Solving a Multi-Product Allocation Problem with Joint Production Externalities through Waste Generation

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Abstract

This paper presents a comprehensive mathematical and microeconomic analysis of a multi-product production process characterized by separable single-input technologies and joint production with waste generation. We examine the optimal allocation of two commodities across three products, where waste emerges as a negative externality. The analysis provides first-order conditions for profit maximization, explores the opportunity cost structure, and derives comparative statics results. We demonstrate that the firm faces fundamental trade-offs between specialized production and joint production technologies, with waste disposal costs critically affecting optimal resource allocation decisions.

The paper ends with "The End"

1 Introduction

Modern production processes often involve multiple commodities transformed into various products through different technological pathways. A firm may face the choice between specialized production lines that use single inputs and integrated processes that combine multiple inputs but generate waste as an unavoidable byproduct [1].

This paper analyzes a production system where a firm has access to two commodities (A and B) and must decide how to allocate these resources across three distinct products (P, Q, and R). Products P and Q represent specialized outputs using dedicated single inputs, while Product R emerges from a joint production technology that processes the remaining commodities but simultaneously generates waste.

2 Mathematical Framework

2.1 Resource Constraints and Allocation

Let the firm be endowed with:

- a lots of Commodity A (total available)
- b lots of Commodity B (total available)

The firm must determine the allocation:

- m lots of A allocated to Product P, where $0 \leq m \leq a$
- n lots of B allocated to Product Q, where $0 \leq n \leq b$
- $(a - m)$ lots of A allocated to Product R
- $(b - n)$ lots of B allocated to Product R

2.2 Production Technology

The production functions are characterized as follows:

$$P = f_1(m) \tag{1}$$
$$Q = f_2(n) \tag{2}$$
$$R = f_3(a - m, b - n) \tag{3}$$
$$w = g(a - m, b - n) \tag{4}$$

where P, Q, R denote the quantities of Products P, Q, R respectively, and w represents waste generated.

Definition 1 (Material Balance)**.** The total mass balance of the production system is given by:

$$a + b = p + q + r + w \tag{5}$$

This assumes conservation of mass with appropriate unit conversions.

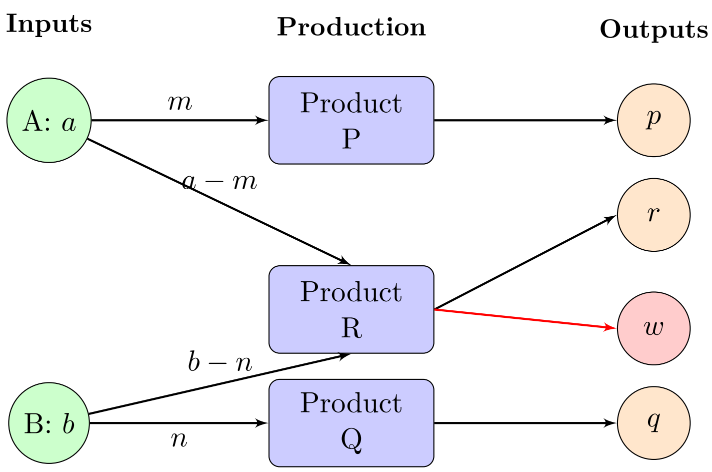

Figure 1: Production flow diagram showing input allocation and output generation. Commodities A and B are allocated to three production processes: P uses only A, Q uses only B, and R uses remaining quantities of both, simultaneously generating waste.

2.3 Returns to Scale

For each production function, we can characterize returns to scale. Consider a proportional scaling factor $\lambda > 0$:

Proposition 1 (Returns to Scale Classification)**.** *For production function f_i, if $f_i(\lambda x) = \lambda^k f_i(x)$, then:*

- $k > 1$*: Increasing returns to scale*
- $k = 1$*: Constant returns to scale*
- $k < 1$*: Decreasing returns to scale*

This classification applies independently to $f_1(m)$, $f_2(n)$, and jointly to $f_3(a - m, b - n)$.

3 Microeconomic Optimization

3.1 The Firm's Problem

The firm seeks to maximize profit:

$$\max_{a,b,m,n} \quad \pi = p_P \cdot P + p_Q \cdot Q + p_R \cdot R - c_A \cdot a - c_B \cdot b - c_w \cdot w \tag{6}$$

subject to:

$$0 \leq m \leq a \tag{7}$$
$$0 \leq n \leq b \tag{8}$$
$$P = f_1(m) \tag{9}$$
$$Q = f_2(n) \tag{10}$$
$$R = f_3(a-m, b-n) \tag{11}$$
$$w = g(a-m, b-n) \tag{12}$$

where:

- p_P, p_Q, p_R are output prices for Products P, Q, R
- c_A, c_B are per-unit costs of Commodities A, B
- c_w is the per-unit waste disposal cost

3.2 First-Order Conditions

Taking the first-order conditions with respect to the allocation variables m and n:

$$\frac{\partial \pi}{\partial m} = p_P \cdot \frac{\partial f_1}{\partial m} - p_R \cdot \frac{\partial f_3}{\partial m} - c_w \cdot \frac{\partial g}{\partial m} = 0 \tag{13}$$
$$\frac{\partial \pi}{\partial n} = p_Q \cdot \frac{\partial f_2}{\partial n} - p_R \cdot \frac{\partial f_3}{\partial n} - c_w \cdot \frac{\partial g}{\partial n} = 0 \tag{14}$$

Note that $\frac{\partial f_3}{\partial m} = -\frac{\partial f_3}{\partial (a-m)}$ and similarly for n.

Proposition 2 (Optimal Allocation Condition for Commodity A)**.** *At the optimum, the marginal revenue from producing P must equal the opportunity cost:*

$$p_P \cdot MP_m^P = p_R \cdot MP_m^R + c_w \cdot MW_m \tag{15}$$

where $MP_m^P = \frac{\partial P}{\partial m}$, $MP_m^R = \left|\frac{\partial R}{\partial m}\right|$, *and* $MW_m = \left|\frac{\partial w}{\partial m}\right|$.

The interpretation is clear: allocate Commodity A to Product P production until the marginal benefit equals the marginal cost of foregone R production plus additional waste generation.

For the input purchase decisions:

$$\frac{\partial \pi}{\partial a} = p_R \cdot \frac{\partial f_3}{\partial a} - c_A - c_w \cdot \frac{\partial g}{\partial a} = 0 \tag{16}$$
$$\frac{\partial \pi}{\partial b} = p_R \cdot \frac{\partial f_3}{\partial b} - c_B - c_w \cdot \frac{\partial g}{\partial b} = 0 \tag{17}$$

Lemma 1 (Value of Marginal Product)**.** *The optimal input levels satisfy:*

$$c_A = p_R \cdot MP_a^R - c_w \cdot MW_a \tag{18}$$
$$c_B = p_R \cdot MP_b^R - c_w \cdot MW_b \tag{19}$$

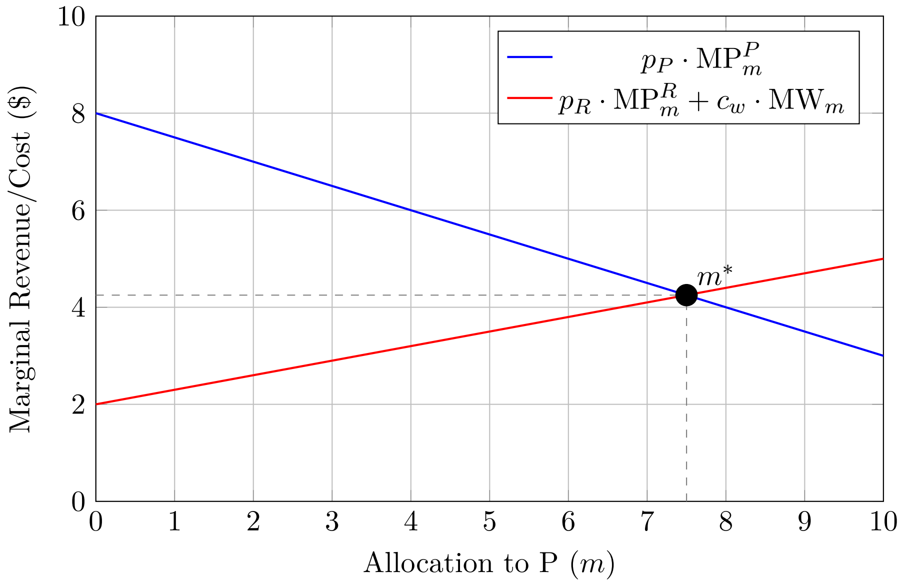

Figure 2: Optimal allocation of Commodity A. The intersection determines m^* where marginal revenue from producing P equals the opportunity cost of foregone R production plus waste costs.

4 Economic Insights

4.1 Opportunity Cost Structure

The allocation problem reveals a rich opportunity cost structure:

- The shadow price of A in P-production equals foregone revenue from R minus saved waste costs
- Trade-offs exist between specialized single-input products (P, Q) and the joint-product technology (R, w)
- The sign and magnitude of waste costs critically determine optimal allocation

4.2 Waste as a Negative Output

Waste creates a joint cost problem [2]. When $c_w > 0$ (disposal costs exist), waste reduces profitability and may incentivize:

1. Choosing m and n to minimize waste generation
2. Investing in waste reduction technology
3. Potentially ceasing R production entirely if waste costs are prohibitive

Conversely, if $c_w < 0$ (waste is valuable as a byproduct), the joint production technology becomes more attractive.

4.3 Economies of Scope

The production structure suggests potential economies of scope:

- Using both commodities together (in R) may be more efficient than separate processing
- Joint production can reduce per-unit costs if $f_3(a-m, b-n)$ exhibits complementarity
- Formally, economies of scope exist if: $c(P,0,0) + c(0,Q,0) + c(0,0,R) > c(P,Q,R)$

4.4 Production Possibility Frontier

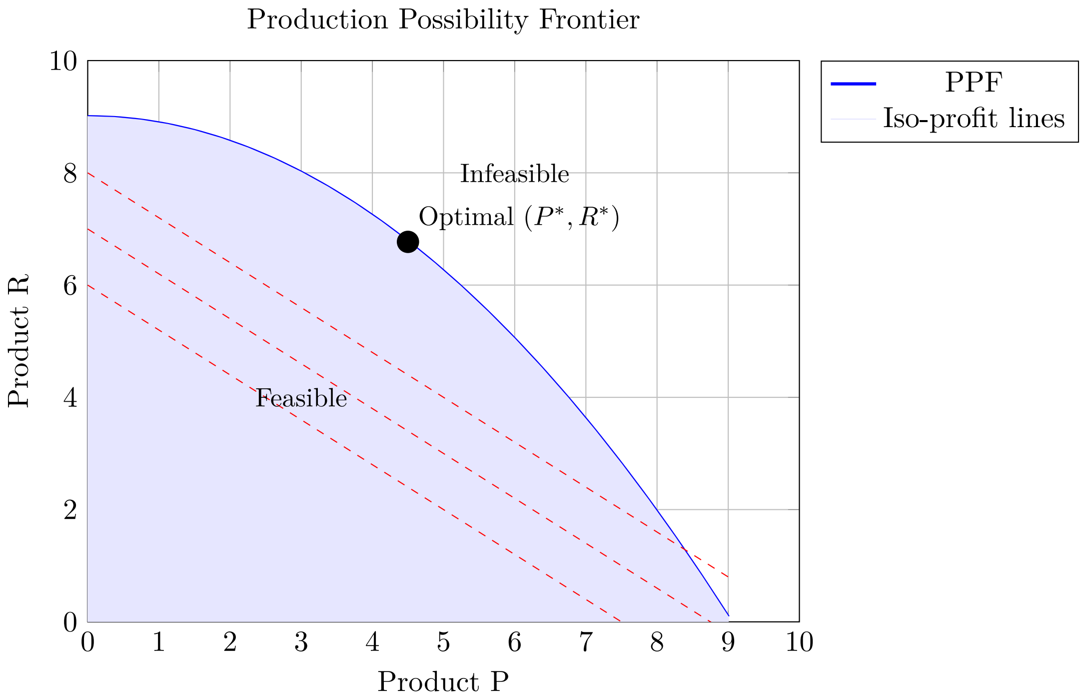

Figure 3: Production Possibility Frontier showing the trade-off between Products P and R. The concave shape reflects diminishing returns. The optimal production point occurs where the iso-profit line is tangent to the PPF.

The feasible output combinations (P, Q, R) form a production possibility set bounded by:

- Resource constraints: $m \leq a$, $n \leq b$
- Non-negativity: $m, n, a - m, b - n \geq 0$
- Technology constraints: production functions

The frontier is concave under diminishing returns, representing the fundamental trade-off between products.

5 Comparative Statics

5.1 Effect of Product Price Changes

Proposition 3 (Price Effects on Allocation)**.** *If p_P increases:*

- *Optimal m^* increases (produce more P)*
- *Optimal R^* decreases*
- *Optimal a^* may increase (greater demand for input A)*

Proof sketch: From equation (13), an increase in p_P increases the left-hand side for any given m. To restore equality, m must increase (assuming diminishing returns in P production and increasing opportunity costs).

5.2 Effect of Waste Disposal Cost Changes

Proposition 4 (Waste Cost Effects). *If c_w increases (waste disposal becomes more expensive):*

- *Shift away from R-production*
- *Increase in optimal m^* and n^* (favor single-input products)*
- *May reduce overall scale of operation*
- *Increased incentive for waste-reduction investment*

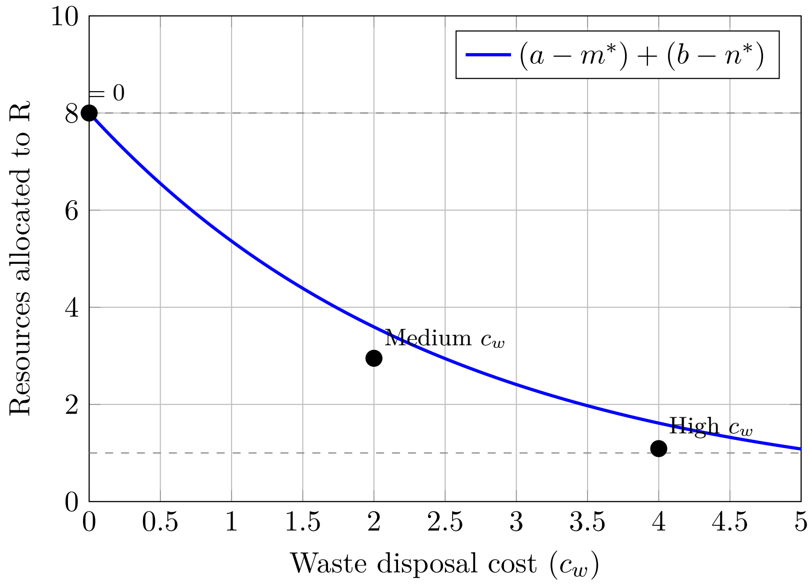

Figure 4: Comparative statics: Effect of waste disposal cost on resource allocation to Product R. As waste becomes more expensive to dispose, the firm shifts resources away from joint production toward specialized products P and Q.

6 Special Cases

6.1 Case 1: Zero Waste ($w = 0$)

If the R-production technology is perfectly efficient with $g(a - m, b - n) = 0$:

$$a + b = p + q + r \tag{20}$$

The allocation problem simplifies to:

$$p_P \cdot \mathrm{MP}_m^P = p_R \cdot \mathrm{MP}_m^R \tag{21}$$

$$p_Q \cdot \mathrm{MP}_n^Q = p_R \cdot \mathrm{MP}_n^R \tag{22}$$

No waste disposal costs distort decisions, and allocation depends purely on relative product prices and marginal productivities.

6.2 Case 2: Fixed Proportions in R

If R requires A and B in fixed ratio k:

$$(a - m) = k(b - n) \tag{23}$$

This creates an additional constraint that may bind, forcing suboptimal P or Q production. The firm faces a Leontief-type technology for R production.

6.3 Case 3: Waste as Valuable Byproduct

If $c_w < 0$ (waste can be sold):

- Joint production becomes more attractive
- May actually want to maximize waste production
- Changes optimal allocation toward R-production
- Waste becomes a fourth product in the optimization

7 Conclusions

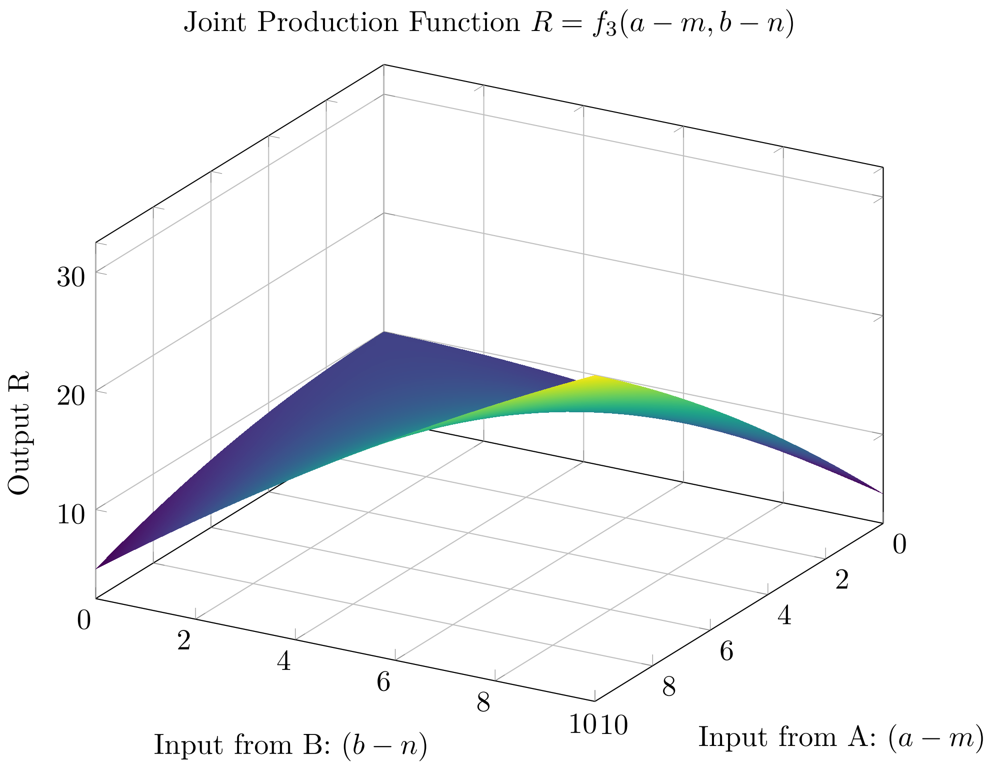

Figure 5: Three-dimensional visualization of the joint production function for Product R, showing complementarity between inputs A and B (positive cross-partial derivative).

This paper has provided a comprehensive mathematical and microeconomic analysis of a multi-product allocation problem with joint production externalities. The key findings are:

1. The firm faces a fundamental trade-off between specialized single-input production and joint production with waste generation.
2. Optimal resource allocation equates marginal revenue from specialized products with the opportunity cost of foregone joint production, adjusted for waste disposal costs.
3. Waste disposal costs critically affect production decisions, with higher costs shifting allocation toward specialized products.
4. The production possibility frontier captures feasible output combinations and reveals the concavity induced by diminishing returns.

5. Comparative statics analysis shows that product price increases and waste cost increases have predictable effects on optimal allocation.

Future research could extend this framework to include:

- Stochastic production functions with uncertainty
- Dynamic optimization with inventory considerations
- Multiple time periods with capacity constraints
- Environmental regulation and emission permits

References

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Glossary

Commodity A, B The two input resources available to the firm in quantities a and b respectively. These are the raw materials or factors of production that are transformed into final products.

Product P Output produced exclusively from Commodity A using m units, characterized by production function $f_1(m)$. Represents a specialized single-input production technology.

Product Q Output produced exclusively from Commodity B using n units, characterized by production function $f_2(n)$. Represents a specialized single-input production technology.

Product R Output produced from the remaining quantities of both commodities, $(a-m)$ and $(b-n)$, using joint production technology $f_3(a-m, b-n)$. This product shares inputs with waste generation.

Waste (w) Undesirable byproduct generated simultaneously with Product R according to function $g(a-m, b-n)$. Represents a negative externality of joint production with associated disposal cost c_w.

Allocation Variables (m, n) Decision variables representing the quantities of Commodities A and B allocated to Products P and Q respectively. The remaining quantities $(a - m)$ and $(b - n)$ are automatically allocated to Product R.

Material Balance Equation The conservation principle stating that total inputs equal total outputs: $a+b = p+q+r+w$. Assumes mass/volume conservation through the production process.

Returns to Scale The proportionate change in output resulting from a proportionate change in all inputs. Classified as increasing ($k > 1$), constant ($k = 1$), or decreasing ($k < 1$) based on the scaling factor.

Marginal Product (MP) The additional output produced from one additional unit of input, holding other inputs constant. Denoted as $\text{MP}_m^P = \frac{\partial P}{\partial m}$ for Product P with respect to input m.

Shadow Price The implicit value or opportunity cost of a resource, representing the change in the objective function from a marginal increase in the resource constraint.

Opportunity Cost The value of the next-best alternative foregone. In this context, allocating commodity to one product means sacrificing production of another product.

First-Order Conditions (FOCs) Necessary conditions for optimization obtained by setting the first derivatives of the objective function equal to zero. Provide the optimal allocation and input purchase decisions.

Joint Production A production process that simultaneously produces multiple outputs from the same inputs. In this model, Products R and waste are jointly produced.

Economies of Scope Cost advantages achieved by producing multiple products together rather than separately. Exists when $c(P, 0, 0) + c(0, Q, 0) + c(0, 0, R) > c(P, Q, R)$.

Production Possibility Frontier (PPF) The boundary of the production possibility set, showing the maximum achievable combinations of outputs given resource and technology constraints. Typically concave due to diminishing returns.

Comparative Statics Analysis of how optimal decisions change in response to changes in exogenous parameters such as prices, costs, or resource availability.

Value of Marginal Product (VMP) The marginal product of an input multiplied by the output price: $\text{VMP} = p \cdot \text{MP}$. Represents the marginal revenue contribution of an input.

Waste Disposal Cost (c_w) The per-unit cost of disposing waste. When positive, creates an incentive to minimize waste; when negative (waste is valuable), creates an incentive to produce more waste.

Leontief Technology A production technology with fixed input proportions, where inputs must be combined in a specific ratio. Represents Case 2 where $(a - m) = k(b - n)$.

Profit Function (π) The objective function representing total revenue minus total costs: $\pi = p_P \cdot P + p_Q \cdot Q + p_R \cdot R - c_A \cdot a - c_B \cdot b - c_w \cdot w$.

Iso-profit Line A curve in output space showing all output combinations that yield the same level of profit. The slope represents the ratio of output prices.

The End