Math 382: Homework 2

Due on Sunday January 16, 2022 at 5:00 PM

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Problem 1

Consider the annulus $U = \{z \in \mathbb{C} | a < |z| < b\}$, where $0 < a < b < \infty$. Show that U is a domain. In showing that any two points of U may be joined by a path, you may exhibit a path that is piecewise differentiable. The original question was to show that you may choose the path to be piecewise linear; if you can do that, you may derive satisfaction for a job well done.

Solution

Proof. To show that U is a domain, we need to ① show that $U \subseteq \mathbb{C}$ is open and ② show that U is path connected. For ①, define the sets

$$S_1 = \{z \in \mathbb{C} \,|\, |z| > a\} \quad \text{and} \quad S_2 = \{z \in \mathbb{C} \,|\, |z| < b\}.$$

We claim that both S_1 and S_2 are open. For S_1, recall that any set is open if and only if its complement is closed. Thus, consider $S_1^C = \{z \in \mathbb{C} \,|\, |z| \leq a\}$, a closed ball of radius a. But note that the boundary $\partial S_1^C = \{z \in \mathbb{C} \,|\, |z| = a\} \subset S_1^C$, i.e., S_1^C contains its boundary so the complement is closed and S_1 is open. For S_1 note that the region is simply an open ball of radius $b < \infty$, so is open by the result shown in class. Now observe the intersection of these two sets is exactly $S_1 \cap S_1 = U$ and the intersection of two open sets is itself open, so in particular U is open. For ②, we need to show that U is path connected. Let $z_1, z_2 \in U$. Then write both in polar form:

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}.$$

Note that since $z_1, z_2 \in U$, we must have $a < r_1, r_2 < b$. Now we claim that the choice of the two following paths, namely

$$\gamma(t) = r_1 e^{i(\theta_1 + t(\theta_2 - \theta_1))} \quad \text{and} \quad \xi(t) = (r_1 + t(r_2 - r_1))e^{i\theta_2}$$

both for $t \in [0, 1]$, give a piecewise differentiable path $\gamma \cup \xi$ that connects z_1 and z_2. First note that the exponential function and affine function are certainly differentiable, so respectively γ and ξ are differentiable functions of t and their concatentaion is then also piecewise differentiable. Then note that geometrically, γ starts at z_1 and traverses along the circle of radius r_1 centered at the origin in a CCW fashion as t runs from $0 \to 1$. When $t = 1$, γ ends at $\gamma(1) = r_1 e^{i\theta_2}$, which is radial with z_2. Then the concatenation with ξ traverses in a straight line along the radial direction until it hits z_2 as t runs from $0 \to 1$, upon which $\xi(1) = r_2 e^{i\theta_2} = z_2$, as claimed. Thus, U is open and path connected, i.e., it is a domain. $\square$

Problem 2

Verify by calculating the partial derivatives with respect to x and y, the real and imaginary parts of z, that the function $\sin(z)$ satisfies the Cauchy-Riemann equation.

Solution

First write $f(z) = \sin z$ in complex exponential form:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Then writing the complex number z as $z = x + iy$ for $x, y \in \mathbb{R}$, we can expand and simplfify the sine as

$$\begin{aligned}
\sin z &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i} \\
&= \frac{(\cos x + i\sin x)e^{-y}}{2i} - \frac{(\cos x - i\sin x)e^{y}}{2i} \\
&= \cos x\frac{e^{-y}}{2i} + \sin x\frac{e^{-y}}{2} - \cos x\frac{e^{y}}{2i} + \sin x\frac{e^{y}}{2} \\
&= \cos x\left(\frac{e^{-y} - e^{y}}{2i}\right) + \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) \\
&= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right) \\
&= \sin x\cosh y + i\cos x\sinh y,
\end{aligned}$$

where we used the identity $e^{ix} = \cos x + i\sin x$. Now let $u = \sin x\cosh y$ and $v = \cos x\sinh y$ so that

$$f(z) = u + iv.$$

Then we can compute

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \cos x\cosh y & \frac{\partial u}{\partial y} &= \sin x\sinh y \\
\frac{\partial v}{\partial y} &= \cos x\cosh y & \frac{\partial v}{\partial x} &= -\sin x\sinh y.
\end{aligned}$$

So, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

so $f(z) = \sin z$ satisfies the Cauchy-Riemann equations.

Problem 3

Consider the square with vertices $\{0, 1, 1+i, i\}$. Let γ be a parametrized path that follows the four sides of this square in a counterclockwise direction.
a) If $g(x,y)dx + h(x,y)dy$ is a differential defined on an open set containing the square, calculate the line integral

$$\int_\gamma g(x,y)dx + h(x,y)dy$$

in terms of explicit definite integrals.
b) Calculate this line integral for the differentials dz, zdz and z^2dz. Do you see a pattern?
c) Calculate the line integral for the differential $\bar{z}dz$.

Solution

Part A