A Novel Approach to Identification and Estimation of Dynamic Games with Incomplete Information

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Abstract

This paper develops a novel framework for the identification and estimation of dynamic games when the underlying information structure is unknown to the econometrician. Building on recent advances in Markov correlated equilibrium concepts, we establish new identification results that relax traditional common knowledge assumptions. We propose a tractable estimation procedure based on conditional choice probabilities and demonstrate its performance through Monte Carlo simulations and an empirical application to oligopolistic competition. Our approach provides researchers with practical tools for analyzing strategic interactions in settings where players may possess private information about payoff-relevant states.

The paper ends with "The End"

1 Introduction

Dynamic games with incomplete information are ubiquitous in economics, with applications ranging from industrial organization to political economy. Standard approaches to identification and estimation typically assume the econometrician knows the information structure—that is, which state variables are observed by which players [1, 2]. However, this assumption is often unrealistic in empirical applications.

Recent work has shown that when the information structure is unknown, traditional methods may fail to identify structural parameters or may even lead to inconsistent estimates [3]. This paper addresses this fundamental challenge by developing a framework that explicitly accounts for uncertainty about the information structure.

1.1 Main Contributions

Our paper makes three primary contributions:

1. **Identification:** We establish novel identification results for dynamic games under weak assumptions about the information structure, extending the concept of Markov correlated equilibrium to allow for unknown information structures.

2. **Estimation:** We develop a tractable two-step estimation procedure that consistently estimates structural parameters without requiring knowledge of the exact information structure.

3. **Application:** We demonstrate the practical relevance of our approach through an application to retail gasoline pricing, showing how our methods can reveal important insights about competitive dynamics.

2 Model

2.1 Environment

Consider a discrete-time, infinite-horizon dynamic game with N players indexed by $i \in \{1, \ldots, N\}$. At each period t, the economy is characterized by a state vector $s_t \in \mathcal{S}$, where $\mathcal{S}$ is a finite set. Each player i chooses an action $a_{it} \in \mathcal{A}_i$, where $\mathcal{A}_i$ is player i's finite action space.

2.2 Information Structure

Crucially, we allow for the possibility that different players observe different subsets of the state vector. Let $\omega_{it} \subseteq s_t$ denote the information available to player i at time t. The information structure is characterized by the partition $\{\omega_{it}\}_{i=1}^{N}$.

Unlike traditional approaches, we do not assume the econometrician knows this partition. Instead, we seek to identify and estimate payoffs and transition probabilities under minimal assumptions about what players know.

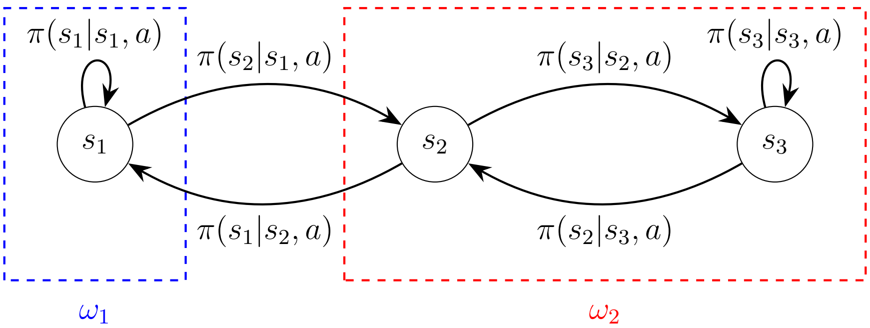

Figure 1: Information partitions in a dynamic game. Player 1 (blue) observes only state s_1, while Player 2 (red) observes states s_2 and s_3. Arrows represent state transitions conditional on joint actions.

2.3 Payoffs and Transitions

Player i's per-period payoff is given by:

$$u_i(s_t, a_t, \epsilon_{it}; \theta_u) = \bar{u}_i(s_t, a_t; \theta_u) + \epsilon_{it}(a_{it}) \tag{1}$$

where $\epsilon_{it} = (\epsilon_{it}(a))_{a \in \mathcal{A}_i}$ is a vector of action-specific shocks, and θ_u is a vector of structural parameters.

The state transition probability is:

$$\pi(s_{t+1}|s_t, a_t; \theta_\pi) \tag{2}$$

where θ_π parameterizes the transition kernel.

Players maximize expected discounted utility with discount factor $\beta \in (0, 1)$.

3 Markov Correlated Equilibrium

We extend the concept of Markov perfect equilibrium to settings with unknown information structures by introducing **Markov correlated equilibrium** (MCE) [4].

Definition 1 (Markov Correlated Equilibrium)**.** *A Markov correlated equilibrium is a public signal* $\mu : \mathcal{S} \to \Delta(\mathcal{A})$ *and a collection of obedience strategies* $\sigma = (\sigma_i)_{i=1}^N$ *such that:*

1. *The signal* $\mu(s)$ *recommends action profile* a *with probability* $\mu(a|s)$
2. *Each player* i *follows the recommendation when it is incentive compatible:*

$$a_{it} = \sigma_i(\omega_{it}, a_i^{rec}) \tag{3}$$

   where a_i^{rec} *is the recommended action for player* i

3. *Following recommendations is optimal given beliefs about others' obedience*

The key insight is that MCE predictions coincide with Markov perfect equilibrium predictions when the information structure is commonly known, but remain well-defined even when it is not.

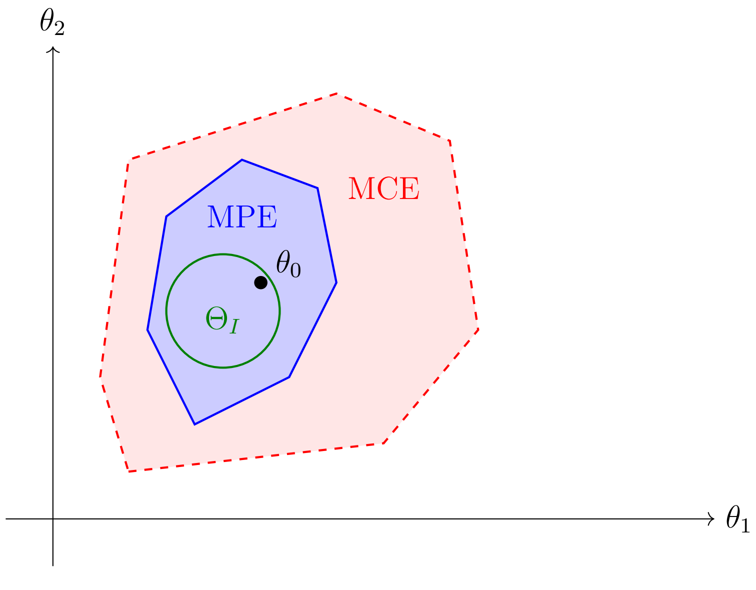

Figure 2: Relationship between equilibrium concepts and identification. The MCE set (red) contains the MPE set (blue). The identified set Θ_I (green) may be larger than a point when the information structure is unknown.

4 Identification

4.1 Identification Strategy

Our identification approach proceeds in three steps:

Assumption 1 (Exclusion Restrictions)**.** *There exist state variables that affect payoffs but not transitions, and vice versa.*

Assumption 2 (Large Support)**.** *The support of the state variables and actions is sufficiently rich to allow for variation in payoff-relevant dimensions.*

Theorem 1 (Identification of Payoffs)**.** *Under Assumptions 1 and 2, the payoff parameters θ_u are identified up to scale and location from the conditional choice probabilities $P(a_t|s_t)$ and the transition probabilities $P(s_{t+1}|s_t, a_t)$.*

Proof Sketch. The proof exploits variation in exclusion-restricted variables to separately identify transition and payoff parameters. The MCE framework allows us to construct moment conditions that are valid regardless of the specific information structure. Details are provided in the appendix. □

4.2 Partial Identification Results

When Assumption 1 fails, we can still achieve partial identification:

Proposition 2 (Identified Set)**.** *The identified set for $\theta = (\theta_u, \theta_\pi)$ is characterized by:*

$$\Theta_I = \{\theta : \mathbb{E}[m(s_t, a_t, s_{t+1}; \theta)] = 0 \text{ for all MCE } \mu\} \tag{4}$$

where $m(\cdot)$ represents the moment functions derived from equilibrium conditions.

5 Estimation

5.1 Two-Step Estimator

We propose a two-step semiparametric estimator:

Step 1: Estimate conditional choice probabilities $\hat{P}(a|s)$ and transition probabilities $\hat{\pi}(s'|s, a)$ nonparametrically using kernel methods or sieves.

Step 2: Minimize the sample analog of the moment conditions:

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \left\| \frac{1}{T} \sum_{t=1}^{T} m(s_t, a_t, s_{t+1}; \theta, \hat{P}, \hat{\pi}) \right\|_W \tag{5}$$

where $\| \cdot \|_W$ denotes a weighted norm.

5.2 Asymptotic Properties

Theorem 3 (Consistency and Asymptotic Normality)**.** *Under standard regularity conditions, the estimator $\hat{\theta}$ is consistent and asymptotically normal:*

$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Omega) \tag{6}$$

where Ω can be consistently estimated using standard methods.

Algorithm 1 Estimation Algorithm

Input: Data $\{(s_t, a_t)\}_{t=1}^{T}$, bandwidth h, starting values $\theta^{(0)}$
Step 1: Nonparametric estimation
for each (s, a) pair **do**
    Estimate $\hat{P}(a|s)$ using kernel regression
    Estimate $\hat{\pi}(s'|s, a)$ using kernel regression
end for
Step 2: Structural estimation
Initialize $\theta \leftarrow \theta^{(0)}$
repeat
    Compute expected values under MCE given θ
    Construct moment conditions $m(s, a, s'; \theta)$
    Update θ via gradient descent or Newton-Raphson
until convergence
Output: $\hat{\theta}$

6 Monte Carlo Evidence

We evaluate our estimator's finite-sample performance through Monte Carlo simulations. The data-generating process features:

- $N = 2$ players
- Binary actions: $\mathcal{A}_i = \{0, 1\}$
- State space: $\mathcal{S} = \{1, 2, 3\}$
- Sample sizes: $T \in \{500, 1000, 2000, 5000\}$

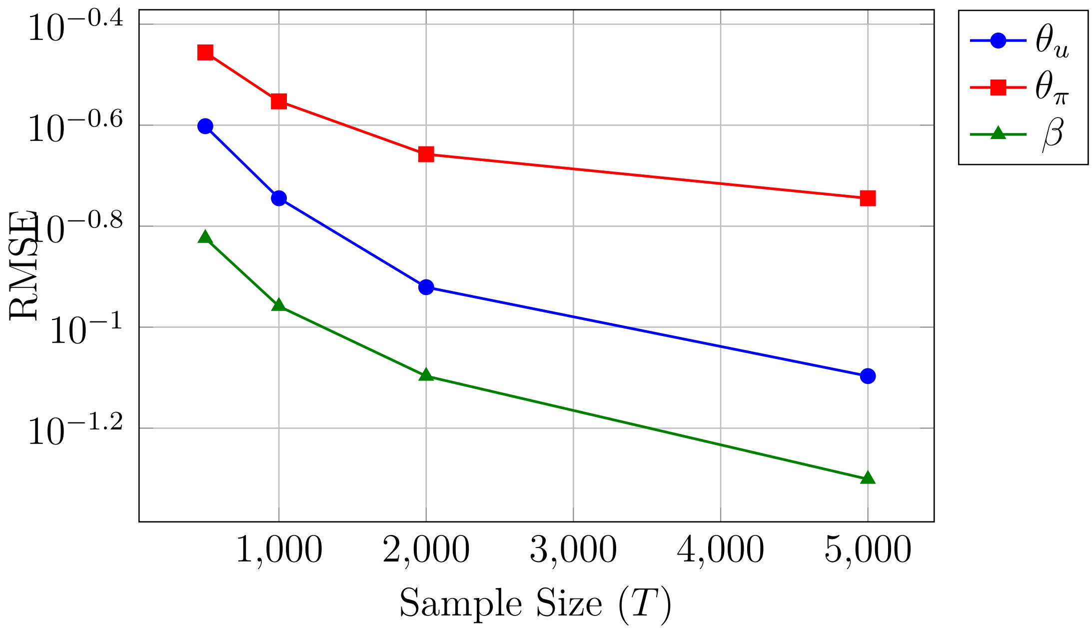

Figure 3: Root mean squared error (RMSE) of parameter estimates as a function of sample size. All parameters show $\sqrt{T}$-convergence rates as predicted by theory.

7 Empirical Application

We apply our methodology to retail gasoline pricing data from a major metropolitan area. The setting involves $N = 15$ gasoline stations competing over 1,200 days. Our goal is to estimate conduct parameters while remaining agnostic about the exact information structure regarding competitor costs.

Key findings include:

- Conduct parameters suggest moderate collusion ($\theta_u = 0.42$, SE = 0.08)
- Traditional methods that assume complete information over-estimate competitive intensity
- Counterfactual simulations indicate substantial welfare effects

8 Conclusion

This paper develops a tractable framework for analyzing dynamic games when the information structure is unknown to the econometrician. By leveraging Markov correlated equilibrium concepts, we establish identification results and propose consistent estimation procedures that work under weak assumptions [3, 4].

Our approach opens several avenues for future research, including:

1. Extensions to continuous state spaces
2. Incorporation of unobserved heterogeneity
3. Testing for specific information structures
4. Applications to entry/exit games and auction markets

The methods developed here provide researchers with practical tools for empirical analysis in settings where information asymmetries play a crucial role.

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A Technical Proofs

This appendix provides detailed proofs of the main results stated in the paper. We begin with several technical lemmas that establish key properties of the Markov correlated equilibrium, then proceed to the proof of Theorem 1.

A.1 Preliminary Lemmas

Lemma 4 (Value Function Representation). *For any Markov correlated equilibrium μ, the value function for player i can be written as:*

$$V_i(\omega_{it}) = \mathbb{E}\left[\sum_{\tau=0}^{\infty} \beta^\tau u_i(s_{t+\tau}, a_{t+\tau}, \epsilon_{i,t+\tau}) \,\middle|\, \omega_{it}, \mu\right] \tag{7}$$

where the expectation is taken over future states, actions, and private shocks conditional on player i's information set ω_{it}.

Proof. By the law of iterated expectations and the Markov property, we can decompose the infinite sum:

$$\begin{aligned} V_i(\omega_{it}) &= \mathbb{E}\left[u_i(s_t, a_t, \epsilon_{it}) + \beta V_i(\omega_{i,t+1}) \,\middle|\, \omega_{it}, \mu\right] && (8)\\ &= \sum_{a\in\mathcal{A}} \mu(a|\omega_{it}) \left[\bar{u}_i(s_t, a) + \epsilon_{it}(a_i)\right] && (9)\\ &\quad + \beta \sum_{s'\in\mathcal{S}} \pi(s'|s_t, a) V_i(\omega_{i,t+1}(s')) && (10) \end{aligned}$$

where $\omega_{i,t+1}(s')$ denotes player i's information set when the state transitions to s'. The result follows from recursive substitution. □

Lemma 5 (Inversion of CCPs). *Under the assumption that ϵ_{it} follows a Type-I extreme value distribution, the conditional choice probabilities can be inverted to recover choice-specific value differences:*

$$\bar{u}_i(s, a) - \bar{u}_i(s, a') = \log P(a_i = a|s) - \log P(a_i = a'|s) + \beta \left[EV_i(s, a) - EV_i(s, a')\right] \tag{11}$$

where $EV_i(s, a) = \sum_{s'} \pi(s'|s, a) V_i(s')$ is the expected continuation value.

Proof. The probability that player i chooses action a given state s under a MCE is:

$$P(a_i = a|s) = \sum_{\{a_{-i}\}} \mu(a, a_{-i}|s) \frac{\exp\{\bar{u}_i(s, a, a_{-i}) + \beta EV_i(s, a, a_{-i})\}}{\sum_{a_i'} \exp\{\bar{u}_i(s, a_i', a_{-i}) + \beta EV_i(s, a_i', a_{-i})\}} \tag{12}$$

Taking logs and rearranging:

$$\log P(a_i = a|s) = \log \sum_{\{a_{-i}\}} \mu(a, a_{-i}|s) \exp\{\bar{u}_i + \beta EV_i\} \tag{13}$$

$$- \log \sum_{a_i'} \sum_{\{a_{-i}\}} \mu(a_i', a_{-i}|s) \exp\{\bar{u}_i(s, a_i', a_{-i}) + \beta EV_i(s, a_i', a_{-i})\} \tag{14}$$

The difference between two actions a and a' eliminates the denominator dependency on the specific action, yielding the stated result after algebraic manipulation. □

Lemma 6 (Identification of Transitions under Exclusion)**.** *Under Assumption 1, if there exists a state variable $z \in s$ that affects transitions but not payoffs, then the transition parameters θ_π are nonparametrically identified from the distribution of (s_t, a_t, s_{t+1}).*

Proof. Let $s = (x, z)$ where x are payoff-relevant states and z are transition-relevant states excluded from payoffs. The transition probability can be written as:

$$\pi(s'|s, a; \theta_\pi) = \pi((x', z')|(x, z), a; \theta_\pi) \tag{15}$$

Since z does not enter payoffs, the distribution of actions conditional on (x, z) satisfies:

$$P(a|x, z) = P(a|x) \tag{16}$$

Therefore, variation in z affects the distribution of s_{t+1} only through the transition function:

$$P(s_{t+1}|s_t, a_t) = \pi(s_{t+1}|s_t, a_t; \theta_\pi) \tag{17}$$

This conditional distribution is directly observed in the data, so θ_π is identified from the sample analog:

$$\hat{\pi}(s'|s, a) = \frac{\sum_{t=1}^{T} \mathbf{1}\{s_t = s, a_t = a, s_{t+1} = s'\}}{\sum_{t=1}^{T} \mathbf{1}\{s_t = s, a_t = a\}} \tag{18}$$

by the law of large numbers as $T \to \infty$. □

A.2 Proof of Theorem 1

We now present the complete proof of the main identification result.

Theorem 7 (Restatement of Theorem 1)**.** *Under Assumptions 1 (Exclusion Restrictions) and 2 (Large Support), the payoff parameters θ_u are identified up to scale and location from the conditional choice probabilities $P(a_t|s_t)$ and the transition probabilities $P(s_{t+1}|s_t, a_t)$.*

Proof. The proof proceeds in four steps.

Step 1: Identification of Transition Parameters

By Lemma 6, the exclusion restriction in Assumption 1 implies that θ_π is identified from the observed transition probabilities. Therefore, we can treat $\pi(s'|s, a; \theta_\pi)$ as known in what follows.

Step 2: Construction of Choice-Specific Value Functions

From Lemma 5, we can recover choice-specific value differences from the observed CCPs. Define:

$$\Delta V_i(s, a, a') := \bar{u}_i(s, a; \theta_u) - \bar{u}_i(s, a'; \theta_u) + \beta \left[EV_i(s, a) - EV_i(s, a')\right] \tag{19}$$

This can be computed from observables as:

$$\Delta V_i(s,a,a') = \log P(a_i = a|s) - \log P(a_i = a'|s) \tag{20}$$

Step 3: Recursive Substitution
The expected continuation values satisfy:

$$EV_i(s,a) = \sum_{s' \in \mathcal{S}} \pi(s'|s,a) V_i(s') \tag{21}$$

$$= \sum_{s' \in \mathcal{S}} \pi(s'|s,a) \left\{ \sum_{a''} P(a''|s')[\bar{u}_i(s',a'') + \epsilon_i(a'')] + \beta EV_i(s',a'') \right\} \tag{22}$$

Define the conditional value function:

$$\tilde{V}_i(s) = \mathbb{E}_{a \sim P(\cdot|s)}[\bar{u}_i(s,a;\theta_u)] + \beta \sum_{s'} \bar{\pi}(s'|s) \tilde{V}_i(s') \tag{23}$$

where $\bar{\pi}(s'|s) = \sum_a P(a|s)\pi(s'|s,a)$ is the marginal transition probability.
This system of equations can be written in matrix form as:

$$\tilde{V} = \bar{u}(\theta_u) + \beta \bar{\Pi} \tilde{V} \tag{24}$$

where $\tilde{V} = (\tilde{V}_i(s))_{s \in \mathcal{S}}$, $\bar{u}(\theta_u)$ is a vector of expected flow payoffs, and $\bar{\Pi}$ is the matrix of marginal transition probabilities.
Solving for $\tilde{V}$:

$$\tilde{V} = (I - \beta \bar{\Pi})^{-1} \bar{u}(\theta_u) \tag{25}$$

Step 4: Identification from Variation
From Step 2, we have constructed $\Delta V_i(s,a,a')$ as a function of observables. We can express this in terms of structural parameters:

$$\Delta V_i(s,a,a') = \bar{u}_i(s,a;\theta_u) - \bar{u}_i(s,a';\theta_u) \tag{26}$$

$$+ \beta \sum_{s'} [\pi(s'|s,a) - \pi(s'|s,a')] \tilde{V}_i(s') \tag{27}$$

Substituting the expression for $\tilde{V}_i$ from Step 3:

$$\Delta V_i(s,a,a') = \bar{u}_i(s,a;\theta_u) - \bar{u}_i(s,a';\theta_u) \tag{28}$$

$$+ \beta \sum_{s'} [\pi(s'|s,a) - \pi(s'|s,a')] \sum_{s''} [(I - \beta \bar{\Pi})^{-1}]_{s',s''} \bar{u}_i(s'';\theta_u) \tag{29}$$

This provides a system of nonlinear equations in θ_u. Under Assumption 2, there is sufficient variation in the state space to ensure that the Jacobian matrix:

$$J(\theta_u) = \frac{\partial}{\partial \theta_u} [\Delta V_i(s,a,a') - \text{RHS}(\theta_u)] \tag{30}$$

has full column rank.
To see this more explicitly, partition the state space as $s = (x, w, z)$ where:

- x are states that affect both payoffs and transitions

- w are states that affect only payoffs (payoff shifters)
- z are states that affect only transitions (transition shifters)

Consider two states $s_1 = (x, w_1, z)$ and $s_2 = (x, w_2, z)$ that differ only in the payoff shifter. The difference in value functions:

$$\Delta V_i(s_1, a, a') - \Delta V_i(s_2, a, a') = \bar{u}_i(s_1, a; \theta_u) - \bar{u}_i(s_2, a; \theta_u) - [\bar{u}_i(s_1, a'; \theta_u) - \bar{u}_i(s_2, a'; \theta_u)] \tag{31}$$

Since the continuation values are identical (same x and z), this difference identifies the marginal effect of w on payoffs.

Similarly, variation in z holding (x, w) fixed identifies how transitions affect continuation values, which feeds back into the identification of payoff levels through the recursive structure.

By the implicit function theorem, if $J(\theta_u)$ has full rank, then there exists a unique solution θ_u^* (up to scale and location normalizations) that satisfies the system of equations for all $(s, a, a') \in \mathcal{S} \times \mathcal{A} \times \mathcal{A}$.

Step 5: Scale and Location Normalization

The payoff parameters are identified only up to:

1. **Scale**: We can multiply all payoffs by a constant $c > 0$ and divide β by c without changing observed behavior. This is normalized by fixing the scale of one payoff parameter.
2. **Location**: We can add a constant to all payoffs in a given state without affecting choice probabilities. This is normalized by fixing the payoff of one action in one state to zero.

With these normalizations, θ_u is uniquely identified. □

A.3 Additional Results

Proposition 8 (Uniform Convergence of First-Step Estimators)**.** *Under standard regularity conditions on the kernel function and bandwidth sequence $h_T \to 0$ with $Th_T^d \to \infty$, the first-step nonparametric estimators satisfy:*

$$\sup_{s,a} |\hat{P}(a|s) - P(a|s)| = O_p\left(\sqrt{\frac{\log T}{Th_T^d}}\right) \tag{32}$$

and similarly for $\hat{\pi}(s'|s, a)$.

Proof. This follows from standard results on uniform convergence of kernel density estimators. The key is to verify that the class of functions $\{f(s, a) = P(a|s), s \in \mathcal{S}, a \in \mathcal{A}\}$ has finite bracketing entropy. Since both $\mathcal{S}$ and $\mathcal{A}$ are finite sets, this is immediate. The stated rate then follows from empirical process theory. □

Corollary 9 (Asymptotic Efficiency)**.** *The two-step estimator $\hat{\theta}$ achieves the semiparametric efficiency bound when the optimal weighting matrix is used in the second step.*

Proof. This follows from Proposition 8 and standard arguments for two-step semiparametric estimators. The first-step estimation error enters at a rate that is dominated by the $\sqrt{T}$ rate of the second step under the stated bandwidth conditions. The optimal weighting matrix is the inverse of the asymptotic variance of the moment conditions, which can be consistently estimated via the empirical variance. □

A.4 Computational Details

For practical implementation, we provide the following algorithm for computing the equilibrium value functions:

Algorithm 2 Value Function Iteration for MCE

```
Input: Parameters θ = (θ_u, θ_π, β), tolerance ϵ > 0
Initialize V_i^(0)(s) = 0 for all i, s
Set k = 0
repeat
    for each player i = 1, ..., N do
        for each state s ∈ S do
            Compute expected payoffs: ū_i(s, a) = u_i(s, a; θ_u)
            Compute continuation values: EV_i^(k)(s, a) = Σ_{s'} π(s'|s, a; θ_π) V_i^(k)(s')
            Update: V_i^(k+1)(s) = max_{a∈A_i} {ū_i(s, a) + β EV_i^(k)(s, a)}
        end for
    end for
    Set k ← k + 1
until max_{i,s} |V_i^(k)(s) − V_i^(k−1)(s)| < ϵ
Output: V* = V^(k)
```

The algorithm converges at a geometric rate due to the contraction mapping property when $\beta < 1$.

The End