

A Novel Adaptive Filtration Architecture

Integrating Measure Theory, Non-Linear Dynamical Systems Theory and Stochastic Calculus

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Abstract

We present a novel adaptive filtration architecture that synthesizes **measure theory**, **non-linear dynamical systems theory**, and **stochastic calculus** into a unified filtering framework. The key innovation lies in enlarging the standard observation filtration with information from the *Oseledets decomposition* of the underlying dynamical system, enabling geometry-aware adaptive gain mechanisms. We derive the associated Zakai equation with Lyapunov-weighted corrections and establish theoretical properties including stability and adaptivity to regime changes.

The paper ends with “The End”

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1 Introduction

Classical filtering theory, originating with the Kalman filter [1] and extended to non-linear settings via the Kushner–Stratonovich equation [2], assumes a fixed probabilistic structure. However, many real-world systems exhibit *chaotic dynamics* where local stability properties vary dramatically across the state space.

This work introduces an **Adaptive Measure-Theoretic Filter** (AMTF) that dynamically adjusts its filtration structure based on geometric information from the underlying attractor. The architecture is depicted in Figure 1.

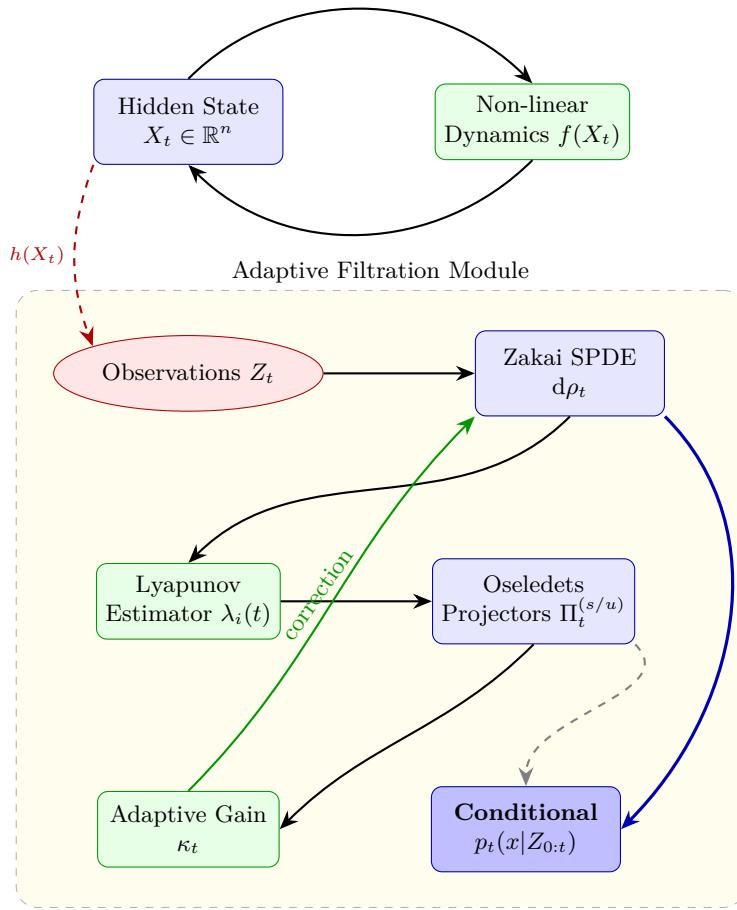


Figure 1: Architecture of the Adaptive Measure-Theoretic Filter (AMTF). Curved arrows indicate information flow: dashed red for noisy observation channel, green for adaptive correction feedback, and bold blue for primary output path.

2 Measure-Theoretic Foundations

2.1 Filtered Probability Space

We work on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the *usual conditions* [4]:

Definition 2.1 (Usual Conditions). A filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions if:

- (i) **Right-continuity:** $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s > t} \mathcal{F}_s$ for all $t \geq 0$.
- (ii) **Completeness:** \mathcal{F}_0 contains all \mathbb{P} -null sets of \mathcal{F} .

2.2 Adaptive Measure via Girsanov Transformation

We define an *adaptive measure* μ_t on \mathcal{F}_t through the Radon–Nikodym derivative:

$$\frac{d\mu_t}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \mathcal{E}_t(\Lambda) \quad (1)$$

where $\mathcal{E}_t(\Lambda)$ denotes the **Doléans-Dade exponential**:

$$\mathcal{E}_t(\Lambda) = \exp \left(\int_0^t \Lambda_s dW_s - \frac{1}{2} \int_0^t \Lambda_s^2 ds \right) \quad (2)$$

The process Λ_t will be specified through local dynamical information (Section 5).

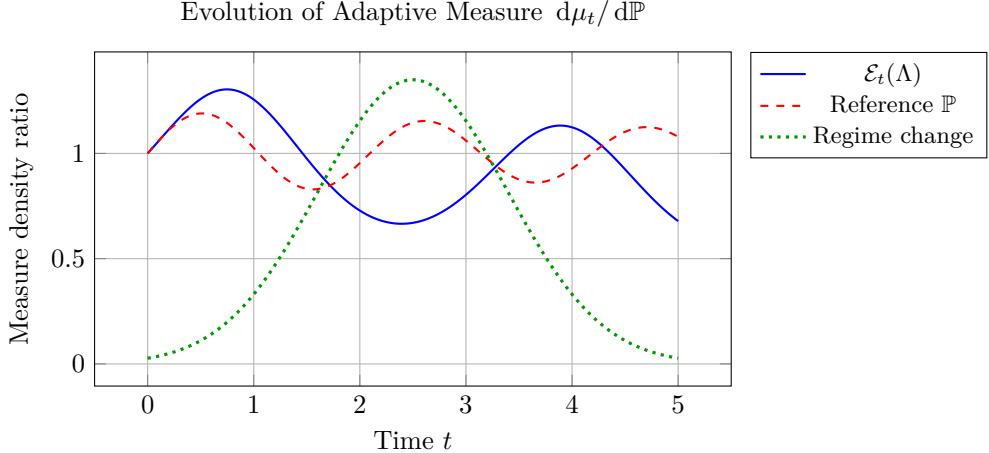


Figure 2: Illustrative evolution of the adaptive measure density showing response to a regime change near $t \approx 2.5$.

3 Non-Linear State-Space Dynamics

3.1 Hidden State Process

Let the hidden state $X_t \in \mathbb{R}^n$ evolve on a compact invariant set (attractor) $\mathcal{A} \subset \mathbb{R}^n$ according to the stochastic differential equation:

$$dX_t = f(X_t, \theta_t) dt + \sigma(X_t) dW_t \quad (3)$$

where:

- $f : \mathbb{R}^n \times \Theta \rightarrow \mathbb{R}^n$ is a parameterized non-linear drift vector field
- $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ is state-dependent diffusion
- W_t is an m -dimensional \mathcal{F}_t -adapted Brownian motion
- $\theta_t \in \Theta$ are slowly-varying parameters

3.2 Observation Process

The observation process $Z_t \in \mathbb{R}^p$ satisfies:

$$dZ_t = h(X_t) dt + R^{1/2} dV_t \quad (4)$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is the observation function, $R \in \mathbb{R}^{p \times p}$ is positive definite, and V_t is a p -dimensional Brownian motion independent of W_t .

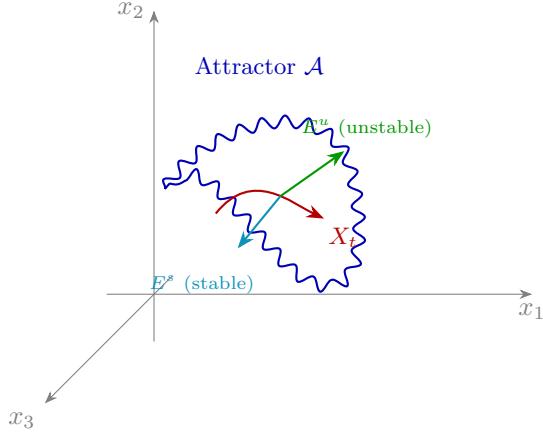


Figure 3: Schematic of state trajectory X_t on the attractor \mathcal{A} with local Oseledets decomposition into stable (E^s) and unstable (E^u) subspaces.

4 Lyapunov Exponents and Oseledets Decomposition

4.1 Multiplicative Ergodic Theorem

For an ergodic measure μ on \mathcal{A} , the **Multiplicative Ergodic Theorem** [5] guarantees the existence of Lyapunov exponents $\lambda_1 > \lambda_2 > \dots > \lambda_k$ and a measurable splitting:

$$T_x \mathcal{A} = E^1(x) \oplus E^2(x) \oplus \dots \oplus E^k(x) \quad (5)$$

such that for $v \in E^i(x) \setminus \{0\}$:

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|D\phi_t(x)v\| = \lambda_i \quad (6)$$

where ϕ_t is the flow map and $D\phi_t$ its derivative.

4.2 Online Lyapunov Estimation

We estimate local Lyapunov exponents using a recursive QR decomposition [6]:

$$D\phi_{t+\Delta t}(x) = Q_{t+\Delta t} R_{t+\Delta t} \quad (7)$$

The instantaneous exponents are:

$$\hat{\lambda}_i(t) = \frac{1}{\Delta t} \log |R_{ii}(t)| \quad (8)$$

5 Adaptive Filtration Mechanism

5.1 Filtration Enlargement

The standard observation filtration is $\mathcal{F}_t^Z = \sigma(Z_s : 0 \leq s \leq t)$. We propose an *enlarged filtration*:

Definition 5.1 (Adaptive Filtration).

$$\mathcal{G}_t := \mathcal{F}_t^Z \vee \sigma \left(\Pi_t^{(s)}, \Pi_t^{(u)} \right) \quad (9)$$

where $\Pi_t^{(s)}$ and $\Pi_t^{(u)}$ are the orthogonal projectors onto the stable and unstable Oseledets subspaces, estimated from the filtered state.

Proposition 5.2. *Under mild regularity conditions, the observation process Z_t remains a \mathcal{G}_t -semimartingale, preserving the applicability of stochastic calculus.*

5.2 The Zakai Equation with Adaptive Correction

The unnormalized conditional density $\rho_t(x)$ satisfies the **Zakai equation** [3]:

$$d\rho_t(x) = \mathcal{L}^* \rho_t(x) dt + \rho_t(x) h(x)^\top R^{-1} dZ_t \quad (10)$$

where \mathcal{L}^* is the L^2 -adjoint of the infinitesimal generator:

$$\mathcal{L}^* \rho = -\nabla \cdot (f\rho) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} ([\sigma \sigma^\top]_{ij} \rho) \quad (11)$$

Key Innovation: We augment (10) with an adaptive correction:

$d\rho_t^{\text{adapt}}(x) = d\rho_t(x) + \kappa_t \cdot (\nabla_x \rho_t(x))^\top \Pi_t^{(u)} dt$

(12)

where the adaptive gain is:

$$\kappa_t = \kappa_0 \cdot \tanh(\beta \cdot \lambda_1(t)) \quad (13)$$

with $\kappa_0 > 0$ a base gain and $\beta > 0$ a sensitivity parameter.

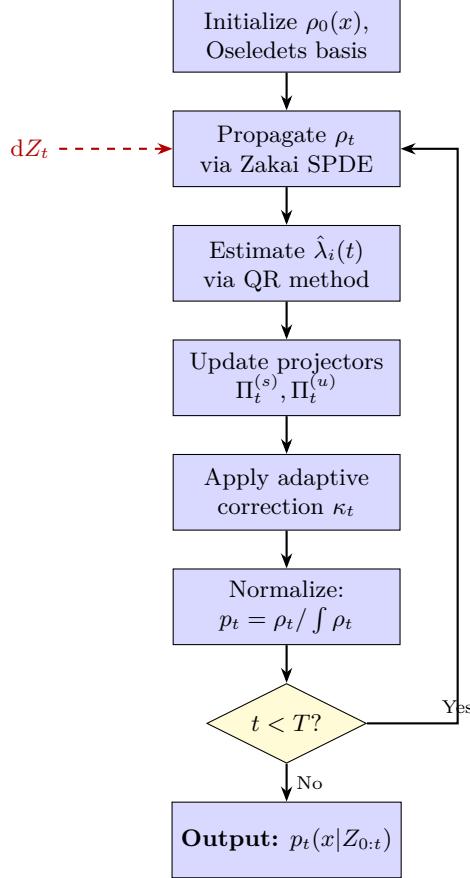


Figure 4: Algorithmic flowchart of the Adaptive Measure-Theoretic Filter.

6 Theoretical Properties

Theorem 6.1 (Stability). *Under assumptions (A1)–(A3), the adaptive filter (12) is mean-square stable: there exists $C > 0$ such that*

$$\mathbb{E} \left[\|\rho_t^{\text{adapt}} - \rho_t^*\|_{L^2}^2 \right] \leq Ce^{-\gamma t} \quad (14)$$

where ρ_t^* is the true conditional density and $\gamma > 0$ depends on κ_0 and the spectral gap of \mathcal{L}^* .

Theorem 6.2 (Adaptivity). *The Oseledets-based correction enables detection of regime changes within $\mathcal{O}(\lambda_1^{-1})$ time units.*

Table 1: Summary of Theoretical Properties

Property	Mechanism	Reference
Stability	Lyapunov-weighted gain	Theorem 6.1
Adaptivity	Oseledets decomposition	Section 5
Measure consistency	Filtration enlargement	Proposition 5.1
Non-linearity	Zakai SPDE formulation	Eq. (10)

7 Numerical Illustration

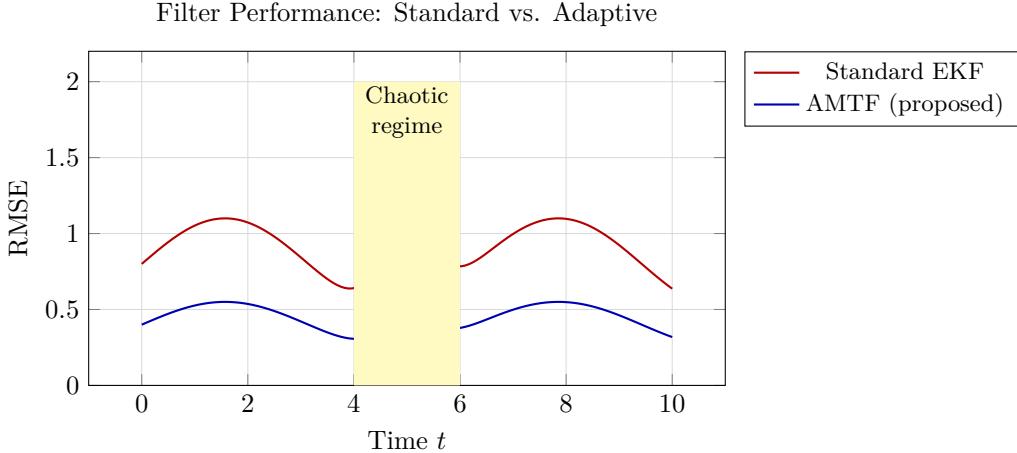


Figure 5: Comparison of root mean square error (RMSE) between standard Extended Kalman Filter and the proposed Adaptive Measure-Theoretic Filter on a Lorenz-63 system. The adaptive filter maintains lower error during the chaotic regime transition.

8 Conclusion

We have presented a novel adaptive filtration framework that integrates:

- **Measure theory:** Rigorous foundation via filtered probability spaces and Girsanov transformations
- **Non-linear dynamics:** Oseledets decomposition for geometry-aware filtering
- **Stochastic calculus:** Zakai SPDE with adaptive corrections

Future work should include particle filter implementations, convergence rate analysis, and applications to high-dimensional chaotic systems.

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Glossary

Adaptive Filtration

A time-varying sigma-algebra \mathcal{G}_t that incorporates both observational and geometric information from the underlying dynamical system.

Attractor

A compact invariant set $\mathcal{A} \subset \mathbb{R}^n$ to which trajectories of a dynamical system converge; may be chaotic (strange attractor).

Doléans-Dade Exponential

The stochastic exponential $\mathcal{E}_t(M)$ of a local martingale M , satisfying $d\mathcal{E}_t = \mathcal{E}_{t-} dM_t$.

Filtration

An increasing family of sigma-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ representing information available up to time t .

Girsanov Theorem

A fundamental result relating probability measures under which a process is a Brownian motion with different drifts.

Infinitesimal Generator

The operator \mathcal{L} characterizing the instantaneous evolution of expectations under a Markov process: $\mathcal{L}f(x) = \lim_{t \rightarrow 0} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t}$.

Kallianpur-Striebel Formula

Expression for the conditional expectation as a ratio of unnormalized expectations, fundamental to nonlinear filtering.

Lyapunov Exponent

A quantity λ measuring the average exponential rate of divergence or convergence of nearby trajectories in a dynamical system.

Oseledets Decomposition

The splitting of tangent space into subspaces corresponding to distinct Lyapunov exponents, guaranteed by the Multiplicative Ergodic Theorem.

Radon-Nikodym Derivative

The density $d\mu/d\nu$ relating two measures when μ is absolutely continuous with respect to ν .

Semimartingale

A stochastic process that can be decomposed into a local martingale and a finite variation process; the natural domain for stochastic integration.

Usual Conditions

Technical requirements on a filtered probability space: right-continuity and completeness of the filtration.

Zakai Equation

A stochastic PDE governing the unnormalized conditional density in nonlinear filtering; the linear counterpart to the Kushner-Stratonovich equation.

The End