Math 382: Homework 2

Due on Sunday January 16, 2022 at 5:00 PM

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Problem 1

Consider the annulus $U = \{z \in \mathbb{C} | a < |z| < b\}$, where $0 < a < b < \infty$. Show that U is a domain. In showing that any two points of U may be joined by a path, you may exhibit a path that is piecewise differentiable. The original question was to show that you may choose the path to be piecewise linear; if you can do that, you may derive satisfaction for a job well done.

Solution

Proof. To show that U is a domain, we need to ① show that $U \subseteq \mathbb{C}$ is open and ② show that U is path connected. For ①, define the sets

$$S_1 = \{z \in \mathbb{C} \,|\, |z| > a\} \text{ and } S_2 = \{z \in \mathbb{C} \,|\, |z| < b\}.$$

We claim that both S_1 and S_2 are open. For S_1, recall that any set is open if and only if its complement is closed. Thus, consider $S_1^C = \{z \in \mathbb{C} \,|\, |z| \leq a\}$, a closed ball of radius a. But note that the boundary $\partial S_1^C = \{z \in \mathbb{C} \,|\, |z| = a\} \subset S_1^C$, i.e., S_1^C contains its boundary so the complement is closed and S_1 is open. For S_1 note that the region is simply an open ball of radius $b < \infty$, so is open by the result shown in class. Now observe the intersection of these two sets is exactly $S_1 \cap S_1 = U$ and the intersection of two open sets is itself open, so in particular U is open. For ②, we need to show that U is path connected. Let $z_1, z_2 \in U$. Then write both in polar form:

$$z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2}.$$

Note that since $z_1, z_2 \in U$, we must have $a < r_1, r_2 < b$. Now we claim that the choice of the two following paths, namely

$$\gamma(t) = r_1 e^{i(\theta_1 + t(\theta_2 - \theta_1))} \text{ and } \xi(t) = (r_1 + t(r_2 - r_1))e^{i\theta_2}$$

both for $t \in [0, 1]$, give a piecewise differentiable path $\gamma \cup \xi$ that connects z_1 and z_2. First note that the exponential function and affine function are certainly differentiable, so respectively γ and ξ are differentiable functions of t and their concatentaion is then also piecewise differentiable. Then note that geometrically, γ starts at z_1 and traverses along the circle of radius r_1 centered at the origin in a CCW fashion as t runs from $0 \to 1$. When $t = 1$, γ ends at $\gamma(1) = r_1 e^{i\theta_2}$, which is radial with z_2. Then the concatenation with ξ traverses in a straight line along the radial direction until it hits z_2 as t runs from $0 \to 1$, upon which $\xi(1) = r_2 e^{i\theta_2} = z_2$, as claimed. Thus, U is open and path connected, i.e., it is a domain.

For the challenge, another way to see that U is path connected but *only* using a *polygonal* path is to draw a closed ball centered at z_1 of radius $b - a$, i.e, $\overline{B_1(b - a, z_1)}$, and now draw any radial line from the center to the edege of the ball in the direction of z_2. This line, call it ℓ_1, is indeed a path since the ball is convex. Then draw another ball $\overline{B_2(b - a, w_1)}$ where w lies on the circle of radius r_1 centered at the origin, i.e., w_1 and z lie on the same circle centered at the origin. Now you can draw any line ℓ_2 that starts from the end of ℓ_1 in the direction of z_2 until you hit the edge of the ball, and by convexity of the ball ℓ_2 is still a path. Continue this process until the ball $\overline{B_i(b - a, w_i)}$ contains the point z_2, upon which you can always draw a straight line from the edge of the ball where ℓ_{i-1} stopped to the point z_2 since the open ball is path connected. The union of all the paths $\gamma = \ell_1 \cup \ell_2 \cdots \ell_i$ gives a polygonal path from z_1 to z_2 and is thus U is path connected. ◻

Problem 2

Verify by calculating the partial derivatives with respect to x and y, the real and imaginary parts of z, that the function $\sin(z)$ satisfies the Cauchy-Riemann equation.

Solution

First write $f(z) = \sin z$ in complex exponential form:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Then writing the complex number z as $z = x + iy$ for $x, y \in \mathbb{R}$, we can expand and simplfify the sine as

$$\begin{aligned}
\sin z &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} \\
&= \frac{(\cos x + i\sin x)e^{-y}}{2i} - \frac{(\cos x - i\sin x)e^{y}}{2i} \\
&= \cos x\frac{e^{-y}}{2i} + \sin x\frac{e^{-y}}{2} - \cos x\frac{e^{y}}{2i} + \sin x\frac{e^{y}}{2} \\
&= \cos x\left(\frac{e^{-y} - e^{y}}{2i}\right) + \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) \\
&= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right) \\
&= \sin x\cosh y + i\cos x\sinh y,
\end{aligned}$$

where we used the identity $e^{ix} = \cos x + i\sin x$. Now let $u = \sin x\cosh y$ and $v = \cos x\sinh y$ so that

$$f(z) = u + iv.$$

Then we can compute

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cos x\cosh y \qquad & \frac{\partial u}{\partial y} &= \sin x\sinh y \\
\frac{\partial v}{\partial y} &= \cos x\cosh y \qquad & \frac{\partial v}{\partial x} &= -\sin x\sinh y.
\end{aligned}$$

So, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

so $f(z) = \sin z$ satisfies the Cauchy-Riemann equations.

Problem 3

Consider the square with vertices $\{0, 1, 1+i, i\}$. Let γ be a parametrized path that follows the four sides of this square in a counterclockwise direction.
a) If $g(x,y)dx + h(x,y)dy$ is a differential defined on an open set containing the square, calculate the line integral

$$\int_\Gamma g(x,y)dx + h(x,y)dy$$

in terms of explicit definite integrals.
b) Calculate this line integral for the differentials dz, zdz and z^2dz. Do you see a pattern?
c) Calculate the line integral for the differential $\bar{z}dz$.

Solution

Part A

We first parametrize the unit square with vertices $\{0, 1, 1+i, i\}$ in a CCW fashion: let Γ be the concatentation of piecewise line segments C_1, C_2, C_3, and C_4, each parametrized respectively by

$$\begin{cases} \gamma_1(t) = t + 0i, & t \in [0,1] \\ \gamma_2(t) = 1 + ti, & t \in [0,1] \\ \gamma_3(t) = (1-t) + i, & t \in [0,1] \\ \gamma_4(t) = 0 + (1-t)i, & t \in [0,1], \end{cases}$$

such that $\Gamma = C_1 \cup C_2 \cup C_3 \cup C_4$ gives the desired piecewise differentiable parametrization. Then the line integral of the 1-form $g(x,y)dx + h(x,y)dy$ over Γ is

$$\begin{aligned} \int_\Gamma g(x,y)dx + h(x,y)dy &= \int_{C_1} g(x,y)dx + h(x,y)dy + \cdots + \int_{C_4} g(x,y)dx + h(x,y)dy \\ &= \int_0^1 [g(\gamma_1(t))\gamma_1'(t) + h(\gamma_1(t))\gamma_1'(t)]\,dt + \cdots + \int_0^1 [g(\gamma_4(t))\gamma_4'(t) + h(\gamma_4(t))\gamma_4'(t)]\,dt. \end{aligned}$$

Then computing

$$\gamma_1'(t) = 1, \gamma_2'(t) = i, \gamma_3'(t) = -1, \text{ and } \gamma_4(t) = -i$$

gives us that

$$\begin{aligned} \int_\Gamma g(x,y)dx + h(x,y)dy = \int_0^1 [g(\gamma_1(t)) + h(\gamma_1(t))]\,dt + i\int_0^1 [g(\gamma_2(t)) + h(\gamma_2(t))]\,dt \\ - \int_0^1 [g(\gamma_3(t)) + h(\gamma_3(t))]\,dt - i\int_0^1 [g(\gamma_4(t)) + h(\gamma_4(t))]\,dt. \end{aligned}$$

Part B

Using the result of **Part A**, for the differential form dz we have

$$\int_\Gamma g(x,y)dx + h(x,y)dy = \int_0^1 \,dt + i\int_0^1 \,dt - \int_0^1 \,dt - i\int_0^1 \,dt = 0.$$

Similarly, for zdz we have

$$\begin{aligned}\int_\Gamma g(x,y)dx + h(x,y)dy &= \int_0^1 t\ dt + i\int_0^1 (1+ti)\ dt - \int_0^1 (1-t+i)\ dt - i\int_0^1 (1-t)i\ dt \\ &= 1 + i - 1 - 1 + 1 - i + 1 - 1 \\ &= 0.\end{aligned}$$

Finally, for z^2dz we have

$$\begin{aligned}\int_\Gamma g(x,y)dx + h(x,y)dy &= \int_0^1 t^2\ dt + i\int_0^1 (1+ti)^2\ dt - \int_0^1 (1-t+i)^2\ dt - i\int_0^1 ((1-t)i)^2\ dt \\ &= \frac{1}{3} - 1 + \frac{2}{3}i + \frac{2}{3} - i + \frac{1}{3}i \\ &= 0.\end{aligned}$$

It seems like the line integral along Γ for all of these differential forms is zero. This seems to be reflective of the classic multivariable result that if $U \subseteq \mathbb{R}^n$ is open and $\mathbf{F} : U \to \mathbb{R}^n$ is a continuous, conservative vector field on U, then the closed line integral vanishes, that is $\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$, for every piecewise oriented closed curve C in U. This follows from the existence of a potential function for $\mathbf{F}$ since it is conservative and the fundamental theorem of line intergals.

Part C

Using the result of **Part A**, for $\overline{z}dz$ we have

$$\begin{aligned}\int_\Gamma g(x,y)dx + h(x,y)dy &= \int_0^1 t\ dt + i\int_0^1 (1-ti)\ dt - \int_0^1 (1-t-i)\ dt - i\int_0^1 -(1-t)i\ dt \\ &= 2i.\end{aligned}$$

If the observation made in **Part B** is correct, then this follows from the fact that $\overline{z}$ does not have an antiderivative, whatever that means.