המרכז האקדמי לב

Calculus I: Exercise 1 (Sequences)

For submission: 1 a, e 2 3 d, e, g, h 4 a 6 a, d 7 b, d 8 (not mandatory, but recommended!)
9 b 10 c, e, f, i, j, l, n, p, r 11 a 12 14 16 (for reading only) 17 a, c 18 19 c, f, h

A. The Real numbers—Rational and Irrational numbers

1. Determine if each of the following claims is correct. Prove or disprove!

   a. The sum of any two rational numbers is rational.
   b. The product of any two rational numbers is rational.
   c. The sum of any two irrational numbers is irrational.
   d. The product of any two irrational numbers is irrational.
   e. For any $x, y \in \mathbb{R}$, if $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$ then $x + y \in \mathbb{R} \setminus \mathbb{Q}$ (in words: the sum of a rational and an irrational number is irrational).
   f. For any $x, y \in \mathbb{R}$, if $x \in \mathbb{Q} \setminus \{0\}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$ then $xy \in \mathbb{R} \setminus \mathbb{Q}$.
   g. There exist $x, y \in \mathbb{R}$ such that $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$ and $y^x \in \mathbb{R} \setminus \mathbb{Q}$.
   h. (challenging) There exist $x, y \in \mathbb{R} \setminus \mathbb{Q}$ such that $y^x \in \mathbb{Q}$ (in words: There exist two irrational numbers such that one to the power of the other is rational)
   i. Determine whether each set is closed under addition and under multiplication: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$. (NOTE: We say that a set is *closed* under some operation if performing the operation on two elements of the set yields another element of the set. For instance, the set $\mathbb{Z}$ of all integers is closed under addition and multiplication but not under division.)

2. Prove that the following numbers are irrational: $\sqrt{3}$, $\sqrt{6}$, $1+\sqrt{2}$, $\sqrt{2}+\sqrt{3}$
   (It is recommended to use the correct claims from question 1.)

B. Convergence and Divergence of Sequences

3. Prove each equation using the definition of a limit:

   a. $\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$ b. $\lim\limits_{n\to\infty} \frac{1}{n^2+1} = 0$ c. $\lim\limits_{n\to\infty} \left(\frac{2}{3n-100}\right)^{1/3} = 0$

   d. $\lim\limits_{n\to\infty} \frac{1}{\sqrt{2n^2-5}} = 0$ e. $\lim\limits_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}$ f. $\lim\limits_{n\to\infty} \frac{n^2}{2n^2+3} = \frac{1}{2}$

   g. $\lim\limits_{n\to\infty} 3^{-n} = 0$ h. $\lim\limits_{n\to\infty} \frac{2}{4^{-n}+1} = 2$ i. $\lim\limits_{n\to\infty} \frac{2}{4^n+1} = 0$ j. $\lim\limits_{n\to\infty} \frac{1}{\log_2 n} = 0$

4. a. Find a real number N such that for all natural $n > N$, $\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| < \frac{1}{1000}$.

   Check your answer for two values of n.

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b. Find a real number N such that for all natural $n > N$, $\left|\frac{2}{4^{-n}+1} - 2\right| < \frac{1}{1000}$.
Check your answer for two values of n.

c. Find a real number N such that for all natural $n > N$, $\left|\frac{2}{4^{n}+1} - 0\right| < \frac{1}{1000}$.
Check your answer for two values of n.

5. a. In question 3 (e) you proved that $\lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{2}{3}$. Try to use the same method to prove that $\lim_{n\to\infty} \frac{2n+1}{3n+2} = \frac{1}{2}$ and determine what goes wrong.

   b. In question 3 (f) you proved that $\lim_{n\to\infty} \frac{n^2}{2n^2+3} = \frac{1}{2}$. Try to use the same method to prove that $\lim_{n\to\infty} \frac{n^2}{2n^2+3} = \frac{2}{3}$ and determine what goes wrong.

6. Use the definition of limit to prove that the following sequences do not converge:

a. $a_n = (-1)^n$ b. $a_n = \sin\frac{n\pi}{2}$ c. $a_n = \cos\frac{n\pi}{4}$ d. $a_n = 2n-1$ e. $a_n = n^2$

Recall: We defined a sequence (a_n) to be **convergent** (and to **converge to l**) when there exists $l \in \Re$ such that: For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{R}$ such that for every natural $n > n_0$ we have: $|a_n - l| < \varepsilon$. A sequence which does not converge is called **divergent**. Thus a sequence (a_n) is divergent if for every $l \in \Re$ there exists $\varepsilon > 0$ such that for every $n_0 \in \mathbb{R}$ there exists a natural $n > n_0$ for which $|a_n - l| \geq \varepsilon$.

7. Prove using the definition of limit the following equations:

a. $\lim_{n\to\infty} \sqrt{n} = +\infty$ b. $\lim_{n\to\infty} 2^n = +\infty$ c. $\lim_{n\to\infty} \sqrt[10]{n} = +\infty$

d. $\lim_{n\to\infty} \log_2 n = +\infty$ e. $\lim_{n\to\infty}\left(\frac{n}{100{,}000} - 10^{10}\right) = +\infty$ f. $\lim_{n\to\infty}\left(\log_{0.25} n\right) = -\infty$

In addition, sketch the graphs of the functions: $f(x) = \sqrt{x}$, $f(x) = 2^x$, $f(x) = \sqrt[10]{x}$, $f(x) = \log_2 x$, $f(x) = \frac{x}{100{,}000} - 10^{10}$, and find in them the significance of the divergence to ∞ of the first five sequences (parts a-e).

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8. Prove the following claims:

   a. If $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = -\infty$ then $\lim_{n\to\infty}(a_n b_n) = -\infty$.

   b. If $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty}(a_n + b_n) = +\infty$.

9. a. Find an example of sequences (a_n) and (b_n) which satisfy $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = -\infty$ and in addition (each part is to be answered separately):

   1) $\lim_{n\to\infty}(a_n + b_n) = +\infty$ 2) $\lim_{n\to\infty}(a_n + b_n) = 0$ 3) $\lim_{n\to\infty}(a_n + b_n) = \pi$ 4) $\lim_{n\to\infty}(a_n + b_n) = -\infty$

   5) $\lim_{n\to\infty}(a_n + b_n)$ does not exist, even in the extended sense.

   b. Find an example of sequences (a_n) and (b_n) which satisfy $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, and in addition (each part is to be answered separately):

   1) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = +\infty$ 2) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 4$ 3) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$ 4) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = -\infty$

   5) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ does not exist, even in the extended sense.

Limit Arithmetic

10. Calculate the following limits using the theorems of limit arithmetic. Do not rely on intuition, and explain every step! You may use the limits proven in class: $\lim_{n\to\infty}\frac{1}{n} = 0$, $\lim_{n\to\infty} n = +\infty$,

    $\lim_{n\to\infty} a^n = \begin{cases} +\infty & : \ a > 1 \\ 0 & : \ 0 < a < 1 \end{cases}$ as well as the theorems in the chart entitled "Limit Arithmetic".

    a. $\lim_{n\to\infty}\frac{n-1}{2n+1}$ b. $\lim_{n\to\infty}\frac{-3n^2+100n+3}{2n^2-5n-7}$ c. $\lim_{n\to\infty}\frac{5n^2+n+3}{10000-2n^2}$ d. $\lim_{n\to\infty}\frac{2n^2+2n+3}{3n-1}$

    e. $\lim_{n\to\infty}\frac{2n+3}{2n^2-n-1}$ f. $\lim_{n\to\infty}\frac{-3n^3-5n^2+8n+1}{5-2n^2}$ g. $\lim_{n\to\infty}\frac{2^n+1}{2^n-1}$ h. $\lim_{n\to\infty}\frac{3^n+2^n}{3^n-2^n}$

    i. $\lim_{n\to\infty}\frac{4^n+3^n}{5^n-2^n}$ j. $\lim_{n\to\infty}(3n^2+5n+1)$ k. $\lim_{n\to\infty}(3n^2-5n+1)$ l. $\lim_{n\to\infty}(-3n^2+5n+1)$

    m. $\lim_{n\to\infty}\left(\sqrt{n^2+1000}+n\right)$ n. $\lim_{n\to\infty}\left(\sqrt{n^2+1000}-n\right)$ o. $\lim_{n\to\infty}\left(\sqrt{3n^2+2n+5}-\sqrt{n^2-6n-4}\right)$

    p. $\lim_{n\to\infty}\left(\sqrt{3n^2+200n+3}-\sqrt{3n^2-100n-7}\right)$ q. $\lim_{n\to\infty}\left(\sqrt{n^3+1}-n^{3/2}\right)$ r. $\lim_{n\to\infty}\frac{n+\sqrt{n}+\sqrt[4]{n}}{n+\sqrt[3]{n}+\sqrt[5]{n}}$

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s. $\lim\limits_{n\to\infty}\left(2n-\sqrt{4n^2+3n+5}\right)$ t. $\lim\limits_{n\to\infty}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$ u. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{n^2+4n+9}-n}{\sqrt{n}+3}\right)$

v. $\lim\limits_{n\to\infty}\left(\dfrac{n\sqrt{n^2+4n+9}-n^2}{n+3}\right)$ w. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{3n^2+4n+9}-n}{n+3}\right)$

11. a. Calculate the limit $\lim\limits_{n\to\infty}\dfrac{1+2+3+\cdots+n}{n^2}$ in two (seemingly) different ways:

Method 1: Calculate the sum $1+2+3+\cdots+n$ and derive from it an expression for $\dfrac{1+2+3+\cdots+n}{n^2}$, and calculate the limit of this expression.

Method 2: Calculate the limit "sum": $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}\right)+\lim\limits_{n\to\infty}\left(\dfrac{2}{n^2}\right)+\lim\limits_{n\to\infty}\left(\dfrac{3}{n^2}\right)+\cdots+\lim\limits_{n\to\infty}\left(\dfrac{n}{n^2}\right)$

How is it possible that you obtained different results? Recall a theorem about the sum of sequences. Hint: Why in method 2 does the word "sum" appear in quotation marks?

b. Calculate the limit $\lim\limits_{n\to\infty}\dfrac{1+2+4+8+\cdots+2^{n-1}}{2^n}$ in three (seemingly) different ways:

Method 1: Calculate the sum $1+2+4+8+\cdots+2^{n-1}$ and derive from it an expression for $\lim\limits_{n\to\infty}\dfrac{1+2+4+\ldots+2^{n-1}}{2^n}$, and calculate the limit of this expression.

Method 2: Calculate the limit "sum": $\lim\limits_{n\to\infty}\left(\dfrac{1}{2^n}\right)+\lim\limits_{n\to\infty}\left(\dfrac{2}{2^n}\right)+\lim\limits_{n\to\infty}\left(\dfrac{3}{2^n}\right)+\cdots+\lim\limits_{n\to\infty}\left(\dfrac{2^{n-1}}{2^n}\right)$.

Method 3: Calculate the limit "sum" from method 2 in reverse order:
$\lim\limits_{n\to\infty}\left(\dfrac{2^{n-1}}{2^n}\right)+\lim\limits_{n\to\infty}\left(\dfrac{2^{n-2}}{2^n}\right)+\lim\limits_{n\to\infty}\left(\dfrac{2^{n-3}}{2^n}\right)+\cdots+\lim\limits_{n\to\infty}\left(\dfrac{2}{2^n}\right)+\lim\limits_{n\to\infty}\left(\dfrac{1}{2^n}\right)$

Recall a theorem about the sum of sequences in order to understand why two of the three methods are problematic.

In some parts of the following questions it is recommended to use induction. For sketches you can use translations and reflections of basic functions or a program such as Mathematica.

12. The sequence (a_n) is defined by $a_1=3$, $a_{n+1}=\sqrt{4a_n+5}$.

a. Find the first five terms of the sequence (you may use a calculator).
b. Prove that all terms of the sequence are positive.

c. Prove that (a_n) is an increasing sequence.
d. Prove that the sequence (a_n) is bounded from above.
e. Prove that the sequence converges and find its limit.
f. Sketch on coordinate axes the graph of $f(x)=\sqrt{4x+5}$ and the line $y=x$. What is the graphical significance of the convergence of the sequence and of its limit?

   Infer from this the behavior of the sequence if the first term is changed to 7. What is the limit in this case?

13. The sequence (a_n) is defined by $a_1=7$, $a_{n+1}=\sqrt{a_n+2}$.

    Answer the same questions as in problem 12, replacing "increasing" with "decreasing" in part (c), "from above" with "from below" in part (d), and in part (f) find a different suitable function.

    How will the sequence behave if the first term is changed to 1? What will be the limit?

14. Given: $a_1=\sqrt{6}$, $a_2=\sqrt{6+\sqrt{6}}$, $a_3=\sqrt{6+\sqrt{6+\sqrt{6}}}$, $a_4=\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}$,...

    a. Define the above sequence recursively.
    b. Prove that the sequence is monotonic and bounded.
    c. Prove that the sequence converges and find its limit.

15. A sequence is defined by the recursive formula: $a_{n+1}=6-\dfrac{5}{a_n}$ with $a_1=2$.

    Prove: a. All elements of the sequence are ≥ 2 and less than 5. b. The sequence is monotonically increasing. c. The sequence converges (find its limit).

    Sketch on coordinate axes the graphs of $f(x)=6-\dfrac{5}{x}$ and $y=x$, and try to understand the significance of the convergence of the sequence and its limit. How would the sequence behave if the first element were changed to 10? To 0.5? What would be the limit of the sequence in each case?

16. (Material learned in class—a sequence raised to the power of a sequence)

17. Use the material learned about "a sequence to the power of a sequence" and the Sandwich Theorem to find the following limits:

    a. $\lim\limits_{n\to\infty}\left(\dfrac{3n+4}{7n+6}\right)^n$ b. $\lim\limits_{n\to\infty}\left(\left|\dfrac{1}{2}+\dfrac{\sin n}{n}\right|\right)^n$ c. $\lim\limits_{n\to\infty}\left(\dfrac{2n+3\sin n}{5n+4\cos n}\right)^n$

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18. Calculate the limit: $\lim\limits_{n\to\infty}\left(\dfrac{5n+7}{2n+4}\right)^n$ (Use the Sandwich Theorem for sequences which diverge to $+\infty$.)

19. Determine if each claim (except (e)) is correct or incorrect. If correct, prove it, and if incorrect, disprove it with a counterexample.

    a. If (a_n) is a convergent sequence and (b_n) diverges then the sequence (a_n+b_n) diverges.
    b. There exist divergent sequences (a_n) and (b_n) such that the sequence (a_n+b_n) converges.
    c. If (a_n) and (b_n) are sequences whose product $(a_n\cdot b_n)$ is a convergent sequence then each of (a_n) and (b_n) is a convergent sequence.
    d. There exist sequences (a_n) and (b_n) such that one converges, the other diverges, and their product $(a_n\cdot b_n)$ converges.
    e. Suppose that the sequence (a_n) converges and the sequence (b_n) diverges. What condition on $\lim\limits_{n\to\infty} a_n$ will guarantee that the sequence $(a_n\cdot b_n)$ diverges?
    f. If (a_n) is a vanishing sequence then $\left(\dfrac{1}{a_n}\right)$ diverges to infinity.
    g. If (a_n) is a sequence diverging to infinity then $\left(\dfrac{1}{a_n}\right)$ is a bounded sequence.
    h. If (a_n) is a positive convergent sequence then its limit is also positive.