CMU Airlab Problem Statement

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Problem Statement

Q1. Name one advantage and one disadvantage of using Euler angles (e.g. Roll-Pitch-Yaw), unit quaternions, rotation matrices and axis/angle representations of rotations.
A1.

- Euler Angles:
  - Advantages:
    - Only 3 parameters need to be stored.
    - It is more human understandable and good for decomposing rotations into individual degrees of freedom.
  - Disadvantages:
    - Interpolation is difficult.
    - There are ambiguities in the order of Matrix Rotations.
    - If the middle rotation is by $90°$, it results in a Gimbal Lock.
- Unit Quaternions:
  - Advantages:
    - There is no ambiguity, no gimbal lock, and interpolation can be done using slerp, producing smooth rotations.
    - It requires only 4 parameters to be stored.
    - Even translations can be accomodated using Dual Quaternions.
    - Computation is efficient for computers.
    - Consecutive rotation is simply multiplication of two quternions.
  - Disadvantages:
    - In general it is not easy to visualize Quaternions, and need to be converted to Axis Angle Representation.
    - The Quaternion-Algebra is quite involved.
    - There exists redundancy, ie two quaternions may imply the same rotation.
- Rotation Matrices
  - Advantages:
    - Rotations can be easily concatenated.
    - They directly give the new axis of rotations, which may be required in some applications.

* With homogenous coordinates it is very easy to incorporate Translation too.

- Disadvantages:
  * Rotation Matrices need 9 parameters to be stored.
  * The matrices must be orthogonal, and often due to floating point inaccuracy, they do not remain orthogonal, and the elements of the matrix may need to be recomputed.

- Axis Angle Representation
  - Advantages:
    * Very easy to visualize.
    * Can be very easily converted to quaternions.
  - Disadvantages:
    * If $\theta = 0$ then axis is arbitrary.
    * For θ and $\theta + 2 * k * \pi$ produce the same result.

Q2. Let $R^a = (\theta^a_{roll}, \theta^a_{pitch}, \theta^a_{yaw})$ and $R^b = (\theta^b_{roll}, \theta^b_{pitch}, \theta^b_{yaw})$ be a rotation corresponding to the following Roll-Pitch-Yaw angles (ZYX conventions):

$$\theta^a_{roll} = \pi/4 \qquad \theta^b_{roll} = -\pi/3$$

$$\theta^a_{pitch} = 0 \qquad \theta^b_{roll} = 0$$

$$\theta^a_{yaw} = \pi/3 \qquad \theta^b_{roll} = 0$$

Compute 3x3 rotation matrices corresponding to R^a and R^b. Then compare R^aR^b and R^bR^a and give physical meaning to these two.
A2.

$$R^a = \begin{bmatrix} 0.707107 & -0.353553 & 0.612372 \\ 0.707107 & 0.353553 & -0.612372 \\ 0 & 0.866025 & 0.5 \end{bmatrix}$$

$$R^b = \begin{bmatrix} 0.5 & 0.866025 & 0 \\ -0.866025 & 0.5 & -0 \\ -0 & 0 & 1 \end{bmatrix}$$

$$R^aR^b = \begin{bmatrix} 0.65974 & 0.435596 & 0.612372 \\ 0.0473672 & 0.789149 & -0.612372 \\ -0.75 & 0.433013 & 0.5 \end{bmatrix}$$

$$R^bR^a = \begin{bmatrix} 0.965926 & 0.12941 & -0.224144 \\ -0.258819 & 0.482963 & -0.836516 \\ 0 & 0.866025 & 0.5 \end{bmatrix}$$

We note that even though matrices are not commutative in general, they are associative. Let $R^{ab} = R^aR^b$. Clearly R^{ab} is inturn a Rotation matrix. When R^{ab} is applied to some other matrix,i.e. pre-multiplied to a matrix (say $R^{ab} * M$) the multiplication can be viewed in two steps, first multiplication with R^b and then with R^a. Similarly with R^{ba}, the transformation can be viewed as successive rotations.

Since rotations in 3D are in general not commutative, here too, $R^{ab} \neq R^{ba}$.

Q3. Compute the quaternions q^a and q^b equivalent to the matrices R^a and R^b . Are these

quaternions a unique representation of these rotations?
A3.
Representing Quaternions in the form of [w x y z]

$$q^a = \begin{bmatrix}0.800103 & 0.46194 & 0.191342 & 0.331414\end{bmatrix}$$

$$q^b = \begin{bmatrix}0.866025 & 0 & 0 & -0.5\end{bmatrix}$$

For Quaternions we note that q and $-q$ both give the same rotation. Hence these Quaternions are not a unique representation of these rotations. Other than this the relationship is unique.

Q4. Compute two compositions of the rotations using quaternions, $q^c = q^a q^b$ and $q^d = q^b q^a$. Are q^c and q^d the same? Compute the relative rotation between q^a and q^b as $q^e = q^a(q^b)^{-1}$. Then compute the composition $q^f = q^e q^b$. Verify that q^f and q^a are the same.
A4.

$$q^c = \begin{bmatrix}0.858616 & 0.304381 & 0.396677 & -0.113039\end{bmatrix}$$

$$q^d = \begin{bmatrix}0.858616 & 0.495722 & -0.0652631 & -0.113039\end{bmatrix}$$

As expected q^c and q^d are not the same, since rotations in 3d are in general not commutative.

$$q^e = \begin{bmatrix}0.527203 & 0.495722 & -0.0652631 & 0.687064\end{bmatrix}$$

$$q^f = \begin{bmatrix}0.800103 & 0.46194 & 0.191342 & 0.331414\end{bmatrix}$$

As expected $q^f = q^a$.

Q5. How does axis/angle representation relate to quaternion? For example, how to convert between quaternion q = (x, y, z, w) and angle/axis (n,)? Is the conversion unique or not? If not, how many?
A5.
Considering the representation of the quaternion as $q = [v \quad s]$, where v is the complex part and s is the scalar part. Therefore $v = [x \quad y \quad z]$ and $s = w$.

The axis angle and quaternions are closely related. Assumption is that the quaternion is a unit quaternion.

- Quaternion $[xyzw]$ to Angle Axis (n, θ):

$$\theta = 2 * \cos^{-1} w$$
$$n_x = \frac{x}{\sqrt{1 - w^2}}$$
$$n_y = \frac{y}{\sqrt{1 - w^2}}$$
$$n_z = \frac{z}{\sqrt{1 - w^2}}$$

- Angle Axis to Quaternion:

$$w = \cos\frac{\theta}{2}$$
$$x = \sin\frac{\theta}{2}$$
$$y = \sin\frac{\theta}{2}$$
$$z = \sin\frac{\theta}{2}$$

Except for the fact that anti-podal Quaternions represent the same rotation there is no other ambiguity, i.e. q and $-q$ represent the same axis angle rotation. But this is to be expected since (n, θ) and $(-n, -\theta)$ also represent the same rotation.

Q6. Assuming Euler angles are small, it is very useful to know the approximation: $R \approx I + [\omega]$, where $[.]_x$ is the skew-symmetric operator and ω is the small rotation. Now can you write down $\frac{\partial Rv}{\partial w}|_{\omega=0}$ where $v \in R^3$

A6.

Given that $R \approx I + [\omega]$ we deduce :

$$Rv \approx v + [\omega]_x v$$

$$Rv \approx v + \omega \times v$$

$$Rv \approx v - v \times \omega$$

$$Rv \approx= v - [v]_x \omega$$

Now we can differentiate easily, using the fact

$$\frac{d(a^T x)}{dx} = a^T$$

Hence we get

$$\frac{\partial Rv}{\partial w}|_{\omega=0} = -[v]_x$$