EE 771 : Recent Topics in Analytical Signal Processing
Assignment 2

Arka Sadhu - 140070011

March 22, 2018

Q1

We are given the bandlimited field $g(x, y)$ as

$$g(x, y) = \sum_{l=-10}^{10} \sum_{k=-5}^{5} a[k, l]exp(j2\pi kx + j2\pi ly)$$

We are also given that we move along the path $y = \sqrt{2}x$ and this path is denoted by L. g is parametrized by time as

$$h(t) = g(t, \sqrt{(2)}t) \qquad 0 \leq t \leq 1/\sqrt{2}$$

1a

We need to find the degrees of freedom along the two axes of the 2d field g. First we consider along the x-axis i.e. y is a constant (here taken to be y_0) and x is variable.

$$g(x, y_0) = \sum_{l=-10}^{10} \sum_{k=-5}^{5} a[k, l]exp(j2\pi kx + j2\pi ly_0)$$

$$g(x, y_0) = \sum_{k=-5}^{5} exp(j2\pi kx) \sum_{l=-10}^{10} a[k, l]exp(j2\pi ly_0)$$

Let $\alpha[k] = \sum_{l=-10}^{10} a[k, l]exp(j2\pi ly_0)$

$$g(x, y_0) = \sum_{k=-5}^{5} exp(j2\pi kx)\alpha[k] \tag{1}$$

Therefore we can reconstruct $g(x, y_0)$ from the values of $\alpha[k]$ and 11 such values are required. Therefore degrees of freedom along x-axis is 11.

For y-axis we have constant x (say x_0) and variable y.

$$\beta[l] = \sum_{k=-5}^{5} a[k, l]exp(j2\pi kx_0)$$

$$g(x_0, y) = \sum_{l=-10}^{10} exp(j2\pi ly)\beta[l] \tag{2}$$

Again, we can reconstruct $g(x_0, y)$ from the values of $\beta[l]$ and 21 such values are required. Therefore degrees of freedom along y-axis is 21.

1b

We note that the representations in 1 and 2 readily suggest the bandlimitness of the 1d representation. For the case of $g(x, 0.5)$ we note that 1 directly implies that the maximum frequency requried would be when $k = +5$ or $k = -5$ both of which correspond to 5Hz. Also we note that this value is independent of the value of y_0. Similarly for the case of $g(0.25, y)$ we have maximum frequency of 10Hz. Therefore both 1d representations are bandlimited.

1c

We first note the formula for $h(t) = g(t, \sqrt{2}t)$ can be given as:

$$h(t) = \sum_{k=-5}^{k=5} \sum_{l=-10}^{l=10} a[k,l]exp(j2\pi kt + j2\pi l\sqrt{2}t) \qquad 0 \leq t \leq 1/\sqrt{2}$$

We further note that we can reconstruct $g(x, y)$ if we are given $a[k, l]$ for all $k = \{-5, -4...5\}$ and $l = \{-10, -9...10\}$. We also note that none of the complex exponentials overlap with the other, that is all the complex exponentials are distinct from each other. This is because the factor with l is an irrational number and therefore the sampled frequency are all distinct from each other. Moreover, we know that the complex exponentials form an orthogonal basis and therefore each of the $a[k, l]$ terms can be recovered.

This indeed proves that h captures all the degrees of freedom of g.

1d

As argued in 1c, h captures all the degrees of freedom in g. To recover g from h we need to extract out the values of $a[k, l]$. We note that we have access to the complete 1d signal $h(t)$.

Further using orthogonality property of complex exponentials we note:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n exp(j2\pi nt)$$

$$c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)exp(-j2\pi nt)dt$$

To recover g from h we do the following. For each $k = \{-5, -4...5\}$ and $l = \{-10, -9...10\}$ we get

$$a[k,l] = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)exp(-j2\pi t(k + l\sqrt{2})dt$$

.

Once we have $a[k, l]$, we can trivially find $g(x, y)$ using its definition.

Q2

We are given that $W(x)$ is a stationary noise process. We are also given that $S_W(\omega) \propto \frac{1}{\omega}$ for $|\omega| \geq \frac{\pi}{X}$ where X is the sampling distance.

From power spectral density theory we know that:

$$S_{W_s}(\omega) = \frac{1}{X} \sum_{k=-\infty}^{\infty} S_W(\omega - \frac{2\pi k}{X}) \qquad |\omega| \leq \frac{\pi}{X}$$

We note that the required variance of W_S is given by $\sigma^2 = \frac{1}{\pi} \int_0^{\infty} S_{W_s}(\omega)d\omega$

$$S_{W_s}(\omega) = \frac{1}{X} \sum_{k=-\infty}^{\infty} S_W(\omega - \frac{2\pi k}{X})$$

$$S_{W_s}(\omega) = \frac{S_W(\omega)}{X} + \frac{1}{X} \sum_{k \in \mathcal{Z}, k \neq 0} S_W(\omega - \frac{2\pi k}{X})$$

Next we note that $\hat{\omega} = |\omega - \frac{2\pi k}{X}| \geq \frac{\pi}{X}$ for all $|\omega| \leq \frac{\pi}{X}$, $k \neq 0$. Moreover, if $|\hat{\omega}| \geq \frac{\pi}{X}$ we have $S_W(\hat{\omega}) = \frac{\alpha}{|\hat{\omega}|}$. This futher implies for $k \geq 1$ we will have $S_W(\hat{\omega}) \geq S_W(|\frac{2\pi k}{X}|)$ and for $k \leq 1$ we will have $S_W(\hat{\omega}) \geq S_W(|\frac{2\pi(K+1)}{X}|)$.

Thus we can equivalently write:

$$S_{W_s}(\omega) = \frac{S_W(\omega)}{X} + \frac{1}{X}\alpha \sum_{k=1}^{\infty} \frac{X}{2\pi k} + \frac{X}{2\pi(k+1)}$$

$$S_{W_s}(\omega) \geq \frac{S_W(\omega)}{X} + \frac{\alpha}{2\pi} \sum_{k=2} \frac{1}{k}$$

Clearly, $S_{W_s}(\omega)$ diverges for all ω. Therefore $\sigma^2 = \int_0^\infty S_{W_s}(\omega)$ also diverges. Thus variance of the sampled noise is ∞.

Q3

We are given that $g(x, y)$ is an infinite support 2d field. Also let $\tilde{g}(\omega, \nu)$ be the 2d fourier transform of thefield g. Moreover, it is known that:

$$\tilde{g}(\omega, \nu) \propto \frac{1}{\omega^3} exp(-|\nu|) \qquad |\omega| > 1, |\nu| > 1$$

We need to find the direction (along x-axis or along y-axis) in which we will find minimum aliasing. We note that when we sample, the frequency component inside the box $[-\pi, \pi]x[-\pi, \pi]$ corresponds to the actual signal (since $\Delta_x = \Delta_y = 1$). Now when we sample along a particular direction (x or y axis), the function is repeated at the rate of sampling frequency along that axis. The contribution of the other copied frequency at intervals of π inside the box represents the aliasing and we would want to minimize it.

We take two cases : along x-axis and the other along y-axis.

Along x-axis: (y is fixed, sampled along x-axis). Denote aliasing in x-axis by A_x We assume that the **??** holds whenever $|\omega| > \pi$ or $|\nu| > \pi$

$$A_x = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}, k \neq 0} \tilde{g}(\omega - 2\pi k, \nu) d\omega d\nu$$

$$A_x = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{\alpha}{|\hat{\omega}|^3} exp(-|\nu|) d\omega d\nu$$

Here $\hat{\omega} = \omega - 2\pi k$ and $|\hat{\omega}| > \pi$, hence we can replace $\tilde{g}$ with the approximation.

$$A_x = 2(1 - exp(-\pi)) \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}, k \neq 0} \frac{\alpha}{|\hat{\omega}|^3} d\omega$$

.

$$A_x = 2(1 - exp(-\pi)) \frac{2\alpha}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{|1 - 2k|^2} - \frac{1}{|1 + 2k|^2}$$

The second term has a telescopic series summing up to 1

$$A_x = (1 - exp(-\pi)) \frac{2\alpha}{\pi^2}$$

Along y-axis: (x is fixed, sampled along y-axis).

$$A_y = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}, k \neq 0} \tilde{g}(\omega, \nu - 2\pi k) d\omega d\nu$$

We would need the value of $\tilde{g}(\omega, \nu)$ when $|\omega| \leq 1$ to compute this (as we can't directly use above formulation since the integral diverges for $\frac{1}{|\omega|^3}$).

Therefore from the given information, it is best to sample along the x-axis as we have guaranteed finite aliasing in that direction.

Q4

We are given a polynomial field $g(x) = a + bx + cx^2$ where $x \in [0,1]$. Denote the legendre polynomial of degree k by $p_k(x)$. We know that the legendre polynomials are orthogonal in $[-1,1]$ but the function g given to us is in $[0,1]$. So we have to first convert it into another polynomial with range $[-1,1]$.

We choose $h(x) = g(\frac{x+1}{2})$. Clearly the domain of definition for h is $[-1,1]$. Let $h(x) = \alpha + \beta x + \gamma x^2$ then we get:

$$\alpha = a + \frac{b}{2} + \frac{c}{2}$$

$$\beta = \frac{b}{2} + c$$

$$\gamma = \frac{c}{2}$$

Clearly, estimating a, b, c is equivalent to estimating α, β, γ. So we now focus on estimating the latter. Denote the coefficients of h in the legendre polynomial basis be given by $A[k]$. We can therefore write:

$$A[k] = \frac{2k+1}{2} \int_{-1}^{1} h(x) p_k(x) dx$$

We know approximate it using reimann sum (M point approximation):

$$A_R[k] = \frac{2k+1}{2} \sum_{i=1}^{M} h(\frac{2i}{M} - 1) p_k(\frac{2i}{M} - 1)$$

Here we have replaced $\frac{i}{M}$ with $\frac{2i}{M} - 1$ which is the transformation from g to h ($g(x) = h(2x-1)$).

We note that we do not have samples at $\frac{2i}{M} - 1$ rather at points $\hat{S}_i = 2S_i - 1$. So we estimate it using

$$\hat{A}[k] = \frac{2k+1}{2} \sum_{i=1}^{M} h(\hat{S}_i) p_k(\frac{2i}{M} - 1)$$

First case we don't consider noise. We want to estimate $A[k]$ from $\hat{A}[k]$ and we try to give a bound for the same. All thefollowing derivations are from the paper on location unaware mobile sensor by Animesh Kumar.

$$\mathbb{E}[|\hat{A}[k] - A[k]|^2] \leq 2\mathbb{E}[|\hat{A}[k] - A_R[k]|^2] + 2\mathbb{E}[|A_R[k] - A[k]|^2]$$

Consider the first term of RHS:

$$|\hat{A}[k] - A_R[k]| = |\frac{1}{M} \sum_{i=1}^{M} [h(\hat{S}_i) - h(\frac{2i}{M} - 1)] p_k(\frac{2i}{M} - 1)|$$

$$|\hat{A}[k] - A_R[k]|^2 \leq ||h||_\infty^2 \frac{1}{M} \sum_{i=1}^{M} |\hat{S}_i - (\frac{2i}{M} - 1)|^2 p_k(\frac{2i}{M} - 1)$$

$$|\hat{A}[k] - A_R[k]|^2 \leq ||h||_\infty^2 \frac{1}{M} \sum_{i=1}^{M} |\hat{S}_i - (\frac{2i}{M} - 1)|^2$$

The last step follows because $p_k(x)$ are bounded between [-1,1].

Following the result from the paper we get:

$$\mathbb{E}[|\hat{A}[k] - A_R[k]|^2] \leq \frac{C_1}{n}$$

Here n is the oversampling rate.

Now consider the second term of RHS:

$$|A_R[k] - A[k]| = |\frac{1}{M} \sum_{i=1}^{M} [h(\frac{2i}{M} - 1) p_k(\frac{2i}{M} - 1) - \int_{\frac{2i}{M} - 1)}^{\frac{2i+2}{M} - 1} h(x) p_k(x) dx]$$

For some constants $Z_{i,m} \in [\frac{2i}{M} - 1), \frac{2i+2}{M} - 1)]$ we have:

$$|A_R[k] - A[k]| = |\frac{1}{M}\sum_{i=1}^{M}[h(\frac{2i}{M} - 1)p_k(\frac{2i}{M} - 1) - h(Z_{i,m})p_k(Z_{i,m})dx]$$

$$|A_R[k] - A[k]| \leq \frac{1}{M}\sum_{i=1}^{M}|Z_{i,m} - \frac{i}{M}|||\frac{d}{dx}h(x)p_k(x)||_\infty$$

$$|A_R[k] - A[k]| \leq \frac{1}{M}\sum_{i=1}^{M}\frac{1}{M}||\frac{d}{dx}h(x)p_k(x)||_\infty$$

$$|A_R[k] - A[k]| \leq \frac{1}{M}||\frac{d}{dx}h(x)p_k(x)||_\infty$$

$$|\frac{d}{dx}h(x)p_k(x)| = |h(x)p_k'(x) + h'(x)p_k(x)| \leq |h(x)| + |h'(x)| \leq C_2$$

$$|A_R[k] - A[k]| \leq \frac{C_2}{M} \leq \frac{C_3}{(n-\lambda)}$$

$$\mathbb{E}[|A_R[k] - A[k]|^2] \leq \frac{C_4}{(n-\lambda)^2}$$

Therefore we get:

$$\mathbb{E}[|\hat{A}[k] - A[k]|^2] \leq 2\frac{C1}{n} + 2\frac{C4}{(n-\lambda)^2}$$

So we see if number of samples are increased the error decreases as an order of $\frac{1}{n}$ and approaches 0. Once $A[k]$ are known we can estimate the original coefficients of g as well.

For the case where additive noise exists, it will lead to an additional term corresponding to the averaged noise given by:

$$\mathbb{E}(|W_{avg}[k]|^2) = \mathbb{E}[|\frac{1}{M}\sum_{i=1}^{M}W(\hat{S}_i)p_k(\frac{2i}{M} - 1)|^2]$$

$$\mathbb{E}(|W_{avg}[k]|^2) = \mathbb{E}[\frac{1}{M^2}\sum_{i=1}^{M}|W(\hat{S}_i)|^2]$$

$$\mathbb{E}(|W_{avg}[k]|^2) = \mathbb{E}[\frac{\sigma^2}{M}] \leq \frac{C_5}{n-\lambda}$$

Even when this term is added the error still decreases as an order of $\frac{1}{n}$ and hence our previous claims still holds.