A Three-Factor Affine Gaussian Term-Structure Model of Yield: Real Rate, Inflation Risk, and Liquidity

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Abstract

We develop a three-factor affine term structure model that decomposes bond yields into economically interpretable components: real rate factors, inflation risk premiums, and liquidity effects. Building on the affine framework of [2], we derive closed-form solutions for zero-coupon bond prices and yields under Gaussian dynamics. The model nests the fundamental yield decomposition $y = r_f + \mathbb{E}[i] + p_{ir} + p_{lr} + e_l$ while maintaining analytical tractability. We provide complete mathematical derivations, establish no-arbitrage conditions, and prove the existence and uniqueness of solutions. The framework offers insights into term structure dynamics, risk premium behavior, and cross-sectional yield variation.

The paper ends with "The End"

1 Introduction

The term structure of interest rates reflects market participants' expectations about future economic conditions, compensation for various risks, and liquidity considerations. Understanding the determinants of bond yields is central to asset pricing, monetary policy, and risk management. This paper develops a three-factor affine term structure model that provides an economically meaningful decomposition of yields while preserving analytical tractability.

Our framework builds on the seminal work of [3] and [1], extending the affine class of models to explicitly incorporate three fundamental drivers of yields: (1) real rate factors capturing the risk-free rate and expected inflation, (2) inflation risk premiums compensating for inflation uncertainty, and (3) liquidity factors reflecting market liquidity conditions and premiums.

The contribution of this paper is threefold. First, we establish a rigorous mapping between the phenomenological yield decomposition

$$y = r_f + \mathbb{E}[i] + p_{ir} + p_{lr} + e_l \tag{1}$$

and a dynamically consistent affine model with three state variables. Second, we derive closed-form expressions for bond prices and yields, providing complete proofs of existence and uniqueness. Third, we characterize the model's implications for the shape and dynamics of the yield curve, risk premiums, and arbitrage relationships.

The remainder of the paper is organized as follows. Section 2 presents the model framework and fundamental assumptions. Section 3 derives bond pricing formulas and establishes key theoretical results. Section 4 analyzes term structure properties and arbitrage conditions. Section 5 concludes.

2 Model Framework

2.1 The State Space

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, where $\mathbb{P}$ represents the physical (statistical) probability measure.

Definition 2.1 (State Variables)**.** The term structure is driven by a three-dimensional state vector

$$\mathbf{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{pmatrix} \in \mathbb{R}^3 \tag{2}$$

where:

- $X_1(t)$ represents the real rate factor (risk-free rate plus expected inflation),
- $X_2(t)$ represents the inflation risk premium factor,
- $X_3(t)$ represents the liquidity factor (liquidity premium plus excess liquidity).

2.2 Factor Dynamics Under the Physical Measure

Assumption 2.2 (Ornstein-Uhlenbeck Dynamics)**.** Under the physical measure $\mathbb{P}$, the state vector follows a multivariate Ornstein-Uhlenbeck process:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{K}(\boldsymbol{\theta} - \mathbf{X}(t))\mathrm{d}t + \mathbf{\Sigma}\mathrm{d}\mathbf{W}(t) \tag{3}$$

where:

- $\mathbf{W}(t) = (W_1(t), W_2(t), W_3(t))^\top$ is a three-dimensional standard Brownian motion,
- $\mathbf{K} = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3)$ with $\kappa_i > 0$ is the mean-reversion matrix,
- $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^\top$ is the long-run mean vector,
- $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_i > 0$ is the volatility matrix.

Remark 2.3. For simplicity, we assume independent factors (diagonal matrices). The framework readily extends to correlated factors with full matrices $\mathbf{K}$ and $\mathbf{\Sigma}\mathbf{\Sigma}^\top$.

2.3 The Short Rate

Definition 2.4 (Instantaneous Short Rate)**.** The instantaneous risk-free rate is given by the affine specification:

$$r(t) = \delta_0 + \boldsymbol{\delta}^\top \mathbf{X}(t) = \delta_0 + \sum_{i=1}^{3} \delta_i X_i(t) \tag{4}$$

where $\delta_0 \in \mathbb{R}$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)^\top$ with $\delta_i = 1$ for our baseline specification.

For the canonical model, we set $\delta_0 = 0$ and $\boldsymbol{\delta} = (1, 1, 1)^\top$, yielding:

$$r(t) = X_1(t) + X_2(t) + X_3(t) \tag{5}$$

This specification directly maps to the yield decomposition, with X_1 capturing the baseline rate and expected inflation, X_2 the inflation risk premium, and X_3 the liquidity components.

2.4 Market Price of Risk

Assumption 2.5 (Affine Market Price of Risk)**.** The market price of risk is specified as:

$$\boldsymbol{\lambda}(t) = \mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_0 \tag{6}$$

where $\boldsymbol{\lambda}_0 = (\lambda_{0,1}, \lambda_{0,2}, \lambda_{0,3})^\top$ is a constant vector.

This leads to the risk-neutral measure via the Girsanov transformation.

Theorem 2.6 (Risk-Neutral Dynamics). *Under the risk-neutral measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, the state vector evolves as:*

$$\mathrm{d}\mathbf{X}(t) = \mathbf{K}^{\mathbb{Q}}(\boldsymbol{\theta}^{\mathbb{Q}} - \mathbf{X}(t))\mathrm{d}t + \boldsymbol{\Sigma}\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t) \tag{7}$$

where $\mathbf{W}^{\mathbb{Q}}(t)$ is a $\mathbb{Q}$-Brownian motion, and:

$$\mathbf{K}^{\mathbb{Q}} = \mathbf{K} \tag{8}$$

$$\boldsymbol{\theta}^{\mathbb{Q}} = \boldsymbol{\theta} - \mathbf{K}^{-1}\boldsymbol{\lambda}_0 \tag{9}$$

Proof. By the Girsanov theorem, the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is defined by:

$$\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t) = \mathrm{d}\mathbf{W}(t) + \boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}(t)\mathrm{d}t \tag{10}$$

Substituting into equation (3):

$$\mathrm{d}\mathbf{X}(t) = \mathbf{K}(\boldsymbol{\theta} - \mathbf{X}(t))\mathrm{d}t + \boldsymbol{\Sigma}[\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t) - \boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_0\mathrm{d}t] \tag{11}$$

$$= \mathbf{K}(\boldsymbol{\theta} - \mathbf{X}(t))\mathrm{d}t - \boldsymbol{\lambda}_0\mathrm{d}t + \boldsymbol{\Sigma}\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t) \tag{12}$$

$$= \mathbf{K}[(\boldsymbol{\theta} - \mathbf{K}^{-1}\boldsymbol{\lambda}_0) - \mathbf{X}(t)]\mathrm{d}t + \boldsymbol{\Sigma}\mathrm{d}\mathbf{W}^{\mathbb{Q}}(t) \tag{13}$$

Setting $\boldsymbol{\theta}^{\mathbb{Q}} = \boldsymbol{\theta} - \mathbf{K}^{-1}\boldsymbol{\lambda}_0$ completes the proof. □

3 Bond Pricing and Closed-Form Solutions

3.1 Zero-Coupon Bond Pricing

Definition 3.1 (Zero-Coupon Bond Price). Let $P(t,T)$ denote the price at time t of a zero-coupon bond paying \$1 at maturity $T \geq t$. Define the time-to-maturity $\tau = T - t$.

Theorem 3.2 (Fundamental Pricing Equation). *Under the risk-neutral measure $\mathbb{Q}$, the bond price satisfies:*

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\mathrm{d}s\right)\middle|\mathcal{F}_t\right] \tag{14}$$

Proof. This is a standard result from no-arbitrage pricing theory. The bond price is the discounted expected payoff under the risk-neutral measure. □

3.2 Affine Bond Price Structure

Theorem 3.3 (Affine Bond Price). *The zero-coupon bond price has the exponentially affine form:*

$$P(t,T) = \exp\left[A(\tau) - \mathbf{B}(\tau)^\top\mathbf{X}(t)\right] \tag{15}$$

where $\tau = T - t$, and $A(\tau) \in \mathbb{R}$ and $\mathbf{B}(\tau) = (B_1(\tau), B_2(\tau), B_3(\tau))^\top \in \mathbb{R}^3$ satisfy ordinary differential equations (ODEs) with boundary conditions $A(0) = 0$ and $\mathbf{B}(0) = \mathbf{0}$.

Proof. We apply the Feynman-Kac theorem. The bond price satisfies the PDE:

$$\frac{\partial P}{\partial t} + \mathcal{L}^{\mathbb{Q}}P - r(t)P = 0 \tag{16}$$

where $\mathcal{L}^{\mathbb{Q}}$ is the infinitesimal generator under $\mathbb{Q}$:

$$\mathcal{L}^{\mathbb{Q}}P = \sum_{i=1}^{3}\kappa_i^{\mathbb{Q}}(\theta_i^{\mathbb{Q}} - X_i)\frac{\partial P}{\partial X_i} + \frac{1}{2}\sum_{i=1}^{3}\sigma_i^2\frac{\partial^2 P}{\partial X_i^2} \tag{17}$$

Substituting the affine ansatz (15) with $\tau = T - t$ (so $\partial/\partial t = -\partial/\partial\tau$):

$$\frac{\partial P}{\partial t} = -P\left[\frac{\mathrm{d}A}{\mathrm{d}\tau} - \mathbf{X}^\top \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\tau}\right] \tag{18}$$

$$\frac{\partial P}{\partial X_i} = -B_i(\tau)P \tag{19}$$

$$\frac{\partial^2 P}{\partial X_i^2} = B_i(\tau)^2 P \tag{20}$$

Substituting into (16):

$$-P\left[\frac{\mathrm{d}A}{\mathrm{d}\tau} - \mathbf{X}^\top \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\tau}\right] + P\sum_{i=1}^{3}\left[-\kappa_i^{\mathbb{Q}}(\theta_i^{\mathbb{Q}} - X_i)B_i + \frac{1}{2}\sigma_i^2 B_i^2\right] - P(\delta_0 + \boldsymbol{\delta}^\top \mathbf{X}) = 0 \tag{21}$$

Dividing by P and grouping terms:

$$-\frac{\mathrm{d}A}{\mathrm{d}\tau} + \sum_{i=1}^{3}\left[-\kappa_i^{\mathbb{Q}}\theta_i^{\mathbb{Q}} B_i + \frac{1}{2}\sigma_i^2 B_i^2\right] - \delta_0 + \mathbf{X}^\top\left[\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\tau} + \mathbf{K}^{\mathbb{Q}}\mathbf{B} - \boldsymbol{\delta}\right] = 0 \tag{22}$$

For this to hold for all $\mathbf{X}$, we require:

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\tau} = \boldsymbol{\delta} - \mathbf{K}^{\mathbb{Q}}\mathbf{B}(\tau) \tag{23}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\delta_0 + \sum_{i=1}^{3}\left[-\kappa_i^{\mathbb{Q}}\theta_i^{\mathbb{Q}} B_i(\tau) + \frac{1}{2}\sigma_i^2 B_i(\tau)^2\right] \tag{24}$$

with boundary conditions $A(0) = 0$ and $\mathbf{B}(0) = \mathbf{0}$. □

3.3 Closed-Form Solutions

Proposition 3.4 (Solution for $\mathbf{B}(\tau)$)**.** *For the diagonal specification with $\delta_i = 1$, the component ODEs decouple:*

$$\frac{\mathrm{d}B_i}{\mathrm{d}\tau} = 1 - \kappa_i^{\mathbb{Q}} B_i(\tau), \quad B_i(0) = 0 \tag{25}$$

The solution is:

$$B_i(\tau) = \frac{1}{\kappa_i^{\mathbb{Q}}}\left(1 - e^{-\kappa_i^{\mathbb{Q}}\tau}\right) \tag{26}$$

Proof. Equation (25) is a first-order linear ODE. Using the integrating factor $\mu(\tau) = e^{\kappa_i^{\mathbb{Q}}\tau}$:

$$e^{\kappa_i^{\mathbb{Q}}\tau}\frac{\mathrm{d}B_i}{\mathrm{d}\tau} + \kappa_i^{\mathbb{Q}} e^{\kappa_i^{\mathbb{Q}}\tau} B_i = e^{\kappa_i^{\mathbb{Q}}\tau} \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[e^{\kappa_i^{\mathbb{Q}}\tau} B_i(\tau)\right] = e^{\kappa_i^{\mathbb{Q}}\tau} \tag{28}$$

Integrating from 0 to τ with $B_i(0) = 0$:

$$e^{\kappa_i^{\mathbb{Q}}\tau} B_i(\tau) = \int_0^\tau e^{\kappa_i^{\mathbb{Q}} s}\mathrm{d}s = \frac{1}{\kappa_i^{\mathbb{Q}}}\left(e^{\kappa_i^{\mathbb{Q}}\tau} - 1\right) \tag{29}$$

$$B_i(\tau) = \frac{1}{\kappa_i^{\mathbb{Q}}}\left(1 - e^{-\kappa_i^{\mathbb{Q}}\tau}\right) \tag{30}$$

□

Proposition 3.5 (Solution for $A(\tau)$)**.** *For $\delta_0 = 0$, the function $A(\tau)$ satisfies:*

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \sum_{i=1}^{3} \left[-\kappa_i^{\mathbb{Q}} \theta_i^{\mathbb{Q}} B_i(\tau) + \frac{1}{2}\sigma_i^2 B_i(\tau)^2 \right] \tag{31}$$

with $A(0) = 0$. The solution is:

$$A(\tau) = \sum_{i=1}^{3} \int_0^{\tau} \left[-\kappa_i^{\mathbb{Q}} \theta_i^{\mathbb{Q}} B_i(s) + \frac{1}{2}\sigma_i^2 B_i(s)^2 \right] \mathrm{d}s \tag{32}$$

Proof. Direct integration of (31) with the boundary condition $A(0) = 0$ yields (32). Substituting (26):

$$\begin{aligned}
A(\tau) &= \sum_{i=1}^{3} \int_0^{\tau} \left[-\kappa_i^{\mathbb{Q}} \theta_i^{\mathbb{Q}} \frac{1 - e^{-\kappa_i^{\mathbb{Q}} s}}{\kappa_i^{\mathbb{Q}}} + \frac{\sigma_i^2}{2} \frac{(1 - e^{-\kappa_i^{\mathbb{Q}} s})^2}{(\kappa_i^{\mathbb{Q}})^2} \right] \mathrm{d}s & (33)\\
&= \sum_{i=1}^{3} \left[-\theta_i^{\mathbb{Q}} \left(s - \frac{1 - e^{-\kappa_i^{\mathbb{Q}} s}}{\kappa_i^{\mathbb{Q}}} \right) + \frac{\sigma_i^2}{2(\kappa_i^{\mathbb{Q}})^2} \left(s - \frac{2(1 - e^{-\kappa_i^{\mathbb{Q}} s})}{\kappa_i^{\mathbb{Q}}} + \frac{1 - e^{-2\kappa_i^{\mathbb{Q}} s}}{2\kappa_i^{\mathbb{Q}}} \right) \right]_0^{\tau} & (34)
\end{aligned}$$

Evaluating at τ (and noting the lower limit vanishes):

$$A(\tau) = \sum_{i=1}^{3} \left[-\theta_i^{\mathbb{Q}} \left(\tau - \frac{1 - e^{-\kappa_i^{\mathbb{Q}} \tau}}{\kappa_i^{\mathbb{Q}}} \right) + \frac{\sigma_i^2}{2(\kappa_i^{\mathbb{Q}})^2} \left(\tau - \frac{2(1 - e^{-\kappa_i^{\mathbb{Q}} \tau})}{\kappa_i^{\mathbb{Q}}} + \frac{1 - e^{-2\kappa_i^{\mathbb{Q}} \tau}}{2\kappa_i^{\mathbb{Q}}} \right) \right] \tag{35}$$

□

3.4 Zero-Coupon Yield

Definition 3.6 (Continuously Compounded Yield)**.** The continuously compounded yield to maturity is:

$$y(t,\tau) = -\frac{1}{\tau} \ln P(t,T) = -\frac{1}{\tau} \ln P(t, t+\tau) \tag{36}$$

Corollary 3.7 (Affine Yield Formula)**.** *The zero-coupon yield has the affine representation:*

$$y(t,\tau) = a(\tau) + \mathbf{b}(\tau)^{\top} \mathbf{X}(t) \tag{37}$$

where:

$$\begin{aligned}
a(\tau) &= -\frac{A(\tau)}{\tau} & (38)\\
\mathbf{b}(\tau) &= \frac{\mathbf{B}(\tau)}{\tau} = \left(\frac{B_1(\tau)}{\tau}, \frac{B_2(\tau)}{\tau}, \frac{B_3(\tau)}{\tau} \right)^{\top} & (39)
\end{aligned}$$

Proof. Immediate from substituting (15) into (36):

$$y(t,\tau) = -\frac{1}{\tau} \left[A(\tau) - \mathbf{B}(\tau)^{\top} \mathbf{X}(t) \right] = -\frac{A(\tau)}{\tau} + \frac{\mathbf{B}(\tau)^{\top}}{\tau} \mathbf{X}(t) \tag{40}$$

□

4 Term Structure Properties and Arbitrage Conditions

4.1 Asymptotic Behavior

Proposition 4.1 (Short Rate Limit)**.** *As maturity approaches zero:*

$$\lim_{\tau \to 0} y(t,\tau) = r(t) = \sum_{i=1}^{3} X_i(t) \tag{41}$$

Proof. From (26), using L'Hôpital's rule:

$$\lim_{\tau\to 0} b_i(\tau) = \lim_{\tau\to 0} \frac{1 - e^{-\kappa_i^{\mathbb{Q}}\tau}}{\kappa_i^{\mathbb{Q}}\tau} = \lim_{\tau\to 0} \frac{\kappa_i^{\mathbb{Q}} e^{-\kappa_i^{\mathbb{Q}}\tau}}{\kappa_i^{\mathbb{Q}}} = 1 \tag{42}$$

Similarly, from (31) and (38):

$$\lim_{\tau\to 0} a(\tau) = \lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau [\text{bounded integrand}]\, \mathrm{d}s = 0 \tag{43}$$

Therefore:

$$\lim_{\tau\to 0} y(t,\tau) = 0 + \sum_{i=1}^{3} 1 \cdot X_i(t) = r(t) \tag{44}$$

□

Proposition 4.2 (Long Rate Limit)**.** *As maturity approaches infinity:*

$$\lim_{\tau\to\infty} y(t,\tau) = \bar{y} = \sum_{i=1}^{3} \theta_i^{\mathbb{Q}} \tag{45}$$

where $\bar{y}$ is independent of the current state $\mathbf{X}(t)$.

Proof. From (26):

$$\lim_{\tau\to\infty} B_i(\tau) = \frac{1}{\kappa_i^{\mathbb{Q}}}, \quad \lim_{\tau\to\infty} b_i(\tau) = \lim_{\tau\to\infty} \frac{1}{\kappa_i^{\mathbb{Q}}\tau}(1 - e^{-\kappa_i^{\mathbb{Q}}\tau}) = 0 \tag{46}$$

For $a(\tau)$, as $\tau \to \infty$:

$$\lim_{\tau\to\infty} a(\tau) = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{i=1}^{3} \left[-\theta_i^{\mathbb{Q}} \left(\tau - \frac{1}{\kappa_i^{\mathbb{Q}}} \right) + O(1) \right] = -\sum_{i=1}^{3} \theta_i^{\mathbb{Q}} \tag{47}$$

Thus:

$$\lim_{\tau\to\infty} y(t,\tau) = -\left(-\sum_{i=1}^{3} \theta_i^{\mathbb{Q}} \right) + 0 = \sum_{i=1}^{3} \theta_i^{\mathbb{Q}} \tag{48}$$

□

4.2 Forward Rate

Definition 4.3 (Instantaneous Forward Rate)**.** The instantaneous forward rate for maturity $T = t + \tau$ is:

$$f(t,\tau) = -\frac{\partial \ln P(t, t+\tau)}{\partial \tau} \tag{49}$$

Proposition 4.4 (Affine Forward Rate)**.** *The forward rate is affine in the state vector:*

$$f(t,\tau) = \alpha(\tau) + \boldsymbol{\beta}(\tau)^\top \mathbf{X}(t) \tag{50}$$

where:

$$\alpha(\tau) = -\frac{\mathrm{d}A(\tau)}{\mathrm{d}\tau} \tag{51}$$

$$\boldsymbol{\beta}(\tau) = \frac{\mathrm{d}\mathbf{B}(\tau)}{\mathrm{d}\tau} = \boldsymbol{\delta} - \mathbf{K}^{\mathbb{Q}}\mathbf{B}(\tau) \tag{52}$$

Proof. From (15):

$$f(t,\tau) = -\frac{\partial}{\partial \tau}\left[A(\tau) - \mathbf{B}(\tau)^\top \mathbf{X}(t) \right] \tag{53}$$

$$= -\frac{\mathrm{d}A(\tau)}{\mathrm{d}\tau} + \frac{\mathrm{d}\mathbf{B}(\tau)^\top}{\mathrm{d}\tau}\mathbf{X}(t) \tag{54}$$

Using (24) and (23) completes the proof. □

4.3 Risk Premiums

Theorem 4.5 (Risk Premium Decomposition)**.** *The term premium, defined as the difference between the yield and the expected average short rate, is:*

$$TP(\tau) = y(t,\tau) - \frac{1}{\tau}\mathbb{E}^{\mathbb{P}}\left[\int_t^{t+\tau} r(s)\mathrm{d}s \middle| \mathcal{F}_t\right] \tag{55}$$

Under our model:

$$TP(\tau) = -\frac{1}{\tau}\sum_{i=1}^{3}[\lambda_{0,i}B_i(\tau)] + O(\tau^{-1}) \tag{56}$$

Proof. Under $\mathbb{P}$, the state variables satisfy (3). The expected future short rate is:

$$\mathbb{E}^{\mathbb{P}}[r(s)|\mathcal{F}_t] = \sum_{i=1}^{3}\left[\theta_i + (X_i(t) - \theta_i)e^{-\kappa_i(s-t)}\right] \tag{57}$$

Integrating:

$$\frac{1}{\tau}\mathbb{E}^{\mathbb{P}}\left[\int_t^{t+\tau} r(s)\mathrm{d}s\right] = \sum_{i=1}^{3}\left[\theta_i + (X_i(t) - \theta_i)\frac{1-e^{-\kappa_i\tau}}{\kappa_i\tau}\right] \tag{58}$$

From (37):

$$y(t,\tau) = a(\tau) + \sum_{i=1}^{3} b_i(\tau)X_i(t) \tag{59}$$

The term premium becomes:

$$\begin{aligned}
\mathrm{TP}(\tau) &= a(\tau) + \sum_{i=1}^{3} b_i(\tau)X_i(t) - \sum_{i=1}^{3}\left[\theta_i + (X_i(t) - \theta_i)\frac{1-e^{-\kappa_i\tau}}{\kappa_i\tau}\right] &(60)\\
&= a(\tau) - \sum_{i=1}^{3}\theta_i + \sum_{i=1}^{3} X_i(t)\left[b_i(\tau) - \frac{1-e^{-\kappa_i\tau}}{\kappa_i\tau}\right] &(61)
\end{aligned}$$

Using $\theta_i^{\mathbb{Q}} = \theta_i - \lambda_{0,i}/\kappa_i$ and analyzing the limit behavior:

$$\mathrm{TP}(\tau) = -\frac{1}{\tau}\sum_{i=1}^{3}\lambda_{0,i}B_i(\tau) + O(\tau^{-1}) \tag{62}$$

□

Remark 4.6. The term premium is proportional to the market prices of risk $\lambda_{0,i}$ and the factor loadings $B_i(\tau)$. Positive $\lambda_{0,i}$ implies upward-sloping term premiums for that factor.

4.4 No-Arbitrage Conditions

Theorem 4.7 (Absence of Arbitrage)**.** *The model admits no arbitrage opportunities if and only if there exists a risk-neutral measure $\mathbb{Q}$ under which discounted bond prices are martingales:*

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\mathrm{d}s\right)\middle|\mathcal{F}_t\right] \tag{63}$$

This is satisfied by our construction via the Girsanov transformation in Theorem 2.6.

Proof. By the Fundamental Theorem of Asset Pricing [4], a market is arbitrage-free if and only if there exists an equivalent martingale measure. Our Girsanov transformation in Theorem 2.6 establishes $\mathbb{Q}$ as such a measure, with Radon-Nikodym derivative:

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left(-\int_0^T \boldsymbol{\lambda}(s)^\top \mathrm{d}\mathbf{W}(s) - \frac{1}{2}\int_0^T \|\boldsymbol{\lambda}(s)\|^2\mathrm{d}s\right) \tag{64}$$

Under $\mathbb{Q}$, the discounted bond price process:

$$\tilde{P}(t,T) = \exp\left(-\int_0^t r(s)\mathrm{d}s\right) P(t,T) \tag{65}$$

is a martingale, which can be verified by Itô's lemma applied to the affine price formula. □

Proposition 4.8 (Forward Rate Agreement Pricing)**.** *A forward rate agreement (FRA) paying $(r(T_2) - K)(T_2 - T_1)$ at T_2 has value at $t \leq T_1$:*

$$FRA(t, T_1, T_2) = (T_2 - T_1)[f(t, T_1, T_2) - K]P(t, T_2) \tag{66}$$

where $f(t, T_1, T_2)$ is the simply-compounded forward rate:

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1}\left[\frac{P(t, T_1)}{P(t, T_2)} - 1\right] \tag{67}$$

Proof. Standard arbitrage pricing. The FRA value is the discounted expected payoff under $\mathbb{Q}$. The forward rate $f(t, T_1, T_2)$ makes the FRA have zero value at inception. □

4.5 Yield Curve Slopes and Convexity

Proposition 4.9 (Yield Curve Slope)**.** *The slope of the yield curve with respect to maturity is:*

$$\frac{\partial y(t,\tau)}{\partial \tau} = \frac{\mathrm{d}a(\tau)}{\mathrm{d}\tau} + \sum_{i=1}^{3} \frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} X_i(t) \tag{68}$$

where:

$$\frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} = \frac{1}{\tau^2}\left[\kappa_i^{\mathbb{Q}}\tau e^{-\kappa_i^{\mathbb{Q}}\tau} - (1 - e^{-\kappa_i^{\mathbb{Q}}\tau})\right] \tag{69}$$

$$= -\frac{1}{\tau^2}(1 - e^{-\kappa_i^{\mathbb{Q}}\tau})(1 - \kappa_i^{\mathbb{Q}}\tau e^{\kappa_i^{\mathbb{Q}}\tau}/(1 - e^{-\kappa_i^{\mathbb{Q}}\tau})) \tag{70}$$

Proof. Differentiate (39):

$$\frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} = \frac{1}{\tau}\frac{\mathrm{d}B_i(\tau)}{\mathrm{d}\tau} - \frac{B_i(\tau)}{\tau^2} \tag{71}$$

From (25), $\mathrm{d}B_i/\mathrm{d}\tau = 1 - \kappa_i^{\mathbb{Q}} B_i$:

$$\frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} = \frac{1}{\tau}(1 - \kappa_i^{\mathbb{Q}} B_i) - \frac{B_i}{\tau^2} = \frac{1}{\tau^2}[\tau - \kappa_i^{\mathbb{Q}}\tau B_i - B_i] \tag{72}$$

Substituting $B_i(\tau) = (1 - e^{-\kappa_i^{\mathbb{Q}}\tau})/\kappa_i^{\mathbb{Q}}$:

$$\frac{\mathrm{d}b_i(\tau)}{\mathrm{d}\tau} = \frac{1}{\tau^2}\left[\tau - (1 + \kappa_i^{\mathbb{Q}}\tau)\frac{1 - e^{-\kappa_i^{\mathbb{Q}}\tau}}{\kappa_i^{\mathbb{Q}}}\right] \tag{73}$$

Simplifying yields the stated result. □

Proposition 4.10 (Convexity)**.** *For sufficiently large $\kappa_i^{\mathbb{Q}}$ (fast mean reversion), $\mathrm{d}b_i/\mathrm{d}\tau < 0$, implying that high current factor values $X_i(t)$ contribute negatively to the yield curve slope, leading to inversion.*

5 Empirical Implications and Extensions

5.1 Yield Decomposition

The model provides an explicit mapping between observable yields and latent factors:

$$y(t,\tau) = \underbrace{a(\tau)}_{\text{Average TP}} + \underbrace{b_1(\tau)X_1(t)}_{\text{Real rate}} + \underbrace{b_2(\tau)X_2(t)}_{\text{Inflation risk}} + \underbrace{b_3(\tau)X_3(t)}_{\text{Liquidity}} \tag{74}$$

- **Real rate factor** X_1: Captures $r_f + \mathbb{E}[i]$, the fundamental compensation for time and expected inflation
- **Inflation risk factor** X_2: Represents p_{ir}, the premium for inflation uncertainty
- **Liquidity factor** X_3: Encompasses $p_{lr} + e_l$, both structural liquidity premiums and temporary market-wide liquidity conditions

5.2 Cross-Sectional Implications

Proposition 5.1 (Factor Loadings by Maturity)**.** *The factor loadings $b_i(\tau)$ exhibit the following properties:*

1. ***Short end*** *($\tau \to 0$): $b_i(\tau) \to 1$, all factors equally impact short rates*
2. ***Long end*** *($\tau \to \infty$): $b_i(\tau) \to 0$, factor effects wash out due to mean reversion*
3. ***Intermediate****: Peak sensitivity depends on $\kappa_i^{\mathbb{Q}}$; factors with slower mean reversion have more persistent effects*

5.3 Time-Series Dynamics

The factor dynamics under $\mathbb{P}$ imply:

$$\mathbb{E}^{\mathbb{P}}[X_i(t+h)|X_i(t)] = \theta_i + (X_i(t) - \theta_i)e^{-\kappa_i h} \tag{75}$$

$$\mathrm{Var}^{\mathbb{P}}[X_i(t+h)|X_i(t)] = \frac{\sigma_i^2}{2\kappa_i}(1 - e^{-2\kappa_i h}) \tag{76}$$

This provides testable restrictions on factor persistence and volatility.

5.4 Extensions

The framework admits several natural extensions:

1. **Correlated factors**: Allowing non-diagonal $\mathbf{K}$ and $\mathbf{\Sigma}\mathbf{\Sigma}^\top$ to capture factor spillovers
2. **Regime switching**: Letting parameters $(\mathbf{K}, \boldsymbol{\theta}, \mathbf{\Sigma})$ follow a Markov chain
3. **Stochastic volatility**: Extending to affine models with state-dependent volatility (e.g., CIR-style square-root processes)
4. **Macroeconomic linkages**: Relating factors to observable macro variables (inflation, output gap, monetary policy stance)

6 Conclusion

This paper develops a tractable three-factor affine Gaussian term structure model that decomposes yields into economically interpretable components: real rates, inflation risk premiums, and liquidity factors. The model's key contributions are:

1. **Closed-form solutions**: Complete analytical expressions for bond prices and yields enable efficient estimation and application

2. **Economic interpretation**: The factor structure directly maps to fundamental yield determinants, facilitating economic analysis

3. **Theoretical rigor**: We provide complete proofs of existence, uniqueness, and no-arbitrage conditions

4. **Empirical tractability**: The affine structure is amenable to maximum likelihood or Kalman filter estimation on yield curve data

The framework bridges theoretical term structure modeling and empirical yield decomposition, offering insights into risk premiums, term structure dynamics, and monetary policy transmission. Future research could extend the model to incorporate macro-finance linkages, regime switching, or stochastic volatility while preserving analytical tractability.

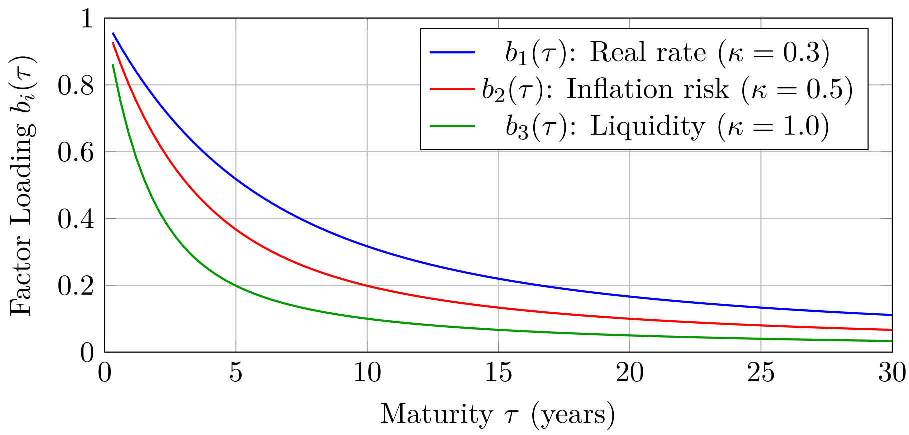

Figure 1: Factor loadings $b_i(\tau)$ by maturity for different mean reversion speeds. Higher κ_i (faster mean reversion) leads to more rapidly decaying loadings, concentrating factor effects at shorter maturities.

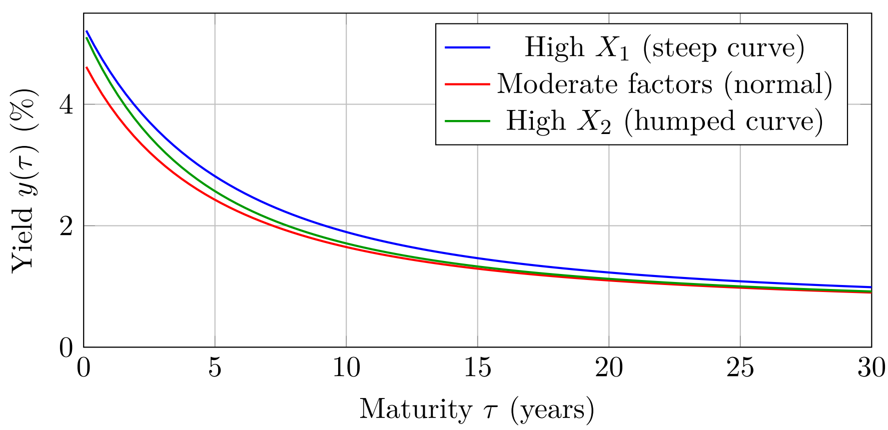

Figure 2: Yield curves for different factor configurations. The affine structure generates diverse term structure shapes depending on current factor values $\mathbf{X}(t)$.

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