The roots of the general septic equation

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Abstract

In this paper, I describe the roots of the general septic equation.
The paper ends with "The End"

Introduction

In a previous paper, I've described how to solve the general quintic equation. In a previous paper, I've described how to solve the general sextic equation. In a previous paper, I've described my monic septic identity. In a previous paper, I've described how the roots of my monic septic equation are expressible in radicals. In this paper, I describe the roots of the general septic equation.

Preliminaries

The general septic equation is

$$ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h = 0$$

where a, b, c, d, e, f, g, h are constants.

If $h = 0$ then the equation reduces to

$$x(ax^6 + bx^5 5 + cx^4 + dx^3 + ex^2 + fx + g) = 0$$

which has the root $x = 0$ and the sextic

$$ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$$

which can be solved.

Similarly, if $a = 0$ then the equation reduces to sextic equation

$$bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h = 0$$

which can be solved.

If $a = h = 0$ then the equation reduces to

$$x(bx^5 + cx^4 + dx^3 + ex^2 + fx + g) = 0$$

which has the root $x = 0$ and the quintic

$$bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$$

whose roots are known.

Therefore, $a \neq 0$ and $g \neq 0$ henceforth. Moreover, we divide the general septic equation by the leading coefficient a to transform the general septic equation to the monic septic equation

$$x^7 + ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0$$

where $g \neq 0$ henceforth.

Comparing coefficients with my monic septic

Recall that my monic septic is

Comparing coefficients we get
(1) $a = a$
(2) $b = b$
(3) $c = (a - P)\left(b - aP + P^2 - Q\right) + \frac{f-g}{Q} + Q + PQ$
(4) $d = \frac{g}{Q} + \frac{(f-g)(a-P)}{Q} + PQ + Q\left(b - aP + P^2 - Q\right)$
(5) $e = f + \frac{g(a-P-Q)}{Q} + \left(b - aP + P^2 - Q\right)Q$
(6) $f = f$
(7) $g = g$
Thus, if suitable P and Q are obtained, then, by my monic septic identity, we can reduce the monic septic equation to the product of a quartic equation and a quadratic equation, whose roots are known.

Choosing P and Q

Eliminating d between equations (4) and (5) gives us the eliminant

$$(8)\quad -ag + aPQ^2 - bQ^2 + eQ - fQ + gP + gQ - P^2Q^2 + Q^3 = 0$$

Eliminating c between equations (3) and (8) gives us the same eliminant

As long as $Q \neq 0$ and we obtain corresponding P and Q, we may choose any value for either P or Q to solve the eliminant. For most septics, $P = a$ is a valid and convenient choice. When $P = a$ is not a valid choice, other valid and convenient choices may be $P = 0$, $Q = P$ etc.

Solving the monic septic

Once we have at least one valid value of P and one valid value of Q, by the right side of my monic septic identity, we obtain 7 roots of the monic septic equation.

Notes

1. Note that by following this procedure, we obtain 7 roots of the general septic equation expressible in radicals.
2. Note that this procedure doesn't invalidate Galois' theory since our procedure is based on expressing the general septic equation as **factored forms** but not on solving the septic equation algebraically.

Exercises for the reader

Find the roots of the following septic equations expressible in radicals:

1. $$3x^6 + 56x^5 - 504x^4 + 39x^3 - 3317x^2 + 9340x - 534 = 0$$

2. $$x^7 - 16x^3 - 25 = 0$$

The End