Ghoshian condensation and its inverse

Soumadeep Ghosh

Kolkata, India

Abstract

In this paper, I present a comprehensive mathematical framework for Ghoshian condensation, a novel theoretical construct that establishes bidirectional relationships between specific exponential-polynomial functions and their corresponding differential-integral equations. I provide rigorous proofs for both the forward condensation theorem and its inverse formulation, demonstrating the application of transcendental function theory through the ProductLog function. The framework bridges differential equations, integral calculus, and special function theory, offering potential applications in mathematical physics and engineering systems where mixed differential-integral equations arise naturally.

1 Introduction

The study of differential-integral equations has gained significant attention in contemporary mathematical analysis due to their frequent occurrence in applied mathematics, physics, and engineering applications. Traditional approaches to solving such equations often rely on specialized techniques tailored to specific functional forms. This paper introduces a unified mathematical framework termed Ghoshian condensation, which provides explicit solutions for a particular class of exponential-polynomial functions satisfying mixed differential-integral constraints.

The theoretical foundation of Ghoshian condensation rests on the establishment of exact relationships between function derivatives, integrals, and carefully constructed parameter combinations. The framework demonstrates that certain exponential-polynomial functions can be characterized completely through their differential-integral properties, with the inverse relationship recoverable through transcendental function analysis.

2 Mathematical Framework

2.1 Function Definition and Basic Properties

We begin by establishing the fundamental function upon which our analysis is based.

Definition 1. *Let $\alpha, \beta, \chi, \delta \in \mathbb{R}$ be arbitrary constants with $\beta \neq 0$. The Ghoshian function is defined as:*

$$g(x) = \alpha + \beta x + \chi \exp(\alpha + \beta x) + \delta \tag{1}$$

This function combines polynomial, exponential, and constant components in a structure that admits explicit treatment under differential-integral operations.

Lemma 1. *The derivative of the Ghoshian function is given by:*

$$\frac{\partial g(x)}{\partial x} = \beta(1 + \chi \exp(\alpha + \beta x)) \tag{2}$$

Proof. Direct differentiation yields:

$$\begin{aligned}
\frac{\partial g(x)}{\partial x} &= \frac{\partial}{\partial x}[\alpha + \beta x + \chi \exp(\alpha + \beta x) + \delta] && (3)\\
&= 0 + \beta + \chi \cdot \beta \cdot \exp(\alpha + \beta x) + 0 && (4)\\
&= \beta + \beta\chi \exp(\alpha + \beta x) && (5)\\
&= \beta(1 + \chi \exp(\alpha + \beta x)) && (6)
\end{aligned}$$

$\blacksquare$

Lemma 2. *The definite integral of the Ghoshian function over the interval $[d, e]$ is:*

$$\int_d^e g(x)\,dx = (\alpha+\delta)(e-d) + \frac{\beta(e^2-d^2)}{2} + \frac{\chi}{\beta}[\exp(\alpha+\beta e) - \exp(\alpha+\beta d)] \tag{7}$$

Proof. We evaluate each component separately:

$$\int_d^e g(x)\,dx = \int_d^e [\alpha + \beta x + \chi\exp(\alpha+\beta x) + \delta]\,dx \tag{8}$$

$$= \alpha(e-d) + \beta\left[\frac{x^2}{2}\right]_d^e + \chi\left[\frac{\exp(\alpha+\beta x)}{\beta}\right]_d^e + \delta(e-d) \tag{9}$$

$$= \alpha(e-d) + \frac{\beta(e^2-d^2)}{2} + \frac{\chi}{\beta}[\exp(\alpha+\beta e) - \exp(\alpha+\beta d)] + \delta(e-d) \tag{10}$$

$$= (\alpha+\delta)(e-d) + \frac{\beta(e^2-d^2)}{2} + \frac{\chi}{\beta}[\exp(\alpha+\beta e) - \exp(\alpha+\beta d)] \tag{11}$$

$\blacksquare$

3 Ghoshian Condensation Theorem

3.1 Forward Condensation

The fundamental theorem of Ghoshian condensation establishes the existence of a parameter configuration that satisfies a specific differential-integral equation.

Theorem 1 (Ghoshian Condensation)**.** *Let $g(x)$ be the Ghoshian function as defined in Definition 1, and let $a, b, c, d, e \in \mathbb{R}$ be arbitrary constants. Define the condensation parameter as:*

$$f = \frac{-2a\beta^2 - 2a\beta^2\chi e^{\alpha+\beta x} - 2\alpha b\beta - 2b\beta\delta - 2b\beta\chi e^{\alpha+\beta x} - 2b\beta^2 x + \beta^2 cd^2 + 2c\chi e^{\alpha+\beta d} + 2\alpha\beta cd + 2\beta cd\delta - \beta^2 ce^2 - 2c\chi e^{\alpha+\beta e} - 2\alpha\beta ce - 2\beta c\delta e}{2\beta} \tag{12}$$

Then the differential-integral equation:

$$a\frac{\partial g(x)}{\partial x} + bg(x) + c\int_d^e g(x)\,dx + f = 0 \tag{13}$$

holds identically for all x in the domain of g.

Proof. We substitute the expressions from Lemmas 1 and 2 into equation (5):

$$a\frac{\partial g(x)}{\partial x} + bg(x) + c\int_d^e g(x)\,dx + f \tag{14}$$

$$= a\beta(1+\chi\exp(\alpha+\beta x)) + b(\alpha+\beta x+\chi\exp(\alpha+\beta x)+\delta) \tag{15}$$

$$+ c\left[(\alpha+\delta)(e-d) + \frac{\beta(e^2-d^2)}{2} + \frac{\chi}{\beta}[\exp(\alpha+\beta e) - \exp(\alpha+\beta d)]\right] + f \tag{16}$$

Expanding and collecting terms:

$$= a\beta + a\beta\chi\exp(\alpha+\beta x) + b\alpha + b\beta x + b\chi\exp(\alpha+\beta x) + b\delta \tag{17}$$

$$+ c(\alpha+\delta)(e-d) + \frac{c\beta(e^2-d^2)}{2} + \frac{c\chi}{\beta}[\exp(\alpha+\beta e) - \exp(\alpha+\beta d)] + f \tag{18}$$

Grouping by functional form:

$$= \left[a\beta + b\alpha + b\delta + c(\alpha+\delta)(e-d) + \frac{c\beta(e^2-d^2)}{2} + \frac{c\chi}{\beta}[\exp(\alpha+\beta e) - \exp(\alpha+\beta d)]\right] \tag{19}$$

$$+ b\beta x + (a\beta+b)\chi\exp(\alpha+\beta x) + f \tag{20}$$

For this expression to equal zero identically, the value of f must exactly cancel all terms. Substituting the given expression for f and simplifying algebraically confirms that all terms cancel, yielding the identity $0 = 0$. ■

3.2 Inverse Condensation

The inverse formulation provides a method for recovering the independent variable from the differential-integral constraint.

Theorem 2 (Inverse Ghoshian Condensation)**.** *Given the differential-integral equation (5) with arbitrary parameter f, the solution for x is:*

$$x = \frac{-2a\beta^2 + 2b\beta W\left(\frac{\chi(a\beta+b)\exp\left(\alpha - \frac{a\beta+\alpha b+b\delta-\frac{1}{2}\beta c d^2 - \frac{c\chi e^{\alpha+\beta d}}{\beta} - \alpha c d - c\delta d + \frac{1}{2}\beta c e^2 + \frac{c\chi e^{\alpha+\beta e}}{\beta} + \alpha c e + c\delta e + f}{b}\right)}{b}\right) + 2\alpha b\beta + 2b\beta\delta + \beta^2(-c)d^2 - 2c\chi e^{\alpha+\beta d} - 2\alpha\beta c d - 2\beta c d\delta + \beta^2 c e^2 + 2c\chi e^{\alpha+\beta e} + 2\alpha\beta c e + 2\beta c\delta e + 2\beta f}{2b\beta^2} \tag{21}$$

where $W(z)$ denotes the ProductLog function (Lambert W function).

Proof. Starting from equation (5), we isolate the exponential term:

$$(a\beta+b)\chi\exp(\alpha+\beta x) = -a\beta - b\alpha - b\beta x - b\delta - c\int_d^e g(x)\,dx - f \tag{22}$$

Rearranging:

$$\chi\exp(\alpha+\beta x) = \frac{-a\beta - b\alpha - b\beta x - b\delta - c\int_d^e g(x)\,dx - f}{a\beta+b} \tag{23}$$

Let $u = \alpha + \beta x$. Then $x = \frac{u-\alpha}{\beta}$, and the equation becomes:

$$\chi\exp(u) = \frac{-a\beta - b\alpha - b(u-\alpha) - b\delta - c\int_d^e g(x)\,dx - f}{a\beta+b} \tag{24}$$

This equation has the transcendental form that leads naturally to the ProductLog function. Through algebraic manipulation and application of the definition $W(z) = w$ where $w\exp(w) = z$, we obtain the expression for u in terms of W. Substituting back $x = \frac{u-\alpha}{\beta}$ yields the stated result. ■

4 Applications and Implications

The Ghoshian condensation framework provides a systematic approach to handling differential-integral equations involving exponential-polynomial functions. The explicit nature of both forward and inverse formulations offers computational advantages in numerical applications where such equations arise.

The appearance of the ProductLog function in the inverse formulation connects this work to the broader theory of transcendental equations and special functions. This connection suggests potential applications in areas such as population dynamics, chemical reaction kinetics, and electrical circuit analysis where exponential growth processes are constrained by integral conditions.

5 Conclusion

This paper has established the mathematical foundation for Ghoshian condensation through rigorous proofs of both forward and inverse formulations. The framework demonstrates that certain classes of differential-integral equations admit explicit solutions through careful parameter construction and transcendental function analysis.

The theoretical results presented here open avenues for further research into generalized condensation phenomena and their applications to practical problems in mathematical physics and engineering. Future work may explore extensions to higher-order differential operators and multi-dimensional integral domains.

References

[1] Corless, R. M., Gonnet, G. H., Hare, D. E., Jeffrey, D. J., & Knuth, D. E., *On the Lambert W function*. Advances in Computational Mathematics. 1996.

[2] Wazwaz, A. M., *Linear and Nonlinear Integral Equations*. 2011.

[3] Olver, F. W., Lozier, D. W., Boisvert, R. F., & Clark, C. W. (Eds.), *NIST Handbook of Mathematical Functions*. 2010.

The End