Linear Algebra 1—Exercise 1: Systems of Linear Equations

To submit: 1 b, e, g   2 b, d, i, j   3 c, d   5   9   10 a, d   11

1) Solve each of the following systems:

a. $$\begin{Bmatrix} 2x+y-3z=0 \\ 6x+3y-8z=0 \\ 2x-y+5z=-4 \end{Bmatrix}$$

b. $$\begin{Bmatrix} x-y+z=2 \\ x+y=1 \\ x+y+z=8 \end{Bmatrix}$$

c. $$\begin{Bmatrix} 3x+y-z=10 \\ x-2y-z=-2 \\ -x+y+z=0 \\ 2x-y-3z=7 \end{Bmatrix}$$

d. $$\begin{Bmatrix} x_1-2x_3+x_4=6 \\ 2x_1-x_2+x_3-3x_4=0 \\ 9x_1-3x_2-x_3-7x_4=4 \end{Bmatrix}$$

e. $$\begin{aligned} 2x+y-z+w &= 1 \\ -x+2y+z+4w &= 0 \end{aligned}$$

f. $$\begin{aligned} x_2+2x_3-x_4+x_5 &= 5 \\ x_1-x_2+x_3+2x_4-x_5 &= 0 \\ x_1+x_2-x_3+x_5 &= -2 \\ x_1+x_3-3x_4-2x_5 &= -3 \end{aligned}$$

g. $$\begin{aligned} 2x+6y-z+w &= -3 \\ x-y+z-w &= 3 \\ -x-3y+3z+2w &= 9 \end{aligned}$$

2) Determine if each claim is correct or incorrect. If it is correct, briefly explain why. If it is incorrect, give a counterexample.

a. The system $\begin{Bmatrix} x+y+z=0 \\ 2x-5y-3z=0 \\ 3x-4y-2z=0 \end{Bmatrix}$ has a unique solution.

b. A linear system with m equations and m+1 variables always has a solution.

c. If a nonhomogeneous system has a unique solution then the corresponding homogeneous system has a unique solution.

d. If the homogeneous system corresponding to a given nonhomogeneous system has infinitely many solutions then the nonhomogeneous system has infinitely many solutions.

e. A system has m equations in n variables. If $n > m$ then the system must have more than one solution.

f. A system has m equations in n variables. If $m > n$ then the system cannot have a unique solution.

g. A homogeneous system of 3 equations in 4 variables must have infinitely many solutions.

h. It is possible that a system of 3 equations in 4 variables has a unique solution.
i. It is possible that a system of 3 equations in 4 variables has no solution.
j. It is possible that a system of equations has exactly two solutions.

3) Find a canonical staircase matrix for each of the following matrices:

a. $\begin{pmatrix} 1 & -3 & 0 & 2 \\ 1 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{pmatrix}$ b. $\begin{pmatrix} 4 & 9 \\ 9 & 5 \end{pmatrix}$ c. $\begin{pmatrix} 3 & 2 & -3 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}$ d. $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

4) (Exam question 5769)
Find the values of a for which the following system has a unique solution, infinitely many solutions, and no solution: $\begin{cases} x + az = 1 \\ y + z = 2 \\ 2x + 8y + 9z = 0 \end{cases}$

5) (Exam question 5771)

Given the restricted coefficient matrix: $A = \begin{pmatrix} 1 & k & k^2 \\ k & k^2 & 1 \\ k^2 & 1 & k \end{pmatrix}$

Find all values of k for which the homogeneous system with this matrix has a unique solution.

6) (Exam question 5777)
Given the system of equations:

$$\begin{gathered} ax_1 + 2x_2 + (a-1)x_3 = 2 \\ ax_1 + (a-1)x_2 + 2x_3 = 2 \\ 2ax_1 + 2x_2 + (a-1)x_3 = 2 \end{gathered}$$

a. Find the values of a for which the system has a unique solution, infinitely many solutions, and no solution.
b. For each value of a for which the system has infinitely many solutions, find the general solution.

7) (Exam question 5772)
Given the system of equations:

$$\begin{aligned} &x + 2y - 3z = 4 \\ &3x - y + 5z = 5 \\ &4x + y + (a^2 - 14)z = a^2 - 5a + 13 \end{aligned}$$

a. Find the values of a for which the system has a unique solution, infinitely many solutions, and no solution.
b. Find the solution of the system for $a = 1$.

8) Given the system of equations:

$$\begin{gathered} x + y - z = 1 \\ x + ky + 3z = 2 \\ 2x + 3y + kz = 3 \end{gathered}$$

a. Find the values of k for which the system has a unique solution, infinitely many solutions, and no solution.
b. For each value of k for which the system has infinitely many solutions, find the general solution.

9) (Exam question 5777)
Given the matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & 1 \\ a & 1-a & 2-a & 1-a \end{pmatrix}$$

a. For which value of a does the system whose restricted coefficient matrix is A have two free variables?
b. For $a=1$, find the general solution of the homogeneous system whose restricted coefficient matrix is A.

10) For each system, find the values of m for which the system has:
i. a unique solution (and find the solution).
ii. infinitely many solutions (and find the general solution).
iii. no solution.

a. $\begin{aligned} x+y+mz&=1 \\ x+my+z&=1 \\ mx+y+z&=1 \end{aligned}$ b. $\begin{aligned} x+y+mz&=2 \\ 3x+4y+2z&=m \\ 2x+3y-z&=1 \end{aligned}$ c. $\begin{aligned} x-3z&=-3 \\ 2x+my-z&=-2 \\ x+2y+mz&=1 \end{aligned}$

d. $\begin{aligned} x_1+(m+1)x_2+x_3&=2m \\ x_1-x_2+mx_3&=-1 \\ x_1+mx_2-x_3&=2m+1 \end{aligned}$ e. $\begin{aligned} x_1+x_2+x_3&=1 \\ x_1+2x_2+6x_3&=m \\ x_1+4x_2+16x_3&=m^2 \end{aligned}$

11) (Exam question 5779)
Given the system:

$$\begin{aligned} 4kx+2k^2y-z &= 4 \\ 4ky-z &= k \\ 16ky-(3k^2+k)z &= k^2-5 \end{aligned}$$

(1) For which values of k does the system have a unique solution?
a. For $k=0$.
b. For $k \notin \{0,1,-\frac{4}{3}\}$.
c. There is no k for which the system has a unique solution.
d. For $k \neq 0$.

(2) For which values of k does the system have no solution?
a. The system has a solution for every $k \in \mathbb{R}$.
b. For $k \in \{1,0-\frac{4}{3}\}$ the system has no solution.
c. For $k=\pm 5$ the system has no solution.
d. Only for $k=0$ does the system have no solution.