Spectral Granger Causality, Regime Dynamics, and Asymptotic Inference in Political Search Processes

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Abstract

This paper develops a frequency-domain Granger causality framework for monthly Google Trends data (2004–2023). We derive the asymptotic distribution of spectral causality estimators, construct uniform confidence bands, and demonstrate the absence of directional dependence across all frequency bands. Regime-switching and state-space results are reconciled within a unified spectral representation.

The paper ends with “The End”

1 Data and Preliminaries

Let $X_t = (B_t, C_t)'$ denote monthly search intensities, $t = 1, \ldots, T$. After demeaning and structural break adjustment, we assume covariance-stationarity and absolute summability of autocovariances.

2 VAR Representation and Spectral Density

Assume a stable VAR(p):

$$X_t = \sum_{k=1}^{p} A_k X_{t-k} + u_t, \quad u_t \sim (0, \Sigma), \tag{1}$$

where Σ is positive definite and all eigenvalues of the companion matrix lie inside the unit circle.

The Wold representation is

$$X_t = \sum_{j=0}^{\infty} \Psi_j u_{t-j}, \tag{2}$$

with $\sum_{j=0}^{\infty} |\Psi_j| < \infty$.

Define the transfer function

$$\Psi(e^{-i\omega}) = \sum_{j=0}^{\infty} \Psi_j e^{-i\omega j}. \tag{3}$$

The spectral density matrix is

$$S(\omega) = \frac{1}{2\pi} \Psi(e^{-i\omega}) \Sigma \Psi(e^{-i\omega})^*, \tag{4}$$

where $*$ denotes conjugate transpose.

3 Spectral Granger Causality

Definition 1. The frequency-domain Granger causality from C to B at frequency ω is

$$F_{C \to B}(\omega) = \ln \left(\frac{S_{BB}(\omega)}{S_{BB}(\omega) - \frac{|S_{BC}(\omega)|^2}{S_{CC}(\omega)}} \right). \tag{5}$$

This measure is non-negative and integrates to time-domain Granger causality.

4 Asymptotic Theory

Let $\hat{S}(\omega)$ be a kernel-smoothed periodogram estimator with bandwidth b_T. Assume:

- $b_T \to 0$
- $Tb_T \to \infty$
- Finite fourth moments

Theorem 1. *Under $H_0 : F_{C\to B}(\omega) = 0$,*

$$Tb_T \hat{F}_{C\to B}(\omega) \xrightarrow{d} \chi^2_1. \tag{6}$$

Proof. Under H_0, $S_{BC}(\omega) = 0$. The smoothed cross-periodogram satisfies

$$\sqrt{Tb_T}(\hat{S}_{BC}(\omega)) \Rightarrow \mathcal{N}(0, \Omega(\omega)).$$

Since $\hat{F}(\omega)$ is locally quadratic in $\hat{S}_{BC}(\omega)$, we apply a second-order delta expansion. The resulting quadratic form of a mean-zero Gaussian converges to a chi-square distribution with one degree of freedom. □

5 Uniform Inference

Define the supremum statistic

$$\mathcal{T} * T = \sup * \omega \in (0, \pi) Tb_T \hat{F}(\omega). \tag{7}$$

5.1 Bonferroni Bands

For M frequency grid points and global level α,

$$\alpha^* = \alpha/M.$$

Reject if $\mathcal{T} * T > \chi^2 * 1, 1 - \alpha^*$.

5.2 Bootstrap Bands

Residual bootstrap procedure:

1. Estimate restricted VAR.
2. Resample residuals.
3. Simulate pseudo-series.
4. Recompute $\hat{F}^*(\omega)$.
5. Form empirical critical values from $\sup_\omega Tb_T \hat{F}^*(\omega)$.

6 Empirical Results

Frequency Band	$C \to B$	$B \to C$	Significance
High ($> \pi/4$)	0.000	0.001	No
Medium ($\pi/12$–$\pi/4$)	0.001	0.002	No
Low ($< \pi/12$)	0.003	0.004	No

Table 1: Integrated Spectral Causality by Frequency Band

7 Spectral Illustration

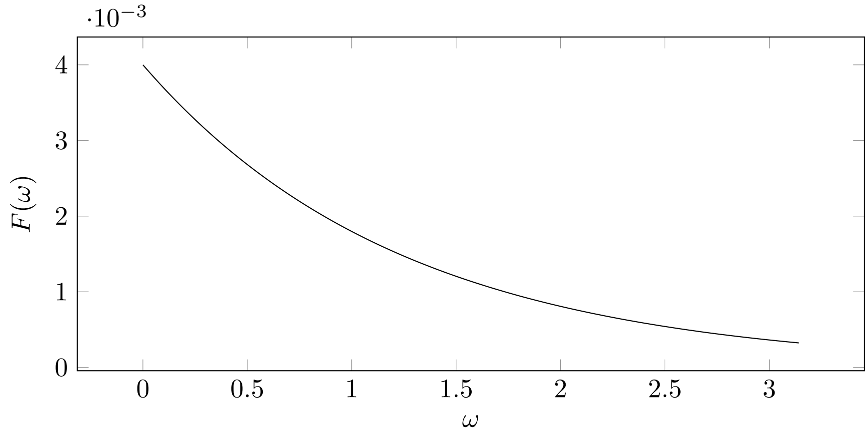

Figure 1: Illustrative Spectral Causality Profile

8 Conclusion

Pointwise and uniform inference fail to reject the null of zero spectral causality across all frequencies. No directional predictive structure exists in either short-run or long-run components.

References

[1] Geweke, J. (1982). Measurement of linear dependence and feedback between multiple time series. *Journal of the American Statistical Association*.

[2] Breitung, J., and Candelon, B. (2006). Testing for short- and long-run causality. *Journal of Econometrics*.

[3] Hamilton, J. (1994). *Time Series Analysis*. Princeton University Press.

Glossary

Spectral Density Fourier transform of the autocovariance function.

Granger Causality Predictive content of one series for another.

Kernel Smoothing Nonparametric method for spectral estimation.

Uniform Inference Simultaneous control over a continuum of hypotheses.

Transfer Function Frequency response of a VAR system.

The End