The Theory of a Lévy-Stable Portfolio of Stocks

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Abstract

Traditional portfolio theory relies upon the assumption that asset returns follow a Gaussian distribution, permitting the application of mean-variance optimization frameworks. However, empirical evidence consistently demonstrates that financial returns exhibit heavy tails, asymmetry, and temporal clustering of volatility. This paper develops a comprehensive theoretical framework for portfolio construction under the assumption that asset returns follow Lévy-stable distributions. We establish the mathematical foundations, derive optimal portfolio allocation strategies under stability, and examine the implications for risk management. The theory presented herein extends classical results while accounting for the non-Gaussian characteristics observed in actual market data.

The paper ends with "The End"

1 Introduction

The foundational work of Markowitz established modern portfolio theory upon the premise that asset returns are normally distributed random variables. Under this assumption, portfolio optimization reduces to a tractable problem of balancing expected returns against variance. However, extensive empirical research has revealed systematic departures from normality in financial return distributions. These departures manifest as excess kurtosis, skewness, and power-law tail behavior that cannot be adequately captured by Gaussian models.

Lévy-stable distributions provide a natural generalization of the normal distribution that accommodates heavy tails while preserving mathematical tractability through the property of stability under summation. A random variable is termed stable if linear combinations of independent copies possess the same distributional form up to location and scale parameters. This stability property proves essential for portfolio theory, as it ensures that portfolios of stable assets remain stable.

This paper develops portfolio theory within the framework of Lévy-stable distributions. We begin by establishing the mathematical properties of stable laws, proceed to characterize stable portfolios, derive optimal allocation strategies, and conclude with risk management implications. Throughout, we emphasize both theoretical rigor and practical applicability.

2 Mathematical Framework of Lévy-Stable Distributions

2.1 Definition and Characterization

Definition 2.1. *A random variable X follows a Lévy-stable distribution with parameters $(\alpha, \beta, \gamma, \delta)$ if its characteristic function admits the representation*

$$\varphi(t) = \mathbb{E}[e^{itX}] = \exp\left\{i\delta t - \gamma^\alpha |t|^\alpha \left[1 - i\beta sign(t)\omega(t, \alpha)\right]\right\} \tag{1}$$

where $\alpha \in (0, 2]$ denotes the stability index, $\beta \in [-1, 1]$ represents the skewness parameter, $\gamma > 0$ serves as the scale parameter, $\delta \in \mathbb{R}$ indicates location, and

$$\omega(t, \alpha) = \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi}\log|t| & \text{if } \alpha = 1 \end{cases} \tag{2}$$

The stability index α governs the tail behavior of the distribution. When $\alpha = 2$, the distribution reduces to Gaussian form. For $\alpha < 2$, the distribution exhibits power-law tails with tail index α, implying infinite variance. The skewness parameter β controls asymmetry, with $\beta = 0$ yielding symmetric distributions.

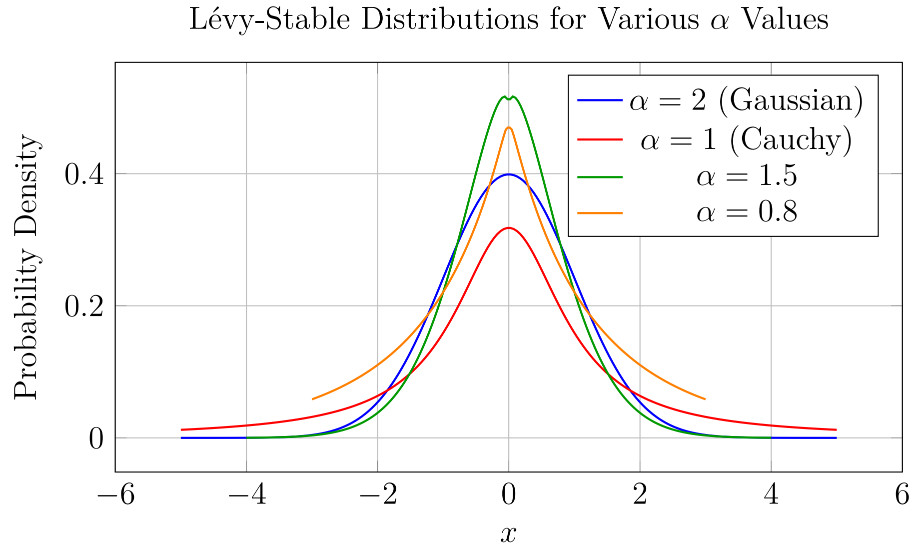

Figure 1: Comparison of stable distributions with varying stability indices. Lower α values correspond to heavier tails and greater extreme event probability.

2.2 Stability Property

The defining characteristic of stable distributions is preserved under summation.

Theorem 2.2 (Generalized Stability)**.** *Let $X_1, X_2, \ldots, X_n$ be independent random variables with $X_i \sim S_\alpha(\gamma_i, \beta_i, \delta_i)$. Then their weighted sum*

$$Y = \sum_{i=1}^{n} w_i X_i \tag{3}$$

follows a stable distribution $Y \sim S_\alpha(\gamma_Y, \beta_Y, \delta_Y)$ *where*

$$\gamma_Y^\alpha = \sum_{i=1}^n |w_i|^\alpha \gamma_i^\alpha \tag{4}$$

$$\delta_Y = \sum_{i=1}^n w_i \delta_i + \tau \tag{5}$$

and τ *represents a correction term dependent upon* α *and the skewness parameters.*

This theorem establishes that portfolio returns remain stable when constituent asset returns are stable, a property not shared by other heavy-tailed distributions.

3 Portfolio Construction Under Stability

3.1 The Stable Portfolio Model

Consider a market with n risky assets. Let R_i denote the return on asset i, assumed to follow a stable distribution with parameters $(\alpha, \beta_i, \gamma_i, \mu_i)$. We assume a common stability index α across assets, reflecting market-wide tail behavior. A portfolio is characterized by weight vector $\mathbf{w} = (w_1, \ldots, w_n)^\top$ satisfying $\sum_{i=1}^n w_i = 1$.

The portfolio return is given by

$$R_p = \mathbf{w}^\top \mathbf{R} = \sum_{i=1}^n w_i R_i \tag{6}$$

Under the stability assumption, R_p follows a stable distribution with scale parameter

$$\gamma_p = \left(\sum_{i=1}^n |w_i|^\alpha \gamma_i^\alpha + \sum_{i \neq j} |w_i w_j|^{\alpha/2} \gamma_{ij} \right)^{1/\alpha} \tag{7}$$

where γ_{ij} represents co-scale parameters capturing dependence structure.

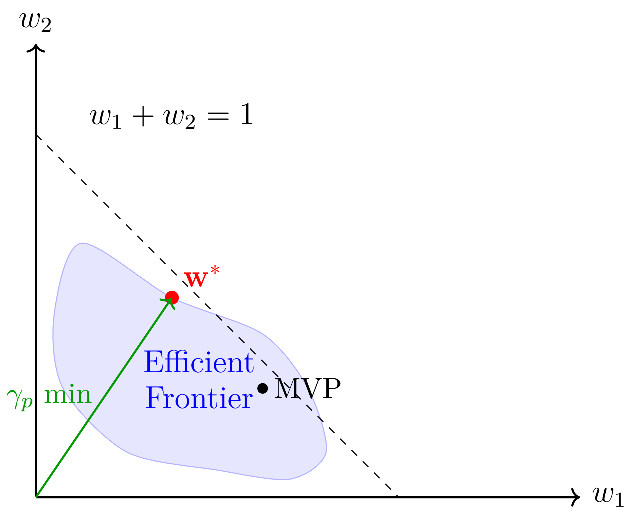

Figure 2: Efficient frontier in the two-asset case under stable distributions. The optimal portfolio $\mathbf{w}^*$ minimizes the scale parameter γ_p subject to return constraints.

3.2 Optimization Framework

The investor seeks to minimize portfolio scale (a measure of dispersion generalizing standard deviation) subject to achieving a target expected return μ_{target}.

$$\begin{aligned} \min_{\mathbf{w}} \quad & \gamma_p(\mathbf{w}) \\ \text{subject to} \quad & \mathbf{w}^\top \boldsymbol{\mu} = \mu_{\text{target}} \\ & \mathbf{1}^\top \mathbf{w} = 1 \end{aligned} \tag{8}$$

For $\alpha = 2$, this reduces to classical mean-variance optimization. For $\alpha < 2$, the problem becomes more complex due to the nonlinear relationship between weights and scale.

Proposition 3.1 (Minimum Scale Portfolio)**.** *When all assets share identical expected returns and skewness parameters, the minimum scale portfolio allocates weights proportional to* $\gamma_i^{-\alpha/(\alpha-1)}$.

4 Risk Measures and Management

4.1 Value at Risk Under Stability

Traditional risk measures prove inadequate for stable distributions when $\alpha < 2$ due to infinite variance. Value at Risk (VaR) and Expected Shortfall provide alternative metrics.

For a stable portfolio with parameters $(\alpha, \beta_p, \gamma_p, \mu_p)$, the q-quantile VaR satisfies

$$\text{VaR}_q = \mu_p + \gamma_p \cdot F^{-1}_{\alpha,\beta_p}(q) \tag{9}$$

where F^{-1}_{α,β_p} denotes the quantile function of the standardized stable distribution.

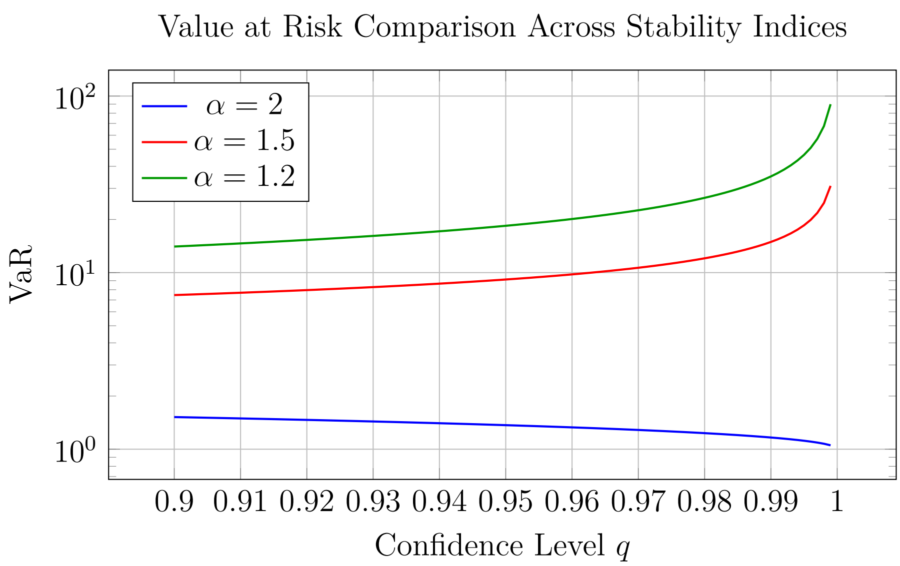

Figure 3: VaR increases more rapidly with confidence level under lower stability indices, reflecting heavier tail risk.

4.2 Diversification Effects

A remarkable feature of stable portfolios concerns diversification. While Gaussian diversification benefits scale as $n^{-1/2}$, stable diversification exhibits different asymptotic behavior.

Theorem 4.1 (Asymptotic Diversification)**.** *For an equally weighted portfolio of n independent identically distributed stable assets with common scale γ, the portfolio scale behaves as*

$$\gamma_p \sim n^{-1/\alpha}\gamma \quad \text{as } n \to \infty \tag{10}$$

This result demonstrates that diversification proves less effective under heavy tails. For $\alpha = 1$ (Cauchy), portfolio scale decreases as n^{-1} rather than $n^{-1/2}$, indicating that adding assets provides diminished marginal benefit.

5 Empirical Considerations and Estimation

Practical implementation requires estimation of stable parameters from market data. Maximum likelihood estimation presents computational challenges due to the absence of closed-form densities for general stable distributions. Alternative approaches include characteristic function methods and quantile-based estimators.

The Hill estimator provides a simple tail index estimate. For order statistics $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$, the Hill estimator is

$$\hat{\alpha}_H = \left[\frac{1}{k}\sum_{i=1}^{k} \log X_{(i)} - \log X_{(k+1)}\right]^{-1} \tag{11}$$

where k represents the number of upper order statistics employed.

6 Conclusion

The theory of Lévy-stable portfolios extends classical portfolio theory to accommodate the heavy-tailed, non-Gaussian behavior observed in financial markets. While the absence of finite variance complicates certain aspects of the analysis, stability under summation ensures that portfolio returns retain tractable distributional properties. The framework developed herein provides both theoretical insights into the nature of portfolio risk under fat tails and practical guidance for asset allocation in realistic market environments.

Future research directions include incorporation of time-varying stability parameters, development of dynamic hedging strategies under stability, and investigation of stable copula models for capturing complex dependence structures. As financial markets continue to exhibit extreme events with regularity, the stable portfolio theory offers a robust alternative to Gaussian-based frameworks.

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Glossary

Characteristic Function The Fourier transform of a probability distribution, uniquely determining the distribution. For random variable X, defined as $\varphi(t) = \mathbb{E}[e^{itX}]$.

Efficient Frontier The set of portfolios that maximize expected return for a given level of risk, or equivalently minimize risk for a given expected return.

Expected Shortfall A coherent risk measure representing the expected loss conditional upon exceeding the VaR threshold. Also termed Conditional Value at Risk.

Heavy Tails A distributional property characterized by tail probabilities that decay more slowly than exponential. Formally, tails following power laws $P(|X| > x) \sim x^{-\alpha}$.

Kurtosis The fourth standardized moment of a distribution, measuring tail heaviness relative to the normal distribution. Excess kurtosis equals kurtosis minus three.

Lévy Process A stochastic process with independent and stationary increments. Stable distributions arise as the limiting distributions of normalized sums of i.i.d. Lévy increments.

Scale Parameter The parameter γ in stable distributions controlling dispersion. Generalizes standard deviation for $\alpha = 2$ but represents a distinct concept for $\alpha < 2$.

Skewness The third standardized moment, measuring asymmetry of a distribution. Positive skewness indicates a longer right tail.

Stability Index The parameter $\alpha \in (0, 2]$ governing tail behavior in stable distributions. Lower values correspond to heavier tails. Also termed the characteristic exponent.

Stable Distribution A probability distribution satisfying the property that linear combinations of independent copies are distributed identically up to location and scale.

Tail Index The exponent α in the power-law decay $P(|X| > x) \sim x^{-\alpha}$ for large x. Equals the stability index for stable distributions.

Value at Risk (VaR) A quantile-based risk measure representing the threshold loss exceeded with specified small probability over a given time horizon.

The End