Linear Algebra 1—Exercise 5: Vector Spaces (II)—Independence, Basis, Dimension

To submit: 1 a, c, f 2 b 3 a, d 4 a 5 a 6 b, d, e 7 b, c 8 a, c 9 b 11

1)

a. Check if the vectors $(4,-2,1),(1,2,4),(5,0,5)$ comprise a basis of $\mathbb{R}^3$.

b. Check if the vectors $(-1,-2,-3),(4,5,6),(7,8,9)$ comprise a basis of $\mathbb{R}^3$.

c. Check if the vectors $(-4,11,12,8),(2,5,6,4),(1,2,3,2),(1,1,-1,1)$ comprise a basis of $\mathbb{R}^4$.

d. Check if the polynomials $6+3x^3, x+5x^2, 2x^2+x^3, x^3$ comprise a basis of $\mathbb{R}_3[x]$.

e. Check if the polynomials $1-x, x+x^2, x^2+x^3, \ldots, x^{n-1}+x^n$ comprise a basis of $\mathbb{R}_n[x]$.

f. Let V denote the vector space of 2x2 matrices with real entries. Check if $A=\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, C=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}, D=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ comprise a basis of V.

g. Check if $A=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, C=\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}, D=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ comprise a basis of V (from part (f)).

h. Check if $A=\begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}, B=\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}, C=\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}, D=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ comprise a basis of V (from part (f)).

2)

a. Express the polynomial $-x^2+3x+11$ as a linear combination of the polynomials x^2-x+6, $2x^2-3x$, and $x-3$.

b. Express the polynomial x^2-2x+1 as a linear combination of the polynomials $3+2x+x^2$, $1+4x+x^2$, and $2x+1$.

3) Prove or disprove:

a. The set $\{(3,7,10),(6,1,4),(-3,2,5)\}$ is linearly independent in $\mathbb{R}^3$.

b. The set $\{(-1,2,1,1),(0,-1,1,1),(0,2,1,1),(1,2,-1,1)\}$ is linearly independent in $\mathbb{R}^4$.

c. The vector $(-3,10,8)$ is a linear combination of $\{(1,2,0),(3,-1,5)\}$.

d. The vector $(0,-2,5,1)$ is a linear combination of $\{(1,1,-3,0),(0,1,-1,0),(2,1,0,-1),(1,0,-1,0)\}$.

4) Consider the vectors $\underline{u}=(1,-3,2), \underline{v}=(2,-1,1)$.

a. 1) For which α is $(1,\alpha,5)$ a linear combination of $\underline{u}$ and $\underline{v}$?

2) Find conditions on a,b,c so that (a,b,c) is a linear combination of of $\underline{u}$ and $\underline{v}$.

b. Let $v_1 = (1, m, -1,3), v_2 = (-1,1,2m, -1), v_3 = (1, -3, -m - 4,1)$.. Find all the values of m for which the vector $(0,8,10,4)$ is a linear combination of v_1, v_2, v_3.

c. Find k such that the set $\{v_1 = (-1,2,3,2), v_2 = (2,2,2,2), v_3 = (-2, -5, -6, -4), v_4 = (2,6, k, 5)\}$ is linearly dependent, or prove that there is no such k.

5) Let V denote the set of all functions from $\mathbb{R}$ to itself.

a. Show that f and g are linearly independent if $g(t) = \sin 2t, f(t) = \cos t$.

b. Show that f, g, h are linearly independent if $f(t) = \sin t, g(t) = \cos t, h(t) = 3 + t$.

6)

a. Find a basis and the dimension of the subspace W of $\mathbb{R}^4$ spanned by $u_1 = (1,4, -3,2), u_2 = (1,3, -1,2), u_3 = (3, -8, -2,7)$. Extend this basis of W to a basis of $\mathbb{R}^4$, and prove that your answer is indeed a basis.

b. Find a basis and the dimension of the subspace W of $\mathbb{R}^4$ spanned by $u_1 = (1, -2, -5,3), u_2 = (2,3, -1, -4), u_3 = (3,6, -3, -5)$ Extend this basis of W to a basis of $\mathbb{R}^4$, and prove that your answer is indeed a basis.

c. Find a basis B and the dimension of the subspace of $\mathbb{R}^3$ spanned by $u_1 = (-1,-2,2)$, $u_2 = (3,1,2)$, $u_3 = (4,-2,7)$, $u_4 = (2,2,0)$, $u_5 = (1,4,3)$, so that $B \subseteq \{u_1, u_2, u_3, u_4, u_5\}$.

d. Find a basis B and the dimension of the subspace of $\mathbb{R}^4$ spanned by $u_1 = (3,-1,-2,1), u_2 = (-2,3,-1,2), u_3 = (1,4,-3,3), u_4 = (2,2,0,4), u_5 = (2,3,-9,7)$, so that $B \subseteq \{u_1, u_2, u_3, u_4\}$.

e. Given the vectors $v_1 = (-4,-5,-10,-8), v_2 = (1,0,2,1), v_3 = (-2,3,4,2), v_4 = (1,1,0,3)$. Denote $U = Sp\{v_1, v_2, v_3, v_4\}$. Find a basis B of U such that $B \subseteq \{v_1, v_2, v_3, v_4\}$..

f. Given the vectors $v_1 = (4,1,2,3), v_2 = (0,0, -1,0), v_3 = (1,2,3,2), v_4 = (-2,2,6,5)$. Denote $U = Sp\{v_1, v_2, v_3, v_4\}$. Find a basis B of U such that $B \subseteq \{v_1, v_2, v_3, v_4\}$

7)

a. Find a basis and the dimension of the solution space to the system: $\begin{cases} 2x - y + 4z = 0 \\ x - 2y + 2z = 0 \\ 5x - 4y + 10z = 0 \end{cases}$

b. Find a basis and the dimension of the solution space to the system: $\begin{cases} -2x - 3y = 0 \\ -10x - 15y = 0 \end{cases}$.

c. Find a basis and the dimension of the solution space to the system: $\begin{cases} 5x - y = 0 \\ -x + 12y = 0 \end{cases}$

d. Find a basis and the dimension of the solution space to the system: $\begin{cases} 2x - 2y - 2z + 2w = 0 \\ x - y + z + 3w = 0 \end{cases}$.

8) Determine in each part if the claim is true or false, and explain:

a. If $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^4$, and $\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$, then the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

b. If $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^4$, and $\vec{v}_3 = \vec{0}$, then the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

c. If $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^4$, and $\vec{v}_3$ is **not** a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_4$,, then the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent.

d. If $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are vectors in $\mathbb{R}^4$, and the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

e. If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly independent, then the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

9)

a. Let $B = \{v_1, v_2, v_3\}$ be a basis of a vector space V of dimension 3. Prove or disprove: $\{v_1 + v_2, v_1 + 5v_2 + 2v_3, v_1 + 2v_2 - v_3\}$ is a basis of V.

b. Let $B = \{u, v, w\}$ be a basis of a vector space V. Prove that $C = \{u, u + v, u - v - 2w\}$ is also a basis of V.

10) Let $V = M_3(\mathbb{R})$ and let $W = \{\begin{pmatrix} a & b & c \\ d & a & b \\ e & d & 0 \end{pmatrix} \in V | a, b, c, d, e \in \mathbb{R}\}$. Let U be the space of symmetric 3x3 matrices with real entries.

a. Find a basis of U and its dimension.

b. Find a basis of W and its dimension.

c. Find a basis of $U \cap W$ and its dimension.

11) Let $V = M_3(\mathbb{R})$ and let $W = \{A \in M_3(\mathbb{R}) \mid A = A^T\}$. Let U be the subspace of V consisting of all matrices in which the sum of each row is 0.

a. Find a basis of U and its dimension. Find a basis of W and its dimension.

b. Find a basis of $U \cap W$ and its dimension.

12) Let $V = \mathbb{R}_3[x]$ and let $W = \{p(x) \in R_3[X] | p(1) = 0\}$.
Let $U = \{p(x) \in R_3[X] | p(2) = p(0)\}$.

a. Find a basis of U and its dimension. Find a basis of W and its dimension.

b. Find a basis of $U \cap W$ and its dimension.

13) Given the matrices $A=\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}$, $C=\begin{pmatrix}3 & 4\\ 2 & 1\end{pmatrix}$ in the vector space $M_2(\mathbb{R})$. Let $U = Span\ \{A,B,C\}$.

a. Is the set $\{A,B,C\}$ linearly dependent? Prove or disprove.

b. Find a basis of U.

c. Does the matrix $D=\begin{pmatrix}3 & 7\\ 1 & -2\end{pmatrix}$ belong to the subspace U?

d. If the answer to part (c) is yes, write D as a linear combination of the basis you found in part (b). Otherwise, find the dimension of the set $Span\ \{A,B,C,D\}$.