CS 753 : Automatic Speech Recognition Assignment 1
Problem 1

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1 Part A

We are given that the probability of sending bit 0 is p and sending bit 1 is $1 - p$. It is also mentioned that the bits are independent and indentically distributed.
Also it is given that the channel is noisy and therefore a bit can be flipped with a probability of δ, and therefore remains unchanged with probability $1 - \delta$.
We need to find the probability that the k^{th} symbol received is 0.

Since the bits are independent the probability of k^{th} symbol is not dependent upon the other symbols. Also it can be assumed that the channel doesn't have a time delaying effect and therefore it affects only the considered bit with a probability given by δ.

The k^{th} symbol can be received as 0 in two ways:

- The sent bit was 0 and no flipping occurred. The probability of this happening is $P1 = p(1 - \delta)$.
- The sent bit was 1 and flipping occurred. The probability of this is $P2 = (1 - p)\delta$.

Since the two events are mutually exclusive, therefore the probabilities simply add up. Therefore

$$Pr(\mathrm{k^{th} symbol received} = 0) = p(1 - \delta) + (1 - p)\delta \tag{1}$$

2 Part B

A repetition code is used such that 0 is encoded as 000 and 1 as 111 and the decoder outputs a majority bit on the receiving 3-bit string.
We need to find the probability of correctly decoding 0.

It is easy to see that 0 will be correctly decoded when at most 1 bit is flipped. This can happen in one of the following ways:

- None of the bits get flipped. $Pr(\mathrm{Nobitflipped}) = (1 - \delta)^3$
- Exactly one of the bits gets flipped. $Pr(\mathrm{Exactlyonebitflipped}) = 3\delta(1 - \delta)^2$

Again the above are mutually exclusive, the probabilities add up. Therefore

$$Pr(\mathrm{0beingcorrectlydecoded}) = (1 - \delta)^3 + 3\delta(1 - \delta)^2 \tag{2}$$

3 Part C

Now it is said that each bit is transmitted unencoded. But the decoder is allowed to output $\{0, 1, \perp\}$.
If the decoder output is incorrect then the penalty points are 2 and if it outputs it to be $\perp$ then it gets penalty 1.
The aim is to find the best decoding strategy which can minimize the expected penalty. We are given $\delta = \frac{1}{8}$ and want the best strategy for $p \in [0, 1]$.

We first do the problem more generally and then impose the conditions given to us. We take that on output of $\perp$ the penalty is 1 and the output on incorrect bit the penalty is X_1. Also we don't put the value of δ till the last step.

The decoder either sees 0 or 1 and chooses on of the three options $\{0, 1, \perp\}$. We assign probabilities to each of the six cases.

	0	1	$\perp$
0	ρ_1	ρ_2	ρ_3
1	ρ_4	ρ_5	ρ_6

Let the expected penalty be denoted by $\mathbb{E}[Pen]$. Clearly there are four ways in which penalty can arise:

- Sender sends 0, bit is not flipped (stays 0). Penalty arises when detector declares it either 1 or $\perp$.
$$\mathbb{E}[Pen|case_1] = X_1\rho_2 + \rho_3$$
$$Pr(case_1) = p(1-\delta)$$
- Sender sends 0, bit is flipped (becomes 1). Penalty arises when detector declares it either 1 or $\perp$.
$$\mathbb{E}[Pen|case_2] = X_1\rho_5 + \rho_6$$
$$Pr(case_2) = p\delta$$
- Sender sends 1, bit is flipped (becomes 0). Penalty arises when detector declares it either 0 or $\perp$.
$$\mathbb{E}[Pen|case_3] = X_1\rho_1 + \rho_3$$
$$Pr(case_3) = (1-p)\delta$$
- Sender sends 1 bit is not flipped (stays 1). Penalty arises when detector declares it either 0 or $\perp$.
$$\mathbb{E}[Pen|case_4] = X_1\rho_4 + \rho_6$$
$$Pr(case_4) = (1-p)(1-\delta)$$

Also
$$\mathbb{E}[Pen] = \sum_i \mathbb{E}[Pen|case_i]Pr(case_i)$$

$$\mathbb{E}[Pen] = \rho_1(X_1\delta(1-p)) + \rho_2(X_1p(1-\delta)) + \rho_3(p(1-\delta)+\delta(1-p)) + \rho_4(X_1(1-p)(1-\delta)) + \rho_5(X_1p\delta) + \rho_6(p\delta+(1-p)(1-\delta)) \tag{3}$$

Now we want to minimize the objective function in 3. The parameters are $\{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$. Moreover the parameters are constrained in the following ways:

$$\begin{aligned} \rho_1 + \rho_2 + \rho_3 &= 1 \\ \rho_4 + \rho_5 + \rho_6 &= 1 \end{aligned} \tag{4}$$

$$\begin{aligned} \rho_1 \geq 0, \quad \rho_2 \geq 0, \quad \rho_3 \geq 0 \\ \rho_4 \geq 0, \quad \rho_5 \geq 0, \quad \rho_6 \geq 0 \end{aligned} \tag{5}$$

We note that the inequality of ≤ 1 is implied when the two constraints 4 and 5 are met. It is quite easy to see that the above can be posed as a linear programming problem with 4 and 5 being the constraints $Ax = b$ and $x \geq 0$. Therefore we can be guarenteed the solution to the above minimization problem lies in on the corners of the vector space $\{\rho_i\}_{i=1}^6$.

Also from the simplex algorithm it can be directly seen that the solutions will be of the form such that one of $\rho_i|_{i=1}^3$ will be 1 and other two 0, and the same for the $\rho_i|_{i=4}^6$. Since our objective is to minimize we can simply chooose the ρ_i whose coefficient is minimum. Therefore calculating the coefficients at once gives us the decoding strategy at once. Also since the probabilities come out to be 0 or 1, the decoding strategy turns out to be deterministic.

Now we simply get the coefficient values for $X_1 = 2$ and $\delta = \frac{1}{8}$.

Table 1: My caption

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
$coeff_i$	$\frac{1-p}{4}$	$\frac{7p}{4}$	$\frac{6p+1}{8}$	$\frac{7(1-p)}{4}$	$\frac{p}{4}$	$\frac{7-6p}{8}$

For ρ_1, ρ_2, ρ_3 we note that when

- When $p \in [0, \frac{1}{8}]$, $coeff_2 \leq coeff_1$ and $coeff_2 \leq coeff_3$. Thus we choose $\rho_2 = 1$ and $\rho_1 = \rho_3 = 0$.
- When $p \in [\frac{1}{8}, 1]$, $coeff_1 \leq coeff_2$ and $coeff_1 \leq coeff_3$. Thus we choose $\rho_1 = 1$ and $\rho_2 = \rho_3 = 0$.

It is interesting to note that ρ_3 is always 0.

For ρ_4, ρ_5, ρ_6 we note that when

- When $p \in [0, \frac{7}{8}]$, $coeff_5 \leq coeff_4$ and $coeff_5 \leq coeff_6$. Thus we choose $\rho_5 = 1$ and $\rho_4 = \rho_6 = 0$.
- When $p \in [\frac{7}{8}, 1]$, $coeff_4 \leq coeff_5$ and $coeff_4 \leq coeff_6$. Thus we choose $\rho_4 = 1$ and $\rho_5 = \rho_6 = 0$.

Again ρ_6 is always 0.

This means that the decoder never chooses the $\perp$ symbol.