Linear Algebra 1—Exercise 4: Vector Spaces (I)—Definition, Examples, Subspaces

To submit: 1 b, c, g, k, n   3 b, d   4 b   5 b   6 b   7 b, c   8 a   10   12

1) Check if each set is a subspace of the vector space of which it is a subset. If it is, prove it; if it isn't, bring a counterexample.

   a. $\left\{ s\begin{pmatrix}2\\2\\0\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} : s,t \in \Re \right\}$

   b. $\left\{ t\begin{pmatrix}2\\-1\\1\end{pmatrix} : t \in \Re \right\}$

   c. $\left\{ t\begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}1\\4\end{pmatrix} : t \in \Re \right\}$

   d. $\{ s\begin{pmatrix}2\\1\\1\end{pmatrix} + t\begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\3\\1\end{pmatrix} \mid s,t \in \mathbb{R}\}$

   e. $\{ax^2 + bx + 2 : a,b \in \mathbb{R}\}$

   f. $\left\{ t^5\begin{pmatrix}1\\1\\2\end{pmatrix} : t \in \Re \right\}$

   g. The set of solutions of the system $\left\{\begin{matrix} x_1 + 4x_2 + 3x_3 + x_4 = 0 \\ 4x_1 + 16x_2 + 12x_3 = 0 \end{matrix}\right\}$..

   h. $\{ax^2 + bx : a,b \in \Re\}$

   i. $\left\{ t^2\begin{pmatrix}1\\-1\\0\end{pmatrix} : t \in \Re \right\}$

   j. The set of solutions of the system $\left\{\begin{matrix} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 3x_2 + 4x_3 = 2 \end{matrix}\right\}$.

   k. The set of solutions of the system $x_1{x_2}^3 + x_3 = 0$.

   l. $\{f \in F[0,1]: \quad f(0.25) = 0\}$, where $F[0,1]$ is the set of all functions from the interval $[0,1]$ to $\mathbb{R}$ where addition and scalar multiplication are defined as in question 6.

   m. The set $K = \{(x,y) \in R^2 : |x| = |y|\}$ as a subset of $\mathbb{R}^2$.

   n. $\left\{ s\begin{pmatrix}2\\0\\1\end{pmatrix} + t\begin{pmatrix}2\\-1\\0\end{pmatrix} : s,t \in \Re \right\}$

2)

   a. Let $U = \{p(x) \in \Re_4[x] | p(2) = 4p(3)\}$. Prove that U is a subspace of $\mathbb{R}_4[x]$.

   b. Let W be the set of 3x3 matrices such that the sum of the second and third rows is zero. Prove that W is a subspace of the space of all 3x3 matrices.

3) Is U a subspace of $\mathbb{R}_3[x]$? Prove or give a counterexample.

a. $U=\{a_0+a_1x+a_2x^2+a_3x^3 \mid a_2=0\}$

b. $U=\{a_0+a_1x+a_2x^2+a_3x^3|a_2=5\}$

c. $U=\{a_0+a_1x+a_2x^2+a_3x^3|a_2=4a_3\}$

d. $U=\{a_0+a_1x+a_2x^2+a_3x^3|a_1={a_0}^2\}$

4) In each part, U_1 and U_2 are subspaces of the same vector space V. Show that the union $U_1 \cup U_2$ is **not** a subspace of V.

a. $V=\mathbb{R}^2$, $U_1=\left\{t\begin{pmatrix}1\\0\end{pmatrix}|t\in R\right\}$, $U_2=\left\{t\begin{pmatrix}0\\-2\end{pmatrix}|t\in R\right\}$

b. $V=\mathbb{R}^4$, $U_1=\{\vec{x}\in\Re^4|x_4=x_3\}$, $U_2=\{\vec{x}\in\Re^4|x_2=0\}$

c. $V=\mathbb{R}_2[x]$, $U_1=\{a_0+a_1x+a_2x^2 \mid a_0=0\}$, $U_2=\{a_0+a_1x+a_2x^2 \mid a_1=0\}$

5)

a. Given a vector space V and two subspaces U_1 and U_2, prove that the union $U_1 \cup U_2$ is a subspace if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

b. Find an example of two subspaces U_1 and U_2 of the same vector space V such that the union $U_1 \cup U_2$ is a subspace of V.

6) Let $F(\mathbb{R})$ denote the set of functions from $\mathbb{R}$ to itself. Define addition on $F(\mathbb{R})$ as follows: For $f,g\in F(\mathbb{R})$, the function $f+g$ is defined by $(f+g)(x)=f(x)+g(x)$ for all $x\in\mathbb{R}$. Scalar multiplication is defined as follows: For $f,g\in F(\mathbb{R})$ and $\alpha\in\mathbb{R}$, the function αf is defined by $(\alpha f)(x)=\alpha\cdot f(x)$ for all $x\in\mathbb{R}$.
For $a\in\mathbb{R}$, let $C(a)$ denote the set of all functions in $F(\mathbb{R})$ which are **continuous** at a.

a. Prove that $C(a)$ is a subspace of $F(\mathbb{R})$.

b. Let $D(a)$ denote the set of all functions in $F(\mathbb{R})$ which are **differentiable** at a. Prove that $D(a)$ is a subspace of $C(a)$. (You may use results from calculus.)

c. Prove that $W=\{f\in D(a) \mid f'(a)=0\}$ is a subspace of $D(a)$.

7) In each part, determine if $\underline{w}$ belongs to the subspace spanned by $\{\underline{v}_1,\underline{v}_2\}$.

a. (in $\mathbb{R}^2$) $\underline{v}_1=\begin{pmatrix}-1\\4\end{pmatrix}$, $\underline{v}_2=\begin{pmatrix}3\\6\end{pmatrix}$, $\underline{w}=\begin{pmatrix}2\\1\end{pmatrix}$

b. (in $\mathbb{R}^3$) $\underline{v}_1=\begin{pmatrix}0\\-2\\-3\end{pmatrix}$, $\underline{v}_2=\begin{pmatrix}2\\1\\6\end{pmatrix}$, $\underline{w}=\begin{pmatrix}1\\2\\1\end{pmatrix}$

c. (in $\mathbb{R}_2[x]$) $\underline{v}_1=1-x$, $\underline{v}_2=5+x^2$, $\underline{w}=x^2+2x+1$

d. (in $\mathbb{R}_3[x]$) $\underline{v}_1=-x^3+x^2+2x+1$, $\underline{v}_2=x^2-3x-1$, $\underline{w}=x^2+x+1$

8) Prove or disprove:

a. $V=\{(x_1,x_2)\,|\,x_1,x_2\in\mathbb{R}\}$ with addition defined by $(x_1,x_2)\oplus(y_1,y_2)=(x_1+y_1,x_2+y_2)$ and scalar multiplication defined by $\lambda(x_1,x_2)=(\lambda x_1,x_2)$ is a vector space.

b. $V=\{(x_1,x_2)\,|\,x_1,x_2\in\mathbb{R}\}$ with addition defined by $(x_1,x_2)\oplus(y_1,y_2)=(x_1+y_1,x_2-y_2)$ and scalar multiplication defined by $\lambda(x_1,x_2)=(\lambda x_1,\lambda x_2)$ is a vector space.

9) Let $M_{3\times3}(\mathbb{R})$ denote the set of 3x3 matrices with real entries. Let $S_{3\times3}(\mathbb{R})$ denote the set of symmetric 3x3 matrices, i.e., matrices in which $a_{ij}=a_{ji}$ for all $1\le i,j\le 3$. For example, $\begin{pmatrix}1&0&2\\0&2&-1\\2&-1&8\end{pmatrix}\in S_{3\times3}(\mathbb{R})$ (a symmetric matrix is one whose rows are equal to its columns).

Prove: $S_{3\times3}(\mathbb{R})$ is a subspace of $M_{3\times3}(\mathbb{R})$.

10) Let $A_{3\times3}(\mathbb{R})$ denote the set of **antisymmetric** 3x3 matrices, i.e., matrices in which $a_{ij}=-a_{ji}$ for all $1\le i,j\le 3$. For example, $\begin{pmatrix}0&-3&2\\3&0&1\\-2&-1&0\end{pmatrix}\in A_{3\times3}(\mathbb{R})$. Prove that $A_{3\times3}(\mathbb{R})$ is a subspace of $M_{3\times3}(\mathbb{R})$.

11) Let $U_{3\times3}(\mathbb{R})$ be the set of all 3x3 matrices in which the first row consists entirely of zeroes. Prove that $U_{3\times3}(\mathbb{R})$ is a subspace of $M_{3\times3}(\mathbb{R})$.

12) The set W is defined as: $W=\left\{\begin{pmatrix}2a&a\\c&-c\end{pmatrix}|a,b,c\in\Re\right\}$ where a,c are any real numbers. Prove that W is a subspace of $M_{2\times2}(\mathbb{R})$.

13) (Exam question 5774) Let $W=\left\{\begin{pmatrix}a+b&a\\a+c&b\end{pmatrix}:\ a,b,c\in\mathbb{R}\right\}$. Prove that W is a vector space with the usual operations of addition and scalar multiplication as defined on matrices.