Calculus I: Exercise 5 (Fundamental Theorems and Graph Sketching)

Submit: 1 4, 12, 16, 18, 23 2 b, d 3 a2, c1, c2 4 b, d, f 5, 7, 8 9 b, d 10 a, b 11, 13 b2, c 14 b,d 15 a, d, e

1) Investigate for each function: a. domain b. zero points, where positive/ negative c. extreme points, regions of ascent/ descent d. inflection points, regions of convexity/ concavity [NOTE: skip in parts 7 and 22] e. asymptotes f. miscellaneous (local discontinuities, differentiability, even/ odd, periodicity)

Based on the above sketch a graph of the function (if periodic then sketch one period).

1. $f(x)=5x^4-4x^5$ 2. $f(x)=2x^4-9x^2+4$ 3. $f(x)=\left|x^3-6x^2\right|$ 4. $f(x)=x^2-3|x|$

5. $f(x)=\dfrac{x}{x^2+4}$ 6. $f(x)=\dfrac{2+x-x^2}{(x-1)^2}$ 7. $f(x)=\dfrac{2x^2}{x^2-x-2}$ 8. $f(x)=\dfrac{x^2}{x-4}$

9. $f(x)=e^{-0.5x^2}$ 10. $f(x)=xe^{-x}$ 11. $f(x)=\dfrac{e^x}{x-3}$ 12. $f(x)=\dfrac{e^x-1}{e^x-2}$

13. $f(x)=x\ln|x|$ 14. $f(x)=\dfrac{x}{\ln x}$ 15. $f(x)=\dfrac{\ln x}{x}$ 16. $f(x)=\ln\left|\dfrac{x-1}{x+1}\right|$

17. $f(x)=x+\sqrt{x}$ 18. $f(x)=x\cdot\sqrt{2x+1}$ 19. $f(x)=x+\sqrt{x^2+1}$ 20. $f(x)=\dfrac{x+2}{|x|-1}$

21. $f(x)=4\cos x+\sin 2x$ 22. $f(x)=\sin^3 x+\cos^3 x$ 23. $f(x)=x-\arctan x$

24. $f(x)=x\arctan x$ 25. $f(x)=\sqrt[3]{(x-1)^2(x+4)}$ (domain is $\mathbb{R}$). 26. $f(x)=\ln\sin x$

27. $f(x)=\begin{cases} x+1, & x\le 1 \\ x^2+1, & x>1 \end{cases}$ 28. $f(x)=\begin{cases} x+1 & : x\le 1 \\ -x^2+2 & : x>1 \end{cases}$ 29. $\begin{cases} y=1-\cos t \\ x=t-\sin t \end{cases}$ $\left(t\in[0,4\pi]\right)$

2) Recall Weierstrass' Theorem: A continuous function on a closed interval attains its maximum and minimum values therein. Find the absolute maximum and minimum of each function in the given domain, or determine that it doesn't exist and explain why.

a. $f(x)=3x^4+4x^3+2$ in $[-2,1]$ b. $f(x)=2x^3-9x^2+12x-2$ in $[0,3]$

c. $f(x)=\dfrac{2}{x}+\dfrac{x}{2}$ in $[-4,1]$ d. (i) $f(x)=x-\ln x$ in (a) $[0.5,4]$ (b) $[-2,4]$

(ii) $f(x)=x-\ln|x|$ in (a) $[0.5,4]$ (b) $[-2,4]$

e. $f(x)=x-\sqrt{|x^2-3x+2|}$ in $[0,3]$ f. $f(x)=x+\sin^2 x$ in (a) $[0,\pi]$ (b) $[-\pi,\pi]$

3) How many solutions does each equation have? Explain fully!

a. 1. $\arctan x+e^x=3$ 2. $\arctan x-e^{-x}=3$

b. 1. $1-4\ln x=6x+x^2$ 2. $1-4\ln|x|=6x+x^2$

c. 1. $1+8\ln x=6x+x^2$ 2. $1+8\ln|x|=6x+x^2$

4) Solve each of the following extreme problems:

   a. Of all pairs of non-negative numbers with sum 60, find the pair the sum of whose cubes is smallest.
   b. The sides of a rectangle have lengths 8 cm and 18 cm. The rectangle is inscribed in a right triangle whose two legs are sides of the rectangle (and their continuations). How long are the legs of the triangle if its area is as small as possible?
   c. From a 30 cm$\times$10 cm cardboard rectangle, identical squares are cut from each corner, and the rectangles protruding on each side are folded upward to make an open-topped box. What is the side length of each corner square if the box has maximal area?
   d. The radius of the base of a cone is R and its height is H. What is the maximum volume of a cylinder inscribed in the cone with one base lying on the base of the cone?
   e. A rectangle is inscribed in a semicircle (of radius R) with two vertices on the diameter and two on the arc. What are the side lengths of the rectangle if its area is maximal?
   f. We wish to connect the graphs of the parabola $y = x^2$ and the line $3x + 4y + 12 = 0$ by a line segment of minimal length. At what point on the parabola will this segment begin?
   g. A river of width *a* meters and a tunnel of width *b* meters (with $a > b$) are perpendicular to each other. A boat (of negligible width) must turn from the river into the tunnel. What is the longest the boat can be and still be able to do so?

5) Given the function $f(x) = \ln(x^2 + 1)$.
   a. Sketch the graph of *f*.
   b. Let *g* denote the restriction of *f* to the domain $[0,+\infty)$. Prove that *g* is invertible as a function from $[0,+\infty)$ to $[0,+\infty)$.
   c. Sketch the graph of g^{-1}.
   d. Find an explicit formula for g^{-1}.

6) Given the function $\begin{cases} f(x) = x\ln x & (x>0) \\ f(0) = 0 \end{cases}$
   a. Sketch the graph of *f*. Check if *f* has a tangent line at the origin.
   b. Let *g* denote the restriction of *f* to the domain $[e^{-1},+\infty)$. Prove that *g* is invertible as a function from $[e^{-1},+\infty)$ to $[-e^{-1},+\infty)$.
   c. Sketch the graph of g^{-1}.
   d. Find the number of solutions to the equation $f(x) = m$ depending on the value of *m*.

7) Given the function $f(x) = \ln(x + \sqrt{x^2 + k})$ with $k > 0$.
   a. Find the largest possible domain *D* in which *f* is differentiable.
   b. Find the derivative $f'(x)$ for all $x \in D$.
   c. Prove that *f* is invertible and find its inverse.
   d. (to solve after question 17) When $k = 1$ what is the inverse function of *f* called?

8) Given the function $f(x) = x^{1/x}$. Answer the following without using a calculator!

a. What is the domain of *f*? Where is it ascending and where is it descending?

b. Conclude from part (a) which is greater: $\sqrt[\pi]{\pi}$ or $\sqrt[e]{e}$.

c. Conclude from part (b) which is greater: e^{π} or π^{e} .

d. How many pairs of distinct numbers *a*, *b* can you think of such that $a^b = b^a$? Use part (a), and the graph of *f* which can be obtained from it, to find how many such pairs exist.

9) Solve the following:

a. The function $f(x) = ax^2 + bx + c$ has an extreme value of 4 obtained when $x = 3$, and we know that $f(2) = 6$. Find *a*, *b*, and *c*.

b. The point $(1,3)$ is an inflection point of $f(x) = ax^3 + bx^2 + cx + d$, and the point $(2,-1)$ is an extreme point of *f*. Find *a*, *b*, *c*, and *d*.

c. The graph of the function $f(x) = \dfrac{ax^2 + bx + c}{x + d}$ has two asymptotes with equations $x = -2$ and $y = 3x + 1$. The *y*-intercept of the graph is $(0,3)$. Find *a*, *b*, *c*, and *d*.

d. The line with equation $y = -2x + 3$ is an asymptote of the graph of $f(x) = \dfrac{ax^2 + bx + c}{x + 3}$, and the line $y = -3x + 2$ intersects the graph of *f* at $x = 1$. Find *a*, *b*, and *c*.

e. An object is thrown upwards. Its height in meters after *t* seconds is given by $f(t) = at^2 + bt + c$. After 8 seconds from the moment it is thrown, its height is 462 m and its velocity is 18 m/s. Two seconds later it reaches is maximal height.

i. What is its maximal height?
ii. What is its height after 12 seconds (from the moment it is thrown)?
iii. What is the starting height and starting velocity of the object?
iv. What will be the velocity of the object when it reaches the ground?
v. What is the acceleration of the object throughout its flight?

[NOTE: Acceleration is the rate of change of velocity, just as velocity is the rate of change of distance.]

f. An infinite set of curves $C_1, C_2, \ldots$ is given in the plane where the equation of C_m is: $y = mx^3 + 3mx^2 - (8m + 5)x - (10m + 3)$, for each $m \neq 0$.

1. Prove that every curve C_m has the same inflection point. What is it?
2. For which values of *m* does the curve have no extreme point?

10) Approximate the following: a. $\sqrt{65}$ b. $\sqrt[3]{26}$ c. $\sqrt[4]{80}$ using one of two methods: approximating a point on a graph by a point on an appropriate tangent line; or using Lagrange's theorem.

11) Given the function $f(x) = \tan x$. Prove that there is no point $c \in (0, \pi)$ with $f'(c) = 0$. Does this contradict Rolle's Theorem?

12) An object starts from rest and after 10 minutes has moved 10 km. Prove that at some moment during those 10 minutes its velocity was exactly 60 km/h. (Note: The velocity of a moving object is the derivative of the distance it has traveled.)

13) For the function and interval in each part, show that the conditions of Lagrange's Theorem are met, and find the value c in the interval for which the equation in the theorem holds.

a. $f(x)=x^3-2x^2+1$ in the interval: (1) $[0,1]$ (2) $[2,3]$

b. $f(x)=\sin x$ in the interval: (1) $\left[0,\frac{\pi}{2}\right]$ (2) $\left[\frac{\pi}{2},\pi\right]$

c. $f(x)=\ln x$ in the interval $[1,e]$

14) Use Lagrange's Theorem to prove the following claims. Explain exactly how you use the theorem in each case, and remember to check that the condition of the theorem are met!

a. For all $x,y\in\mathbb{R}$ we have $|\sin x-\sin y|\le|x-y|$.

b. For all $x,y\in\mathbb{R}$ we have $|\arctan x-\arctan y|\le|x-y|$.

c. For all $x,y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ we have $|\tan x+\tan y|\ge|x+y|$.

d. For all $x,y\in(0,+\infty)$ with $x>y$ we have $\frac{x-y}{x}<\ln\frac{x}{y}<\frac{x-y}{y}$.

15) Find examples of the following:

a. a function *f* defined on an interval $[a,b]$ and continuous and differentiable on (a,b) with $f(a)=f(b)$, without any $c\in(a,b)$ such that $f'(c)=0$.

b. a function *f* continuous on an interval $[a,b]$ with $f(a)=f(b)$, without any $c\in(a,b)$ such that $f'(c)=0$.

c. a function *f* defined on an interval $[a,b]$ and continuous and differentiable on (a,b), without any $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

d. a function *f* continuous on an interval $[a,b]$ without any $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

e. Write out Rolle's Theorem and Lagrange's Theorem exactly, and in light of parts (a)-(d) understand the significance of all the conditions.

16) a. Find an example of a function defined on $[a,b]$, continuous and differentiable on (a,b) and with positive derivative in (a,b), but not increasing on $[a,b]$.

b. Use Lagrange's Theorem to prove: If *f* is continuous on $[a,b]$ and differentiable on (a,b), and with positive derivative in (a,b), then *f* is increasing on $[a,b]$.
[Hint: Note what you need to prove: If $x<y$ then $f(x)<f(y)$.]

17) <u>The Hyperbolic Functions</u> (NOTE: This question is not for submission but you are responsible for the functions and their basic properties.)

Define: $\sinh t = \frac{1}{2}\left(e^t - e^{-t}\right)$ (hyperbolic sine) from $\mathbb{R}$ to $\mathbb{R}$.

$\cosh t = \frac{1}{2}\left(e^t + e^{-t}\right)$ (hyperbolic cosine) from $\mathbb{R}$ to $[1,+\infty)$.

$\tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ (hyperbolic tangent) from $\mathbb{R}$ to $(-1,1)$.

$\coth t = \frac{\cosh t}{\sinh t} = \frac{e^t + e^{-t}}{e^t - e^{-t}}$ (hyperbolic cotangent) from $\mathbb{R}-\{0\}$ to $(-\infty,-1)\cup(1,+\infty)$.

Sketch the graph of each function.

Properties:

a. cosh is even, the rest are odd functions (check!)

b. cosh is positive (check!)

c. Identities:

$\cosh^2 t - \sinh^2 t = 1$ (Prove!) $\qquad 1-\tanh^2 t = \frac{1}{\cosh^2 t} \qquad \coth^2 t - 1 = \frac{1}{\sinh^2 t}$

$\sinh(t_1 \pm t_2) = \sinh t_1 \cosh t_2 \pm \sinh t_2 \cosh t_1 \qquad \cosh(t_1 \pm t_2) = \cosh t_1 \cosh t_2 \pm \sinh t_1 \sinh t_2$

$$\tanh(t_1 \pm t_2) = \frac{\tanh t_1 \pm \tanh t_2}{1 \pm \tanh t_1 \tanh t_2}$$

$\sinh 2t = 2\sinh t \cosh t$ (Prove!)

$$\cosh 2t = \cosh^2 t + \sinh^2 t = 1 + 2\sinh^2 t = 2\cosh^2 t - 1$$

$$\tanh 2t = \frac{2\tanh t}{1+\tanh^2 t}$$

d. The hyperbolic functions are continuous and differentiable in their domains, and:
$[\sinh]'(t) = \cosh t$, $[\cosh]'(t) = \sinh t$, $[\tanh]'(t) = \frac{1}{\cosh^2 t}$, $[\coth]'(t) = \frac{-1}{\sinh^2 t}$. (Check!)

e. Inverses (check all three claims!):

1. sinh is invertible, with inverse (from $\mathbb{R}$ to $\mathbb{R}$): arcsinh $t = \ln\left(t + \sqrt{t^2+1}\right)$.
2. cosh is invertible as a function from $[0,+\infty)$ to $[1,+\infty)$, with inverse (from $[1,+\infty)$ to $[0,+\infty)$): arccosh $t = \ln\left(t + \sqrt{t^2-1}\right)$.

(cosh is also invertible as a function from $(-\infty,0]$ to $[1,+\infty)$, with inverse (from $[1,+\infty)$ to $(-\infty,0]$): $-\text{arccosh}\, t = \ln\left(t-\sqrt{t^2-1}\right)$

3. tanh is invertible, with inverse (from $(-1,1)$ to $\mathbb{R}$): $\text{arctanh}\, t = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$.

f. The inverse hyperbolic functions are continuous and differentiable in their domains and:

$$[\text{arcsinh}]'(t) = \frac{1}{\sqrt{t^2+1}} \qquad [\text{arccosh}]'(t) = \frac{1}{\sqrt{t^2-1}} \qquad [\text{arctanh}]'(t) = \frac{1}{1-t^2}$$

(Check all three derivatives.)

Further explanation: The trigonometric functions are also called circular functions because using a parameter *t*, the pair $(\cos t, \sin t)$ gives a point on the unit circle $x^2+y^2=1$ (*t* is the "circular angle").

If we change the circle to the hyperbola $x^2-y^2=1$ then each point in the right half of the hyperbola can be denoted $(\cosh t, \sinh t)$, where the parameter *t* denotes double the area between the *x*-axis, the graph of the hyperbola, and the interval connecting the origin to the point $(\cosh t, \sinh t)$.