Homework 1: graph signal processing

EE 771: Recent Topics in Analytical Signal Processing, Spring 2018
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Instructions: The homeworks are to be submitted in class on 13/02/18. Please do not copy. Copied homework (in full or part) will attract -5 marks for the entire submission. The negative marking is without a warning.

1. Consider a graph $\mathcal{G}$ with 7 nodes $\{1, 2, \ldots, 7\}$, and a weight matrix given by

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

   Let $\vec{f} = \{3, 1, 2, 2, 2, 1, 1\}$ be a signal defined on $\mathcal{G}$. Answer the following questions.

   (a) Find the orthonormal eigenvector basis $\vec{u}_1, \ldots, \vec{u}_7$, ordered by their eigenvalues, corresponding to the Laplacian $\mathcal{L} = D - W$.

   (b) Find the coordinates of $\vec{f}$ in this basis of eigenvectors.

   (c) Plot the number of zero-crossings of $\vec{u}_1, \ldots, \vec{u}_7$ as a function of their index.

   (d) Emulate ideal lowpass filtering of $\vec{f}$ by retaining only first four coefficients in $\vec{u}_1, \ldots, \vec{u}_7$ basis. Compare $\vec{f}$ against its filtered version.

2. Generate a 100 node Erdos-Renyi graph and attach a weight of 1 to every edge. (Pick the probability in ER graph as $p = 0.1$ and $p = 0.2$.) Let $\mathcal{L} = D - W$ be the Laplacian matrix. Find the orthonormal eigenvectors and corresponding (ordered) eigenvalues for the Laplacian matrix. Plot the number of zero crossings as a function of 1 to 100. Comment on the obtained result.

3. For the graph generated in the previous question, plot the eigenvalues for the normalized Laplacian $\tilde{\mathcal{L}} = I - D^{-1/2} W D^{-1/2}$. Comment on the result.

4. Consider the images in the file denoise.zip. For gull-noise.jpg and sfo-noise.jpg, use the graph signal processing based image denoising algorithm (using Tikhonov regularization) to find a denoised version of the images. Compare the PSNR before and after denoising, with respect to the provided original.

   Use $\theta = 0.1$, and two different values of $\kappa = 0, 1$.

5. Consider a bipartite graph with nodes $\{1, 2, \ldots, 6\}$ where the nodes $\{1, 2, 3\}$ and $\{4, 5, 6\}$ form the bipartitions. The weight matrix is given by

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2}$$

   Let $\vec{f} = \{1, 2, 3, 1, 2, 3\}$ defined on this graph. Verify the symmetry properties of the eigenvalues and eigenvectors of $\tilde{L}$. Then, find the downsampled versions of the signal $\vec{f}_L$ and $\vec{f}_H$.