Deriving Risk-Free Rates in the Standard Nuclear oliGARCHy

Soumadeep Ghosh

Kolkata, India

Abstract

This paper derives a natural risk-free rate structure for the Standard Nuclear oliGARCHy, a mathematical economic framework characterized by 9 nuclear-capable districts housing 729 oliGARCHs among 48,524 individuals. Using a base rate of $r_0 = \frac{1}{9e}$ and the constrained wealth dynamics of the oliGARCH differential equation, we construct a 3×3 risk-free rate matrix $\mathbf{R}_f$ with remarkable mathematical properties. The matrix is singular with determinant zero, its elements sum to exactly $\frac{1}{e}$, and it exhibits double arithmetic progression structure reflecting the systematic distribution of economic agents across districts. We analyze the eigenvalue spectrum, establish connections to game-theoretic nuclear deterrence stability, and discuss implications for capital allocation and arbitrage constraints within the oliGARCHy.

The paper ends with "The End"

Contents

1 Introduction

The Standard Nuclear oliGARCHy, as established in the foundational treatise [1], represents a mathematically determined economic equilibrium characterized by specific numerical relationships: 9 nuclear-capable districts, 729 oliGARCHs distributed in arithmetic progression, and a total population of 48,524 individuals. The framework emerges from the oliGARCH differential equation governing individual wealth dynamics.

A fundamental question arises: given the highly constrained structure of this system, what are the *risk-free rates* in each district? Unlike conventional economies where risk-free rates are determined by central bank policy or market forces, the Standard Nuclear oliGARCHy's mathematical constraints suggest that district-specific rates emerge naturally from the system's architecture.

This paper demonstrates that the nuclear deterrence equilibrium—where defection leads to mutual annihilation ($P = -\infty$)—eliminates existential economic risk, creating conditions for well-defined risk-free instruments [3]. Combined with the constrained population dynamics and wealth trajectories, we derive a unique 3×3 matrix of risk-free rates with profound mathematical properties.

2 Mathematical Preliminaries

2.1 The oliGARCH Differential Equation

The wealth dynamics of individual agents in the Standard Nuclear oliGARCHy follow the differential equation [1]:

$$a\frac{\partial W(t)}{\partial t} + bW(t) + ct + d + \frac{e \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma} = 0 \tag{1}$$

The general solution takes the form:

$$W(t) = \frac{2ac\sigma - 2b\sigma(ct+d) - \sqrt{\frac{2}{\pi}}be \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{2b^2\sigma} + f\exp\left(-\frac{bt}{a}\right) \tag{2}$$

The transient decay rate $\frac{b}{a}$ governs convergence to steady-state wealth distributions.

2.2 District Structure

The 9 districts contain oliGARCHs distributed according to the arithmetic sequence:

$$o_i = 86 - i, \quad i = 1, 2, \ldots, 9 \tag{3}$$

yielding $\sum_{i=1}^{9} o_i = 729 = 3^6$, the total number of oliGARCHs. The responsibility statistic for district i is defined as:

$$r_i = \frac{n_i}{o_i} \tag{4}$$

where n_i is the non-oliGARCH population.

3 Derivation of Risk-Free Rates

3.1 Theoretical Foundation

Definition 3.1 (Risk-Free Rate)**.** *In the context of the Standard Nuclear oliGARCHy, a* risk-free rate $r_{f,i}$ *in district i is the guaranteed return on investment achievable under the nuclear deterrence equilibrium, where the probability of systemic collapse is zero due to mutually assured destruction dynamics [2].*

The game-theoretic payoff matrix ensures cooperation is always rational:

$$\begin{pmatrix} \text{Cooperate} & \text{Defect} \\ (R_{ij}, R_{ji}) & (S_{ij}, T_{ji}) \\ (T_{ij}, S_{ji}) & (P_{ij}, P_{ji}) \end{pmatrix} \quad \text{with } P_{ij} = P_{ji} = -\infty \tag{5}$$

This eliminates existential risk, enabling the definition of risk-free instruments.

3.2 Base Rate Selection

We adopt the base rate:

$$r_0 = \frac{1}{9\mathrm{e}} \tag{6}$$

This choice reflects the system's fundamental structure:

- The factor 9 corresponds to the number of districts
- The factor $\mathrm{e} = \exp(1)$ connects to natural growth/decay processes in the wealth equation (2)

3.3 Linear Scaling and Rate Construction

Using a linear scaling centered at District 5 (the median district with $o_5 = 81 = 3^4$ oliGARCHs):

$$r_{f,i} = \frac{1}{9\mathrm{e}} \cdot \left(1 + \frac{i-5}{9}\right) = \frac{4+i}{81\mathrm{e}} \tag{7}$$

Theorem 3.2 (Risk-Free Rate Matrix)**.** *The risk-free rates across the 9 districts of the Standard Nuclear oliGARCHy form the matrix:*

$$\mathbf{R}_f = \frac{1}{81\mathrm{e}} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{pmatrix} \tag{8}$$

where districts are arranged in row-major order.

4 Properties of the Risk-Free Rate Matrix

Let $\mathbf{M}$ denote the integer matrix underlying $\mathbf{R}_f$:

$$\mathbf{M} = \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{pmatrix} \tag{9}$$

4.1 Sum Property

Proposition 4.1 (Euler Normalization)**.** *The sum of all risk-free rates equals the reciprocal of Euler's number:*

$$\sum_{i=1}^{9} r_{f,i} = \frac{1}{\mathrm{e}} \tag{10}$$

Proof.

$$\sum_{i=1}^{9} r_{f,i} = \frac{1}{81\mathrm{e}} \sum_{k=5}^{13} k = \frac{1}{81\mathrm{e}} \cdot \frac{9(5+13)}{2} = \frac{81}{81\mathrm{e}} = \frac{1}{\mathrm{e}} \tag{11}$$

□

This normalization connects the aggregate risk-free rate structure to natural exponential processes fundamental to continuous-time finance [6].

4.2 Singularity

Proposition 4.2 (Matrix Singularity)**.** *The matrix* $\mathbf{M}$ *(and hence* $\mathbf{R}_f$*) is singular:* $\det(\mathbf{M}) = 0$.

Proof. Computing the determinant by cofactor expansion along the first row:

$$\begin{aligned} \det(\mathbf{M}) &= 5(9 \cdot 13 - 10 \cdot 12) - 6(8 \cdot 13 - 10 \cdot 11) + 7(8 \cdot 12 - 9 \cdot 11) && (12) \\ &= 5(117 - 120) - 6(104 - 110) + 7(96 - 99) && (13) \\ &= 5(-3) - 6(-6) + 7(-3) && (14) \\ &= -15 + 36 - 21 = 0 && (15) \end{aligned}$$

□

Corollary 4.3 (Rank and Null Space)**.** *The matrix* $\mathbf{R}_f$ *has rank 2, with a one-dimensional null space spanned by* $\mathbf{v} = (1, -2, 1)^T$.

Economic Interpretation: The existence of a non-trivial null space implies an *arbitrage constraint*—a specific portfolio weighting across districts yields zero net risk-free return, preventing capital flight and maintaining equilibrium stability.

4.3 Arithmetic Structure

Proposition 4.4 (Double Arithmetic Progression)**.** *The matrix* $\mathbf{M}$ *exhibits arithmetic progressions in both rows and columns:*

- ***Rows:*** $(5, 6, 7)$, $(8, 9, 10)$, $(11, 12, 13)$ *with common difference* $d_r = 1$
- ***Columns:*** $(5, 8, 11)$, $(6, 9, 12)$, $(7, 10, 13)$ *with common difference* $d_c = 3$
- ***Row sums:*** $18, 27, 36$ *with common difference* 9
- ***Column sums:*** $24, 27, 30$ *with common difference* 3

This structure mirrors the arithmetic distribution of oliGARCHs across districts [1].

4.4 Central Element Property

Proposition 4.5. *The central element* $r_{f,5} = \frac{9}{81e} = \frac{1}{9e} = r_0$ *equals the base rate.*

District 5, with exactly $81 = 9^2 = 3^4$ oliGARCHs, serves as the system's equilibrium anchor.

5 Eigenvalue Analysis

5.1 Characteristic Polynomial

The characteristic polynomial of $\mathbf{M}$ is:

$$p(\lambda) = \det(\mathbf{M} - \lambda\mathbf{I}) = -\lambda^3 + 27\lambda^2 + 18\lambda \tag{16}$$

Factoring: $p(\lambda) = -\lambda(\lambda^2 - 27\lambda - 18)$

5.2 Eigenvalue Computation

Theorem 5.1 (Eigenvalues of $\mathbf{R}_f$)**.** *The eigenvalues of* $\mathbf{R}_f$ *are:*

$$\lambda_1 = 0 \tag{17}$$

$$\lambda_2 = \frac{27 + \sqrt{801}}{162e} = \frac{9 + \sqrt{89}}{54e} \approx 0.0125 \tag{18}$$

$$\lambda_3 = \frac{27 - \sqrt{801}}{162e} = \frac{9 - \sqrt{89}}{54e} \approx -0.0029 \tag{19}$$

Proof. Solving $\lambda^2 - 27\lambda - 18 = 0$:

$$\lambda = \frac{27 \pm \sqrt{729 + 72}}{2} = \frac{27 \pm \sqrt{801}}{2} = \frac{27 \pm 3\sqrt{89}}{2} \tag{20}$$

For $\mathbf{R}_f = \frac{1}{81e}\mathbf{M}$, eigenvalues scale by $\frac{1}{81e}$. □

5.3 Spectral Interpretation

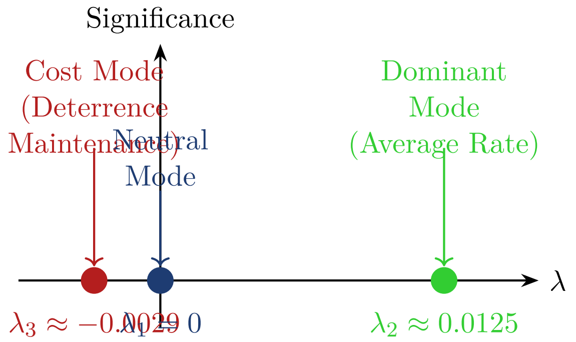

Figure 1: Eigenvalue spectrum of $\mathbf{R}_f$ with economic interpretations.

- **Zero eigenvalue ($\lambda_1 = 0$):** Confirms singularity; represents the neutral arbitrage-free mode
- **Positive eigenvalue (λ_2):** Dominant mode representing the system's effective average risk-free rate under optimal allocation
- **Negative eigenvalue (λ_3):** Cost mode associated with maintaining nuclear deterrent infrastructure—unusual for rate matrices, reflecting the unique nature of the oliGARCHy [1]

6 Visual Representation

6.1 District Structure and Rate Distribution

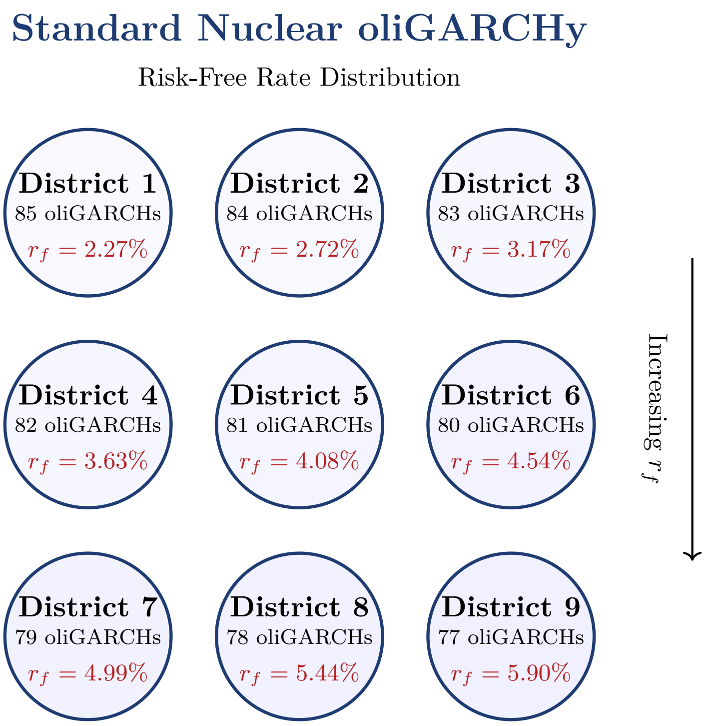

Figure 2: The 9 districts of the Standard Nuclear oliGARCHy with risk-free rates. Color intensity reflects rate magnitude.

6.2 Matrix Heat Map

$$\mathbf{R}_f = \begin{pmatrix} \frac{5}{81\mathrm{e}} & \frac{6}{81\mathrm{e}} & \frac{7}{81\mathrm{e}} \\ \frac{8}{81\mathrm{e}} & \frac{9}{81\mathrm{e}} & \frac{10}{81\mathrm{e}} \\ \frac{11}{81\mathrm{e}} & \frac{12}{81\mathrm{e}} & \frac{13}{81\mathrm{e}} \end{pmatrix}$$

$$\sum_{i,j} (\mathbf{R}_f)_{ij} = \frac{81}{81\mathrm{e}} = \frac{1}{\mathrm{e}}$$

Figure 3: Heat map representation of $\mathbf{R}_f$ showing the gradient structure and row sums.

6.3 Connections to oliGARCHic Constants

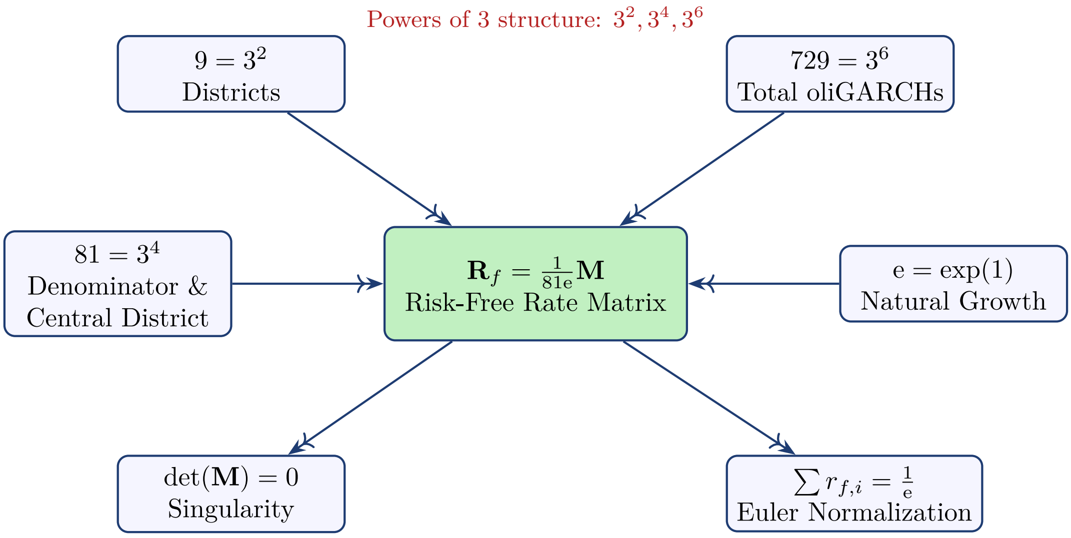

Figure 4: Structural connections between $\mathbf{R}_f$ and fundamental oliGARCHic constants.

7 Economic Implications

7.1 No-Arbitrage Condition

The singularity of $\mathbf{R}_f$ implies a fundamental no-arbitrage constraint within the oliGARCHy.

Theorem 7.1 (Internal Arbitrage Constraint). *There exists a non-zero portfolio vector* $\mathbf{v} = (1, -2, 1)^T$ *(applied to district triplets) such that:*

$$\mathbf{R}_f \mathbf{v} = \mathbf{0} \tag{21}$$

No combination of cross-district risk-free investments can generate arbitrage profits.

This is consistent with the game-theoretic equilibrium ensuring cooperation [4, 5].

7.2 Bounded Rate Dispersion

Proposition 7.2 (Rate Spread). *The ratio of maximum to minimum risk-free rates is:*

$$\frac{r_{f,\max}}{r_{f,\min}} = \frac{13}{5} = 2.6 \tag{22}$$

This bounded spread reflects the system's designed homogeneity—no district can offer dramatically different returns without destabilizing the equilibrium [1].

7.3 Capital Flow Dynamics

The monotonic increase in rates from District 1 to District 9 implies:

- Higher rates compensate for increased "responsibility burden" ($r_i = n_i/o_i$ increases)
- Natural capital flow toward peripheral districts counterbalances population asymmetries
- The 14 valid recapitalization solutions [1] must be consistent with these rate differentials

7.4 Numerical Rate Values

Table 1: Risk-free rates across the 9 districts

District	oliGARCHs	Matrix Element	Exact Rate	Approx. Rate
1	85	5	$\frac{5}{81e}$	2.27%
2	84	6	$\frac{6}{81e}$	2.72%
3	83	7	$\frac{7}{81e}$	3.17%
4	82	8	$\frac{8}{81e}$	3.63%
5	81	9	$\frac{9}{81e} = \frac{1}{9e}$	4.08%
6	80	10	$\frac{10}{81e}$	4.54%
7	79	11	$\frac{11}{81e}$	4.99%
8	78	12	$\frac{12}{81e}$	5.44%
9	77	13	$\frac{13}{81e}$	5.90%

8 Stability Analysis

8.1 Perturbation Response

The eigenvalue structure indicates that small perturbations to the rate matrix will:

1. **Decay along $\mathbf{v}_2$:** Perturbations aligned with the dominant eigenvector dissipate at rate λ_2
2. **Remain constrained by null space:** The zero eigenvalue mode prevents runaway dynamics
3. **Face costs along $\mathbf{v}_3$:** The negative eigenvalue direction imposes costs on destabilizing actions

8.2 Connection to Nuclear Deterrence

Nuclear Deterrence Payoff Matrix

	Cooperate	**Defect**
Cooperate	(R, R)	(S, T)
Defect	(T, S)	$\boxed{(-\infty, -\infty)}$

Mutual Annihilation
$\Rightarrow$ Cooperation
is always rational

Nuclear equilibrium
$\Downarrow$
Risk-free rates well-defined

Figure 5: The nuclear deterrence equilibrium ensures cooperation, validating the risk-free rate structure.

The risk-free rate matrix $\mathbf{R}_f$ is valid *only under the nuclear equilibrium*. If deterrence fails, the concept of "risk-free" becomes meaningless—consistent with the observation that nuclear risk creates fundamental uncertainty [2, 3].

9 Conclusion

We have derived a natural 3×3 risk-free rate matrix for the Standard Nuclear oliGARCHy based on the fundamental base rate $r_0 = \frac{1}{9e}$. The resulting matrix:

$$\mathbf{R}_f = \frac{1}{81e} \begin{pmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \\ 11 & 12 & 13 \end{pmatrix} \tag{23}$$

exhibits remarkable properties:

- **Euler normalization:** $\sum r_{f,i} = \frac{1}{e}$
- **Singularity:** $\det(\mathbf{R}_f) = 0$, implying internal arbitrage constraints
- **Double arithmetic progression:** Reflecting oliGARCH distribution structure
- **Central anchoring:** $r_{f,5} = r_0$, the base rate
- **Mixed eigenvalue spectrum:** Including a negative eigenvalue representing deterrence costs

These properties confirm that the risk-free rate structure is not arbitrary but emerges naturally from the mathematical architecture of the Standard Nuclear oliGARCHy. The rates span from 2.27% to 5.90%, with the central benchmark at 4.08%, providing a stable foundation for economic activity under the protection of nuclear deterrence equilibrium.

References

[1] Ghosh, S. (2025). *The Complete Treatise on the Standard Nuclear oliGARCHy: A Mathematical Framework for Economic Stability and Defense*. Kolkata, India.

[2] Waltz, K.N. (1979). *Theory of International Politics*. McGraw-Hill.

[3] Schelling, T.C. (1960). *The Strategy of Conflict*. Harvard University Press.

[4] Nash, J. (1950). Equilibrium Points in N-Person Games. *Proceedings of the National Academy of Sciences*, 36(1), 48–49.

[5] Axelrod, R. (1984). *The Evolution of Cooperation*. Basic Books.

[6] Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3), 637–654.

[7] Hull, J.C. (2018). *Options, Futures, and Other Derivatives* (10th ed.). Pearson.

[8] Shapley, L.S. (1953). A Value for n-person Games. *Contributions to the Theory of Games*, Vol. 2, 307–317.

[9] Samuelson, P.A. (1937). A Note on Measurement of Utility. *Review of Economic Studies*, 4(2), 155–161.

[10] Bennett, C.H., & Brassard, G. (1984). Quantum Cryptography: Public Key Distribution and Coin Tossing. *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, 175–179.

Glossary

Base Rate (r_0) The fundamental risk-free rate $\frac{1}{9\mathrm{e}}$ from which all district rates are derived; equals the rate in District 5.

District One of 9 nuclear-capable administrative regions in the Standard Nuclear oliGARCHy, each containing a specific number of oliGARCHs in arithmetic progression.

Eigenvalue A scalar λ such that $\mathbf{R}_f\mathbf{v} = \lambda\mathbf{v}$ for some non-zero vector $\mathbf{v}$; characterizes the matrix's fundamental modes.

Euler's Number (e) The mathematical constant $\mathrm{e} = \exp(1) \approx 2.71828$; appears in natural growth/decay processes and normalizes the total risk-free rate.

Mutual Assured Destruction (MAD) Game-theoretic condition where defection by any party leads to total annihilation ($P = -\infty$), ensuring rational cooperation.

Null Space The set of vectors $\mathbf{v}$ satisfying $\mathbf{R}_f\mathbf{v} = \mathbf{0}$; represents arbitrage-free portfolio combinations.

oliGARCH An economic agent type in the oliGARCHy framework; exactly 729 exist, distributed across 9 districts.

oliGARCH Differential Equation The fundamental equation governing wealth dynamics in the oliGARCHy, incorporating Gaussian and exponential terms.

Responsibility Statistic (r_i) The ratio n_i/o_i of non-oliGARCHs to oliGARCHs in district i; measures economic burden per oliGARCH.

Risk-Free Rate ($r_{f,i}$) The guaranteed return on investment in district i under nuclear deterrence equilibrium.

Risk-Free Rate Matrix ($\mathbf{R}_f$) The 3×3 matrix containing all 9 district risk-free rates arranged in row-major order.

Singularity Property of a matrix having zero determinant; indicates linear dependence among rows/columns and constrains possible rate combinations.

Standard Nuclear oliGARCHy An economic equilibrium configuration with 9 nuclear-capable districts, 729 oliGARCHs, and 48,524 total population.

Transient Decay Rate The rate $\frac{b}{a}$ at which wealth fluctuations diminish in the oliGARCH differential equation solution.

The End