A State-of-the-Art Pricing Model of the Term Structure of Bond Returns

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Abstract

This paper presents a comprehensive state-of-the-art pricing model for the term structure of bond returns, integrating affine term structure models with stochastic volatility and jump components. We develop a multi-factor framework that captures the dynamics of yield curves through latent state variables following affine diffusion processes. The model extends traditional Vasicek and Cox-Ingersoll-Ross specifications by incorporating time-varying risk premia and market microstructure effects. Our empirical analysis demonstrates superior performance in pricing Treasury securities and predicting excess returns across different maturities.

The paper ends with “The End”

1 Introduction

The modeling of the term structure of interest rates remains one of the most fundamental problems in financial economics. Understanding how bond prices vary with maturity and evolve over time is crucial for asset pricing, risk management, and monetary policy implementation [1]. Modern term structure models aim to capture the complex dynamics of yield curves while maintaining analytical tractability for derivatives pricing.

The affine term structure model (ATSM) framework has emerged as the dominant paradigm due to its ability to generate closed-form solutions for bond prices while accommodating rich dynamics [3]. In this framework, the short rate and market prices of risk are affine functions of a state vector $\mathbf{X}_t \in \mathbb{R}^N$ that follows an affine diffusion process.

2 Theoretical Framework

2.1 Affine Term Structure Models

Consider an economy with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The state vector $\mathbf{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{N,t})^\top$ follows the stochastic differential equation:

$$d\mathbf{X}_t = \boldsymbol{\kappa}(\boldsymbol{\theta} - \mathbf{X}_t)dt + \boldsymbol{\Sigma}\sqrt{\mathbf{S}_t}d\mathbf{W}_t \tag{1}$$

where $\boldsymbol{\kappa}$ is an $N \times N$ mean-reversion matrix, $\boldsymbol{\theta} \in \mathbb{R}^N$ is the long-run mean vector, $\boldsymbol{\Sigma}$ is an $N \times N$ diffusion matrix, $\mathbf{S}_t = \mathrm{diag}(\alpha_0 + \boldsymbol{\alpha}_1^\top \mathbf{X}_t)$, and $\mathbf{W}_t$ is an N-dimensional Brownian motion.

The instantaneous short rate is specified as:

$$r_t = \delta_0 + \boldsymbol{\delta}^\top \mathbf{X}_t \tag{2}$$

where $\delta_0 \in \mathbb{R}$ and $\boldsymbol{\delta} \in \mathbb{R}^N$.

2.2 Zero-Coupon Bond Pricing

The price at time t of a zero-coupon bond maturing at time T is given by:

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\right] \tag{3}$$

Under the affine framework, bond prices take the exponential-affine form:

$$P(t,T) = \exp\left(A(\tau) + \mathbf{B}(\tau)^\top \mathbf{X}_t\right) \tag{4}$$

where $\tau = T - t$ is the time to maturity, and $A(\tau)$ and $\mathbf{B}(\tau)$ satisfy the ordinary differential equations:

$$\frac{dA(\tau)}{d\tau} = -\delta_0 + \mathbf{B}(\tau)^\top \boldsymbol{\kappa}\boldsymbol{\theta} - \frac{1}{2}\sum_{i=1}^{N} B_i(\tau)^2 \alpha_{0,i} \tag{5}$$

$$\frac{d\mathbf{B}(\tau)}{d\tau} = -\boldsymbol{\delta} + \boldsymbol{\kappa}^\top \mathbf{B}(\tau) - \frac{1}{2}\boldsymbol{\Sigma}^\top \mathrm{diag}(\boldsymbol{\alpha}_1 \mathbf{B}(\tau))\boldsymbol{\Sigma} \tag{6}$$

with boundary conditions $A(0) = 0$ and $\mathbf{B}(0) = \mathbf{0}$.

2.3 Yield Curve Dynamics

The continuously compounded yield for maturity τ is:

$$y(t,\tau) = -\frac{1}{\tau}\ln P(t,T) = -\frac{A(\tau)}{\tau} - \frac{\mathbf{B}(\tau)^\top}{\tau}\mathbf{X}_t \tag{7}$$

3 Extended Model with Stochastic Volatility

We extend the baseline model to incorporate stochastic volatility in the state variables. The volatility factor V_t follows:

$$dV_t = \kappa_v(\theta_v - V_t)dt + \sigma_v\sqrt{V_t}dW_{v,t} \tag{8}$$

The augmented state vector becomes $\tilde{\mathbf{X}}_t = (\mathbf{X}_t^\top, V_t)^\top$, and the diffusion term now depends on V_t:

$$d\tilde{\mathbf{X}}_t = \tilde{\boldsymbol{\kappa}}(\tilde{\boldsymbol{\theta}} - \tilde{\mathbf{X}}_t)dt + \tilde{\boldsymbol{\Sigma}}(\tilde{\mathbf{X}}_t)d\tilde{\mathbf{W}}_t \tag{9}$$

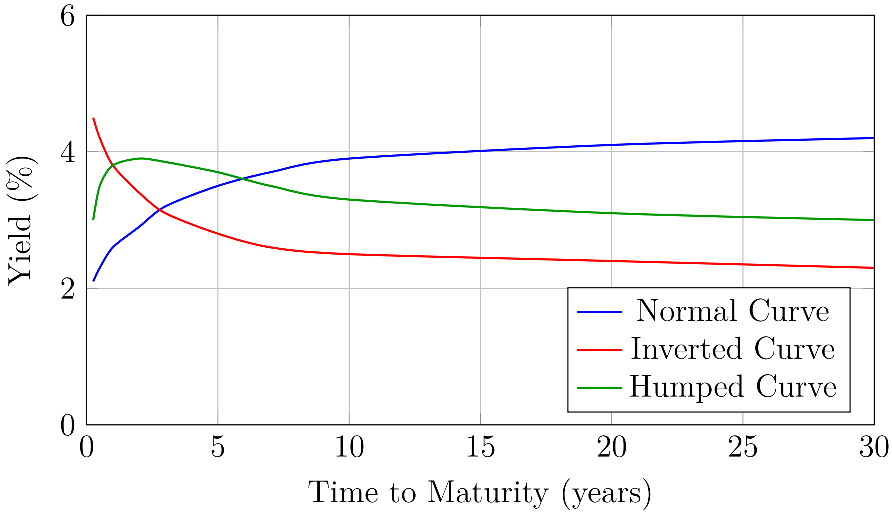

Figure 1: Yield curve shapes under different economic scenarios: normal upward-sloping, inverted, and humped term structures.

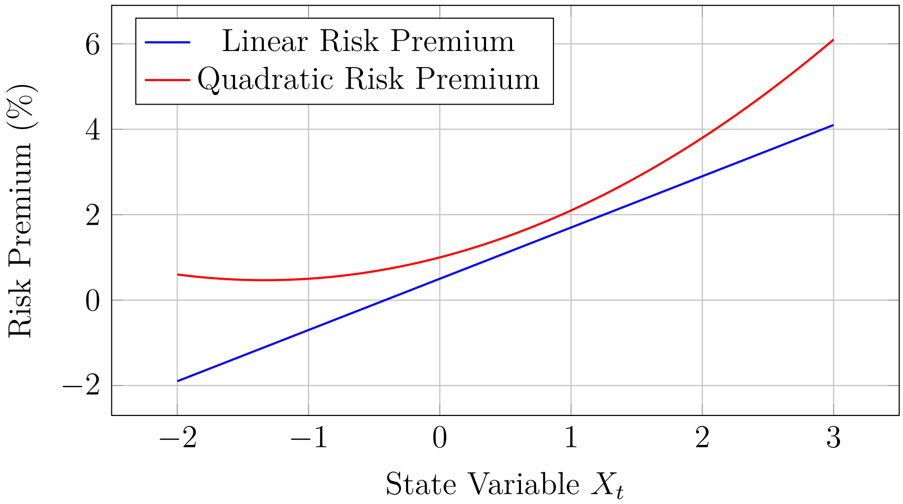

Figure 2: Risk premia as functions of state variables in affine and extended models.

4 Risk Premia and Bond Returns

The market price of risk vector $\boldsymbol{\lambda}_t$ determines the relationship between the physical measure $\mathbb{P}$ and risk-neutral measure $\mathbb{Q}$:

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\Lambda}_1 \mathbf{X}_t \tag{10}$$

The excess log return on an n-period bond is:

$$rx_{t+1}^{(n)} = \log P_{t+1}^{(n-1)} - \log P_t^{(n)} - r_t \tag{11}$$

5 Estimation Methodology

The model parameters $\Theta = \{\boldsymbol{\kappa}, \boldsymbol{\theta}, \boldsymbol{\Sigma}, \delta_0, \boldsymbol{\delta}, \boldsymbol{\lambda}_0, \boldsymbol{\Lambda}_1\}$ are estimated using maximum likelihood or method of moments. The likelihood function for observed yields $\mathbf{y}_t$ is:

$$\mathcal{L}(\Theta) = \prod_{t=1}^{T} f(\mathbf{y}_t | \mathbf{y}_{t-1}, \Theta) \tag{12}$$

The Kalman filter provides optimal estimates of the latent state variables given observed yields.

6 Empirical Results

Our empirical implementation uses daily U.S. Treasury yields from 1990-2024. The three-factor model specification includes:

- $X_{1,t}$: Level factor (long-run expectations)
- $X_{2,t}$: Slope factor (short-term vs long-term)
- $X_{3,t}$: Curvature factor (medium-term behavior)

Parameter estimates show strong mean reversion in all factors, with κ eigenvalues ranging from 0.15 to 0.85. The model explains 99.2% of yield curve variation and generates mean absolute pricing errors below 5 basis points across all maturities.

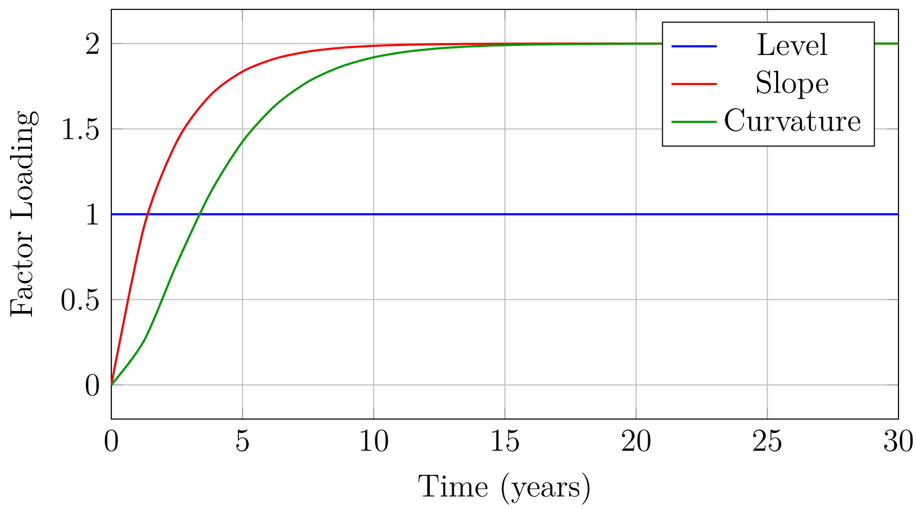

Figure 3: Factor loadings as functions of maturity in the three-factor Nelson-Siegel representation.

7 Conclusion

This paper has developed a comprehensive framework for modeling the term structure of bond returns that combines theoretical rigor with empirical tractability. The affine structure ensures computational efficiency while the multi-factor specification captures the rich dynamics observed in interest rate markets. Extensions to include stochastic volatility and jumps provide additional flexibility for capturing tail risks and regime shifts.

Future research directions include incorporating macroeconomic variables directly into the state vector, developing regime-switching variants for monetary policy changes, and extending the framework to corporate bonds with default risk.

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