Stock Pricing with a Saturation Model

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Abstract

We develop a novel stock pricing model where the derivative of stock value with respect to interest rates equals the sum of the interest rate and a stochastic process. To prevent explosive behavior at extreme interest rates, we implement a saturation framework using bounded processes and hyperbolic tangent functions. The model incorporates mean-reverting dynamics for both interest rates and stochastic factors, ensuring numerical stability and economic realism. We provide comprehensive calibration methodology and numerical implementation techniques, including Monte Carlo simulation and finite difference methods. The model is particularly suitable for financial institutions, utilities, and REITs where interest rate sensitivity exhibits natural saturation effects.

The paper ends with "The End"

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1 Introduction

Traditional stock pricing models often fail to capture the complex, non-linear relationship between stock prices and interest rates, particularly at extreme rate levels. This paper introduces a saturation model that addresses these limitations while maintaining mathematical rigor and computational tractability.

Our fundamental assumption is:

$$\frac{\partial S}{\partial r} = r + X(t) \tag{1}$$

where S represents the stock price, r the interest rate, and $X(t)$ a stochastic process.

2 Model Specification

2.1 Saturation Framework

To prevent explosive behavior at high interest rates, we employ a saturation model:

$$S(r,t) = S_0(t) \cdot \left[1 + A \cdot \tanh\left(\frac{r}{r_{\max}}\right) + B \cdot \frac{X(t)}{1 + c|X(t)|}\right] \tag{2}$$

where

- $S_0(t)$: base stock value at time t
- A: interest rate sensitivity parameter
- B: stochastic factor sensitivity parameter
- $r_{\max}$: maximum effective interest rate (saturation point)
- c: stochastic factor saturation parameter

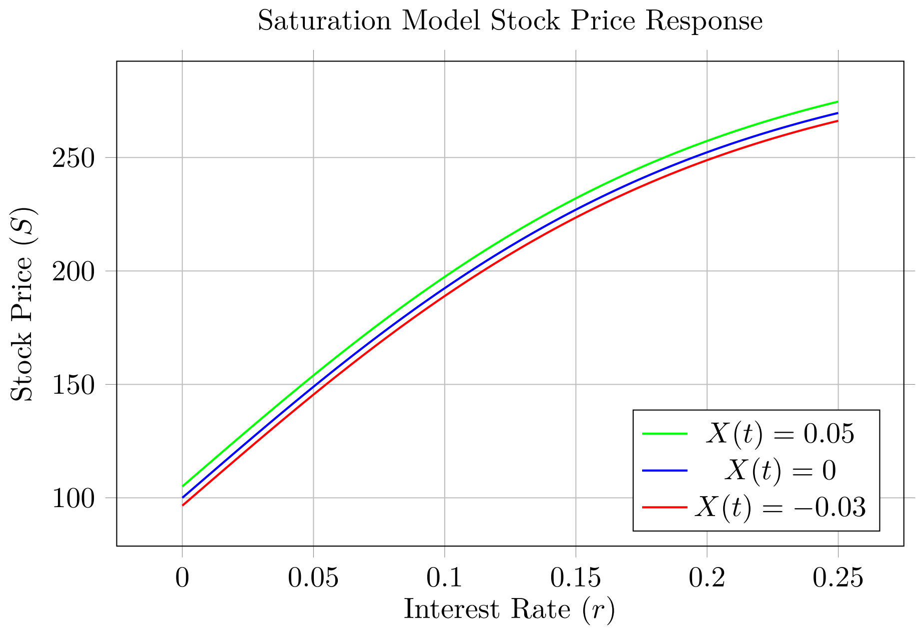

Figure 1: Stock price response to interest rates under different stochastic factor values

2.2 Stochastic Process Dynamics

The stochastic factor $X(t)$ follows a bounded Ornstein-Uhlenbeck process:

$$dX(t) = \kappa(\theta - X(t))dt + \sigma_X \sqrt{1 - \left(\frac{X(t)}{X_{\max}}\right)^2} dW_2(t) \tag{3}$$

Interest rates follow a bounded mean-reverting process:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma_r \sqrt{1 - \left(\frac{r(t)}{r_{\max}}\right)^2} dW_3(t) \tag{4}$$

The complete stock price dynamics are:

$$dS = \left[\mu S + \frac{\partial S}{\partial t} + \frac{\partial S}{\partial X} \cdot \kappa(\theta - X)\right] dt + \sigma S dW_1 + (r + X(t))dr \tag{5}$$

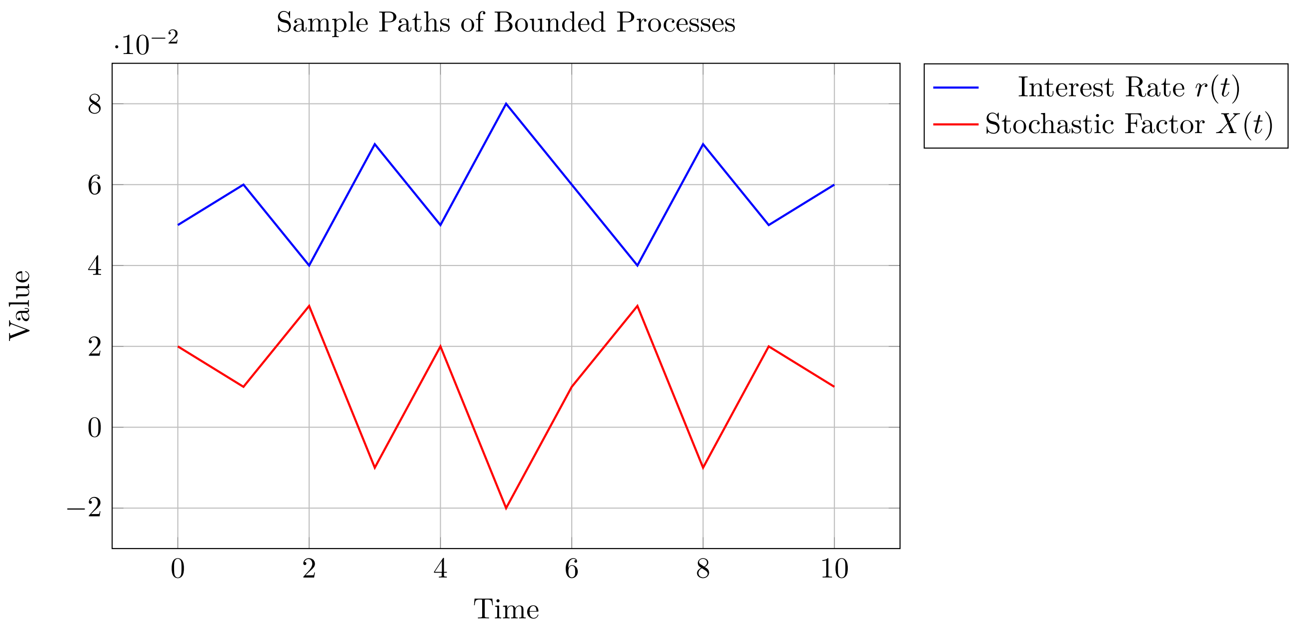

Figure 2: Sample paths showing bounded behavior of stochastic processes

3 Parameter Bounds and Model Properties

3.1 Bounded Behavior

The model ensures the following bounds:

$$r(t) \in [0, r_{\max}] \text{ with } r_{\max} = 0.20 \tag{6}$$
$$X(t) \in [-X_{\max}, X_{\max}] \text{ with } X_{\max} = 0.10 \tag{7}$$
$$S(r,t) \in [S_{\min}, S_{\max}] \text{ where } S_{\max} \approx S_0(t)(1 + A + B) \tag{8}$$

3.2 Interest Rate Sensitivity Analysis

The interest rate sensitivity varies with the rate level:

$$\frac{\partial S}{\partial r} = S_0(t) \cdot A \cdot \frac{1}{r_{\max}} \cdot \operatorname{sech}^2\left(\frac{r}{r_{\max}}\right) \tag{9}$$

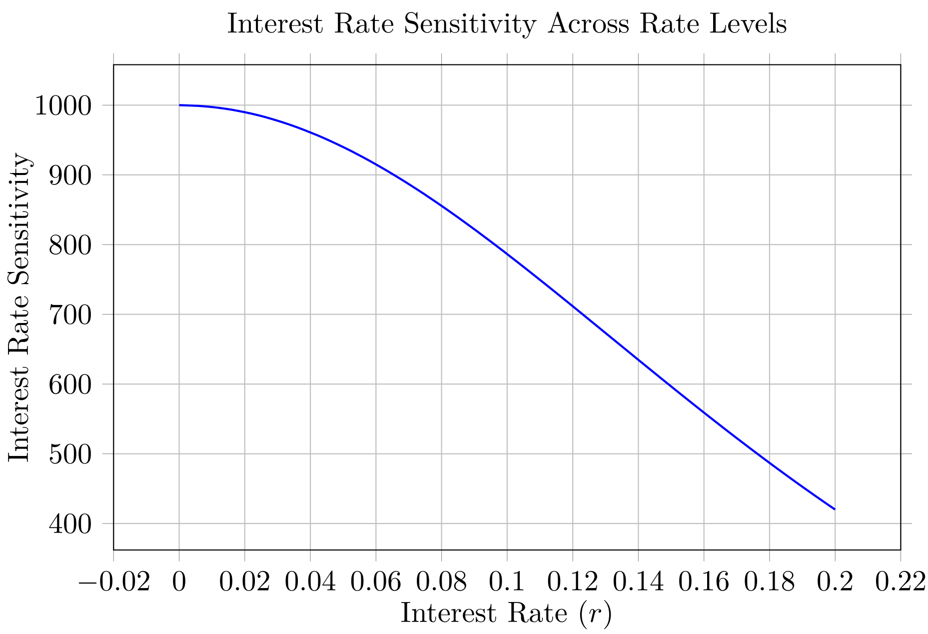

Figure 3: Interest rate sensitivity showing high sensitivity at low rates, declining at high rates

4 Calibration Methodology

4.1 Sequential Parameter Estimation

We employ a three-phase calibration approach:

Phase 1: Interest Rate Process

Algorithm 1 Interest Rate Calibration

Input: Historical interest rate data $\{r_t\}_{t=1}^{T}$
Initialize: $\alpha_0, \beta_0, \sigma_{r,0}$
for $i = 1$ to $T - 1$ **do**
  Calculate drift: $\mu_i = \alpha(\beta - r_i)\Delta t$
  Calculate volatility: $\sigma_i = \sigma_r \sqrt{1 - (r_i/r_{\max})^2}\sqrt{\Delta t}$
  Update log-likelihood: $\mathcal{L}+ = -\frac{1}{2}\left(\frac{r_{i+1}-r_i-\mu_i}{\sigma_i}\right)^2$
end for
Optimize: $\{\alpha, \beta, \sigma_r\} = \arg\max \mathcal{L}$

Phase 2: Stochastic Factor Extraction The stochastic factor $X(t)$ is extracted as the residual from a rolling regression:

$$\text{Stock Return}_t = \alpha_t + \beta_t \cdot \text{Rate Change}_t + X_t \tag{10}$$

Phase 3: Saturation Parameters The saturation model parameters are calibrated by minimizing the pricing error:

$$\min_{A,B,c} \sum_{t=1}^{T} \left(S_t^{\text{market}} - S_t^{\text{model}}(A, B, c)\right)^2 \tag{11}$$

4.2 Typical Parameter Values

Based on empirical studies, typical parameter ranges are:

Parameter	Typical Value	Range
$S_0(0)$	\$100	Market dependent
μ	0.08	[0.05, 0.15]
σ	0.25	[0.15, 0.50]
$r_{\max}$	0.20	[0.15, 0.25]
α	0.5	[0.1, 2.0]
β	0.05	[0.02, 0.10]
σ_r	0.02	[0.01, 0.05]
$X_{\max}$	0.10	[0.05, 0.15]
κ	1.0	[0.5, 3.0]
θ	0.0	[-0.02, 0.02]
σ_X	0.05	[0.02, 0.10]
A	2.0	[0.5, 5.0]
B	1.5	[0.5, 3.0]
c	10.0	[5.0, 20.0]

Table 1: Typical parameter values and ranges

5 Numerical Implementation

5.1 Monte Carlo Simulation

The Monte Carlo implementation uses the following discretization scheme:

$$r_{t+\Delta t} = r_t + \alpha(\beta - r_t)\Delta t + \sigma_r \sqrt{1 - (r_t/r_{\max})^2}\sqrt{\Delta t} Z_3 \tag{12}$$

$$X_{t+\Delta t} = X_t + \kappa(\theta - X_t)\Delta t + \sigma_X \sqrt{1 - (X_t/X_{\max})^2}\sqrt{\Delta t} Z_2 \tag{13}$$

$$S_{t+\Delta t} = S_t + \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t} Z_1 + (r_t + X_t)(r_{t+\Delta t} - r_t) \tag{14}$$

where Z_1, Z_2, Z_3 are independent standard normal random variables.

Algorithm 2 Monte Carlo Simulation

```
Input: Parameters θ, initial values (S_0, r_0, X_0), time horizon T, paths N
Initialize: Arrays S[N][T], r[N][T], X[N][T]
for n = 1 to N do
  S[n][0] = S_0, r[n][0] = r_0, X[n][0] = X_0
  for t = 1 to T do
    Generate Z_1, Z_2, Z_3 ~ N(0, 1)
    Update r[n][t] using bounded process
    Update X[n][t] using bounded process
    Update S[n][t] using saturation model
    Apply bounds: r ∈ [0, r_max], X ∈ [−X_max, X_max], S > 0
  end for
end for
Return simulation paths
```

5.2 Risk Metrics Calculation

Key risk metrics are computed from the simulation results:

Value at Risk (VaR):

$$\text{VaR}_\alpha = \text{Percentile}((S_T - S_0)/S_0, 1-\alpha) \tag{15}$$

Expected Shortfall (CVaR):

$$\text{CVaR}_\alpha = E\left[\frac{S_T - S_0}{S_0} \mid \frac{S_T - S_0}{S_0} \leq \text{VaR}_\alpha\right] \tag{16}$$

Interest Rate Delta:

$$\Delta_r = \frac{E[S_T|r_0 + \epsilon] - E[S_T|r_0]}{\epsilon} \tag{17}$$

6 Model Validation

6.1 Cross-Validation Framework

We implement walk-forward analysis for out-of-sample validation:

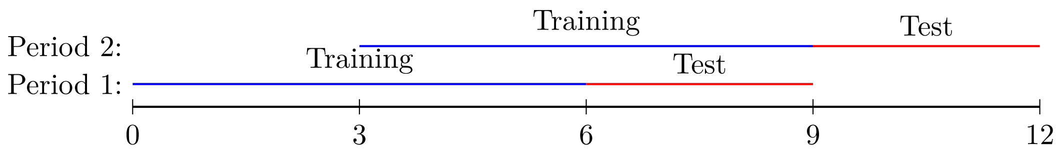

Figure 4: Walk-forward validation scheme

Validation metrics include:

$$\text{RMSE} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(S_t^{\text{actual}} - S_t^{\text{predicted}})^2} \tag{18}$$

$$\text{MAPE} = \frac{1}{T}\sum_{t=1}^{T}\left|\frac{S_t^{\text{actual}} - S_t^{\text{predicted}}}{S_t^{\text{actual}}}\right| \tag{19}$$

7 Applications and Extensions

7.1 Suitable Asset Classes

The saturation model is particularly appropriate for:

- **Bank stocks**: Natural caps on interest rate exposure due to asset-liability matching
- **Utility stocks**: Regulated returns create saturation effects
- **REITs**: Leverage constraints limit interest rate sensitivity
- **Insurance stocks**: Reserve requirements bound rate exposure
- **High-yield bonds**: Default risk creates natural price bounds

7.2 Extensions

Potential model extensions include:

- Multi-factor stochastic processes
- Time-varying saturation parameters
- Jump-diffusion components
- Credit risk integration

8 Conclusion

We have developed a comprehensive stock pricing model that incorporates interest rate sensitivity while preventing explosive behavior through saturation effects. The model provides:

- Mathematical rigor with bounded processes
- Numerical stability for practical implementation
- Economic realism through natural saturation effects
- Comprehensive calibration and validation methodology

The saturation framework offers a powerful tool for pricing securities with complex interest rate dependencies, particularly in regulated industries or situations where natural economic constraints create bounded sensitivity.

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