A useful three-matrix identity with special properties

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Abstract

We exhibit an explicit triple of 9×9 matrices M, N, O of separable form $M_{ij} = \sqrt{x_i x_j}$, $N_{ij} = \sqrt{y_i y_j}$ and $O_{ij} = \sqrt{z_i z_j}$ with all underlying scalars in $(0, 1)$ and satisfying the exact identity $M + N = O$. We show that such an identity is possible if and only if the generating vectors are collinear, and we provide a fully worked numerical example together with its spectral and geometric interpretation.

The paper ends with "The End"

1 General structure

Let

$$X = (\sqrt{x_1}, \dots, \sqrt{x_9})^\top, \qquad x_i \in (0, 1), \tag{1}$$

and define

$$M = XX^\top. \tag{2}$$

We seek matrices

$$N = YY^\top, \qquad O = ZZ^\top, \tag{3}$$

with

$$Y = (\sqrt{y_1}, \dots, \sqrt{y_9})^\top, \qquad Z = (\sqrt{z_1}, \dots, \sqrt{z_9})^\top, \tag{4}$$

and $y_i, z_i \in (0, 1)$, such that

$$M + N = O. \tag{5}$$

1.1 Rank and collinearity

Since M and N are rank–one matrices, $M + N$ can be rank one if and only if the generating vectors X and Y are collinear. Therefore the identity $M + N = ZZ^\top$ is possible if and only if

$$Y = cX \tag{6}$$

for some $c > 0$. In that case,

$$N = c^2 XX^\top, \tag{7}$$

and

$$M + N = (1 + c^2) XX^\top. \tag{8}$$

Defining

$$Z = \sqrt{1 + c^2}, X \tag{9}$$

produces

$$O = ZZ^\top = (1 + c^2) XX^\top = M + N. \tag{10}$$

Componentwise, all solutions are therefore of the form

$$y_i = c^2 x_i, \qquad z_i = (1 + c^2) x_i, \qquad i = 1, \dots, 9. \tag{11}$$

The admissibility constraints $0 < y_i < 1$ and $0 < z_i < 1$ are equivalent to

$$(1 + c^2) x_i < 1, \qquad i = 1, \dots, 9. \tag{12}$$

2 Worked numerical triple

We now construct a concrete triple (M, N, O). Let

$$x_i = \left(\frac{i}{10}\right)^2, \qquad i = 1, \ldots, 9. \tag{13}$$

Then

$$X = \frac{1}{10}(1, 2, 3, 4, 5, 6, 7, 8, 9)^\top. \tag{14}$$

Since $\max_i x_i = 0.81$, choose

$$c = 0.4. \tag{15}$$

This ensures $(1 + c^2)x_i < 1$ for all i.

The matrices are

$$M = XX^\top = \frac{1}{100}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{pmatrix}, \tag{16}$$

$$N = YY^\top = c^2 M = \frac{1}{625}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{pmatrix}, \tag{17}$$

$$O = ZZ^\top = (1 + c^2)M = \frac{29}{2500}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 \end{pmatrix}. \tag{18}$$

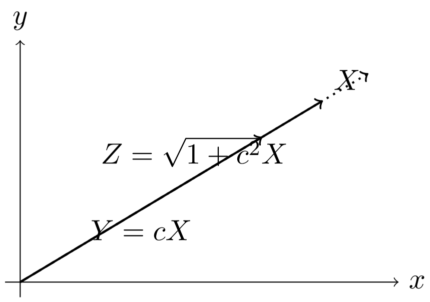
By construction,

$$M + N = O. \tag{19}$$

3 Spectral and geometric interpretation

All three matrices share the same one–dimensional eigenspace spanned by X. Their only nonzero eigenvalues are

$$\lambda_M = |X|^2, \qquad \lambda_N = c^2|X|^2, \qquad \lambda_O = (1 + c^2)|X|^2. \tag{20}$$



The identity $M + N = O$ is therefore a direct consequence of all three matrices being generated by the same latent direction.

References

[1] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.

[2] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th ed., Johns Hopkins University Press, 2013.

[3] H. Lütkepohl, *New Introduction to Multiple Time Series Analysis*, Springer, 2005.

Glossary

Rank–one matrix A matrix that can be written as an outer product $uv^\top$ of two nonzero vectors.

Positive semi–definite matrix A symmetric matrix A such that $v^\top Av \geq 0$ for all vectors v.

Outer product For vectors u and v, the matrix $uv^\top$.

Gram matrix A matrix of inner products of a set of vectors.

Collinearity The property that two vectors differ only by a scalar multiple.