Linear Algebra 1—Exercise 7: Determinants

To submit: 1 a, d 2 b, d 3 b 4 d, h 6 a 7 a 8 a 9 11 a

1) Evaluate the determinant of each of the following matrices:

a. $\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$ b. $\begin{vmatrix} 3 & -2 \\ 0 & -2 \end{vmatrix}$ c. $\begin{vmatrix} 3-m & 1 \\ 4 & 2-m \end{vmatrix}$ d. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 7 & 1 \\ 1 & 3 & 5 \end{vmatrix}$ e. $\begin{vmatrix} 4 & m & -1 \\ 5 & -1 & 0 \\ 1 & 1 & -m \end{vmatrix}$

2) Evaluate the determinant of each of the following matrices: (Use elementary operations on the **columns** to bring each matrix to triangular form.)

a. $\begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -3 & 4 \\ 1 & 2 & -3 & 4 \\ 2 & 1 & 1 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 4 & 2 \\ 1 & 5 & 5 & 0 \\ -1 & 1 & 1 & 1 \end{bmatrix}$

3) In each sequence of matrices below, the transition from each matrix to the next consists of one or two elementary operations. Write the operations and how they influence the determinant of the matrix. Derive from these the determinant of the original matrix.

a. $\begin{pmatrix} 5 & 1 & 1 \\ 5 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 6 & 1 & 1 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b. $\begin{pmatrix} 8 & 1 & 1 \\ -4 & 0 & 0 \\ 6 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 8 & 1 & 1 \\ 1 & 0 & 0 \\ 6 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4) Given the two matrices $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}$, calculate the determinants of the following matrices. (Note: For any square matrix A, $A^{-n} = \left(A^{-1}\right)^n = \left(A^n\right)^{-1}$.)

a. $|A|$ b. $|B|$ c. $|3A|$ d. $|AB|$ e. $|A^4|$ f. $|A^{-1}|$ g. $|4B^{-1}|$ h. $|4A^2B^{-2}|$ i. $|5A^2B^3|$

5) Given the two matrices: $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 5 & 3 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & -2 \\ 1 & 2 & 0 \\ 1 & 4 & 0 \end{pmatrix}$. Find the following determinants, using properties of determinants and appropriate results:

a. $|A|$ b. $|B|$ c. $|4A|$ d. $|AB|$ e. $|A^4|$ f. $|3A^{-1}|$ g. $|B^{-1}A^2|$ h. $|2A^{-3}B^4|$

6) Evaluate the following determinants using elementary row operations:

a. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 8 & 64 \end{bmatrix}$ c. $\begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$ (assume x, y, z are all different)

(Answers: a. 6 b. 30 c. $(y-x)(y-z)(x-z)$)

7) a. Find a 3x3 matrix such that no two rows are multiples of one another, and no two columns are multiples of one another, yet the determinant of the matrix is 0.

b. Find a 4x4 matrix such that no two rows are multiples of one another, and no two columns are multiples of one another, yet the determinant of the matrix is 0.

8) Solve the equation $\det(A) = 0$ when $A = \ldots$

a. $\begin{bmatrix} 1-x & 2 \\ 3 & 2-x \end{bmatrix}$ b. $\begin{bmatrix} x-2 & 4 & 3 \\ 1 & x+1 & -2 \\ 0 & 0 & x-4 \end{bmatrix}$ c. $\begin{bmatrix} x-3 & 3 & 2 \\ 3 & x-3 & 2 \\ 2 & 3 & x-3 \end{bmatrix}$

(Note: In part (c) one obtains a third-degree polynomial, you may use Mathematica to help factor it.)

9) Given the system of equations: $\begin{cases} x + (a-1)z = a \\ -y + z = a \\ (a-2)x + y + z = 0 \end{cases}$

a. Find the determinant of the coefficient matrix.
b. Based on your answer to part (a), determine for which values of a the system has: a unique solution; infinitely many solutions; no solution.
c. Find the inverse of the coefficient matrix for the values of a for which it is invertible.
d. Use part (c) to find the solution to the system for those values of a.

10) (question missing)

11) (Exam question 5771)

a. Let A, B, C be 4x4 matrices satisfying the following:
$\det A = 5$, $B + 2A^{-1} = 0$, $C - A^3 B^T = 0$
Use properties of determinants to calculate det C.

b. Given the matrix: $A = \begin{pmatrix} 5-x & 5+x & 5-x & 3 \\ 7+x & 7-x & 7+x & 3 \\ 5 & 5 & 5 & 3 \\ 3 & 3 & 2 & 4 \end{pmatrix}$

True or false? Explain!

i. A is invertible for every integer value of x.

ii. For every integer value of x, det A is divisible by 6.

(NOTE: The determinant of A need not be calculated in order to answer the question!)

12) (Exam question 5775)

True or false? Explain your answers.

a. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 2$ then $\begin{vmatrix} a_1+3b_1 & a_2+3b_2 & a_3+3b_3 \\ 6b_1 & 6b_2 & 6b_3 \\ 5c_1 & 5c_2 & 5c_3 \end{vmatrix} = 60$.

b. If A, B are 3x3 matrices and $|A| = 3, |2AB| = 48$, then $|B| = 2$.

13) Calculate the determinant of A $= \begin{pmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{pmatrix}$.