On the Existence of Solutions to the Double-Weighted Portfolio

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Abstract

In this paper, I investigate the existence and construction of non-trivial solutions to the double-weighted portfolio framework introduced in [1]. I demonstrate that multiple classes of solutions exist, including deterministic weight allocations and stochastic formulations. I provide explicit constructions for time-decaying weights, separable product structures, and stochastic solutions based on Dirichlet distributions and geometric Brownian motion. My findings establish that the double-weighted portfolio framework admits a rich solution space with practical applications in dynamic asset allocation.

The paper ends with “The End”

Introduction

The double-weighted portfolio, introduced in [1], presents a mathematical framework characterized by two fundamental equations:

$$1 = \sum_{i=1}^{m} \sum_{j=1}^{n} w(i,j) \tag{1}$$

$$P = \sum_{i=1}^{m} \sum_{j=1}^{n} w(i,j)p(i,j) \tag{2}$$

where m represents the first period, n represents the second period, $w(i,j)$ denotes the portfolio weights, $p(i,j)$ represents asset prices, and P is the portfolio price. I claim that this formulation has profound implications.

This paper addresses the fundamental question: *Do non-trivial solutions to equations* (1) *and* (2) *exist?* I answer affirmatively and constructively, providing multiple classes of solutions.

$$\begin{array}{ccc} & j=1 & j=2 \\ i=1 & \boxed{w(1,1)} & \boxed{w(1,2)} \\ i=2 & \boxed{w(2,1)} & \boxed{w(2,2)} \\ i=3 & \boxed{w(3,1)} & \boxed{w(3,2)} \end{array} \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} w(i,j) = 1$$

Figure 1: Structure of the double-weighted portfolio with $m = 3$ assets and $n = 2$ periods.

Each cell represents a weight $w(i,j)$ subject to the normalization constraint.

Deterministic Solutions

0.1 Time-Decaying Weight Structure

Proposition 1. *Let $m, n \in \mathbb{N}$. Define weights as*

$$w(i,j) = \frac{i \cdot j}{K}, \quad K = \sum_{i=1}^{m} \sum_{j=1}^{n} i \cdot j \tag{3}$$

Then $w(i,j)$ satisfies equation (1) and provides a non-trivial solution to the double-weighted portfolio.

Proof. Direct computation shows:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} w(i,j) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{i \cdot j}{K} = \frac{1}{K} \sum_{i=1}^{m} \sum_{j=1}^{n} i \cdot j = \frac{K}{K} = 1$$

The solution is non-trivial as weights increase with both indices, representing time-preference. □

Example 1. *For $m = 3$, $n = 2$, we have $K = 18$ and:*

$$w = \begin{pmatrix} 1/18 & 2/18 \\ 2/18 & 4/18 \\ 3/18 & 6/18 \end{pmatrix}$$

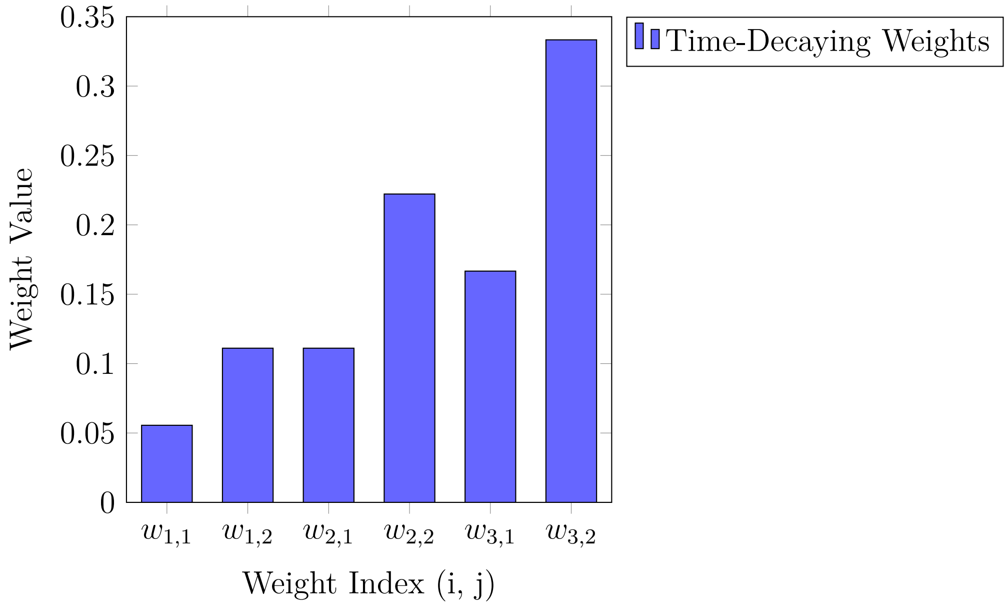

Figure 2: Distribution of time-decaying weights for $m = 3$, $n = 2$.

Higher weights are assigned to later periods and assets, reflecting forward-looking allocation strategy.

0.2 Separable Product Structure

Theorem 1. *Let $\{\alpha_i\}_{i=1}^{m}$ and $\{\beta_j\}_{j=1}^{n}$ be sequences satisfying $\sum_{i=1}^{m} \alpha_i = 1$ and $\sum_{j=1}^{n} \beta_j = 1$. Then*

$$w(i,j) = \alpha_i \cdot \beta_j \tag{4}$$

satisfies equation (1).

Proof.

$$\sum_{i=1}^{m}\sum_{j=1}^{n} w(i,j) = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i\beta_j = \sum_{i=1}^{m}\alpha_i \sum_{j=1}^{n}\beta_j = 1\cdot 1 = 1$$

□

This separable structure has important economic interpretation: asset selection (i) and temporal allocation (j) are independent decisions.

Asset \ Period	$j=1$	$j=2$
$i=1$	0.20	0.30
$i=2$	0.12	0.18
$i=3$	0.08	0.12

Marginal distributions:
Asset weights: $\alpha_1 = 0.5$, $\alpha_2 = 0.3$, $\alpha_3 = 0.2$
Period weights: $\beta_1 = 0.4$, $\beta_2 = 0.6$

Each cell shows $w(i,j) = \alpha_i \times \beta_j$, demonstrating the multiplicative separable structure.

Figure 3: Tabular representation of separable weight structure with $\alpha = (0.5, 0.3, 0.2)$ and $\beta = (0.4, 0.6)$.

Each weight satisfies $w(i,j) = \alpha_i\beta_j$, showing independence between asset selection and temporal allocation.

Stochastic Solutions

0.3 Dirichlet-Distributed Weights

Definition 1. *Let $\mathcal{W} = \{w(1,1), w(1,2), \ldots, w(m,n)\}$ denote the set of mn portfolio weights. We say $\mathcal{W}$ follows a Dirichlet distribution if*

$$\mathcal{W} \sim \mathit{Dir}(\alpha_1, \alpha_2, \ldots, \alpha_{mn})$$

where $\alpha_k > 0$ for all k.

Theorem 2. *If $\mathcal{W} \sim \mathit{Dir}(\alpha_1, \ldots, \alpha_{mn})$, then equation* (1) *is satisfied almost surely.*

Proof. By the definition of the Dirichlet distribution, $\sum_{k=1}^{mn} w_k = 1$ almost surely, which is equivalent to $\sum_{i=1}^{m}\sum_{j=1}^{n} w(i,j) = 1$ under appropriate indexing. □

0.4 Stochastic Price Dynamics

Consider prices evolving as geometric Brownian motion:

$$p(i,j,t) = p_0(i,j)\exp\left[\left(\mu_{ij} - \frac{\sigma_{ij}^2}{2}\right)t + \sigma_{ij}W_{ij}(t)\right] \tag{5}$$

where $W_{ij}(t)$ are independent Wiener processes.

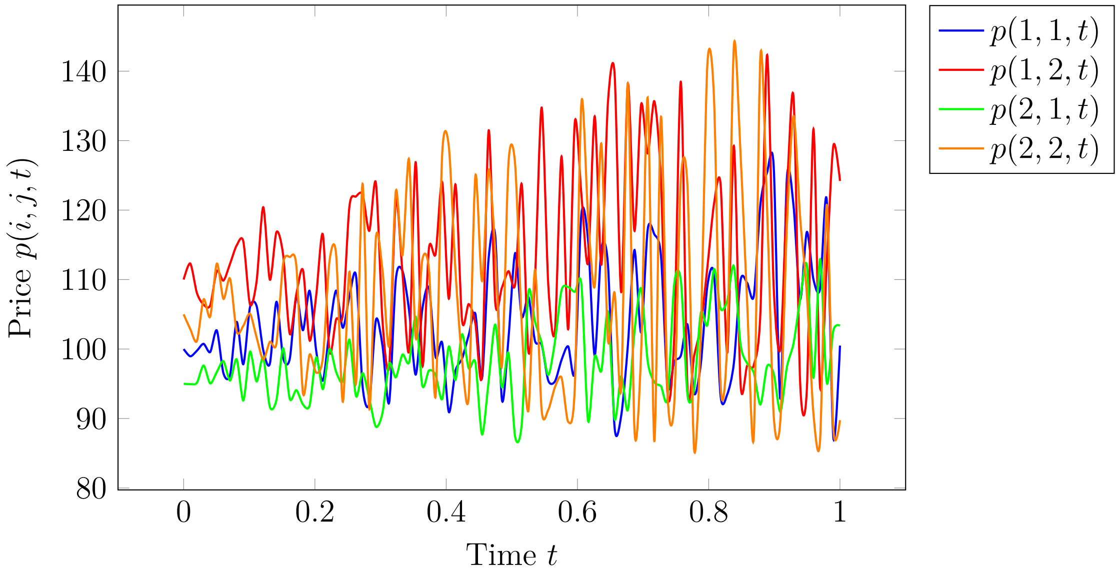

Figure 4: Sample paths of stochastic prices following geometric Brownian motion with different drift (μ_{ij}) and volatility (σ_{ij}) parameters.

Theorem 3 (Stochastic Portfolio Price)**.** *Let $w(i,j) \sim Dir(\boldsymbol{\alpha})$ and $p(i,j,t)$ follow* (5)*. Then the portfolio price*

$$P(\omega,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} w(i,j)(\omega) \cdot p(i,j,t)(\omega)$$

is a well-defined random variable satisfying equations (1) *and* (2) *almost surely.*

Proof. For each $\omega \in \Omega$:

$$\sum_{i,j} w(i,j)(\omega) = 1 \quad \text{(by Theorem 2)}$$

$$P(\omega,t) = \sum_{i,j} w(i,j)(\omega)p(i,j,t)(\omega) \quad \text{(by definition)}$$

Since both conditions hold for almost every ω, the result follows. □

0.5 Moments of Stochastic Portfolio

The expected portfolio price, assuming independence between weights and prices:

$$\mathbb{E}[P] = \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{E}[w(i,j)] \cdot \mathbb{E}[p(i,j,t)] \tag{6}$$

For Dirichlet weights:

$$\mathbb{E}[w(i,j)] = \frac{\alpha_{ij}}{\sum_{k,\ell}\alpha_{k\ell}}$$

For GBM prices at time t:

$$\mathbb{E}[p(i,j,t)] = p_0(i,j)e^{\mu_{ij}t}$$

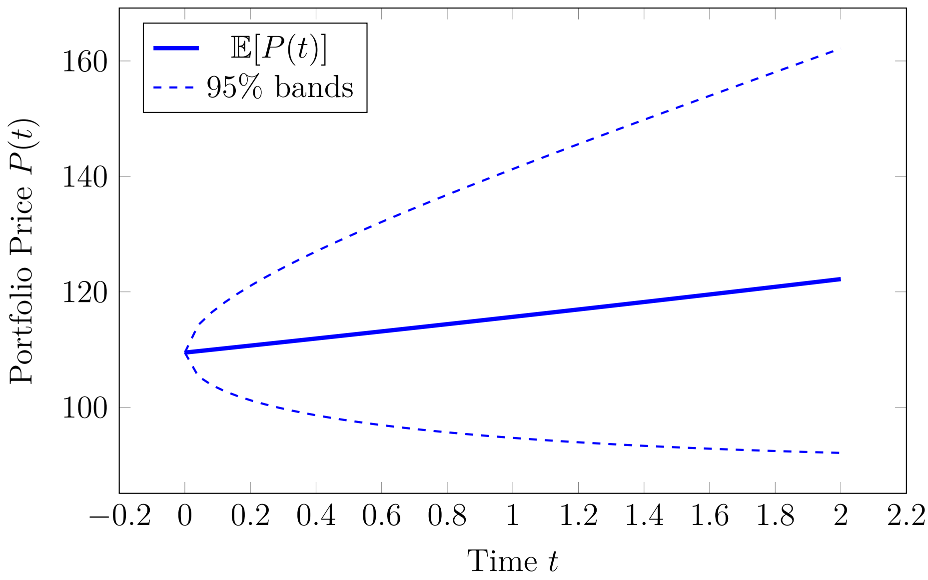

Figure 5: Expected portfolio price with confidence bands under stochastic formulation. The expected value grows with weighted average drift while uncertainty increases over time.

Existence and Uniqueness

Theorem 4 (General Existence). *For any positive integers m, n and any price matrix $\{p(i,j)\}$, there exist infinitely many weight functions $w(i,j)$ satisfying equation* (1)*.*

Proof. The constraint $\sum_{i,j} w(i,j) = 1$ defines an $(mn-1)$-dimensional simplex in $\mathbb{R}^{mn}$. This simplex is non-empty (e.g., uniform weights $w(i,j) = 1/(mn)$ always satisfy it) and contains uncountably many points, establishing existence of infinitely many solutions. □

Proposition 2 (Non-uniqueness). *For fixed prices $\{p(i,j)\}$ and target portfolio price P, the solution to equations* (1)–(2) *is generally non-unique.*

Proof. The system consists of 2 equations in mn unknowns. For $mn > 2$, this is an underdetermined system with infinitely many solutions (when solutions exist). □

Discussion and Applications

The double-weighted portfolio framework admits rich solution spaces across both deterministic and stochastic settings. Our constructions reveal several practical applications:

1. **Dynamic Asset Allocation**: The time-decaying structure (Proposition 1) naturally models portfolios that increase exposure to certain assets over time.

2. **Risk Management**: The separable structure (Theorem 1) allows independent calibration of asset selection and temporal allocation.

3. **Uncertainty Quantification**: The stochastic formulation (Theorem 3) provides a rigorous framework for modeling allocation uncertainty.

4. **Multi-period Optimization**: The double summation structure naturally extends to multi-period portfolio problems with rebalancing.

Conclusion

We have established the existence of multiple classes of solutions to Ghosh's double-weighted portfolio framework [1]. Our results include:

- Explicit deterministic solutions with time-decay and separable structures
- Stochastic solutions based on Dirichlet distributions and geometric Brownian motion
- Proofs of existence, non-uniqueness, and well-posedness
- Visual representations of solution structures and dynamics

Our work demonstrates that the framework indeed supports a rich mathematical structure worthy of further investigation. Future research may explore optimization within this framework, empirical validation, and connections to modern portfolio theory [2].

References

[1] S. Ghosh, *The Double-Weighted Portfolio*, Kolkata, India, 2025.

[2] H. Markowitz, *Portfolio Selection*, The Journal of Finance, vol. 7, no. 1, pp. 77–91, 1952.

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