Calculus 1 Exercise 1

13) $a_1 = 7$

$a_{n+1} = \sqrt{a_n + 2}$

b. Claim: $a_n > 0$ for all n.

Pf: Induction.

Base case: $a_1 = 7 > 0$.

Assume $a_k > 0$.

Then $a_{k+1} = \sqrt{a_k + 2} > \sqrt{0+2} = \sqrt{2} > 0$.

c. Claim: $\{a_n\}$ is decreasing

(i.e., $a_{n+1} < a_n$ for n)

Pf: Induction

Base case: $a_2 = \sqrt{7+2} = 3 < 7 = a_1$

Assume $a_{k+1} < a_k$.

Then $a_{k+1} + 2 < a_k + 2$

$$\sqrt{a_{k+1} + 2} < \sqrt{a_k + 2}$$

$$a_{k+2} < a_{k+1}$$

d. Already proved in part (b) -
Since every term is positive, $\{a_n\}$
is bounded from below by 0.

e. Since $\{a_n\}$ is monotonically decreasing and bounded from below, it must converge.

We find the limit, which we denote L:

$$a_{n+1} = \sqrt{a_n + 2}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 2}$$

$$= \sqrt{\left(\lim_{n\to\infty} a_n\right) + 2}$$

$$\circledast \quad L = \sqrt{L+2}$$

$$L^2 = L + 2$$

$$L^2 - L - 2 = 0 \Rightarrow (L-2)(L+1) = 0$$

$$L = 2 \quad \text{or} \quad \underbrace{L = -1}_{\text{impossible b/c } a_n > 0 \text{ for all } n.}$$

f. The graph of $f(x) = \sqrt{x+2}$ and the line $y = x$ intersect at the point $(2,2)$. This is a direct consequence of the equation $\circledast$ from part e. (Think about why!)

Questions 12 and 14 follow the same pattern as question 13.

For the upper bounds (to prove by induction, like the lower bound of 0 in question 13) use a bound of 5 in 12(d) and a bound of 3 in 14(b).

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17) Example similar to 17 a:

Claim: $\lim_{n\to\infty} \left(\frac{2n+3}{5n+4}\right)^n = 0.$

Proof: For all n:

$$\frac{2n}{5n+4} < \frac{2n+3}{5n+4} < \frac{2n+3}{5n}$$

For $n>4$: $\frac{2n}{6n} < \frac{2n}{5n+4}$ (since $6n > 5n+4$)

i.e: $\frac{1}{3} < \frac{2n}{5n+4} < \frac{2n+3}{5n+4}$

For $n>3$: $\dfrac{2n+3}{5n} < \dfrac{3n}{5n}$ (since $2n+3<3n$)

i.e.: $$\frac{2n+3}{5n+4} < \frac{2n+3}{5n} < \frac{3}{5}$$

Therefore, for all $n>4$ we have:

$$\frac{1}{3} < \frac{2n+3}{5n+4} < \frac{3}{5}$$

and hence: $$\left(\frac{1}{3}\right)^n < \left(\frac{2n+3}{5n+4}\right)^n < \left(\frac{3}{5}\right)^n.$$

Since $0<\frac{1}{3}<1$ and $0<\frac{3}{5}<1$ we have:

$$\lim_{n\to\infty}\left(\frac{1}{3}\right)^n = \lim_{n\to\infty}\left(\frac{3}{5}\right)^n = 0$$

and so by the Sandwich Theorem

we conclude that:

$$\lim_{n\to\infty}\left(\frac{2n+3}{5n+4}\right)^n = 0.$$