המרכז האקדמי לב

Calculus I: Exercise 2 (Limit of a Function)

For submission: 1 a, c, f, h, n 2 e, g, j, l, m, p, s, u, z 3 a, c, g, j, l, o, p, r, t, u 4 b, d 5 b, e, g, i, j, m, q, r, t, v, w 6 a, c 7 b 8 a

Definition of a limit

1. In each part: a. Determine the limit based on estimation/ intuition. b. Write out the definition of the limit appropriate to that part c. Use the definition of the limit to prove that your answer from part (a) is correct.

a. $\lim\limits_{x\to 1}(4x-1)$ b. $\lim\limits_{x\to -2}(3x+11)$ c. $\lim\limits_{x\to 0^-} e^{\frac{1}{x}}$ d. $\lim\limits_{x\to 0^+} e^{\frac{1}{x}}$ e. $\lim\limits_{x\to 0} e^{\frac{1}{x^2}}$

f. $\lim\limits_{x\to 0^+}(\ln x)$ g. $\lim\limits_{x\to \infty}(\ln x)$ h. $\lim\limits_{x\to -\infty}\frac{1}{\ln\left(x^2\right)}$ i. $\lim\limits_{x\to +\infty}\frac{1-2^x}{1+2^x}=-1$ j. $\lim\limits_{x\to -\infty}\frac{1-e^x}{1+e^x}=1$ \

k. $\lim\limits_{x\to 1}\left(\frac{1}{1+2^{\frac{1}{x}}}\right)$ l. $\lim\limits_{x\to \frac{1}{2}}\left(\frac{1}{1+2^{\frac{1}{x}}}\right)$ m. $\lim\limits_{x\to \infty}\left(\frac{1}{1+2^{\frac{1}{x}}}\right)$ n. $\lim\limits_{x\to 0^+}\left(\frac{1}{1+2^{\frac{1}{x}}}\right)$ o. $\lim\limits_{x\to 0^-}\left(\frac{1}{1+2^{\frac{1}{x}}}\right)$

Limit Arithmetic

2. Calculate each (extended) limit or explain why it doesn't exist. If the limit doesn't exist, check if either or both of the one-sided limits exist, and if so calculate them.

(solve parts (a) – (d) where: (i) $a=\infty$ (ii) $a=-\infty$ (iii) $a=1$)

a. $\lim\limits_{x\to a}\frac{x+3}{(x-1)^2}$ b. $\lim\limits_{x\to a}\frac{x^2-x+3}{(x-1)^2}$ c. $\lim\limits_{x\to a}\frac{x^3-x+3}{(x-1)^2}$ d. $\lim\limits_{x\to a}\frac{x^3-3x+2}{(x-1)^2}$

e. $\lim\limits_{x\to a}\frac{7x^2-2x-5}{x^2-3x+2}$ where: (i) $a=1$ (ii) $a=2^+$ (iii) $a=2$ (iv) $a=-1$

f. $\lim\limits_{x\to a}\frac{x^3+27}{x^2-9}$ where: (i) $a=3$ (ii) $a=-3$ (iii) $a=\infty$ (iv) $a=-\infty$

g. $\lim\limits_{x\to a}\frac{x^4-7x^3+2x^2+5x-1}{x^3+2x^2-x-2}$ where: (i) $a=2$ (ii) $a=1$ (iii) $a=-1$ (iv) $a=-2$ (v) $a=-\infty$

h. $\lim\limits_{x\to a}\frac{2x^3+12x^2+5x-22}{-3x^4-2x^3+25x^2+28x-12}$ where: (i) $a=-2$ (ii) $a=3$ (iii) $a=\frac{1}{3}$ (iv) $a=-\infty$

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i. $\lim\limits_{x\to0}\frac{\sqrt{x^2+1}-1}{x}$ j. $\lim\limits_{x\to0}\frac{\sqrt{x^2+3x+1}-1}{x}$ k. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+1}+x\right)$ l. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+3x}+x\right)$

m. $\lim\limits_{x\to-3}\frac{\sqrt{x^2+2x+6}-\sqrt{2x^2-9}}{1-\sqrt{x+4}}$ n. $\lim\limits_{x\to+\infty}\frac{\sqrt{x^2+2x+6}-\sqrt{2x^2-9}}{1-\sqrt{x+4}}$ o. $\lim\limits_{x\to+\infty}\frac{\sqrt{x^2+2x+6}-\sqrt{x^2+x+3}}{1-\sqrt{x+4}}$

p. $\lim\limits_{x\to-\infty}\frac{\sqrt{x^2+3x+8}}{2x+3}$ q. $\lim\limits_{x\to-\infty}\frac{\sqrt{3x^2+5x+8}-\sqrt{x^2-3x+6}}{4-5x}$ r. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+3x+8}+\sqrt{x^2+1}\right)$

s. $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+3x+8}-\sqrt{x^2+1}\right)$ t. $\lim\limits_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt[3]{1+x}-\sqrt[3]{1-x}}$ (Use the identity: $A^3-B^3=(A-B)(A^2+AB+B^2)$) u. $\lim\limits_{x\to+\infty}\left(\sqrt{(x+a)(x+b)}-x\right)$ $a,b\in\Re$ v. $\lim\limits_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

w. $\lim\limits_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$ x. $\lim\limits_{x\to+\infty}\frac{7+6\log x}{-4+3\log x}$ y. $\lim\limits_{x\to0+}\frac{7+6\log x}{-4+3\log x}$ z. Given the function f:

$$f(x)=\begin{cases}2x^3-5x^2-\frac{3}{x} & : \ x<3\\ 4 & : \ x=3\\ \frac{x+5}{(x-4)^2} & : \ x>3\end{cases}$$

calculate the limits: $\lim\limits_{x\to0}f(x)$, $\lim\limits_{x\to2}f(x)$, $\lim\limits_{x\to3}f(x)$, $\lim\limits_{x\to4}f(x)$, $\lim\limits_{x\to+\infty}f(x)$, $\lim\limits_{x\to-\infty}f(x)$

3. Use the limit $\lim\limits_{x\to0}\frac{\sin x}{x}=1$ and prove that $\lim\limits_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$. Using these two limits and trigonometric identities, calculate the following limits:

a. $\lim\limits_{x\to0}\frac{\sin 3x}{\sin 5x}$ b. $\lim\limits_{x\to0}\frac{1-\cos 3x}{1-\cos 5x}$ c. $\lim\limits_{x\to0}\frac{\tan 4x\cos 5x}{\sin 3x}$ d. $\lim\limits_{x\to0}\frac{\sin 8x-\sin 3x}{\sin 6x}$

e. $\lim\limits_{x\to0}\frac{1-\cos 10x}{\sin 3x}$ f. $\lim\limits_{x\to0}\frac{1-\cos 10x}{4x\sin 3x}$ g. $\lim\limits_{x\to0}\frac{1-\cos 10x}{\sin^2 3x}$ h. $\lim\limits_{x\to0}\frac{\sin 3x-\sin 3x\cos 4x}{7x^2}$

i. $\lim\limits_{x\to+\infty}\left(x\cdot\tan\frac{1}{x}\right)$ j. $\lim\limits_{x\to0}\frac{x^2\cdot\sin\frac{1}{x}}{\sin x}$ k. $\lim\limits_{x\to-2}\frac{x^2-4}{\tan(x+2)}$ l. $\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos 3x}{\left(\frac{\pi}{2}-x\right)}$ m. $\lim\limits_{x\to0}\frac{\cos^2 x}{\sin 3x}$

n. $\lim\limits_{x\to0}\frac{\sqrt{1+\sin 3x}-1}{\sin 5x}$ o. $\lim\limits_{x\to+\infty}\frac{2x+5\sin x}{5x-2}$ p. $\lim\limits_{x\to0}\frac{2x+5\sin x}{5x-2}$ q. $\lim\limits_{x\to\pi}\frac{2x+5\sin x}{5x-2}$

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r. $\lim_{x\to\pi} \frac{\sin 3x}{\sin 4x}$ s. $\lim_{x\to+\infty} \frac{\sin x}{x}$ t. $\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$ u. $\lim_{x\to 0} \tan 4x \cot 7x$

v. $\lim_{x\to\frac{\pi}{4}} \tan(2x)\tan\left(\frac{\pi}{4} - x\right)$

4) Use the Sandwich Theorem to calculate (NOTE: square brackets indicate the whole number function, e.g., $[2.4]=2$, $[5]=5$, $[-1.4]=-2$):

a. $\lim_{x\to 0} x[x]$ b. $\lim_{x\to 0}\left(x\left[\frac{7}{x}\right]\right)$ c. $\lim_{x\to 0}\left(x\sin\frac{1}{x}\right)$ d. $\lim_{x\to 0}\left(x\sqrt{3-2\cos\frac{1}{x}}\right)$

5) Calculate the following limits. If a limit doesn't exist, calculate the one-sided limits if they exist. You may use the limits: $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$, $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$, and

$$\lim_{x\to+\infty} a^x = \begin{cases} +\infty & : a>1 \\ 0 & : 0<a<1 \end{cases}$$

Note that you can use substitutions (see question 8) to reduce each part to applications of the rules of limit arithmetic to one of the above limits.

a. $\lim_{x\to 0}(1+2x)^{\frac{3}{4x}}$ b. $\lim_{x\to 0}(1-2x)^{\frac{3}{4x}}$ c. $\lim_{x\to\infty}\left(1+\frac{3}{4x}\right)^{5x}$ d. $\lim_{x\to\infty}\left(1-\frac{7}{4x}\right)^{5x}$

e. $\lim_{x\to-\infty}\left(1-\frac{4}{2x-1}\right)^{6x-3}$ f. $\lim_{x\to\infty}\left(1+\frac{m}{x}\right)^{nx}$ $(m,n\in\mathbb{Q})$ g. Prove that: $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x+1} = e$ and $\lim_{x\to+\infty}\left(1-\frac{1}{x}\right)^{x+1} = e^{-1}$ [Calculate each of the following limits after writing the base in the form $1+\frac{1}{t}$.] h. $\lim_{x\to+\infty}\left(\frac{x+4}{x+3}\right)^x$ i. $\lim_{x\to+\infty}\left(\frac{x+4}{x+5}\right)^x$ j. $\lim_{x\to+\infty}\left(\frac{2x+5}{2x+1}\right)^{2x-3}$

k. $\lim_{x\to+\infty}\left(\frac{2x+3}{2x+6}\right)^{2x-1}$ l. $\lim_{x\to+\infty}\left(\frac{2x+3}{2x-1}\right)^{3x+4}$ m. $\lim_{x\to+\infty}\left(2+\frac{1}{x}\right)^x$ (HINT: Use the equation $\left(2+\frac{1}{x}\right)^x = \left(2\left(1+\frac{1}{2x}\right)\right)^x = 2^x\left(1+\frac{1}{2x}\right)^x$) n. $\lim_{x\to+\infty}\left(1.00001+\frac{1}{x}\right)^x$

o. $\lim_{x\to+\infty}\left(0.99999+\frac{1}{x}\right)^x$ p. Prove that if for some real number a, $\lim_{x\to a} g(x) = +\infty$, and there exists a real number l such that $\lim_{x\to a}(f(x)) = l$ then: $\lim_{x\to a}\left(f(x)^{g(x)}\right) = \begin{cases} +\infty & : l>1 \\ 0 & : 0\le l<1 \end{cases}$
(HINT: Use the Sandwich Theorem.)

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Use part (p) to calculate the limits: q. $\lim_{x\to+\infty}\left(\frac{2x+3}{x-4}\right)^{3x-5}$ r. $\lim_{x\to+\infty}\left(\frac{2x+1}{3x-2}\right)^{4x-3}$ s. $\lim_{x\to0^+}\left(\frac{2x+3}{x+4}\right)^{\frac{3}{x}}$

t. $\lim_{x\to0^-}\left(\frac{2x+3}{x+4}\right)^{\frac{3}{x}}$ u. $\lim_{x\to1^+}2^{\frac{1}{1-x}}$ v. $\lim_{x\to1^-}2^{\frac{1}{1-x}}$

w. Note that in part (p) we didn't address the case $\lim_{x\to a}\left(f(x)\right)=1$ and $\lim_{x\to a}g(x)=+\infty$. That's because in this case, computing $\lim_{x\to a}\left(f(x)^{g(x)}\right)$ requires additional work, as we saw in parts (a)-(o). Find three examples of functions *f* and *g* for which this limit takes three different values. HINT: Use the definition of the number *e*, and parts (a)-(o).

6) Prove the following limits:

a. $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\quad(n\in\mathbb{N},\ a\in\mathbb{R})$

(Use the factorization $x^n-a^n=(x-a)\left(x^{n-1}+x^{n-2}a+\ldots+xa^{n-2}+a^{n-1}\right)$)

b. $\lim_{x\to a}\frac{x^{-n}-a^{-n}}{x-a}=-na^{-n-1}\quad(n\in\mathbb{N},\ a\in\mathbb{R})$

c. $\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\cos a$

(Use an identity to turn the difference of sines into a product, or substitute $x=a+t$.)

d. $\lim_{x\to a}\frac{\cos x-\cos a}{x-a}=-\sin a$

e. $\lim_{x\to a}\frac{\tan x-\tan a}{x-a}=\frac{1}{\cos^2 a}$

f. $\lim_{x\to a}\frac{\cot x-\cot a}{x-a}=-\frac{1}{\sin^2 a}$

7) Prove the following rules of limit arithmetic:

Let $l\in\mathbb{R}\setminus\{0\}$ and $a\in\mathbb{R}$. Prove:

a. If $\lim_{x\to a}f(x)=l<0$ and $\lim_{x\to a}g(x)=-\infty$ then $\lim_{x\to a}[f\cdot g](x)=+\infty$.

b. If $\lim_{x\to-\infty}f(x)=l<0$ and $\lim_{x\to-\infty}g(x)=-\infty$ then $\lim_{x\to-\infty}[f\cdot g](x)=+\infty$.

c. If $\lim_{x\to a}f(x)=+\infty$ and $\lim_{x\to a}f(x)=-\infty$ then $\lim_{x\to a}\frac{1}{f(x)}=0$.

Is it also true that if $\lim_{x\to a}f(x)=0$ then $\lim_{x\to a}\frac{1}{f(x)}=+\infty$ or $\lim_{x\to a}\frac{1}{f(x)}=-\infty$?

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8) (Challenge) Read and understand the claims even if you don't prove them.

In each part, $a \in \mathbb{R}$ and $l \in \mathbb{R}$ or $l = \pm\infty$. In proving each claim it suffices to assume $l \in \mathbb{R}$.

a. From the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ we concluded that: $\lim\limits_{x\to\pi}\frac{\sin(x-\pi)}{x-\pi} = 1$. Is this valid?

Prove: If $\lim\limits_{x\to a} f(x) = l$ then $\lim\limits_{x-a\to 0} f(x) = l$, i.e., we can substitute $t = x - a$ and obtain $\lim\limits_{t\to 0} f(t+a) = l$.

b. From the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ we concluded that: $\lim\limits_{x\to 0}\frac{\sin 7x}{7x} = 1$. Is this valid?

Prove: If $\lim\limits_{x\to a} f(x) = l$ then $\lim\limits_{x\to 0} f(ax) = l$ when $a \neq 0$.

c. From the limit $\lim\limits_{x\to 0}(1+x)^{1/x} = e$ we concluded that $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$ and also $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$. Is this valid?

Prove: $\lim\limits_{x\to 0+} f(x) = l$ iff $\lim\limits_{x\to+\infty} f\left(\frac{1}{x}\right) = l$, and $\lim\limits_{x\to 0-} f(x) = l$ iff $\lim\limits_{x\to-\infty} f\left(\frac{1}{x}\right) = l$.