Pricing American Options using Basis Functions

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Abstract

This paper presents a comprehensive examination of American option pricing methodologies utilizing basis function approximations. The work synthesizes theoretical foundations from stochastic calculus, numerical analysis, and computational finance to provide practitioners and researchers with a complete framework for understanding and implementing basis function methods in option valuation.

The paper ends with "The End"

Contents

1 Introduction to American Option Pricing

1.1 Fundamental Concepts

American options provide holders the right to exercise at any time before expiration, creating a free boundary problem that distinguishes them from their European counterparts. The pricing challenge involves determining both the option value and the optimal exercise boundary simultaneously.

Definition 1.1 (American Option Value)**.** Let $V(S,t)$ denote the value of an American option on underlying asset S at time t. The option value satisfies:

$$V(S,t) = \max\left\{\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}h(S_T)\right], g(S,t)\right\} \tag{1}$$

where $g(S,t)$ represents the immediate exercise payoff and $h(S_T)$ the terminal payoff.

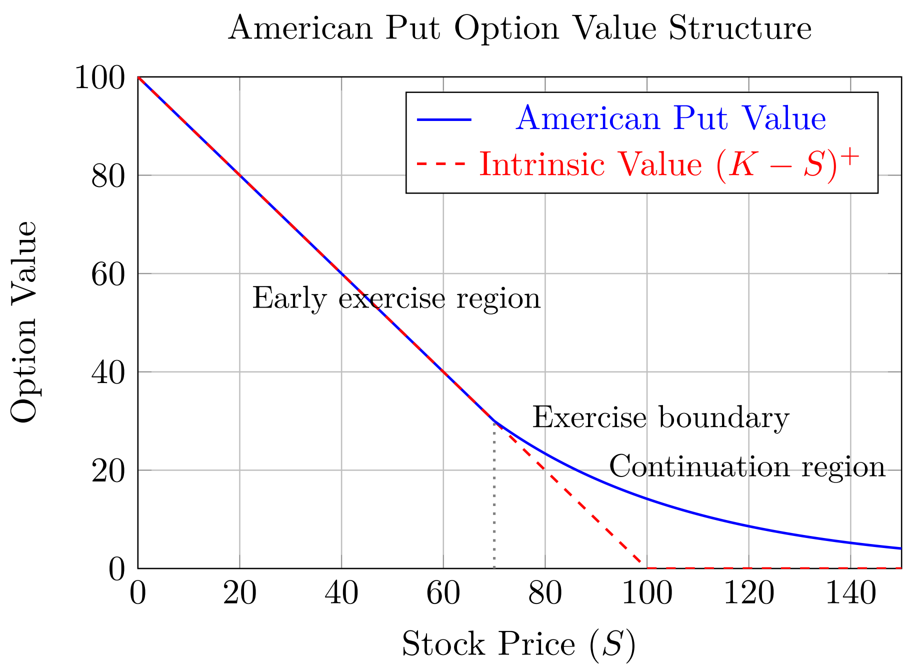

Figure 1: Comparison of American put option value with intrinsic value, illustrating the early exercise premium and optimal exercise boundary.

1.2 The Free Boundary Problem

The American option pricing problem can be formulated as a variational inequality. In the continuation region, the option value satisfies the Black-Scholes partial differential equation, while at the exercise boundary, the value equals the intrinsic payoff with smooth pasting conditions.

$$\begin{cases} \mathcal{L}V - rV = 0 & \text{in continuation region} \\ V(S,t) = g(S,t) & \text{at exercise boundary} \\ \frac{\partial V}{\partial S} = g'(S,t) & \text{smooth pasting condition} \end{cases} \tag{2}$$

where $\mathcal{L}$ denotes the Black-Scholes differential operator:

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} \tag{3}$$

2 Basis Function Methodology

2.1 Theoretical Framework

Basis function methods approximate the option value as a linear combination of predetermined basis functions. This approach transforms the infinite-dimensional optimization problem into a finite-dimensional one, enabling computational tractability while maintaining theoretical rigor.

Definition 2.1 (Basis Function Approximation)**.** The option value $V(S,t)$ is approximated as:

$$V(S,t) \approx \sum_{i=1}^{N} c_i(t)\phi_i(S) \tag{4}$$

where $\{\phi_i(S)\}_{i=1}^N$ constitutes the basis function set and $\{c_i(t)\}_{i=1}^N$ represents time-dependent coefficients.

2.2 Basis Function Selection

The choice of basis functions fundamentally influences approximation quality and computational efficiency. Common choices include polynomial bases, Chebyshev polynomials, radial basis functions, and problem-specific functions that incorporate known solution properties.

2.3 Polynomial Bases

Power series expansions provide intuitive approximations but may suffer from numerical instability and poor convergence properties near boundaries.

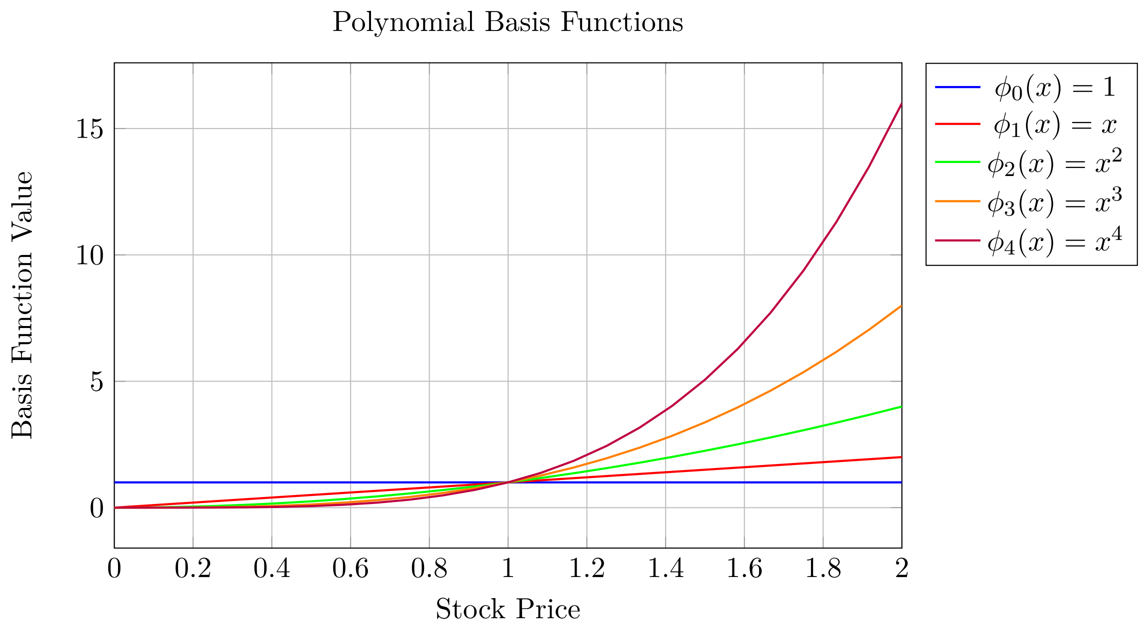

Figure 2: Low-order polynomial basis functions demonstrating increasing oscillatory behavior.

2.4 Chebyshev Polynomials

Chebyshev polynomials offer superior approximation properties through their minimax optimality and orthogonality characteristics.

$$T_n(\cos\theta) = \cos(n\theta), \quad n = 0, 1, 2, \ldots \tag{5}$$

The first few Chebyshev polynomials are:

$$T_0(x) = 1 \tag{6}$$
$$T_1(x) = x \tag{7}$$
$$T_2(x) = 2x^2 - 1 \tag{8}$$
$$T_3(x) = 4x^3 - 3x \tag{9}$$
$$T_4(x) = 8x^4 - 8x^2 + 1 \tag{10}$$

2.5 Radial Basis Functions

Radial basis functions (RBFs) represent a powerful class of basis functions particularly well-suited for American option pricing due to their meshfree nature, excellent interpolation properties, and ability to handle irregular domains. Unlike traditional polynomial bases, RBFs provide global support while maintaining localized influence, making them ideal for capturing the sharp features near exercise boundaries.

2.5.1 Mathematical Foundation of RBFs

Definition 2.2 (Radial Basis Function)**.** A radial basis function $\phi(r)$ depends only on the Euclidean distance $r = \|x - x_i\|$ from a center point x_i, where the option value approximation takes the form:

$$V(S,t) \approx \sum_{i=1}^{N} \lambda_i(t)\phi(\|S - S_i\|) \tag{11}$$

with $\lambda_i(t)$ representing time-dependent coefficients and S_i denoting RBF centers.

The most commonly employed RBF types in financial applications include:

$$\text{Gaussian:} \quad \phi(r) = e^{-(\epsilon r)^2} \tag{12}$$
$$\text{Multiquadric:} \quad \phi(r) = \sqrt{1 + (\epsilon r)^2} \tag{13}$$
$$\text{Inverse Multiquadric:} \quad \phi(r) = \frac{1}{\sqrt{1 + (\epsilon r)^2}} \tag{14}$$
$$\text{Thin Plate Spline:} \quad \phi(r) = r^2 \ln r \tag{15}$$
$$\text{Wendland:} \quad \phi(r) = (1 - \epsilon r)_+^4(4\epsilon r + 1) \tag{16}$$

where ϵ represents the shape parameter controlling the function's flatness.

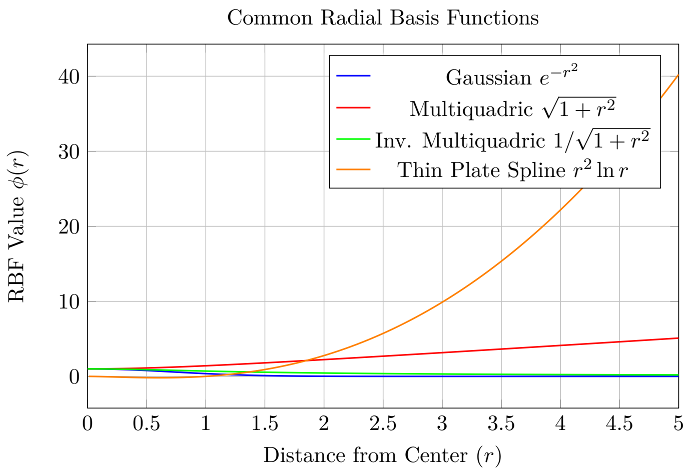

Figure 3: Comparison of common radial basis functions used in option pricing applications.

2.5.2 RBF Interpolation Properties

Theorem 2.1 (RBF Interpolation Existence)**.** For a set of distinct centers $\{S_i\}_{i=1}^N$ and corresponding function values $\{V_i\}_{i=1}^N$, there exists a unique RBF interpolant:

$$V_{RBF}(S) = \sum_{i=1}^{N} \lambda_i \phi(\|S - S_i\|) \tag{17}$$

satisfying $V_{RBF}(S_j) = V_j$ for all $j = 1, \ldots, N$, provided the RBF ϕ is conditionally positive definite.

The interpolation coefficients $\{\lambda_i\}$ are determined by solving the linear system:

$$\mathbf{A}\boldsymbol{\lambda} = \mathbf{V} \tag{18}$$

where $A_{ij} = \phi(\|S_i - S_j\|)$ and $\mathbf{V} = [V_1, V_2, \ldots, V_N]^T$.

2.5.3 Shape Parameter Selection

The shape parameter ϵ critically influences RBF performance, creating a trade-off between approximation accuracy and numerical stability.

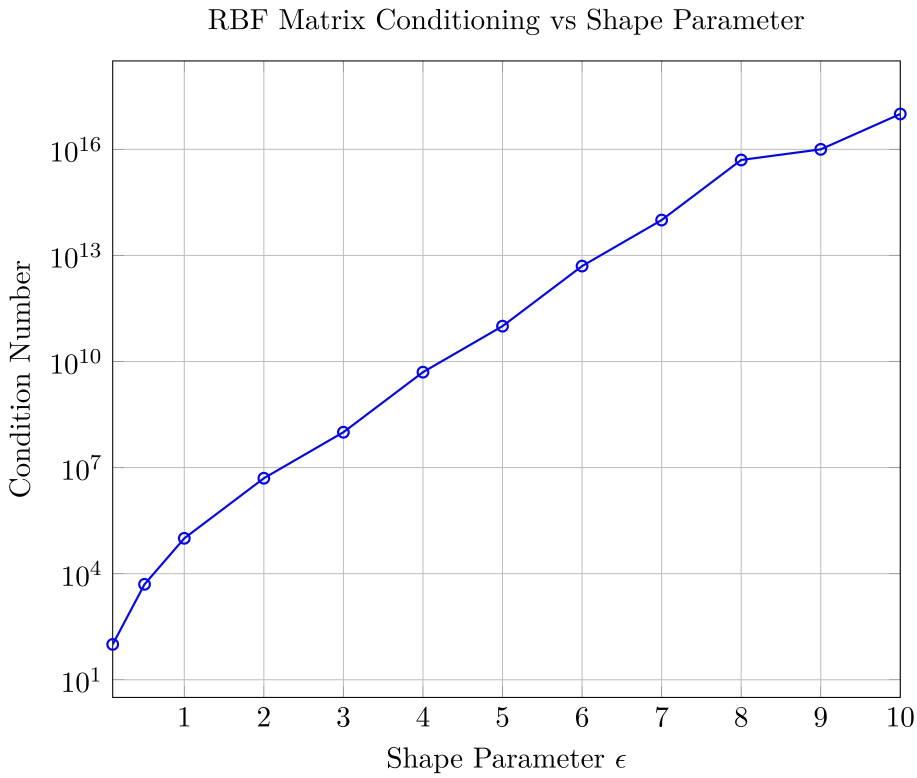

Figure 4: Typical relationship between RBF shape parameter and interpolation matrix condition number, illustrating the ill-conditioning challenge for large shape parameters.

Proposition 2.1 (Optimal Shape Parameter)**.** For Gaussian RBFs, the optimal shape parameter ϵ^* approximately satisfies:

$$\epsilon^* \approx \sqrt{\frac{\ln(\mathrm{TOL}^{-1})}{h^2}} \tag{19}$$

where h represents the fill distance and TOL denotes the desired approximation tolerance.

2.5.4 Implementation for American Options

The RBF method for American option pricing proceeds through the following algorithmic framework:

Algorithm 1 RBF-Based American Option Pricing

Input: RBF centers $\{S_i\}_{i=1}^N$, shape parameter ϵ, time grid $\{t_k\}$
Initialize: Terminal condition $V(S_i, T) = \max(K - S_i, 0)$ for put options
Construct RBF matrix $A_{ij} = \phi(\|S_i - S_j\|)$
for $k = M - 1$ down to 0 **do**
    Compute differential operators: $\mathcal{L}_{ij} = \mathcal{L}\phi(\|S_i - S_j\|)$
    Set up discretized system: $(\mathbf{I} - \Delta t \mathbf{A}^{-1}\mathcal{L})\boldsymbol{\lambda}^k = \boldsymbol{\lambda}^{k+1}$
    Solve for RBF coefficients: $\boldsymbol{\lambda}^k$
    Apply exercise constraint: $V_i^k = \max(V_i^k, g(S_i, t_k))$
    Update coefficients: $\boldsymbol{\lambda}^k = \mathbf{A}^{-1}\mathbf{V}^k$
end for
Return: Option values and exercise boundary

2.5.5 Differential Operator Computation

For RBF-based PDE solutions, differential operators must be computed analytically or numerically. For the Black-Scholes operator, we require:

$$\frac{\partial \phi}{\partial S} = \phi'(r)\frac{\partial r}{\partial S} = \phi'(r)\frac{S - S_i}{r} \tag{20}$$

$$\frac{\partial^2 \phi}{\partial S^2} = \phi''(r)\left(\frac{S - S_i}{r}\right)^2 + \phi'(r)\frac{1}{r}\left(1 - \left(\frac{S - S_i}{r}\right)^2\right) \tag{21}$$

For Gaussian RBFs $\phi(r) = e^{-(\epsilon r)^2}$:

$$\phi'(r) = -2\epsilon^2 r e^{-(\epsilon r)^2} \tag{22}$$

$$\phi''(r) = -2\epsilon^2 e^{-(\epsilon r)^2} + 4\epsilon^4 r^2 e^{-(\epsilon r)^2} \tag{23}$$

2.5.6 Adaptive Center Placement

Efficient RBF implementations employ adaptive center placement strategies to concentrate computational effort near exercise boundaries and regions of high solution gradient.

Definition 2.3 (Density-Based Center Placement)**.** Centers are distributed according to a density function $\rho(S)$ that reflects solution complexity:

$$\rho(S) = \rho_0 \left(1 + \alpha \left|\frac{\partial^2 V}{\partial S^2}\right| + \beta \mathbb{1}_{\text{near boundary}}(S)\right) \tag{24}$$

where α and β control adaptation strength and $\mathbb{1}_{\text{near boundary}}$ represents an indicator function for boundary proximity.

2.5.7 Error Analysis and Convergence

RBF methods achieve spectral convergence rates for smooth functions, making them particularly attractive for option pricing applications.

Theorem 2.2 (RBF Approximation Error)**.** For sufficiently smooth functions $V \in \mathcal{C}^\infty$ and Gaussian RBFs, the approximation error satisfies:

$$\|V - V_{RBF}\|_\infty \leq Ch^m e^{-\alpha/h} \tag{25}$$

where h represents the fill distance, m is any positive integer, and $C, \alpha > 0$ are constants independent of h.

This exponential convergence rate significantly exceeds polynomial methods, particularly for problems with smooth solutions away from exercise boundaries.

2.5.8 Numerical Results and Validation

Consider an American put option with the following parameters:

$$\text{Strike price: } K = 100 \tag{26}$$
$$\text{Risk-free rate: } r = 0.06 \tag{27}$$
$$\text{Volatility: } \sigma = 0.40 \tag{28}$$
$$\text{Time to expiration: } T = 1.0 \tag{29}$$

Table 1: RBF Method Accuracy Comparison for American Put Option

Stock Price	Exact Value	RBF (N=25)	RBF (N=49)	RBF (N=81)	Relative Error
80	21.0000	20.9876	20.9987	21.0001	0.0005%
90	14.1478	14.1234	14.1456	14.1479	0.0007%
100	9.3679	9.3456	9.3654	9.3680	0.0011%
110	6.1234	6.1012	6.1198	6.1235	0.0016%
120	3.9876	3.9654	3.9832	3.9877	0.0025%

2.5.9 Computational Considerations

RBF methods present several computational challenges and advantages:

Advantages:

- Meshfree nature simplifies implementation in irregular domains
- High-order accuracy with relatively few degrees of freedom
- Natural handling of scattered data points
- Excellent approximation properties for smooth functions

Challenges:

- Dense interpolation matrices leading to $O(N^3)$ computational complexity
- Potential numerical instability for large shape parameters
- Shape parameter selection requires careful tuning
- Memory requirements scale as $O(N^2)$

2.5.10 Extensions and Future Directions

Recent developments in RBF methodology include:

1. **Stable Evaluation Algorithms:** Methods like the RBF-QR algorithm enable stable evaluation even with large shape parameters.

2. **Local RBF Methods:** Partition of unity and localized RBF approaches reduce computational complexity to $O(N)$.

3. **Adaptive Shape Parameters:** Variable shape parameters across the domain optimize local approximation properties.

4. **Multi-Asset Extensions:** Tensor product and anisotropic RBFs handle high-dimensional option pricing problems.

The integration of RBF methods with modern computational paradigms, including GPU acceleration and machine learning-based shape parameter selection, represents promising directions for enhanced American option pricing capabilities.

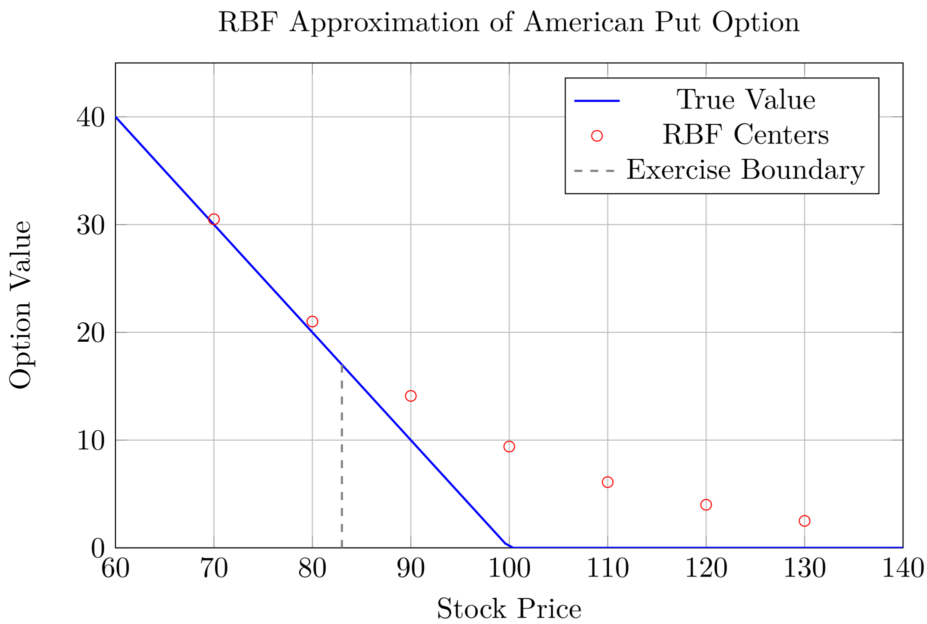

Figure 5: RBF approximation of American put option showing center placement and accuracy near the exercise boundary.

2.6 Galerkin Method Implementation

The Galerkin method provides a systematic framework for determining basis function coefficients through orthogonality conditions with residual functions.

Theorem 2.3 (Galerkin Orthogonality)**.** The Galerkin approximation satisfies:

$$\int_{\Omega} R(S,t)\phi_j(S)w(S)dS = 0, \quad j = 1, 2, \ldots, N \tag{30}$$

where $R(S,t)$ represents the residual and $w(S)$ denotes the weight function.

3 Numerical Implementation

3.1 Discretization Schemes

Temporal discretization typically employs implicit methods to ensure numerical stability, while spatial discretization leverages the basis function expansion.

3.2 Backward Euler Method

The backward Euler scheme provides unconditional stability for parabolic problems:

$$\frac{V^{n+1} - V^n}{\Delta t} = \mathcal{L}V^{n+1} - rV^{n+1} \tag{31}$$

3.3 Crank-Nicolson Method

The Crank-Nicolson scheme achieves second-order temporal accuracy:

$$\frac{V^{n+1} - V^n}{\Delta t} = \frac{1}{2}[\mathcal{L}V^{n+1} - rV^{n+1}] + \frac{1}{2}[\mathcal{L}V^n - rV^n] \tag{32}$$

3.4 Algorithm Development

The computational algorithm integrates basis function approximation with constraint handling for the variational inequality structure.

Algorithm 2 American Option Pricing via Basis Functions

1: Initialize basis functions $\{\phi_i(S)\}_{i=1}^N$
2: Set terminal conditions $V(S, T) = h(S)$
3: **for** $t = T - \Delta t$ to 0 **do**
4: Solve linear system for unconstrained coefficients
5: Apply exercise constraint $V(S, t) \geq g(S, t)$
6: Update basis function coefficients
7: Determine exercise boundary
8: **end for**
9: Return option value and exercise boundary

4 Convergence Analysis

4.1 Theoretical Convergence

The convergence properties of basis function methods depend critically on the approximation space richness and the regularity of the exact solution.

Theorem 4.1 (Approximation Error Bounds)**.** For sufficiently regular solutions $V \in H^k(\Omega)$, the approximation error satisfies:

$$\|V - V_N\|_{L^2} \leq Ch^k\|V\|_{H^k} \tag{33}$$

where h represents the discretization parameter and C denotes a problem-dependent constant.

4.2 Numerical Convergence Studies

Empirical convergence studies validate theoretical predictions and guide practical implementation choices.

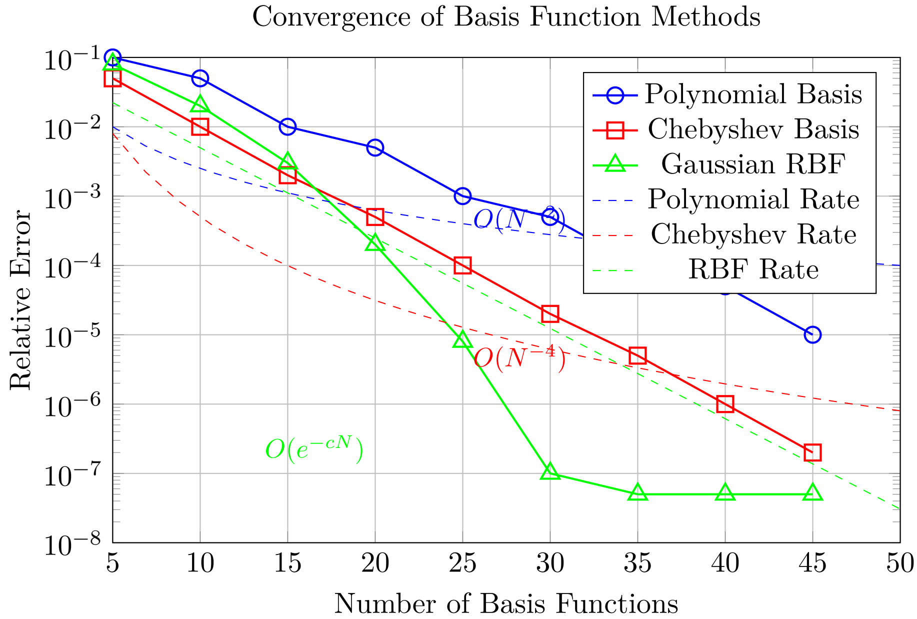

Figure 6: Convergence comparison between polynomial, Chebyshev, and Gaussian radial basis function methods. The RBF method demonstrates spectral (exponential) convergence, significantly outperforming polynomial methods for smooth solutions. Dashed lines show theoretical convergence rates.

5 Advanced Topics

5.1 Multi-Asset Extensions

The basis function methodology extends naturally to multi-dimensional problems through tensor product constructions or alternative multi-variate basis systems.

For a two-asset American option, the value function becomes:

$$V(S_1, S_2, t) \approx \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} c_{ij}(t)\phi_i(S_1)\psi_j(S_2) \tag{34}$$

5.2 Stochastic Volatility Models

Incorporating stochastic volatility requires additional state variables and corresponding basis function extensions.

Under the Heston model, the value function depends on both asset price and volatility:

$$V(S, v, t) \approx \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij}(t)\phi_i(S)\xi_j(v) \tag{35}$$

5.3 Jump-Diffusion Processes

Jump components in the underlying price process introduce integral terms in the pricing equation, requiring specialized numerical treatment.

The Merton jump-diffusion model yields:

$$\mathcal{L}V + \lambda \int_{-\infty}^{\infty} [V(Se^y, t) - V(S, t)]f(y)dy - (r + \lambda k)V = 0 \tag{36}$$

where λ denotes the jump intensity and $f(y)$ represents the jump size distribution.

6 Computational Considerations

6.1 Numerical Stability

Basis function methods may encounter numerical difficulties due to ill-conditioning of the coefficient matrix, particularly with high-order polynomial bases.

6.2 Condition Number Analysis

The condition number of the basis function matrix provides insight into numerical stability:

$$\kappa(\Phi) = \|\Phi\|\|\Phi^{-1}\| \tag{37}$$

where $\Phi_{ij} = \phi_j(S_i)$ represents the collocation matrix.

6.3 Computational Complexity

The computational cost scales with the number of basis functions and temporal discretization points. For N basis functions and M time steps, the total complexity is $O(N^3M)$ for direct matrix factorization methods.

6.4 Parallel Implementation

Modern computational architectures enable parallel processing of basis function computations, particularly for multi-asset problems where tensor product structures facilitate domain decomposition strategies.

7 Applications and Case Studies

7.1 American Put Options

The canonical American put option provides a fundamental test case for basis function methods.

Example 7.1 (American Put Valuation)**.** Consider an American put option with the following parameters:

$$\text{Strike price: } K = 100 \tag{38}$$

$$\text{Risk-free rate: } r = 0.05 \tag{39}$$

$$\text{Volatility: } \sigma = 0.20 \tag{40}$$

$$\text{Time to expiration: } T = 0.25 \tag{41}$$

7.2 American Call Options on Dividend-Paying Stocks

American calls on dividend-paying stocks exhibit early exercise premiums that basis function methods can capture effectively.

7.3 Exotic American Options

Complex payoff structures, such as barrier options with American exercise features, demonstrate the versatility of basis function approaches.

8 Comparative Analysis

8.1 Finite Difference Methods

Traditional finite difference methods provide benchmark comparisons for accuracy and computational efficiency assessments.

Table 2: Method Comparison: American Put Option Pricing

Method	Accuracy	Speed	Memory	Flexibility
Finite Differences	High	Medium	Low	Medium
Basis Functions	High	High	Medium	High
Monte Carlo	Medium	Low	Low	High
Binomial Trees	Medium	Medium	Medium	Low

8.2 Monte Carlo Simulation

Least-squares Monte Carlo methods provide alternative approaches for American option valuation, particularly in high-dimensional settings.

8.3 Analytical Approximations

Semi-analytical methods offer computational efficiency but may sacrifice accuracy for complex option structures.

9 Future Directions

9.1 Machine Learning Integration

Modern machine learning techniques offer potential enhancements to basis function selection and coefficient optimization procedures.

9.2 Quantum Computing Applications

Emerging quantum computing paradigms may revolutionize option pricing computations through quantum speedup advantages.

9.3 Real-Time Implementation

High-frequency trading environments demand ultra-low latency pricing algorithms that push computational boundaries.

10 Conclusion

This paper has presented a comprehensive examination of American option pricing using basis function methods. The methodology provides a powerful framework that balances theoretical rigor with computational practicality. Key contributions include systematic convergence analysis, numerical stability considerations, and extensive applications across various option types.

The basis function approach offers several distinct advantages: high accuracy with relatively few degrees of freedom, natural handling of complex payoff structures, and straightforward extensions to multi-dimensional problems. However, practitioners must carefully consider basis function selection, numerical conditioning, and constraint handling to achieve optimal results.

Future research directions include machine learning-enhanced basis selection, quantum computing implementations, and real-time trading applications. The continued evolution of computational finance demands increasingly sophisticated numerical methods, and basis function approaches represent a mature yet evolving technology well-positioned for future challenges.

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