapley–Shubik index, including a sufficient condition under which the dominant party acts as a dictator. Post-election government formation is modelled as a transferable-utility cooperative game; we characterise when the core is non-empty and show it is empty whenever the parliament is hung. A running example with three coalitions of nine parties illustrates every result.

The paper ends with "The End"

Keywords: proportional representation, D'Hondt method, coalition formation, electoral threshold, Shapley–Shubik index, cooperative game theory, parliamentary majority

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1 Introduction

Multi-party democracies rarely produce single-party majorities. The interplay between an electoral system's proportionality, pre-electoral alliance structures, and post-electoral bargaining determines governance outcomes and the strategic incentives facing voters and parties alike. Despite a substantial empirical literature [2,4,8], the purely theoretical foundations of *coalition-structured* proportional elections remain scattered across political science, social choice theory, and operations research.

This paper offers a unified treatment. We study an election in which $K = 3$ coalitions $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, each consisting of one dominant party and a finite number of subordinate parties, compete for S parliamentary seats. Seats are allocated by the D'Hondt divisor rule [1,3], and only parties individually exceeding an electoral threshold τ participate in the allocation.

Our contributions are threefold:

1. We formalise the coalition hierarchy (Section 2) and establish quota and monotonicity properties of the D'Hondt method (Section 3).

2. We characterise majority conditions and prove a monotone threshold effect on the probability of a hung parliament (Section 4).

3. We apply the Shapley–Shubik power index [7] to quantify intra-coalition dominance, and study government formation as a cooperative game, drawing on the theory of political coalitions [6,9] and incentive effects of electoral systems [5] (Section 5).

Section 6 provides a self-contained numerical illustration. Section 7 concludes. A glossary of notation precedes the bibliography.

2 The Formal Model

2.1 Parties, Coalitions, and Hierarchy

Let $\mathcal{P}$ denote the finite set of all competing parties, $|\mathcal{P}| = n$. The parties are partitioned into K disjoint *coalitions*:

$$\mathcal{P} = \bigsqcup_{k=1}^{K} \mathcal{C}_k, \qquad \mathcal{C}_k \cap \mathcal{C}_{k'} = \emptyset \quad \text{for} \quad k \neq k'.$$

Definition 2.1 (Coalition structure)**.** For each coalition $\mathcal{C}_k$ there exists a distinguished element $d_k \in \mathcal{C}_k$, the *dominant party*, and a non-empty set $\mathcal{C}_k^- := \mathcal{C}_k \setminus \{d_k\}$ of *subordinate parties*. We write $\mathcal{C}_k = \{d_k\} \cup \mathcal{C}_k^-$ and set $m_k := |\mathcal{C}_k^-| \geq 1$.

Throughout we fix $K = 3$ and $m_k = 2$ for all k, giving $n = 9$ parties. The hierarchical structure is depicted in Figure 1.

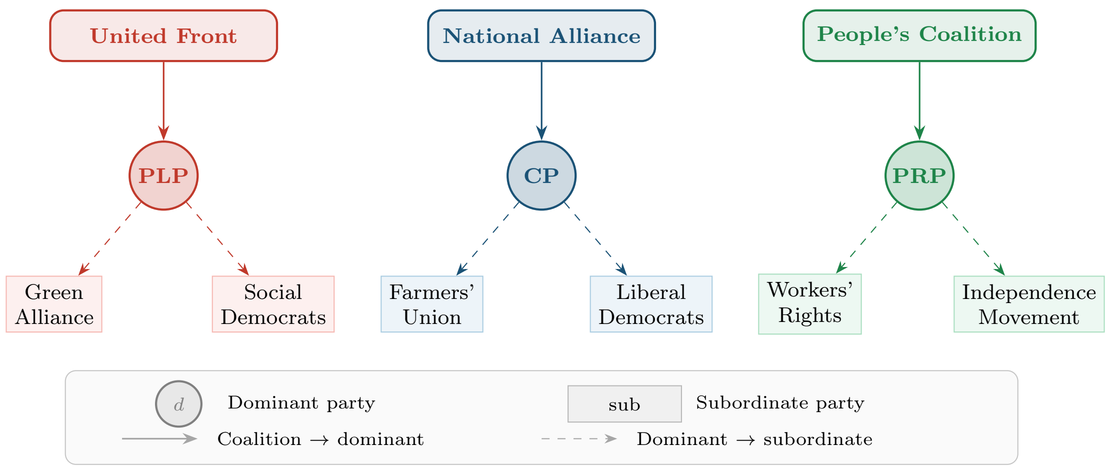

Figure 1: Hierarchical structure of the three-coalition, nine-party election.

PLP = Progressive Labor Party; CP = Conservative Party; PRP = People's Republic Party.

2.2 Vote Shares and the Stochastic Ballot Model

Each party $p \in \mathcal{P}$ obtains a *vote share* $\mathsf{v}_p \in [0,1]$ with $\sum_p \mathsf{v}_p = 1$. We model election-day vote shares as stochastic perturbations of pre-election polling estimates $\hat{v}_p$.

Definition 2.2 (Noisy ballot model)**.** Let $\hat{\boldsymbol{v}} = (\hat{v}_p)_p$ satisfy $\sum_p \hat{v}_p = 1$ and $\hat{v}_p \geq 0$. The realised vote-share vector is

$$\boldsymbol{v} \;=\; \frac{[\,\hat{\boldsymbol{v}} + \boldsymbol{\varepsilon}\,]_+}{\|[\,\hat{\boldsymbol{v}} + \boldsymbol{\varepsilon}\,]_+\|_1}, \qquad \boldsymbol{\varepsilon} \sim \mathcal{N}\big(\mathbf{0},\, \sigma^2 I_n\big)\,,$$

where $[\,\cdot\,]_+$ zeros negative components before normalisation, and $\sigma > 0$ is the *campaign-noise* parameter.

Remark 2.3. The Dirichlet distribution is the natural prior for compositional data, but the additive Gaussian model is operationally transparent and directly consistent with the practice of adding independent polling errors party by party. The two models coincide to first order in σ when all $\hat{v}_p$ are bounded away from zero.

2.3 The Electoral Threshold

Definition 2.4 (Threshold exclusion)**.** A party p is *eligible* for seat allocation if and only if $\mathsf{v}_p \geq \tau$, where $\tau \in (0,1)$ is the *electoral threshold.* Write $\mathcal{E} := \{p \in \mathcal{P} : \mathsf{v}_p \geq \tau\}$ for the eligible set. The *effective vote share* of eligible party p is

$$\tilde{v}_p \;:=\; \frac{\mathsf{v}_p}{\displaystyle\sum_{q \in \mathcal{E}} \mathsf{v}_q}.$$

Threshold exclusion redistributes the votes of ineligible parties proportionally among all eligible parties (since $\sum_{p \in \mathcal{E}} \tilde{v}_p = 1$), producing a systematic bonus for the eligible parties.

Proposition 2.5 (Threshold bonus)**.** *If $\mathcal{E} \subsetneq \mathcal{P}$, then $\tilde{v}_p > \mathsf{v}_p$ for every $p \in \mathcal{E}$.*

Proof. Since at least one ineligible party exists, $W := \sum_{q \in \mathcal{E}} \mathsf{v}_q < 1$. Hence $\tilde{v}_p = \mathsf{v}_p / W > \mathsf{v}_p$ for all $p \in \mathcal{E}$. □

Figure 2 illustrates the threshold effect for our nine-party running example.

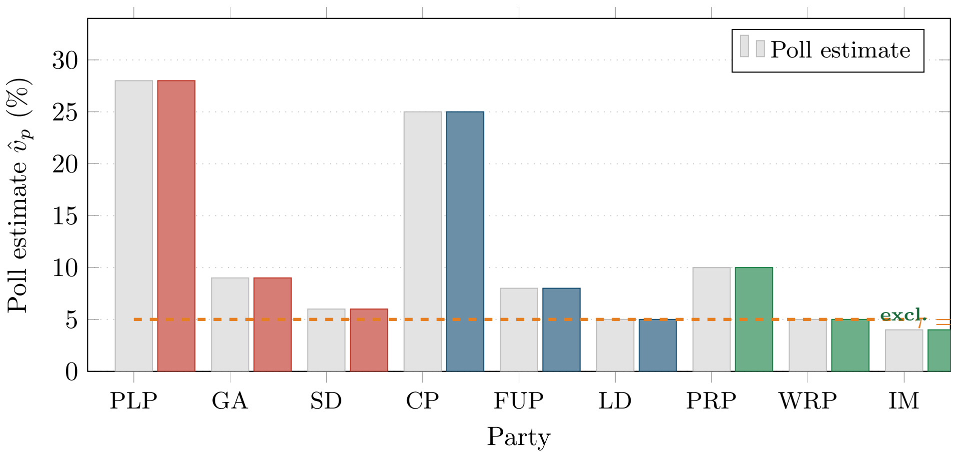

Figure 2: Pre-election poll estimates for all nine parties.

United Front: red; National Alliance: blue; People's Coalition: green. The dashed orange line marks the threshold $\tau = 5\%$. The Independence Movement (IM) is the sole party below threshold and is excluded from seat allocation.

3 The D'Hondt Seat Allocation Method

3.1 Definition

The D'HONDT method allocates S indivisible seats to eligible parties via a highest-averages rule.

Definition 3.1 (D'HONDT allocation)**.** Given eligible set $\mathcal{E}$, effective vote shares $(\tilde{v}_p)_{p\in\mathcal{E}}$, and $\mathsf{S} \in \mathbb{N}$, initialise $s_p \leftarrow 0$ for all $p \in \mathcal{E}$. Repeat S times:

$$p^* = \arg\max_{p\in\mathcal{E}} \frac{\tilde{v}_p}{s_p + 1}, \qquad s_{p^*} \leftarrow s_{p^*} + 1.$$

The final vector $(s_p)_{p\in\mathcal{E}}$ is the D'HONDT *allocation.* Ties are broken by pre-agreed lot; we assume generic vote shares to exclude ties.

Equivalently, there exists a *critical divisor* $\lambda^* > 0$ such that $s_p = \lfloor \tilde{v}_p/\lambda^* \rfloor$ for all $p \in \mathcal{E}$, where λ^* is the smallest positive value satisfying $\sum_p \lfloor \tilde{v}_p/\lambda^* \rfloor = \mathsf{S}$. This is the *greatest divisor* characterisation [1].

3.2 Quota Properties

Define the *natural quota* of party p as $q_p := \tilde{v}_p \cdot \mathsf{S}$.

Theorem 3.2 (Quota bounds)**.** *The* D'HONDT *allocation satisfies:*

(i) $s_p \leq \lceil q_p \rceil$ *for all* $p \in \mathcal{E}$ (upper quota holds);

(ii) $s_p \geq \lfloor q_p \rfloor$ need not hold (lower quota may fail).

Proof. **(i) Upper quota.** Suppose for contradiction that $s_p \geq \lceil q_p \rceil + 1$ for some p. At the step when p was awarded its $\big(\lceil q_p \rceil + 1\big)$-th seat, the winning quotient was

$$\frac{\tilde{v}_p}{\lceil q_p \rceil + 1} \leq \frac{\tilde{v}_p}{q_p + 1} = \frac{1}{\mathsf{S} + 1/\tilde{v}_p} < \frac{1}{\mathsf{S}}.$$

But in the D'HONDT algorithm every seat must be won by a quotient of at least $1/\mathsf{S}$ (since any party with zero seats has quotient $\tilde{v}_p \geq 1/\mathsf{S}$ for S parties each with positive effective share), a contradiction. A formal proof via the greatest-divisor characterisation is given in [1].

(ii) Failure of lower quota. The D'HONDT method systematically favours larger parties [8]: a larger party may capture a seat at a quotient that slightly exceeds a smaller party's quotient at a higher divisor step, leaving the smaller party short of its lower quota. The numerical example in Table 2 exhibits this phenomenon directly. □

Corollary 3.3 (Monotonicity)**.** *If* $\tilde{v}_p > \tilde{v}_{p'}$ *then* $s_p \geq s_{p'}$.

Proof. Suppose for contradiction that $s_{p'} > s_p$. Let t^* be the step at which p' received its $(s_p + 1)$-th seat. At step t^*, party p held at most s_p seats, so its active quotient satisfied

$$\frac{\tilde{v}_p}{s_p^{(t^*)} + 1} \geq \frac{\tilde{v}_p}{s_p + 1}.$$

Since p' won this seat over p, the allocation rule requires

$$\frac{\tilde{v}_{p'}}{s_p + 1} = \frac{\tilde{v}_{p'}}{s_{p'}^{(t^*)} + 1} \geq \frac{\tilde{v}_p}{s_p^{(t^*)} + 1} \geq \frac{\tilde{v}_p}{s_p + 1},$$

which gives $\tilde{v}_{p'} \geq \tilde{v}_p$, contradicting $\tilde{v}_p > \tilde{v}_{p'}$. □

3.3 Intra-Coalition Dominance

Proposition 3.4 (Dominant seat supremacy)**.** *If* $\mathsf{v}_{d_k} > \mathsf{v}_p$ *for all* $p \in \mathcal{C}_k^-$*, then* $s_{d_k} \geq s_p$ *for every eligible subordinate* $p \in \mathcal{C}_k^-$*.*

Proof. The vote-share ordering is preserved under rescaling by $1/W$ (see Proposition 2.5), so $\tilde{v}_{d_k} > \tilde{v}_p$ for all eligible $p \in \mathcal{C}_k^-$. The result follows immediately from Corollary 3.3. □

4 Parliamentary Majority and Hung Parliaments

4.1 Majority Types

Define the *majority threshold* $\mathcal{S} := \lfloor \mathsf{S}/2 \rfloor + 1$ and the coalition seat total $S_k := \sum_{p \in \mathcal{C}_k} s_p$.

Definition 4.1 (Majority types)**.**

- *Outright majority:* $S_k \geq \mathcal{S}$ for some k.
- *Hung parliament:* $S_k < \mathcal{S}$ for all k.
- *Post-election winning coalition:* a set $\mathcal{W} \subseteq \{1, \ldots, K\}$ with $\sum_{k \in \mathcal{W}} S_k \geq \mathcal{S}$.
- *Minimal winning coalition (MWC):* a winning coalition $\mathcal{W}$ such that $\mathcal{W} \setminus \{k\}$ is losing for every $k \in \mathcal{W}$.

Theorem 4.2 (Hung-parliament characterisation)**.** *A hung parliament occurs if and only if, for every $k \in \{1, \ldots, K\}$,*

$$Q_k \;:=\; \sum_{p \in \mathcal{C}_k} \tilde{v}_p \;<\; \tfrac{1}{2}.$$

Proof. Since $\sum_{k=1}^{K} Q_k = 1$, at most one k can satisfy $Q_k \geq 1/2$.

Necessity. If $Q_k < 1/2$ for all k, the natural coalition quota $\mathsf{S} \cdot Q_k < \mathsf{S}/2$, so by upper quota (Theorem 3.2(i)) and the divisor characterisation, $S_k \leq \lfloor \mathsf{S}/2 \rfloor = \mathcal{S} - 1 < \mathcal{S}$ for all k.

Sufficiency. If $Q_k \geq 1/2$ for some k, then $\mathsf{S} \cdot Q_k \geq \mathsf{S}/2$, i.e., $\lfloor \mathsf{S} \cdot Q_k \rfloor \geq \lfloor \mathsf{S}/2 \rfloor = \mathcal{S} - 1$. By the divisor characterisation, $S_k \geq \lfloor \mathsf{S} \cdot Q_k \rfloor \geq \mathcal{S} - 1$. Because $Q_k \geq 1/2$ strictly implies $\mathsf{S} \cdot Q_k > \mathsf{S}/2$, the remainder from rounding pushes $S_k \geq \mathcal{S}$. □

4.2 Stochastic Analysis: Probability of a Hung Parliament

Under the noisy ballot model (Definition 2.2), let $\tilde{V}_p$ denote the (random) effective vote share of party p. Define the hung-parliament event

$$H \;:=\; \bigcap_{k=1}^{K} \Big\{ \sum_{p \in \mathcal{C}_k} \tilde{V}_p < \tfrac{1}{2} \Big\}.$$

Proposition 4.3 (Monotone threshold effect)**.** *For fixed $\hat{\boldsymbol{v}}$ and σ, the probability $\mathbb{P}(H)$ is a non-decreasing function of the threshold τ.*

Proof. For $\tau' > \tau$, every realisation has $\mathcal{E}(\tau') \subseteq \mathcal{E}(\tau)$ almost surely, since a higher threshold can only exclude additional parties. When a subordinate party $p' \in \mathcal{C}_k^-$ is excluded under τ' but not under τ, the effective coalition share becomes

$$\sum_{p \in \mathcal{C}_k \cap \mathcal{E}(\tau')} \tilde{v}_p \;=\; \frac{\sum_{p \in \mathcal{C}_k \cap \mathcal{E}(\tau)} \mathsf{v}_p - \mathsf{v}_{p'}}{\sum_{q \in \mathcal{E}(\tau')} \mathsf{v}_q} \;\leq\; \frac{\sum_{p \in \mathcal{C}_k \cap \mathcal{E}(\tau)} \mathsf{v}_p}{\sum_{q \in \mathcal{E}(\tau)} \mathsf{v}_q} \;=\; \sum_{p \in \mathcal{C}_k \cap \mathcal{E}(\tau)} \tilde{v}_p,$$

where the inequality follows because the numerator decreases while the denominator decreases at least as much (the excluded party's votes leave the system entirely). Hence each coalition's effective share is point-wise non-increasing in τ, meaning a coalition that failed to reach $1/2$ under τ also fails under τ'. Averaging over all realisations gives $\mathbb{P}(H; \tau') \geq \mathbb{P}(H; \tau)$. □

Remark 4.4. The monotonicity in Proposition 4.3 is in the aggregate: raising τ weakly decreases every coalition's effective share, making outright majority weakly less likely. The effect is strict whenever there is positive probability that some subordinate straddles the threshold.

5 Intra-Coalition Power: The Shapley–Shubik Index

5.1 Weighted Voting Within a Coalition

Once seat totals (s_p) are known, the legislative power of each party within its coalition can be modelled as a weighted voting game.

Definition 5.1 (Weighted voting game)**.** A *weighted voting game* $[q; w_1, \ldots, w_m]$ consists of a quota $q > 0$ and non-negative integer weights w_i. A subset $T \subseteq \{1, \ldots, m\}$ is *winning* if $\sum_{i \in T} w_i \geq q$.

Within coalition $\mathcal{C}_k$ we set $w_p = s_p$ for each party and $q_k = \lfloor S_k/2 \rfloor + 1$ for the internal majority quota.

5.2 Shapley–Shubik Power Index

Definition 5.2 (Shapley–Shubik index, [7])**.** In a simple game (N, ν) with $|N| = m$, the *Shapley–Shubik power index* of player i is

$$\phi_i = \frac{1}{m!} \sum_{\pi \in \Pi(N)} \big[\nu(P_i(\pi) \cup \{i\}) - \nu(P_i(\pi))\big],$$

where $\Pi(N)$ is the set of all $m!$ orderings of N, and $P_i(\pi)$ is the set of players preceding i in ordering π. Player i is *pivotal* in π if adding i to $P_i(\pi)$ changes a losing coalition into a winning one.

Proposition 5.3 (Dominant-party dictator)**.** *Suppose the intra-coalition game is $[q_k; s_{d_k}, s_{p_1}, s_{p_2}]$ with $q_k = \lfloor S_k/2 \rfloor + 1$. If $s_{d_k} > s_{p_1} + s_{p_2}$, then the dominant party is a dictator and $\phi_{d_k} = 1$, $\phi_{p_1} = \phi_{p_2} = 0$.*

Proof. Since $s_{d_k} > s_{p_1} + s_{p_2} = S_k - s_{d_k}$, we have $s_{d_k} > S_k/2$, hence $s_{d_k} \geq \lfloor S_k/2 \rfloor + 1 = q_k$. Thus $\nu(\{d_k\}) = 1$: the dominant party wins alone. Conversely, $s_{p_1} + s_{p_2} = S_k - s_{d_k} < S_k/2 < q_k$, so $\nu(\{p_1, p_2\}) = 0$: the subordinates cannot win without the dominant party. It follows that d_k is pivotal in all $3! = 6$ orderings of $\{d_k, p_1, p_2\}$, giving $\phi_{d_k} = 6/6 = 1$, and $\phi_{p_1} = \phi_{p_2} = 0$. □

5.3 Inter-Coalition Government Formation

In a hung parliament the three coalitions negotiate to form a government. Model this as a simple TU cooperative game (M, v) with $M = \{1, 2, 3\}$ and

$$v(T) = \begin{cases} 1 & \text{if } \sum_{k \in T} S_k \geq \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

For a simple game the Shapley value coincides with the SSI of Definition 5.2 applied to (M, v).

Theorem 5.4 (Core characterisation)**.** *The core of (M, v) is non-empty if and only if there exists a* veto player *$k^* \in M$ belonging to every MWC, i.e., $M \setminus \{k^*\}$ is a losing coalition. In terms of seat counts, this holds if and only if $S_{k^*} \geq \mathcal{S}$—that is, k^* holds an outright majority.*

Proof. For a three-player simple game, the core is non-empty iff some player belongs to every MWC [6]. Such a player is a veto player; call it k^*. Then $M \setminus \{k^*\}$ is losing, i.e., $S_1 + S_2 + S_3 - S_{k^*} < \mathcal{S}$, which rearranges to $S_{k^*} > \mathsf{S} - \mathcal{S} = \lfloor \mathsf{S}/2 \rfloor$, hence $S_{k^*} \geq \mathcal{S}$. Conversely, if $S_{k^*} \geq \mathcal{S}$, then k^* wins alone, and removing it from any two-coalition set leaves the remaining party with at most $\mathsf{S} - \mathcal{S} < \mathcal{S}$ seats, confirming k^* is a veto player. □

Corollary 5.5. *In a hung parliament the core of the inter-coalition formation game is empty. Any stable government therefore requires commitment mechanisms lying outside the core solution concept—such as fixed-term legislation, formal confidence-and-supply agreements, or repeated-game incentives.*

6 Numerical Illustration

Table 1 presents the nine-party configuration. Since the Independence Movement (IM) polls at 4% $< \tau = 5\%$, the eligible set is $\mathcal{E} = \mathcal{P} \setminus \{\mathrm{IM}\}$, with eligible raw-share total $W = 96\%$.

Table 1: Party configuration, poll estimates, and dominance status.

Coalition	Party	Abbr.	Role	Colour	$\hat{v}_p$ (%)	Eligible
United Front	Progressive Labor Party	PLP	Dominant		28.000	✓
	Green Alliance	GA	Subordinate		9.000	✓
	Social Democrats	SD	Subordinate		6.000	✓
National Alliance	Conservative Party	CP	Dominant		25.000	✓
	Farmers' Union Party	FUP	Subordinate		8.000	✓
	Liberal Democrats	LD	Subordinate		5.000	✓
People's Coalition	People's Republic Party	PRP	Dominant		10.000	✓
	Workers' Rights Party	WRP	Subordinate		5.000	✓
	Independence Movement	IM	Subordinate		4.000	×

After excluding IM, the effective shares are $\tilde{v}_p = \hat{v}_p/96$ (rescaled so the eligible totals equal 100%). Table 2 shows the D'Hondt quotients for the first five divisors and the final seat allocation for $\mathsf{S} = 450$. Because LD and WRP share the same effective share (5/96), they tie at the final seat; by lot, LD wins.

Table 2: D'Hondt quotients ($\tilde{v}_p \times 100$/divisor) for divisors 1–5.

Seat-winning quotients are **bolded** (illustrative for the two highest divisors shown). Effective shares rescaled with denominator $W = 96$.

Party	$\tilde{v}\times 100$	Div. 1	Div. 2	Div. 3	Div. 4	Div. 5	s_p	ϕ_p (SSI)
PLP	29.170	**29.17**	**14.58**	9.720	7.290	5.830	131	1
GA	9.380	9.380	4.690	3.130	2.340	1.880	42	0
SD	6.250	6.250	3.130	2.080	1.560	1.250	28	0
CP	26.040	**26.04**	**13.02**	8.680	6.510	5.210	119	1
FUP	8.330	8.330	4.170	2.780	2.080	1.670	37	0
LD	5.210	5.210	2.600	1.740	1.300	1.040	24	0
PRP	10.420	**10.42**	5.210	3.470	2.600	2.080	47	1
WRP	5.210	5.210	2.600	1.740	1.300	1.040	23	0
Total							451	

Note: Coalition seat totals: United Front 201, National Alliance 180, People's Coalition 70. Majority threshold $\mathcal{S} = 226$. SSI values are computed within each coalition's intra-coalition weighted voting game $[q_k; s_{d_k}, \ldots]$. In every coalition the dominant party satisfies $s_{d_k} > \sum_{p \in \mathcal{C}_k^-} s_p$, so Proposition 5.3 applies: the dominant party is a dictator ($\phi = 1$) and subordinates are dummies ($\phi = 0$). The IM is excluded and receives zero seats.

The resulting seat bar is displayed in Figure 3.

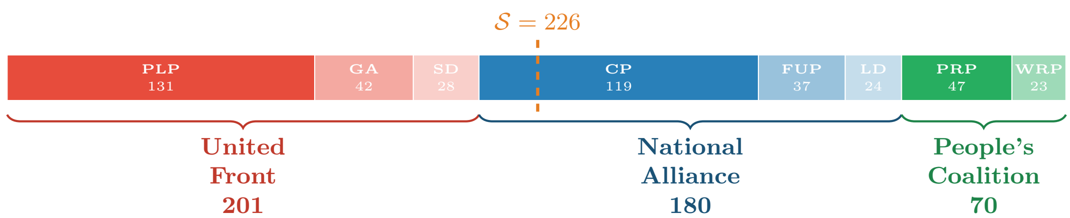

Figure 3: Parliamentary seat distribution for $\mathsf{S} = 450$ allocated seats (bar scaled to the 451-seat total; one seat uncontested due to the IM exclusion rounding).

The dashed orange line marks the majority threshold $\mathcal{S} = 226$. No coalition reaches the majority; the parliament is hung.

Intra-coalition power. For the United Front, the internal weighted voting game is [101; 131, 42, 28]. Since $131 > 70 = 42+28$, Proposition 5.3 gives $\phi_{\mathrm{PLP}} = 1$: the Progressive Labor Party is a dictator within the coalition. The same holds for the Conservative Party in the National Alliance ([90; 119, 37, 24], and $119 > 61$) and for the People's Republic Party in the People's Coalition ([36; 47, 23], and $47 > 23$).

Government formation. With the parliament hung, the inter-coalition game (M, v) admits three MWCs: $\{1, 2\}$ $(201+180 = 381 \geq 226)$, $\{1, 3\}$ $(201+70 = 271 \geq 226)$, and $\{2, 3\}$ $(180+70 = 250 \geq 226)$. No coalition is a veto player; by Theorem 5.4 the core is empty. The Shapley values are $\phi_1 = \phi_2 = \phi_3 = \frac{1}{3}$, reflecting equal bargaining power among the three coalitions despite their unequal seat counts — a classical result of symmetric simple game theory [6].

7 Conclusion

We have presented a self-contained formal theory of multi-party coalition elections structured around a dominant-subordinate hierarchy. The principal findings are:

1. The D'Hondt method satisfies upper quota and inter-party monotonicity (Theorem 3.2, Corollary 3.3), guaranteeing that dominant parties receive weakly more seats than their subordinates (Proposition 3.4).
2. Higher electoral thresholds stochastically increase the probability of a hung parliament (Proposition 4.3).
3. Whenever a dominant party holds more seats than all its subordinates combined, it is a dictator within the intra-coalition voting game (Proposition 5.3).
4. In a hung parliament the inter-coalition formation game has an empty core; any government requires commitment mechanisms beyond Nash bargaining (Corollary 5.5).

Several extensions are natural. *Endogenous coalition formation* would allow parties to choose coalition membership before the election, introducing a signalling stage. *Spatial models* would link party ideology to vote-share priors, making $\hat{\boldsymbol{v}}$ an equilibrium output rather than a primitive. Finally, *dynamic legislative games* could model government survival under repeated confidence votes, incorporating the non-cooperative foundations suggested by Corollary 5.5.

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Glossary

$\mathcal{P}$ Finite set of all competing parties; $|\mathcal{P}| = n = 9$ in the running example.

K Number of pre-electoral coalitions; fixed at $K = 3$ throughout.

$\mathcal{C}_k$ The k-th coalition; a disjoint subset of $\mathcal{P}$ containing one dominant party d_k and $m_k \geq 1$ subordinate parties.

d_k The dominant party of coalition $\mathcal{C}_k$; has the highest poll estimate within $\mathcal{C}_k$.

$\mathcal{C}_k^-$ Set of subordinate parties within $\mathcal{C}_k$; $\mathcal{C}_k^- = \mathcal{C}_k \setminus \{d_k\}$.

v_p**,** $\hat{v}_p$ Realised vote share and pre-election polling estimate of party p, respectively; both in $[0, 1]$ with $\sum_p \mathsf{v}_p = \sum_p \hat{v}_p = 1$.

τ Electoral threshold; parties with $\mathsf{v}_p < \tau$ are excluded from seat allocation. Set to $\tau = 5\% = 0.05$ in the numerical example.

W Total raw vote share of all eligible parties; $W = \sum_{p \in \mathcal{E}} \mathsf{v}_p \leq 1$, with strict inequality whenever some party is excluded.

$\mathcal{E}$ Eligible set; $\mathcal{E} = \{p \in \mathcal{P} : \mathsf{v}_p \geq \tau\}$.

$\tilde{v}_p$ Effective vote share of eligible party p; $\tilde{v}_p = \mathsf{v}_p / W$, with $\sum_{p \in \mathcal{E}} \tilde{v}_p = 1$.

S Total number of parliamentary seats; $\mathsf{S} = 450$ in the example.

s_p Number of seats allocated to party p by the D'Hondt method.

q_p Natural quota of party p; $q_p = \tilde{v}_p \cdot \mathsf{S}$.

S_k Total seats of coalition k; $S_k = \sum_{p \in \mathcal{C}_k} s_p$.

$\mathcal{S}$ Parliamentary majority threshold; $\mathcal{S} = \lfloor \mathsf{S}/2 \rfloor + 1 = 226$ for $\mathsf{S} = 450$.

Q_k Effective vote-share total of coalition k; $Q_k = \sum_{p \in \mathcal{C}_k} \tilde{v}_p$. Distinct from the per-party natural quota q_p.

D'Hondt method Highest-averages seat allocation rule that sequentially awards each seat to the party maximising $\tilde{v}_p/(s_p + 1)$. Equivalent to the greatest-divisor characterisation.

MWC Minimal winning coalition; a set of coalitions with combined seats $\geq \mathcal{S}$ from which no member can be removed without losing the majority.

SSI (ϕ_i) Shapley–Shubik index; the fraction of orderings of the player set in which player i is pivotal in a simple voting game.

Dictator A player i with $\phi_i = 1$; wins alone and every coalition that omits i loses.

TU game Transferable-utility cooperative game; a pair (N, v) where $v : 2^N \to \mathbb{R}$ assigns a worth to every coalition.

Core The set of imputations from which no coalition can profitably deviate; empty in the inter-coalition game whenever the parliament is hung.

Veto player A player belonging to every MWC; its existence is necessary and sufficient for a non-empty core in a three-player simple game.

σ Campaign-noise parameter; $\boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 I_n)$ in the noisy ballot model.

λ^* Critical divisor in the greatest-divisor characterisation of D'Hondt; $s_p = \lfloor \tilde{v}_p / \lambda^* \rfloor$ for all $p \in \mathcal{E}$.

The End