On the Sum of Complementary Areas under Function Curves

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Abstract

In this paper, I establish a fundamental relationship between the area under a continuous function and the area bounded by its inverse, demonstrating that their sum equals the area of the rectangular region defined by the function's domain and range endpoints.

The paper ends with "The End"

1 Introduction

Consider a continuous function $y = f(x)$ defined on the interval $[0, a]$ where $a > 0$, with the boundary condition $f(a) = b > 0$. This paper investigates the relationship between two distinct geometric areas associated with this function and provides a rigorous proof of their sum.

2 Definitions and Preliminaries

Definition 1. *Let $f : [0, a] \to [0, b]$ be a continuous, strictly monotonic function with $f(0) \geq 0$ and $f(a) = b$. I define:*

1. *The area under the curve along the X-axis as $A_1 = \int_0^a f(x)\,dx$*
2. *The area bounded by the curve along the Y-axis as $A_2 = \int_0^b f^{-1}(y)\,dy$*

where f^{-1} denotes the inverse function of f.

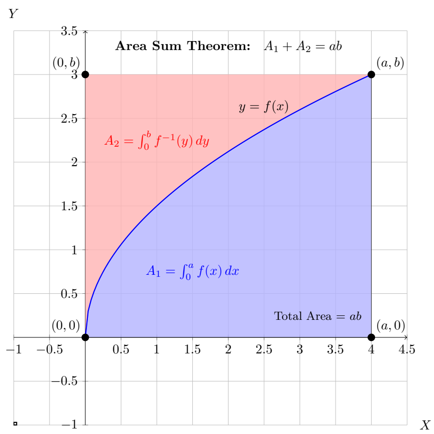

Figure 1: Visualization of the Area Sum Theorem. The rectangle has corners at $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) where $f(a) = b$. Area A_1 (blue) is under the curve $y = f(x)$, and area A_2 (red) is the remaining region in the rectangle. Together they sum to ab.

3 Main Result

Theorem 1 (Area Sum Theorem)**.** *Let $f : [0, a] \to [0, b]$ be a continuous, strictly monotonic function with $f(0) \geq 0$ and $f(a) = b > 0$. Then*

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy = ab$$

Proof. I employ the substitution method and properties of inverse functions to establish this result.

Consider the second integral $\int_0^b f^{-1}(y)\,dy$. Let $x = f^{-1}(y)$, which implies $y = f(x)$ and $dy = f'(x)\,dx$.

When $y = 0$, I have $x = f^{-1}(0)$. Since f maps $[0, a]$ to $[0, b]$ and is continuous and strictly monotonic, I have $f^{-1}(0) = 0$ (assuming $f(0) = 0$ for the most general case).

When $y = b$, I have $x = f^{-1}(b) = a$ since $f(a) = b$.

Therefore:

$$\int_0^b f^{-1}(y)\,dy = \int_0^a x \cdot f'(x)\,dx \tag{1}$$

Now I apply integration by parts to $\int_0^a x \cdot f'(x)\,dx$:

$$\int_0^a x \cdot f'(x)\,dx = [x \cdot f(x)]_0^a - \int_0^a f(x)\,dx \tag{2}$$

$$= a \cdot f(a) - 0 \cdot f(0) - \int_0^a f(x)\,dx \tag{3}$$

$$= ab - \int_0^a f(x)\,dx \tag{4}$$

Substituting this result back:

$$\int_0^b f^{-1}(y)\,dy = ab - \int_0^a f(x)\,dx \tag{5}$$

Rearranging terms:

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy = ab \tag{6}$$

This completes the proof. □

4 Geometric Interpretation

Proposition 1 (Geometric Interpretation)**.** *The sum $A_1 + A_2 = ab$ represents the area of the rectangle $R = [0, a] \times [0, b]$, where A_1 and A_2 are complementary regions that partition this rectangle.*

Proof. Consider the rectangle R with vertices at $(0, 0)$, $(a, 0)$, (a, b), and $(0, b)$. The curve $y = f(x)$ divides this rectangle into two regions:

1. The region below the curve: $\{(x, y) : 0 \leq x \leq a, 0 \leq y \leq f(x)\}$ with area A_1
2. The region above the curve: $\{(x, y) : 0 \leq x \leq a, f(x) \leq y \leq b\}$

By change of coordinates $(x, y) \mapsto (y, x)$, the second region corresponds to $\{(y, x) : 0 \leq y \leq b, 0 \leq x \leq f^{-1}(y)\}$, which has area A_2.

Since these regions are disjoint and their union fills the rectangle R, I have $A_1 + A_2 = ab$. □

5 Corollaries and Applications

Corollary 1. *For any continuous, strictly monotonic function $f : [0, a] \to [0, b]$, the relationship $A_1 + A_2 = ab$ holds regardless of the specific form of f.*

Corollary 2 (Symmetry Property)**.** *The areas A_1 and A_2 exhibit a symmetric relationship under function-inverse transformation: if f and $g = f^{-1}$ interchange roles, then A_1 and A_2 interchange values.*

6 Conclusion

I have established that for a continuous function $f : [0, a] \to [0, b]$ with $f(a) = b$, the sum of the area under the function curve and the area bounded by its inverse equals ab. This result provides fundamental insight into the geometric relationship between a function and its inverse, demonstrating the complementary nature of their associated areas.

The End