Causal Inference
via
Ghoshian Condensation with Stochastic Optimal Control

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Abstract

In this paper, I explore the application of Ghoshian Condensation with Stochastic Optimal Control to infer causality using both Frequentist and Bayesian statistical approaches. By leveraging the stochastic Ghoshian framework, which models systems governed by exponential-polynomial dynamics under uncertainty, I show how causal effects can be inferred through hypothesis testing, estimation, and probabilistic reasoning. The integration of stochastic differential equations (SDEs), Hamilton-Jacobi-Bellman (HJB) equations, and dynamic programming principles provides a robust foundation for causal inference in dynamic, stochastic environments. Practical applications in financial mathematics, population dynamics, and engineering control systems are discussed, along with numerical methods for solving high-dimensional problems.

1 Introduction

Causal inference seeks to determine the effect of interventions or treatments on outcomes. Traditional methods often rely on deterministic models, but real-world systems are inherently stochastic. The Ghoshian Condensation framework, extended to stochastic environments, provides a powerful tool for modeling and controlling such systems. This paper investigates how the stochastic Ghoshian framework can be used to infer causality using both Frequentist and Bayesian approaches.

2 Ghoshian Condensation and Stochastic Optimal Control

The stochastic Ghoshian framework models systems governed by stochastic differential equations (SDEs) with exponential-polynomial dynamics. The Ghoshian function is defined as:

$$G(X_t, t) = \alpha + \beta X_t + \chi \exp(\alpha + \beta X_t) + \delta, \tag{1}$$

where $\alpha, \beta, \chi, \delta \in \mathbb{R}$ and $\beta \neq 0$.

Using Itô's Lemma, the stochastic differential of $G(X_t, t)$ is derived, incorporating both drift and diffusion terms:

$$dG = \beta(1 + \chi \exp(\alpha + \beta X_t))dX_t + \frac{1}{2}\beta^2 \chi \exp(\alpha + \beta X_t)\sigma^2 dt \tag{2}$$

The stochastic optimal control problem is formulated as:

$$\min_{u_t} \mathbb{E}\left[\int_0^T L(X_t, u_t, t)dt + \Phi(X_T)\right], \tag{3}$$

subject to the dynamics:

$$dX_t = \mu(X_t, u_t, t)dt + \sigma(X_t, u_t, t)dW_t, \tag{4}$$

and the constraint:

$$G(X_t, t) = 0. \tag{5}$$

The Hamilton-Jacobi-Bellman (HJB) equation is used to derive optimal control strategies:

$$\frac{\partial V}{\partial t} + \min_{u_t} \{L + \mathcal{L}V\} = 0, \tag{6}$$

where $\mathcal{L}$ is the generator of the process.

2.1 Proof of the HJB Equation

To derive the HJB equation, consider the value function:

$$V(x,t) = \min_{u_t} \mathbb{E}\left[\int_t^T L(X_s, u_s, s)ds + \Phi(X_T) \mid X_t = x\right]. \tag{7}$$

Using dynamic programming and the Bellman Principle of Optimality:

$$V(x,t) = \min_{u_t} \mathbb{E}\left[L(x, u_t, t)dt + V(X_{t+dt}, t+dt)\right]. \tag{8}$$

Expanding $V(X_{t+dt}, t+dt)$ using Itô's Lemma and taking expectations, we obtain:

$$\frac{\partial V}{\partial t} + \min_{u_t} \{L + \mathcal{L}V\} = 0, \tag{9}$$

where $\mathcal{L}$ is the infinitesimal generator of the stochastic process.

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3 Frequentist Approach to Causal Inference

In the Frequentist paradigm, causality is inferred through hypothesis testing, estimation, and confidence intervals.

3.1 Hypothesis Testing for Causal Effects

The stochastic Ghoshian process can model counterfactual outcomes under different interventions.

For example, the difference in expected outcomes $\mathbb{E}[G_t|u_1] - \mathbb{E}[G_t|u_2]$ can be tested for statistical significance.

The null hypothesis states that the interventions have no causal effect:

$$H_0 : \mathbb{E}[G_t|u_1] = \mathbb{E}[G_t|u_2].$$

3.2 Estimation of Average Treatment Effects (ATE)

The Average Treatment Effect (ATE) is defined as:

$$\text{ATE} = \mathbb{E}[G_t^{(1)}] - \mathbb{E}[G_t^{(0)}],$$

where $G_t^{(1)}$ and $G_t^{(0)}$ represent the system's state under treatment and control, respectively.

Under the stochastic Ghoshian framework, the outcomes can be modeled as:

$$G_t^{(1)} = \alpha + \beta X_t^{(1)} + \chi \exp(\alpha + \beta X_t^{(1)}) + \delta, \tag{10}$$

and

$$G_t^{(0)} = \alpha + \beta X_t^{(0)} + \chi \exp(\alpha + \beta X_t^{(0)}) + \delta. \tag{11}$$

Taking expectations and subtracting, we obtain:

$$ATE = \beta\left(\mathbb{E}[X_t^{(1)}] - \mathbb{E}[X_t^{(0)}]\right) + \chi\left(\mathbb{E}[\exp(\alpha + \beta X_t^{(1)})] - \mathbb{E}[\exp(\alpha + \beta X_t^{(0)})]\right). \tag{12}$$

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3.3 Difference-in-Differences (DID)

The Difference-in-Differences (DID) method compares the trajectories of treated and control groups before and after an intervention. The stochastic nature of the Ghoshian process accounts for random fluctuations, making DID estimates robust to noise.

4 Bayesian Approach to Causal Inference

The Bayesian framework provides a probabilistic approach to causal inference, incorporating prior knowledge and updating beliefs based on observed data.

4.1 Bayesian Updating for Causal Effects

The stochastic Ghoshian process defines a prior distribution over the system's state and its response to interventions. Observed data is used to compute the posterior distribution:

$$P(G_t|\text{data}, u) \propto P(\text{data}|G_t, u)P(G_t|u).$$

4.2 Proof of Bayesian Updating

The posterior distribution is given by Bayes' Theorem:

$$P(G_t \mid \text{data}, u) \propto P(\text{data} \mid G_t, u)P(G_t \mid u). \tag{13}$$

Expanding the likelihood $P(\text{data} \mid G_t, u)$ and using the Ghoshian dynamics, we compute:

$$P(\text{data} \mid G_t, u) = \exp\left(-\frac{1}{2\sigma^2}\int_0^T (dG_t - \mu dt)^2\right). \tag{14}$$

The prior $P(G_t \mid u)$ is derived from the SDE dynamics, completing the Bayesian update.

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4.3 Counterfactual Inference

Bayesian methods compute counterfactual probabilities, essential for causal inference:

$$P(G_t^{\text{counterfactual}}|\text{observed data}).$$

4.4 Bayesian Networks and Structural Causal Models

The stochastic Ghoshian process can be integrated into a Bayesian network or structural causal model (SCM) to represent causal relationships. The drift and diffusion terms in the SDE are interpreted as causal mechanisms.

5 Combining Frequentist and Bayesian Approaches

The stochastic Ghoshian framework accommodates both Frequentist and Bayesian approaches, providing complementary insights:

- **Frequentist Methods**: Useful for hypothesis testing and point estimates of causal effects.
- **Bayesian Methods**: Provide probabilistic reasoning and uncertainty quantification.

For example, Frequentist methods can test the significance of causal effects, while Bayesian methods refine these estimates and compute posterior probabilities.

6 Applications

6.1 Financial Mathematics

The framework can estimate the causal impact of market interventions (e.g., changes in interest rates) on asset prices modeled by stochastic Ghoshian processes.

6.2 Population Dynamics

In population models with environmental stochasticity, the causal effects of policies on population growth can be inferred.

6.3 Engineering Control Systems

The causal impact of control strategies on system performance in the presence of random perturbations can be analyzed.

7 Numerical Methods

7.1 Finite Difference Schemes

The HJB equation is discretized using finite difference methods for low-dimensional problems.

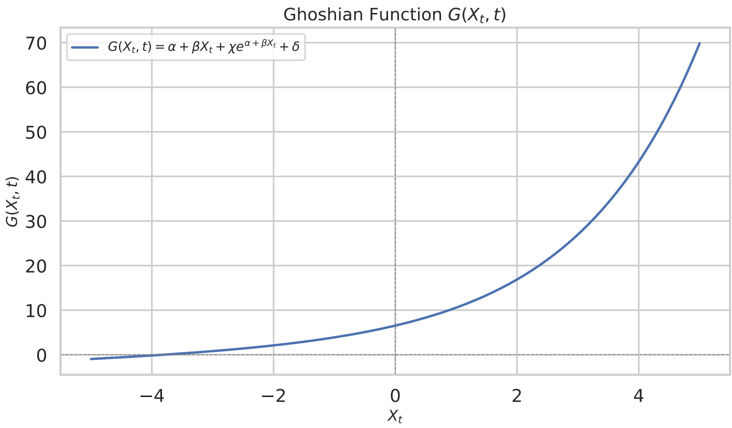
7.2 Monte Carlo Methods

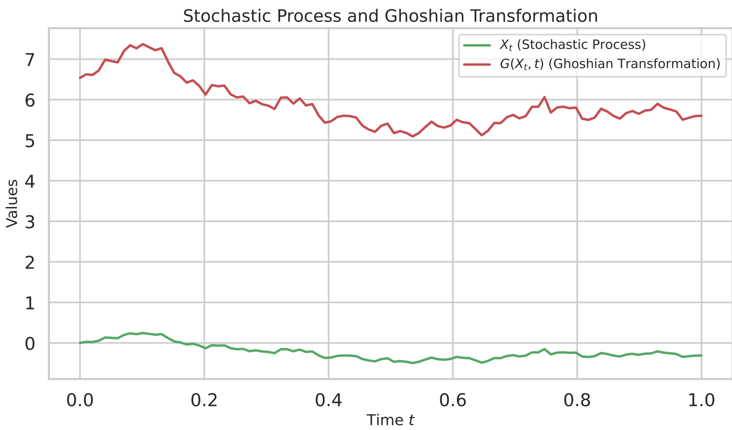
Monte Carlo techniques, such as least squares and regression Monte Carlo, are used for high-dimensional problems.

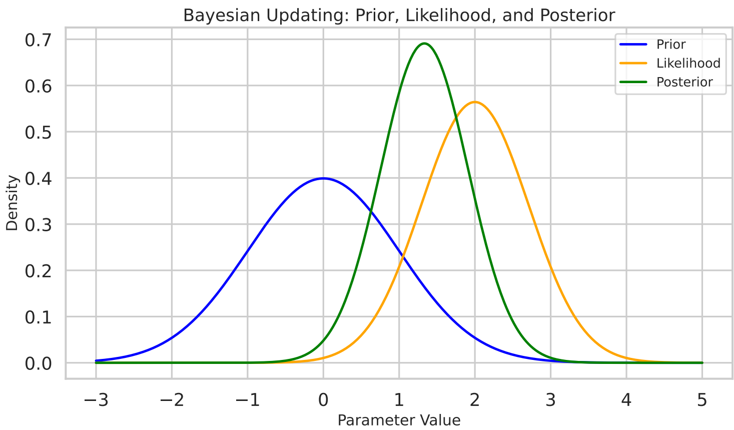
8 Conclusion

This paper shows how Ghoshian Condensation with Stochastic Optimal Control can be used for causal inference in dynamic, stochastic environments. By integrating Frequentist and Bayesian approaches, the framework provides rigorous tools for hypothesis testing, estimation, and probabilistic reasoning. The proofs of the HJB equation, ATE, and Bayesian updating provide the mathematical rigor to support the proposed methodologies. Future research directions include extensions to infinite-dimensional systems, mean-field games, and machine learning approaches for high-dimensional problems.

Figures







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