Math 382: Homework 4

Due on Sunday January 30, 2022 at 5:00 PM

Prof. Ezra Getzler

Anthony Tam

Problem 1

The function $f(z) = \tan z$ solves the first-order ordinary differential equation $f'(z) = 1 + f(z)^2$, with initial condition $f(0) = 0$.
1) Show that the solution of this equation is odd. (Hint: the solution of a first-order ordinary differential equation is determined by its value at $z = 0$. But $g(z) = -f(-z)$ solves the same equation.)
2) Consider the Taylor series of $f(z)$ around $z = 0$:

$$\sum_{k=0}^{\infty} \frac{a_{2k+1} z^{2k+1}}{(2k+1)!}$$

(This is the most general Taylor series for an odd function.) Convert the differential equation into an equation for a_{2k+1} in terms of $a_{2j+1}, 0 \leq j < k$.
3) Write pseudocode for a program to calculate the coefficient a_{2k+1}, given a positive integer k. This may be an outline of a computer program in any of your favorite structured languages (for example Python), together with a series of lines that explain the sequence of instructions in such a program.
4) Calculate the first three nonzero terms of the Taylor series. How accurate is the resulting polynomial as an approximation for $\tan z$, when $z = 1/100$?
5) Calculate the first three nonzero coefficients of the power series $f(z) \cos z$, and show that they equal the first three nonzero coefficients of the power series $\sin z$.

Solution

Part A

Proof. Note that if $f(z)$ solves the differential equation, the function $g(z) = -f(-z)$ solves it too:

$$g'(z) = 1 + g(z)^2 \iff f'(-z) = 1 + [f(-z)]^2.$$

But, we know that linear combinations of particular solutions also give a solution, so in particular

$$\frac{f(x) + g(x)}{2} = \frac{f(x) - f(-x)}{2},$$

is also a solution. Given the initial condition $f(0) = 0$, we know that this solution is unique, so we must have

$$f(x) = \frac{f(x) - f(-x)}{2} \iff f(-x) = -f(x),$$

and thus the solution $f(z)$ is odd. □

Part B

Plugging in the Taylor series, we get the following after differentiating term by term and expanding the Cauchy product

$$\left(\sum_{k=0}^{\infty} a_{2k+1} \frac{z^{2k+1}}{(2k+1)!} \right)' = 1 + \left(\sum_{k=0}^{\infty} a_{2k+1} \frac{z^{2k+1}}{(2k+1)!} \right)^2$$
$$\iff \sum_{k=0}^{\infty} a_{2k+1} \frac{z^{2k}}{(2k)!} - \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k-1} \frac{a_{2j+1}}{(2j+1)!} \cdot \frac{a_{2(k-j)-1}}{(2(k-j)-1)!} \right) z^{2k} = 1.$$

Problem 1 continued on next page. . .

Comparing coefficients, we must have that

$$a_1 = 1 \text{ and for } k > 1,\ a_{2k+1} = (2k)! \left(\sum_{j=0}^{k-1} \frac{a_{2j+1}}{(2j+1)!} \cdot \frac{a_{2(k-j)-1}}{(2(k-j)-1)!} \right).$$

Part C

Using the result of **Part B**, we have

$$\tan(x) \approx x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7).$$

The error is then approximately

$$\tan\left(\frac{1}{100}\right) - \left(\frac{1}{100} + \frac{\left(\frac{1}{100}\right)^3}{3} + \frac{2\left(\frac{1}{100}\right)^5}{15}\right) \approx 1 \times 10^{-15}.$$

Part D

Multiplying the first three nonzero terms of the Taylor series we found in **Part D** and recalling that the Taylor series for $\cos z$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ gives

$$\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots .$$

Indeed, recall that the Taylor series of $\sin z$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, so the result follows.

Problem 2

Calculate the Taylor series of the function $f_k(z) = (1-z)^{-k}$ at $z = 0$. What is the radius of convergence of this series? Justify.

Solution

Proof. First let us compute the n^{th} derivative of f_k at zero:

$$f_k^{(1)}(0) = k,\ f_k^{(1)}(0) = k(k+1)\,, \ldots, f_k^{(n)}(0) = k(k+1)\cdots(k+n-1) = k^{(n)},$$

where $k^{(n)}$ denotes the rising factorial. Then the Taylor series of $f_k(z)$ centered at $z = 0$ is given by

$$f_k(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{k^{(n)}}{n!} z^n = \sum_{n=0}^{\infty} \binom{k+n-1}{n} z^n,$$

where we use the relation $k^{(n)} = \frac{(k+n-1)!}{(k-1)!}$ and thus $\frac{k^{(n)}}{n!} = \binom{k+n-1}{n}$. For the radius of convergence, let us apply the ratio test to $a_k = \binom{k+n-1}{n}$:

$$\begin{aligned} R = \lim_{n\to\infty} \left| \frac{\binom{k+n-1}{n}}{\binom{k+n}{n+1}} \right| &= \lim_{n\to\infty} \frac{(k+n-1)!}{n!(k-1)!} \cdot \frac{(n+1)!(k-1)!}{(k+n)!} \\ &= \lim_{n\to\infty} \frac{(k+n-1)!}{n!} \cdot \frac{(n+1)(n!)}{(k+n)(k+n-1)!} \\ &= \lim_{n\to\infty} \frac{n+1}{k+n} \\ &= 1. \end{aligned}$$

Hence, the radius of convergence is $R = 1$. $\square$

Problem 3

Gameline §V.3 Exercise 6: Show the series $\sum a_k z^k$, the differentiated series $\sum k a_k z^{k-1}$, and the integrated series $\sum \frac{a_k}{k+1} z^{k+1}$ all have the same radius of convergence.

Solution

Proof. Consider the power series $\sum_{k\geq 0} a_k z^k$. The Cauchy Hadamard formula gives taht this power series has a radius of convergence

$$R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}.$$

Using the fact that $\limsup_{k\to\infty} \sqrt[k]{k} = 1$, we have that the differentiated series $\sum_{k\geq 1} k a_k z^{k-1}$ has radius of convergence

$$\begin{aligned} R' &= \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|k a_k|}} \\ &= \frac{1}{\limsup_{k\to\infty} \sqrt[k]{k}\,\sqrt[k]{|a_k|}} \\ &= \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} \\ &= R. \end{aligned}$$

Similarly, using the fact that $\limsup_{k\to\infty} \sqrt[k]{1/(k+1)} = 1$, we have that the integrated series $\sum_{k\geq 0} \frac{a_k}{k+1} z^{k+1}$ has radius of convergence

$$\begin{aligned} R'' &= \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\left|\frac{a_k}{k+1}\right|}} \\ &= \frac{1}{\limsup_{k\to\infty} \sqrt[k]{\frac{1}{k+1}}\,\sqrt[k]{|a_k|}} \\ &= \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} \\ &= R. \end{aligned}$$

Thus all series have the same radius of convergence $R = R' = R''$. $\square$