EE 771 : Recent Topics in Analytical Signal Processing Assignment 2

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Q1

Results are taken from the paper : M. Vetterli, P. Marziliano, and T. Blu, "Sampling Signals with Finite Rate of Innovation", IEEE Trans. on Signal Processing, Jun 2002.

We are given a 1 dimensional periodic signal $x(t)$ with period $T = 1$ as:

$$x(t) = a_1 u(t - t_1) + a_2 u(t - t_2) - (a_1 + a_2) u(t - t_3)$$

with $0 < t_1 < t_2 < t_3$

1a

We want to find the number of samples of $x(t) * sinc(Bt)$ which are sufficient for the reconstruction of $x(t)$ and a suitable value for B. We note that $x(t)$ is an example of a non-uniform spline and therefore we can directly use Theorem 2 from the paper.

The minimum number of samples required are $N = 2M + 1$ where $M = \lfloor \frac{B\tau}{2} \rfloor$. We need $B \geq \rho$ where ρ is the rate of innovation. In this case $\rho = \frac{2K}{\tau}$ and $K = 3$, $\tau = 1$. Therefore $\rho = 6$. But we note that $c_1 + c_2 + c_3 = 0$ which reduces one degree of freedom and hence $\tau = 5$. Thus, we need $B \geq 5Hz$. Choosing $B = 5Hz$ gives us $M = 2$ and $N = 5$, i.e. we can reconstruct the signal given 5 samples and choosing $B = 5Hz$.

1b

We are now given a new 1d periodic signal $x_1(t)$ with the same period $T = 1$ as:

$$x_1(t) = x(t) + b_1 \delta(t - t_1) - b_1 \delta(t - t_3)$$

Clearly, this is an example of the stream of derivatives of dirac deltas and we can use Theorem 3 from the paper. First we find the rate of innovation. The degrees of freedom increases by only from the previous part. This gives us $\rho = \frac{6}{1} = 6Hz$. Choosing $B = 6Hz$ we get $M = 3$ and correspondingly $N = 7$.

Q2

We want to prove that the Yule Walker system in the algorithm mentioned in the paper is invertible.

Denote the Yule Walker system matrix as A. We consider a $3x3$ matrix and note that the proof can be easily extended to any other nxn matrix.

$$A = \begin{bmatrix} X[0] & X[-1] & X[-2] \\ X[1] & X[0] & X[-1] \\ X[2] & X[1] & X[0] \end{bmatrix}$$

Here $X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k exp(-i2\pi m t_k/\tau)$. Denote $u_k = exp(-i2\pi t_k/\tau)$. We can re-write $X[m] = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k u_k^m$. In this case $K = 3$. Also it is clear that the value of τ wouldn't make a difference in the invertibility of the matrix A. Thus we can write the following:

$$X[0] = c_1 + c_2 + c_3$$

$$X[1] = c_1u_1 + c_2u_2 + c_3u_3$$
$$X[2] = c_1u_1^2 + c_2u_2^2 + c_3u_3^2$$
$$X[-1] = c_1{u_1}^{-1} + c_2{u_2}^{-1} + c_3{u_3}^{-1}$$
$$X[-2] = c_1{u_1}^{-2} + c_2{u_2}^{-2} + c_3{u_3}^{-2}$$

We further note that we can write A as $A = [A_1c|A_2c|A_3c]$. Here:

$$A_3 = \begin{bmatrix} u_1^{-2} & u_2^{-2} & u_3^{-2} \\ u_1^{-1} & u_1^{-1} & u_3^{-1} \\ 1 & 1 & 1 \end{bmatrix}$$

$$A_2 = UA_3$$
$$A_1 = U^2A_3$$

$$U = \begin{bmatrix} u1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix}$$

Moreover, A_3 is a permutation of a vander monde matrix. Therefore A_3 is invertible and therefore is non-singular. Now denote the determinant of A by $det(A)$. We have

$$det(A) = det(A_3[U^2c|Uc|c]) = det(A_3)det([U^2c|Uc|c])$$

The first term on the rhs is non-zero since A_3 is non-singular. The second term on the rhs is

$$det([U^2c|Uc|c]) = c_1c_2c_3det(B)$$

$$B = \begin{bmatrix} u1^2 & u1 & 1 \\ u2^2 & u2 & 1 \\ u3^2 & u3 & 1 \end{bmatrix}$$

Clearly B is also a vander monde matrix and therefore, B is also non-singular. Also, c_1, c_2, c_3 are also non-zero (else there will be no diracs at those places and the dimension of the matrix will reduce). Therefore we have:

$$det(A) \neq 0$$

Consequently, we have proved that A is invertible.

Q3

Paper followed : M. Vetterli, P. Marziliano, and T. Blu, "Sampling Signals with Finite Rate of Innovation", IEEE Trans. on Signal Processing, Jun 2002. We are given u_1, u_2 as the roots of unity. We want to construct the annihilation filter for the Fourier Series coefficients

$$X[m] = \sum_{r=0}^{3} c_r m^r u_1^m + \sum_{r=0}^{1} d_r m^r u_2^m$$

This can be re-written as:

$$X[m] = u_1^m(c_0 + c_1m + c_2m^2 + c_3m^3) + u_2^m(d_0 + d_1m)$$
$$X[m] = u_1^m * poly(m, 3) + u_2^m * poly(m, 1)$$

Here $poly(m, r)$ denotes a polynomial in m of degree r. The annihilation filter $A(z)$ can be constructed in the following way (as noted in the paper):

- For $u_1^m poly(m, 3)$ we need the annihilation filter $A_1(z) = (1 - u_1z^{-4})$ (as stated in the paper). This is because, say $A_1(z) = \sum_{l=0}^{4} A_1[l]z^{-l}$, then $A_1^{(n)}(u_1) = 0 = \sum_{l=0}^{4} A_1[l] * l * (l-1) * .. * (l-n+1) * z^{-l}$ and this would be true for all $n \leq 3$. Thus $\sum_{l=0}^{4} A_1[l]P[l]u_1^{-l} = 0$. Here $P[l]$ is any polynomial of degree less than 3. Moreover, this is smallest possible annihilation filter as we need to annihilate polynomials of degree 3.

- Similarly, to annihilate $u_2^m poly(m,1)$ we require $A_2(z) = (1-u_2z^{-1})^2$. This is the smallest filter to annihilate the components of u_2.
- To annihilate the sum we simply take the product of the two filters. $A(z) = A_1(z)A_2(z)$.

$$A(z) = (1-u_1z^{-1})^4(1-u_2z^{-1})^2$$

Q4

Paper followed : A Generalized Sampling Method for Finite-Rate-of-Innovation-Signal Reconstruction by Chandra Sekhar Seelamantula, Member, IEEE, and Michael Unser, Fellow, IEEE.

We are given

$$x(t) = \sum_{k=1}^{K} b_k \delta(t-t_k)$$

Here all t_k are in $(0,1)$. Two RC filters with values (R_1, C_1) and (R_2, C_2) are used in parallel to filter out $x(t)$. We are also given the impulse response of the RC filters as:

$$h_i = exp(-R_iC_i)u(t)$$

where $i = 1, 2$ and $u(t)$ is the unit time step signal. We need to find the conditions on sampling interval T so that the sampled signal $x(t) * h_i(t)|_{t=nT}$ are sufficient to reconstruct the parameters of $x(t)$.

Denote $\alpha_i = R_iC_i$

$$y_i(t) = x * h_i(t) = \sum_{k=1}^{K} b_k exp(-\alpha_i(t-t_k))u(t-t_k)$$

.

Let the sampling period be T. Then we get:

$$y_i(nT) = \sum_{k=1}^{K} b_k exp(-\alpha_i(nT-t_k))u(nT-t_k), n \in \mathcal{Z}$$

Now we consider a discrete time finite impulse response filter specified by the Z-Transform:

$$G_i(z) = (1 - exp(-\alpha_i T)z^{-T})$$

It is noted that $G_i(z)$ is the convolutional inverse of the discrete time exponential $exp(-\alpha nT)u(nT)$.

Therefore, when the sequence $y_i[n] = y_i(nT)$ is processed by $G_i(z)$ it gives rise to a stream of kronecker impulses.

$$p_i(nT) = \sum_{k=1}^{K} b_k exp(-\alpha_i nT + \alpha_i t_k)(u(nT-t_k) - u((n-1)T - t_k))$$

$$p_i(nT) = \sum_{k=1}^{K} b_k exp(-\alpha_i r(t_k)T + \alpha_i t_k)\delta[n - r(t_k)]$$

Here $r(t_l) = \lceil t_k/T \rceil$ which indicates the ceiling operation.

If we assume that there is at most one dirac impulse in a sampling interval then the amplitude of the kronecker delta signal carries information about the position as well as the amplitude of the corresponding dirac impulse in a separable fashion. The condition boils down to

$$\min_{2 \le k \le K} \{t_k - t_{k-1}\} > T$$

Now we note that the k^{th} non zero value in $p_i(nT)$ (i =1, 2) occur at the time instant $r(t_k)T$ just after k^{th} dirac impulse has excited the respective analog system. We compute for i=1,2:

$$q_i[k] = p_i(r(t_k)T)exp(\alpha_i r(t_k)T) = b_k exp(\alpha_i t_k)$$

We solve the above (two equations for i=1,2) and get:

$$t_k = \frac{1}{\alpha_1 - \alpha_2} ln(\frac{q_1[k]}{q_2[k]}))$$

$$a_k = exp(-\frac{\alpha_1}{\alpha_1 - \alpha_2} ln(\frac{q_1[k]}{q_2[k]}))$$