The Complete Treatise on Pricing American Options

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Abstract

This comprehensive treatise examines the theoretical foundations and practical methodologies for pricing American-style options. We present a unified framework encompassing the Black-Scholes-Merton partial differential equation approach, optimal stopping theory, binomial and trinomial tree methods, Monte Carlo simulation techniques, and finite difference methods. Special attention is given to the free boundary problem, early exercise premiums, and computational challenges inherent in American option valuation. The treatment integrates mathematical rigor with practical implementation considerations, providing both theoretical insights and numerical algorithms for practitioners and researchers in quantitative finance.

The treatise ends with "The End"

Contents

List of Figures

1 Introduction

American options represent one of the most challenging and practically important problems in mathematical finance. Unlike European options, which can only be exercised at expiration, American options grant the holder the right to exercise at any time up to and including the expiration date. This additional flexibility creates a complex optimization problem that has spawned decades of theoretical development and computational innovation.

The fundamental challenge in American option pricing lies in determining the optimal exercise boundarythe evolving threshold that separates the continuation region (where holding the option is optimal) from the exercise region (where immediate exercise maximizes value). This free boundary problem cannot be solved analytically except in very special cases, necessitating sophisticated numerical methods.

2 Mathematical Framework

2.1 The Stochastic Foundation

We consider a complete market with a risk-free asset and a risky underlying asset. Under the risk-neutral measure $\mathbb{Q}$, the underlying asset price S_t follows a geometric Brownian motion:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t \tag{1}$$

where r is the risk-free rate, q is the dividend yield, σ is the volatility, and W_t is a standard Brownian motion.

Definition 2.1 (American Option Value)**.** The value of an American option at time t with underlying price $S_t = s$ is given by:

$$V(s,t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(\tau - t)} \Phi(S_\tau) \mid S_t = s \right] \tag{2}$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times between t and T, and $\Phi(S)$ is the payoff function.

2.2 The Variational Inequality Formulation

The American option pricing problem can be formulated as a variational inequality. In the continuation region, the option value satisfies the Black-Scholes PDE, while in the exercise region, the option value equals the intrinsic value.

Theorem 2.1 (American Option Variational Inequality)**.** The American option value $V(s,t)$ satisfies:

$$\max \left\{ \frac{\partial V}{\partial t} + \mathcal{L}V, \Phi(s) - V(s,t) \right\} = 0 \tag{3}$$

$$V(s,T) = \Phi(s) \tag{4}$$

where $\mathcal{L}$ is the Black-Scholes differential operator:

$$\mathcal{L}V = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + (r - q)s\frac{\partial V}{\partial s} - rV \tag{5}$$

3 The Free Boundary Problem

3.1 Characterization of the Exercise Boundary

The exercise boundary $S^*(t)$ separates the continuation region from the exercise region. For an American put option with strike K:

$$S^*(t) = \sup\{s \geq 0 : V(s,t) > (K - s)^+\} \tag{6}$$

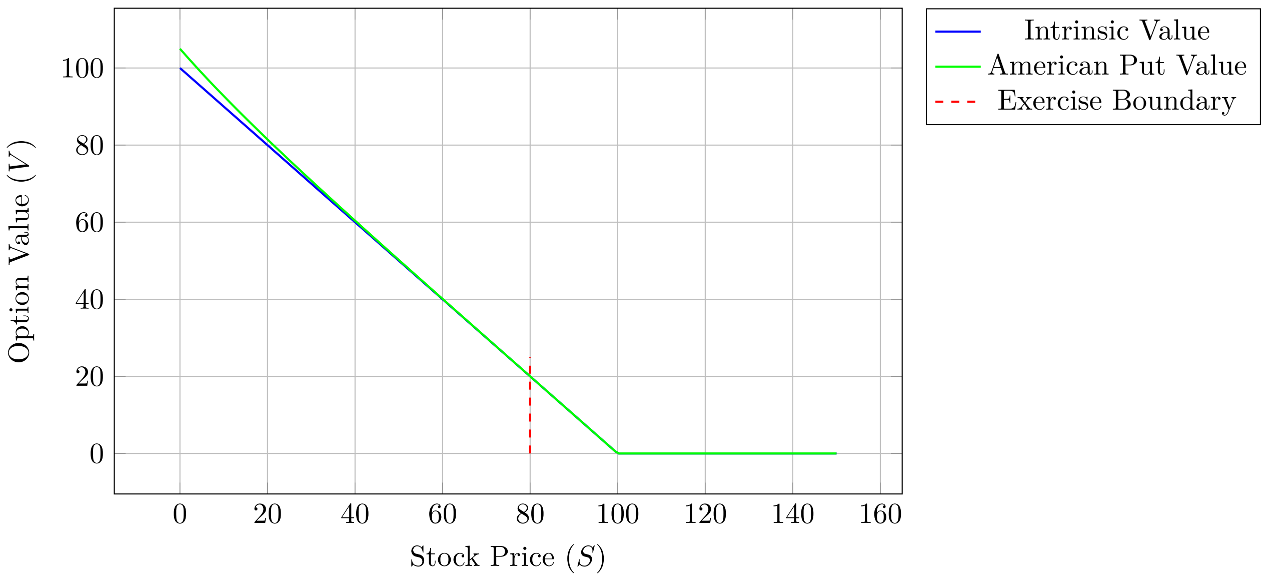

Figure 1: American Put Option Value and Exercise Boundary

3.2 Properties of the Exercise Boundary

Proposition 3.1 (Monotonicity of Exercise Boundary)**.** For an American put option, the exercise boundary $S^*(t)$ is non-increasing in time: $S^*(t_1) \geq S^*(t_2)$ for $t_1 < t_2$.

Proposition 3.2 (Boundary Conditions)**.** The exercise boundary satisfies the following conditions:

$$V(S^*(t), t) = \Phi(S^*(t)) \quad \text{(value matching)} \tag{7}$$

$$\frac{\partial V}{\partial s}(S^*(t), t) = \Phi'(S^*(t)) \quad \text{(smooth pasting)} \tag{8}$$

4 Binomial and Trinomial Tree Methods

4.1 The Cox-Ross-Rubinstein Binomial Model

The binomial tree method provides a discrete-time approximation to the continuous-time problem. At each node, we solve the optimal stopping problem by comparing the continuation value with the exercise value.

Algorithm 1 Binomial Tree for American Options

```
Initialize parameters: u = e^{σ√Δt}, d = 1/u, p = (e^{(r−q)Δt} − d)/(u − d)
for i = 0 to N do
    S(N, i) = S_0 · u^i · d^{N−i}
    V(N, i) = Φ(S(N, i))
end for
for n = N − 1 down to 0 do
    for i = 0 to n do
        C = e^{−rΔt}[pV(n + 1, i + 1) + (1 − p)V(n + 1, i)]
        E = Φ(S(n, i))
        V(n, i) = max(C, E)
    end for
end for
return V(0, 0)
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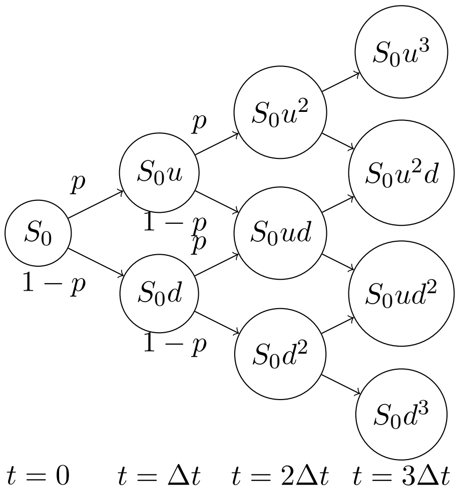

Figure 2: Binomial Tree Structure for American Option Pricing

4.2 Trinomial Trees

Trinomial trees offer improved stability and accuracy, especially for barrier options and when dividends are present.

$$
\begin{aligned}
u &= e^{\sigma\sqrt{3\Delta t}} \\
d &= e^{-\sigma\sqrt{3\Delta t}} \\
p_u &= \frac{1}{6} + \frac{(r-q)\sqrt{\Delta t}}{6\sigma\sqrt{3}} \\
p_m &= \frac{2}{3} \\
p_d &= \frac{1}{6} - \frac{(r-q)\sqrt{\Delta t}}{6\sigma\sqrt{3}}
\end{aligned}
\tag{9}
$$

5 Finite Difference Methods

5.1 Explicit Finite Difference Scheme

The American option PDE can be discretized using finite differences. The explicit scheme updates option values using:

$$V_i^{n+1} = \max\left\{a_i V_{i-1}^n + b_i V_i^n + c_i V_{i+1}^n, \Phi(S_i)\right\} \tag{10}$$

where:

$$a_i = \frac{\Delta t}{2}\left(\sigma^2 i^2 - (r-q)i\right) \tag{11}$$

$$b_i = 1 - \Delta t(\sigma^2 i^2 + r) \tag{12}$$

$$c_i = \frac{\Delta t}{2}\left(\sigma^2 i^2 + (r-q)i\right) \tag{13}$$

5.2 Implicit and Crank-Nicolson Schemes

The implicit scheme provides better stability:

$$\max\left\{-a_i V_{i-1}^{n+1} + (1+b_i)V_i^{n+1} - c_i V_{i+1}^{n+1} + V_i^n, \Phi(S_i) - V_i^{n+1}\right\} = 0 \tag{14}$$

6 Monte Carlo Methods

6.1 Longstaff-Schwartz Algorithm

The Least Squares Monte Carlo (LSM) method approximates the continuation value using regression on basis functions.

Algorithm 2 Longstaff-Schwartz Algorithm

Generate M paths of the underlying asset
Initialize cash flows at expiration: $CF_m = \Phi(S_{N,m})$
for $n = N-1$ down to 1 **do**
  Identify in-the-money paths: $\mathcal{I} = \{m : \Phi(S_{n,m}) > 0\}$
  if $|\mathcal{I}| > 0$ **then**
    Regress discounted cash flows on basis functions: $\hat{C}_m = \sum_k \beta_k \phi_k(S_{n,m})$
    for $m \in \mathcal{I}$ **do**
      if $\Phi(S_{n,m}) > \hat{C}_m$ **then**
        $CF_m = \Phi(S_{n,m})$ {Exercise}
      else
        $CF_m = CF_m \cdot e^{-r\Delta t}$ {Continue}
      end if
    end for
  end if
end for
return $\frac{1}{M}\sum_{m=1}^{M} CF_m$

6.2 Convergence Analysis

Theorem 6.1 (LSM Convergence)**.** Under appropriate regularity conditions, the Longstaff-Schwartz estimator converges to the true American option value as the number of paths $M \to \infty$ and the number of basis functions increases appropriately.

7 Advanced Topics

7.1 Dividend-Paying Stocks

For stocks paying discrete dividends, the asset price follows:

$$S_{t^-} = S_{t^+} + D_t \tag{15}$$

at dividend times, requiring careful treatment of the jump discontinuity.

7.2 Multi-Asset American Options

For options on multiple underlying assets, the curse of dimensionality becomes severe. Advanced techniques include:

- Sparse grid methods
- Monte Carlo with variance reduction
- Neural network approximations
- Fourier methods

8 Computational Comparison

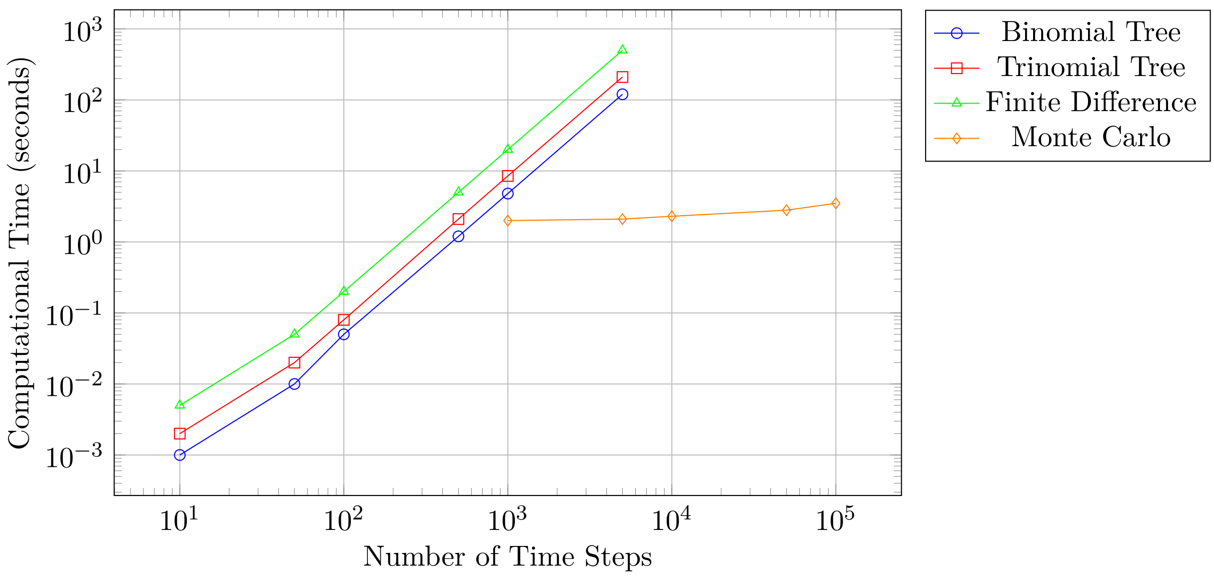

Figure 3: Computational Complexity Comparison of American Option Pricing Methods

9 Practical Implementation Considerations

9.1 Numerical Stability

Key considerations for robust implementation include:

- Grid spacing and time step selection
- Boundary condition treatment
- Interpolation methods for irregular grids
- Handling of extreme moneyness cases

9.2 Greeks Calculation

The sensitivities (Greeks) of American options require special attention:

$$\Delta = \frac{\partial V}{\partial S} \tag{16}$$

$$\Gamma = \frac{\partial^2 V}{\partial S^2} \tag{17}$$

$$\Theta = \frac{\partial V}{\partial t} \tag{18}$$

$$\mathcal{V} = \frac{\partial V}{\partial \sigma} \tag{19}$$

$$\rho = \frac{\partial V}{\partial r} \tag{20}$$

10 Market Applications and Extensions

10.1 American Style Exotic Options

Extensions to barrier options, Asian options, and basket options present additional computational challenges and require specialized techniques.

10.2 Credit Risk Considerations

When counterparty risk is present, the American option valuation must account for default probabilities and recovery rates, leading to coupled PDE systems.

11 Conclusions

American option pricing remains one of the most computationally challenging problems in quantitative finance. While no single method dominates across all scenarios, the combination of theoretical understanding and numerical sophistication has enabled practitioners to value these complex instruments with increasing accuracy and efficiency.

The choice of method depends critically on the specific requirements: binomial/trinomial trees excel for simple options and educational purposes, finite differences offer flexibility for complex boundary conditions, and Monte Carlo methods scale well to high dimensions. Modern implementations often combine multiple approaches to achieve optimal performance.

Future research directions include machine learning applications, quantum computing algorithms, and improved high-dimensional techniques for multi-asset American options.

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