EE 771 : Recent Topics in Analytical Signal Processing Assignment 2

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Q1

We are given the bandlimited field $g(x, y)$ as

$$g(x, y) = \sum_{l=-10}^{10} \sum_{k=-5}^{5} a[k, l] exp(j2\pi kx + j2\pi ly)$$

We are also given that we move along the path $y = \sqrt{2}x$ and this path is denoted by L. g is parametrized by time as

$$h(t) = g(t, \sqrt{(}2)t) \qquad 0 \leq t \leq 1/\sqrt{2}$$

Let the samples taken along the path L be separated by distances of Δ. Slope of the line is $\sqrt{2}$ and thus its projections of x-axis and y-axis are $\Delta_x = \Delta/\sqrt{3}$ and $\Delta_y = \Delta\sqrt{2}/\sqrt{3}$

1a

We need to find the degrees of freedom along the two axes of the 2d field g. First we consider along the x-axis i.e. y is a constant (here taken to be y_0) and x is variable.

$$g(x, y_0) = \sum_{l=-10}^{10} \sum_{k=-5}^{5} a[k, l] exp(j2\pi kx + j2\pi ly_0)$$

$$g(x, y_0) = \sum_{k=-5}^{5} exp(j2\pi kx) \sum_{l=-10}^{10} a[k, l] exp(j2\pi ly_0)$$

Let $\alpha[k] = \sum_{l=-10}^{10} a[k, l] exp(j2\pi ly_0)$

$$g(x, y_0) = \sum_{k=-5}^{5} exp(j2\pi kx)\alpha[k] \tag{1}$$

Therefore we can reconstruct $g(x, y_0)$ from the values of $\alpha[k]$ and 11 such values are required. Therefore degrees of freedom along x-axis is 11.

For y-axis we have constant x (say x_0) and variable y.

$$\beta[l] = \sum_{k=-5}^{5} a[k, l] exp(j2\pi kx_0)$$

$$g(x_0, y) = \sum_{l=-10}^{10} exp(j2\pi ly)\beta[l] \tag{2}$$

Again, we can reconstruct $g(x_0, y)$ from the values of $\beta[l]$ and 21 such values are required. Therefore degrees of freedom along y-axis is 21.

1b

We note that the representations in 1 and 2 readily suggest the bandlimitness of the 1d representation. For the case of $g(x, 0.5)$ we note that 1 directly implies that the maximum frequency requried would be when $k = +5$ or $k = -5$ both of which correspond to 5Hz. Also we note that this value is independent of the value of y_0. Similarly for the case of $g(0.25, y)$ we have maximum frequency of 10Hz. Therefore both 1d representations are bandlimited.

1c

We first note that $g(x, y)$ is bandlimited to $[-\rho_x, \rho_x]x[-\rho_y, \rho_y]$ where $\rho_x = 10\pi$ and $\rho_y = 20\pi$. For $h(t)$ to cover all degrees of freedom of $g(x, y)$ we need that there should be no aliasing in either dimension. That is, we need $\Delta_x \leq \pi/\rho_x$ and $\Delta_y \leq \pi/\rho_y$. That is:

$$\Delta/\sqrt{3} \leq 1/10$$

$$\Delta\sqrt{2}/\sqrt{3} \leq 1/20$$

The above reduces to $\Delta \leq \sqrt{3}/10$ and $\Delta \leq \sqrt{3}/(20\sqrt{2})$. Clearly the second inequality is stronger and therefore we need $\Delta \leq \sqrt{3}/(20 * \sqrt{2})$ so that $h(t)$ can capture all degrees of freedom of $g(x, y)$.

1d

As shown in above part, if $\Delta \leq \sqrt{3}/(20/\sqrt{2})$ then we can reconstruct $g(x, y)$ from $h(t)$.

Q2

We are given that $W(x)$ is a white noise process. We are also given that $S_W(\omega) \propto \frac{1}{\omega}$ for $|\omega| \geq \frac{\pi}{X}$ where X is the sampling distance.

From power spectral density theory we know that:

$$S_{W_s}(\omega) = \frac{1}{X} \sum_{k=-\infty}^{\infty} S_W(\omega - \frac{2\pi k}{X}) \qquad |\omega| \leq \tfrac{\pi}{X}$$

We note that the required variance of W_S is given by $\sigma^2 = S_{W_s}(0)$

$$S_{W_s}(0) = \frac{1}{X} \sum_{k=-\infty}^{\infty} S_W(\frac{2\pi k}{X})$$

$$\sigma^2 = S_{W_s}(0) = \frac{S_W(0)}{X} + \frac{1}{X} \sum_{k \in \mathcal{Z}, k \neq 0} S_W(\frac{2\pi k}{X})$$

Further, we know, power spectral density is always positive. Also the power spectral density of a real valued process is a real and even function of frequency. Therefore $S_W(\omega) = S_W(-\omega)$.

This gives us:

$$\sigma^2 = \frac{S_W(0)}{X} + \frac{2}{X} \sum_{k=1}^{\infty} S_W(\frac{2\pi k}{X})$$

Using the fact that X is sufficiently small we get:

$$\sigma^2 = \frac{S_W(0)}{X} + \frac{2}{X} \sum_{k=1}^{\infty} \frac{\alpha X}{2\pi k}$$

$$\sigma^2 = \frac{S_W(0)}{X} + \frac{\alpha}{\pi} \sum_{k=1}^{\infty} \frac{1}{k}$$

Clearly, this divergest to ∞ and therefore the variance of sampled white noise is ∞.

Q3

Q4