Math 382: Homework 1

Due on Sunday January 9, 2022 at 5:00 PM

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Problem 1

Show that the triangle with vertices $0, z$ and w is equilateral if and only if $|z|^2 = |w|^2 = 2\,\mathrm{Re}(\bar{z}w)$.

Solution

Proof. The three sides of the triangle formed by $0, z,$ and w considered as points in $\mathbb{C}$ are the line segments connecting 0 and z, 0 and w, and z and w. It follows then that the lengths of these sides are then $|z-0| = |z|$, $|w-0| = |w|$, and $|z-w|$ respectively. Note that we could have just as easily chosen $|0-z|$, $|0-w|$, and $|w-z|$ as representatives of the length. So, the equivalent statement to prove is

$$|z| = |w| = |z-w| \text{ if and only if } |z|^2 = |w|^2 = 2\mathrm{Re}(\overline{z}w).$$

Suppose then that $|z| = |w| = |z-w|$. This is true if and only if by squaring everything,

$$\begin{aligned}
|z|^2 = |w|^2 &= |z-w|^2 \\
&= (z-w)\overline{(z-w)} \\
&= (z-w)(\overline{z}-\overline{w}) \\
&= z\overline{z} - w\overline{z} - \overline{w}z + w\overline{w} \\
&= |z|^2 + |w|^2 - w\overline{z} - \overline{w}z \\
&= |z|^2 + |w|^2 - 2\mathrm{Re}(\overline{z}w),
\end{aligned}$$

where in the last line we used the fact that $\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z})$ and that $\overline{\overline{z}w} = \overline{\overline{z}}\,\overline{w} = z\overline{w}$. So overall, we have

$$\begin{aligned}
|z| = |w| = |z-w| &\iff |z|^2 = |w|^2 = |z|^2 + |w|^2 - 2\mathrm{Re}(\overline{z}w) \\
&\iff |z|^2 = |w|^2 = 2\mathrm{Re}(\overline{z}w),
\end{aligned}$$

as claimed. $\square$

Problem 2

Let n be an integer. Show that the function $\cos nz$ is a polynomial in the function $\cos z$. For example, $\cos 2z = 2(\cos z)^2 - 1$.

Solution

Proof. Recall that the complex exponential e^{inz} for $n \in \mathbb{Z}$ can be rewritten as

$$e^{inz} = \cos nz + i \sin nz,$$

so in particular $\operatorname{Re}(e^{inz}) = \cos nz$. To rewrite cosine then, we expand the following complex binomial using the binomial formula:

$$\begin{aligned}
\cos nz &= \operatorname{Re}(e^{inz}) \\
&= \operatorname{Re}((e^{iz})^n) \\
&= \operatorname{Re}((\cos z + i \sin z)^n) \\
&= \operatorname{Re}\left[\sum_{k=0}^{n} \binom{n}{k} \cos^{n-k} z (i^k \sin^k z)\right].
\end{aligned}$$

Note now that i^k is real if and only if k is even, so make a substitution $k \mapsto 2m$. Summing over even indicies is equivalent to taking a sum from $m = 0$ to $m = \lfloor n/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function which returns the greatest integer less than or equal to the argument. To see why this is, note that if n is even, then $\lfloor n/2 \rfloor = n/2$ so the last term is $2 \cdot n/2$, but if n is odd, then $\lfloor n/2 \rfloor = (n-1)/2$ so the last term is $2 \cdot (n-1)/2 = n - 1$. Further, i^{2m} will oscillate between $i^2 = -1$ and $i^4 = 1$, so all together we have

$$\begin{aligned}
\cos nz &= \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} \cos^{n-2m} z \sin^{2m} z (-1)^m \\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} \cos^{n-2m} z (\sin^2 z)^m (-1)^m \\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} \cos^{n-2m} z (1 - \cos^2 z)^m (-1)^m \\
&= \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{2m} \cos^{n-2m} z (\cos^2 z - 1)^m,
\end{aligned}$$

where we used the identity $\sin^2 z + \cos^2 z = 1$ in the second to last equality. Note that the highest power of cosine in the binomial in parentheses will be $2m$, which times the cosine factor outside of the parentheses will give us a polynomial in $\cos z$, in fact of degree $n - 2m + 2m = n$. $\square$

Problem 3

Gamelin Exercise I.8.5. Let S denote the two slits along the imaginary axis in the complex plane, one running from i to $+i\infty$, the other from $-i$ to $-i\infty$. Show that $(1+iz)/(1-iz)$ lies on the negative real axis $(-\infty, 0]$ if and only if $z \in S$. Show that the principal branch

$$\mathrm{Tan}^{-1} z = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$$

maps the slit plane $\mathbb{C}\backslash S$ one-to-one onto the vertical strip $\{|\mathrm{Re}\, w| < \pi/2\}$.

Solution

Proof. Let $\frac{1+iz}{1-iz} = v$, where $v \in \mathbb{C}$. We want to show that $v \in (-\infty, 0]$ if and only if $z \in S$. An easy way to show this is to construct a one-to-one mapping from the sections of interest in the z-plane to the desired sections of the v-plane. First solving for z in v by clearing the denominator gives

$$\begin{aligned} 1 + iz &= v - ivz \\ iz + ivz &= v - 1 \\ iz(v+1) &= v - 1 \\ \iff z &= \frac{1}{i}\frac{v-1}{v+1} = i\frac{1-v}{1+v}, \end{aligned}$$

which implies that this is a one-to-one mapping. Now we compute the real limits of the right hand side of the last equality to draw some conclusions about z. Take careful note that we are only taking real values of v and carefully choosing straight line paths along the real-axis, so we evaluate the limits in the usual real-valued function way:

$$\lim_{v\to-\infty}\frac{1-v}{1+v} = -1\ , \ \lim_{v\to-1^-}\frac{1-v}{1+v} \text{ tends to } -\infty,$$
$$\lim_{v\to-1^+}\frac{1-v}{1+v} = \text{ tends to } +\infty, \text{ and } \lim_{v\to 0}\frac{1-v}{1+v} = 1.$$

Hence, $\frac{1-v}{1+v}i$ goes from $-i$ to $-i\infty$ along the **Im**-axis as v goes from $-\infty$ to -1^-. Further, $\frac{1-v}{1+v}i$ goes from i to $i\infty$ along the **Im**-axis as v goes from -1^+ to 0. Since this map is one-to-one, there is a correspondence between the subsets

$$\underbrace{[-i, -i\infty)}_{\subset z-\text{plane}} \longleftrightarrow \underbrace{(-\infty, -1^-]}_{\subset v-\text{plane}} \text{ and } \underbrace{[i, i\infty)}_{\subset z-\text{plane}} \longleftrightarrow \underbrace{[-1^+, 0]}_{\subset v-\text{plane}}.$$

Note that this is slight abuse of notation with -1^+ and -1^- in the set notation, but the important remark here is that the union of the subsets of the v-plane give $(-\infty, 0]$ and the union of the subsets of the z-plane give S so that $z \in S$ if and only if $v \in (-\infty, 0]$ by this one-to-one mapping as claimed.

To show the next claim, it is best to build up the function in question as the composition of one-to-one mappings, where we use the fact that the composition of two one-to-one maps is another one-to-one map. We have already determined that $v = \frac{1+iz}{1-iz}$ is one-to-one from $\mathbb{C} \setminus S \rightarrowtail \mathbb{C} \setminus (-\infty, 0]$. So, let $u = \mathrm{Log}\ v$ and note that now this is a one-to-one (and onto) map from $\mathbb{C} \setminus (-\infty, 0] \rightarrowtail \{|\mathrm{Im}\, u| < \pi\}$ since Log is defined to be a single valued inverse for e^v. Let $w = \frac{1}{2i}u$ and then remark that this composition now maps $\{|\mathrm{Im}\, u| < \pi\} \rightarrowtail \left\{|\mathrm{Re}\, w| < \frac{\pi}{2}\right\}$ in a one-to-one manner. To summarize,

$$\mathbb{C} \setminus S \overset{v}{\rightarrowtail} \mathbb{C} \setminus (-\infty, 0] \overset{u}{\rightarrowtail} \{|\mathrm{Im}\, u| < \pi\} \overset{w}{\rightarrowtail} \{|\mathrm{Re}\, w| < \pi/2\},$$

with $\mathrm{Tan}^{-1} z = w \circ u \circ v : \mathbb{C} \setminus S \rightarrowtail \left\{|\mathrm{Re}\, w| < \frac{\pi}{2}\right\}$ a one-to-one map. $\square$