

CS 753 : Automatic Speech Recognition Assignment 1

Problem 1

Arka Sadhu - 140070011

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1 Part A

We are given that the probability of sending bit 0 is p and sending bit 1 is $1 - p$. It is also mentioned that the bits are independent and identically distributed.

Also it is given that the channel is noisy and therefore a bit can be flipped with a probability of δ , and therefore remains unchanged with probability $1 - \delta$.

We need to find the probability that the k^{th} symbol received is 0.

Since the bits are independent the probability of k^{th} symbol is not dependent upon the other symbols. Also it can be assumed that the channel doesn't have a time delaying effect and therefore it affects only the considered bit with a probability given by δ .

The k^{th} symbol can be received as 0 in two ways:

- The sent bit was 0 and no flipping occurred. The probability of this happening is $P1 = p(1 - \delta)$.
- The sent bit was 1 and flipping occurred. The probability of this is $P2 = (1 - p)\delta$.

Since the two events are mutually exclusive, therefore the probabilities simply add up. Therefore

$$Pr(k^{th} \text{ symbol received} = 0) = p(1 - \delta) + (1 - p)\delta \quad (1)$$

2 Part B

A repetition code is used such that 0 is encoded as 000 and 1 as 111 and the decoder outputs a majority bit on the receiving 3-bit string.

We need to find the probability of correctly decoding 0.

It is easy to see that 0 will be correctly decoded when at most 1 bit is flipped. This can happen in one of the following ways:

- None of the bits get flipped. $Pr(\text{Nobitflipped}) = (1 - \delta)^3$
- Exactly one of the bits gets flipped. $Pr(\text{Exactlyonebitflipped}) = 3\delta(1 - \delta)^2$

Again the above are mutually exclusive, the probabilities add up. Therefore

$$Pr(0 \text{ being correctly decoded}) = (1 - \delta)^3 + 3\delta(1 - \delta)^2 \quad (2)$$

3 Part C

Now it is said that each bit is transmitted unencoded. But the decoder is allowed to output $\{0, 1, \perp\}$.

If the decoder output is incorrect then the penalty points are 2 and if it outputs it to be \perp then it gets penalty 1.

The aim is to find the best decoding strategy which can minimize the expected penalty. We are given $\delta = \frac{1}{8}$ and want the best strategy for $p \in [0, 1]$.

We first do the problem more generally and then impose the conditions given to us. We take that on output of \perp the penalty is 1 and the output on incorrect bit the penalty is X_1 . Also we don't put the value of δ till the last step.

The decoder either sees 0 or 1 and chooses on of the three options $\{0, 1, \perp\}$. We assign probabilities to each of the six cases.

	0	1	\perp
0	ρ_1	ρ_2	ρ_3
1	ρ_4	ρ_5	ρ_6

Let the expected penalty be denoted by $\mathbb{E}[Pen]$. Clearly there are four ways in which penalty can arise:

- Sender sends 0, bit is not flipped (stays 0). Penalty arises when detector declares it either 1 or \perp .

$$\mathbb{E}[Pen|case_1] = X_1\rho_2 + \rho_3$$

$$Pr(case_1) = p(1 - \delta)$$

- Sender sends 0, bit is flipped (becomes 1). Penalty arises when detector declares it either 1 or \perp .

$$\mathbb{E}[Pen|case_2] = X_1\rho_5 + \rho_6$$

$$Pr(case_2) = p\delta$$

- Sender sends 1, bit is flipped (becomes 0). Penalty arises when detector declares it either 0 or \perp .

$$\mathbb{E}[Pen|case_3] = X_1\rho_1 + \rho_3$$

$$Pr(case_3) = (1 - p)\delta$$

- Sender sends 1 bit is not flipped (stays 1). Penalty arises when detector declares it either 0 or \perp .

$$\mathbb{E}[Pen|case_4] = X_1\rho_4 + \rho_6$$

$$Pr(case_4) = (1 - p)(1 - \delta)$$

Also

$$\mathbb{E}[Pen] = \sum_i \mathbb{E}[Pen|case_i]Pr(case_i)$$

$$\mathbb{E}[Pen] = \rho_1(X_1\delta(1-p)) + \rho_2(X_1p(1-\delta)) + \rho_3(p(1-\delta) + \delta(1-p)) + \rho_4(X_1(1-p)(1-\delta)) + \rho_5(X_1p\delta) + \rho_6(p\delta + (1-p)(1-\delta)) \quad (3)$$

Now we want to minimize the objective function in 3. The parameters are $\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$. Moreover the parameters are constrained in the following ways:

$$\begin{aligned} \rho_1 + \rho_2 + \rho_3 &= 1 \\ \rho_4 + \rho_5 + \rho_6 &= 1 \end{aligned} \quad (4)$$

$$\begin{aligned} \rho_1 \geq 0, \quad \rho_2 \geq 0, \quad \rho_3 \geq 0 \\ \rho_4 \geq 0, \quad \rho_5 \geq 0, \quad \rho_6 \geq 0 \end{aligned} \quad (5)$$

We note that the inequality of ≤ 1 is implied when the two constraints 4 and 5 are met. It is quite easy to see that the above can be posed as a linear programming problem with 4 and 5 being the constraints $Ax = b$ and $x \geq 0$. Therefore we can be guaranteed the solution to the above minimization problem lies in on the corners of the vector space $\{\rho_i\}_{i=1}^6$.

Also from the simplex algorithm it can be directly seen that the solutions will be of the form such that one of $\rho_i|_{i=1}^3$ will be 1 and other two 0, and the same for the $\rho_i|_{i=4}^6$. Since our objective is to minimize we can simply choose the ρ_i whose coefficient is minimum. Therefore calculating

the coefficients at once gives us the decoding strategy at once. Also since the probabilities come out to be 0 or 1, the decoding strategy turns out to be deterministic.

Now we simply get the coefficient values for $X_1 = 2$ and $\delta = \frac{1}{8}$.

Table 1: ρ_i, coeff_i

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
coeff_i	$\frac{1-p}{4}$	$\frac{7p}{4}$	$\frac{6p+1}{8}$	$\frac{7(1-p)}{4}$	$\frac{p}{4}$	$\frac{7-6p}{8}$

For ρ_1, ρ_2, ρ_3 we note that when

- When $p \in [0, \frac{1}{8}]$, $\text{coeff}_2 \leq \text{coeff}_1$ and $\text{coeff}_2 \leq \text{coeff}_3$. Thus we choose $\rho_2 = 1$ and $\rho_1 = \rho_3 = 0$.
- When $p \in [\frac{1}{8}, 1]$, $\text{coeff}_1 \leq \text{coeff}_2$ and $\text{coeff}_1 \leq \text{coeff}_3$. Thus we choose $\rho_1 = 1$ and $\rho_2 = \rho_3 = 0$.

It is interesting to note that ρ_3 is always 0.

For ρ_4, ρ_5, ρ_6 we note that when

- When $p \in [0, \frac{7}{8}]$, $\text{coeff}_5 \leq \text{coeff}_4$ and $\text{coeff}_5 \leq \text{coeff}_6$. Thus we choose $\rho_5 = 1$ and $\rho_4 = \rho_6 = 0$.
- When $p \in [\frac{7}{8}, 1]$, $\text{coeff}_4 \leq \text{coeff}_5$ and $\text{coeff}_4 \leq \text{coeff}_6$. Thus we choose $\rho_4 = 1$ and $\rho_5 = \rho_6 = 0$.

Again ρ_6 is always 0.

This means that the decoder never chooses the \perp symbol.

Further we can get the decoding strategy for each of $p \in [0, 1]$.

- For $p \in [0, \frac{1}{8}]$

Table 2: $p \in [0, \frac{1}{8}]$

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
0	1	0	0	1	0

This means that when the decoder knows that $p \in [0, \frac{1}{8}]$ then it chooses the symbol to be 1 irrespective of what it sees.

- For $p \in [\frac{1}{8}, \frac{7}{8}]$

Table 3: $p \in [\frac{1}{8}, \frac{7}{8}]$

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
1	0	0	0	1	0

When decoder knows that $p \in [\frac{1}{8}, \frac{7}{8}]$ then it chooses the same symbol as it is appearing.

- For $p \in [\frac{7}{8}, 1]$

Table 4: $p \in [\frac{7}{8}, 1]$

ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
1	0	0	1	0	0

When decoder knows that $p \in [\frac{7}{8}, 1]$ then it chooses the symbol 0 irrespective of what it sees.