Calculus I: Exercise 7 (The Definite Integral)

Submit: 1 a,d 2 c, f, i, l, m 3 b, e 4 b, d, g 6 7 b, c--2, 3 8 11 13 a, c, f, g, i, j 15 a, c 16 a--2, 4, 6, 7, 11 b--2, 5

1) Calculate each integral: (a) by the definition, i.e., as the limit of a Riemann sum (b) by the Newton-Leibniz formula

a. $\int_2^5 (3x^2-4x)dx$ b. $\int_{-1}^4 (x^3-x)dx$ c. $\int_{-2}^3 (2x^2-3x-4)dx$ d. $\int_3^7 e^x dx$

You may use the usual formulas for summing arithmetic and geometric sequences as well as:

$$\sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6} \qquad \sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4} \qquad \sum_{i=1}^k i^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$$

2) Calculate the following definite integrals using the Newton-Leibniz formula:

a. $\int_1^4 \frac{dx}{\sqrt{x}}$ b. $\int_{-\pi/4}^{\pi/4} \sin 2x dx$ c. $\int_1^5 (x^2-6x+8)dx$ d. $\int_{\frac{\pi}{12}}^0 \tan^2(3x)\,dx$

e. $\int_{-4}^4 |x|\,dx$ f. $\int_1^5 |x^2-6x+8|\,dx$ g. $\int_1^2 \frac{x^7-2x+1}{4x^3}dx$ h. $\int_1^{1.5} (4x-5)^9 dx$

i. $\int_1^4 \sqrt{x^2-6x+9}\,dx$ j. $\int_0^8 \frac{x}{(x+1)^{\frac{3}{2}}}dx$ k. $\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}dx$ l. $\int_e^{e^3} \frac{dx}{x\ln^2 x}$

m. $\int_1^{16} \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ n. $\int_0^1 \frac{5+e^x}{5+5x+e^x}dx$ o. $\int_0^1 \frac{20}{(2x+1)(x-2)}dx$

3) Evaluate each integral using integration by parts for the definite integral:

a. $\int_1^e \ln x dx$ b. $\int_1^e x^2 \ln x dx$ c. $\int_0^{\ln 2} xe^{3x}dx$ d. $\int_0^{\pi} t^2 \sin 2t dt$

e. $\int_0^{\pi/4} \frac{x}{\cos^2 x}dx$ f. $\int_0^{\sqrt{3}} \arctan x dx$

4) Evaluate each integral using the method of substitution for the definite integral:

a. $\int_1^3 (3x-4)^4 dx$ b. $\int_0^9 \frac{dx}{\sqrt{x}+4}$ c. $\int_1^5 \frac{\sqrt{x-1}}{x}dx$ d. $\int_1^e \frac{\ln x}{x(1+\ln x)}dx$

e. $\int_0^{0.5\pi} \cos x\sqrt{\sin x}dx$ f. $\int_0^8 e^{\sqrt[3]{x}}dx$ g. $\int_0^{\pi/2} \frac{\sin 2x}{(2\sin x+1)(\sin x+2)}dx$

5) Given the sequence: $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ (for $n \in \mathbb{N}$).

a. Compute I_0 and I_1, and express I_{n+2} in terms of I_n.

b. Compute I_9 and I_{10} using the formula you found in part (a).

6) Given the sequence: $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ (for $n \in \mathbb{N}$).

a. Compute I_0 and I_1, and express I_{n+2} in terms of I_n.

b. Compute I_9 and I_{10} using the formula you found in part (a).

c. From the graph of $f(x) = \tan^n(x)$ in the interval $\left[0, \frac{\pi}{4}\right]$, what do you expect to be $\lim_{n\to\infty} I_n$ (without formal proof)? Use part (a) to obtain approximations to π and $\ln 2$.

7) a. Prove: If *f* is integrable and even on the interval $[-a,a]$ then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

b. Prove: If *f* is integrable and odd on the interval $[-a,a]$ then $\int_{-a}^{a} f(x)\,dx = 0$.

c. Prove: (1) $\int_{-\pi/8}^{\pi/8} x^{10} \sin^9 x dx = 0$ (2) $\int_{-1}^{1} \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x} dx = 0$

(3) $\int_{-1}^{1} e^{\cos x} dx = 2\int_0^1 e^{\cos x} dx$ (4) $\int_{-1/2}^{1/2} \cos x \ln \frac{1+x}{1-x} dx = 0$

8) If *f* is continuous and periodic with period *T*, prove that the integral $\int_a^{a+T} f(x)dx$ does not depend on *a*.

9) Given a function *f* integrable on the interval $[0,c]$.

a. Show that $\int_0^c f(c-x)\,dx$ exists.

b. Prove that $\int_0^c f(c-x)\,dx = \int_0^c f(x)\,dx$.

c. Conclude from part (b) that $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$ for all $n \in \mathbb{N}$.

d. Prove from part (c), without computing the integrals: $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4}$.

10) Given a function f continuous on the interval $[-1,1]$.

a. Prove: $\int\limits_0^{\pi/2} f(\cos x)dx = \int\limits_0^{\pi/2} f(\sin x)dx$ (you must also show that each side exists.)

b. Using part (a) compute $\int\limits_0^{\pi/2} \cos^2 x \text{ dx}$ and $\int\limits_0^{\pi/2} \sin^2 x \text{ dx}$ (this is similar to question 9.)

11) Given a function f integrable on the interval $[a,b]$.

a. Explain why both the integrals $\int_a^b f(x)\,dx$ and $\int_a^b f(a+b-x)\,dx$ exist.

b. Prove: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Note that this is a generalization of q. 9b.

c. Use part (b) to evaluate the integral $\int_0^1 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{1-x}}dx$.

d. Define for every rational r: $I_r = \int_0^{\pi/2} \frac{\cos^r x}{\cos^r x + \sin^r x}dx$. Prove that for all r, $I_r = \frac{\pi}{4}$.

12) Solve the following:

a. Prove: $\int f(x)dx = xf(x) - \int xf'(x)dx$.

b. Given $f(0) = g(0) = 0$, prove:

$\int\limits_0^a f(x)g''(x)dx = f(a)g'(a) - f'(a)g(a) + \int\limits_0^a f''(x)g(x)dx$.

c. The function f is continuously differentiable in the interval $[a,b]$, and g is the inverse of f there. Prove: $\int\limits_a^b f(x)dx = bf(b) - af(a) - \int\limits_{f(a)}^{f(b)} g(y)dy$.

d. Use part (c) to evaluate $\int\limits_1^e \ln x \, dx$.

e. For $0 < a < b$ and if f and g are positive, explain geometrically the equation from part (c).

13) Calculate the areas of the regions bound by the following lines and curves. Sketch each carefully on a pair of coordinate axes, and note that some are split into several parts. Note whether it is better to integrate with respect to x or to y (especially in parts f-i).

a. $y = 2-\sqrt{x}$, $\text{y} = \sqrt{4-x}$

b. $y = 0$, $\text{y} = \text{x}^3 - 6\text{x}^2 + 8\text{x}$

c. $y = 5x^2 - 3x + 2$, $\text{y} = \text{x}^3 + 2\text{x}^2 - 4\text{x} + 5$

d. $y = \left(x^2+1\right)^{-1}, y = x^2/2$

e. $y = \cos x$, $y = \sin 2x$ in the interval $[0, \pi]$.

f. $y = x^3 + 1$, the x-axis, and the tangent to $y = x^3 + 1$ at the point (1,2):
   1. the region above the x-axis. 2. the region below the x-axis. (each in one integral)

g. $y = 0$, $x = 1$, $y = \arcsin x$

h. $y = 0$, $x = 1$, $y = \arctan x$

i. $y = 2x - 4$, $y^2 = 4x$

j. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (this is the canonical equation of an ellipse.)

14)

a. Prove: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$

b. Prove: $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} > \ln(n+1)$

(Hint: Sketch the graphs of $f(x) = \frac{1}{\sqrt{x}}, g(x) = \frac{1}{x}$ and build rectangles with bases $[i, i+1]$ for $i = 1, \ldots, n$.)

15) Questions on rates/ velocities:

a. The velocity of an object t seconds after it starts to move is $v(t) = 30 - 2t$ m/s. How far does it travel in the first 10 seconds of its movement?

b. An object travels with an acceleration of $a(t) = 12t^2 \ {}^{m}\!/_{s^2}$, where t is the number of seconds since it started to move. Its initial velocity is zero. How far does it travel in the first 10 seconds of its movement? [Note: When the acceleration is a constant a, then in t seconds the object's velocity increases by at.]

c. Water flows into an aquifier at a rate of $100 - 3t$ liters per second, where t denotes the number of seconds from the start of the flow. How much water flows into the aquifier between $t = 10$ and $t = 20$ seconds?

d. A student reads a textbook at a rate of $10 - t$ pages per hour, where t denotes the number of hours since he started reading. How many pages does he read… (a) … in the first three hours? (b) … in the following three hours?

e. A solid travels 10 meters under the influence of a force measuring $0.2(x-5)^2$ Newtons, where x denotes the distance of the solid from the starting point. Calculate the work performed by the solid during its travel. [Note: The work performed by an object under a constant force F (measured in Newtons) as it travels s meters is $F \cdot s$ joules (a unit measuring work or energy)]

16)

a. Determine if each improper integral converges or diverges. If it converges, compute its value.

1. $\displaystyle\int_e^{\infty} \frac{dx}{x\sqrt{\ln x}}$ 2. $\displaystyle\int_0^{\infty} xe^{-x^2}dx$ 3. $\displaystyle\int_0^{\infty} xe^{-x}dx$ 4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+16}$

5. $\displaystyle\int_0^{2} \frac{dx}{\sqrt[3]{x}}$ 6. $\displaystyle\int_0^{0.5} \frac{dx}{\sqrt{1-4x^2}}$ 7. $\displaystyle\int_1^{3} \frac{dx}{x-2}$ 8. $\displaystyle\int_0^{2} \frac{dx}{(x-1)^2}$

9. $\displaystyle\int_0^{\ln 2} \frac{e^{1/x}dx}{x^2}$ 10. $\displaystyle\int_1^{e} \frac{dx}{x\sqrt{\ln x}}$ 11. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2-4x}$ 12. $\displaystyle\int_{-1}^{1} \ln|x|\,dx$

b. For each integral, determine for which values of p it converges and for which it diverges. When it converges, find its value in terms of p.

1. $\displaystyle\int_1^{+\infty} x^p dx$ 2. $\displaystyle\int_0^{1} x^p dx$ 3. $\displaystyle\int_0^{+\infty} x^p dx$ 4. $\displaystyle\int_1^{2} \frac{dx}{x\,(\ln x)^p}$ 5. $\displaystyle\int_2^{\infty} \frac{dx}{x\,(\ln x)^p}$