CS 753 : Automatic Speech Recognition Assignment 1 Problem 3

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1 Part A

We are given a machine which has internally two coins G, S. Probability of heads for Gold coin is $p_1 = \frac{1}{2}$ and probability of heads for Silver coin is $p_2 = \frac{1}{4}$.

Also it is given that when we press the button on the machine, it probabilistically chooses the Gold coin with probability ρ and the silver coin $1 - \rho$.

We are let $Z = \{G, S\}$ for Gold and Silver, and the $X = \{H, T\}$.
We then define $\gamma_\rho(z, x) = Pr(Z = z|X = x)$. We need to find γ_ρ for all values of $z \in \{G, S\}$ and $x \in \{H, T\}$.
Clearly there are only four possible cases.

- $z = G$ and $x = H$.

$$\gamma_\rho(G, H) = Pr(z = G|x = H) = \frac{Pr(z = G, x = H)}{Pr(x = H)}$$

$$Pr(x = H) = Pr(x = H|z = G)Pr(z = G) + Pr(x = H|z = S)Pr(z = S)$$

$$Pr(x = H) = p_1\rho + p_2(1 - \rho)$$

$$\gamma_\rho(G, H) = \frac{p_1\rho}{p_1\rho + p_2(1 - \rho)}$$

- $z = S$ and $x = H$

$$\gamma_\rho(S, H) = Pr(z = S|x = H) = \frac{Pr(z = S, x = H)}{Pr(x = H)} = \frac{Pr(x = H|z = S)Pr(z = S)}{Pr(x = H)}$$

$$\gamma_\rho(S, H) = \frac{p_2(1 - \rho)}{p_1\rho + p_2(1 - \rho)}$$

- $z = G$ and $x = T$

$$\gamma_\rho(G, T) = Pr(z = G|x = T) = \frac{Pr(z = G, x = T)}{Pr(x = T)}$$

$$Pr(x = T) = Pr(x = T|z = G)Pr(z = G) + Pr(x = T|z = S)Pr(z = S)$$

$$Pr(x = T) = (1 - p_1)\rho + (1 - p_2)(1 - \rho)$$

$$\gamma_\rho(G, T) = \frac{(1 - p_1)\rho}{(1 - p_1)\rho + (1 - p_2)(1 - \rho)}$$

- $z = S$ and $x = T$

$$\gamma_\rho(S, T) = Pr(z = S|x = T) = \frac{Pr(z = S, x = T)}{Pr(x = T)} = \frac{Pr(x = T|z = S)Pr(z = S)}{Pr(x = T)}$$

$$\gamma_\rho(S, T) = \frac{(1 - p_2)(1 - \rho)}{(1 - p_1)\rho + (1 - p_2)(1 - \rho)}$$

2 Part B

We note that ρ can be estimated using the Expectation Maximization (EM) algorithm from the set of observations $x_1...x_N$ ($x_i \in H, T$).
EM is an iterative algorithm and at each round it updates the current estimate of ρ to a new estimate ρ' which maximizes the expression 1

$$\mathcal{L}_\rho(\rho') = \sum_{i=1}^{N} \sum_{z\in\{G,S\}} \gamma_\rho(z, x_i) \cdot log(Pr(z, x_i; \rho')) \tag{1}$$

We would like to find

$$\operatorname*{argmax}_{\rho'} \mathcal{L}_\rho(\rho')$$

$$\mathcal{L}_\rho(\rho') = \sum_{i=1}^{N} [\gamma_\rho(G, x_i) \cdot log(Pr(G, x_i; \rho')) + \gamma_\rho(S, x_i) \cdot log(Pr(S, x_i; \rho'))] \tag{2}$$

$$\begin{aligned}\mathcal{L}_\rho(\rho') = &\sum_{i=1}^{N_H} [\gamma_\rho(G, H) \cdot log(Pr(G, H; \rho')) + \gamma_\rho(S, H) \cdot log(Pr(S, H; \rho'))] \\ &+ \sum_{j=1}^{N_T} [\gamma_\rho(G, T) \cdot log(Pr(G, T; \rho')) + \gamma_\rho(S, T) \cdot log(Pr(S, T; \rho'))]\end{aligned} \tag{3}$$

For convenience, we use a shortform for the variables given below:

$$c_1 = \gamma_\rho(G, H)$$
$$c_2 = \gamma_\rho(S, H)$$
$$c_3 = \gamma_\rho(G, T)$$
$$c_4 = \gamma_\rho(S, T)$$

Also we note

$$Pr(G, H; \rho') = Pr(H|G; \rho')Pr(G; \rho') = p_1\rho'$$
$$Pr(S, H; \rho') = Pr(H|S; \rho')Pr(S; \rho') = p_2(1 - \rho')$$
$$Pr(G, T; \rho') = Pr(T|G; \rho')Pr(G; \rho') = (1 - p_1)\rho'$$
$$Pr(S, T; \rho') = Pr(T|S; \rho')Pr(S; \rho') = (1 - p_2)(1 - \rho')$$

$$\mathcal{L}_\rho(\rho') = \sum_{i=1}^{N_H} [c_1 \cdot log(p_1\rho') + c_2 \cdot log(p_2(1-\rho'))] + \sum_{j=1}^{N_T} [c_3 \cdot log((1-p_1)\rho') + c_4 \cdot log((1-p_2)(1-\rho'))] \tag{4}$$

To get $\operatorname{argmax}_{\rho'} \mathcal{L}_\rho(\rho')$ we differentiate $\mathcal{L}_\rho(\rho')$ w.r.t. ρ'.

$$\frac{d\mathcal{L}_\rho(\rho')}{d\rho'} = \sum_{i=1}^{N_H} [\frac{c_1}{\rho'} - \frac{c_2}{1-\rho'}] + \sum_{j=1}^{N_T} [\frac{c_3}{\rho'} - \frac{c_4}{1-\rho'}] = 0 \tag{5}$$

$$[\frac{c_1}{\rho'} - \frac{c_2}{1-\rho'}]N_H + [\frac{c_3}{\rho'} - \frac{c_4}{1-\rho'}]N_T = 0 \tag{6}$$

Rearranging the equation we get

$$\rho' = \frac{c_1N_H + c_3N_T}{(c_1 + c_2)N_H + (c_3 + c_4)N_T} \tag{7}$$

From the definitions of c_1, c_2, c_3, c_4 we know $c_1 + c_2 = 1$ and $c_3 + c_4 = 1$. Therefore 7 reduces to

$$\rho' = \frac{c_1 N_H + c_3 N_T}{N_H + N_T} \tag{8}$$

Putting the values of c_1 and c_3 we get

$$\rho' = \frac{p_1\rho}{p_1\rho + p_2(1-\rho)} \cdot \frac{N_H}{N_H + N_T} + \frac{(1-p_1)\rho}{(1-p_1)\rho + (1-p_2)(1-\rho)} \cdot \frac{N_T}{N_H + N_T} \tag{9}$$

We can now simply put in the values of $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{4}$.

$$\rho' = \frac{2\rho}{1+\rho} \cdot \frac{N_H}{N_H + N_T} + \frac{2\rho}{3-\rho} \cdot \frac{N_T}{N_H + N_T} \tag{10}$$