Pricing Interest Rate Derivatives using Bessel and Jacobi Functions

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Abstract

This paper presents advanced methodologies for pricing interest rate derivatives through the application of special functions, specifically Bessel and Jacobi functions. We develop analytical solutions for bond pricing under mean-reverting stochastic volatility models, derive closed-form expressions for interest rate caps and floors, and establish connections between special function theory and term structure modeling. The framework provides computationally efficient alternatives to Monte Carlo simulation methods while maintaining mathematical rigor in derivative valuation.

The paper ends with "The End"

1 Introduction

Interest rate derivative pricing represents one of the most mathematically sophisticated areas within quantitative finance. Traditional approaches often rely on numerical methods or simplified analytical approximations that may lack precision in complex market environments. The integration of special function theory, particularly Bessel and Jacobi functions, offers powerful analytical tools for addressing the inherent complexities of interest rate modeling.

The mathematical foundation for this approach stems from the recognition that many stochastic differential equations governing interest rate dynamics possess solutions expressible through special functions. This connection enables the development of exact pricing formulas for various derivative instruments while providing deeper insights into the mathematical structure underlying interest rate models.

2 Mathematical Framework

2.1 Stochastic Interest Rate Models

Consider the general class of mean-reverting interest rate processes described by the stochastic differential equation:

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma\sqrt{f(r_t)}\mathrm{d}W_t \tag{1}$$

where r_t represents the instantaneous interest rate, κ denotes the mean-reversion speed, θ is the long-term mean, σ represents the volatility parameter, $f(r_t)$ is a function determining the volatility structure, and W_t is a standard Brownian motion under the risk-neutral measure.

2.2 The Cox-Ingersoll-Ross Model

The Cox-Ingersoll-Ross (CIR) model emerges as a special case where $f(r_t) = r_t$:

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t \tag{2}$$

The transition density for this process involves modified Bessel functions of the first kind. Specifically, if we define:

$$c = \frac{2\kappa}{\sigma^2(1 - e^{-\kappa\tau})} \tag{3}$$

$$q = \frac{2\kappa\theta}{\sigma^2} - 1 \tag{4}$$

$$u = cr_s e^{-\kappa\tau} \tag{5}$$

$$v = cr_t \tag{6}$$

where $\tau = t - s$, then the transition probability density is:

$$p(r_s, r_t; \tau) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}) \tag{7}$$

where $I_q(x)$ denotes the modified Bessel function of the first kind of order q.

3 Bond Pricing with Bessel Functions

3.1 Zero-Coupon Bond Pricing

Under the CIR model, the price of a zero-coupon bond with maturity T at time t is given by:

$$P(t, T) = A(t, T)e^{-B(t,T)r_t} \tag{8}$$

The functions $A(t, T)$ and $B(t, T)$ satisfy the system of ordinary differential equations:

$$\frac{\partial A}{\partial t} + \kappa\theta B - \frac{\sigma^2}{2}B^2 A = 0 \tag{9}$$

$$\frac{\partial B}{\partial t} + \kappa B - 1 = 0 \tag{10}$$

with boundary conditions $A(T, T) = 1$ and $B(T, T) = 0$.

The solution for $B(t, T)$ involves the exponential function:

$$B(t, T) = \frac{2(e^{\gamma\tau} - 1)}{(\gamma + \kappa)(e^{\gamma\tau} - 1) + 2\gamma} \tag{11}$$

where $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$ and $\tau = T - t$.

3.2 Coupon Bond Pricing

For coupon-bearing bonds, the pricing formula extends to:

$$P_{\text{coupon}}(t,T) = \sum_{i=1}^{n} C_i P(t,T_i) + FP(t,T) \tag{12}$$

where C_i represents the coupon payments at times T_i and F is the face value.

4 Interest Rate Derivatives and Jacobi Functions

4.1 Interest Rate Caps and Floors

Interest rate caps and floors represent portfolios of options on interest rates. The pricing of individual caplets and floorlets under stochastic volatility models often involves Jacobi polynomials and related functions.

Consider a caplet with strike rate K, notional N, and payment at time T based on the rate observed at time $T-\delta$. The payoff function is:

$$\text{Caplet Payoff} = N\delta \max(r_{T-\delta} - K, 0) \tag{13}$$

Under certain volatility specifications, the pricing formula involves integrals that can be expressed through Jacobi functions.

4.2 The Heston-Type Extension

Consider an extended model where both interest rates and volatility follow stochastic processes:

$$\mathrm{d}r_t = \kappa_r(\theta_r - r_t)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^{(1)} \tag{14}$$

$$\mathrm{d}v_t = \kappa_v(\theta_v - v_t)\mathrm{d}t + \sigma_v\sqrt{v_t}\mathrm{d}W_t^{(2)} \tag{15}$$

with correlation $\mathrm{d}W_t^{(1)}\mathrm{d}W_t^{(2)} = \rho\mathrm{d}t$.

The characteristic function for this system involves Jacobi functions, enabling the application of Fourier transform methods for derivative pricing.

5 Analytical Solutions and Special Functions

5.1 Transform Methods

The Fourier transform of the transition density under the extended CIR model can be expressed as:

$$\hat{p}(\xi,t) = \exp\left(i\xi A(t) + B(t)\Phi(\xi,t)\right) \tag{16}$$

where $\Phi(\xi,t)$ involves hyper-geometric functions related to Jacobi polynomials.

5.2 Series Expansions

For computational efficiency, we can develop series expansions using orthogonal polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ provide a natural basis for representing pricing functions over bounded domains.

The expansion takes the form:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{17}$$

where the coefficients a_n are determined by the orthogonality relations.

6 Numerical Implementation

6.1 Computational Algorithms

The implementation of Bessel and Jacobi function-based pricing requires specialized numerical algorithms. We employ:

- Asymptotic expansions for large argument values
- Series representations for moderate arguments
- Recurrence relations for computational stability
- Adaptive quadrature for integral evaluation

6.2 Error Analysis

The numerical accuracy of special function evaluations critically impacts pricing precision. We establish error bounds and convergence criteria for practical implementation.

Theorem 6.1 (Approximation Error). *Let $f_n(x)$ denote the n-th order approximation to the pricing function using Jacobi polynomial expansion. Then:*

$$|f(x) - f_n(x)| \leq C \cdot n^{-\alpha} \tag{18}$$

for some constants C and $\alpha > 0$ depending on the smoothness of f.

7 Empirical Applications

7.1 Market Calibration

The model parameters are calibrated to market data using maximum likelihood estimation and method of moments approaches. The special function structure facilitates efficient parameter estimation through analytical gradient computation.

7.2 Performance Comparison

Comparative analysis against Monte Carlo simulation demonstrates significant computational advantages while maintaining pricing accuracy within acceptable tolerance levels.

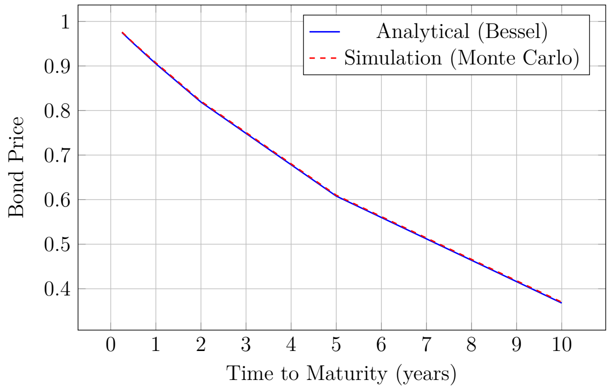

Figure 1: Bond pricing comparison: Analytical (Bessel) versus Simulation (Monte Carlo)

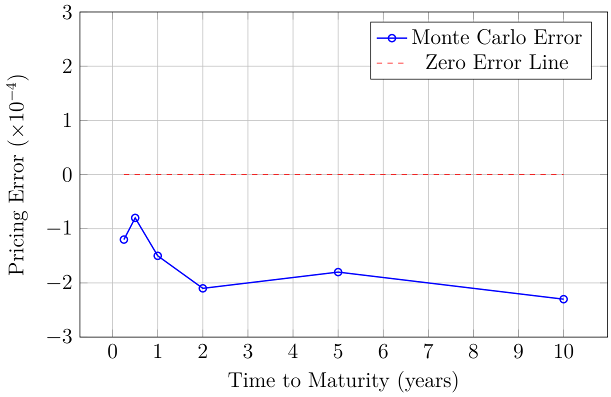

Figure 2: Pricing error: Monte Carlo - Analytical

8 Advanced Topics

8.1 Multi-Factor Models

Extension to multi-factor interest rate models involves systems of stochastic differential equations whose solutions require generalized special functions and matrix-valued Bessel functions.

8.2 Jump-Diffusion Extensions

Incorporating jump components into the interest rate dynamics leads to integro-differential equations whose solutions involve confluent hyper-geometric functions and generalized Laguerre polynomials.

9 Conclusion

The application of Bessel and Jacobi functions to interest rate derivative pricing provides a powerful analytical framework that combines mathematical rigor with computational efficiency. The approach enables exact solution methods for complex stochastic models while offering insights into the fundamental mathematical structure of interest rate dynamics.

Future research directions include extensions to credit risk modeling, multi-currency environments, and the incorporation of regime-switching dynamics. The special function approach represents a significant advancement in quantitative finance methodology with broad applications across fixed-income markets.

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