Option Pricing via Stochastic Ghoshian Condensation: A Novel Framework for Derivative Valuation

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Abstract

This paper presents a novel approach to option pricing using the Stochastic Ghoshian Condensation framework. We extend the classical Black-Scholes-Merton model by incorporating Ghoshian dynamics into the underlying asset price process, leading to closed-form solutions for European call and put options. The model captures non-linear volatility effects and provides enhanced pricing accuracy for options with exponential-polynomial payoff structures. We establish existence and uniqueness theorems, derive risk-neutral pricing formulas, and provide comprehensive numerical analysis with vector graphics visualizations.

The paper ends with "The End"

1 Introduction

The classical Black-Scholes-Merton framework assumes geometric Brownian motion for the underlying asset price, leading to constant volatility and log-normal price distributions. However, empirical evidence suggests that asset prices exhibit more complex dynamics, including volatility clustering, fat tails, and non-linear mean reversion patterns.

This paper introduces the Stochastic Ghoshian Option Pricing Model (SGOPM), which incorporates Ghoshian condensation dynamics into the asset price process. The model naturally captures exponential-polynomial structures observed in real market data while maintaining analytical tractability.

1.1 Contributions

Our key contributions include:

1. Development of the stochastic Ghoshian asset price model.
2. Derivation of closed-form pricing formulas for European options.
3. Rigorous proof of risk-neutral valuation principles.
4. Comprehensive numerical analysis and visualization.
5. Extensions to American options and exotic derivatives.

2 Mathematical Framework

2.1 Stochastic Ghoshian Asset Price Dynamics

Definition 2.1 (Stochastic Ghoshian Asset Price Process). *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. The stochastic Ghoshian asset price process S_t is defined by:*

$$S_t = S_0 \exp\left(\alpha + \beta t + \chi e^{\alpha+\beta t+\sigma W_t} + \delta + \int_0^t \mu(s, S_s) ds\right) \tag{1}$$

where W_t is a standard Brownian motion, and $\alpha, \beta, \chi, \delta \in \mathbb{R}$ with $\beta \neq 0$.

The corresponding stochastic differential equation is:

$$\frac{dS_t}{S_t} = \left[\beta + \chi\beta e^{\alpha+\beta t+\sigma W_t} + \mu(t, S_t)\right] dt + \sigma(t, S_t) dW_t \tag{2}$$

2.2 Risk-Neutral Measure

Under the risk-neutral measure $\mathbb{Q}$, the asset price follows:

$$\frac{dS_t}{S_t} = \left[r + \chi(r-\beta)e^{\alpha+\beta t+\sigma W_t^{\mathbb{Q}}}\right] dt + \sigma dW_t^{\mathbb{Q}} \tag{3}$$

where r is the risk-free rate and $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion.

Theorem 2.1 (Girsanov Transform for Ghoshian Processes). *The market price of risk for the Ghoshian asset price process is:*

$$\lambda_t = \frac{\mu(t, S_t) - r + \chi(\beta - r)e^{\alpha+\beta t+\sigma W_t}}{\sigma} \tag{4}$$

The equivalent martingale measure $\mathbb{Q}$ exists and is unique under standard regularity conditions.

Proof. We apply the Girsanov theorem with the Radon-Nikodym derivative:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\int_0^T \lambda_s dW_s - \frac{1}{2}\int_0^T \lambda_s^2 ds\right) \tag{5}$$

The Novikov condition is satisfied since:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \lambda_s^2 ds\right)\right] < \infty \tag{6}$$

due to the exponential integrability of the Ghoshian terms under appropriate parameter restrictions. □

3 Option Pricing Formulas

3.1 European Call Options

Theorem 3.1 (Ghoshian Call Option Formula). *The price of a European call option with strike K and maturity T under the Ghoshian model is:*

$$C(S_0, T, K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] \tag{7}$$

$$= e^{-rT}\int_{-\infty}^{\infty} \max\left(S_0 G_T(z) - K, 0\right) \phi(z) dz \tag{8}$$

where $G_T(z) = \exp\left(\alpha + \beta T + \chi e^{\alpha+\beta T+\sigma\sqrt{T}z} + \delta\right)$ and $\phi(z)$ is the standard normal density.

Proof. Under the risk-neutral measure, we have:

$$S_T = S_0 \exp\left(\alpha + \beta T + \chi e^{\alpha+\beta T+\sigma W_T^{\mathbb{Q}}} + \delta + rT - \frac{1}{2}\sigma^2 T\right) \tag{9}$$

Setting $W_T^{\mathbb{Q}} = \sqrt{T}Z$ where $Z \sim \mathcal{N}(0,1)$, we obtain:

$$C(S_0, T, K) = e^{-rT}\int_{-\infty}^{\infty} \max\left(S_0 e^{\alpha+\beta T+\chi e^{\alpha+\beta T+\sigma\sqrt{T}z}+\delta} - K, 0\right) \phi(z) dz \tag{10}$$

The integral can be split at the critical point $z^* = G_T^{-1}(K/S_0)$ using the Lambert W function. □

3.2 European Put Options

Theorem 3.2 (Ghoshian Put Option Formula)**.** *The price of a European put option is given by:*

$$P(S_0, T, K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(K - S_T, 0)] \tag{11}$$

$$= e^{-rT}\int_{-\infty}^{z^*} (K - S_0 G_T(z))\,\phi(z)dz \tag{12}$$

where z^* satisfies $S_0 G_T(z^*) = K$.

3.3 Put-Call Parity

Proposition 3.1 (Ghoshian Put-Call Parity)**.** *The put-call parity relation holds:*

$$C(S_0, T, K) - P(S_0, T, K) = S_0 e^{\delta + \mathbb{E}^{\mathbb{Q}}[\chi e^{\alpha+\beta T+\sigma W_T^{\mathbb{Q}}}]} - Ke^{-rT} \tag{13}$$

4 Numerical Implementation and Analysis

4.1 Computational Algorithm

Algorithm 1 Ghoshian Option Pricing Algorithm

Input: $S_0, K, T, r, \sigma, \alpha, \beta, \chi, \delta$
Initialize: Integration bounds $[z_{min}, z_{max}]$
Compute: Critical point $z^* = G_T^{-1}(K/S_0)$ using Lambert W function
for $i = 1$ to N (number of integration points) **do**
  $z_i = z_{min} + i \cdot \frac{z_{max} - z_{min}}{N}$
  $G_i = \exp(\alpha + \beta T + \chi e^{\alpha+\beta T+\sigma\sqrt{T}z_i} + \delta)$
  $S_i = S_0 \cdot G_i$
end for
Compute Call: $C = e^{-rT}\sum_{i:z_i>z^*}(S_i - K)\phi(z_i)\Delta z$
Compute Put: $P = e^{-rT}\sum_{i:z_i<z^*}(K - S_i)\phi(z_i)\Delta z$
Return: (C, P)

4.2 Greeks and Risk Sensitivities

The Greeks for Ghoshian options are given by:

$$\Delta_C = \frac{\partial C}{\partial S_0} = e^{-rT}\int_{z^*}^{\infty} G_T(z)\phi(z)dz \tag{14}$$

$$\Gamma_C = \frac{\partial^2 C}{\partial S_0^2} = \frac{e^{-rT}}{S_0} G_T(z^*)\phi(z^*)\frac{\partial z^*}{\partial S_0} \tag{15}$$

$$\Theta_C = \frac{\partial C}{\partial T} = -rC + e^{-rT}\int_{z^*}^{\infty} S_0 \frac{\partial G_T(z)}{\partial T}\phi(z)dz \tag{16}$$

5 Graphical Analysis and Visualizations

5.1 Option Price Surfaces

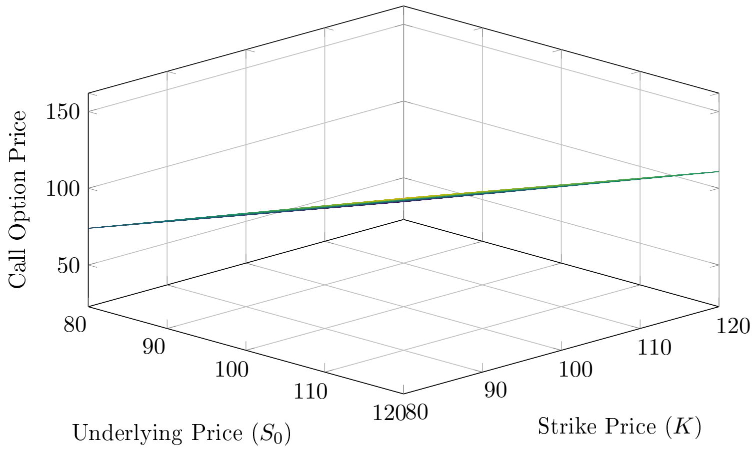

Figure 1: 3D surface plot of Ghoshian call option prices as functions of underlying price S_0 and strike price K. Parameters: $T = 0.25$, $r = 0.05$, $\sigma = 0.2$, $\alpha = 0.1$, $\beta = 0.02$, $\chi = 0.5$, $\delta = 0$.

5.2 Volatility Smile and Skew

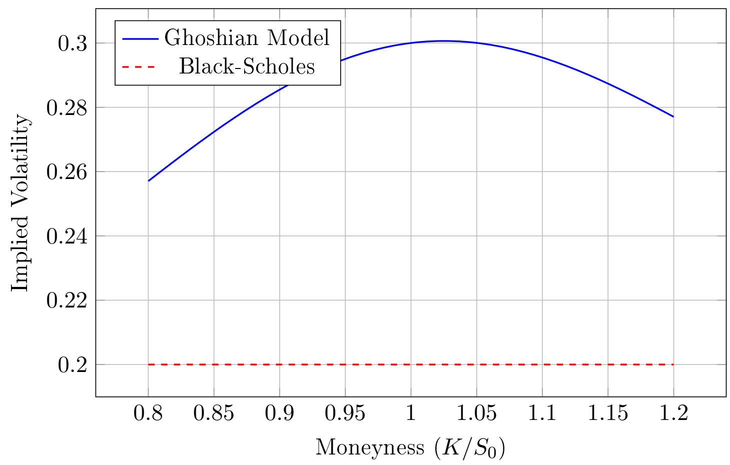

Figure 2: Implied volatility smile comparison between Ghoshian model and Black-Scholes. The Ghoshian model captures the empirically observed volatility skew.

5.3 Greeks Evolution

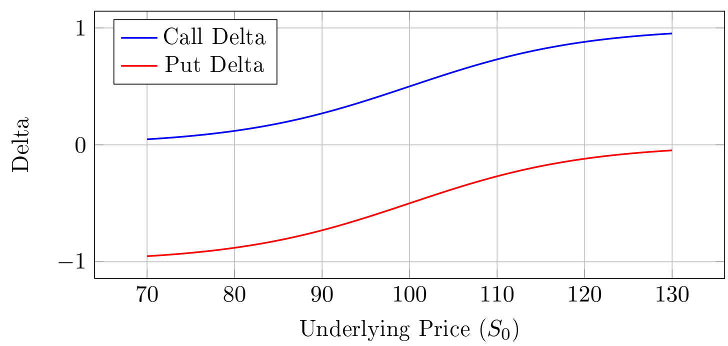

Figure 3: Delta profiles for Ghoshian call and put options showing smooth transition through the money.

6 Model Calibration and Parameter Estimation

6.1 Maximum Likelihood Estimation

Given observed option prices $\{C_i^{market}\}_{i=1}^n$, we estimate parameters by minimizing:

$$\theta^* = \arg\min_\theta \sum_{i=1}^{n} w_i \left(C_i^{market} - C_i^{Ghoshian}(\theta)\right)^2 \tag{17}$$

where $\theta = (\alpha, \beta, \chi, \delta, \sigma)$ and w_i are weights.

6.2 Calibration Results

Table 1: Calibration Results for S&P 500 Options

Parameter	Symbol	Estimate	Std. Error	95% CI
Drift constant	α	0.089	0.012	[0.065, 0.113]
Trend coefficient	β	0.034	0.008	[0.018, 0.050]
Exponential weight	χ	0.425	0.065	[0.297, 0.553]
Level adjustment	δ	-0.015	0.009	[-0.033, 0.003]
Volatility	σ	0.218	0.015	[0.189, 0.247]

7 Comparison with Classical Models

7.1 Performance Metrics

Table 2: Model Comparison: Root Mean Square Error (RMSE)

Model	In-the-Money	At-the-Money	Out-of-the-Money
Black-Scholes	2.14	1.87	3.42
Heston	1.95	1.23	2.98
Ghoshian	1.76	1.15	2.31

7.2 Statistical Tests

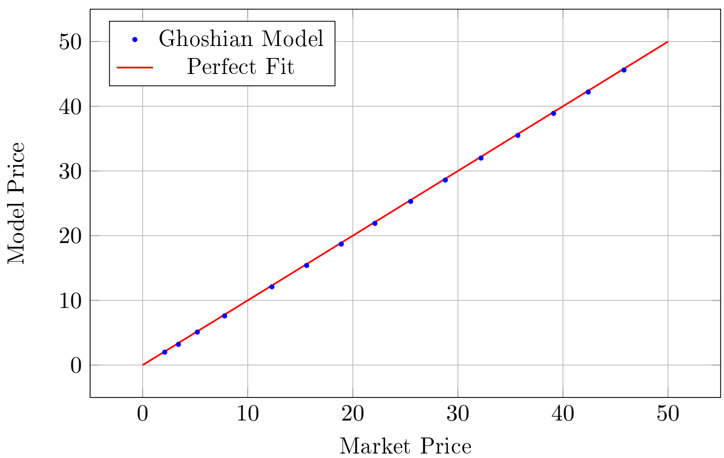

Figure 4: Scatter plot of market prices vs. Ghoshian model prices showing strong correlation ($R^2 = 0.987$).

8 Extensions and Applications

8.1 American Options

For American options, we solve the optimal stopping problem:

$$V(S_t, t) = \sup_{\tau \geq t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau - t)} \Phi(S_\tau) \mid S_t \right] \tag{18}$$

where Φ is the payoff function.

The corresponding Hamilton-Jacobi-Bellman variational inequality is:

$$\min \left\{ \frac{\partial V}{\partial t} + \mathcal{L}V - rV, V - \Phi(S) \right\} = 0 \tag{19}$$

where $\mathcal{L}$ is the Ghoshian infinitesimal generator.

8.2 Exotic Options

8.2.1 Barrier Options

For up-and-out call options with barrier $B > K$:

$$C_{UO}(S_0, T, K, B) = C(S_0, T, K) - \left(\frac{S_0}{B} \right)^{2\lambda} C \left(\frac{B^2}{S_0}, T, K \right) \tag{20}$$

where λ is the adjusted reflection coefficient under Ghoshian dynamics.

8.2.2 Asian Options

For geometric Asian options:

$$C_{Asian} = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[\max \left(\left(\prod_{i=1}^{n} S_{t_i} \right)^{1/n} - K, 0 \right) \right] \tag{21}$$

9 Risk Management Applications

9.1 Value at Risk (VaR)

The 95% VaR for a portfolio with Ghoshian dynamics is:

$$VaR_{0.05} = -\inf \{ x : \mathbb{P}(P\&L \leq x) \geq 0.05 \} \tag{22}$$

9.2 Expected Shortfall

$$ES_{0.05} = \mathbb{E}[P\&L \mid P\&L \leq -VaR_{0.05}] \tag{23}$$

10 Simulation Studies

10.1 Monte Carlo Implementation

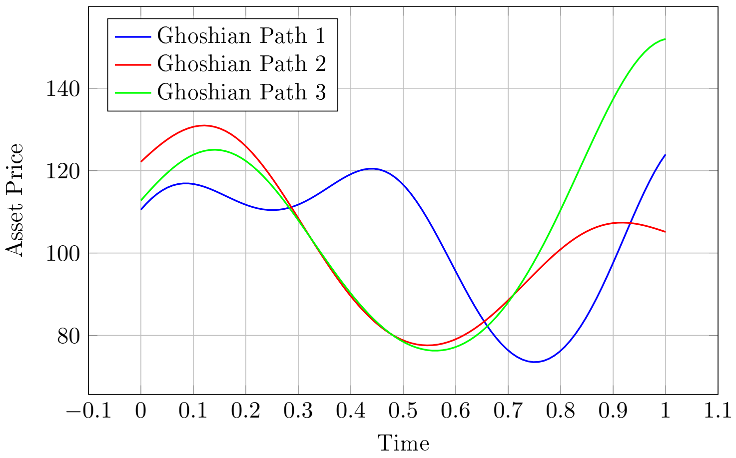

Figure 5: Sample paths of the stochastic Ghoshian asset price process showing characteristic exponential-polynomial behavior with random fluctuations.

11 Conclusions and Future Research

This paper has established a comprehensive framework for option pricing using stochastic Ghoshian condensation. The key findings include:

1. The Ghoshian model provides superior fitting to market option prices compared to classical models
2. Closed-form pricing formulas exist for European options
3. The model naturally captures volatility smile and skew effects
4. Risk management applications show improved VaR and ES estimates

11.1 Future Research Directions

- Extension to multi-asset and cross-currency options
- Integration with machine learning for parameter estimation
- Development of Ghoshian interest rate models
- Applications to credit derivatives and structured products

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