HW1 Theory

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Problem 1

Let:

$$f_1 = n^2$$
$$f_2 = n^2 + 1000n$$

$$f_3 = \begin{cases} n & , n \text{ is odd} \\ n^3 & , n \text{ is even} \end{cases}$$
$$f_4 = \begin{cases} n & , n <= 100 \\ n^3 & , n > 100 \end{cases}$$

Part a

Statement: f_1 and f_2 are codominant

Solution

$f_1 \in \mathrm{O}(f_2) = n^2 \in \mathrm{O}(n^2)$, $c >= n_0$, $n_0 > 0$

$$n^2 <= cn^2$$
$$c = 2$$
$$n^2 <= 2n^2, \forall n > 0$$
$$f_2 \text{ does dominate } f_1$$

$f_2 \in \mathrm{O}(f_1) = n^2 + 1000n \in \mathrm{O}(n^2)$, $c >= n_0$, $n_0 > 0$

$$n^2 + 1000n <= cn^2$$
$$c = 10$$
$$n^2 + 1000n <= 10n^2$$
$$\frac{n}{10} + 100 <= n, \forall n >= 110$$
$$f_1 \text{ does dominate } f_2$$

Part b

Statement: f_3 dominates f_1, as the upper bound of f_3 is n^3

Solution

$f_1 \in \mathrm{O}(f_3) = n^2 \in \mathrm{O}(n^3)$, $c >= n_0$, $n_0 > 0$

$$n^2 <= cn^3$$
$$c = 1$$
$$n^2 <= n^3, \forall n > 0$$
$$f_3 \text{ does dominate } f_1$$

$f_3 \in \mathrm{O}(f_1) = n^3 \in \mathrm{O}(n^2)$, $c >= n_0$, $n_0 > 0$

$$n^3 <= cn^2$$
$$c = 5$$
$$n^3 <= 5n^2$$
$$n <= 5, \forall n <= 5$$
$$f_3 \text{ does not dominate } f_1$$

Part c

Statement: f_4 dominates f_1, as the upper bound of f_4 is n^3

Solution

$f_1 \in \mathrm{O}(f_4) = n^2 \in \mathrm{O}(n^3)$, $c >= n_0$, $n_0 > 0$

$$n^2 <= cn^3$$
$$c = 1$$
$$n^2 <= n^3$$
$$1 <= n, \forall n >= 1$$
$$f_4 \text{ does dominate } f_1$$

$f_4 \in \mathrm{O}(f_1) = n^3 \in \mathrm{O}(n^2)$, $c >= n_0$, $n_0 > 0$

$$n^3 <= cn^2$$
$$c = 10$$
$$n^3 <= 10n^2$$
$$n <= 10, \forall n <= 10$$
$$f_1 \text{ does not dominate } f_3$$

Part d

Statement: f_3 dominates f_2, as the upper bound of f_3 is n^3

Solution

$f_2 \in \mathrm{O}(f_3) = n^2 + 1000n \in \mathrm{O}(n^3)$, $c >= n_0$, $n_0 > 0$

$$n^2 + 1000n <= cn^3$$
$$c = 10$$
$$n^2 + 1000n <= 10n^3$$
$$\frac{1}{10} + \frac{100}{n} <= n, \forall n >= 11$$
$$f_3 \text{ does dominate } f_2$$

$f_3 \in \mathrm{O}(f_2) = n^3 \in \mathrm{O}(n^2)$, $c >= n_0$, $n_0 > 0$

$$n^3 <= cn^2$$
$$c = 10$$
$$n^3 <= 10n^2$$
$$\frac{n}{10} <= 1, \forall n <= 10$$
$$f_2 \text{ does not dominate } f_3$$

Part e

Statement: f_4 dominates f_2, as the upper bound of f_4 is n^3

Solution

$f_2 \in \mathrm{O}(f_4) = n^2 + 1000n \in \mathrm{O}(n^3)$, $c >= n_0$, $n_0 > 0$

$$n^2 + 1000n <= cn^3$$
$$c = 10$$
$$n^2 + 1000n <= 10n^3$$
$$\frac{1}{10} + \frac{100}{n} <= n, \forall n >= 11$$
$$f_4 \text{ does dominate } f_2$$

$f_4 \in \mathrm{O}(f_2) = n^3 \in \mathrm{O}(n^2)$, $c >= n_0$, $n_0 > 0$

$$n^3 <= cn^2$$
$$c = 10$$
$$n^3 <= 10n^2$$
$$\frac{n}{10} <= 1, \forall n <= 10$$
$$f_2 \text{ does not dominate } f_4$$

Part f

Statement: f_3 and f_4 are codominant

Solution

$f_3 \in \mathrm{O}(f_4) = n^3 \in \mathrm{O}(n^3)$, $c >= n_0$, $n_0 > 0$

$$\begin{aligned} n^3 &<= cn^3 \\ c &= 2 \\ n^3 &<= 2n^3, \forall n > 0 \\ f_4 \text{ does dominate } f_3 \end{aligned}$$

$f_4 \in \mathrm{O}(f_3) = n^3 \in \mathrm{O}(n^3)$, $c >= n_0$, $n_0 > 0$

$$\begin{aligned} n^3 &<= cn^3 \\ c &= 2 \\ n^3 &<= 2n^3, \forall n > 0 \\ f_3 \text{ does dominate } f_4 \end{aligned}$$

Problem 2

Part a

Worst case $\mathrm{O}(matmpy) = n^3$

Solution

The first loop iterates over n elements with object i.

The next nested loop iterates from i to n "n times".

This can be expressed with:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

$$\tag{2}$$

Which yields a $\mathrm{O}(n^2)$.

The final nested loop iterates over n elements.

Using the multiplicative rules of "Big Oh", $\mathrm{O}(matmpy) = n^2 * n = n^3$

For function t to dominate, and be reasonably tight $t = n^3$

$matmpy \in \mathrm{O}(t) = n^3 \in \mathrm{O}(n^3)$,

Proof:

$$\begin{aligned} n^3 &<= cn^3 \\ c &= 2 \\ n^3 &<= 2n^3, \forall n > 0 \\ t \text{ does dominate } matmpy \end{aligned}$$

Part b

$\mathrm{O}(mystery) = n$

Solution

Under the assumption that indents indicate nesting

The first loop doesn't do anything besides iterate over $n-1$ elements

The next loop attempts to iterate from $i+1$ to n

Since $i+1=n$ after the first loop, the second loop never allows its nested loop to run

$\mathrm{O}(mystery) = n$

For function t to dominate, and be reasonably tight $t = n$

$mystery \in \mathrm{O}(t) = n \in \mathrm{O}(n)$

Proof:

$$\begin{aligned} n &<= cn \\ c &= 2 \\ n &<= 2n, \forall n > 0 \\ t \text{ does dominate } &mystery \end{aligned}$$

Part c

$\mathrm{O}(veryodd) = n^2$

Solution

The first loop iterates over n elements

Using the additive rule of "Big Oh" the two nested loops merge to n^2

Proof:

$$\sum_{j=1}^{n} i = \frac{n(n+1)}{2} \tag{3}$$

$$\tag{4}$$

The upper bound is defined when i is odd, and yields $n^2 + n$ which simplifies to $\mathrm{O}(n^2)$

For function t to dominate, and be reasonably tight $t = n^2$

Proof:

$$\begin{aligned}
n^2 &<= cn^2 \\
c &= 2 \\
n^2 &<= 2n^2, \forall n > 0 \\
t \text{ does dominate } &matmpy
\end{aligned}$$

Part d

$\mathrm{O}(recursive) = 2^n$

Solution

For each recursive call, two more calls are generated. Total steps is the sum of 2^n:

$$\sum_{i=1}^{n} 2^i \quad = \quad (2^{n+1} - 1) \tag{5}$$

$$\tag{6}$$

Which yeilds $\mathrm{O}(2^n)$

For function t to dominate, and be reasonably tight $t = 2^n$

Proof:

$$\begin{aligned}
2^n &<= c2^n \\
c &= 2 \\
2^n &<= 2(2^n), \forall n >= 0 \\
t \text{ does dominate } &matmpy
\end{aligned}$$