Linear Algebra 1—Exercise 2: Vectors in $\mathbb{R}^n$

To submit: 2 7 8 9 10b 11a 12a 15 16b

1) In rectangle ABCD denote: $\overrightarrow{AB} = \underline{v}$, $\overrightarrow{AC} = \underline{u}$.
   a. Express in terms of $\underline{u}$ and $\underline{v}$ the following vectors: $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$, $\overrightarrow{BD}$.
   b. The point F satisfies $\overrightarrow{AF} = \underline{v} - \underline{u}$. Find where F is located.

2) In the cube $ABCDA'B'C'D'$ denote: $\overrightarrow{AD'} = \underline{v}$, $\overrightarrow{D'B} = \underline{w}$, $\overrightarrow{AC} = \underline{u}$. Express in terms of $\underline{u}$, $\underline{v}$, $\underline{w}$ the following vectors: $\overrightarrow{AB}$, $\overrightarrow{AA'}$, $\overrightarrow{BA'}$, $\overrightarrow{AC'}$, $\overrightarrow{DB}$, $\overrightarrow{DA}$.

3) In the parallelogram ABCD:
   M is the midpoint of AB, N the midpoint of BC, and P the midpoint of MN
   $\overrightarrow{AB} = \underline{v}$ and $\overrightarrow{BC} = \underline{u}$
   a. Express $\overrightarrow{BP}$ and $\overrightarrow{BD}$ using $\underline{u}$ and $\underline{v}$.
   b. Prove that P is on the diagonal BD and find the ratio into which P splits this diagonal.

4) In the trapezoid $ABCD$ with $AB \| CD$, $CD = 3AB$, M is a point on BC such that: $\overrightarrow{BM} = \frac{1}{2}\overrightarrow{MC}$.
   a. Express $\overrightarrow{AC}$ and $\overrightarrow{DM}$ using $\underline{u}$ and $\underline{v}$, where:
      i. $\overrightarrow{AB} = \underline{u}$ and $\overrightarrow{AD} = \underline{v}$
      ii. $\overrightarrow{AB} = \underline{u}$ and $\overrightarrow{BM} = \underline{v}$
      iii. $\overrightarrow{AD} = \underline{u}$ and $\overrightarrow{MC} = \underline{v}$
   b. Given: $\angle BAD = 120^\circ$, $AB = 3$ cm , $AD = 6$ cm
      Calculate: $\left\|\overrightarrow{AC}\right\|$, $\left\|\overrightarrow{DM}\right\|$ and the angle between $\overrightarrow{AC}$ and $\overrightarrow{DM}$.

5) Given ΔABC.
   a. Let D be the midpoint of BC. Prove that: $\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.
   b. Let M be the intersection of the medians of ΔABC (it is a theorem in geometry that the three medians always meet at one point). Prove that: $\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM} = \underline{0}$.
   c. Let P be any point. Prove that: $\overrightarrow{PM} = \frac{1}{3}\left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}\right)$.

6) Given ΔABC with M and N the midpoints of the edges AB and AC respectively.
   Prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$ and note which theorem can be proven based on this equation.

7) Given the trapezoid $ABCD$ with: $AB||CD$, E the midpoint of AB, F the midpoint of CD, and M the intersection of the continuations of the legs AD and BC.

   a. Prove that: $\overrightarrow{ME} = \frac{1}{2}\left(\overrightarrow{MA} + \overrightarrow{MB}\right)$ and $\overrightarrow{MF} = \frac{1}{2}\left(\overrightarrow{MD} + \overrightarrow{MC}\right)$

   b. Prove that E, M, and F are all on one line.
   (NOTE: You may use similar triangles.)

8) Given points $A = (2,-4,-3)$, $B = (7,2,7)$, $C = (4,5,9)$, $D = (-1,-1,-1)$
   Prove using vectors that $ABCD$ is a parallelogram.

9) Given the three points: $A = (3,1,0)$, $B = (-4,5,3)$, $C = (1,-4,6)$.
   a. Prove that these points are the vertices of a triangle.
   b. Calculate the angles of this triangle.

10) Find a vector orthogonal to both $\underline{u}$ and $\underline{v}$:
   a. $\underline{u} = (1,2,6)$, $\underline{v} = (0,2,3)$
   b. $\underline{u} = (4,2,3)$, $\underline{v} = (0,0,2)$

11) $\underline{u}$ and $\underline{v}$ are two vectors of equal length.
   a. If $\underline{u}$ and $\underline{v}$ are orthogonal, calculate the angle between $\underline{u} + 2\underline{v}$ and $\underline{v} - \underline{u}$.
   b. If the angle between $\underline{u}$ and $\underline{v}$ is 30°, calculate the angle between $\underline{u} + 3\underline{v}$ and $4\underline{u} - 2\underline{v}$.

12) Find the area of the triangle ΔPQR with the vertices:
   a. $P = (1,5,-2)$, $Q = (0,0,0)$, $R = (3,5,1)$
   b. $P = (2,1,3)$, $Q = (-1,4,5)$, $R = (4,2,0)$

13) True or False: Given $a,b,c \in \mathbb{R}^n$:
   a. If $a \cdot b = a \cdot c$ and $a \neq 0$ then $b = c$
   b. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
   c. $a \cdot b + c \cdot b = a \cdot c + b$

14) Prove that if the vector $\underline{u}$ is orthogonal to vectors $\underline{v}, \underline{w}$ then $\underline{u}$ is orthogonal to the vector $4\underline{v} - 2\underline{w}$

15) Let $\underline{u}$ and $\underline{v}$ be two vectors representing two adjacent sides of a parallelogram. Use them to prove the following geometric statements:
   a. The length of all sides of a parallelogram are equal if and only if the diagonals are perpendicular to one another.

   b. The lengths of the diagonals of a parallelogram are equal if and only if the adjacent sides of the parallelogram are perpendicular to one another.

16)

a. Find a vector in $\mathbb{R}^2$ at an angle of $\frac{\pi}{4}$ with the positive x-axis.

b. In three-dimensional space $\mathbb{R}^3$, find two (non-parallel) unit vectors at an angle of $\frac{\pi}{3}$ with the positive x-axis.

c. In three-dimensional space, find two (non-parallel) vectors of length 4, at an angle of $\frac{\pi}{3}$ with the positive y-axis.

17) Given the trapezoid $ABCD$ with: $AB \parallel CD$, $\overrightarrow{AB} = k\overrightarrow{DC}$ (for some $k \in \mathbb{R}$), $\overrightarrow{AB} = \underline{u}$, $\overrightarrow{AD} = \underline{v}$.

a. Express the vectors: $\overrightarrow{BC}, \overrightarrow{AC}, \overrightarrow{DC}, \overrightarrow{DB}$ using $\underline{u}$, $\underline{v}$, k.

b. Find k if $\overrightarrow{BC} = \underline{v} - \frac{2}{3}\underline{u}$.

18) Show using vectors that: $A = (2,-1,1)$, $B = (3,2,1)$, $C = (-1,0,-2)$ are the vertices of a right triangle.