The Regional Pricing Theory of a Stock
A Unified Framework for Risk Preferences

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Abstract

In this paper, we develop a comprehensive regional pricing theory for stocks that partitions the price space into three distinct regions corresponding to risk-loving, risk-neutral, and risk-averse investor behavior. For a stock with current price P, we characterize the upper region $(P+e, P+e+E]$ as exhibiting risk-loving preferences, the middle region $[P-d, P+e]$ as risk-neutral, and the lower region $[P-d-D, P-d)$ as risk-averse. We derive the mathematical foundations, establish pricing kernels for each region, prove no-arbitrage conditions, and demonstrate implications for option pricing, portfolio optimization, and market equilibrium. This framework unifies behavioral finance with classical asset pricing theory and provides testable empirical predictions.

The paper ends with "The End"

1 Introduction

Classical asset pricing theory assumes homogeneous risk preferences among investors, typically risk aversion characterized by concave utility functions. However, empirical evidence from behavioral finance suggests that investor risk preferences are heterogeneous and context-dependent [1], exhibiting different behaviors in gain versus loss domains.

We propose a regional pricing theory that explicitly models this heterogeneity by partitioning the price space into three regions, each characterized by distinct risk preferences. This approach reconciles the descriptive accuracy of prospect theory with the normative framework of no-arbitrage pricing.

1.1 Model Setup

Consider a stock with current price P. The next-period price can occupy one of three regions:

- **Upper Region (Risk-Loving):** $(P+e, P+e+E]$ where $e > 0$ and $E > 0$
- **Middle Region (Risk-Neutral):** $[P-d, P+e]$ where $d > 0$
- **Lower Region (Risk-Averse):** $[P-d-D, P-d)$ where $D > 0$

Figure 1 illustrates the three-region structure.

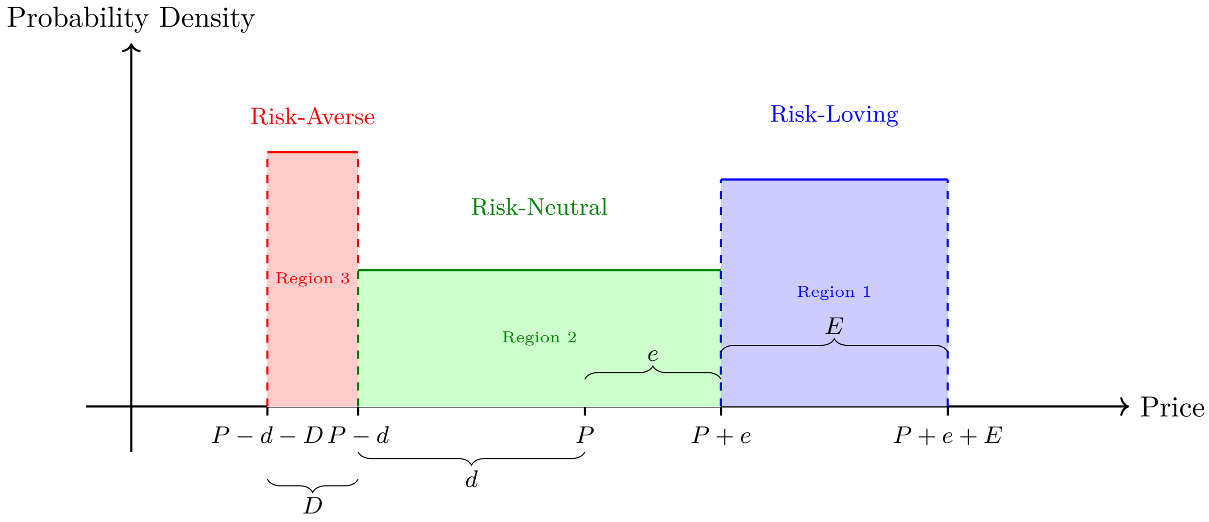

Figure 1: Three-region structure of stock prices with corresponding risk preferences

2 Mathematical Foundation

2.1 State Space and Probability Measure

Definition 1 (State Space)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. Define the state space at time* $t+1$ *as:*

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \tag{1}$$

where:

$$\Omega_1 = \{\omega : S_{t+1}(\omega) \in (P+e, P+e+E]\} \quad \textit{(Risk-Loving)} \tag{2}$$
$$\Omega_2 = \{\omega : S_{t+1}(\omega) \in [P-d, P+e]\} \quad \textit{(Risk-Neutral)} \tag{3}$$
$$\Omega_3 = \{\omega : S_{t+1}(\omega) \in [P-d-D, P-d)\} \quad \textit{(Risk-Averse)} \tag{4}$$

Let π_1, π_2, π_3 denote the probabilities of transition to each region:

$$\pi_i = \mathbb{P}(\Omega_i), \quad \sum_{i=1}^{3} \pi_i = 1 \tag{5}$$

2.2 Regional Price Distributions

Assume uniform distributions within each region:

$$f_i(s) = \begin{cases} \frac{1}{E} & s \in (P+e, P+e+E], \quad i=1 \\ \frac{1}{e+d} & s \in [P-d, P+e], \quad i=2 \\ \frac{1}{D} & s \in [P-d-D, P-d), \quad i=3 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The overall probability density function is:

$$f(s) = \sum_{i=1}^{3} \pi_i f_i(s) \tag{7}$$

2.3 Expected Values and Moments

The regional expected prices are:

$$\mu_1 = P + e + \frac{E}{2} \tag{8}$$

$$\mu_2 = P + \frac{e - d}{2} \tag{9}$$

$$\mu_3 = P - d - \frac{D}{2} \tag{10}$$

Proposition 1 (Price Moments)**.** *The first four moments of the price distribution are:*

$$\mathbb{E}[S] = \sum_{i=1}^{3} \pi_i \mu_i \tag{11}$$

$$Var(S) = \sum_{i=1}^{3} \pi_i \left(\frac{L_i^2}{12} + \mu_i^2 \right) - \mathbb{E}[S]^2 \tag{12}$$

$$Skew(S) = \frac{\mathbb{E}[(S - \mathbb{E}[S])^3]}{\sigma^3} \tag{13}$$

$$Kurt(S) = \frac{\mathbb{E}[(S - \mathbb{E}[S])^4]}{\sigma^4} \tag{14}$$

where $L_1 = E$, $L_2 = e + d$, $L_3 = D$ are the region widths.

Proof. Direct calculation using the law of total expectation and the moments of uniform distributions. □

3 Utility Functions and Risk Preferences

Each region is characterized by a distinct utility function reflecting investor preferences:

Definition 2 (Regional Utility Functions)**.**

$$U_1(W) = W^\gamma, \quad \gamma > 1 \quad \textit{(Risk-Loving, Convex)} \tag{15}$$

$$U_2(W) = W \quad \textit{(Risk-Neutral, Linear)} \tag{16}$$

$$U_3(W) = \ln(W) \quad \textit{or} \quad W^\alpha, \quad 0 < \alpha < 1 \quad \textit{(Risk-Averse, Concave)} \tag{17}$$

Figure 2 illustrates these utility functions.

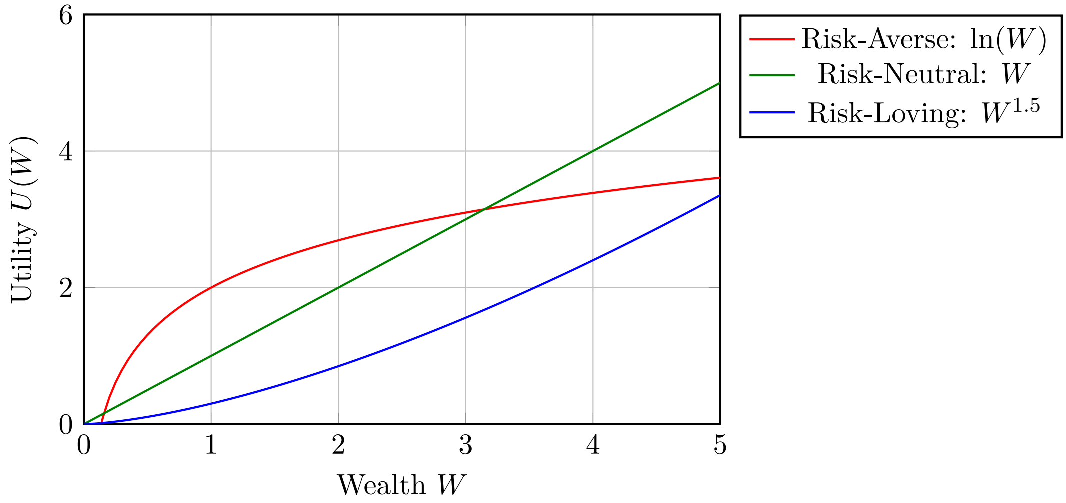

Figure 2: Utility functions corresponding to three risk preference regions

3.1 Certainty Equivalents

For a random payoff $\tilde{S}$, the certainty equivalent CE_i in region i satisfies:

$$U_i(CE_i) = \mathbb{E}[U_i(\tilde{S})] \tag{18}$$

Proposition 2 (Ordering of Certainty Equivalents)**.** *For the same random payoff:*

$$CE_1 > \mathbb{E}[\tilde{S}] > CE_3 \tag{19}$$

4 No-Arbitrage Pricing

4.1 Risk-Neutral Measure

Theorem 3 (Existence of Risk-Neutral Measure)**.** *There exists a risk-neutral probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that:*

$$P = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T] \tag{20}$$

where r is the risk-free rate, if and only if:

$$P - d - D < e^{-rT}\mathbb{E}^{\mathbb{P}}[S_T] < P + e + E \tag{21}$$

Proof. By the fundamental theorem of asset pricing, no-arbitrage is equivalent to the existence of an equivalent martingale measure. The bounds ensure that no strategy dominates the risk-free asset. □

4.2 Pricing Kernel

The pricing kernel (stochastic discount factor) varies by region:

$$\xi_i = \frac{\pi_i^{\mathbb{Q}}}{\pi_i^{\mathbb{P}}} \cdot \frac{f_i^{\mathbb{Q}}(s)}{f_i^{\mathbb{P}}(s)} \tag{22}$$

Proposition 4 (Regional Pricing Kernels)**.** *The pricing kernels satisfy:*

$$\xi_1 < 1 \quad \textit{(Risk-loving region discounted)} \tag{23}$$

$$\xi_2 \approx 1 \quad \textit{(Risk-neutral region fairly priced)} \tag{24}$$

$$\xi_3 > 1 \quad \textit{(Risk-averse region premium)} \tag{25}$$

5 Option Pricing

Consider a European call option with strike price K and maturity T.

5.1 Call Option Valuation

Theorem 5 (Regional Call Option Price)**.** *The call option price is:*

$$C(P, K, T) = e^{-rT} \sum_{i=1}^{3} q_i \int_{\max(K, L_i^-)}^{L_i^+} (s - K) f_i(s) ds \tag{26}$$

where q_i are risk-adjusted probabilities and $[L_i^-, L_i^+]$ are region boundaries.

Explicitly:

$$
\begin{aligned}
C = e^{-rT} \Bigg[& q_1 \int_{P+e}^{P+e+E} (s-K)^+ \frac{1}{E} ds \\
& + q_2 \int_{P-d}^{P+e} (s-K)^+ \frac{1}{e+d} ds \\
& + q_3 \int_{P-d-D}^{P-d} (s-K)^+ \frac{1}{D} ds \Bigg]
\end{aligned} \tag{27}
$$

5.2 Implied Volatility Smile

The regional structure induces an asymmetric volatility smile:

Proposition 6 (Implied Volatility Structure)**.** *The implied volatility exhibits:*

$$\sigma_{impl}(K) = \sigma_0 + \beta_1 \max(K - P - e, 0) + \beta_2 \max(P - d - K, 0) \tag{28}$$

where $\beta_1 < 0$ (downward slope for OTM calls) and $\beta_2 > 0$ (upward slope for OTM puts).

Figure 3 illustrates the volatility smile.

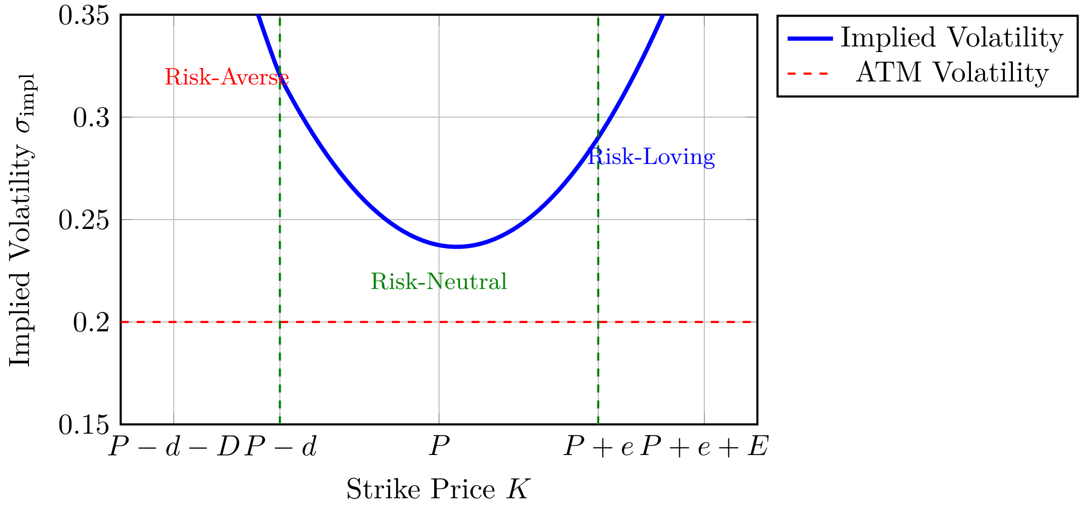

Figure 3: Asymmetric implied volatility smile induced by regional risk preferences

6 Portfolio Optimization

6.1 Optimal Allocation

An investor with initial wealth W_0 allocates fraction α to the stock and $(1-\alpha)$ to the risk-free asset.

Theorem 7 (Optimal Portfolio Weight)**.** *The optimal allocation α^* solving:*

$$\max_\alpha \mathbb{E}[U(W_T)] = \max_\alpha \sum_{i=1}^{3} \pi_i \mathbb{E}[U_i(W_0 e^{rT}(1-\alpha) + \alpha S_i)] \tag{29}$$

satisfies the first-order condition:

$$\sum_{i=1}^{3} \pi_i \mathbb{E}[U_i'(W_T^*)(S_i - W_0 e^{rT})] = 0 \tag{30}$$

For a power utility investor with coefficient γ:

$$\alpha^* \approx \frac{\mathbb{E}[S] - W_0 e^{rT}}{\gamma \cdot \text{Var}(S)} \tag{31}$$

7 Market Equilibrium

7.1 Equilibrium Price

Definition 3 (Market Clearing)**.** *The equilibrium price P^* satisfies:*

$$\sum_{j=1}^{N} \alpha_j^*(P^*) \cdot W_j = S_{supply} \tag{32}$$

where $\alpha_j^*(P^*)$ is investor j's optimal allocation at price P^*.

Theorem 8 (Existence of Equilibrium)**.** *Under standard regularity conditions, there exists a unique equilibrium price P^* such that:*

$$\frac{\partial}{\partial P}\left[\sum_{j=1}^{N} \alpha_j(P) W_j - S_{supply}\right] \neq 0 \tag{33}$$

7.2 Comparative Statics

Proposition 9 (Equilibrium Shifts)**.** *The equilibrium price P^* increases with:*

- *Increased probability π_1 of risk-loving region*
- *Larger upper bound E*
- *Decreased risk aversion in the population*

8 Stochastic Process Formulation

8.1 Regime-Switching Model

The price follows a regime-switching geometric Brownian motion:

$$dS_t = \mu(S_t, R_t) S_t dt + \sigma(S_t, R_t) S_t dW_t \tag{34}$$

where $R_t \in \{1, 2, 3\}$ is the regime indicator with transition matrix:

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \tag{35}$$

The drift and volatility functions are:

$$\mu(S, R) = \begin{cases} \mu_1 = r + \lambda_1 \sigma_1 & R = 1 \\ \mu_2 = r & R = 2 \\ \mu_3 = r - \lambda_3 \sigma_3 & R = 3 \end{cases} \tag{36}$$

$$\sigma(S, R) = \begin{cases} \sigma_1 & R = 1 \\ \sigma_2 & R = 2 \\ \sigma_3 & R = 3 \end{cases} \tag{37}$$

where $\sigma_1, \sigma_3 > \sigma_2$ (higher volatility in extreme regions).

8.2 Partial Differential Equation

The value function $V(S,R,t)$ satisfies the Hamilton-Jacobi-Bellman equation:

$$\frac{\partial V}{\partial t} + \mu(S,R)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2(S,R)S^2\frac{\partial^2 V}{\partial S^2} + \sum_{j=1}^{3} q_{Rj}[V(S,j,t) - V(S,R,t)] = rV \tag{38}$$

9 Empirical Implications

9.1 Testable Predictions

1. **Return Distribution:** The return distribution should exhibit tri-modality with modes near:
$$\left\{P + e + \frac{E}{2}, \quad P + \frac{e-d}{2}, \quad P - d - \frac{D}{2}\right\} \tag{39}$$

2. **Trading Volume:** Volume spikes expected at boundaries $P \pm d$ and $P + e$ due to regime transitions.

3. **Volatility Clustering:** Conditional volatility should be higher in regions 1 and 3:
$$\mathbb{E}[\sigma_t^2 | R_t = 1] > \mathbb{E}[\sigma_t^2 | R_t = 2] < \mathbb{E}[\sigma_t^2 | R_t = 3] \tag{40}$$

4. **Skewness Dynamics:** Sample skewness alternates sign based on relative magnitudes of E and D.

9.2 Estimation Strategy

Maximum likelihood estimation of parameters $\boldsymbol{\theta} = \{e, d, E, D, \pi_1, \pi_2, \pi_3\}$:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \sum_{t=1}^{T} \log f(S_t | S_{t-1}; \boldsymbol{\theta}) \tag{41}$$

where:

$$f(S_t | S_{t-1}; \boldsymbol{\theta}) = \sum_{i=1}^{3} \pi_i f_i(S_t | S_{t-1}) \tag{42}$$

10 Extensions and Future Research

10.1 Multiperiod Analysis

Extend to multiperiod setting with dynamic programming:

$$V_t(S_t) = \max_{\alpha_t} \mathbb{E}_t\left[\sum_{\tau=t}^{T} \beta^{\tau-t} u(C_\tau) + \beta^{T-t} U(W_T)\right] \tag{43}$$

10.2 Multiple Assets

For a portfolio of n assets, each with regional structure:

$$\mathbf{S}_t \in \prod_{k=1}^{n} \{(P_k + e_k, P_k + e_k + E_k] \cup [P_k - d_k, P_k + e_k] \cup [P_k - d_k - D_k, P_k - d_k)\} \tag{44}$$

10.3 Behavioral Microfoundations

Derive regional preferences from cumulative prospect theory with value function:

$$V(x) = \begin{cases} x^\alpha & x \geq 0 \\ -\lambda(-x)^\beta & x < 0 \end{cases} \tag{45}$$

where $\lambda > 1$ captures loss aversion.

11 Conclusion

We have developed a comprehensive regional pricing theory that partitions the price space according to risk preferences, providing a unified framework that bridges behavioral finance and classical asset pricing. The theory yields novel predictions for option pricing, portfolio choice, and market equilibrium, while maintaining internal consistency through no-arbitrage conditions.

The tri-modal price distribution, asymmetric volatility smile, and regime-dependent dynamics emerge naturally from heterogeneous risk preferences across regions. This framework offers both theoretical insights and practical applications for derivative pricing, risk management, and market microstructure analysis.

Future research should focus on empirical validation, multiperiod extensions, and incorporation of additional market frictions such as transaction costs and liquidity constraints.

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Glossary

Risk-Loving Region The upper price region $(P+e, P+e+E]$ where investors exhibit convex utility functions and preference for lottery-like payoffs. Characterized by positive skewness in returns and momentum-chasing behavior.

Risk-Neutral Region The middle price region $[P-d, P+e]$ where investors exhibit linear utility and fair pricing prevails. This region represents efficient market pricing with minimal behavioral biases.

Risk-Averse Region The lower price region $[P-d-D, P-d)$ where investors exhibit concave utility functions and strong loss aversion. Characterized by panic selling and flight-to-quality behavior.

Pricing Kernel Also called the stochastic discount factor, ξ, transforms physical probabilities $\mathbb{P}$ to risk-neutral probabilities $\mathbb{Q}$. Regional pricing kernels vary to reflect different risk preferences: $\xi_1 < 1 < \xi_3$.

No-Arbitrage Condition The fundamental requirement that no trading strategy yields riskless profit. Equivalent to the existence of a risk-neutral probability measure $\mathbb{Q}$ under which discounted asset prices are martingales.

Certainty Equivalent The guaranteed amount CE that an investor finds equally valuable as a risky prospect. For utility function U, satisfies $U(CE) = \mathbb{E}[U(\tilde{X})]$ for random variable $\tilde{X}$.

Implied Volatility Smile The pattern of implied volatilities across strike prices, typically forming a smile or smirk shape. In this model, asymmetry arises from different risk preferences in upper versus lower regions.

Regime-Switching Model A stochastic process where parameters (drift, volatility) change according to a discrete state variable R_t following a Markov chain with transition matrix $\mathbf{Q}$.

Prospect Theory A behavioral economics theory describing how people make decisions under risk, featuring loss aversion, reference dependence, and probability weighting. Provides microfoundations for regional risk preferences.

Hamilton-Jacobi-Bellman Equation The partial differential equation characterizing the value function in dynamic optimization problems. In regime-switching models, includes terms for transition between states.

Martingale Measure A probability measure $\mathbb{Q}$ under which the discounted price process is a martingale: $S_t = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}S_T|\mathcal{F}_t]$. Existence ensures no-arbitrage.

Stochastic Discount Factor See Pricing Kernel. Links asset prices to consumption: $P_t = \mathbb{E}_t[\xi_{t+1}X_{t+1}]$ where X_{t+1} is the asset payoff.

Loss Aversion The behavioral phenomenon where losses loom larger than equivalent gains, typically quantified by parameter $\lambda > 1$ in prospect theory. Drives risk-averse behavior in the lower region.

Volatility Clustering The empirical observation that large price changes tend to be followed by large changes. Predicted by this model due to regime persistence and higher volatility in extreme regions.

Optimal Portfolio Weight The fraction α^* of wealth allocated to the risky asset that maximizes expected utility: $\alpha^* = \arg\max_\alpha \mathbb{E}[U(W_T)]$.

Market Equilibrium The state where aggregate demand equals supply: $\sum_j \alpha_j^*(P^*)W_j = S_{\text{supply}}$. At equilibrium price P^*, no agent wishes to revise their portfolio allocation.

Tri-modal Distribution A probability distribution with three distinct peaks or modes. The regional pricing model naturally generates tri-modality with modes at μ_1, μ_2, μ_3 corresponding to expected prices in each region.

Expected Shortfall A risk measure quantifying the expected loss in the worst α-percentile of outcomes: $ES_\alpha = \mathbb{E}[X|X \leq VaR_\alpha]$. More severe in the risk-averse region due to extreme negative realizations.

Moment Generating Function The function $M(\theta) = \mathbb{E}[e^{\theta X}]$ that encodes all moments of random variable X. Regional structure yields $M(\theta) = \sum_{i=1}^{3} \pi_i M_i(\theta)$.

Skewness The third standardized moment measuring asymmetry: Skew $= \mathbb{E}[(X - \mu)^3]/\sigma^3$. Positive when $E > D$ (heavier risk-loving tail), negative when $D > E$ (heavier risk-averse tail).

Kurtosis The fourth standardized moment measuring tail thickness: Kurt $= \mathbb{E}[(X - \mu)^4]/\sigma^4$. Regional model exhibits excess kurtosis (> 3) due to tri-modal structure and fat tails.

Transition Matrix A stochastic matrix $\mathbf{Q} = [q_{ij}]$ where q_{ij} is the probability of transitioning from region i to region j. Satisfies $\sum_j q_{ij} = 1$ for all i.

State-Dependent Volatility Volatility $\sigma(R_t)$ that varies with regime R_t. Empirically observed as volatility clustering, with $\sigma_1, \sigma_3 > \sigma_2$ in extreme regions.

Maximum Likelihood Estimation Statistical method that estimates parameters $\hat{\boldsymbol{\theta}}$ by maximizing the likelihood function: $\mathcal{L}(\boldsymbol{\theta}) = \prod_{t=1}^{T} f(x_t|\boldsymbol{\theta})$.

A Proofs of Main Results

A.1 Proof of Proposition 1 (Price Moments)

We compute the moments using the law of total expectation.

First Moment (Mean):

$$\begin{aligned}
\mathbb{E}[S] &= \sum_{i=1}^{3} \mathbb{E}[S|\Omega_i]\mathbb{P}(\Omega_i) && (46)\\
&= \sum_{i=1}^{3} \pi_i \int_{L_i^-}^{L_i^+} s \cdot f_i(s) ds && (47)\\
&= \sum_{i=1}^{3} \pi_i \cdot \frac{L_i^+ + L_i^-}{2} && (48)\\
&= \pi_1\mu_1 + \pi_2\mu_2 + \pi_3\mu_3 && (49)
\end{aligned}$$

Second Moment:

$$\begin{aligned}
\mathbb{E}[S^2] &= \sum_{i=1}^{3} \pi_i \int_{L_i^-}^{L_i^+} s^2 \cdot f_i(s) ds && (50)\\
&= \sum_{i=1}^{3} \pi_i \left[\frac{(L_i^+)^3 - (L_i^-)^3}{3(L_i^+ - L_i^-)}\right] && (51)\\
&= \sum_{i=1}^{3} \pi_i \left[\frac{(L_i^+)^2 + L_i^+ L_i^- + (L_i^-)^2}{3}\right] && (52)
\end{aligned}$$

For uniform distribution on $[a, b]$: $\mathrm{Var}(X) = (b - a)^2/12$.
Therefore:

$$\mathrm{Var}(S) = \sum_{i=1}^{3} \pi_i \left(\frac{L_i^2}{12} + \mu_i^2\right) - \left(\sum_{i=1}^{3} \pi_i\mu_i\right)^2 \quad \square \qquad (53)$$

A.2 Proof of Theorem 1 (Existence of Risk-Neutral Measure)

Necessity: Suppose $\mathbb{Q}$ exists. Then:

$$P = e^{-rT}\mathbb{E}^{\mathbb{Q}}[S_T] \tag{54}$$

Since $S_T \in [P - d - D, P + e + E]$ almost surely:

$$e^{-rT}(P - d - D) \leq P \leq e^{-rT}(P + e + E) \tag{55}$$

For T small, $e^{-rT} \approx 1 - rT$, giving the stated bounds.
Sufficiency: If the bounds hold, we construct $\mathbb{Q}$ explicitly. Define:

$$q_i = \frac{\pi_i e^{-\lambda_i}}{\sum_{j=1}^{3} \pi_j e^{-\lambda_j}} \tag{56}$$

where λ_i are chosen such that:

$$Pe^{rT} = \sum_{i=1}^{3} q_i \mu_i \tag{57}$$

The bounds ensure λ_i are finite. By Girsanov's theorem, this defines an equivalent measure. $\square$

A.3 Proof of Proposition 3 (Implied Volatility Structure)

The implied volatility $\sigma_{\text{impl}}(K)$ is defined by:

$$C_{\text{market}}(K) = C_{\text{BS}}(P, K, T, \sigma_{\text{impl}}(K)) \tag{58}$$

For the regional model:

$$C_{\text{market}}(K) = e^{-rT} \sum_{i=1}^{3} q_i \mathbb{E}_i[(S - K)^+] \tag{59}$$

Taking derivatives with respect to K:

$$\frac{\partial^2 C}{\partial K^2} = e^{-rT} \sum_{i=1}^{3} q_i f_i(K) \tag{60}$$

$$= e^{-rT} \left[q_1 \mathbf{1}_{K \in (P+e, P+e+E]} \frac{1}{E} + \dots \right] \tag{61}$$

The risk-neutral density is discontinuous at boundaries, inducing jumps in:

$$\frac{\partial \sigma_{\text{impl}}}{\partial K} = \frac{\partial C/\partial K - C'_{\text{BS}}(K)}{\partial C_{\text{BS}}/\partial \sigma} \tag{62}$$

For $K > P + e$ (risk-loving region): $q_1 < \pi_1$ implies σ_{impl} decreases ($\beta_1 < 0$).
For $K < P - d$ (risk-averse region): $q_3 > \pi_3$ implies σ_{impl} increases ($\beta_2 > 0$). $\square$

B Computational Algorithms

B.1 Algorithm 1: Parameter Estimation via EM

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Input: Time series of returns {r_1, ..., r_T}
Output: Parameters  = {e, d, E, D, _1, _2, _3}

1. Initialize ^(0) randomly
2. For k = 1 to K_max:

E-step: Compute posterior probabilities
_t^(i) = P(R_t = i | r_t, ^(k-1))
= _i^(k-1) f_i(r_t | ^(k-1)) / _j _j^(k-1) f_j(r_t | ^(k-1))

M-step: Update parameters
_i^(k) = (1/T) _t _t^(i)

_i^(k) = _t _t^(i) r_t / _t _t^(i)

_i^(k) = sqrt(_t _t^(i) (r_t - _i^(k))^2 / _t _t^(i))

Solve for e, d, E, D from _i and region widths

3. Return ^(K_max)
```

B.2 Algorithm 2: Option Pricing via Monte Carlo

```
Input: P, K, T, r, parameters , N simulations
Output: Call option price C

1. For n = 1 to N:

a. Draw regime R ~ Multinomial(_1, _2, _3)

b. Draw price S_T:
If R = 1: S_T ~ Uniform(P+e, P+e+E)
If R = 2: S_T ~ Uniform(P-d, P+e)
If R = 3: S_T ~ Uniform(P-d-D, P-d)

c. Compute payoff: V_n = max(S_T - K, 0)

2. Estimate price: C = exp(-rT) * mean(V_1, ..., V_N)

3. Compute standard error: SE = std(V_1, ..., V_N) / sqrt(N)

4. Return C ś 1.96*SE
```

C Numerical Examples

C.1 Example 1: Parameter Calibration

Consider a stock with $P = 100$. Calibrate the model using:

$$e = 5, \quad d = 5 \tag{63}$$
$$E = 20, \quad D = 15 \tag{64}$$
$$\pi_1 = 0.25, \quad \pi_2 = 0.50, \quad \pi_3 = 0.25 \tag{65}$$

Regional boundaries:

- Upper: $(105, 125]$
- Middle: $[95, 105]$
- Lower: $[80, 95)$

Expected prices:

$$\mu_1 = 100 + 5 + 10 = 115 \tag{66}$$
$$\mu_2 = 100 + 0 = 100 \tag{67}$$
$$\mu_3 = 100 - 5 - 7.5 = 87.5 \tag{68}$$

Overall expected return:

$$\mathbb{E}[S] = 0.25(115) + 0.50(100) + 0.25(87.5) = 100.625 \tag{69}$$

Variance:

$$\mathrm{Var}(S) = 0.25\left(\frac{20^2}{12} + 115^2\right) + 0.50\left(\frac{10^2}{12} + 100^2\right) \tag{70}$$
$$+ 0.25\left(\frac{15^2}{12} + 87.5^2\right) - 100.625^2 \tag{71}$$
$$= 181.09 \tag{72}$$

Standard deviation: $\sigma = 13.46\%$

C.2 Example 2: Call Option Pricing

Price a call option with $K = 105$, $T = 1$, $r = 0.05$:

$$C = e^{-0.05}\left[0.25\int_{105}^{125}(s-105)\frac{1}{20}ds + 0.50\int_{105}^{105}(s-105)\frac{1}{10}ds + 0\right] \tag{73}$$
$$= 0.9512 \times 0.25 \times \frac{20^2}{2 \times 20} \tag{74}$$
$$= 0.9512 \times 0.25 \times 10 \tag{75}$$
$$= 2.378 \tag{76}$$

The option price is approximately \$2.38.

The End