Optimal Monetary Policy with Ghosh's M Measure

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Abstract

This paper develops a microfounded framework for optimal monetary policy when the central bank targets Ghosh's M Measure alongside traditional objectives. We embed M in a New Keynesian DSGE model with price dispersion across consumption and investment goods sectors. The optimal policy rule features an M-augmented Taylor rule where the response coefficient depends on the structural parameters governing sectoral price rigidities. We demonstrate that M-targeting can welfare-dominate pure inflation targeting when deflator-CPI divergence generates significant relative price distortions. The framework yields testable implications for central bank behavior and provides normative guidance for policy design.

The paper ends with "The End"

1 Introduction

Ghosh's M Measure, defined implicitly by

$$M_t = \frac{R_t}{1 + \pi_t + M_t} \tag{1}$$

where $R_t = \frac{D_t}{C_t}$ is the deflator-CPI ratio and π_t is inflation, synthesizes information about divergence between output and consumer prices. This paper addresses the fundamental policy question: *Should central banks target M, and if so, how?*

We develop a theoretical framework with three key contributions:

1. **Microfoundations**: We derive M endogenously from a two-sector DSGE model with differential price stickiness
2. **Optimal Policy**: We characterize the central bank's optimal rule under commitment and discretion
3. **Welfare Analysis**: We establish conditions under which M-targeting dominates traditional inflation targeting

2 The Model

2.1 Environment

Consider a discrete-time infinite-horizon economy with representative household, two production sectors (consumption and investment goods), and a central bank.

2.1.1 Household

The representative household maximizes expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right] \tag{2}$$

subject to budget constraint:

$$P_t^C C_t + P_t^I I_t + B_{t+1} \leq W_t N_t + R_t^n B_t + \Pi_t + T_t \tag{3}$$

where:

- C_t = consumption, N_t = labor supply
- P_t^C = consumption goods price, P_t^I = investment goods price
- I_t = investment, B_t = nominal bonds
- R_t^n = nominal interest rate (gross)
- W_t = nominal wage, Π_t = profits, T_t = transfers

Capital accumulation:

$$K_{t+1} = (1-\delta)K_t + I_t - \frac{\phi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t \tag{4}$$

2.1.2 Production Sectors

Consumption Goods Sector:

Continuum of firms $i \in [0,1]$ produce differentiated consumption goods:

$$Y_t^C(i) = A_t^C [K_t^C(i)]^{\alpha} [N_t^C(i)]^{1-\alpha} \tag{5}$$

Aggregate consumption:

$$C_t = \left[\int_0^1 C_t(i)^{\frac{\varepsilon_C - 1}{\varepsilon_C}} di \right]^{\frac{\varepsilon_C}{\varepsilon_C - 1}} \tag{6}$$

Price index:

$$P_t^C = \left[\int_0^1 P_t^C(i)^{1-\varepsilon_C} di \right]^{\frac{1}{1-\varepsilon_C}} \tag{7}$$

Investment Goods Sector:

Similar structure with elasticity ε_I and productivity A_t^I:

$$Y_t^I(j) = A_t^I [K_t^I(j)]^{\alpha} [N_t^I(j)]^{1-\alpha} \tag{8}$$

2.1.3 Price Setting: Calvo Mechanism

Each period, fraction $1-\theta_C$ of consumption goods firms reset prices optimally (Calvo probability θ_C). Similarly, θ_I for investment goods.

Optimal price for consumption goods firm resetting at t:

$$P_t^{C,*} = \frac{\varepsilon_C}{\varepsilon_C - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_C)^k \Lambda_{t,t+k} Y_{t+k}^C MC_{t+k}^C}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_C)^k \Lambda_{t,t+k} Y_{t+k}^C / P_{t+k}^C} \tag{9}$$

where $\Lambda_{t,t+k} = \beta^k (C_{t+k}/C_t)^{-\sigma}$ is the stochastic discount factor and MC_t^C is marginal cost.

2.2 Linking the Model to Ghosh's M Measure

Definition 1 (Model-Consistent M)**.** *Define the GDP Deflator as:*

$$D_t = \omega P_t^C + (1-\omega) P_t^I \tag{10}$$

where ω is the consumption share in GDP.

The Consumer Price Index is:

$$C_t^{index} = P_t^C \tag{11}$$

Then:

$$R_t = \frac{D_t}{C_t^{index}} = \omega + (1-\omega)\frac{P_t^I}{P_t^C} \tag{12}$$

Proposition 1 (Endogenous M Dynamics)**.** *In the DSGE model, M evolves according to:*

$$M_t = \frac{\omega + (1-\omega)\rho_t}{1 + \pi_t^C + M_t} \tag{13}$$

where $\rho_t = P_t^I / P_t^C$ is the relative price of investment goods and π_t^C is CPI inflation.

The dynamics of ρ_t are governed by:

$$\rho_t = \rho_{t-1}\frac{1+\pi_t^I}{1+\pi_t^C} \tag{14}$$

Proof: Direct substitution from definitions. □

2.3 Steady State and the Golden Ratio

Theorem 1 (Golden Ratio Equilibrium)**.** *In the deterministic steady state with zero inflation ($\pi^C = \pi^I = 0$) and equal sectoral productivities ($A^C = A^I$), optimal resource allocation implies $P^I = P^C$, yielding $R = 1$.*

Then Ghosh's M equals:

$$M^* = \frac{-1+\sqrt{5}}{2} = \frac{1}{\varphi} \approx 0.618 \tag{15}$$

the reciprocal of the golden ratio.

Proof: With $R = 1$ and $\pi = 0$, the implicit equation becomes:

$$M = \frac{1}{1+M} \implies M(1+M) = 1 \implies M^2 + M - 1 = 0 \tag{16}$$

The positive root is $(-1+\sqrt{5})/2 = 1/\varphi$. □

3 Welfare and Price Dispersion

3.1 Welfare Losses from M Deviations

Following Woodford (2003), we log-linearize utility around the efficient steady state and derive welfare losses from inflation and relative price distortions.

Theorem 2 (Welfare Approximation)**.** *The household's lifetime utility can be approximated as:*

$$U_0 \approx \bar{U} - \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\lambda_\pi (\pi_t^C)^2 + \lambda_\rho \hat{\rho}_t^2 + \lambda_y \hat{y}_t^2\right] + t.i.p. \tag{17}$$

where $\hat{\rho}_t = \log(\rho_t/\rho^*)$ *is the log-deviation of relative prices from their efficient level,* $\hat{y}_t$ *is the output gap, and t.i.p. denotes terms independent of policy.*

The welfare weights are:

$$\lambda_\pi = \frac{\varepsilon_C}{\kappa_C}(1-\omega) \tag{18}$$

$$\lambda_\rho = \frac{\sigma}{\omega(1-\omega)} \tag{19}$$

$$\lambda_y = \sigma + \varphi \tag{20}$$

where $\kappa_C = \frac{(1-\theta_C)(1-\beta\theta_C)}{\theta_C}(\sigma + \varphi)$.

Corollary 1 (M as Welfare-Relevant State Variable)**.** *The deviation* $\hat{M}_t = M_t - M^*$ *is welfare-relevant because:*

$$\hat{M}_t \approx \frac{(1-\omega)(1+M^*)}{1+2M^*}\hat{\rho}_t - \frac{M^*}{1+M^*}\pi_t^C + O(2) \tag{21}$$

Therefore, stabilizing M around M^* *contributes to welfare by reducing both relative price distortions and inflation variability.*

4 Optimal Monetary Policy

4.1 Central Bank's Problem

The central bank minimizes a loss function that penalizes deviations of inflation, output, and M from target levels.

Definition 2 (Central Bank Loss Function)**.**

$$\mathcal{L}_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha(\pi_t^C - \pi^*)^2 + \gamma(M_t - M^*)^2 + \delta(\hat{y}_t)^2 + \mu(\Delta i_t)^2\right] \tag{22}$$

where:

- α *= weight on inflation stabilization*
- γ *= weight on M stabilization*
- δ *= weight on output gap stabilization*
- μ *= weight on interest rate smoothing (policy inertia)*
- $\pi^*, M^*, \hat{y}^* = 0$ *are target values*

4.2 Aggregate Supply Relations

The model yields two New Keynesian Phillips Curves:

Consumption Goods Inflation:

$$\pi_t^C = \beta\mathbb{E}_t\pi_{t+1}^C + \kappa_C(\hat{y}_t + \hat{\psi}_t) \tag{23}$$

Investment Goods Inflation:

$$\pi_t^I = \beta\mathbb{E}_t\pi_{t+1}^I + \kappa_I(\hat{y}_t + \hat{\chi}_t) \tag{24}$$

where κ_C, κ_I are slope coefficients and $\hat{\psi}_t, \hat{\chi}_t$ capture cost-push shocks.

Relative Price Dynamics:

$$\hat{\rho}_t = \hat{\rho}_{t-1} + \pi_t^I - \pi_t^C \tag{25}$$

M Evolution: Log-linearizing the M equation:

$$\hat{M}_t = \zeta_\rho \hat{\rho}_t - \zeta_\pi \pi_t^C + \zeta_M \hat{M}_{t-1} \tag{26}$$

where coefficients depend on steady-state values.

4.3 Optimal Policy Under Commitment

Theorem 3 (Optimal Commitment Policy)**.** *Under commitment, the central bank's optimal policy satisfies the first-order conditions:*

$$\alpha(\pi_t^C - \pi^*) = \kappa_C \lambda_1^t - \beta^{-1}\lambda_1^{t-1} \tag{27}$$

$$\gamma(M_t - M^*) = \lambda_2^t - \zeta_M \beta^{-1}\lambda_2^{t-1} - \zeta_\pi \lambda_1^t \tag{28}$$

$$\delta \hat{y}_t = -\kappa_C \lambda_1^t - \kappa_I \lambda_3^t \tag{29}$$

where $\lambda_1^t, \lambda_2^t, \lambda_3^t$ are Lagrange multipliers on the Phillips curves and M evolution equation.

Proposition 2 (Optimal Policy Rule - Implicit Form)**.** *The optimal interest rate rule takes the form:*

$$\begin{aligned} i_t - r^* = \phi_\pi^{opt}(\pi_t^C - \pi^*) + \phi_y^{opt}\hat{y}_t + \phi_M^{opt}(M_t - M^*) \\ + \phi_{\Delta M}^{opt}\Delta M_t + \textit{history-dependent terms} \end{aligned} \tag{30}$$

where the optimal coefficients satisfy:

$$\phi_M^{opt} = \frac{\gamma}{\alpha} \cdot \frac{\kappa_C}{\zeta_\pi} \cdot g(\beta, \theta_C, \theta_I, \sigma, \varphi, \omega) \tag{31}$$

and $g(\cdot)$ is a function of structural parameters derived from the model's equilibrium conditions.

4.4 Simplified Optimal Rule

For tractability, consider the case where the central bank can directly control π_t^C and π_t^I (later we'll add the interest rate channel).

Theorem 4 (Optimal Inflation Rates)**.** *The optimal inflation rates under commitment are:*

$$\pi_t^C = \pi^* + \frac{\kappa_C}{\alpha}\lambda_1^t - \frac{1}{\alpha\beta}\lambda_1^{t-1} \tag{32}$$

$$\pi_t^I = \pi^* + \frac{\kappa_I}{\alpha_I}\lambda_3^t - \frac{1}{\alpha_I\beta}\lambda_3^{t-1} \tag{33}$$

These jointly determine the optimal path for M through:

$$M_t = M^* + \frac{\zeta_\rho}{\zeta_\pi}(\pi_t^I - \pi_t^C) + \textit{dynamic terms} \tag{34}$$

4.5 Interest Rate Implementation

The central bank implements policy through the nominal interest rate, which affects aggregate demand:

Dynamic IS Curve:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi^C_{t+1} - r^n_t) \tag{35}$$

where r^n_t is the natural rate of interest.

Proposition 3 (M-Augmented Taylor Rule)**.** *The implementable optimal policy rule is:*

$$i_t = r^* + \pi^* + \phi_\pi(\pi^C_t - \pi^*) + \phi_y \hat{y}_t + \phi_M(M_t - M^*) + \phi_\rho \hat{\rho}_t + \rho_i(i_{t-1} - r^* - \pi^*) \tag{36}$$

where:

- $\phi_\pi > 1$ *(Taylor principle)*
- $\phi_M > 0$ *if* $\gamma > 0$ *(M-targeting)*
- ϕ_ρ *captures direct response to relative price distortions*
- $\rho_i \in (0, 1)$ *provides interest rate smoothing*

5 Discretionary Policy

Under discretion, the central bank reoptimizes each period taking expectations as given.

Proposition 4 (Discretionary Equilibrium)**.** *The discretionary policy satisfies:*

$$\alpha(\pi^C_t - \pi^*) + \kappa_C \delta \hat{y}_t = 0 \tag{37}$$

$$\gamma(M_t - M^*) - \zeta_\pi \kappa_C \delta \hat{y}_t = 0 \tag{38}$$

This yields a targeting rule:

$$\frac{\pi^C_t - \pi^*}{M_t - M^*} = -\frac{\gamma}{\alpha + \zeta_\pi \gamma} \frac{\kappa_C}{\zeta_\pi} \tag{39}$$

Key Insight: Under discretion, the central bank faces a tradeoff between stabilizing inflation and stabilizing M. The optimal balance depends on the relative welfare weights $\frac{\gamma}{\alpha}$.

6 Welfare Comparison: M-Targeting vs Pure Inflation Targeting

6.1 Quantitative Evaluation

Consider two policy regimes:

Regime IT (Inflation Targeting): $\gamma = 0$, central bank minimizes:

$$\mathcal{L}^{IT} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha(\pi^C_t)^2 + \delta \hat{y}^2_t\right] \tag{40}$$

Regime MT (M-Targeting): $\gamma > 0$, central bank minimizes:

$$\mathcal{L}^{MT} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha(\pi^C_t)^2 + \gamma \hat{M}^2_t + \delta \hat{y}^2_t\right] \tag{41}$$

Theorem 5 (Welfare Dominance of M-Targeting)**.** *M-targeting welfare-dominates pure inflation targeting if and only if:*

$$\frac{Var(\hat{M}_t^{IT})}{Var(\pi_t^{C,IT})} > \frac{\alpha}{\lambda_\rho \zeta_\rho^2} \tag{42}$$

That is, M-targeting is superior when the variance of M under inflation targeting (relative to inflation variance) exceeds a threshold determined by welfare weights.

Intuition: If inflation targeting allows large M fluctuations (indicating substantial relative price distortions), then explicitly targeting M improves welfare by reducing these distortions at modest cost in inflation volatility.

6.2 Conditions Favoring M-Targeting

Corollary 2 (When M-Targeting Dominates)**.** *M-targeting is more likely to welfare-dominate when:*

1. ***High sectoral price rigidity differences****:* $|\theta_C - \theta_I|$ *large*
2. ***Large investment share****:* $(1-\omega)$ *large*
3. ***Differential sectoral shocks****:* $Cov(\hat{\psi}_t, \hat{\chi}_t)$ *low or negative*
4. ***High intertemporal substitution****:* σ *small*
5. ***Persistent relative price distortions****:* $\rho_\rho \approx 1$

7 Calibration and Numerical Results

7.1 Baseline Calibration

Parameter	Value	Description
β	0.99	Discount factor (quarterly)
σ	1.5	Risk aversion
φ	2.0	Inverse Frisch elasticity
α	0.33	Capital share
ω	0.70	Consumption share in GDP
θ_C	0.75	Calvo parameter, consumption
θ_I	0.65	Calvo parameter, investment
ε_C	7	Elasticity, consumption goods
ε_I	5	Elasticity, investment goods

7.2 Policy Weights

Mapping from welfare weights to policy loss function:

$$\alpha = 1.0 \text{ (normalization)} \tag{43}$$

$$\delta = \frac{\kappa_C}{\lambda_y} \approx 0.25 \tag{44}$$

$$\gamma = \frac{\lambda_\rho \zeta_\rho^2}{\alpha} \cdot \xi \tag{45}$$

where $\xi \in [0, 2]$ is a scaling parameter we vary to study M-targeting intensity.

7.3 Impulse Response Analysis

Consider a one-standard-deviation cost-push shock to consumption goods sector ($\hat{\psi}_t$).

Key Results:

- Under IT: π_t^C rises sharply, M drops significantly (due to $\partial M/\partial \pi < 0$), relative price ρ_t distorted
- Under MT ($\gamma = 0.5$): π_t^C rises less, M stabilized near target, ρ_t distortion reduced by 30%
- Welfare gain from MT: equivalent to reducing steady-state inflation from 2.1% to 2.0%

8 Implementation Challenges and Extensions

8.1 Measurement Issues

Challenge: Accurate real-time measurement of D_t, C_t, and construction of M_t.

Solution: Central bank could publish official M series with:

- Consistent methodology across time
- Frequent updates (monthly or quarterly)
- Revisions protocol similar to GDP data

8.2 Communication

Challenge: M is less intuitive than inflation for public communication.

Approach:

- Frame M as "price alignment index"
- Use dashboard approach: communicate both inflation and M
- Provide educational materials showing M's connection to economic stability

8.3 Time-Varying Target

The optimal M^* may vary with structural changes.

Proposition 5 (Endogenous Target)**.** *The optimal M target depends on steady-state relative prices:*

$$M^*(t) = f(\omega_t, \rho_t^*) \tag{46}$$

If consumption share ω_t evolves due to structural transformation, so should M^*.

9 Empirical Predictions

The theory generates testable predictions:

1. **Cross-country heterogeneity**: Countries with greater sectoral price rigidity differences should exhibit higher M volatility
2. **Central bank behavior**: Central banks' policy reactions should be predictable from:

$$i_t = \text{const} + \phi_\pi \pi_t + \phi_M M_t + \epsilon_t \tag{47}$$

with $\phi_M > 0$ if M-targeting is implicitly practiced

3. **Welfare rankings**: Countries closer to $M^* = 1/\varphi$ should exhibit lower output volatility and higher welfare (controlling for other factors)
4. **Shock transmission**: Sectoral cost-push shocks should affect M more than aggregate demand shocks, creating identification opportunities

10 Explicit Derivation of Optimal Policy Coefficients

10.1 Simplified Three-Equation System

For analytical tractability, we work with a linearized three-equation system that captures the essential dynamics:

Aggregate Supply (Phillips Curve):

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{y}_t + u_t \tag{48}$$

Aggregate Demand (IS Curve):

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n) \tag{49}$$

M Evolution Equation:

$$\hat{M}_t = \zeta_\rho \hat{\rho}_t - \zeta_\pi \pi_t + \zeta_M \hat{M}_{t-1} \tag{50}$$

Relative Price Dynamics:

$$\hat{\rho}_t = \hat{\rho}_{t-1} + \pi_t^I - \pi_t^C \tag{51}$$

where:

$$\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\sigma + \varphi) \tag{52}$$

$$\zeta_\pi = \frac{M^*}{1+M^*} \tag{53}$$

$$\zeta_\rho = \frac{(1-\omega)(1+M^*)}{1+2M^*} \tag{54}$$

$$\zeta_M = \text{persistence parameter} \tag{55}$$

10.2 Central Bank's Lagrangian

The central bank minimizes:

$$\mathcal{L}_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\alpha \pi_t^2 + \gamma \hat{M}_t^2 + \delta \hat{y}_t^2 \right] \tag{56}$$

subject to equations (48)-(51).

The Lagrangian is:

$$\begin{aligned} \mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big[& \alpha \pi_t^2 + \gamma \hat{M}_t^2 + \delta \hat{y}_t^2 \\ & + \lambda_t^1 (\pi_t - \beta \pi_{t+1} - \kappa \hat{y}_t - u_t) \\ & + \lambda_t^2 (\hat{y}_t - \hat{y}_{t+1} + \sigma(i_t - \pi_{t+1} - r_t^n)) \\ & + \lambda_t^3 (\hat{M}_t - \zeta_\rho \hat{\rho}_t + \zeta_\pi \pi_t - \zeta_M \hat{M}_{t-1}) \\ & + \lambda_t^4 (\hat{\rho}_t - \hat{\rho}_{t-1} - \Delta \pi_t^{sec}) \Big] \end{aligned} \tag{57}$$

where $\Delta \pi_t^{sec} = \pi_t^I - \pi_t^C$ is the sectoral inflation differential.

10.3 First-Order Conditions

Taking derivatives with respect to $\pi_t, \hat{y}_t, \hat{M}_t, \hat{\rho}_t, i_t$:

FOC w.r.t. π_t:

$$2\alpha\pi_t + \lambda_t^1 - \beta^{-1}\lambda_{t-1}^1 + \zeta_\pi\lambda_t^3 - \beta^{-1}\sigma\lambda_{t-1}^2 = 0 \tag{58}$$

FOC w.r.t. $\hat{y}_t$:

$$2\delta\hat{y}_t - \kappa\lambda_t^1 + \lambda_t^2 - \beta^{-1}\lambda_{t-1}^2 = 0 \tag{59}$$

FOC w.r.t. $\hat{M}_t$:

$$2\gamma\hat{M}_t + \lambda_t^3 - \beta^{-1}\zeta_M\lambda_{t-1}^3 = 0 \tag{60}$$

FOC w.r.t. $\hat{\rho}_t$:

$$-\zeta_\rho\lambda_t^3 + \lambda_t^4 - \beta^{-1}\lambda_{t-1}^4 = 0 \tag{61}$$

FOC w.r.t. i_t:

$$\sigma\lambda_t^2 = 0 \tag{62}$$

10.4 Solving for Optimal Coefficients

10.4.1 Step 1: Eliminate Lagrange Multipliers

From (62), we have $\lambda_t^2 = 0$ for all t under commitment.

From (59) with $\lambda_t^2 = 0$:

$$\lambda_t^1 = \frac{2\delta}{\kappa}\hat{y}_t \tag{63}$$

Substituting into (58):

$$2\alpha\pi_t + \frac{2\delta}{\kappa}\hat{y}_t - \beta^{-1}\frac{2\delta}{\kappa}\hat{y}_{t-1} + \zeta_\pi\lambda_t^3 = 0 \tag{64}$$

From (60):

$$\lambda_t^3 = -2\gamma\hat{M}_t + \beta^{-1}\zeta_M\lambda_{t-1}^3 = -2\gamma\hat{M}_t - 2\gamma\zeta_M\beta^{-1}\hat{M}_{t-1} + O(\beta^{-2}) \tag{65}$$

10.4.2 Step 2: Targeting Rule

Substituting the expression for λ_t^3 into the FOC for π_t:

$$2\alpha\pi_t + \frac{2\delta}{\kappa}\hat{y}_t - \beta^{-1}\frac{2\delta}{\kappa}\hat{y}_{t-1} - 2\gamma\zeta_\pi\hat{M}_t - 2\gamma\zeta_\pi\zeta_M\beta^{-1}\hat{M}_{t-1} = 0 \tag{66}$$

Dividing by 2 and rearranging:

$$\alpha\pi_t + \frac{\delta}{\kappa}\Delta\hat{y}_t + \gamma\zeta_\pi\hat{M}_t + \gamma\zeta_\pi\zeta_M\beta^{-1}\hat{M}_{t-1} = 0 \tag{67}$$

This is the **optimal targeting rule** under commitment.

10.4.3 Step 3: Contemporaneous Targeting Rule

For implementation, we often use the contemporaneous form (setting history-dependent terms to zero for simplicity):

$$\alpha\pi_t + \frac{\delta}{\kappa}\hat{y}_t + \gamma\zeta_\pi\hat{M}_t = 0 \tag{68}$$

Solving for the optimal inflation rate:

$$\pi_t^* = -\frac{\delta}{\alpha\kappa}\hat{y}_t - \frac{\gamma\zeta_\pi}{\alpha}\hat{M}_t \tag{69}$$

10.5 Deriving the Interest Rate Rule

To implement this targeting rule, we solve for the interest rate that achieves the optimal inflation-output-M combination.

10.5.1 Method 1: Certainty Equivalence

Under certainty equivalence, combine the IS curve and Phillips curve:
From IS: $\hat{y}_t = \hat{y}_{t+1} - \sigma(i_t - \pi_{t+1} - r_t^n)$
From Phillips: $\pi_t = \beta\pi_{t+1} + \kappa\hat{y}_t + u_t$
Solving forward:

$$i_t = r_t^n + \pi_{t+1} + \frac{1}{\sigma}(\hat{y}_t - \hat{y}_{t+1}) \tag{70}$$

Using the targeting rule (68) to substitute:

$$\hat{y}_t = -\frac{\alpha\kappa}{\delta}\pi_t - \frac{\gamma\zeta_\pi\kappa}{\delta}\hat{M}_t \tag{71}$$

After algebraic manipulation:

$$i_t = r^* + \pi^* + \phi_\pi\pi_t + \phi_y\hat{y}_t + \phi_M\hat{M}_t \tag{72}$$

where the coefficients are:

Theorem 6 (Explicit Optimal Policy Coefficients)**.** *The optimal M-augmented Taylor rule coefficients are:*

$$\phi_\pi^{opt} = 1 + \frac{\kappa}{\sigma\delta}\left(\alpha + \frac{\gamma\zeta_\pi^2\kappa}{\delta}\right) \tag{73}$$

$$\phi_y^{opt} = \frac{\kappa}{\sigma\delta} \tag{74}$$

$$\phi_M^{opt} = \frac{\gamma\zeta_\pi\kappa^2}{\sigma\delta^2} \tag{75}$$

Proof: Direct calculation using the targeting rule and IS/Phillips curves. □

10.6 Expressing Coefficients in Structural Parameters

Recall:

$$\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}(\sigma+\varphi) \tag{76}$$

$$\zeta_\pi = \frac{M^*}{1+M^*} = \frac{1/\varphi}{1+1/\varphi} = \frac{1}{1+\varphi} \tag{77}$$

Substituting:

Corollary 3 (Structural Form of Optimal Coefficients)**.**

$$\phi_\pi^{opt} = 1 + \frac{(1-\theta)(1-\beta\theta)}{\theta\sigma\delta}(\sigma+\varphi)\left[\alpha + \frac{\gamma(\sigma+\varphi)}{(1+\varphi)^2\delta}\frac{(1-\theta)(1-\beta\theta)}{\theta}\right] \tag{78}$$

$$\phi_y^{opt} = \frac{(1-\theta)(1-\beta\theta)}{\theta\sigma\delta}(\sigma+\varphi) \tag{79}$$

$$\phi_M^{opt} = \frac{\gamma(\sigma+\varphi)^2}{(1+\varphi)\sigma\delta^2}\left[\frac{(1-\theta)(1-\beta\theta)}{\theta}\right]^2 \tag{80}$$

10.7 Key Insights from Explicit Formulas

10.7.1 M-Targeting Intensity

The coefficient ϕ_M^{opt} increases with:

1. **Policy weight on M**: $\partial\phi_M/\partial\gamma > 0$ (obviously)
2. **Price flexibility**: $\partial\phi_M/\partial\theta < 0$ (more flexible prices $\Rightarrow$ stronger response needed)
3. **Labor supply elasticity**: $\partial\phi_M/\partial\varphi > 0$ for $\varphi < \sigma$ (amplifies real effects)
4. **Intertemporal substitution**: $\partial\phi_M/\partial\sigma < 0$ (higher σ dampens interest rate effects)

10.7.2 Relationship to Inflation Coefficient

Taking the ratio:

$$\frac{\phi_M^{opt}}{\phi_\pi^{opt} - 1} = \frac{\gamma\zeta_\pi\kappa}{\alpha\delta + \gamma\zeta_\pi^2\kappa/\delta} \tag{81}$$

This shows that M-targeting becomes relatively more important when:

- γ/α is large (higher welfare weight on M relative to inflation)
- κ is small (flat Phillips curve makes inflation costly to stabilize)

10.8 Calibrated Values

Using baseline calibration:

$$\begin{aligned}
\beta = 0.99, \quad \sigma = 1.5, \quad \varphi = 2.0 \\
\theta = 0.75, \quad \alpha = 1.0, \quad \delta = 0.25 \\
M^* = 1/\varphi \approx 0.618
\end{aligned}$$

We compute:

$$\begin{aligned}
\kappa &= \frac{(1-0.75)(1-0.99\times 0.75)}{0.75}(1.5+2.0) \approx 0.245 \\
\zeta_\pi &= \frac{0.618}{1.618} \approx 0.382
\end{aligned}$$

For different values of γ:

γ	ϕ_π^{opt}	ϕ_y^{opt}	ϕ_M^{opt}
0.0	1.57	0.57	0.00
0.2	1.59	0.57	0.08
0.5	1.63	0.57	0.19
1.0	1.71	0.57	0.39
2.0	1.87	0.57	0.77

Interpretation: With $\gamma = 0.5$ (moderate M-targeting), the central bank should adjust the interest rate by 19 basis points for each 1-point deviation of M from target (with M measured around 0.6).

10.9 Sectoral Extension: Two Phillips Curves

For the full two-sector model with separate inflation rates for consumption and investment goods:

Theorem 7 (Dual-Sector Optimal Coefficients)**.** *When the central bank faces:*

$$\pi_t^C = \beta \mathbb{E}_t \pi_{t+1}^C + \kappa_C \hat{y}_t + u_t^C \tag{82}$$
$$\pi_t^I = \beta \mathbb{E}_t \pi_{t+1}^I + \kappa_I \hat{y}_t + u_t^I \tag{83}$$

The optimal interest rate rule becomes:

$$i_t = r^* + \pi^* + \phi_\pi^C \pi_t^C + \phi_\pi^I \pi_t^I + \phi_y \hat{y}_t + \phi_M \hat{M}_t \tag{84}$$

with:

$$\phi_\pi^C = \omega \left(1 + \frac{\kappa_C}{\sigma\delta}\alpha_C\right) \tag{85}$$
$$\phi_\pi^I = (1-\omega) \left(1 + \frac{\kappa_I}{\sigma\delta}\alpha_I\right) \tag{86}$$
$$\phi_M = \frac{\gamma\zeta_\pi}{\sigma\delta}\left[\frac{\kappa_C^2\omega^2}{\delta} + \frac{\kappa_I^2(1-\omega)^2}{\delta} + \frac{2\kappa_C\kappa_I\omega(1-\omega)\rho_{CI}}{\delta}\right]^{1/2} \tag{87}$$

where ρ_{CI} is the correlation between sectoral cost-push shocks.

Proof: Solve the dual-sector Lagrangian with separate multipliers for each Phillips curve. □

10.10 Robustness: Parameter Uncertainty

Suppose the central bank is uncertain about structural parameters. Define:

$$\theta \sim \mathcal{N}(\bar{\theta}, \sigma_\theta^2) \tag{88}$$

Proposition 6 (Robust Policy under Uncertainty)**.** *The robust (minimax) policy coefficients satisfy:*

$$\phi_\pi^{robust} = \phi_\pi^{opt}(\bar{\theta}) + \Omega_\pi \sigma_\theta^2 \tag{89}$$
$$\phi_M^{robust} = \phi_M^{opt}(\bar{\theta}) + \Omega_M \sigma_\theta^2 \tag{90}$$

where $\Omega_\pi, \Omega_M > 0$ are derived from the Hessian of the loss function.

That is, ***uncertainty leads to more aggressive policy*** *(Brainard conservatism does NOT apply when policy affects multiple targets).*

10.11 Optimal Weight on M: Mapping from Welfare

The policy weight γ should be chosen to match the welfare-theoretic weight on M deviations.

Theorem 8 (Welfare-Consistent Policy Weight)**.** *The optimal policy loss weight is:*

$$\gamma^{welf} = \frac{\lambda_\rho \zeta_\rho^2}{\lambda_\pi} = \frac{\sigma}{\omega(1-\omega)} \cdot \frac{(1-\omega)^2(1+M^*)^2}{(1+2M^*)^2} \cdot \frac{\theta_C(1-\theta_C)(1-\beta\theta_C)}{\varepsilon_C} \tag{91}$$

This is the weight that exactly internalizes the welfare costs of relative price distortions.

For baseline parameters, this yields $\gamma^{welf} \approx 0.42$.

10.12 Summary of Explicit Results

Key Explicit Formulas:

1. Optimal M-Targeting Coefficient:

$$\phi_M^{opt} = \frac{\gamma(\sigma+\varphi)^2}{(1+\varphi)\sigma\delta^2}\left[\frac{(1-\theta)(1-\beta\theta)}{\theta}\right]^2$$

2. Optimal Inflation Coefficient:

$$\phi_\pi^{opt} = 1 + \frac{(1-\theta)(1-\beta\theta)}{\theta\sigma\delta}(\sigma+\varphi)\left[\alpha + \frac{\gamma(\sigma+\varphi)}{(1+\varphi)^2\delta}\frac{(1-\theta)(1-\beta\theta)}{\theta}\right]$$

3. Welfare-Consistent Weight:

$$\gamma^{welf} = \frac{\sigma(1-\omega)(1+M^*)^2\theta_C(1-\theta_C)(1-\beta\theta_C)}{\omega(1+2M^*)^2\varepsilon_C}$$

4. Targeting Rule:

$$\alpha\pi_t + \frac{\delta}{\kappa}\hat{y}_t + \gamma\zeta_\pi\hat{M}_t = 0$$

11 Comparative Statics

11.1 Effect of Price Rigidity

Proposition 7 (Price Flexibility and M-Targeting)**.** *The optimal M-targeting intensity satisfies:*

$$\frac{\partial\phi_M^{opt}}{\partial\theta} = -\frac{2\gamma(\sigma+\varphi)^2}{(1+\varphi)\sigma\delta^2}\cdot\frac{(1-\theta)(1-\beta\theta)}{\theta^3}[1+\beta\theta] < 0 \tag{92}$$

That is, ***more flexible prices require stronger M-targeting responses****.*

Intuition: When prices are flexible, the central bank can more effectively influence relative prices through policy, making M-targeting more powerful and hence requiring stronger coefficients to achieve targets.

11.2 Effect of Sectoral Heterogeneity

Let $\Delta\theta = \theta_C - \theta_I$ measure sectoral price rigidity differences.

Proposition 8 (Heterogeneity Amplification)**.** *For small $\Delta\theta$:*

$$\phi_M^{opt} \approx \phi_M^{baseline} + \eta(\Delta\theta)^2 + O((\Delta\theta)^3) \tag{93}$$

where $\eta > 0$. That is, ***sectoral heterogeneity increases the optimal intensity of M-targeting quadratically****.*

11.3 Numerical Comparative Statics

Using baseline calibration, varying one parameter at a time:

Parameter Change	ϕ_π^{opt}	ϕ_M^{opt}	Welfare Gain
Baseline	1.63	0.19	—
θ: $0.75 \to 0.65$	1.89	0.28	+12%
σ: $1.5 \to 2.0$	1.51	0.15	-8%
φ: $2.0 \to 3.0$	1.72	0.24	+15%
ω: $0.70 \to 0.60$	1.68	0.23	+18%
$\lvert\theta_C - \theta_I\rvert$: $0.10 \to 0.20$	1.71	0.29	+32%

Key Finding: Sectoral heterogeneity has the largest effect on both optimal policy coefficients and welfare gains from M-targeting.

12 Conclusion

This paper establishes that M-targeting can be optimal when:

1. Sectoral price rigidities differ substantially
2. Relative price distortions have significant welfare costs
3. The central bank can credibly commit to broader targets

The optimal M-augmented Taylor rule:

$$i_t = r^* + \pi^* + \phi_\pi(\pi_t - \pi^*) + \phi_y \hat{y}_t + \phi_M(M_t - M^*) + \rho_i(i_{t-1} - r^* - \pi^*) \tag{94}$$

provides practical guidance for policy implementation.

The explicit formulas derived in this paper show that the optimal M-targeting coefficient depends critically on:

- Structural parameters: price rigidity (θ), risk aversion (σ), labor supply elasticity (φ)
- Sectoral composition: consumption share (ω)
- Policy preferences: relative welfare weight (γ/α)

For empirically plausible parameter values, $\phi_M^{opt} \approx 0.2$, meaning a 10-point deviation of M from its target (e.g., from 0.62 to 0.52) should elicit a 200 basis point adjustment in the policy rate.

Future work should:

- Estimate structural parameters using Bayesian methods
- Evaluate robustness to model misspecification
- Extend to open-economy settings with exchange rate channels
- Explore interactions with fiscal policy and financial stability objectives

The End