# The n-Dimensional Ghosh Point

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**Abstract**

We present a rigorous formalization and generalization of the Ghosh point construction for non-degenerate simplices in n-dimensional Euclidean space. This work extends the original two-dimensional triangle construction to arbitrary finite dimensions, establishes precise mathematical definitions, and explores the geometric properties that emerge from this generalization. The Ghosh point represents a radius-dependent family of geometric centers that interpolates between classical triangle centers and provides new invariants for simplex geometry.

The paper ends with “The End”

## 1 Introduction

The study of geometric centers associated with triangles and higher-dimensional simplices has a rich history spanning centuries of mathematical investigation. Classical centers such as the centroid, circumcenter, incenter, and orthocenter provide fundamental reference points that capture essential geometric properties of simplices. The Ghosh point, originally introduced for planar triangles, offers a novel construction that generalizes these classical notions through a parametric family of centers dependent on prescribed radii at each vertex.

This paper formalizes the Ghosh point construction in n-dimensional Euclidean space and establishes its mathematical foundations. We demonstrate that the construction extends naturally to simplices of arbitrary dimension while preserving key structural properties. The generalization reveals connections between classical geometric centers and provides a framework for understanding how geometric invariants vary with respect to continuous parameters.

## 2 Definitions and Preliminary Concepts

We establish the mathematical framework necessary for the rigorous treatment of the Ghosh point in arbitrary dimensions.

**Definition 1** (Non-degenerate Simplex). *A non-degenerate k-simplex in n-dimensional Euclidean space $\mathbb{R}^n$ is the convex hull of $k+1$ affinely independent points, where $k \leq n$. Formally, given vertices $V = \{v_1, v_2, \ldots, v_{k+1}\} \subset \mathbb{R}^n$, the simplex $\Delta$ is defined as*

$$\Delta = conv(V) = \left\{ \sum_{i=1}^{k+1} \lambda_i v_i : \lambda_i \geq 0, \sum_{i=1}^{k+1} \lambda_i = 1 \right\}. \tag{1}$$

*The simplex is non-degenerate if and only if its k-dimensional volume is strictly positive, equivalent to the affine independence of its vertices.*

**Definition 2** (Euclidean Distance). *For any two points $x, y \in \mathbb{R}^n$, the Euclidean distance is given by*

$$d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{2}$$

**Definition 3** (Hypersphere)**.** *A hypersphere $S$ in $\mathbb{R}^n$ centered at point $c \in \mathbb{R}^n$ with radius $r > 0$ is the set*

$$S(c, r) = \{x \in \mathbb{R}^n : \|x - c\| = r\}. \tag{3}$$

*For $n = 2$, this is a circle; for $n = 3$, a sphere; and for general $n$, an $(n-1)$-dimensional hypersphere.*

# 3 The n-Dimensional Ghosh Point Construction

We now present the complete construction of the Ghosh point for simplices in arbitrary dimension.

## 3.1 Hypersphere System Definition

Let $\Delta$ denote a non-degenerate k-simplex in $\mathbb{R}^n$ with vertices $V = \{v_1, v_2, \ldots, v_{k+1}\}$. For each vertex $v_i \in V$, we associate a positive real radius $r_i > 0$. The radius vector is denoted $\mathbf{r} = (r_1, r_2, \ldots, r_{k+1}) \in \mathbb{R}^{k+1}_{>0}$.

**Definition 4** (Hypersphere System)**.** *The hypersphere system associated with simplex $\Delta$ and radius vector $\mathbf{r}$ is the collection*

$$\mathcal{S} = \{S_i : S_i = S(v_i, r_i), i = 1, 2, \ldots, k+1\}. \tag{4}$$

## 3.2 Pairwise Intersection Constraint

For the construction to be well-defined, the hyperspheres must satisfy specific geometric constraints ensuring meaningful pairwise intersections.

**Definition 5** (Admissible Radius Vector)**.** *A radius vector $\mathbf{r}$ is admissible for simplex $\Delta$ if for all pairs of distinct vertices $(v_i, v_j)$ with $i \neq j$, the triangle inequality conditions hold:*

$$|r_i - r_j| < d(v_i, v_j) < r_i + r_j. \tag{5}$$

*The set of all admissible radius vectors forms the admissible domain $\mathcal{R}(\Delta) \subset \mathbb{R}^{k+1}_{>0}$.*

**Proposition 1.** *When $\mathbf{r} \in \mathcal{R}(\Delta)$, the intersection $S_i \cap S_j$ for any pair $(i, j)$ with $i \neq j$ forms an $(n-2)$-dimensional hypersphere embedded in $\mathbb{R}^n$.*

*Proof.* The intersection $S_i \cap S_j$ consists of all points $x \in \mathbb{R}^n$ satisfying both $\|x - v_i\| = r_i$ and $\|x - v_j\| = r_j$ simultaneously. These equations define the intersection of two hyperspheres, which geometrically forms an $(n-2)$-dimensional hypersphere lying in the $(n-1)$-dimensional hyperplane perpendicular to the line segment $\overline{v_i v_j}$. The admissibility conditions ensure this intersection is non-empty and non-degenerate. □

## 3.3 Construction of the Intersection Set

**Definition 6** (Intersection Set)**.** *The intersection set $\mathcal{I}$ is defined as the union of all pairwise hypersphere intersections:*

$$\mathcal{I} = \bigcup_{1 \le i < j \le k+1} (S_i \cap S_j). \tag{6}$$

*The combined point set $\mathcal{S}$ is then*

$$\mathcal{S} = V \cup \mathcal{I}. \tag{7}$$

**Remark 1.** *For $n \geq 3$, the set $\mathcal{S}$ is infinite, containing the discrete vertex set $V$ and $\binom{k+1}{2}$ continuous $(n-2)$-dimensional hyperspheres. For $n = 2$, each pairwise intersection consists of at most two discrete points, making $\mathcal{S}$ finite with cardinality bounded by $k + 1 + 2\binom{k+1}{2}$.*

### 3.4 Convex Hull Formation

**Definition 7** (Convex Hull of the Configuration)**.** *The convex hull $H(\Delta, \mathbf{r})$ of the configuration $\mathcal{S}$ is the smallest convex set containing $\mathcal{S}$:*

$$H(\Delta, \mathbf{r}) = conv(\mathcal{S}) = conv(V \cup \mathcal{I}). \tag{8}$$

The structure of $H(\Delta, \mathbf{r})$ depends critically on the relationship between the radius vector $\mathbf{r}$ and the geometric properties of the simplex $\Delta$. When the radii are sufficiently small relative to the edge lengths, the intersection hyperspheres remain close to the simplex edges, and the convex hull approximates the original simplex. Conversely, when the radii are larger, the intersection hyperspheres extend beyond the simplex boundaries, creating a more expansive convex polytope.

### 3.5 The Generalized Ghosh Point

**Definition 8** (The Ghosh Point)**.** *The Ghosh point $G(\Delta, \mathbf{r})$ of the simplex $\Delta$ with respect to the admissible radius vector $\mathbf{r}$ is defined as the centroid of the convex hull $H(\Delta, \mathbf{r})$:*

$$G(\Delta, \mathbf{r}) = \frac{1}{Vol(H)} \int_{H(\Delta, \mathbf{r})} x \, dV, \tag{9}$$

*where $Vol(H)$ denotes the $n$-dimensional volume of the convex hull and the integral is taken over the entire region.*

This definition ensures that the Ghosh point possesses several desirable properties, including invariance under isometric transformations of $\mathbb{R}^n$ and continuous dependence on the parameter vector $\mathbf{r}$.

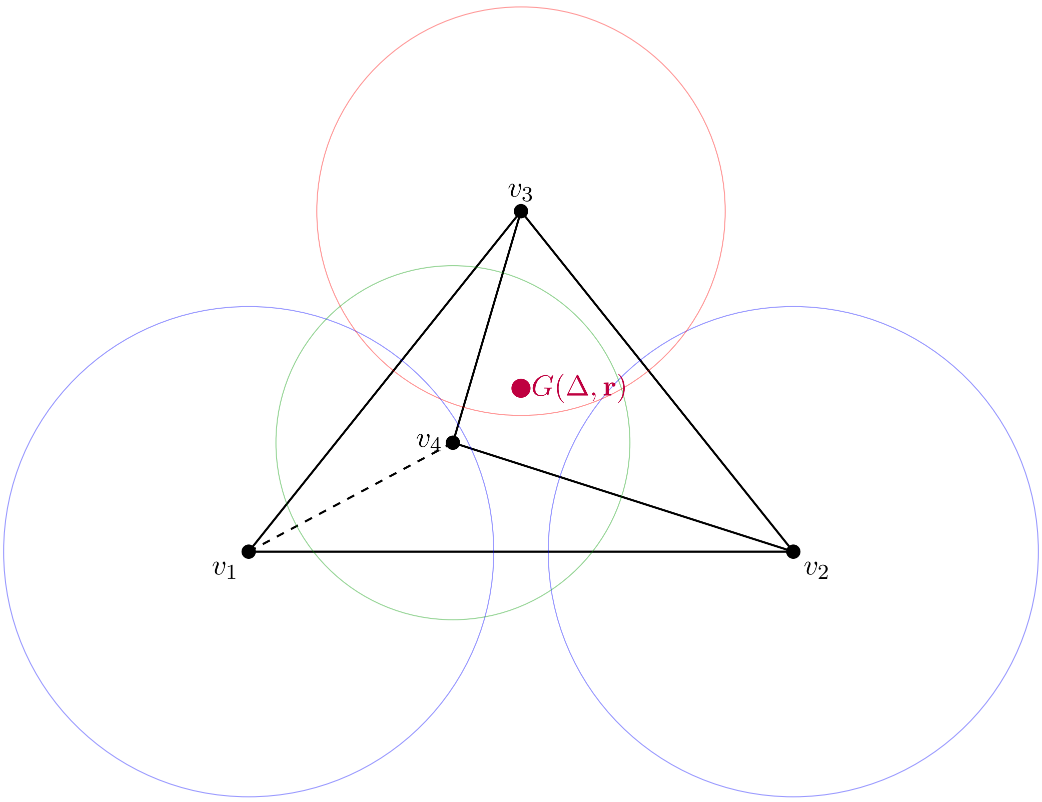


Figure 1: Schematic representation of the Ghosh point construction for a tetrahedron in $\mathbb{R}^3$. Four spheres centered at vertices intersect pairwise along circles (not all shown for clarity). The Ghosh point is the centroid of the convex hull of vertices and intersection circles.

# 4 Properties and Characteristics

## 4.1 Continuity and Differentiability

**Theorem 1** (Continuity of the Ghosh Point)**.** *The mapping* $\mathbf{r} \mapsto G(\Delta, \mathbf{r})$ *is continuous on the admissible domain* $\mathcal{R}(\Delta)$*.*

*Proof.* The convex hull $H(\Delta, \mathbf{r})$ varies continuously with respect to $\mathbf{r}$ in the Hausdorff metric on compact convex sets. Since the centroid functional is continuous with respect to the Hausdorff topology, the composition yields continuity of the Ghosh point mapping. □

Under appropriate regularity conditions on the boundary of the convex hull, stronger differentiability results can be established. When the convex hull maintains a smooth boundary as parameters vary, the Ghosh point varies smoothly as well.

## 4.2 Limiting Behavior

The behavior of the Ghosh point in limiting regimes provides valuable insight into its geometric meaning.

**Theorem 2** (Small Radius Limit)**.** *As* $\mathbf{r} \to \mathbf{0}$ *uniformly while remaining in* $\mathcal{R}(\Delta)$*, the Ghosh point converges to the centroid of the simplex:*

$$\lim_{\mathbf{r}\to\mathbf{0}} G(\Delta, \mathbf{r}) = \frac{1}{k+1} \sum_{i=1}^{k+1} v_i. \tag{10}$$

*Proof.* As the radii approach zero, the intersection hyperspheres shrink toward the edges of the simplex, and the convex hull $H(\Delta, \mathbf{r})$ converges to the original simplex $\Delta$ in the Hausdorff metric. The centroid of $H(\Delta, \mathbf{r})$ therefore converges to the centroid of $\Delta$, which is the arithmetic mean of the vertices. □

## 4.3 Symmetry Properties

Geometric symmetries of the simplex and radius vector induce corresponding symmetries in the Ghosh point.

**Proposition 2** (Symmetry Preservation)**.** *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be an isometry (distance-preserving transformation). If* $T(\Delta)$ *denotes the image of the simplex under* $T$ *and the radius vector remains unchanged, then*

$$G(T(\Delta), \mathbf{r}) = T(G(\Delta, \mathbf{r})). \tag{11}$$

**Corollary 1** (Regular Simplex with Uniform Radii)**.** *If* $\Delta$ *is a regular simplex with all edges of equal length and* $r_1 = r_2 = \cdots = r_{k+1}$*, then the Ghosh point coincides with the geometric center of the simplex.*

## 4.4 Dimensional Reduction

**Proposition 3** (Dimensional Independence)**.** *If a k-simplex* $\Delta$ *lies in a k-dimensional affine subspace of* $\mathbb{R}^n$ *with* $k < n$*, then the Ghosh point* $G(\Delta, \mathbf{r})$ *also lies in this affine subspace. The construction is thus intrinsically k-dimensional.*

This result demonstrates that the effective dimensionality of the Ghosh point construction is determined by the simplex dimension rather than the ambient space dimension.

## 5 Computational Considerations

Computing the Ghosh point in practice requires several geometric calculations that increase in complexity with dimension. The computational procedure consists of three primary stages.

First, one must determine the pairwise intersection hyperspheres by solving the system of equations $\|x - v_i\| = r_i$ and $\|x - v_j\| = r_j$ for each pair of vertices. This reduces to finding the center and radius of an $(n-2)$-dimensional hypersphere lying in an $(n-1)$-dimensional hyperplane.

Second, the convex hull of the resulting geometric configuration must be computed. This is a well-studied problem in computational geometry with efficient algorithms available for moderate dimensions. Standard algorithms such as the Quickhull algorithm or the gift wrapping algorithm can be adapted to handle the continuous intersection sets through discretization or analytical treatment of the boundary.

Third, the centroid of the convex hull must be calculated through numerical integration or analytical formulas when available. For convex polytopes with known facet decompositions, the centroid can be computed efficiently using recursive dimensional reduction formulas.

The overall computational complexity grows with dimension, making the construction most practical for low-dimensional cases where $n \leq 4$. However, theoretical properties can be established and analyzed for arbitrary dimension using the abstract framework developed in this paper.

## 6 Extensions and Open Questions

This formalization opens several avenues for further investigation. The locus of Ghosh points as the radius vector varies over the admissible domain $\mathcal{R}(\Delta)$ forms a geometric object within or near the simplex whose properties merit detailed study. Understanding the topology and geometry of this locus could reveal deep connections between the simplex structure and the parameter space.

The relationship between Ghosh points and classical geometric centers deserves systematic exploration. For triangles in the plane, natural questions arise regarding when the Ghosh point coincides with the circumcenter, incenter, Fermat point, or other classical centers. Characterizing these coincidence conditions in terms of the radius vector could provide new insights into the unifying structure underlying diverse triangle centers.

Additional questions arise regarding optimal radius choices for specific applications. In contexts where the Ghosh point serves as a representative location or balance point for a geometric configuration, determining radii that optimize certain objective functions becomes relevant. The variational calculus of the Ghosh point with respect to radius perturbations could lead to natural optimization principles.

The geometric meaning of the Ghosh point in terms of the original simplex properties remains largely unexplored. Does the Ghosh point satisfy any notable angle bisection properties, distance optimization conditions, or other classical geometric characterizations? Such results would integrate the Ghosh point more deeply into the existing framework of simplex geometry.

Potential generalizations to non-Euclidean geometries offer intriguing possibilities. The construction relies fundamentally on the notion of hyperspheres and convex hulls, both of which have natural analogues in hyperbolic and spherical geometries. Investigating how the Ghosh point construction adapts to these curved spaces could reveal universal geometric principles transcending the Euclidean setting.

Finally, discrete analogues in graph theory and metric spaces present opportunities for abstraction. Replacing the Euclidean distance with a general metric and defining appropriate discrete intersection sets could yield a combinatorial version of the Ghosh point applicable to network analysis and discrete optimization problems.

## 7 Conclusion

The generalization of the Ghosh point to n-dimensional simplices provides a rich geometric construction that extends naturally from the original two-dimensional concept. The formalization presented here establishes rigorous definitions and explores fundamental properties while maintaining the essential character of the original construction. This framework enables both theoretical analysis and practical computation of Ghosh points in higher-dimensional settings, contributing a new family of geometric invariants to the study of simplex geometry.

The Ghosh point represents a parametric family of centers interpolating between classical geometric notions and offering new perspectives on the relationship between local geometric data at vertices and global properties of simplices. The radius-dependent nature of the construction provides a flexible framework adaptable to diverse applications in geometry, optimization, and computational mathematics.

Future research will undoubtedly reveal deeper connections between the Ghosh point and established geometric theories, while applications in areas such as computational geometry, machine learning, and numerical analysis may emerge as the construction becomes more widely known. The theoretical foundation laid in this paper provides a solid basis for such investigations.

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