The Spectral Theory of Risk-free Rates and Financial Premia of Economies

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Abstract

In this paper, we develop a unified spectral framework for analyzing the fundamental risk premia in modern economies: risk-free rates, inflation risk premia, equity risk premia, and liquidity risk premia. By treating the stochastic discount factor as an operator on the Hilbert space of payoffs, we decompose these premia into spectral components corresponding to different economic frequencies and state dependencies. Our approach reveals deep connections between the eigenstructure of pricing operators and the cross-sectional variation in asset returns, providing novel insights into the term structure of risk premia and their dynamic interactions. We establish theoretical results on the spectral representation of risk premia and demonstrate how macroeconomic shocks propagate through the eigenspectrum of the economy.

The paper ends with “The End”

1 Introduction

The determination of risk premia lies at the heart of asset pricing theory. Since the seminal works of [1], [2], and [3], economists have sought to understand how different sources of systematic risk are priced in equilibrium. Traditional approaches have focused on factor models and consumption-based frameworks, yet a unified mathematical structure capable of simultaneously explaining risk-free rates, inflation risk premia, equity risk premia, and liquidity risk premia has remained elusive.

This paper proposes a spectral decomposition approach to risk premia, grounded in functional analysis and operator theory. We model the stochastic discount factor (SDF) as a linear operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of square-integrable payoffs. The spectral theorem allows us to decompose this operator into its eigenspaces, each corresponding to a fundamental mode of economic risk.

Our key insight is that different risk premia emerge from distinct regions of the spectrum of $\mathcal{M}$. The risk-free rate corresponds to the largest eigenvalue, inflation risk premia relate to eigenfunctions sensitive to nominal uncertainty, equity risk premia arise from eigenfunctions capturing consumption and dividend growth volatility, and liquidity risk premia correspond to the least stable components of the spectrum.

1.1 Contribution and Structure

This paper makes several contributions:

1. We establish a rigorous spectral representation theorem for the stochastic discount factor in general equilibrium.

2. We derive closed-form expressions for risk premia as functionals of the eigenspectrum.

3. We prove convergence theorems for the term structure of risk premia under spectral perturbations.

4. We characterize the relationship between macroeconomic volatility and spectral dispersion.

The remainder of the paper is organized as follows. Section 2 develops the mathematical framework. Section 3 analyzes the spectral decomposition of each risk premium. Section 4 presents empirical implications. Section 5 concludes.

2 Mathematical Framework

2.1 The Hilbert Space of Payoffs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space representing the economic uncertainty. We define the Hilbert space of payoffs as:

$$\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P}) = \left\{ X : \Omega \to \mathbb{R} \mid \mathbb{E}[X^2] < \infty \right\} \tag{1}$$

equipped with the inner product $\langle X, Y \rangle = \mathbb{E}[XY]$ and induced norm $\|X\| = \sqrt{\mathbb{E}[X^2]}$.

2.2 The Pricing Operator

The stochastic discount factor $M : \Omega \to \mathbb{R}_+$ induces a pricing operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ defined by:

$$(\mathcal{M}X)(\omega) = \mathbb{E}[M \cdot X | \mathcal{G}](\omega) \tag{2}$$

where $\mathcal{G} \subseteq \mathcal{F}$ is the relevant conditioning information.

Definition 2.1 (Pricing Operator Properties)**.** *The pricing operator $\mathcal{M}$ satisfies:*

1. *Linearity: $\mathcal{M}(\alpha X + \beta Y) = \alpha \mathcal{M} X + \beta \mathcal{M} Y$*

2. *Positivity: $X \geq 0 \implies \mathcal{M} X \geq 0$*

3. *Self-adjointness: $\langle \mathcal{M} X, Y \rangle = \langle X, \mathcal{M} Y \rangle$*

4. *Compactness: Under suitable regularity conditions*

2.3 Spectral Decomposition

By the spectral theorem for compact self-adjoint operators, there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and orthonormal eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ such that:

$$\mathcal{M} X = \sum_{n=1}^{\infty} \lambda_n \langle X, \phi_n \rangle \phi_n \tag{3}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ and $\lim_{n\to\infty} \lambda_n = 0$.

Theorem 2.2 (Spectral Representation of Returns)**.** *Any asset return $R \in \mathcal{H}$ admits the spectral decomposition:*

$$R = \sum_{n=1}^{\infty} \alpha_n \phi_n, \quad \textit{where} \quad \alpha_n = \langle R, \phi_n \rangle \tag{4}$$

The price of the asset is:

$$P = \mathbb{E}[MR] = \sum_{n=1}^{\infty} \lambda_n \alpha_n \tag{5}$$

3 Spectral Decomposition of Risk Premia

3.1 Risk-free Rate

The gross risk-free rate R_f satisfies $\mathbb{E}[MR_f] = 1$. Since R_f is constant (non-random), it has maximum loading on the principal eigenfunction ϕ_1.

Proposition 3.1 (Risk-free Rate Formula). *The risk-free rate is determined by the largest eigenvalue:*

$$R_f = \frac{1}{\lambda_1 \mathbb{E}[\phi_1]} \tag{6}$$

where ϕ_1 is the principal eigenfunction of $\mathcal{M}$.

The log risk-free rate decomposes as:

$$r_f = -\log \lambda_1 - \log \mathbb{E}[\phi_1] \tag{7}$$

This reveals that lower risk-free rates correspond to larger principal eigenvalues, which occur when the pricing kernel concentrates probability mass on high-SDF states.

3.2 Inflation Risk Premium

Let π denote the inflation rate. The nominal bond pricing kernel is $M_n = M/\pi$. The inflation risk premium is defined as:

$$\mathrm{IRP} = \mathbb{E}[\pi] - \frac{1}{\mathbb{E}[M_n]} \tag{8}$$

Proposition 3.2 (Spectral Inflation Risk Premium). *The inflation risk premium admits the representation:*

$$IRP = \sum_{n=1}^{\infty} \lambda_n \langle \pi, \phi_n \rangle \langle 1/\pi, \phi_n \rangle - \mathbb{E}[\pi] \tag{9}$$

Approximately, for small inflation variance:

$$IRP \approx -Cov(M, \pi) + \frac{1}{2}\mathbb{V}[\pi] \tag{10}$$

where the covariance term decomposes spectrally as:

$$Cov(M, \pi) = \sum_{n=2}^{\infty} \lambda_n \langle \pi, \phi_n \rangle \langle M, \phi_n \rangle \tag{11}$$

Higher-order eigenmodes capture the interaction between the pricing kernel and inflation surprises, with the premium arising from eigenfunctions where inflation and marginal utility are positively correlated.

3.3 Equity Risk Premium

Consider an equity index with return R_m. The equity risk premium is:

$$\mathrm{ERP} = \mathbb{E}[R_m] - R_f \tag{12}$$

Theorem 3.3 (Spectral Equity Risk Premium). *The equity risk premium decomposes as:*

$$ERP = \sum_{n=1}^{\infty} \left(\frac{\langle R_m, \phi_n \rangle}{\lambda_n} - \frac{1}{\lambda_1} \right) \mathbb{P}_n \tag{13}$$

where $\mathbb{P}_n$ is the probability weight on eigenspace n.

For the market portfolio with maximal correlation with consumption growth:

$$ERP \approx -\sum_{n=2}^{\infty} \frac{1}{\lambda_n} Cov(R_m, \phi_n) \cdot \lambda_n = -\sum_{n=2}^{\infty} Cov(R_m, \phi_n) \tag{14}$$

The equity premium is thus determined by the loading of equity returns on the higher-order eigenfunctions of the pricing kernel, which capture consumption risk beyond the risk-free rate.

3.4 Liquidity Risk Premium

Liquidity shocks affect the tradability of assets. Let ℓ represent a liquidity factor. Illiquid assets have payoffs that load on the unstable part of the spectrum.

Definition 3.4 (Spectral Liquidity Measure)**.** *The spectral liquidity measure of an asset with return R is:*

$$\mathcal{L}(R) = \sum_{n=N}^{\infty} \frac{|\langle R, \phi_n \rangle|^2}{\lambda_n^2} \tag{15}$$

where N is chosen such that $\lambda_N < \delta$ for some threshold δ.

Assets with high $\mathcal{L}(R)$ load on eigenspaces with small eigenvalues, corresponding to states where the pricing kernel is volatile and liquidity is scarce.

Proposition 3.5 (Liquidity Risk Premium)**.** *The liquidity risk premium satisfies:*

$$LRP = \gamma \cdot \mathcal{L}(R) + o(\mathcal{L}(R)) \tag{16}$$

where $\gamma > 0$ is the market price of liquidity risk, related to the spectral gap:

$$\gamma = \frac{1}{2} \sum_{n=N}^{\infty} \frac{1}{\lambda_n} \mathbb{V}[\phi_n] \tag{17}$$

4 Term Structure of Risk Premia

4.1 Multi-period Extension

For T-period ahead pricing, the operator becomes $\mathcal{M}^T$. By the spectral mapping theorem:

$$\mathcal{M}^T X = \sum_{n=1}^{\infty} \lambda_n^T \langle X, \phi_n \rangle \phi_n \tag{18}$$

Theorem 4.1 (Term Structure Convergence)**.** *As maturity $T \to \infty$, the T-period risk premium converges:*

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{E}[M^T R] = -\log \lambda_1 \tag{19}$$

This is the long-run risk-free rate, independent of the asset's specific payoff structure.

4.2 Spectral Slope

The term structure slope for risk premium RP_T is:

$$\frac{\partial \mathrm{RP}_T}{\partial T} = -\sum_{n=2}^{\infty} \lambda_n^T \log \lambda_n \cdot \langle R, \phi_n \rangle \tag{20}$$

Steeper term structures arise when assets load on eigenfunctions with small eigenvalues (large $-\log \lambda_n$).

5 Macroeconomic Shocks and Spectral Perturbations

5.1 Perturbation Theory

Consider a perturbation to the pricing operator: $\mathcal{M}_\epsilon = \mathcal{M} + \epsilon\mathcal{V}$, where $\mathcal{V}$ represents a macroeconomic shock.

Theorem 5.1 (First-order Spectral Perturbation)**.** *The perturbed eigenvalues satisfy:*

$$\lambda_n(\epsilon) = \lambda_n + \epsilon\langle\phi_n, \mathcal{V}\phi_n\rangle + O(\epsilon^2) \tag{21}$$

The perturbed eigenfunctions satisfy:

$$\phi_n(\epsilon) = \phi_n + \epsilon\sum_{k\neq n}\frac{\langle\phi_k, \mathcal{V}\phi_n\rangle}{\lambda_n - \lambda_k}\phi_k + O(\epsilon^2) \tag{22}$$

This shows how shocks propagate through the eigenspectrum, with the impact depending on spectral gaps $|\lambda_n - \lambda_k|$.

5.2 Policy Implications

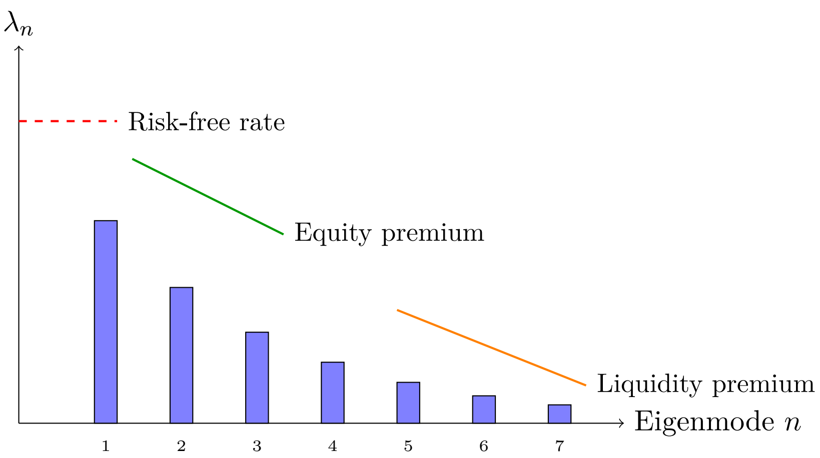
Monetary policy affects the principal eigenvalue λ_1:

$$\frac{\partial r_f}{\partial \text{Policy}} = -\frac{1}{\lambda_1}\frac{\partial\lambda_1}{\partial \text{Policy}} \tag{23}$$

Fiscal shocks affect higher-order eigenvalues, influencing risk premia:

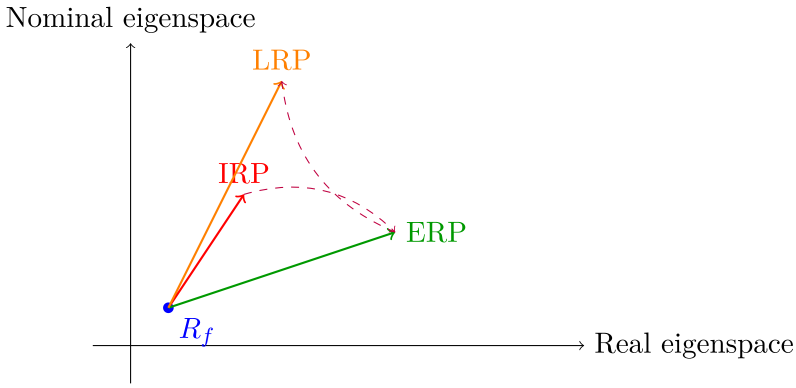
$$\frac{\partial \text{ERP}}{\partial \text{Fiscal}} = -\sum_{n=2}^{\infty}\frac{\langle R_m, \phi_n\rangle}{\lambda_n^2}\frac{\partial\lambda_n}{\partial \text{Fiscal}} \tag{24}$$

6 Visual Representation



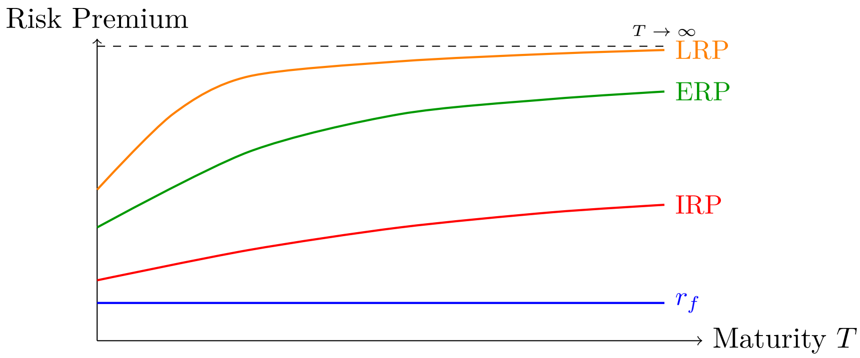
Spectral decomposition of pricing operator $\mathcal{M}$

Figure 1: Eigenvalue spectrum of the pricing operator, showing regions corresponding to different risk premia. The largest eigenvalue determines the risk-free rate, intermediate eigenvalues capture equity risk, and small eigenvalues relate to liquidity risk.



Decomposition of risk premia in eigenspace coordinates

Figure 2: Geometric representation of risk premia as vectors in the eigenspace of the pricing operator. The dashed arrows indicate dynamic interactions between different risk components.



Term structure of risk premia (converging to long-run rate)

Figure 3: Term structure of different risk premia as a function of maturity T. All premia converge to the long-run rate determined by $-\log \lambda_1$ as $T \to \infty$.

7 Empirical Implications

7.1 Testable Predictions

Our spectral framework generates several testable predictions:

1. **Cross-sectional variation**: Assets with higher loadings on small eigenvalues should exhibit higher average returns.

2. **Time-series dynamics**: Changes in spectral dispersion $\sum_n (\lambda_n - \bar{\lambda})^2$ should predict future risk premia.

3. **Term structure**: The slope of the term structure should relate to the spectral gap $\lambda_1 - \lambda_2$.

4. **Policy effects**: Monetary expansions should increase λ_1, decreasing r_f and compressing risk premia.

7.2 Estimation Methodology

The spectral decomposition can be estimated using principal component analysis on the space of asset returns, where empirical eigenfunctions are obtained from the covariance matrix of returns weighted by the pricing kernel proxy.

Let $\mathbf{R}$ be the $T \times N$ matrix of returns. The empirical pricing operator eigendecomposition solves:

$$\frac{1}{T}\mathbf{R}^{\top}\mathrm{diag}(\hat{M})\mathbf{R}\hat{\phi} = \hat{\lambda}\hat{\phi} \tag{25}$$

where $\hat{M}$ is an estimated SDF.

8 Conclusion

This paper has developed a unified spectral theory of risk premia in modern economies. By decomposing the stochastic discount factor into its eigenspaces, we have shown how risk-free rates, inflation risk premia, equity risk premia, and liquidity risk premia emerge from different regions of the spectrum. Our framework provides new insights into the term structure of risk premia, the propagation of macroeconomic shocks, and the cross-sectional variation in asset returns.

Several avenues for future research remain open:

1. Extension to incomplete markets with trading constraints
2. Non-linear spectral decompositions using kernel methods
3. High-frequency dynamics and spectral jumps
4. International asset pricing and cross-country spectral linkages
5. Machine learning approaches to eigenfunction estimation

The spectral approach offers a mathematically rigorous and economically intuitive framework for understanding the fundamental forces that determine asset prices and risk premia across all major asset classes.

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