The parametric quadratic spline

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Abstract

In this paper, I describe the parametric quadratic spline.
The paper ends with "The End"

Introduction

The parametric quadratic spline is useful in many fields including analytical geometry, economics, finance, ship-building, port-building, data analysis, signal processing and computer graphics.
In this paper, I describe the parametric quadratic spline.

The n-dimensional parametric quadratic spline

The **n-dimensional parametric quadratic spline** is

$$x_1(t) = a_2^1 t^2 + a_1^1 t + a_0^1$$

$$x_2(t) = a_2^2 t^2 + a_1^2 t + a_0^2$$

$$\vdots$$

$$x_n(t) = a_2^n t^2 + a_1^n t + a_0^n$$

where
$n \geq 2$ is the number of dimensions
a_j^i is the j^{th} coefficient of the i^{th} spline
The a_j^is are to be determined from data but
not all sets of data yield a n-dimensional parametric quadratic spline.

Therefore, to demonstrate the mathematics of the parametric quadratic spline, we reduce the scope of this paper to the 2-dimensional parametric quadratic spline.

The 2-dimensional parametric quadratic spline

The **2-dimensional parametric quadratic spline** is

$$x(t) = at^2 + bt + c$$

$$y(t) = et^2 + ft + g$$

There are many ways to determine a, b, c, e, f, and g and we demonstrate a few:

1. **3 points**

   Here, we have

   $t_1 \neq t_2 \neq t_3$

   $x(t_1) = X_1$

   $y(t_1) = Y_1$

   $x(t_2) = X_2$

   $y(t_2) = Y_2$

   $x(t_3) = X_3$

   $y(t_3) = Y_3$

   which can be solved to yield

$$a = \frac{t_3\left(X_2 - X_1\right) + t_2\left(X_1 - X_3\right) + t_1\left(X_3 - X_2\right)}{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(t_2 - t_3\right)}$$

$$b = \frac{t_1^2\left(X_2 - X_3\right) + t_3^2\left(X_1 - X_2\right) + t_2^2\left(X_3 - X_1\right)}{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(t_2 - t_3\right)}$$

$$c = \frac{t_3\left(t_2\left(t_2 - t_3\right)X_1 + t_1\left(t_3 - t_1\right)X_2\right) + t_1\left(t_1 - t_2\right)t_2 X_3}{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(t_2 - t_3\right)}$$

$$e = \frac{t_3\left(Y_2 - Y_1\right) + t_2\left(Y_1 - Y_3\right) + t_1\left(Y_3 - Y_2\right)}{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(t_2 - t_3\right)}$$

$$f = \frac{t_1^2\left(Y_2 - Y_3\right) + t_3^2\left(Y_1 - Y_2\right) + t_2^2\left(Y_3 - Y_1\right)}{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(t_2 - t_3\right)}$$

$$g = \frac{t_3\left(t_2\left(t_2 - t_3\right)Y_1 + t_1\left(t_3 - t_1\right)Y_2\right) + t_1\left(t_1 - t_2\right)t_2 Y_3}{\left(t_1 - t_2\right)\left(t_1 - t_3\right)\left(t_2 - t_3\right)}$$

2. **2 points, each with 1 slope**

Here, we have

$$t_1 \neq t_2$$
$$x(t_1) = X_1$$
$$y(t_1) = Y_1$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_1} = \tan\theta_1$$
$$x(t_2) = X_2$$
$$y(t_2) = Y_2$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_2} = \tan\theta_2$$

which can be solved to yield

$$a = \frac{-\left(X_1 - X_2\right)\left(\tan\left(\theta_1\right) + \tan\left(\theta_2\right)\right) + 2Y_1 - 2Y_2}{\left(t_1 - t_2\right)^2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}$$

$$b = \frac{2\left(t_1\left(\left(X_1 - X_2\right)\tan\left(\theta_1\right) - Y_1 + Y_2\right) + t_2\left(\left(X_1 - X_2\right)\tan\left(\theta_2\right) - Y_1 + Y_2\right)\right)}{\left(t_1 - t_2\right)^2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}$$

$$c = \frac{t_1^2 X_2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right) + t_2^2 X_1\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right) + 2t_2t_1\left(-X_1\tan\left(\theta_1\right) + X_2\tan\left(\theta_2\right) + Y_1 - Y_2\right)}{\left(t_1 - t_2\right)^2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}$$

$$e = \frac{\tan\left(\theta_2\right)\left(2\left(X_2 - X_1\right)\tan\left(\theta_1\right) + Y_1 - Y_2\right) + \left(Y_1 - Y_2\right)\tan\left(\theta_1\right)}{\left(t_1 - t_2\right)^2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}$$

$$f = \frac{2\tan\left(\theta_2\right)\left(\left(t_1 + t_2\right)\left(X_1 - X_2\right)\tan\left(\theta_1\right) + t_1\left(Y_2 - Y_1\right)\right) + 2t_2\left(Y_2 - Y_1\right)\tan\left(\theta_1\right)}{\left(t_1 - t_2\right)^2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}$$

$$g = \frac{2t_2t_1\left(\tan\left(\theta_2\right)\left(\left(X_2 - X_1\right)\tan\left(\theta_1\right) + Y_1\right) - Y_2\tan\left(\theta_1\right)\right) + t_1^2 Y_2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right) + t_2^2 Y_1\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}{\left(t_1 - t_2\right)^2\left(\tan\left(\theta_1\right) - \tan\left(\theta_2\right)\right)}$$

3. **1 point with 1 slope, and 2 more slopes**

Here, we have

$$t_1 \neq t_2 \neq t_3$$
$$x(t_1) = X_1$$
$$y(t_1) = Y_1$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_1} = \tan\theta_1$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_2} = \tan\theta_2$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_3} = \tan\theta_3$$

which is left as an exercise for the reader.

4. **1 point, and 3 more slopes**

Here, we have

$$t_1 \neq t_2 \neq t_3 \neq t_4$$
$$x(t_1) = X_1$$
$$y(t_1) = Y_1$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_2} = \tan\theta_1$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_3} = \tan\theta_2$$
$$\frac{\partial y(t)}{\partial x(t)}|_{t_4} = \tan\theta_3$$

which is left as an exercise for the reader.

The End