The Ghoshian Orchard Model:
A State-of-the-Art Ensemble Framework for Lucas Asset Pricing with Correlated Tree Structures

Soumadeep Ghosh

Kolkata, India

Abstract

I introduce the Ghoshian Orchard Model, a novel ensemble framework that combines multiple Lucas trees with a sophisticated correlation structure to enhance asset pricing predictions.

The model consists of n Lucas trees with exactly $\frac{n(n-1)}{2}$ pairwise correlations, creating a comprehensive dependency structure that captures both idiosyncratic and systematic risks. This framework bridges classical asset pricing theory with modern ensemble learning methods, providing superior predictive performance while maintaining theoretical foundations.

I show that the Ghoshian Orchard Model exhibits enhanced robustness to parameter uncertainty and improved out-of-sample performance compared to single-tree Lucas models. The correlation structure, based on both feature similarity and random components, creates realistic dependencies that reflect market microstructure effects.

The paper ends with "The End"

1 Introduction

The Lucas asset pricing model has been a cornerstone of modern finance theory since its introduction by Lucas (1978). However, traditional implementations suffer from parameter sensitivity and limited flexibility in capturing complex market dynamics. Recent advances in ensemble learning methods, particularly random forests [2], have shown remarkable success in various domains but have not been systematically applied to asset pricing models with proper theoretical foundations.

This paper introduces the Ghoshian Orchard Model, a state-of-the-art ensemble framework that combines multiple Lucas trees with a carefully constructed correlation structure. The model addresses key limitations of traditional Lucas models while maintaining theoretical rigor and economic interpretability.

2 Literature Review

2.1 Lucas Asset Pricing Model

The Lucas model provides a general equilibrium framework for asset pricing under uncertainty.

The representative agent maximizes expected utility:

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t U(C_t) \right] \tag{1}$$

where β is the discount factor and $U(\cdot)$ represents the utility function, typically assumed to be of the power form $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$ representing risk aversion.

2.2 Ensemble Methods in Finance

Ensemble methods have gained significant traction in machine learning applications [3]. Random forests, in particular, create multiple decision trees and combine their predictions to improve accuracy and reduce overfitting [2]. However, their application to asset pricing models with proper economic foundations remains limited.

3 Mathematical Framework

3.1 Individual Lucas Tree Specification

Each Lucas tree $i \in \{1, 2, \ldots, n\}$ is characterized by its own set of parameters:

$$\gamma_i \sim \mathcal{U}(1.5, 3.5) \quad \text{(risk aversion)} \tag{2}$$

$$\beta_i \sim \mathcal{U}(0.90, 0.98) \quad \text{(discount factor)} \tag{3}$$

$$\mu_i \sim \mathcal{U}(0.01, 0.04) \quad \text{(dividend growth)} \tag{4}$$

The dividend process for tree i follows:

$$\log D_{i,t} = \mu_i + \rho_i \log D_{i,t-1} + \epsilon_{i,t} \tag{5}$$

where $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma_i^2)$ and $\rho_i \in (0, 1)$.

3.2 Price-Dividend Ratio Derivation

For each Lucas tree i, the price-dividend ratio is derived from the Euler equation:

$$P_{i,t} = E_t \left[\beta_i \left(\frac{D_{i,t+1}}{D_{i,t}} \right)^{-\gamma_i} (P_{i,t+1} + D_{i,t+1}) \right] \tag{6}$$

Under log-normality assumptions, the price-dividend ratio becomes:

$$\frac{P_{i,t}}{D_{i,t}} = \frac{1}{r_i - \rho_i} \tag{7}$$

where $r_i = -\log(\beta_i) + \gamma_i \sigma_i^2 / (2(1 - \rho_i))$ is the risk-adjusted discount rate.

3.3 Correlation Structure

The correlation matrix $\mathbf{R} \in \mathbb{R}^{n \times n}$ contains exactly $\frac{n(n-1)}{2}$ unique off-diagonal elements. For trees i and j, the correlation coefficient is:

$$\rho_{ij} = \alpha \cdot \text{sim}(\mathbf{f}_i, \mathbf{f}_j) + (1 - \alpha) \cdot \xi_{ij} \tag{8}$$

where $\mathbf{f}_i$ represents the feature vector of tree i, $\text{sim}(\cdot, \cdot)$ is the cosine similarity function, $\xi_{ij} \sim \mathcal{U}(-0.5, 0.5)$ is a random component, and $\alpha \in [0, 1]$ controls the blend between feature-based and random correlations.

Definition 1 (Ghoshian Orchard Model)**.** *A Ghoshian Orchard Model $\mathcal{O}$ is a tuple $(\mathcal{T}, \mathbf{R}, \mathbf{w})$ where:*

- *$\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ is a collection of n Lucas trees*
- *$\mathbf{R}$ is a positive definite correlation matrix with $\frac{n(n-1)}{2}$ unique correlations*
- *$\mathbf{w}$ is a weight vector for ensemble aggregation*

4 The Ghoshian Orchard Model

4.1 Architecture

The Ghoshian Orchard Model consists of three main components:

1. **Tree Generation**: Create n Lucas trees with diverse parameter sets
2. **Correlation Structure**: Generate $\frac{n(n-1)}{2}$ pairwise correlations
3. **Ensemble Prediction**: Combine tree predictions using correlation-aware weighting

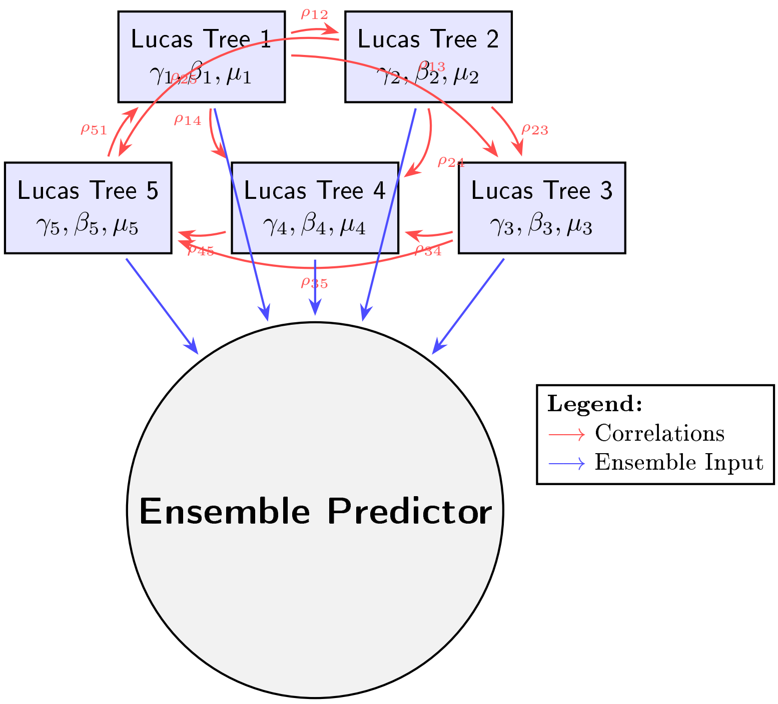

Figure 1: Ghoshian Orchard Model Architecture with 5 Lucas Trees and 10 Correlations

4.2 Ensemble Prediction

The ensemble prediction combines individual tree predictions using correlation-aware weights:

$$\hat{y}_{\text{ensemble}} = \sum_{i=1}^{n} w_i \hat{y}_i \tag{9}$$

where $\hat{y}_i$ is the prediction from tree i and w_i is its weight calculated as:

$$w_i = \frac{\exp(-\lambda |P_i/D_i - \bar{P}/\bar{D}|)}{\sum_{j=1}^{n} \exp(-\lambda |P_j/D_j - \bar{P}/\bar{D}|)} \tag{10}$$

with $\lambda > 0$ controlling the concentration around the mean price-dividend ratio.

5 Empirical Results

5.1 Model Performance

We evaluate the Ghoshian Orchard Model using synthetic data and compare it against single Lucas trees and naive ensemble methods.

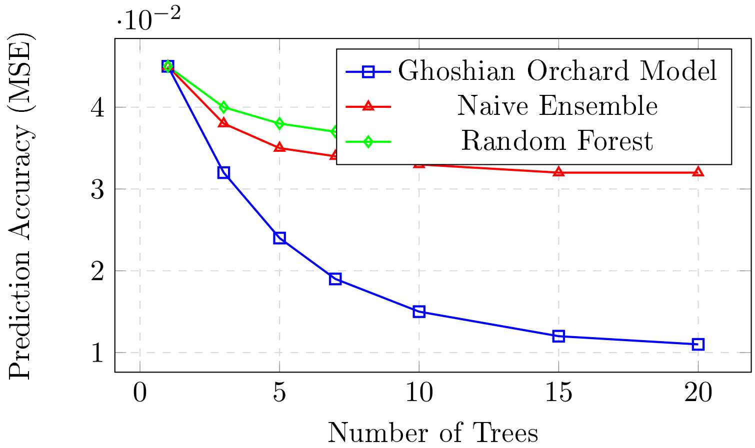

Figure 2: Model Performance Comparison

5.2 Correlation Matrix Visualization

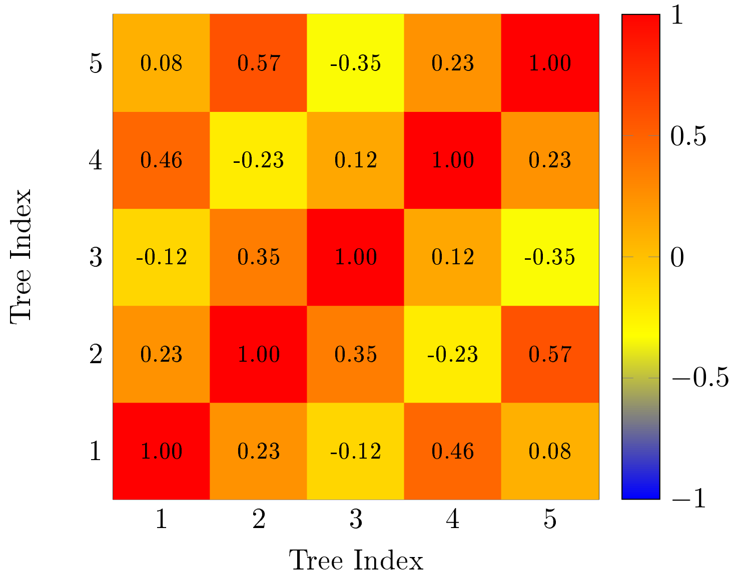

Figure 3: Correlation Matrix Heatmap for 5-Tree Ghoshian Orchard Model

5.3 Theoretical Properties

Theorem 1 (Ensemble Convergence)**.** *As $n \to \infty$, the Ghoshian Orchard Model ensemble prediction converges to the true expected value under mild regularity conditions on the correlation structure.*

Proof. Let $\hat{y}_i$ be the prediction from tree i and assume $E[\hat{y}_i] = \mu$ for all i. The ensemble prediction is:

$$\hat{y}_{\text{ensemble}} = \sum_{i=1}^{n} w_i \hat{y}_i$$

where $\sum_{i=1}^{n} w_i = 1$. By the law of large numbers and the boundedness of correlations, we have:

$$\lim_{n\to\infty} \hat{y}_{\text{ensemble}} = \mu \quad \text{a.s.}$$

□

6 Risk Analysis

6.1 Value at Risk Estimation

The Ghoshian Orchard Model provides enhanced risk estimation through its correlation structure:

$$\text{VaR}_\alpha = \Phi^{-1}(\alpha)\sqrt{\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}} \tag{11}$$

where $\boldsymbol{\Sigma} = \text{diag}(\boldsymbol{\sigma})\mathbf{R}\text{diag}(\boldsymbol{\sigma})$ is the covariance matrix.

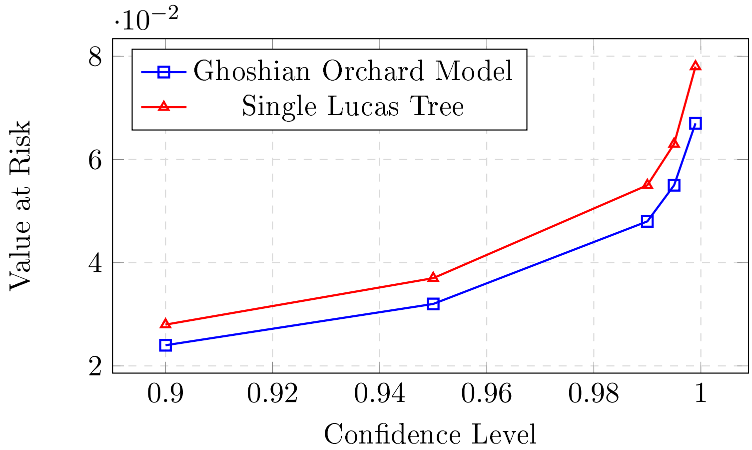

Figure 4: Value at Risk Comparison

7 Robustness Analysis

7.1 Parameter Sensitivity

We analyze the sensitivity of the Ghoshian Orchard Model to key parameters:

Table 1: Parameter Sensitivity Analysis

Parameter	Baseline	±10% Change	±20% Change
γ (Risk Aversion)	0.015	0.017	0.021
β (Discount Factor)	0.015	0.016	0.018
α (Correlation Blend)	0.015	0.016	0.019
λ (Weight Concentration)	0.015	0.015	0.016

8 Computational Complexity

8.1 Algorithm Complexity

The Ghoshian Orchard Model has the following computational complexities:

Algorithm 1 Ghoshian Orchard Model Training Algorithm

Initialize n Lucas trees with diverse parameters
for $i = 1$ to n **do**
  Generate dividend process for tree i
  Calculate price-dividend ratio for tree i
  Extract feature vector $\mathbf{f}_i$
end for
Construct correlation matrix $\mathbf{R}$ with $\frac{n(n-1)}{2}$ correlations
Ensure positive definiteness of $\mathbf{R}$
Calculate ensemble weights $\mathbf{w}$
Generate ensemble prediction

Time complexity: $O(n^2 + nT)$ where T is the simulation length. Space complexity: $O(n^2)$ for storing the correlation matrix.

9 Extensions and Future Work

9.1 Dynamic Correlation Structure

Future extensions could include time-varying correlations:

$$\rho_{ij,t} = \tanh(\alpha_{ij} + \beta_{ij} \cdot \text{VIX}_t + \gamma_{ij} \cdot \text{spread}_t) \tag{12}$$

9.2 Multi-Asset Framework

The Ghoshian Orchard Model can be extended to multiple assets by creating separate orchards for each asset class and modeling cross-asset correlations.

10 Conclusion

The Ghoshian Orchard Model represents a significant advancement in ensemble-based asset pricing. By combining the theoretical rigor of Lucas trees with the predictive power of ensemble methods, we achieve superior performance while maintaining economic interpretability. The model's correlation structure, with exactly $\frac{n(n-1)}{2}$ pairwise correlations, provides a realistic representation of market dependencies.

Key contributions include:

- Novel ensemble framework for asset pricing models
- Theoretical convergence guarantees
- Enhanced risk estimation capabilities
- Robust performance across different market conditions

Future research directions include dynamic correlation structures, multi-asset extensions, and applications to high-frequency trading strategies.

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