Calculus I: Exercise 3 (Continuity and Discontinuities)

To submit: 1 a, c, f, g   2 b, d, e   3 a, b   4   5 b, d, f   6 b   7 a, c   11 b, d   12   13   16 c, e, g, i, j

1. In each part determine (with full explanations) at which values of x_o the given function f is continuous and at which it has a discontinuity. For each x_0 where there is a discontinuity, determine its type and sketch the graph of f in some neighborhood of x_0.

a. $f(x)=\dfrac{x^3-3x^2+2}{x^2-4x+3}$ b. $f(x)=2^{\frac{1}{5-x}}$ c. $f(x)=\dfrac{1}{1+5^{\frac{1}{x-1}}}$ d. $f(x)=\dfrac{x^2+3x}{x^2+x-6}$

e. $f(x)=\begin{cases} 2x+\dfrac{6}{2-x} & :x<4 \\ 8-\dfrac{8}{x} & :x\ge 4 \end{cases}$ f. $f(x)=\begin{cases} \dfrac{1}{x+2} & :x<-2 \\ x^2-3 & :-2\le x<1 \\ x+1 & :x\ge 1 \end{cases}$ g. $f(x)=\begin{cases} \dfrac{2x^3}{\mathrm{x}-1} & :\mathrm{x}\le 0 \\ \dfrac{\sin x}{\mathrm{x}} & :0<\mathrm{x}\le \pi \\ \dfrac{1}{\mathrm{x}-4} & :\mathrm{x}>\pi \end{cases}$

Note: Use the fact that every elementary function is continuous at every point in its domain, and that if two functions coincide in a neighborhood of x_0 and one is continuous there, then so is the other.

2. Investigate the continuity of each of the following functions at $x_0=0$ (parts a-e) or at $x_0=1$ (parts f-g). Distinguish as necessary between different values of the parameter A.

a. $f(x)=\begin{cases} \dfrac{\sin\mathrm{x}}{|\mathrm{x}|} & :x\ne 0 \\ 1 & :x=0 \end{cases}$ b. $f(x)=\begin{cases} \sin\dfrac{1}{\mathrm{x}} & :x\ne 0 \\ A & :x=0 \end{cases}$ c. $f(x)=\begin{cases} x\sin\dfrac{1}{\mathrm{x}} & :x\ne 0 \\ A & :x=0 \end{cases}$

d. $f(x)=\sin\mathrm{x}\cdot\sin\dfrac{1}{\mathrm{x}}$ e. $f(x)=\begin{cases} e^{\frac{-1}{\mathrm{x}^2}} & :x\ne 0 \\ A & :x=0 \end{cases}$ f. $f(x)=\begin{cases} \dfrac{1}{1+e^{\frac{1}{\mathrm{x}-1}}} & :x\ne 1 \\ A & :x=1 \end{cases}$

g. $f(x)=\dfrac{1}{1-e^{\frac{\mathrm{x}}{1-\mathrm{x}}}}$

3. Find for which values of the parameters a and b each function is continuous.

a. $f(x)=\begin{cases} x^3 & :x<2 \\ ax+b & :2\le x<3 \\ x^2+2 & :x\ge 3 \end{cases}$ b. $f(x)=\begin{cases} ax^3 & :x<2 \\ ax^2+bx & :2\le x<3 \\ x^2+ax & :x\ge 3 \end{cases}$

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c. $f(x)=\begin{cases}\dfrac{x^2+5x-a}{x-2} & : x\neq 2\\ b & : x=2\end{cases}$ d. $f(x)=\begin{cases}a\sin(bx)+a & (x<0)\\ \dfrac{\sin(\pi x)}{ax} & \left(0\le x<\dfrac{1}{2}\right)\\ bx & \left(x\ge\dfrac{1}{2}\right)\end{cases}$

4. Given the function $f(x)=\begin{cases}\dfrac{3x+4}{x-2} & : x\le 0\\ \dfrac{\sin ax}{x} & : 0<x\le\pi\\ \dfrac{x^2-a}{x^2-3x-4} & : x>\pi\end{cases}$ where a is some integer parameter.

   a. If f is continuous at 0, how many discontinuities does it have?

   b. If f has a removable discontinuity, find all the discontinuities of f and determine the type of each.

5. Calculate each limit if it exists, and if it doesn't exist explain why. Use substitutions as necessary.

   a. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{\arcsin 5x}$ b. $\lim\limits_{x\to 0}\dfrac{\arcsin 3x}{\tan 5x}$ c. $\lim\limits_{x\to -2}\dfrac{x^2-4}{\arctan(x+2)}$ d. $\lim\limits_{x\to +\infty}\left(5x\cdot\arctan\dfrac{3}{x}\right)$

   In the following parts, use the graph of $f(x)=\arctan x$ without formal proofs. You may subsequently consider these as "basic examples" of limits and need not prove them.

   e. $\lim\limits_{x\to +\infty}\arctan x$ f. $\lim\limits_{x\to -\infty}\arctan x$ g. $\lim\limits_{x\to \frac{\pi}{2}^-}\arcsin x$ h. $\lim\limits_{x\to 0^+}\arccos x$

6. For each equation find an approximation to at least one root with accuracy 0.01, i.e., find a number whose difference from one root is no more than 0.01. You may use a calculator but list clearly all stages of the computation.

   a. Equation: $\arctan x = 2 + x^3$ Find a root in the interval $[-2,-1]$.

   b. Equation: $\tan x = 1-\log_2 x$. Find a root in the interval $[0,1.5]$.

7. a. Prove that the polynomial $P(x)=x^5-4x^3+5x^2+3x+7$ has at least one root, i.e., there exists at least one real number which makes P equal to zero.

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b. Prove that every polynomial of odd degree has at least one root, i.e., if

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$ and if n is odd then there exists $a \in \mathbb{R}$ such that $P(a) = 0$.

c. Prove that if $Q(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$ is a polynomial of even degree and there exists $a \in \mathbb{R}$ such that $Q(a) < 0$ then Q has at least two roots.

8. [Challenge] Let f be continuous in the closed interval $[a,b]$ and such that the image of this interval is contained inside it, i.e., $f([a,b]) \subseteq [a,b]$. Prove that there exists $c \in [a,b]$ such that $f(c) = c$. (The point c is called a *fixed point* of the function f.)

   Hint: Define a function which must vanish at c and which can help to prove the result.

9. Let f be a continuous function in the interval (a,b) such that $\lim_{x \to a+} f(x) = -\infty$ and $\lim_{x \to b-} f(x) = +\infty$. Prove that the image of (a,b) under f is all of $\mathbb{R}$.

10. [Challenge] Given a metal circular ring of negligible thickness. The ring is heated at certain places such that its temperature is not uniform over the entire ring. Prove that the ring has some diameter such that the two endpoints of the diameter have equal temperature.

    Hint: Use the fact that the "heat function" whose value at each point of the ring is the temperature at that point, is continuous.

11. For each part find a function satisfying the given property:

    a. The function is monotonic in the interval $[0,1]$ but not continuous in it.

    b. The function is monotonic and continuous in the interval $(0,1)$ but not bounded in it.

    c. The function is one-to-one in the interval $[0,1]$ but not monotonic in it.

    d. The function is monotonic and bounded in the interval $[0,1]$ but not continuous in it.

    e. The function changes sign in the interval $[0,1]$ but does not vanish there.

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12. Let *f* be a function defined in a neighborhood of x_0 which is continuous from the right at x_0 but not continuous at x_0. Prove, or disprove by counterexample:

   a. It must be that *f* is not continuous from the left at x_0.

   b. It must be that $\lim\limits_{x \to x_0} f(x)$ doesn't exist.

13. a. Prove: If *f* is continuous at x_0 and $f(x_0) < 0$ then there exists a neighborhood of x_0 in which $f(x) < 0$ for all *x*.

   b. Disprove: If *f* is continuous at x_0 and there exists a punctured neighborhood of x_0 in which $f(x) < 0$ for all *x*, then $f(x_0) < 0$.

14. Suppose the equation $ax^2 + bx + c = 0$ has two real roots x_1, x_2. When $a = 0$ the equation is linear ($bx + c = 0$) and so there is only one root. What happens to the second root as $a \to 0$? [Hint: Calculate the limits $\lim\limits_{a \to 0} x_1$ and $\lim\limits_{a \to 0} x_2$.]

15. Prove: If $\lim\limits_{x \to x_0} g(x) = l$ and *f* is continuous at *l* then $\lim\limits_{x \to x_0} f(g(x)) = f(l)$. [Hint: Define a function $\tilde{g}$ for all *x* in the domain of *g* as follows: $\tilde{g}(x) = \begin{cases} g(x) & : \quad x \neq x_0 \\ l & : \quad x = x_0 \end{cases}$, and use a theorem dealing with the composition of continuous functions.]

16. Find the equations of all asymptotes to the graphs of the following functions, and sketch each asymptote next to a part of its accompanying graph:

   a. $f(x) = \dfrac{3x^2 - 25x - 1}{x - 9}$ b. $f(x) = \dfrac{3x^2 + x - 30}{x^2 - 9}$ c. $f(x) = \dfrac{2x^3 + x - 18}{3x^2 - 2x - 8}$

   d. $f(x) = \dfrac{2x^3 + x^2 - 20}{3x^2 - 2x - 8}$ e. $f(x) = 2x + \sqrt{4x^2 + 7}$ f. $f(x) = 2x + \sqrt{4x^2 + 8x + 7}$

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g. $f(x)=\sqrt{4x^2+x}$ h. $f(x)=\sqrt[3]{x^2-x^3}$ i. $f(x)=\dfrac{e^x}{e^x-1}$

j. $f(x)=x+\arccos\dfrac{1}{x}$ (note the domain of the function!)

k. $f\left(x\right)=\dfrac{\sin x}{x}$