Asset pricing and inflation when the inflation risk premium is zero

Soumadeep Ghosh

Kolkata, India

Abstract

In a sequence of earlier notes I constructed examples in which (i) the inflation risk premium is zero at all points in time, (ii) the implied path of inflation is characterized, and (iii) the capital asset pricing model (CAPM) is satisfied through several explicit solutions for market returns and betas. In this paper I unify those ideas into a single theoretical framework with more explicit economic foundations. I define the inflation risk premium using a nominal stochastic discount factor and an inflation index, derive general conditions under which it is identically zero, and then provide a constructive continuous–time example. Within this environment I characterize inflation dynamics and show how a continuum of CAPM–consistent equilibria can be generated.

The paper ends with "The End"

1 Introduction

A central question in asset pricing is how inflation risk is compensated in financial markets. When nominal payoffs are uncertain and inflation is stochastic, investors typically demand an inflation risk premium. However, it is logically possible that the inflation risk premium is zero, either because inflation is perfectly predictable or because it is spanned and hedged by traded assets in a way that renders investors effectively neutral to it.

In previous short papers I showed that (i) specific functional forms for asset returns, the risk–free rate and expected inflation can be chosen so that the inflation risk premium is zero at all times, (ii) inflation in such an environment can be described by simple time profiles, and (iii) the CAPM can be satisfied by multiple explicit combinations of market returns and betas.[1]

The purpose of this paper is threefold. First, I give a more systematic and rigorous derivation of the concept of an inflation risk premium within a standard stochastic discount factor (SDF) framework. Second, I characterise a class of economies in which the inflation risk premium is identically zero and provide a concrete continuous–time example. Third, I embed this example into the CAPM and show that the zero inflation risk premium is compatible with a multiplicity of CAPM equilibria.

The contribution is theoretical and constructive. I am not arguing that the world is exactly of this form; rather, the goal is to delineate a limiting case in which inflation uncertainty does not generate an additional premium in nominal expected returns, and to understand the implications of this case for equilibrium asset pricing.

[1] Those notes are unified, extended and replaced by the present paper.

2 Economic environment and notation

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. All random variables and processes are assumed to be adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

2.1 Nominal and real quantities

Let I_t denote the level of the price index at time $t \geq 0$ and define the (instantaneous) inflation rate i_t by

$$i_t \; := \; \frac{\mathrm{d}}{\mathrm{d}t} \log I_t,$$

whenever the derivative exists. For any horizon $h > 0$ we can also define average inflation between t and $t + h$ by

$$i(t, t+h) := \frac{1}{h} \log \frac{I_{t+h}}{I_t}.$$

Let B_t denote the nominal value at t of a money market account with instantaneous risk–free rate r_t^f, so that

$$\frac{\mathrm{d}B_t}{B_t} = r_t^f \,\mathrm{d}t.$$

Let S_t^A be the nominal price process of a risky asset A, with instantaneous expected nominal return r_t^A:

$$\frac{\mathrm{d}S_t^A}{S_t^A} = r_t^A \,\mathrm{d}t + (\text{martingale terms}).$$

Real quantities are obtained by deflating nominal ones by the price level. For example, the real price of the money market account is $\tilde{B}_t := B_t / I_t$.

2.2 Stochastic discount factor and inflation risk

We assume the existence of a strictly positive nominal stochastic discount factor (SDF) or pricing kernel $(M_t)_{t\geq 0}$ such that for any payoff X_T at date T,

$$\frac{S_t^X}{B_t} = \mathbb{E}_t \left[\frac{M_T}{M_t} \frac{X_T}{B_T} \right],$$

where $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]$.

A convenient way to think about inflation risk is via the real SDF $\hat{M}_t := M_t I_t$. When $\hat{M}_t$ prices real payoffs, nominal pricing can be decomposed into a real component and an inflation component. This allows us to define the inflation risk premium.

3 The inflation risk premium

Fix a short horizon $h > 0$. Consider the nominal log return on asset A between t and $t + h$, defined as

$$R^A(t, t+h) := \log \frac{S_{t+h}^A}{S_t^A}.$$

Similarly, let $R^f(t, t+h)$ be the log risk–free return over the same horizon,

$$R^f(t, t+h) := \log \frac{B_{t+h}}{B_t} = \int_t^{t+h} r_s^f \,\mathrm{d}s,$$

and let $i(t, t+h)$ be average inflation as defined above.

Definition 1 (Inflation risk premium at horizon h)**.** *For any asset A and horizon $h > 0$, the inflation risk premium at time t is*

$$\pi^i_A(t,h) := \mathbb{E}_t\big[R^A(t,t+h)\big] - R^f(t,t+h) - \mathbb{E}_t\big[i(t,t+h)\big].$$

The first term is the expected nominal excess return on the asset; the second term subtracts the risk–free rate; the third term subtracts the expected inflation over the horizon. We thus interpret $\pi^i_A(t,h)$ as the portion of the expected nominal excess return that compensates investors for bearing inflation risk.

Remark 1. *If an asset is a perfectly indexed real bond, then its nominal payoff is $X_T = \tilde{X}_T I_T$ with $\tilde{X}_T$ non–stochastic. In that case the inflation risk premium reduces to the usual real risk premium, and $\pi^i_A(t,h)$ is zero by construction. The interesting case is where nominal payoffs are not perfectly indexed.*

4 General conditions for a zero inflation risk premium

I now derive conditions under which the inflation risk premium $\pi^i_A(t,h)$ is identically zero for all t and all h.

Assumption 1 (Real pricing kernel independent of inflation)**.** *The real SDF $\hat{M}_t := M_t I_t$ is independent of the inflation process $(I_t)_{t\geq 0}$ under $\mathbb{P}$, and nominal payoffs can be written as*

$$X_T = \tilde{X}_T \cdot g(I_T),$$

where $\tilde{X}_T$ is $\sigma(\hat{M}_s : s \leq T)$–measurable and g is a real–valued function.

Assumption 1 is a stylized way of saying that real risk (which drives $\hat{M}_t$ and $\tilde{X}_T$) is statistically independent of inflation risk (which enters only through I_T). This is a strong assumption but leads to a transparent benchmark.

Proposition 1 (Zero inflation risk premium)**.** *Suppose Assumption 1 holds and that inflation is conditionally deterministic given current information, i.e. for every t and $h > 0$,*

$$i(t,t+h) = \mathbb{E}_t\big[i(t,t+h)\big].$$

Then for any asset A and any horizon $h > 0$,

$$\pi^i_A(t,h) = 0, \qquad \text{for all } t \geq 0.$$

Proof. Under Assumption 1, the real payoff $\tilde{X}_T := X_T/I_T$ and the real SDF $\hat{M}_T$ are independent of inflation. Nominal pricing gives

$$S^A_t = \mathbb{E}_t[M_T X_T] = \mathbb{E}_t\left[\hat{M}_T \tilde{X}_T\right],$$

so the real payoff is priced entirely by the real SDF. The expected real return over $[t, t+h]$ is therefore determined by real risk only; inflation does not enter.

Now write the nominal log return as

$$R^A(t, t+h) = R^{A,\text{real}}(t, t+h) + i(t, t+h),$$

where $R^{A,\text{real}}(t, t+h)$ is the log real return and $i(t, t+h)$ is average inflation. Similarly,

$$R^f(t, t+h) = R^{f,\text{real}}(t, t+h) + i(t, t+h),$$

because a risk–free nominal bond is a risk–free real bond plus inflation.

Taking conditional expectations at time t and using conditional determinism of inflation,

$$\mathbb{E}_t\big[R^A(t, t+h)\big] = \mathbb{E}_t\big[R^{A,\text{real}}(t, t+h)\big] + \mathbb{E}_t\big[i(t, t+h)\big],$$

and

$$R^f(t, t+h) = R^{f,\text{real}}(t, t+h) + \mathbb{E}_t\big[i(t, t+h)\big].$$

Subtracting,

$$\mathbb{E}_t\big[R^A(t, t+h)\big] - R^f(t, t+h) - \mathbb{E}_t\big[i(t, t+h)\big] = \mathbb{E}_t\big[R^{A,\text{real}}(t, t+h)\big] - R^{f,\text{real}}(t, t+h),$$

which is purely a real risk premium. By definition, the inflation risk premium is the part of the nominal excess return that compensates for inflation risk, not real risk. Hence $\pi^i_A(t, h) = 0$ under the stated assumptions. □

Proposition 1 gives conditions for a zero inflation risk premium that are agnostic about the specific time profile of inflation or returns. In the next section I present a simple continuous–time specification that satisfies these conditions and is convenient for explicit calculation.

5 A constructive continuous–time example

We now construct an example in which the inflation risk premium is zero for all $t \geq 0$. Time t is taken to be continuous on $[0, \infty)$.

5.1 Functional forms for returns and inflation

Let the instantaneous expected nominal return on a particular asset A be given by a Gaussian density in time:

$$r^A(t) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right), \qquad t \geq 0, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are parameters.

Let inflation be deterministic and exponentially decaying:

$$i(t) := \lambda e^{-\lambda t}\, \mathbb{1}_{\{t \geq 0\}}, \qquad \lambda > 0. \tag{2}$$

Because $i(t)$ is deterministic, $\mathbb{E}_t[i(s)] = i(s)$ for all $s \geq t$; thus inflation is conditionally deterministic and satisfies the assumption of Proposition 1.

Define the instantaneous risk–free rate by

$$r^f(t) := r^A(t) - i(t), \qquad t \geq 0. \tag{3}$$

This implies

$$r^A(t) = r^f(t) + i(t), \qquad \text{for all } t \geq 0. \tag{4}$$

5.2 Zero inflation risk premium in the example

For a small horizon $h > 0$, approximate log returns by their instantaneous rates:

$$R^A(t, t+h) \approx r^A(t)h, \quad R^f(t, t+h) \approx r^f(t)h, \quad i(t, t+h) \approx i(t)h.$$

Since inflation is deterministic, $\mathbb{E}_t[i(t, t+h)] = i(t, t+h)$. Using (4), we get

$$r^A(t)h - r^f(t)h - \mathbb{E}_t[i(t, t+h)] \approx \big(r^A(t) - r^f(t) - i(t)\big)h = 0.$$

Hence, at leading order in h, the inflation risk premium is zero at all times $t \geq 0$.

For finite horizons, one can integrate the instantaneous relations over time. For example, for any $T > 0$ we have

$$\int_0^T r^A(s)\,\mathrm{d}s = \int_0^T r^f(s)\,\mathrm{d}s + \int_0^T i(s)\,\mathrm{d}s,$$

so that the cumulative nominal excess return equals the risk–free return plus realized inflation and therefore does not contain an additional inflation risk premium.

6 CAPM with a zero inflation risk premium

We now embed the example above into a CAPM structure and show how to construct many CAPM–consistent equilibria.

6.1 Continuous–time CAPM

Let $r^M(t)$ denote the instantaneous expected return on the market portfolio at time $t \geq 0$ and let $\beta_A(t)$ denote the (possibly time–varying) beta of asset A with respect to the market.

Definition 2 (Instantaneous CAPM)**.** *Asset A is said to satisfy the instantaneous CAPM if*

$$r^A(t) = r^f(t) + \beta_A(t)\big(r^M(t) - r^f(t)\big), \qquad \textit{for all } t \geq 0. \tag{5}$$

In our example, $r^A(t)$ and $r^f(t)$ are given by (1) and (3), so the CAPM relation (5) imposes a restriction linking $\beta_A(t)$ and $r^M(t)$.

6.2 A continuum of CAPM solutions

For any choice of (non–zero) beta function $\beta_A(t)$, the CAPM equation (5) can be solved for the market return:

$$r^M(t) = r^f(t) + \frac{r^A(t) - r^f(t)}{\beta_A(t)}. \tag{6}$$

Conversely, for any choice of $r^M(t)$ such that $r^M(t) \neq r^f(t)$, one can solve for $\beta_A(t)$. This leads to a continuum of CAPM–consistent equilibria.

Proposition 2 (Multiplicity of CAPM equilibria)**.** *Suppose $r^A(t)$ and $r^f(t)$ are given by* (1)–(3)*, with inflation defined by* (2)*. For any measurable function $\beta_A : [0, \infty) \to \mathbb{R}$ satisfying $\beta_A(t) \neq 0$ for all t, define $r^M(t)$ by* (6)*. Then the instantaneous CAPM relation* (5) *holds identically for all $t \geq 0$. In particular, there exist infinitely many pairs $(\beta_A(t), r^M(t))$ that satisfy CAPM.*

Proof. The result follows by direct substitution. Given $\beta_A(t) \neq 0$, define $r^M(t)$ by (6). Then

$$r^f(t) + \beta_A(t)\big(r^M(t) - r^f(t)\big) = r^f(t) + \beta_A(t)\left(\frac{r^A(t) - r^f(t)}{\beta_A(t)}\right) = r^A(t),$$

so (5) holds for all t. $\square$

6.3 Five explicit CAPM solutions

To illustrate Proposition 2, we present five specific choices of $\beta_A(t)$ and the corresponding market return functions.

Let $\Delta(t) := r^A(t) - r^f(t) = i(t)$ denote the instantaneous nominal excess return of asset A over the risk–free rate; in our example, $\Delta(t) = i(t) = \lambda e^{-\lambda t}\, \mathbb{1}_{\{t \geq 0\}}$.

Solution 1: Constant negative beta $\beta_A(t) \equiv -2$. Set

$$\beta_A^{(1)}(t) := -2, \qquad r^{M,(1)}(t) := r^f(t) - \tfrac{1}{2}\Delta(t).$$

Then (5) holds.

Solution 2: Constant negative beta $\beta_A(t) \equiv -1$. Set

$$\beta_A^{(2)}(t) := -1, \qquad r^{M,(2)}(t) := r^f(t) - \Delta(t).$$

Solution 3: Constant positive beta $\beta_A(t) \equiv \frac{1}{2}$. Set

$$\beta_A^{(3)}(t) := \tfrac{1}{2}, \qquad r^{M,(3)}(t) := r^f(t) + 2\Delta(t).$$

Solution 4: Constant positive beta $\beta_A(t) \equiv 1$. Set

$$\beta_A^{(4)}(t) := 1, \qquad r^{M,(4)}(t) := r^A(t).$$

Here the market portfolio has the same instantaneous return as asset A; the asset is literally the market.

Solution 5: Time–varying beta. Let $M > 0$ and $\Sigma > 0$ and define a heavy–tailed, time–varying beta by

$$\beta_A^{(5)}(t) := \frac{1}{\sqrt{2\pi\Sigma}} \frac{\exp\big(-\frac{(t-M)^2}{2\Sigma}\big)}{(t-M)^{3/2}}\, \mathbb{1}_{\{t>M\}}.$$

For $t \leq M$ we can set $\beta_A^{(5)}(t)$ to any non–zero constant. Then we obtain

$$r^{M,(5)}(t) := r^f(t) + \frac{\Delta(t)}{\beta_A^{(5)}(t)},$$

and again (5) holds. This last solution illustrates that CAPM is compatible with highly non–trivial time variation in beta and market returns when the underlying asset and risk–free rate are specified as above.

7 Discussion and extensions

The model developed here is intentionally stylized. The independence between real risk and inflation and the conditional determinism of inflation are extreme assumptions designed to isolate the mechanics of a zero inflation risk premium. Relaxing these assumptions would produce richer and more realistic implications.

One extension is to allow inflation to be stochastic but spanned by traded assets. In a complete markets setting where the inflation index is perfectly replicable, inflation risk can be hedged at zero cost and the inflation risk premium may again be negligible. Another extension is to embed the analysis into a consumption–based asset pricing model where the representative agent has preferences over real consumption but cares about the level of nominal wealth for reasons such as nominal rigidities or borrowing constraints.

Empirically, the zero inflation risk premium benchmark can be used as a null hypothesis when studying the pricing of inflation–linked securities and the decomposition of nominal yields into real yields and inflation expectations.

8 Conclusion

This paper has developed a unified theoretical framework for thinking about asset pricing when the inflation risk premium is zero at all points in time. Starting from a stochastic discount factor representation, I defined the inflation risk premium, derived conditions under which it vanishes, and constructed a continuous–time example in which it is identically zero. Within that environment I showed how multiple CAPM–consistent equilibria can be generated.

The main message is conceptual: a world with zero inflation risk premium is internally coherent and compatible with standard asset pricing relations such as the CAPM. It provides a useful benchmark for understanding how inflation risk is or is not compensated in more realistic settings.

References

[1] Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics*, 7(3), 265–296.

[2] Campbell, J. Y., & Viceira, L. M. (2002). *Strategic Asset Allocation*. Oxford University Press.

[3] Cochrane, J. H. (2005). *Asset Pricing* (Revised ed.). Princeton University Press.

[4] Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19(3), 425–442.

[5] Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, 47(1), 13–37.

The End