EE 771 : Recent Topics in Analytical Signal Processing Assignment 2

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Q1

We are given the bandlimited field $g(x,y)$ as

$$g(x,y) = \sum_{l=-10}^{10} \sum_{k=-5}^{5} a[k,l]exp(j2\pi kx + j2\pi ly)$$

We are also given that we move along the path $y = \sqrt{2}x$ and this path is denoted by L. g is parametrized by time as

$$h(t) = g(t, \sqrt(2)t) \qquad 0 \leq t \leq 1/\sqrt{2}$$

Let the samples taken along the path L be separated by distances of Δ_L. Also suppose that we shift this line above by Δ_p (only in the y-direction). Slope of the line is $\sqrt{2}$ and thus its projections of x-axis and y-axis are $\Delta_x = \Delta_L/\sqrt{3}$ and $\Delta_y = \Delta_p \Delta_L \sqrt{2}/\sqrt{3}$

1a

We need to find the degrees of freedom along the two axes of the 2d field g. First we consider along the x-axis i.e. y is a constant (here taken to be y_0) and x is variable.

$$g(x,y_0) = \sum_{l=-10}^{10} \sum_{k=-5}^{5} a[k,l]exp(j2\pi kx + j2\pi ly_0)$$

$$g(x,y_0) = \sum_{k=-5}^{5} exp(j2\pi kx) \sum_{l=-10}^{10} a[k,l]exp(j2\pi ly_0)$$

Let $\alpha[k] = \sum_{l=-10}^{10} a[k,l]exp(j2\pi ly_0)$

$$g(x,y_0) = \sum_{k=-5}^{5} exp(j2\pi kx)\alpha[k] \tag{1}$$

Therefore we can reconstruct $g(x,y_0)$ from the values of $\alpha[k]$ and 11 such values are required. Therefore degrees of freedom along x-axis is 11.

For y-axis we have constant x (say x_0) and variable y.

$$\beta[l] = \sum_{k=-5}^{5} a[k,l]exp(j2\pi kx_0)$$

$$g(x_0,y) = \sum_{l=-10}^{10} exp(j2\pi ly)\beta[l] \tag{2}$$

Again, we can reconstruct $g(x_0,y)$ from the values of $\beta[l]$ and 21 such values are required. Therefore degrees of freedom along y-axis is 21.

1b

We note that the representations in 1 and 2 readily suggest the bandlimitness of the 1d representation. For the case of $g(x, 0.5)$ we note that 1 directly implies that the maximum frequency requried would be when $k = +5$ or $k = -5$ both of which correspond to 5Hz. Also we note that this value is independent of the value of y_0. Similarly for the case of $g(0.25, y)$ we have maximum frequency of 10Hz. Therefore both 1d representations are bandlimited.

1c

We first note that $g(x, y)$ is bandlimited to $[-\rho_x, \rho_x]x[-\rho_y, \rho_y]$ where $\rho_x = 10\pi$ and $\rho_y = 20\pi$. For $h(t)$ to cover all degrees of freedom of $g(x, y)$ we need that there should be no aliasing in either dimension. That is, we need $\Delta_x \leq \pi/\rho_x$ and $\Delta_y \leq \pi/\rho_y$. That is:

$$\Delta/\sqrt{3} \leq 1/10$$

$$\Delta\sqrt{2}/\sqrt{3} \leq 1/20$$

The above reduces to $\Delta \leq \sqrt{3}/10$ and $\Delta \leq \sqrt{3}/(20\sqrt{2})$. Clearly the second inequality is stronger and therefore we need $\Delta \leq \sqrt{3}/(20 * \sqrt{2})$ so that $h(t)$ can capture all degrees of freedom of $g(x, y)$.

1d

As shown in above part, if $\Delta \leq \sqrt{3}/(20/\sqrt{2})$ then we can reconstruct $g(x, y)$ from $h(t)$.

Q2

We are given that $W(x)$ is a white noise process. We are also given that $S_W(\omega) \propto \frac{1}{\omega}$ for $|\omega| \geq \frac{\pi}{X}$ where X is the sampling distance.

From power spectral density theory we know that:

$$S_{W_s}(\omega) = \frac{1}{X}\sum_{k=-\infty}^{\infty} S_W(\omega - \frac{2\pi k}{X}) \qquad |\omega| \leq \tfrac{\pi}{X}$$

We note that the required variance of W_S is given by $\sigma^2 = S_{W_s}(0)$

$$S_{W_s}(0) = \frac{1}{X}\sum_{k=-\infty}^{\infty} S_W(\frac{2\pi k}{X})$$

$$\sigma^2 = S_{W_s}(0) = \frac{S_W(0)}{X} + \frac{1}{X}\sum_{k\in\mathcal{Z},k\neq 0} S_W(\frac{2\pi k}{X})$$

Further, we know, power spectral density is always positive. Also the power spectral density of a real valued process is a real and even function of frequency. Therefore $S_W(\omega) = S_W(-\omega)$.

This gives us:

$$\sigma^2 = \frac{S_W(0)}{X} + \frac{2}{X}\sum_{k=1}^{\infty} S_W(\frac{2\pi k}{X})$$

Using the fact that X is sufficiently small we get:

$$\sigma^2 = \frac{S_W(0)}{X} + \frac{2}{X}\sum_{k=1}^{\infty} \frac{\alpha X}{2\pi k}$$

$$\sigma^2 = \frac{S_W(0)}{X} + \frac{\alpha}{\pi}\sum_{k=1}^{\infty} \frac{1}{k}$$

Clearly, this divergest to ∞ and therefore the variance of sampled white noise is ∞.

Q3

Q4

We are given a polynomial field $g(x) = a + bx + cx^2$ where $x \in [0, 1]$. Denote the legendre polynomial of degree k by $p_k(x)$. We know that the legendre polynomials are orthogonal in $[-1, 1]$ but the function g given to us is in $[0, 1]$. So we have to first convert it into another polynomial with range $[-1, 1]$.

We choose $h(x) = g(\frac{x+1}{2})$. Clearly the domain of definition for h is $[-1, 1]$. Let $h(x) = \alpha + \beta x + \gamma x^2$ then we get:

$$\alpha = a + \frac{b}{2} + \frac{c}{2}$$

$$\beta = \frac{b}{2} + c$$

$$\gamma = \frac{c}{2}$$

Clearly, estimating a, b, c is equivalent to estimating α, β, γ. So we now focus on estimating the latter. Denote the coefficients of h in the legendre polynomial basis be given by $A[k]$. We can therefore write:

$$A[k] = \frac{2k+1}{2} \int_{-1}^{1} h(x) p_k(x) dx$$

We know approximate it using reimann sum (M point approximation):

$$A_R[k] = \frac{2k+1}{2} \sum_{i=1}^{M} h(\frac{2i}{M} - 1) p_k(\frac{2i}{M} - 1)$$

Here we have replaced $\frac{i}{M}$ with $\frac{2i}{M} - 1$ which is the transformation from g to h ($g(x) = h(2x - 1)$).

We note that we do not have samples at $\frac{2i}{M} - 1$ rather at points $\hat{S}_i = 2S_i - 1$. So we estimate it using

$$\hat{A}[k] = \frac{2k+1}{2} \sum_{i=1}^{M} h(\hat{S}_i) p_k(\frac{2i}{M} - 1)$$

First case we don't consider noise. We want to estimate $A[k]$ from $\hat{A}[k]$ and we try to give a bound for the same. All thefollowing derivations are from the paper on location unaware mobile sensor by Animesh Kumar.

$$\mathbb{E}[|\hat{A}[k] - A[k]|^2] \leq 2\mathbb{E}[|\hat{A}[k] - A_R[k]|^2] + 2\mathbb{E}[|A_R[k] - A[k]|^2]$$

Consider the first term of RHS:

$$|\hat{A}[k] - A_R[k]| = |\frac{1}{M} \sum_{i=1}^{M} [h(\hat{S}_i) - h(\frac{2i}{M} - 1)] p_k(\frac{2i}{M} - 1)|$$

$$|\hat{A}[k] - A_R[k]|^2 \leq ||h||_\infty^2 \frac{1}{M} \sum_{i=1}^{M} |\hat{S}_i - (\frac{2i}{M} - 1)|^2 p_k(\frac{2i}{M} - 1)$$

$$|\hat{A}[k] - A_R[k]|^2 \leq ||h||_\infty^2 \frac{1}{M} \sum_{i=1}^{M} |\hat{S}_i - (\frac{2i}{M} - 1)|^2$$

The last step follows because $p_k(x)$ are bounded between [-1,1].

Following the result from the paper we get:

$$\mathbb{E}[|\hat{A}[k] - A_R[k]|^2] \leq \frac{C_1}{n}$$

Here n is the oversampling rate.

Now consider the second term of RHS:

$$|A_R[k] - A[k]| = |\frac{1}{M}\sum_{i=1}^{M}[h(\frac{2i}{M}-1)p_k(\frac{2i}{M}-1) - \int_{\frac{2i}{M}-1)}^{\frac{2i+2}{M}-1} h(x)p_k(x)dx]$$

For some constants $Z_{i,m} \in [\frac{2i}{M}-1), \frac{2i+2}{M}-1)]$ we have:

$$|A_R[k] - A[k]| = |\frac{1}{M}\sum_{i=1}^{M}[h(\frac{2i}{M}-1)p_k(\frac{2i}{M}-1) - h(Z_{i,m})p_k(Z_{i,m})dx]$$

$$|A_R[k] - A[k]| \leq \frac{1}{M}\sum_{i=1}^{M}|Z_{i,m} - \frac{i}{M}|||\frac{d}{dx}h(x)p_k(x)||_\infty$$

$$|A_R[k] - A[k]| \leq \frac{1}{M}\sum_{i=1}^{M}\frac{1}{M}||\frac{d}{dx}h(x)p_k(x)||_\infty$$

$$|A_R[k] - A[k]| \leq \frac{1}{M}||\frac{d}{dx}h(x)p_k(x)||_\infty$$

$$|\frac{d}{dx}h(x)p_k(x)| = |h(x)p_k'(x) + h'(x)p_k(x)| \leq |h(x)| + |h'(x)| \leq C_2$$

$$|A_R[k] - A[k]| \leq \frac{C_2}{M} \leq \frac{C_3}{(n-\lambda)}$$

$$\mathbb{E}[|A_R[k] - A[k]|^2] \leq \frac{C_4}{(n-\lambda)^2}$$

Therefore we get:

$$\mathbb{E}[|\hat{A}[k] - A[k]|^2] \leq 2\frac{C1}{n} + 2\frac{C4}{(n-\lambda)^2}$$

So we see if number of samples are increased the error decreases as an order of $\frac{1}{n}$ and approaches 0. Once $A[k]$ are known we can estimate the original coefficients of g as well.

For the case where additive noise exists, it will lead to an additional term corresponding to the averaged noise given by:

$$\mathbb{E}(|W_{avg}[k]|^2) = \mathbb{E}[|\frac{1}{M}\sum_{i=1}^{M}W(\hat{S}_i)p_k(\frac{2i}{M}-1)|^2]$$

$$\mathbb{E}(|W_{avg}[k]|^2) = \mathbb{E}[\frac{1}{M^2}\sum_{i=1}^{M}|W(\hat{S}_i)|^2]$$

$$\mathbb{E}(|W_{avg}[k]|^2) = \mathbb{E}[\frac{\sigma^2}{M}] \leq \frac{C_5}{n-\lambda}$$

Even when this term is added the error still decreases as an order of $\frac{1}{n}$ and hence our previous claims still holds.