Closed-form values of $sin\, A^\circ$ and $cos\, A^\circ$ for whole A in the first quadrant

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Abstract

In this paper, I describe closed-form values of $sin\ A^\circ$ and $cos\ A^\circ$ for whole $A \in \{0, 1, 2, \ldots, 88, 89, 90\}$. The paper ends with "The End"

Introduction

In this paper, I describe closed-form values of $sin\ A^\circ$ and $cos\ A^\circ$ for whole $A \in \{0, 1, 2, \ldots, 88, 89, 90\}$.

Preliminaries

Recall the following trigonometric identities:

$$sin^2\ A^\circ + cos^2\ A^\circ = 1$$

$$sin(A^\circ + B^\circ) = sin\ A^\circ\ cos\ B^\circ + cos\ A^\circ\ sin\ B^\circ$$

$$sin(A^\circ - B^\circ) = sin\ A^\circ\ cos\ B^\circ - cos\ A^\circ\ sin\ B^\circ$$

$$cos(A^\circ + B^\circ) = cos\ A^\circ\ cos\ B^\circ - sin\ A^\circ\ sin\ B^\circ$$

$$cos(A^\circ - B^\circ) = cos\ A^\circ\ cos\ B^\circ + sin\ A^\circ\ sin\ B^\circ$$

whence

$$sin\ 2A^\circ = 2\ sin\ A^\circ\ cos\ A^\circ$$

$$sin\ 3A^\circ = 3\ sin\ A^\circ - 4\ sin^3\ A^\circ$$

and so on...

and

$$cos\ 2A^\circ = 2\ cos^2\ A^\circ - 1$$

$$cos\ 3A^\circ = 4\ cos^3\ A^\circ - 3\ cos\ A^\circ$$

and so on...

whence

$$sin^2\ \frac{A}{2}^\circ = \frac{1 - cos\ A}{2}$$

and

$$cos^2\ \frac{A}{2}^\circ = \frac{1 + cos\ A}{2}$$

Deriving closed-form values of $cos\, A^\circ$

For $0^\circ \leq A^\circ \leq 90^\circ$, we have

$$cos\ A^\circ = \sqrt{1 - sin^2\ A^\circ}$$

Therefore, for $0^\circ \leq A^\circ \leq 90^\circ$, to derive closed-form values of $cos\ A^\circ$, knowing the value of $sin\ A^\circ$ is **sufficient**.

Commonly-known closed-form values of $sin\, A^\circ$

Recall the following commonly-known closed-form values of $sin\, A^\circ$:

$$sin\, 90^\circ = 1$$

$$sin\, 84^\circ = \frac{1}{4}\sqrt{7 + \sqrt{5} + \sqrt{6\left(\sqrt{5} + 5\right)}}$$

$$sin\, 78^\circ = \frac{1}{8}\left(\sqrt{5} + \sqrt{30 + 6\sqrt{5}} - 1\right)$$

$$sin\, 75^\circ = \frac{1 + \sqrt{3}}{2\sqrt{2}}$$

$$sin\, 72^\circ = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$$

$$sin\, 60^\circ = \frac{\sqrt{3}}{2}$$

$$sin\, 54^\circ = \frac{1 + \sqrt{5}}{4}$$

$$sin\, 45^\circ = \frac{1}{\sqrt{2}}$$

$$sin\, 36^\circ = \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$$

$$sin\, 30^\circ = \frac{1}{2}$$

$$sin\, 18^\circ = \frac{\sqrt{5} - 1}{4}$$

$$sin\, 15^\circ = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

$$sin\, 12^\circ = \frac{1}{8}\left(\sqrt{3} + \sqrt{10 + 2\sqrt{5}} - \sqrt{15}\right)$$

$$sin\, 6^\circ = \frac{1}{8}\left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)$$

$$sin\, 0^\circ = 0$$

Therefore, for $A \in \{0, 1, 2, \ldots, 88, 89, 90\}$,
the value of $sin\, 1^\circ$ is **necessary and sufficient** to
derive $sin\, A^\circ$ and $cos\, A^\circ$.

Derivation of the closed-form values of $sin\,1°$ and $cos\,1°$

We derive the closed-form values of $sin\,1°$ and $cos\,1°$ by finite descent:

For $0° \leq A° \leq 45°$, from the system

$$sin\,2A° = 2\,sin\,A° cos\,A°$$

$$cos\,2A° = cos^2\,A° - sin^2\,A°$$

$$sin^2\,A° + cos^2\,A° = 1$$

$$0 \leq sin\,A° \leq cos\,A° \leq \frac{1}{\sqrt{2}}$$

we can derive $sin\,A°$ and $cos\,A°$

Similarly, for $0° \leq A° \leq 30°$, from the system

$$sin\,3A° = 3\,sin\,A° - 4\,sin^3\,A°$$

$$cos\,3A° = 4\,cos^3\,A° - 3\,cos\,A°$$

$$sin^2\,A° + cos^2\,A° = 1$$

$$0 \leq sin\,A° \leq cos\,A° \leq \frac{1}{2}$$

we can derive $sin\,A°$ and $cos\,A°$

We begin with $sin\,18°$ and $cos\,18°$ to obtain

$$sin\,9° = \frac{1}{2}\sqrt{2 - \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}$$

and

$$cos\,9° = \frac{1}{2}\sqrt{2 + \sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}$$

We continue with $sin\,15°$ and $cos\,15°$ and eliminate $sin\,5°$ and $cos\,5°$ to obtain

$$sin\,10° = \sqrt{1 - \frac{\left(2\sqrt[3]{2} + \left(2\left(\sqrt{3}+i\right)\right)^{2/3}\right)^2}{16\left(\sqrt{3}+i\right)^{2/3}}}$$

and

$$cos\,10° = \frac{2\sqrt[3]{2} + \left(2\left(\sqrt{3}+i\right)\right)^{2/3}}{4\sqrt[3]{\sqrt{3}+i}}$$

We continue with the subtraction identities above with $sin\,10°$, $cos\,10°$, $sin\,9°$ and $cos\,9°$ to obtain

$$sin\,1° = \frac{\sqrt[18]{-1}\left(2\left((-1)^{8/9}-1\right)\sqrt{4-\sqrt{2\left(\sqrt{5}+5\right)}} + \left(-2i+(-1)^{13/18}+(-1)^{2/9}\sqrt{3}\right)\sqrt{\sqrt{2\left(\sqrt{5}+5\right)}+4}\right)}{8\sqrt{2}}$$

and

$$cos\,1° = \frac{\sqrt[18]{-1}\left(-2\left((-1)^{8/9}-1\right)\sqrt{\sqrt{2\left(\sqrt{5}+5\right)}+4} + \left(-2i+(-1)^{13/18}+(-1)^{2/9}\sqrt{3}\right)\sqrt{4-\sqrt{2\left(\sqrt{5}+5\right)}}\right)}{8\sqrt{2}}$$

Derivation of closed-form values of $sin\, A^\circ$ and $sin\, A^\circ$ for $A \in \{0, 1, 2, \ldots, 88, 89, 90\}$

From the closed-form values of $sin\, 1^\circ$ and $cos\, 1^\circ$,
closed-form values of $sin\, A^\circ$ and $cos\, A^\circ$ for $A \in \{0, 1, 2, \ldots, 88, 89, 90\}$
can be derived by repeatedly using the addition and/or subtraction and/or multiple-angle identities above.

The End