A State-of-the-Art Approach to Modelling Interest Rates

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Abstract

This paper surveys and unifies modern approaches to modelling interest rates with an emphasis on no-arbitrage foundations, empirical tractability, multi-curve post-crisis realities, negative-rate regimes, and data-driven methods that preserve economic structure. We review affine term structure models (ATSMs), Heath–Jarrow–Morton (HJM) and LIBOR Market Models (LMM), arbitrage-free Nelson–Siegel (AFNS) and dynamic factor approaches, shadow-rate models for effective lower bounds, and recent machine-learning methods including neural SDEs and sequence models constrained by arbitrage. We provide estimation strategies, practical implementation details, and vector-graphics visualizations to illustrate term-structure dynamics.

The paper ends with "The End"

1 Introduction

The term structure of interest rates is central to asset pricing, monetary policy, and risk management. Modern practice balances: (i) structural, no-arbitrage consistency; (ii) flexibility to fit cross-sections and dynamics; (iii) multi-curve markets with collateralization and basis; (iv) regime features like negative rates; and (v) the use of high-capacity statistical learning while enforcing economic constraints.

We synthesize classical models—Vasicek and CIR one-factor models [1,2]—general affine frameworks [3,14], HJM and LMM [6,7], dynamic Nelson–Siegel and AFNS [10,11], shadow-rate models for effective lower bounds [12,13], and recent ML-based approaches that enforce no-arbitrage via drift constraints, shape restrictions, or representation learning [15, 16]. Our focus is on implementable, estimation-ready specifications.

2 Preliminaries and Notation

Let $P(t,T)$ denote the time-t price of a zero-coupon bond maturing at $T \geq t$, with continuously compounded spot rate $y(t,T)$ and instantaneous forward curve $f(t,T)$:

$$P(t,T) = \exp\left(-\int_t^T y(t,u)\,\mathrm{d}u\right), \tag{2.1}$$

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T), \qquad y(t,T) = \frac{1}{T-t}\int_t^T f(t,u)\,\mathrm{d}u. \tag{2.2}$$

Under a risk-neutral measure $\mathbb{Q}$ with short rate r_t, the discounted bond price is a martingale:

$$\frac{P(t,T)}{B_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{B_T}\right], \qquad B_t := \exp\left(\int_0^t r_s\,\mathrm{d}s\right). \tag{2.3}$$

3 No-Arbitrage Foundations: HJM

In the HJM framework [6], the forward-rate curve evolves as

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t + \sum_{k=1}^{d}\sigma_k(t,T)\,\mathrm{d}W_t^k, \tag{3.1}$$

with W a d-dimensional Brownian motion under $\mathbb{Q}$. No-arbitrage imposes the HJM drift condition

$$\alpha(t,T) = \sum_{k=1}^{d}\sigma_k(t,T)\int_t^T \sigma_k(t,u)\,\mathrm{d}u. \tag{3.2}$$

Finite-dimensional Markovian HJM specifications obtain if the volatility structure admits a factor representation (e.g., exponential-polynomial forms) leading to state-space dynamics amenable to filtering and option pricing.

4 Affine Term Structure Models (ATSMs)

Affine models posit $P(t,T) = \exp\{A(T-t) + B(T-t)^\top X_t\}$ where X_t follows an affine diffusion [3–5]:

$$\mathrm{d}X_t = K(\theta - X_t)\,\mathrm{d}t + \Sigma(X_t)\,\mathrm{d}W_t, \qquad \Sigma(X_t)\Sigma(X_t)^\top = S_0 + \sum_{i=1}^{n} X_{t,i}S_i. \tag{4.1}$$

Then

$$\partial_\tau A(\tau) = \phi\Big(B(\tau)\Big), \qquad A(0) = 0, \tag{4.2}$$

$$\partial_\tau B(\tau) = \psi\Big(B(\tau)\Big), \qquad B(0) = 0, \tag{4.3}$$

with $\tau = T - t$ and ϕ, ψ determined by the generator. Vasicek and CIR are one-factor special cases [1, 2], with closed forms for A, B and cap/floor or swaption prices available via Fourier or Jamshidian decompositions in one-factor settings [8].

5 Dynamic Nelson–Siegel and AFNS

The Nelson–Siegel (NS) family approximates the yield curve via level, slope, and curvature loadings [9]:

$$y(t, \tau) = \beta_{0,t} + \beta_{1,t} \frac{1 - e^{-\lambda\tau}}{\lambda\tau} + \beta_{2,t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right). \tag{5.1}$$

Diebold–Li advocate a dynamic factor representation with AR dynamics and Kalman filtering [10]. Arbitrage-free Nelson–Siegel (AFNS) embeds NS-style loadings in a no-arbitrage affine system ensuring internally consistent discounting and derivative pricing [11].

6 Multi-Curve Modelling Post-Crisis

Post-2008, collateralization and basis spreads led to multiple curves: an OIS discount curve and separate forwarding curves for different tenors. The LMM naturally accommodates forward rate dynamics per tenor with collateral-consistent discounting [7]. Affine or HJM multi-curve extensions model basis spreads as additional factors while preserving no-arbitrage across discounting/forwarding relationships.

7 Negative Rates and Shadow-Rate Models

In the presence of effective lower bounds, shadow-rate models map a latent Gaussian (or affine) short rate through a nonlinearity to enforce constraints, enabling tractable filtering and option pricing while reconciling near-zero yields [12, 13].

8 Derivatives Pricing and Measures

Pricing swaptions and caps/floors depends on the model choice: in one-factor affine models, Jamshidian decomposition reduces a swaption to a bond option [8]. In multi-factor settings, characteristic-function methods (Fourier inversion) or PDE/FEM methods apply. Measure changes (e.g., to bond or annuity measures) simplify drift terms for forward rates in LMM/HJM.

9 Machine Learning Under No-Arbitrage

High-capacity learners can fit rich cross-sectional and time-series patterns if constrained to avoid arbitrage. Approaches include:

- Neural SDEs where drift/volatility are learned but regularized to satisfy HJM drift (Equation (3.2)) or affine structure [15].
- Sequence models (e.g., transformers) mapping macro/market observables to factor dynamics with shape constraints on $f(t,T)$ to prevent butterfly arbitrage.
- Gaussian-process term-structure models with covariance kernels enforcing smoothness and monotonic discount factors.
- Physics-informed losses and dual networks to calibrate under $\mathbb{Q}$ and reconcile with $\mathbb{P}$ via pricing-kernel consistency [16].

10 Estimation Strategies

Depending on the specification:

- State-space (ATSM/AFNS): maximum likelihood with Kalman filter; nonlinearity or constraints via extended/unscented Kalman or particle filters.
- HJM/LMM: quasi-maximum likelihood using discretized forward dynamics; cap/floor/swaption implied vol matching via least-squares or likelihood-based calibration.
- GMM/efficiency: moment matching of yields, forwards, and excess returns; risk premium identification via macro-finance instruments.
- Bayesian inference for parameter regularization and density forecasts, enabling coherent uncertainty quantification for risk management.

11 Empirical Illustration with Vector Graphics

We illustrate NS/AFNS loadings and example yield curves via PGFPlots (vector graphics). Figure 1 shows the NS basis; Figure 2 shows plausible yield curves on three hypothetical dates.

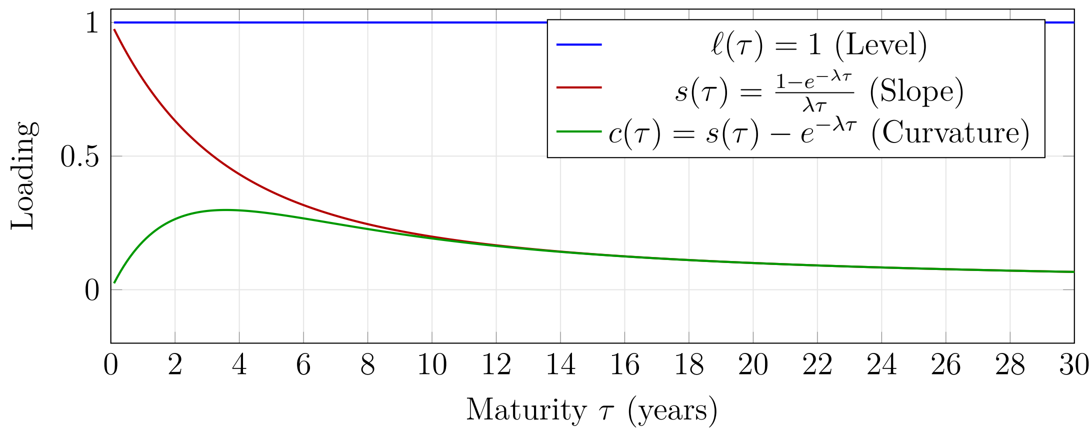

Figure 1: Nelson–Siegel basis functions with $\lambda = 0.5$.

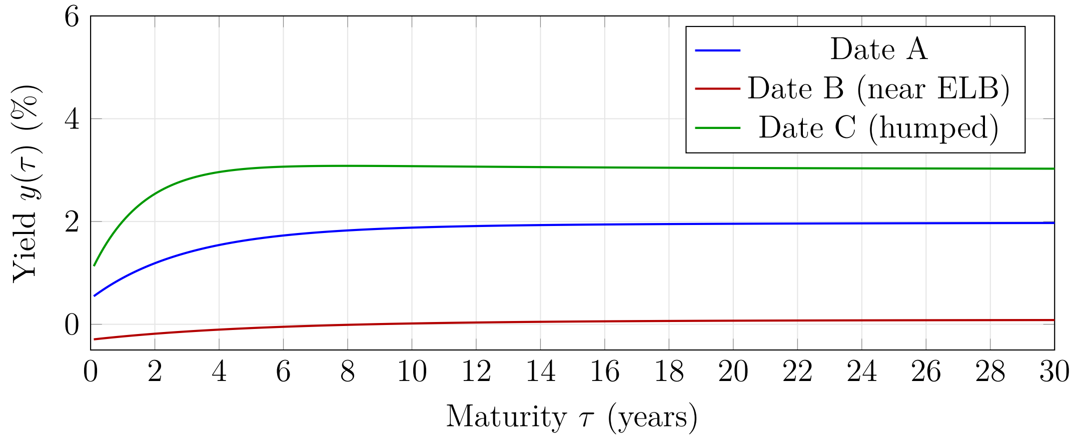

Figure 2: Illustrative yield curves using NS parameterizations.

12 Conclusion

A pragmatic, state-of-the-art interest rate modelling stack combines: an arbitrage-consistent backbone (affine/HJM/LMM), flexible cross-sectional fit (AFNS, factor loadings), multi-curve treatment with collateral/basis, robust handling of lower bounds (shadow-rate), and data-driven dynamics constrained by economic structure. Estimation should leverage state-space and simulation-based methods with careful identification of risk premia, and validation should include both curve-fit diagnostics and derivative pricing consistency.

A Additional Implementation Notes

A.1 Numerical Stability and Identification

Ensure positivity constraints where required (e.g., CIR variance), use square-root transforms for variances, and regularize poorly identified parameters (long-horizon mean reversion vs. level). For AFNS, fix λ or treat it as identified by curvature location; avoid drifting λ without strong priors.

A.2 Calibration Workflows

For LMM/HJM, calibrate to caps/floors for marginal vol and to swaptions for correlation structure. For ATSM/AFNS, jointly fit yields and options or use a two-step approach with correction for measurement error.

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