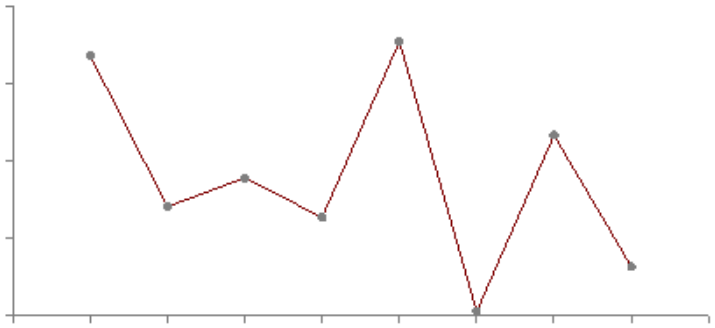
Interpolation methods

Written by Paul Bourke
December 1999

Discussed here are a number of interpolation methods, this is by no means an exhaustive list but the methods shown tend to be those in common use in computer graphics. The main attributes is that they are easy to compute and are stable. Interpolation as used here is different to "smoothing", the techniques discussed here have the characteristic that the estimated curve passes through all the given points. The idea is that the points are in some sense correct and lie on an underlying but unknown curve, the problem is to be able to estimate the values of the curve at any position between the known points.

Linear

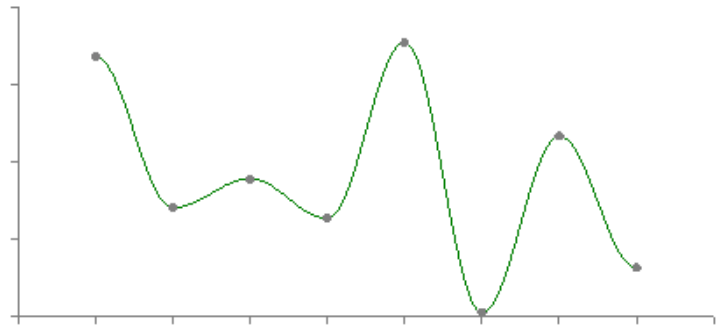
Linear interpolation is the simplest method of getting values at positions in between the data points. The points are simply joined by straight line segments. Each segment (bounded by two data points) can be interpolated independently. The parameter mu defines where to estimate the value on the interpolated line, it is 0 at the first point and 1 and the second point. For interpolated values between the two points mu ranges between 0 and 1. Values of mu outside this range result in extrapolation. This convention is followed for all the subsequent methods below. As with subsequent more interesting methods, a snippet of plain C code will server to describe the mathematics.



```
double LinearInterpolate(
   double y1,double y2,
   double mu)
{
   return(y1*(1-mu)+y2*mu);
}
```

Cosine

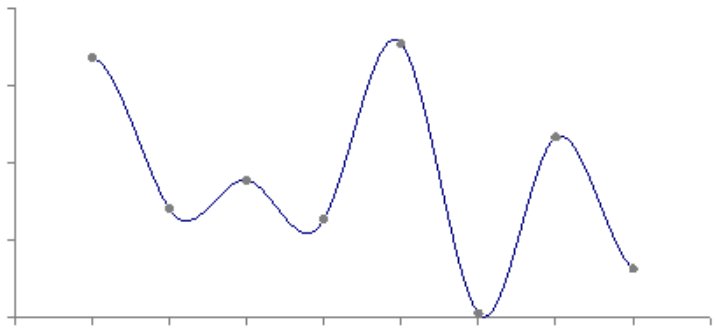
Linear interpolation results in discontinuities at each point. Often a smoother interpolating function is desirable, perhaps the simplest is cosine interpolation. A suitable orientated piece of a cosine function serves to provide a smooth transition between adjacent segments.



```
double CosineInterpolate(
   double y1,double y2,
   double mu)
{
   double mu2;

   mu2 = (1-cos(mu*PI))/2;
   return(y1*(1-mu2)+y2*mu2);
}
```

Cubic

Cubic interpolation is the simplest method that offers true continuity between the segments. As such it requires more than just the two endpoints of the segment but also the two points on either side of them. So the function requires 4 points in all labelled y0, y1,



Hermite

y2, and y3, in the code below. mu still behaves the same way for interpolating between the segment y1 to y2. This does raise issues for how to interpolate between the first and last segments. In the examples here I just haven't bothered. A common solution is the dream up two extra points at the start and end of the sequence, the new points are created so that they have a slope equal to the slope of the start or end segment.

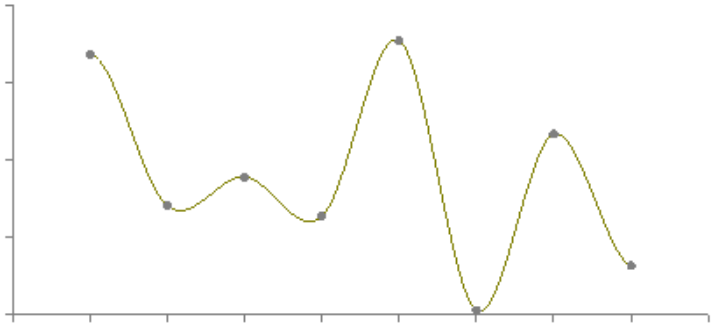
```
double CubicInterpolate(
   double y0,double y1,
   double y2,double y3,
   double mu)
{
   double a0,a1,a2,a3,mu2;

   mu2 = mu*mu;
   a0 = y3 - y2 - y0 + y1;
   a1 = y0 - y1 - a0;
   a2 = y2 - y0;
   a3 = y1;

   return(a0*mu*mu2+a1*mu2+a2*mu+a3);
}
```

Paul Breeuwsma proposes the following coefficients for a smoother interpolated curve, which uses the slope between the previous point and the next as the derivative at the current point. This results in what are generally referred to as Catmull-Rom splines.

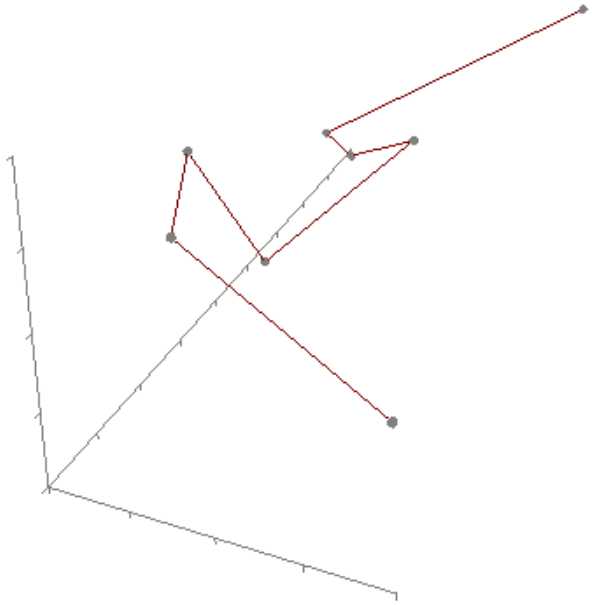
```
   a0 = -0.5*y0 + 1.5*y1 - 1.5*y2 + 0.5*y3;
   a1 = y0 - 2.5*y1 + 2*y2 - 0.5*y3;
   a2 = -0.5*y0 + 0.5*y2;
   a3 = y1;
```

Hermite interpolation like cubic requires 4 points so that it can achieve a higher degree of continuity. In addition it has nice tension and biasing controls. Tension can be used to tighten up the curvature at the known points. The bias is used to twist the curve about the known points. The examples shown here have the default tension and bias values of 0, it will be left as an exercise for the reader to explore different tension and bias values.

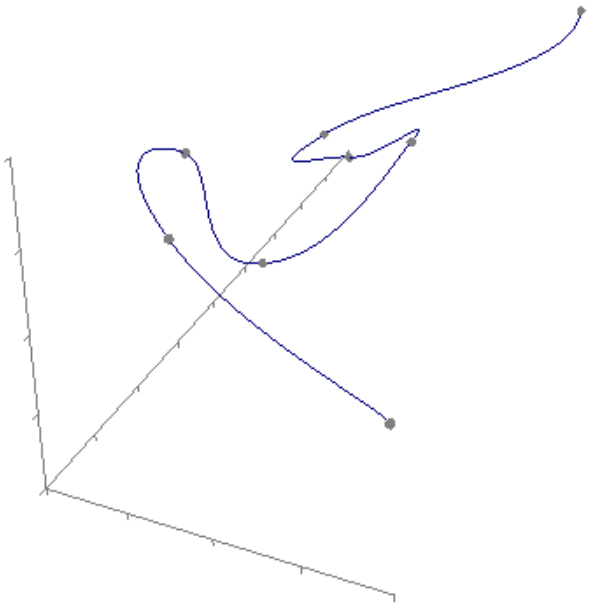
```
/*
   Tension: 1 is high, 0 normal, -1 is low
   Bias: 0 is even,
         positive is towards first segment,
         negative towards the other
*/
double HermiteInterpolate(
   double y0,double y1,
   double y2,double y3,
   double mu,
   double tension,
   double bias)
{
   double m0,m1,mu2,mu3;
   double a0,a1,a2,a3;

        mu2 = mu * mu;
        mu3 = mu2 * mu;
   m0  = (y1-y0)*(1+bias)*(1-tension)/2;
   m0 += (y2-y1)*(1-bias)*(1-tension)/2;
```



3D linear

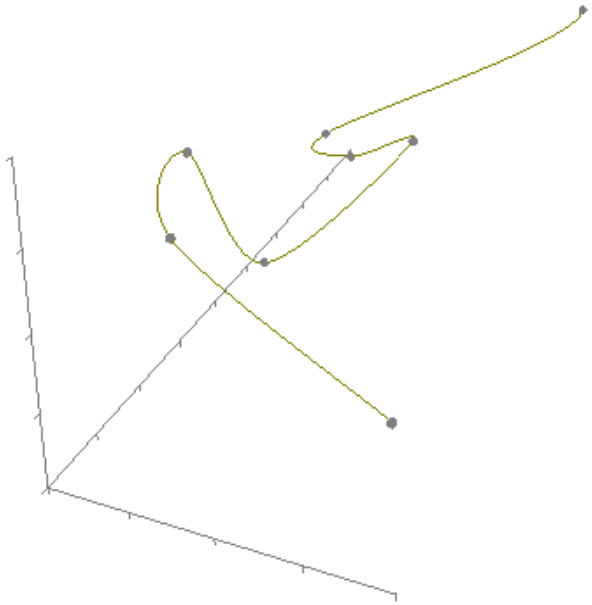


3D cubic



3D Hermite

```
   m1  = (y2-y1)*(1+bias)*(1-tension)/2;
   m1 += (y3-y2)*(1-bias)*(1-tension)/2;
   a0 =  2*mu3 - 3*mu2 + 1;
   a1 =    mu3 - 2*mu2 + mu;
   a2 =    mu3 -   mu2;
   a3 = -2*mu3 + 3*mu2;

   return(a0*y1+a1*m0+a2*m1+a3*y2);
}
```



While you may think the above cases were 2 dimensional, they are just 1 dimensional interpolation (the horizontal axis is linear). In most cases the interpolation can be extended into higher dimensions simply by applying it to each of the x,y,z coordinates independently. This is shown on the right for 3 dimensions for all but the cosine interpolation. By a cute trick the cosine interpolation reverts to linear if applied independently to each coordinate.

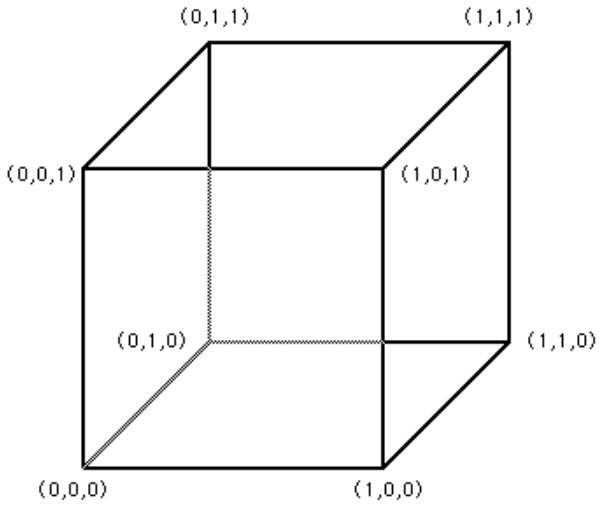
For other interpolation methods see the Bezier, Spline, and piecewise Bezier methods here.

Trilinear Interpolation

Written by Paul Bourke
July 1997

Trilinear interpolation is the name given to the process of linearly interpolating points within a box (3D) given values at the vertices of the box. Perhaps its most common application is interpolating within cells of a volumetric dataset.

Consider a unit cube with the lower/left/base vertex at the origin as shown here on the right.
The values at each vertex will be denoted V_{000}, V_{100}, V_{010}, ...etc....V_{111}



The value at position (x,y,z) within the cube will be denoted V_{xyz} and is given by

$$V_{xyz} = V_{000}\,(1 - x)\,(1 - y)\,(1 - z) +$$
$$V_{100}\,x\,(1 - y)\,(1 - z) +$$
$$V_{010}\,(1 - x)\,y\,(1 - z) +$$

V_{001} (1 - x) (1 - y) z +
V_{101} x (1 - y) z +
V_{011} (1 - x) y z +
V_{110} x y (1 - z) +
V_{111} x y z

In general the box will not be of unit size nor will it be aligned at the origin. Simple translation and scaling (possibly of each axis independently) can be used to transform into and then out of this simplified situation.

Linear Regression

Written by Paul Bourke
October 1998

Linear regression is a method to best fit a linear equation (straight line) of the form y(x) = **a** + **b** x to a collection of N points (x_i,y_i). Where **b** is the slope and **a** the intercept on the y axis.

The result will be stated below without derivation, that requires minimisation of the sum of the squared distance from the data points and the proposed line. This function is minimised by calculating the derivative with respect to **a** and **b** and setting these to zero. For a more complete derivation see the "Numerical Recipes in C".

The solution is clearer if we define the following

$$s_{xx} = \sum_{i=0}^{N-1} (x_i - \overline{x})^2$$

$$s_{yy} = \sum_{i=0}^{N-1} (y_i - \overline{y})^2$$

$$s_{xy} = \sum_{i=0}^{N-1} (x_i - \overline{x})(y_i - \overline{y})$$

Then

$$\text{slope } (b) = \frac{s_{xy}}{s_{xx}}$$

and

$$\text{y intercept } (a) = \overline{y} - b\,\overline{x}$$

And finally the regression coefficient is

$$r = \frac{s_{xy}}{\sqrt{s_{xx}\ s_{yy}}}$$

This is 0 if there is no linear trend, 1 for perfect linear fit.

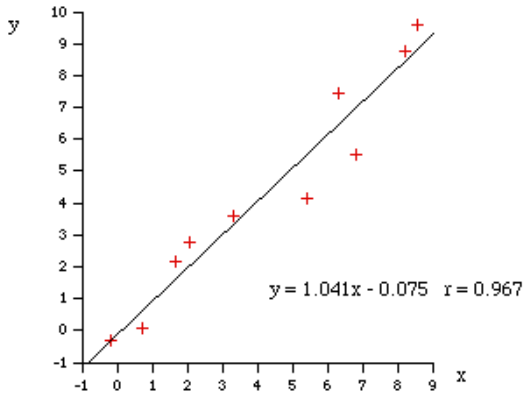
Note

- This discussion assumes there is no known variance for the x and y values. There are solutions which can take this into account, this is particularly important if some values are known with less error than others.
- The solution above requires that the slope is not infinite, S_{xx} is not zero.

Example

The following example shows the points and the best fit line as determined using the techniques discussed here.

x	y
-0.20707	-0.319029
0.706672	0.0931669
1.63739	2.17286
2.03117	2.76818
3.31874	3.56743
5.38201	4.11772
6.79971	5.52709
6.31814	7.46613
8.20829	8.7654
8.53994	9.58096



Source

C
linregress.c

C++ contributed by Charles Brown
RegressionLine.cpp
RegressionLine.hpp

Curve Fit Through Arbitrary Points

Written by Paul Bourke
August 1991

The following introduces a method of immediately deriving a polynomial that passes through an arbitrary number of points. That is, find a polynomial f(x) that passes through the N points

$$(x_0,y_0), (x_1,y_1), (x_2,y_2), ... (x_{N-1},y_{N-1})$$

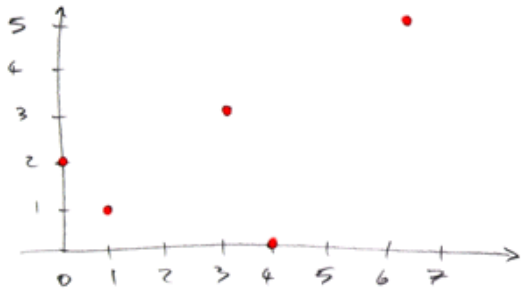
The key to this solution is that we want an exact fit at the points given *and we don't care what happens in between those points*. The general solution is

$$f(x) = \sum_{i=0}^{N-1} y_i \prod_{j=0,\, j\neq i}^{N-1} \frac{x - x_j}{x_i - x_j}$$

To see how this works, consider the product term. When $x = x_i$ the product term has the same denominator and numerator and thus equals 1 and therefore contributes y_i to the sum. All other terms in the summation contribute 0 since there exists a $(x_j - x_j)$ in the numerator. Thanks to Simon Stegmaier for pointing out that this is known as a Lagrange Polynomial.

For a numerical example consider the polynomial that passes through the following points

(0,2)
(1,1)
(3,3)
(4,0)
(6,5)



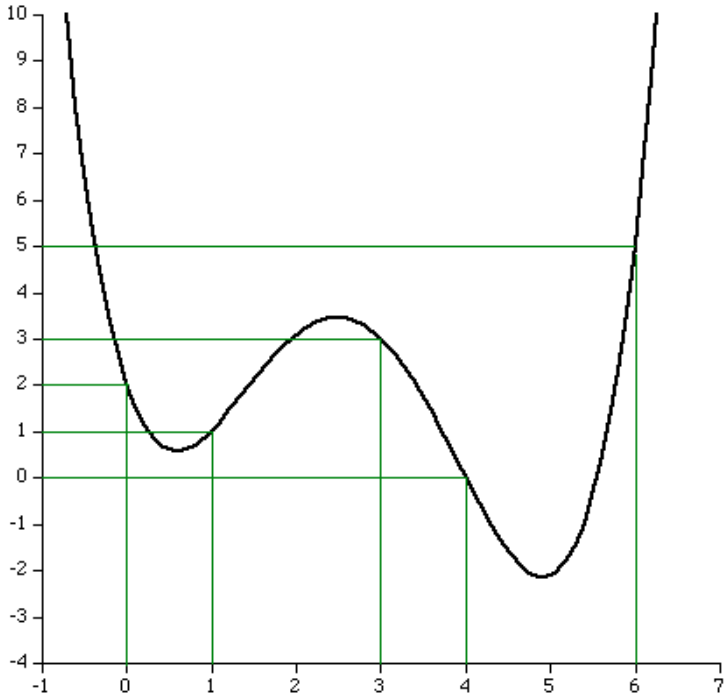
The function using the above formula is

```
f(x) = 2 * (x-1) * (x-3) * (x-4) * (x-6) / [ (0-1) * (0-3) * (0-4) * (0-6) ]
     + 1 * (x-0) * (x-3) * (x-4) * (x-6) / [ (1-0) * (1-3) * (1-4) * (1-6) ]
     + 3 * (x-0) * (x-1) * (x-4) * (x-6) / [ (3-0) * (3-1) * (3-4) * (3-6) ]
     + 0 * (x-0) * (x-1) * (x-3) * (x-6) / [ (4-0) * (4-1) * (4-3) * (4-6) ]
     + 5 * (x-0) * (x-1) * (x-3) * (x-4) / [ (6-0) * (6-1) * (6-3) * (6-4) ]

f(x) = (x-1) * (x-3) * (x-4) * (x-6) / 36
     - (x-0) * (x-3) * (x-4) * (x-6) / 30
     + (x-0) * (x-1) * (x-4) * (x-6) / 6
     + (x-0) * (x-1) * (x-3) * (x-4) / 36

f(x) = 17*x^4/90 - 181*x^3/90 + 563*x^2/90 - 163*x/30 + 2
```

By substituting the values of x for the points the function must pass through (x=0,1,3,4,6) it is easy to see that the expression above achieves the result, namely y=2,1,3,0,5 respectively.

What happens at other points?

All bets are off regarding the behaviour between the fixed points. The polynomial is of degree N and could violently fly off anywhere. The continuous curve for the numerical example above is shown below.



A competition in the Mathematics news groups in October 1998

```
From: "John Santos" <santos_john@hotmail.com>
Newsgroups: alt.sci.math.probability,alt.tv.mathnet,aus.mathematics
Subject: $100.00 prize for the solution
Date: Tue, 15 Sep 1998 20:56:50 -0700
X-Newsreader: Microsoft Outlook Express 4.72.3110.1
X-MimeOLE: Produced By Microsoft MimeOLE V4.72.3110.3
NNTP-Posting-Host: 209.239.197.111
X-NNTP-Posting-Host: 209.239.197.111
Message-ID: <35ff36d1.0@blushng.jps.net>
X-NNTP-Posting-Host: 209.63.114.134

Hi everyone,
My name is John Santos and I am willing to pay anyone $100.00 for the first
person to solve this problem. I am providing some hints. This works as follows:
you will see nine digits - the tenth digit or letter is the answer.
What I need is the formula or mathematical equation to get there...

   Number          Answer
---------------------------
   749736637         1
   713491024         8
   523342792         D
   749236871         P
   727310078         E
   746261832         4
   733237527         L
   743510589         9
   715240338         K
   722592910         1
   739627071         R

The first one with the answer and emails it to me wins the $100.00
Email address: santos_john@hotmail.com
Good Luck !!!!
```

They refused to pay up for this solution !!!!

My reply to this posting was

```
The following is the solution to the posted problem, although it probably
doesn't offer the insight you are seeking, it certainly falls within the
scope of competition. To illustrate my solution I will use the following
symbols instead of the numbers you used simply to save space. For your
second column with letters and numbers use their ASCII values instead,
this has no loss of generality as it is a simple 1 to 1 mapping.
```

```
x1 y1
x2 y2
x3 y3
x4 y4
etc

The following is a general method of making a function that passes
through any pair of values (xi,yi).

        y1 (x-x2) (x-x3) (x-x4)
f(x) = ---------------------------
         (x1-x2) (x1-x3) (x1-x4)

        y2 (x-x1) (x-x3) (x-x4)
     + ---------------------------
         (x2-x1) (x2-x3) (x2-x4)

        y3 (x-x1) (x-x2) (x-x4)
     + ---------------------------
         (x3-x1) (x3-x2) (x3-x4)

        y4 (x-x1) (x-x2) (x-x3)
     + ---------------------------
         (x4-x1) (x4-x2) (x4-x3)

etc etc. As you can see, at x=x1 all the terms disappear except the
first which equals y1, at x=x2 all the terms disappear except the
second which equals y2, etc etc.
```

X Number	Answer	Y ASCII
749736637	1	49
713491024	8	56
523342792	D	68
749236871	P	80
727310078	E	69
746261832	4	52
733237527	L	76
743510589	9	57
715240338	K	75
722592910	1	49
739627071	R	82

The "lovely" expression in this case is

f(x) = + 49 ((x - 713491024) / 36245613) ((x - 523342792) / 226393845) ((x - 749236871) / 499766) ((x - 727310078) / 22426559) ((x - 746261832) / 3474805) ((x - 733237527) / 16499110) ((x - 743510589) / 6226048) ((x - 715240338) / 34496299) ((x - 722592910) / 27143727) ((x - 739627071) / 10109566) + 56 ((x - 749736637) / -36245613) ((x - 523342792) / 190148232) ((x - 749236871) / -35745847) ((x - 727310078) / -13819054) ((x - 746261832) / -32770808) ((x - 733237527) / -19746503) ((x - 743510589) / -30019565) ((x - 715240338) / -1749314) ((x - 722592910) / -9101886) ((x - 739627071) / -26136047) + 68 ((x - 749736637) / -226393845) ((x - 713491024) / -190148232) ((x - 749236871) / -225894079) ((x - 727310078) / -203967286) ((x - 746261832) / -222919040) ((x - 733237527) / -209894735) ((x - 743510589) / -220167797) ((x - 715240338) / -191897546) ((x - 722592910) / -199250118) ((x - 739627071) / -216284279) + 80 ((x - 749736637) / -499766) ((x - 713491024) / 35745847) ((x - 523342792) / 225894079) ((x - 727310078) / 21926793) ((x - 746261832) / 2975039) ((x - 733237527) / 15999344) ((x - 743510589) / 5726282) ((x - 715240338) / 33996533) ((x - 722592910) / 26643961) ((x - 739627071) / 9609800) + 69 ((x - 749736637) / -22426559) ((x - 713491024) / 13819054) ((x - 523342792) / 203967286) ((x - 749236871) / -21926793) ((x - 746261832) / -18951754) ((x - 733237527) / -5927449) ((x - 743510589) / -16200511) ((x - 715240338) / 12069740) ((x - 722592910) / 4717168) ((x - 739627071) / -12316993) + 52 ((x - 749736637) / -3474805) ((x - 713491024) / 32770808) ((x - 523342792) / 222919040) ((x - 749236871) / -2975039) ((x - 727310078) / 18951754) ((x - 733237527) / 13024305) ((x - 743510589) / 2751243) ((x - 715240338) / 31021494) ((x - 722592910) / 23668922) ((x - 739627071) / 6634761) + 76 ((x - 749736637) / -16499110) ((x - 713491024) / 19746503) ((x - 523342792) / 209894735) ((x - 749236871) / -15999344) ((x - 727310078) / 5927449) ((x - 746261832) / -13024305) ((x - 743510589) / -10273062) ((x - 715240338) / 17997189) ((x - 722592910) / 10644617) ((x - 739627071) / -6389544) + 57 ((x - 749736637) / -6226048) ((x - 713491024) / 30019565) ((x - 523342792) / 220167797) ((x - 749236871) / -5726282) ((x - 727310078) / 16200511) ((x - 746261832) / -2751243) ((x - 733237527) / 10273062) ((x - 715240338) / 28270251) ((x - 722592910) / 20917679) ((x - 739627071) / 3883518) +

75 ((x - 749736637) / -34496299) ((x - 713491024) / 1749314) ((x - 523342792) / 191897546) ((x - 749236871) / -33996533) ((x - 727310078) / -12069740) ((x - 746261832) / -31021494) ((x - 733237527) / -17997189) ((x - 743510589) / -28270251) ((x - 722592910) / -7352572) ((x - 739627071) / -24386733) + 49 ((x - 749736637) / -27143727) ((x - 713491024) / 9101886) ((x - 523342792) / 199250118) ((x - 749236871) / -26643961) ((x - 727310078) / -4717168) ((x - 746261832) / -23668922) ((x - 733237527) / -10644617) ((x - 743510589) / -20917679) ((x - 715240338) / 7352572) ((x - 739627071) / -17034161) + 82 ((x - 749736637) / -10109566) ((x - 713491024) / 26136047) ((x - 523342792) / 216284279) ((x - 749236871) / -9609800) ((x - 727310078) / 12316993) ((x - 746261832) / -6634761) ((x - 733237527) / 6389544) ((x - 743510589) / -3883518) ((x - 715240338) / 24386733) ((x - 722592910) / 17034161) f(749736637) = 49 (1) f(713491024) = 56 (8) f(523342792) = 68 (D) f(749236871) = 80 (P) f(727310078) = 69 (E) f(746261832) = 52 (4) f(733237527) = 76 (L) f(743510589) = 57 (9) f(715240338) = 75 (K) f(722592910) = 49 (1) f(739627071) = 82 (R)

Nearest neighbour weighted interpolation

Written by Paul Bourke
April 1998

The following describes perhaps the simplest method of "smoothly" approximating height values on a surface given a collection of randomly distributed samples. It is often used to derive estimates of the surface height at the vertices of a regular grid from irregularly spaced samples. While the example given here is based on determining the height of a surface (x,y), the same general technique can be used in higher dimensions. For example, estimating the density with a volume (x,y,z) given irregular density measurements.

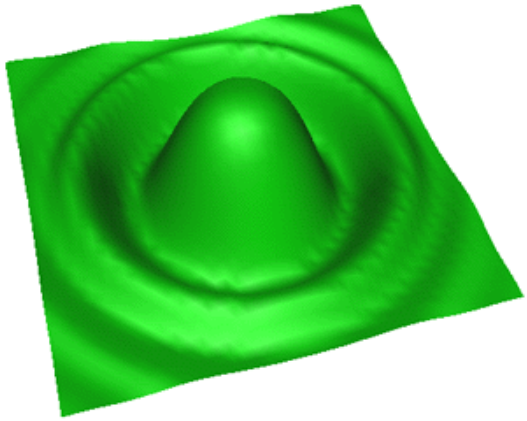
Consider N height samples, that is, we have N triples (x_i, y_i, z_i). We want to estimate the height z given a position on the plane (x,y). The general form of the so called "nearest neighbour weighted interpolation" also sometimes called the "inverse distance method" for estimating z is given by the following.

$$z = \begin{cases} \dfrac{\sum_{i=0}^{N-1} \dfrac{z_i}{[(x_i - x)^2 + (y_i - y)^2]^{p/2}}}{\sum_{i=0}^{N-1} \dfrac{1}{[(x_i - x)^2 + (y_i - y)^2]^{p/2}}} & x_i \neq x \text{ or } y_i \neq y \\ z_i & x_i = x \text{ and } y_i = y \end{cases}$$

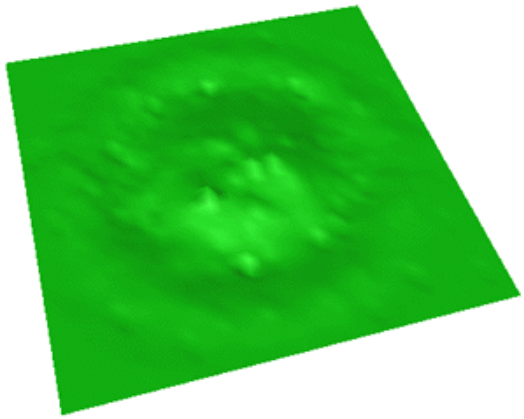
where p generally determines relative importance of distant samples. Note the denominator above gives a measure of how close the point being estimated is from the samples. Naturally if a sample is close then it has a greater influence on the estimate than if the sample is distant.

The following shows an example of reconstructing a surface from 1000 samples. The approximation is generally better with increased values of p.

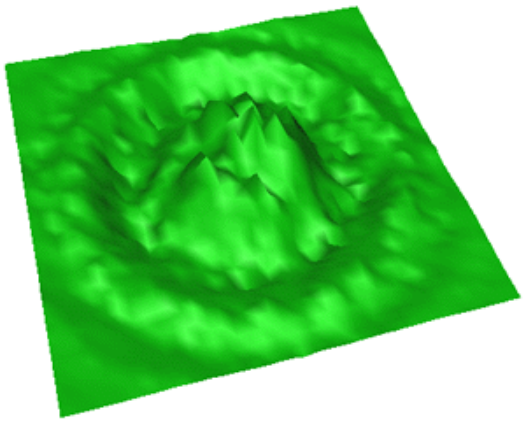
The original surface from which samples are taken for this example is shown on the right.



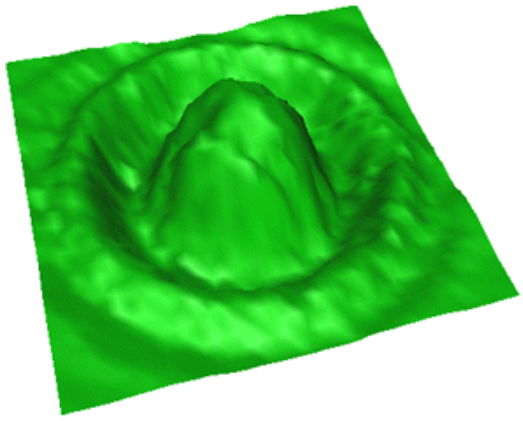
p = 1



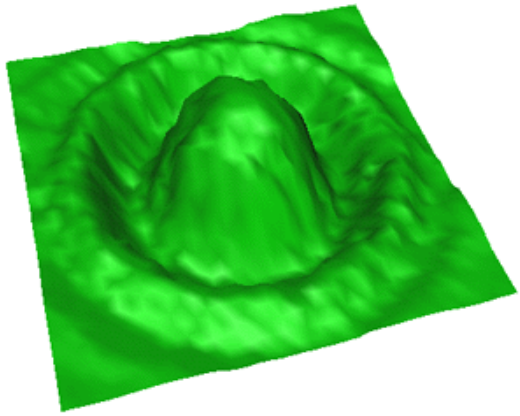
p = 2



p = 4



p = 6



Colour and Normal Interpolation

As it applies to triangles and quadrilaterals in the rendering of 3D surfaces

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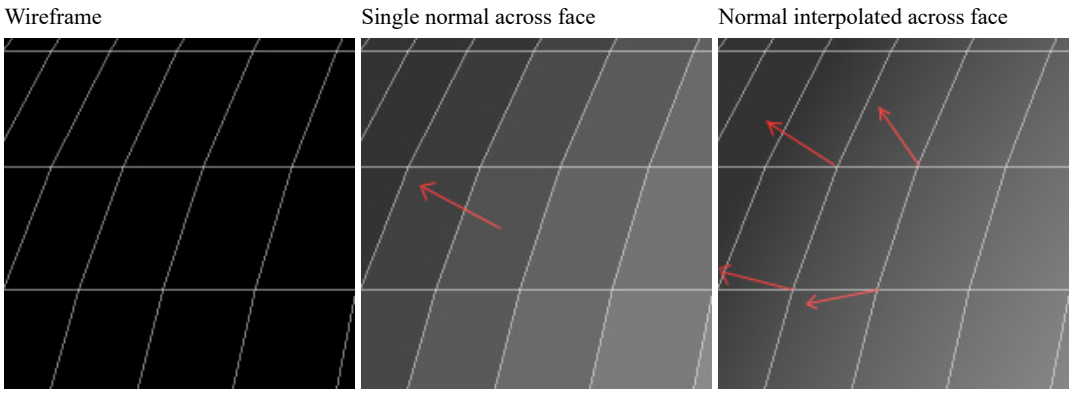
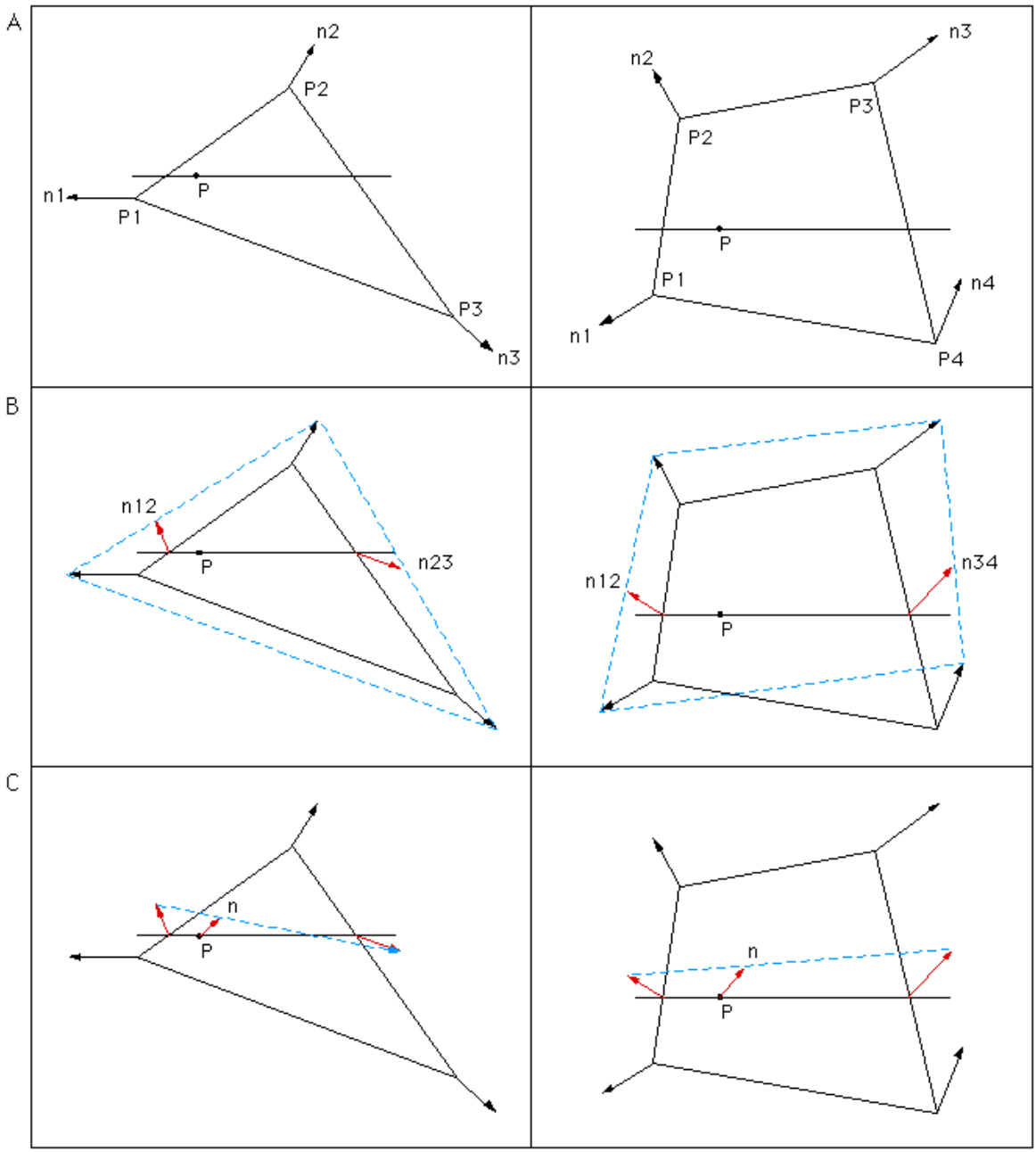
It is frequently desirable to estimate the colour or normal at a point in the interior of a 3 or 4 vertex planar polygon given only the colour and normal at each of the vertices. The most common application of this is smooth rendering of surfaces approximated by a finite number of triangular facets or quadrilaterals. Without colour and/or normal interpolation each planar piece of the surface has the same colour and normal, this results in a visible discontinuity between adjacent faces. The following illustrates a part of a sphere made up of quadrilaterals and rendered using a single normal applied to the whole face or 4 normals at each vertex interpolated across the face.



The approach most commonly used by 3D rendering packages, both real-time such as OpenGL and more CPU intensive algorithms such as raytracing, is called **Phong normal interpolation**. A often used efficient implementation is called **barycentric interpolation**. The idea is the same for both colour and normal interpolation, a line is extended from the point in question to two edges of the polygon. The estimate of the colour or normal at those points is made by linear interpolation between the values at the vertices of the edge. The estimate at the point in question is linearly interpolated from the estimates at the ends of the extended line.

This is illustrated in the sequence below, while this is for normals the method is identical for colours which are after all generally a (r,g,b) triple instead of a (x,y,z) triple. In (A) the point P is where the colour or normal is to be estimated, a line is extended (in any direction but shown as horizontal in this diagram) until it intersects two edges. In (B) the normals at the intersection points of the extended line are shown in red, they are calculated by linear interpolation. In (C) the two normals in (B) are linearly interpolated to give the estimate of the normal at point P.



Note

- The colour or normal estimate at the vertices is always the same as the vertex value.
- The colour or normals along the edges only depends on the colour or normal at the edge vertices and not on the values at the other vertices. It is this that ensures that adjacent faces with the same colour or normal along a joining edge will join smoothly even though their other vertices may have very different values.
- The direction in which the line is extended out from the point being estimated doesn't matter except that it must be the same for all points within a face. One way is to choose a major axis by specifying a normal. The plane with this normal that passes though the point in question cuts two of the polygon edges, this is used as the extended line.
- One difference between interpolation of normals and colours is that the normals estimated at the end of the extended lines and the final normal at P are normalised to unit length. In colour interpolation each r,g,b component is treated independently.