Features and Properties of the Complete Graph K_{33}

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Abstract

This paper presents a comprehensive overview of the complete graph K_{33}, a fundamental structure in graph theory consisting of 33 vertices where every pair of distinct vertices is connected by a unique edge. We explore its structural, combinatorial, and algebraic properties, accompanied by visual representations using TikZ.

The paper ends with “The End”

1 Introduction

A **complete graph** K_n is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. The complete graph K_{33} is the complete graph on 33 vertices and serves as an important object of study in combinatorics, network theory, and algebraic graph theory.

2 Basic Structural Properties

2.1 Vertices and Edges

The complete graph K_{33} has:

- **Vertices:** $n = 33$
- **Edges:** The number of edges is given by the binomial coefficient:

$$|E(K_{33})| = \binom{33}{2} = \frac{33 \times 32}{2} = 528$$

2.2 Degree Sequence

Every vertex in K_{33} has degree $n - 1 = 32$. Thus, K_{33} is a **32-regular graph**. The degree sequence is:

$$\underbrace{(32, 32, 32, \ldots, 32)}_{33 \text{ times}}$$

The sum of all degrees satisfies the Handshaking Lemma:

$$\sum_{v \in V} \deg(v) = 33 \times 32 = 1056 = 2 \times 528 = 2|E|$$

3 Distance and Connectivity Properties

3.1 Distance Metrics

- **Diameter:** $\mathrm{diam}(K_{33}) = 1$, since any two distinct vertices are adjacent.
- **Radius:** $\mathrm{rad}(K_{33}) = 1$, as the eccentricity of every vertex is 1.
- **Girth:** $g(K_{33}) = 3$, the length of the shortest cycle (a triangle).

3.2 Connectivity

The complete graph K_{33} is **maximally connected**:

- **Vertex connectivity:** $\kappa(K_{33}) = 32$
- **Edge connectivity:** $\lambda(K_{33}) = 32$

By Menger's theorem, there exist 32 vertex-disjoint paths between any pair of vertices.

4 Combinatorial Properties

4.1 Counting Substructures

Substructure	**Formula**	**Count in K_{33}**
Triangles (K_3)	$\binom{n}{3}$	$\binom{33}{3} = 5456$
Complete subgraphs K_4	$\binom{n}{4}$	$\binom{33}{4} = 40920$
Edges	$\binom{n}{2}$	528
Spanning trees	n^{n-2}	33^{31}

Table 1: Counting substructures in K_{33}.

4.2 Hamiltonian Decomposition

Since $n = 33$ is odd, K_{33} can be decomposed into $\frac{n-1}{2} = 16$ edge-disjoint Hamiltonian cycles. This is a well-known result for complete graphs on an odd number of vertices.

5 Algebraic and Chromatic Properties

5.1 Chromatic Number

The **chromatic number** of K_{33} is:

$$\chi(K_{33}) = 33$$

This is because every vertex is adjacent to every other vertex, requiring each vertex to have a distinct color in any proper coloring.

5.2 Chromatic Polynomial

The chromatic polynomial is:

$$P(K_{33}, k) = k(k-1)(k-2)\cdots(k-32) = \prod_{i=0}^{32}(k-i) = \frac{k!}{(k-33)!}$$

5.3 Spectrum of the Adjacency Matrix

The adjacency matrix A of K_{33} has eigenvalues:

- $\lambda_1 = 32$ with multiplicity 1
- $\lambda_2 = -1$ with multiplicity 32

The characteristic polynomial is:

$$p_A(\lambda) = (\lambda - 32)(\lambda + 1)^{32}$$

6 Planarity and Embedding

6.1 Non-Planarity

The graph K_{33} is **non-planar**. By Kuratowski's theorem, any complete graph K_n with $n \geq 5$ contains K_5 as a subgraph and hence is non-planar.

6.2 Genus

The **genus** γ of a complete graph is given by:

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

For K_{33}:

$$\gamma(K_{33}) = \left\lceil \frac{30 \times 29}{12} \right\rceil = \left\lceil \frac{870}{12} \right\rceil = \lceil 72.5 \rceil = 73$$

7 Visual Representation

Below is a TikZ representation of the complete graph K_{33} with all 528 edges.

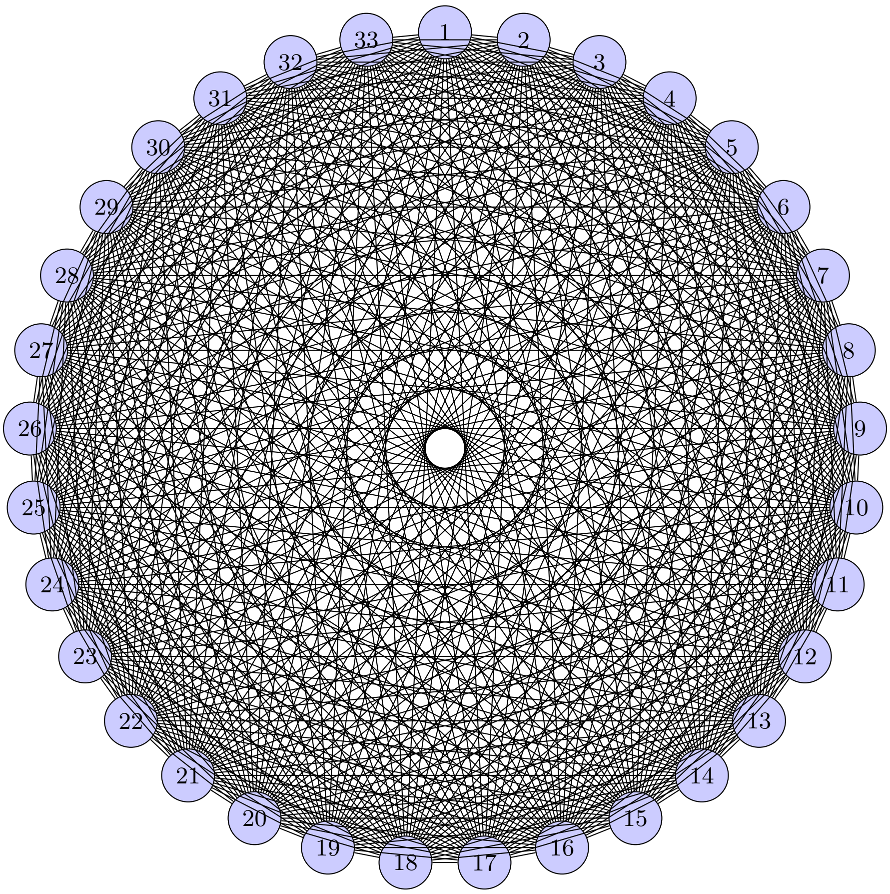

Figure 1: The complete graph K_{33} drawn with vertices arranged on a circle.

8 Summary Table

Property	**Value for** K_{33}
Number of vertices	33
Number of edges	528
Degree of each vertex	32
Diameter	1
Radius	1
Girth	3
Chromatic number	33
Clique number	33
Vertex connectivity	32
Edge connectivity	32
Genus	73
Hamiltonian cycles (decomposition)	16
Planar	No

Table 2: Summary of properties of K_{33}.

References

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Glossary

Complete Graph K_n A simple undirected graph on n vertices in which every pair of distinct vertices is connected by a unique edge.

Chromatic Number $\chi(G)$ The minimum number of colors needed to properly color the vertices of graph G such that no two adjacent vertices share the same color.

Clique Number $\omega(G)$ The size of the largest complete subgraph contained in G.

Degree The number of edges incident to a vertex.

Diameter The greatest distance between any pair of vertices in a graph.

Girth The length of the shortest cycle in a graph.

Genus The minimum number of handles that must be added to a sphere so that the graph can be embedded without edge crossings.

Hamiltonian Cycle A cycle that visits every vertex of the graph exactly once.

Planar Graph A graph that can be embedded in the plane without any edges crossing.

Regular Graph A graph where every vertex has the same degree.

Spanning Tree A subgraph that is a tree and includes all vertices of the original graph.

Vertex Connectivity $\kappa(G)$ The minimum number of vertices whose removal disconnects the graph.

The End