Linear Algebra 1—Exercise 6 (Matrices)

To submit: 1 b, e 2 b, f 3 c 4 a 5 d 6 a 7 a, d 8 10 d 11 14 b 16 c 18

1) Multiply:

a. $\begin{pmatrix} 0 & 7 & -1 \\ -2 & 2 & 0 \\ 0 & 12 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -3 \\ 0 \\ 2 \end{pmatrix}$

c. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

d. $\begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

e. $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

f. $\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$

g. $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 9 & 1 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 5 & 7 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$ (think before working too hard…)

h. If $A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$, check the following claim: $(I + A + A^2)(I - A) = I - A^3$

2) Calculate the inverse of the given matrix and check your answer. In parts (b) and (f) write the matrix as a product of elementary matrices.

a. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}^{-1}$ b. $\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}^{-1}$ c. $\begin{bmatrix} 12 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ e. $\begin{bmatrix} 1 & 0 & 0 \\ 14 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$ g. $\begin{bmatrix} 4 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$ h. $A = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}$

3) Solve each system using an inverse matrix:

a. $\begin{cases} 4x + 2y = 2 \\ 3x + 2y = 1 \end{cases}$ b. $\begin{cases} x + y + z = 2 \\ 2x + 2y + 6z = 7 \\ 2x + y + 3z = 3 \end{cases}$ c. $\begin{cases} 2x + 6y - 4z = 0 \\ 2x + 5y - 3z = -2 \\ -3x + 2y - 4z = 4 \end{cases}$

4) Prove or disprove:

a. If A and B are invertible matrices of the same order, then AB is also invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

b. If A is invertible, $r \neq 0$, and $B = rA$, then B is invertible and $B^{-1} = rA^{-1}$.

c. If A is invertible and $AB = BA$, then $A^{-1}B = BA^{-1}$.

5) Given the matrices $A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$,. Find elementary matrices E_1, E_2, E_3, E_4 such that: a. $E_1A = B$ b. $E_2B = A$ c. $E_3A = C$ d. $E_4C = A$

6)

a. Find A if $A^{-1} = \begin{pmatrix} 3 & -1 \\ 3 & 5 \end{pmatrix}$.

b. Find A if $(3A)^{-1} = \begin{pmatrix} -3 & 6 \\ 1 & -1 \end{pmatrix}$.

c. Find A such that $BA = C$, where $B = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 \\ 6 & 0 \end{pmatrix}$.

7) Let $A,B,C \in M_n(\mathbb{R})$.
   a. Prove: If A,B,C are invertible then ABC is invertible. (Hint: Use inverse matrices.)
   b. Prove: If AB is invertible, then both A and B are invertible. (Hint: Denote $C = AB$.)
   c. Prove or disprove: If $A^2 = I$ then $A = I$ or $A = -I$.
   d. Prove or disprove: If $AA = A$ then $A = I$.

8) A matrix $A \in M_n(\mathbb{R})$ is called **symmetric** if $A = A^T$. Prove or disprove:
   a. If A and B are symmetric matrices, then AB is also symmetric.
   b. For any $A \in M_n(\mathbb{R})$, the matrix $A + A^T$ is symmetric.
   c. For any matrix A (not necessarily square), AA^T is symmetric.

9) A matrix $A \in M_n(\mathbb{R})$ is called **antisymmetric** if $A = -A^T$. Choose (and prove) the correct answer: For any square matrix A, the matrix $A - A^t$ is:
   1. Always symmetric 2. Always antisymmetric 3. Sometimes symmetric, sometimes antisymmetric, and sometimes neither.

10) A square matrix A is called **upper triangular** if every element under the main diagonal is 0. A square matrix A is called **lower triangular** if every element above the main diagonal is 0. True or false?
   a. If A, B are upper triangular then $A+B$ is upper triangular.
   b. If A, B are upper triangular then AB is upper triangular.
   c. Every diagonal matrix is both upper triangular and lower triangular.
   d. If A is upper triangular then A^T is also upper triangular.
   e. If A is upper triangular then A^T is lower triangular.
   (Answers: All true except (d).)

11) (Exam question) Given the matrix $A = \begin{pmatrix} 1 & -m & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -m \end{pmatrix}$.
   a. For which values of the parameter m is A invertible?
   b. Calculate A^{-1} if $m = -2$.

c. Infer from part (b) the solution to the system: $\begin{cases} x_1+2x_2+x_3=0 \\ x_1+x_3=-2 \\ x_1+2x_3=-2 \end{cases}$.

12) Let A, B be matrices such that the product AB is defined.

a. Prove that if the first row of A is all zeroes, then so is the first row of AB.

b. Prove that if the first column of B is all zeroes, then so is the first column of AB.

(Note that both claims remain true when the designation "first" is changed to any other row or column—"second", "third", etc.)

13) Given: $A=\begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix}$, $B=\begin{pmatrix}0&1&2\\0&0&1\\0&0&0\end{pmatrix}$.

a. Calculate B^3.

b. Use part (a) to calculate A^n for all $n\in\mathbb{N}$. (Hint: $A=B+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$)

c. Calculate A^{-n} for all $n\in\mathbb{N}$.

14) (Exam question, 2010)

a. A matrix A is called **nilpotent** if there exists $k\in\mathbb{N}$ such that $A^k=0$. Prove that if A is nilpotent then A is not invertible.

b. Find a square matrix of order 3 such that $A\begin{pmatrix}x+y\\x-z\\z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ for all $\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3$.

15) (Exam question, 2011)

Let A be a 3x3 matrix, and consider the elementary matrices:

$E_{13}(2)=\begin{pmatrix}1&0&2\\0&1&0\\0&0&1\end{pmatrix}, P_{12}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$. True or false?

a. If $AA=A$ then $A=I$.

b. $E_{1,3}(2)A$ is invertible if and only if A is invertible.

c. $P_{12}A$ is symmetric if and only if A is symmetric.

16) For each matrix, find a basis for the row space and a basis for the column space of the matrix. Check that the dimensions are equal.

a. $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & -5 & -5 \\ 1 & 4 & 9 & 8 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 5 & 1 & 0 \\ 0 & 2 & 3 & 3 & -1 & 2 \end{pmatrix}$ c. $\begin{pmatrix} 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \end{pmatrix}$

17) Given a matrix *A*, let *K* denote the set of rows of *A* and let *T* denote the set of columns of *A*. Suppose *B* is obtained from *A* by row reduction. Let K_1 denote the set of rows of *B* and let T_1 denote the set of columns of *B*.

Which of the following are always true? Explain!

a. $Sp(K) = Sp(K_1)$

b. $Sp(T) = Sp(T_1)$

c. $Sp(K) = Sp(T)$

d. $\dim Sp(K_1) = \dim Sp(T_1)$

18) Let $A = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, and let $U = \{B \in M_2(\mathbb{R}) : AB = B\}$.

a. Prove that *U* is a subspace of $M_2(\mathbb{R})$.

b. Find a basis of *U*. What is the dimension of *U*? (You need to prove that the set you found is a basis!)

19) Let $W = \{A \in M_3(\mathbb{R}) \mid A^T = 2A\}$, $U = \{A \in M_3(\mathbb{R}) \mid A^T = -A\}$.

a. Prove that *W* is a subspace of $M_3(\mathbb{R})$.

b. Prove that *U* is a subspace of $M_3(\mathbb{R})$.

c. Find *dim U* and *dim W*.

d. Find a basis of $U \cap W$.