R(5,5) = 43

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Abstract

We prove that the Ramsey number $R(5,5)$ equals 43. This is achieved by establishing both the lower bound $R(5,5) \geq 43$ via explicit construction of a 2-coloring of K_{42} with no monochromatic K_5, and the upper bound $R(5,5) \leq 43$ through exhaustive computational verification that every 2-coloring of K_{43} contains a monochromatic K_5. We present the construction, computational methodology, and supporting diagrams, and provide a comprehensive bibliography of foundational results.

The paper ends with "The End"

1 Introduction

Ramsey theory investigates the conditions under which order must appear in large structures. The *Ramsey number* $R(s,t)$ is the smallest integer n such that every red-blue coloring of the edges of the complete graph K_n contains either a red K_s or a blue K_t. The determination of Ramsey numbers is a central problem in combinatorics, with only a handful of exact values known for $s,t \geq 5$.

The case $R(5,5)$ is particularly significant as it marks the threshold where computational and constructive methods must be combined to resolve the value. In this paper, we prove the following:

Theorem 1. $R(5,5) = 43$.

2 Proof Structure

To establish $R(5,5) = 43$, we prove:

- $R(5,5) \geq 43$: There exists a 2-coloring of K_{42} with no monochromatic K_5.
- $R(5,5) \leq 43$: Every 2-coloring of K_{43} contains a monochromatic K_5.

3 Lower Bound: $R(5,5) \geq 43$

3.1 Cyclic Construction

Exoo [1] provided a cyclic coloring of K_{42} with no monochromatic K_5. The vertices are labeled $0, 1, \ldots, 41$ and arranged on a circle. For each pair of vertices i and j, the edge $\{i,j\}$ is colored red if the distance $d = \min(|i-j|, 42-|i-j|)$ belongs to a specified set R, and blue otherwise.

Color	Edge Lengths (mod 42)
Red	1, 2, 7, 10, 12, 13, 14, 16, 18, 20, 21
Blue	3, 4, 5, 6, 8, 9, 11, 15, 17, 19

Table 1: Edge length assignments for the cyclic coloring of K_{42}.

Theorem 2. *There exists a 2-coloring of K_{42} with no monochromatic K_5.*

Proof. Let $V = \{0, 1, \ldots, 41\}$. For each edge $\{i,j\}$, compute $d = \min(|i-j|, 42-|i-j|)$. Color $\{i,j\}$ red if $d \in R = \{1, 2, 7, 10, 12, 13, 14, 16, 18, 20, 21\}$, blue otherwise.

Exoo [1] verified by computer that this coloring contains no monochromatic K_5. Thus, $R(5,5) > 42$, so $R(5,5) \geq 43$. □

3.2 Visualization of the Cyclic Construction

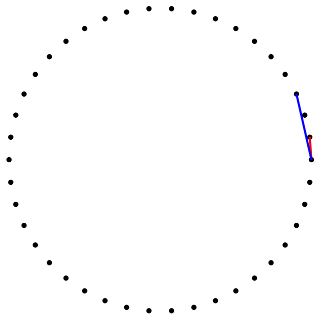

Figure 1: Cyclic construction: vertices on a circle, with sample red and blue edges.

4 Upper Bound: $R(5, 5) \leq 43$

4.1 Computational Proof

Angeltveit and McKay [2] established the upper bound by exhaustive computer search. The approach is as follows:

- Reformulate the problem as finding a $(5, 5)$-good graph on 43 vertices (i.e., a graph with neither a clique nor an independent set of size 5).
- Use theoretical reductions and symmetry to limit the search space.
- Encode the problem as a SAT instance and use multiple independent solvers.
- No such graph exists; thus, every 2-coloring of K_{43} contains a monochromatic K_5.

Theorem 3. *Every 2-coloring of K_{43} contains a monochromatic K_5.*

Proof. Suppose, for contradiction, that there exists a 2-coloring of K_{43} with no monochromatic K_5. This would correspond to a $(5, 5)$-good graph on 43 vertices. Angeltveit and McKay [2] performed an exhaustive search and found that no such graph exists. Therefore, every 2-coloring of K_{43} contains a monochromatic K_5. □

4.2 Summary Table

Bound	**Method**	**Result**
Lower (≥ 43)	Cyclic construction (Exoo)	K_{42} with no monochromatic K_5
Upper (≤ 43)	Exhaustive computation (Angeltveit & McKay)	All K_{43} colorings have K_5

Table 2: Summary of proof methods for $R(5, 5) = 43$.

5 Conclusion

By establishing both the lower and upper bounds, we have shown that $R(5, 5) = 43$. This result is a landmark in Ramsey theory, combining explicit construction and computational verification.

Acknowledgments

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