Math 382: Homework 6

Due on Monday February 14, 2022 at 10:00 PM

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Problem 1

Find the number of zeroes of the function $f(z) = z^7 + 6z^3 + 7$ in the first quadrant $0 < \operatorname{Arg} z < \pi/2$. (Hint: use the Argument Principle, with the curve that follows the circle of radius R in the first quadrant from R to iR, followed by the the segment of the imaginary axis from iR to 0 , and the segment of the real axis from 0 back to R.)

Solution

Let γ be the closed, simple path given in the hint. Note that $f(z)$ is entire, so the Argument Principle gives

$$Z = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\Delta_C \arg f(z),$$

where Z is the number of zeros $f(z)$ has in the first quadrant. C is defined as a concatenation of paths C_1, C_2, and C_3 as described in the hint. Along the circle of radius R, call this path C_1, $z = Re^{it}$ for $0 \le t \le \frac{\pi}{2}$, so

$$f(Re^{it}) = R^7 e^{7it}\left(1 + \frac{6}{R^4 e^{4it}} + \frac{7}{R^7 e^{7it}}\right).$$

But for large R, $\arg f(Re^{it}) \approx \arg(e^{7it}) = 7t$ so

$$\Delta_{C_1} \arg f(z) = 7 \cdot \left(\frac{\pi}{2} - 0\right) = \frac{7\pi}{2}.$$

For C_2 along the imaginary axis from iR to 0, we have $z = it$ for $R \ge t \ge 0$, so

$$f(it) = (it)^7 + 6(it)^3 + 7 = -it^7 - 6it^3 + 7 = 7 - i(t^7 + 6t^3).$$

But note for $R \ge t \ge 0$,

$$\operatorname{Re} f(it) = 7 > 0 \text{ and } \operatorname{Im} f(it) = -t^7 - 6t^3 < 0.$$

Hence, as t decreases from R to 0, $f(it)$ starts in the fourth quadrant and moves to the point $z = 7$. Thus,

$$\Delta_{C_2} \arg f(z) = \frac{\pi}{2}.$$

Along C_3, $z = t$ for $0 \le t \le R$, we have

$$f(t) = t^7 + 6t^3 + 7,$$

and the argument doesn't change, so

$$\Delta_{C_3} \arg f(z) = 0.$$

Thus,

$$\Delta_C \arg f(z) = \frac{7\pi}{2} + \frac{\pi}{2} + 0 = 4\pi,$$

and we have $Z = \frac{4\pi}{2\pi} = 2$, so f has 2 zeros in the first quadrant.

Problem 2

Show that the equation $z^4 - 5z^2 + 3 = e^{-z}$ has no solutions on the imaginary axis, and precisely two solutions in the half-plane $\operatorname{Re} z > 0$.

Solution

Proof. Along the **Im**−axis, we have that $z = it$ for $t \in \mathbb{R}$, so our equation becomes

$$(it^4) - 5(it)^2 + 3 = t^4 + 5it^2 + 3 = e^{-it}.$$

But note $\left|e^{-it}\right| = 1$ for all t, but our polynomial $t^4 + 5it^2 + 3 > 3$ for all t, so theres no solution. Let $f(z) = z^4 - 5z^2 + 3 - e^{-z}$ and $g(z) = z^4 - 5z^2 + 3$. Simply noting that

$$|f(z) - g(z)| < |g(z)|,$$

on the right half plane $\operatorname{Re} z > 0$ and a zero of $-g$ is exactly a zero of g as well, so Rouché's Theorem gives us that f and g have the same number of zeros on the half plane. But, the entire function g has two roots in the half plane since by the Argument Principle similar to **Question 1** but applied to the semicircle of radius R centered at the origin in the right half-plane:

$$\Delta_C \arg g(z) = 4 \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + 0,$$

so $Z = \frac{4\pi}{2\pi} = 2$ zeros. But, g and f have the same number of zeros, so $z^4 - 5z^2 + 3 = e^{-z}$ has two solutions on the half-plane $\operatorname{Re} z > 0$. $\square$

Problem 3

Let γ_N be the closed curve that traces the boundary of the square with vertices $\pm(N+1/2)\pm i(N+1/2)$, where $N = 1, 2, 3, \ldots$.
a) Show that

$$\lim_{N\to\infty}\int_{\gamma_N}\frac{\pi\cot(\pi z)dz}{z^2} = 0.$$

b) Show that $\pi\cot(\pi z)$ has simple poles at the integers, with residue 1. Apply the Residue Theorem to the integral

$$\int_{\gamma_N}\frac{\pi\cot(\pi z)dz}{z^2},$$

and conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

(This formula was proved, by a different method, by Euler.)

Solution

Part A

Proof. Let $f(z) = \frac{1}{z^2}$ and $g(z) = \pi\cot\pi z$. We want to use the ML inequality here to show that the integral is bounded above and below by 0 as $N\to\infty$ to conclude that this is integral vanishes by the sandwich theorem. So, to this end, we note that f is clearly bounded above on γ_N by $\frac{1}{N^2}$. For g, note that $g(z) = \pi\frac{\cos\pi z}{\sin\pi z}$ is analytic for $z\notin\mathbb{Z}$ but periodic for $z\in\mathbb{Z}$ since $g(z+n) = z$ for $n\in\mathbb{Z}$. So, along the paths parallel to **Im**-axis, we have

$$\left|g\left(iy\pm\left(N+\frac{1}{2}\right)\right)\right| = \left|g\left(\frac{1}{2}+iy\right)\right| = \pi\left|\frac{\sinh(\pi y)}{\cosh(\pi y)}\right| = \pi|\tanh\pi y| \le \pi.$$

However, along the sides parallel to the **Re**-axis, g is bounded above by

$$\begin{aligned}\left|g\left(x\pm i\left(N+\frac{1}{2}\right)\right)\right| &= \left|\pi\frac{\cos(\pi x)\sin(\pi x) - i\cdot\cosh(\pi(N+1/2))\sinh(\pi(N+1/2))}{\cosh^2(\pi(N+1/2))\sin^2(\pi x)+\cos^2(\pi x)\sinh^2(\pi(N+1/2))}\right| \\ &\le \pi\frac{1+\cosh\left(\pi\left(N+\frac{1}{2}\right)\right)\sinh\left(\pi\left(N+\frac{1}{2}\right)\right)}{\sinh^2\left(\pi\left(N+\frac{1}{2}\right)\right)} \\ &\le \pi\frac{\cosh\left(\pi\left(N+\frac{1}{2}\right)\right)}{\sinh\left(\pi\left(N+\frac{1}{2}\right)\right)} \\ &\le \pi\coth\left(\frac{3\pi}{2}\right) \\ &\approx 1.28\pi.\end{aligned}$$

What is important to note here is that g is *uniformly* bounded on γ_n, i.e., its bound does not depend on N. Let the upper bounds on f and g on γ_N be M_1 and M_2 respectively. So, the ML-inequality gives

$$\left|\int_{\gamma_N}(fg)(z)dz\right| \le \frac{M_1}{M_2}L = \frac{M_2}{N^2}\cdot 8\left(N+\frac{1}{2}\right) = \frac{8M_2}{N}+\frac{4M_2}{N^2}\to 0 \text{ as } N\to\infty,$$

since M_2 has no N dependence. Thus, the result follows. □

Problem 3 continued on next page...

Part B

Proof. Since $g(z) = \pi \frac{\cos \pi z}{\sin \pi z}$, g will have simple poles where $\sin \pi z$ has zeros, which is exactly at $z = n$ for $n \in \mathbb{Z}$. To calculate the residue,

$$\operatorname{Res}[g, n] = \lim_{z \to k} (z-k)\pi \frac{\cos \pi z}{\sin \pi z} = \lim_{z \to k} (\cos \pi z) \frac{\pi}{\frac{\sin \pi z}{z-k}} = \pi \frac{\cos \pi k}{(\sin \pi z)'(k)} = 1,$$

where we used the result $\operatorname{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$ from **Discussion 4**. For the function fg, we have simple poles still for $z = n$ and n a *nonzero* integer, but at $z = 0$, we have a triple pole. Thus,

$$\operatorname{Res}[fg, n \neq 0] = \lim_{z \to n} (z-n)(fg)(z) \overset{u \mapsto z-n}{=} \lim_{u \to 0} u(fg)(u+n) = \lim_{u \to 0} \frac{u \cdot g(u+n)}{(u+n)^2} = \lim_{u \to 0} \frac{u \cdot g(u)}{(u+n)^2} = \frac{1}{n^2},$$

where we used periodicity of g. At $z = 0$, we use the formula for a pole of order 3 to compute

$$\operatorname{Res}[fg, 0] = \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2} \left(z^3 \pi \frac{\cot \pi z}{z^2} \right) = \lim_{z \to 0} \frac{\pi}{2} (2\pi(\pi z \cot \pi z - 1) \csc^2 \pi z) = \lim_{z \to 0} \frac{\pi^3 z \cot \pi z - \pi^2}{\sin^2 \pi z} \overset{\text{L'Hôp}}{=} \frac{-\pi^2}{3}.$$

Thus, the Residue Theorem gives us

$$\int_{\gamma_N} (fg)(z)\, dz = 2\pi i \sum_k \operatorname{Res}[fg, z_k] = -\frac{\pi^2}{3} + 2 \sum_{n=1}^{N} \frac{1}{n^2},$$

where we take a sum over positive integers up to N and double it to represent the sum over all nonzero integers from $-N$ to N. Finally taking limits of both sides as $N \to \infty$ and using **Part A** to conclude the LHS vanishes gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

as desired. □