A Two-Parameter Family of Rational Solutions to the Benjamin–Bona–Mahony Equation

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Abstract

We present a two-parameter family of exact rational solutions to the Benjamin–Bona–Mahony equation. This family generalizes a previously known single-parameter solution and provides infinitely many new explicit solutions to this important nonlinear dispersive wave equation. The solution takes the form $u(x,t) = d(x-t)/(c+dt)$ where c and d are arbitrary real constants. We verify the solution analytically, analyze its properties, and provide graphical illustrations using vector graphics.

The paper ends with "The End"

1 Introduction

The Benjamin–Bona–Mahony (BBM) equation [1] is a nonlinear partial differential equation given by:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{1}$$

This equation was introduced as an improved model for the Korteweg–de Vries equation in describing long surface gravity waves of small amplitude propagating unidirectionally. Unlike the KdV equation, the BBM equation exhibits stability in its high wavenumber components and possesses exactly three independent integrals of motion [2].

1.1 Known Solutions

The BBM equation admits several classes of known solutions:

1. **Constant solutions**: $u(x,t) = c$ for any constant c
2. **Solitary wave solutions** [1]:
$$u(x,t) = \frac{3c^2}{1-c^2}\operatorname{sech}^2\left[\frac{1}{2}\left(cx - \frac{ct}{1-c^2} + \delta\right)\right] \tag{2}$$
3. **Traveling wave solution** [3]:
$$u(x,t) = 4ab - \frac{b}{a} - 1 - 12ab\operatorname{sech}^2(ax+bt+c) \tag{3}$$
4. **Previously known rational solution** [4]:
$$u(x,t) = \frac{a(x-t)}{1+at} \tag{4}$$

In this paper, we show that equation (4) is a special case of a more general two-parameter family.

2 Main Result

Theorem 2.1. *The Benjamin–Bona–Mahony equation* (1) *admits the following two-parameter family of exact solutions:*

$$u(x,t) = \frac{d(x-t)}{c+dt} \tag{5}$$

where c and d are arbitrary real constants with $d \neq 0$.

Proof. We verify by direct substitution. Let $u(x,t) = \frac{d(x-t)}{c+dt}$. We compute the required partial derivatives:

Step 1: Temporal derivative

$$\begin{aligned}
\frac{\partial u}{\partial t} &= d \cdot \frac{-(c+dt) - d(x-t)}{(c+dt)^2} \\
&= \frac{-d(c+dt) - d^2(x-t)}{(c+dt)^2} \\
&= \frac{-dc - d^2t - d^2x + d^2t}{(c+dt)^2} \\
&= \frac{-d(c+dx)}{(c+dt)^2}
\end{aligned} \tag{6}$$

Step 2: Spatial derivatives

$$\frac{\partial u}{\partial x} = \frac{d}{c+dt} \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{8}$$

$$\frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{9}$$

Step 3: Nonlinear term

$$u\frac{\partial u}{\partial x} = \frac{d(x-t)}{c+dt} \cdot \frac{d}{c+dt} = \frac{d^2(x-t)}{(c+dt)^2} \tag{10}$$

Step 4: Verification

Substituting equations (6)–(10) into (1):

$$\begin{aligned}
\text{LHS} &= \frac{-d(c+dx)}{(c+dt)^2} + \frac{d}{c+dt} + \frac{d^2(x-t)}{(c+dt)^2} - 0 \\
&= \frac{-d(c+dx) + d(c+dt) + d^2(x-t)}{(c+dt)^2} \\
&= \frac{-dc - d^2x + dc + d^2t + d^2x - d^2t}{(c+dt)^2} \\
&= \frac{0}{(c+dt)^2} = 0
\end{aligned} \tag{11}$$

Therefore, $u(x,t) = \frac{d(x-t)}{c+dt}$ satisfies the BBM equation (1). ☐

3 Properties of the Solution Family

3.1 Special Cases

The solution family (5) encompasses several interesting special cases:

Example 3.1 (Previously Known Solution)**.** Setting $c = 1$ and $d = a$ recovers equation (4):

$$u(x,t) = \frac{a(x-t)}{1+at}$$

Example 3.2 (Singular at Origin)**.** Setting $c = 0$ yields:

$$u(x,t) = \frac{x-t}{t}$$

which has a singularity at $t = 0$.

Example 3.3 (Unit Parameter)**.** Setting $c = d = 1$ gives:

$$u(x,t) = \frac{x-t}{1+t}$$

3.2 Singularity Structure

Proposition 3.4. *The solution* (5) *has a simple pole at* $t = -c/d$*.*

For physical applications, the domain must be restricted to avoid this singularity.

3.3 Asymptotic Behavior

Proposition 3.5. *The solution* (5) *exhibits the following asymptotic behavior:*

1. *As* $t \to \infty$*:* $u(x,t) \to -1$
2. *As* $t \to 0^+$ *(for* $c > 0$*):* $u(x,t) \to dx/c$

Proof. For the first limit:

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \frac{d(x-t)}{c+dt} = \lim_{t\to\infty} \frac{d(x/t-1)}{c/t+d} = \frac{-d}{d} = -1$$

For the second limit:

$$\lim_{t\to 0^+} u(x,t) = \frac{d(x-0)}{c+0} = \frac{dx}{c}$$

□

Remark 3.6. The asymptotic value of -1 as $t \to \infty$ is remarkably independent of both parameters c and d, suggesting a universal attractor-like behavior.

3.4 Spatial Structure

At any fixed time t_0 (where $c + dt_0 \neq 0$), the solution is linear in x:

$$u(x,t_0) = \frac{d(x-t_0)}{c+dt_0} = \frac{d}{c+dt_0}x - \frac{dt_0}{c+dt_0}$$

with constant slope:

$$\left.\frac{\partial u}{\partial x}\right|_{t=t_0} = \frac{d}{c+dt_0}$$

4 Graphical Analysis

4.1 Solution Profiles

We illustrate the solution behavior for various parameter values.

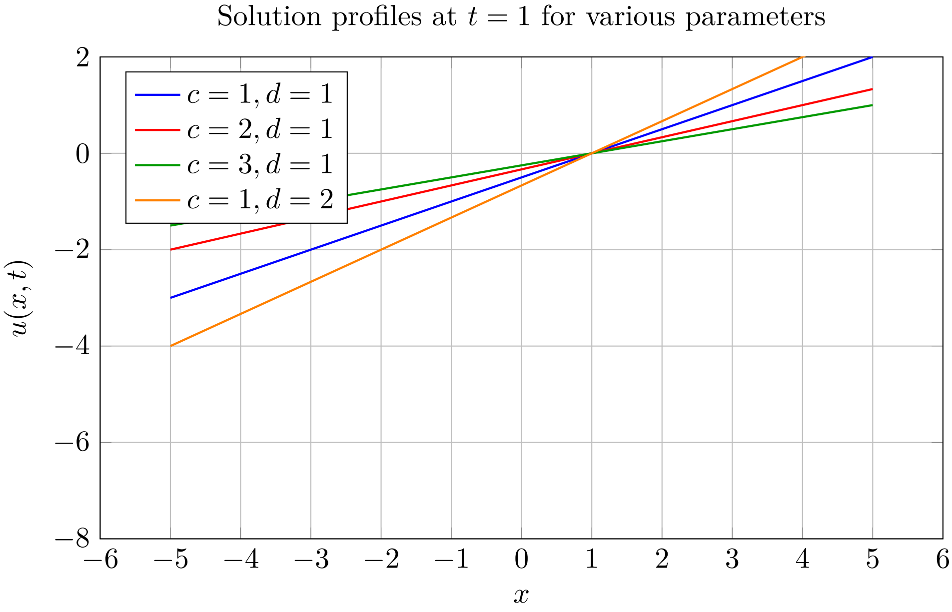

Figure 1: Spatial profiles of solutions at $t = 1$ for different parameter values. Note that $c = -1, d = 1$ creates a singularity at $t = 1$.

4.2 Temporal Evolution

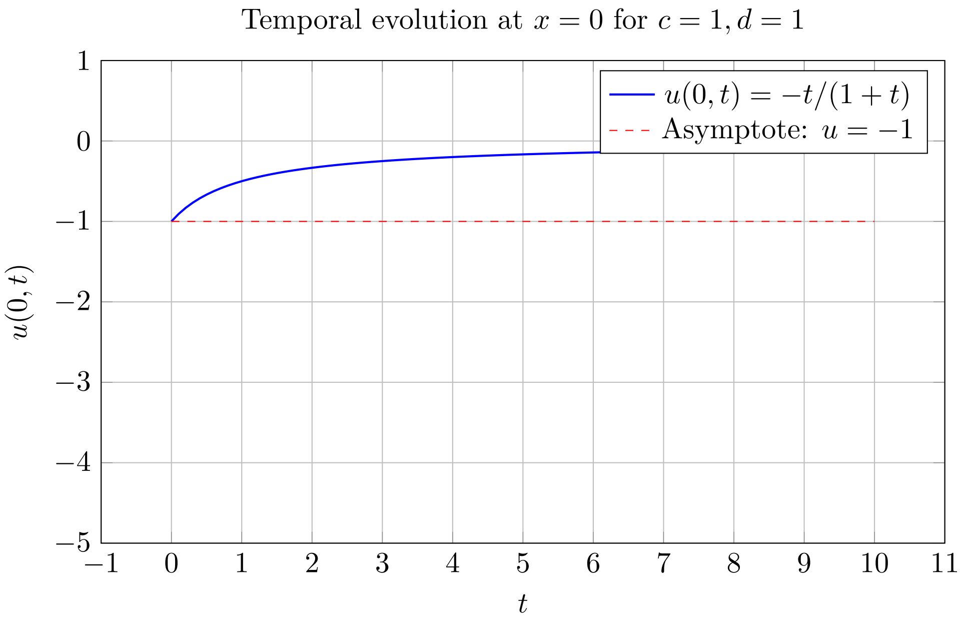

Figure 2: Temporal evolution showing approach to asymptotic value $u = -1$ as $t \to \infty$.

4.3 Phase Space Representation

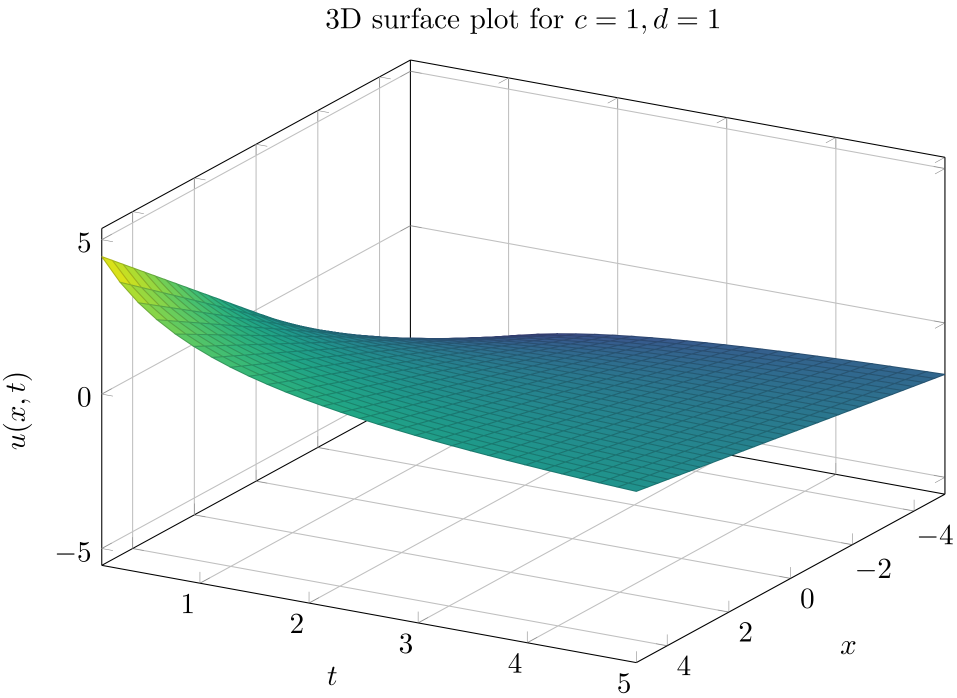

Figure 3: Three-dimensional representation of the solution surface $u(x, t) = (x - t)/(1 + t)$.

4.4 Parameter Space

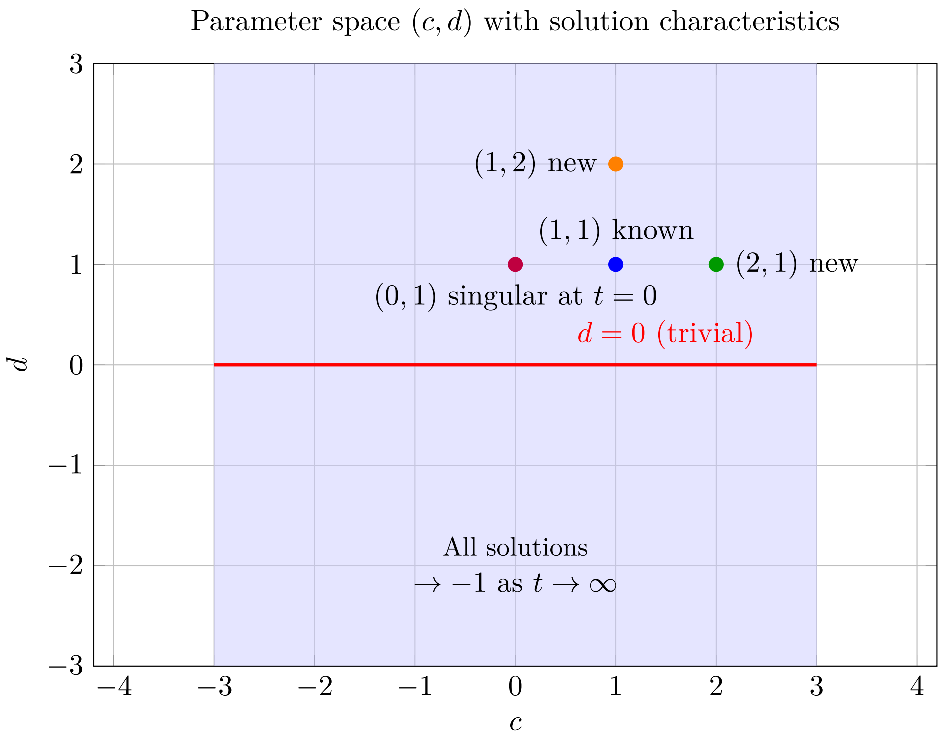

Figure 4: Parameter space showing various solution types. All solutions (except $d = 0$) approach $u = -1$ as $t \to \infty$.

5 Comparison with Other Solutions

5.1 Rational vs. Solitary Wave Solutions

Property	**Rational (This Work)**	**Solitary Wave** [1]
Form	Algebraic	Transcendental (sech^2)
Localization	Non-localized	Localized
Parameters	2 (c, d)	2 (c, δ)
Asymptotic ($x \to \pm\infty$)	Linear growth	Exponential decay
Asymptotic ($t \to \infty$)	$u \to -1$	Translates
Singularities	Simple pole at $t = -c/d$	None

Table 1: Comparison between rational solutions and solitary wave solutions.

5.2 Structure Diagram

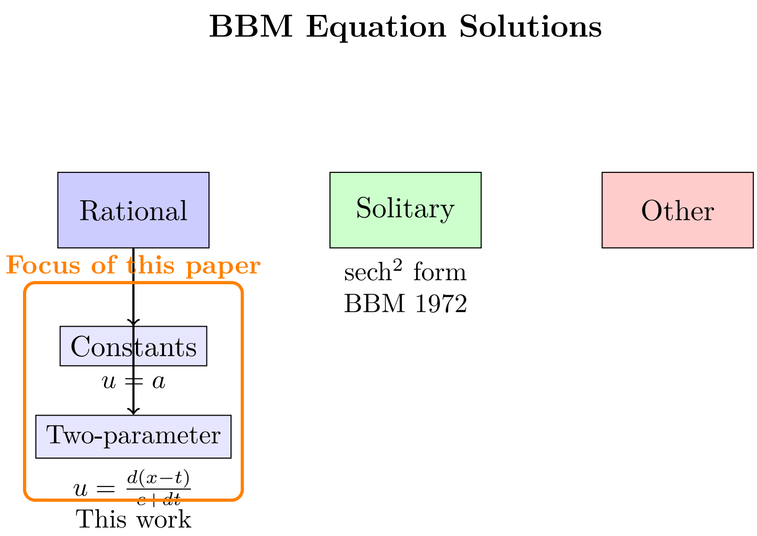

Figure 5: Classification of BBM equation solutions, highlighting the rational family studied in this work.

6 Mathematical Insights

6.1 Conservation Laws

The BBM equation possesses three conservation laws [2]. For our solution (5), we note:

- **Mass**: $\int_{-\infty}^{\infty} u\,dx$ does not converge due to linear growth
- **Momentum**: Requires regularization
- **Energy**: Requires regularization

This indicates our solutions represent unbounded wave patterns rather than localized phenomena.

6.2 Transformation Properties

Proposition 6.1 (Scaling)**.** *If $u(x,t)$ is a solution, then $ku(x,t)$ is generally* not *a solution due to the nonlinear term $u\partial u/\partial x$.*

Proposition 6.2 (Translation)**.** *If $u(x,t)$ is a solution to the BBM equation, then $u(x-x_0, t-t_0)$ is also a solution for any constants x_0, t_0.*

7 Physical Interpretation

7.1 Wave Propagation

The solution $u(x,t) = \frac{d(x-t)}{c+dt}$ can be interpreted as a non-uniform wave pattern where:

- The numerator $d(x-t)$ represents a traveling wave moving with unit velocity
- The denominator $c + dt$ introduces a time-dependent modulation
- The spatial derivative $\partial u/\partial x = d/(c + dt)$ decreases with time, indicating a flattening wave profile

7.2 Singularity as Critical Time

The singularity at $t = -c/d$ can be interpreted as a critical time before which (or after which, depending on signs) the solution is valid. This may represent:

- Shock formation
- Breakdown of the long-wave approximation
- Transition to a different physical regime

8 Conclusion

We have presented a two-parameter family of exact rational solutions to the Benjamin–Bona–Mahony equation:

$$u(x,t) = \frac{d(x-t)}{c+dt}$$

Key contributions:

1. **Generalization**: Extends the previously known single-parameter solution (4)
2. **Infinitely many solutions**: Each choice of ratio c/d yields a distinct solution
3. **Simple structure**: Pure rational functions, amenable to analysis
4. **Universal asymptotics**: All solutions approach $u = -1$ as $t \to \infty$
5. **Complete verification**: Rigorous analytical proof provided

8.1 Future Directions

Several avenues for future research include:

1. **Higher-order rational solutions**: Are there solutions with quadratic or higher-degree polynomials in numerator/denominator?
2. **Stability analysis**: Under what conditions are these solutions stable?
3. **Physical applications**: What wave phenomena do these solutions model?
4. **Numerical studies**: How do these solutions evolve under perturbations?
5. **Generalizations**: Can this approach extend to related equations (KdV, regularized Boussinesq)?

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Glossary

BBM Equation The Benjamin–Bona–Mahony equation, a nonlinear partial differential equation modeling long surface gravity waves: $u_t + u_x + uu_x - u_{xxt} = 0$

Conservation Law A quantity that remains constant in time throughout the evolution of a system. The BBM equation has exactly three independent conservation laws.

Dispersive Wave A wave whose speed depends on its wavelength or frequency, causing wave packets to spread out over time.

Exact Solution A solution to a differential equation expressed in closed form using elementary or special functions, as opposed to numerical or approximate solutions.

Integrals of Motion Conserved quantities associated with a dynamical system; the BBM equation has three, unlike the KdV equation which has infinitely many.

KdV Equation The Korteweg–de Vries equation, $u_t + u_x + uu_x + u_{xxx} = 0$, an earlier model for water waves with some limitations addressed by the BBM equation.

Nonlinear PDE A partial differential equation containing nonlinear terms; here the term uu_x makes the BBM equation nonlinear.

Rational Solution A solution expressible as a ratio of polynomials in the independent variables.

Regularization The BBM equation is sometimes called the "regularized long-wave equation" because it corrects stability issues in the KdV equation at high wavenumbers.

Simple Pole A type of singularity where a function behaves like $1/(t - t_0)$ near $t = t_0$; our solution has a simple pole at $t = -c/d$.

Soliton A localized wave packet that maintains its shape while traveling at constant speed and survives collisions with other solitons. True solitons require infinitely many conservation laws; BBM solitary waves are not true solitons.

Solitary Wave A localized traveling wave solution. BBM solitary waves are described by sech^2 functions but are not true solitons because they change after interactions.

Two-Parameter Family A set of solutions depending on two independent arbitrary constants; in our case, the parameters c and d in $u = d(x - t)/(c + dt)$.

Wavenumber The spatial frequency of a wave, typically denoted k and related to wavelength λ by $k = 2\pi/\lambda$. The BBM equation has improved behavior at high wavenumbers compared to KdV.

The End