Math 382: Homework 5

Due on Wednesday February 9, 2022 at 10:00 PM

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Problem 1

a) What is the radius of convergence of the Taylor series of the function $z \coth z$ at $z = 0$?
b) What is the radius of convergence of the Taylor series of the function $\left(z^3+1\right) / \left(z^4+1\right)$ at $z = 1$?

Solution

Part A

Let $f(z) = z \coth z$. The radius of convergence of $f(z)$ centered at $z = 0$ is the distance from z to the nearest singularity z_0. Write $f(z)$ as

$$f(z) = \frac{z \cosh z}{\sinh z},$$

and note that $\sinh z$ has zeroes $k\pi i$ for $k \in \mathbb{Z}$ and $z \cosh z$ has only one zero at $z = 0$. So we compute

$$\lim_{z \to 0} (z-0) f(z) = \lim_{z \to 0} \frac{z^2 \cosh z}{\sinh z} = 0,$$

so at $z = 0$, f is actually analytic and thus has a removable singularity there. Let us now check the other singularities, so compute

$$\lim_{z \to i\pi} (z - i\pi) f(z) = \lim_{z \to i\pi} z(z-i\pi) \frac{\cosh z}{\sinh z} \overset{u \mapsto z - i\pi}{=} \lim_{u \to 0} u(u+i\pi) \frac{\cosh(u+i\pi)}{\sinh(u+i\pi)}.$$

Using the periodicity of sinh and cosh then gives us

$$\lim_{z \to i\pi} (z - i\pi) f(z) = \lim_{u \to 0} \frac{u^2 \cosh u}{\sinh u} + i\pi \lim_{u \to 0} \frac{u \cosh u}{\sinh u} = i\pi.$$

Since this limit exists and is finite an nonzero, f has simple poles for every non-zero integer multiple of $i\pi$. So, the closest poles are $\pm i\pi$, which are in fact equidistant from $z = 0$ of distance π, so the radius of convergence is $R = \pi$.

Part B

Let $f(z) = \frac{z^3+1}{z^4+1}$. Like **Part A**, the radius of convergence of $f(z)$ centered at $z = 1$ is the distance from z to the nearest singularity. So write $f(z)$ as

$$f(z) = \frac{z^3+1}{z^4+1}.$$

Factoring the denominator gives $z^4 + 1 = \left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z - \frac{-1+i}{\sqrt{2}}\right)\left(z - \frac{-1-i}{\sqrt{2}}\right)$, so it has zeroes

$$z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, z_2 = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, z_3 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, z_4 = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}.$$

Factoring the numerator gives $z^3 + 1 = (z+1)(z^2 - z + 1)$, so it has zeroes at

$$z_5 = -1, z_6 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, z_7 = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Hence, the numerator is continuous and holomorphic at z_1, z_2, z_3, and z_4, so f has simple poles at those points, of which z_1 and z_2 are closest since

$$|1 - z_1| = |1 - z_2| = \sqrt{2 - \sqrt{2}} < \sqrt{2 + \sqrt{2}} = |1 - z_3| = |z - z_4|,$$

and hence the radius of convergence is $R = \sqrt{2 - \sqrt{2}}$.

Problem 2

Locate the poles of the function $\frac{z}{\sin^2 z}$, and calculate its principal parts and residues.

Solution

Let $f(z) = \frac{z}{\sin^2 z}$. The numerator z has a zero at $z = 0$ and the denominator $\sin^2 z$ has zeroes $z = k\pi$ for $k \in \mathbb{Z}$. Now, we compute

$$\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{z^2}{\sin^2 z} = \lim_{z\to 0}\left(\frac{z}{\sin^2 z}\right)^2 = 1^2 = 1,$$

so since this limit exists and is nonzero and finite, f has a simple pole at $z = 0$ with residue $\operatorname{Res}[f, 0] = 1$. At $z = k\pi$ for nonzero $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\lim_{z\to k\pi}\frac{d}{dz}\left[(z-k\pi)^2\frac{z}{\sin^2 z}\right] &\overset{u\mapsto z-k\pi}{=} \lim_{u\to 0}\frac{d}{du}\left[u^2\cdot\frac{u+k\pi}{\sin^2(u+k\pi)}\right] \\
&= \lim_{u\to 0}\frac{-u[2u(u+k\pi)\cos u-(3u+2k\pi)\sin u]}{\sin^3 u} \\
&= \lim_{u\to 0}\frac{-u^2(u+k\pi)+u^2(3u+2k\pi)}{u^3} \\
&= 1.
\end{aligned}$$

Since this limit exists and is nonzero and finite, we have that f has a pole of order 2 at $z = k\pi$ for nonzero integers k with residue $\operatorname{Res}[f, k\pi] = 1 = b_1$. To find the b_2 term in this Laurent series, simply note that we have

$$f(z) = \frac{z}{\sin^2 z} = \frac{1}{z-k\pi} + \frac{b_2}{(z-k\pi)^2} + \cdots,$$

which multiplying through by $(z-k\pi)^2$ and taking the limit as $z \to k\pi$ gives

$$b_2 = \lim_{z\to k\pi}(z-k\pi)^2\frac{z}{\sin^2 z} = k\pi.$$

In summary, expanding f around $z = 0$ gives

$$f(z) = \frac{z}{\sin^2 z} = \underbrace{\frac{1}{z}}_{\text{principal}} + \cdots.$$

Expanding around $z = k\pi$ for any nonzero k gives

$$f(z) = \frac{z}{\sin^2 z} = \underbrace{\frac{k\pi}{(z-k\pi)^2} + \frac{1}{z-k\pi}}_{\text{principal}} + \cdots.$$

Problem 3

Gamelin §V.7 Exercise 9: Show that if the analytic function $f(z)$ has a zero of order N at z_0, then $f(z) = g(z)^N$ for some function $g(z)$ analytic near z_0 and satisfying $g'(z_0) \neq 0$.

Solution

Proof. If f has a zero at $z = z_0$ of order N, we can write f as

$$f(z) = (z - z_0)^N \tilde{f}(z),$$

where $\tilde{f}(z_0) \neq 0$ and has a convergent power series. I claim that we can take

$$g(z) = (z - z_0)e^{\frac{\ln \tilde{f}(z)}{N}}$$

such that $f(z) = g(z)^N$. Indeed, note that by construction of our g and for a suitable choice of branch for the logarithm, we have

$$f(z) = g(z)^N = (z - z_0)^N e^{\ln \tilde{f}(z)} = (z - z_0)^N \tilde{f}(z).$$

Further,

$$g'(z) = e^{\frac{\ln \tilde{f}(z)}{N}} + (z - z_0)\left(\frac{\tilde{f}'(z)}{\tilde{f}(z)}\right)e^{\frac{\ln \tilde{f}(z)}{N}} = e^{\frac{\ln \tilde{f}(z)}{N}}\left(1 + (z - z_0)\frac{\tilde{f}'(z)}{\tilde{f}(z)}\right),$$

so we have $g'(z_0) = e^{\frac{\ln \tilde{f}(z)}{N}}$ since $\tilde{f}(z_0) \neq 0$. The exponential is always greater than zero, so indeed we have $g'(z_0) \neq 0$, and our constructed g works. $\square$