Calculus I: Exercise 4 (Differentiability and Derivatives)

To submit: 1 b, d 2 14, 19, 24, 25, 27, 30, 33, 35, 37 3 b, d, f, i, j 4 c, d, e 5 a, c 6 a, b, c1 7 b, d 9 b, d, f 10, 11, 12, 14 c, f, h, l, o, p 15

NOTE: It is highly recommended to solve as many exercises as possible and not only those for submission.

1. Use the definition of the derivative to compute the derivatives of the following functions:

   a. $f(x)=\cot x$ b. $f(x)=\dfrac{3}{2x-5}$ c. $f(x)=\dfrac{3}{2x^5}$ d. $f(x)=\sqrt{3-4x}$

   (Recall question 6 of Exercise 2 and what you proved there without realizing it…)

2. Calculate the derivative of each function using derivatives of elementary functions and theorems of the arithmetic of derivatives. Note that there may be values of x where each function is not differentiable (and hence the derivative is undefined there).

   1. $f(x)=x^3+3x^2-7$ 2. $f(x)=\dfrac{x^5}{3}+\dfrac{3x^3}{4}-\dfrac{5}{2x}+\dfrac{e}{x^2}$ 3. $f(x)=3\sqrt[5]{x}-\dfrac{1}{2\sqrt{x}}+2\cot x$

   4. $f(x)=x^3\cdot\sin x$ 5. $f(x)=\sqrt{x}\cdot\tan x$ 6. $f(x)=\dfrac{1}{x}\cdot\arctan x$

   7. $f(x)=\operatorname{arccot} x\cdot\arctan x$ 8. $f(x)=\sin x\cdot\arccos x$ 9. $f(x)=\dfrac{2x+5}{\sqrt{3x-x^2}}$

   10. $f(x)=\dfrac{\cos x}{1+2\sqrt{x}}$ 11. $f(x)=e^{-x^2}+xe^{-2x}$ 12. $f(x)=\ln(\ln(\ln x))$

   13. $f(x)=\sin\left(x^4\right)$ 14. $f(x)=\sin^5 3x$ 15. $f(x)=\tan^5\left(x^3\right)$

   16. $f(x)=\tan\sin x$ 17. $f(x)=\cos\left(\cot\sqrt{x}\right)$ 18. $f(x)=\cos\sqrt{\cot x}$

   19. $f(x)=\sqrt{\cos(\cot x)}$ 20. $f(x)=\cos x\cdot\sqrt{\cot x}$ 21. $f(x)=\dfrac{\cos x}{2\sqrt{\sin 3x}}$

   22. $f(x)=\arccos\sqrt{1-x^2}$ 23. $f(x)=\operatorname{arccot}\dfrac{2\sqrt{x}}{x-1}$ 24. $f(x)=\arcsin\dfrac{\sqrt{x^2-1}}{x}$

   25. $f(x)=\begin{cases} x^2-3x & x\in[1,+\infty) \\ x^3-2x^2-1 & x\in(-2,1) \\ -2x^2+12x+9 & x\in(-\infty,-2] \end{cases}$ 26. $f(x)=x|x|$ 27. $f(x)=\dfrac{\left|x^2-1\right|}{x-1}$

28. $f(x)=\sin|x|$ 29. $f(x)=|x|\sin|x|$ 30. $f(x)=\ln(1+|x|)$

In 31-33 find the derivative in two ways: using the product/ quotient rules, and using the fact that $f(x)=e^{\ln(f(x))}$ when $f(x)>0$, and check that the result holds even for $f(x)\le 0$.

31. $f(x)=x^2\cdot 2^x\sin x$ 32. $f(x)=x^5e^x\log_2 x\cdot\tan x$ 33. $f(x)=\dfrac{\sqrt{x}\cdot\cos x}{\sqrt[5]{x^2-3x}\cdot\ln x}$

In the next parts use the fact that a function expressed as $f(x)=g(x)^{h(x)}$ can also be expressed as $f(x)=e^{h(x)\ln g(x)}$ (the domain of *f* is wherever both *g* and *h* exist and $g>0$).

34. $f(x)=x^x$ 35. $f(x)=(3x^2-2x)^{5x+3}$ 36. $f(x)=(\sin x)^{\cos x}$

37. $f(x)=(\ln x)^{\arctan x}$ 38. $f(x)=e^2x^ee^{\sin x}$ 39. $f(x)=\sin^x(e^x)$

3. Show that the derivative of each function *f* is the function *f'* appearing in the same line. In addition specify at which points each function is differentiable. In parts e,f,g,i find also the second derivative *f''*.

a. $f(x)=(3x-5)^7(7-2x)^{12}$ $f'(x)=3(89-52x)(3x-5)^6(7-2x)^{11}$

b. $f(x)=\dfrac{(3x^2-5x+2)^5}{(4-3x)^7}$ $f'(x)=\dfrac{(3x^2-5x+2)^4(-27x^2+90x-58)}{(4-3x)^8}$

c. $f(x)=\dfrac{x}{(1-x)^2(1+x)^3}$ $f'(x)=\dfrac{4x^2-x+1}{(1-x)^3(1+x)^4}$

d. $f(x)=(2x-3)\sqrt{4x-x^2}$ $f'(x)=\dfrac{-4x^2+15x-6}{\sqrt{4x-x^2}}$

e. $f(x)=\tan\dfrac{x}{2}-\cot\dfrac{x}{2}$ $f'(x)=\dfrac{2}{\sin^2 x}$

f. $f(x)=\tan x-\dfrac{1}{3}\tan^3 x+\dfrac{1}{5}\tan^5 x$ $f'(x)=1+\tan^6 x$

g. $f(x)=e^{-x}\left(\dfrac{1}{2}(1-x^2)\sin x-\dfrac{1}{2}(1+x)^2\cos x\right)$ $f'(x)=x^2e^{-x}\sin x$

h. $f(x)=\ln(\ln^2(\ln^3(x^4)))$ $f'(x)=\dfrac{6}{x\ln|x|\cdot\ln(64\ln^3(|x|))}$

i. $f(x)=x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}$ $f'(x)=\ln\left(x+\sqrt{1+x^2}\right)$

j. $f(x)=\arcsin(1+x^2)$ *f* isn't differentiable anywhere in $\mathbb{R}$.

4. In each of the following functions, note that *f* is differentiable for all $x_0\neq 0$, and the derivative *f'* is continuous at x_0.

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a. Check if *f* is continuous at 0.
b. Check if *f* is differentiable at 0 and if so, find $f'(0)$.
c. If *f* is differentiable at 0 check if *f'* is continuous at 0. If *f* is not differentiable at 0, check if is right- or left-differentiable there.

[Introductory exercise: Prove that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$ and $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)=0$.]

a. $f(x)=\begin{cases}\frac{\sin^2 x}{x} & : x\neq 0\\ 0 & : x=0\end{cases}$ b. $f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & : x\neq 0\\ 0 & : x=0\end{cases}$ c. $f(x)=\begin{cases}x\sin\left(\frac{1}{x}\right) & : x\neq 0\\ 0 & : x=0\end{cases}$

d. $f(x)=\begin{cases}x^2\sin\left(\frac{1}{x}\right) & : x\neq 0\\ 0 & : x=0\end{cases}$ e. $f(x)=\begin{cases}x^3\sin\left(\frac{1}{x}\right) & : x\neq 0\\ 0 & : x=0\end{cases}$

If you solved those correctly then you've seen a range of different possibilities regarding the continuity of *f* at 0, the differentiability of *f* at 0, and the continuity of the derivative at 0.

Moreover, you can prove: Define $f(x)=\begin{cases}x^n\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$ for some natural *n*.

Then: Only for $n>0$ is *f* continuous at 0. Only for $n>1$ is *f* differentiable at 0. Only for $n>2$ is the derivative *f'* continuous at 0. Only for $n>3$ is *f* twice differentiable at 0 (i.e., *f'* is differentiable at 0). And so on…

5. We proved in class (Chain Rule): If *f* is differentiable at x_0 and *g* is differentiable at $y_0=f(x_0)$ then $h=g\circ f$ is also differentiable at x_0 and $h'(x_0)=g'(f(x_0))f'(x_0)$. Each of the following functions is differentiable for all x_0. Also, as we know, $g(x)=|x|$ is differentiable for all $x_0\neq 0$. So based on the Chain Rule, for which x_0 can we conclude that each *f* is differentiable at x_0? Where we can't prove it from the Chain Rule, is $h=g\circ f$ necessarily non-differentiable at x_0? Check explicitly if $g\circ f$ is differentiable at each such x_0 and if it is, compute $(g\circ f)'(x_0)$.

   a. $f(x)=x^2-3x$ b. $f(x)=x^3-3x^2$ c. $f(x)=x^3-6x^2+9x$

6. a. Let *f* be a function invertible in some neighborhood of $x_0=3$, with $f(3)=5$ and $f'(3)=-2$. Compute $(f^{-1})'(5)$ (use the theorem on inverse functions proved in class).

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b. Let $f(x)=\dfrac{3}{2x-4}$ (domain $\mathbb{R}-\{2\}$). Find the inverse function f^{-1} and calculate its derivative in two ways: using the theorem from class, and differentiating f^{-1} directly.

c. Prove that each function is monotonic and continuous (hence invertible), find the domain of the inverse function and compute $\left(f^{-1}\right)'(e+1)$: 1. $f(x)=x+\ln x$ 2. $f(x)=x+e^x$

d. Find each linear function ($f(x)=ax+b$) which is inverse to itself ("self-inverse"), and check that for every $x_0\in\mathbb{R}$, if $f'(x_0)\neq 0$ and $f(x_0)=y_0$ then $\left(f^{-1}\right)'(y_0)=\dfrac{1}{f'(x_0)}$.

e. Find the inverse of the function $f(x)=\dfrac{ax+b}{cx+d}$, and check that for every $x_0\in\mathbb{R}-\left\{\dfrac{-d}{c}\right\}$, if $f'(x_0)\neq 0$ and $f(x_0)=y_0$ then $\left(f^{-1}\right)'(y_0)=\dfrac{1}{f'(x_0)}$.

f. Check that $f(x)=\dfrac{ax-b}{cx-a}$ is self-inverse and using part (e) show that $\left(f^{-1}\right)'=f'$.

7. Each equation (a-c) defines y as an implicit function $f(x)$. Find f' whenever f is differentiable.

a. $x^2+2x^3y^5+e^y+\ln x=0$ b. $y=x+\arctan\left(xy^2\right)$ c. $x\tan y=x^2+y^2$

d. Find the equation of the tangent line to the curve defined by the equation $(x+2y)^3-3x^2\ln y+x=y^2-3e^{(x+2)y}$ at the point $(-2,1)$.

8. (This question may be skipped in the 5780 academic year.)

Find the derivatives $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ (as functions of the parameter t) of each of the following functions defined parametrically. [Recall: If $\begin{cases} y=\psi(t) \\ x=\varphi(t)\end{cases}$ for $t\in[a,b]$ where φ,ψ are differentiable and $\varphi(t)\neq 0$ in $[a,b]$ then for all $t\in[a,b]$ we have $\dfrac{dy}{dx}(t)=\dfrac{\frac{d\psi}{dt}}{\frac{d\varphi}{dt}}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$].

a. $\begin{cases} y=t+e^t \\ x=t^2+\ln t\end{cases}$ b. $\begin{cases} y=1-\cos t \\ x=t-\sin t\end{cases}$ c. $\begin{cases} y=5t+8 \\ x=t^5+2t-1\end{cases}$ d. $\begin{cases} y=4\cos t \\ x=3\sin t\end{cases}$

9. The *normal* to a graph at a point is the line perpendicular to the tangent line at that point.

a. Find the equations of the tangent and the normal to the graph $y=\dfrac{8a^3}{4a^2+x^2}$ where $x=2a$.

b. Find the equation of a line parallel to $y=x$ which is a normal to $y=x\ln x$.

c. At what angle to the graphs of $y=\tan x$ and $y=\cos x$ intersect in the domain $\left(0,\frac{\pi}{2}\right)$?

(The angle at which two graphs intersect is defined as the angle at which their tangents intersect at the point of intersection of the graphs.)

d. Sketch the graph of $y=e^{-|x|}$ and find the angle made by the graph at $x=0$. (The angle made by a graph is the angle at which the tangents from the right and from the left intersect. Note: At a point where a function is differentiable, this angle is a straight line.)

e. For what value of *c* is the line $y=1-2x$ tangent to the graph of $f(x)=cx^3$?

f. For what value of *c* is the line $y=2x+3$ tangent to the graph of $f(x)=\frac{c}{x}$?

10. Given the function $f(x)=\arcsin(\sin x)$ defined on all $\mathbb{R}$.

a. Find the derivative function f'. Where is *f* differentiable (i.e., what is the domain of f')? (Recall that the function $\arcsin x$ is differentiable only in the interval $(-1,1)$.)

b. In part (a) you should have found a derivative with Type I discontinuities. Sketch the graph of f'.

c. Recall that arcsin is the inverse of the sine function where $\sin:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$. Given that, what is $f(x)=\arcsin(\sin x)$ on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and what is its derivative in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$? Try to find the graph on other intervals like $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2},\frac{5\pi}{2}\right]$, and check that the graph matches what you found in parts (a)-(b).

d. f is a composition of continuous functions (the first on all of $\mathbb{R}$ and the second on the image of the first) and is therefore continuous on $\mathbb{R}$. Based on that and parts (a)-(b), sketch the graph of *f*.

11. Given the functions $f(x)=\arctan\left(\frac{1+x}{1-x}\right)$ and $g(x)=\arctan x$.

a. Find the domain of *f* and sketch the graph of *g*.
b. Find the derivative functions of *f* and of *g*.
c. Using a theorem you learned, infer from parts (a) and (b) a connection between the functions *f* and *g*. Did you derive an identity connecting $\arctan\left(\frac{1+x}{1-x}\right)$ to $\arctan x$?
d. In the identity you derived, the value $x=1$ is in the domain of only one side. Why is that?
e. Find the graph of *f* (e.g., using a graphing calculator) and find a connection between the graphs of *f* and *g*. Does this observation match the identity from part (c)? (It should…)

[Hints: Distinguish between the intervals $(1,+\infty)$ and $(-\infty,1)$. The theorem you use in (c) implies that the identity you derive is really two identities, one for each of these intervals.]

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12. Given the function $f(x)=\arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right)$, compute f' and infer, using a theorem you learned, an identity connecting $\arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right)$ to $\arcsin x$ for all $x\in(-1,1)$.

13. Find the derivative functions of $f(x)=\sin^2 x$ and of $g(x)=-\cos^2 x$ and infer (using a theorem you learned) a well-known identity.

14. Evaluate the following limits. If you use l'Hopital's rule, you must justify doing so.

a. $\lim\limits_{x\to 0}\frac{1-e^{12x}}{x}$ b. $\lim\limits_{x\to\infty}\frac{x^7}{\ln x}$ c. $\lim\limits_{x\to+\infty}\frac{(\ln x)^2}{\sqrt[3]{x}}$ d. $\lim\limits_{x\to 0}\frac{1-\cos 8x}{1-\cos 3x}$ e. $\lim\limits_{x\to+\infty}\frac{e^{\sqrt{x}}}{x^2}$

f. $\lim\limits_{x\to 0^+}\left(\sqrt{x}\cdot\ln x\right)$ g. $\lim\limits_{x\to 0}\frac{\tan x-x}{x-\sin x}$ h. $\lim\limits_{x\to 0}\left(\frac{x+\sin x}{2x-3\sin x}\right)$ i. $\lim\limits_{x\to 0^+}x^x$ j. $\lim\limits_{x\to 0^+}(\arctan x)^x$

k. $\lim\limits_{x\to+\infty}(\ln x)^{\frac{1}{x}}$ l. $\lim\limits_{x\to+\infty}\left(\frac{x+\sin x}{2x-3\sin x}\right)$ m. $\lim\limits_{x\to 0^+}\left((\sin x)^{\ln x}\right)$ n. $\lim\limits_{x\to 0^+}(5-4\cos x)^{\frac{1}{x}}$

o. $\lim\limits_{x\to 0^+}(1+\sin x)^{\ln x}$ p. $\lim\limits_{x\to 0^+}(1-\sin x)^{1/x^2}$ q. $\lim\limits_{x\to 0^-}(1-\sin x)^{1/x^2}$

15. True or False:

a. If $f(x)=\begin{cases}2x+1 & : \quad x\neq 3\\ 5 & : \quad x=3\end{cases}$ then $f'(x)=\begin{cases}2 & : \quad x\neq 3\\ 0 & : \quad x=3\end{cases}$.

b. If $f(x)=\begin{cases}2x+1 & : \quad x\neq 3\\ 7 & : \quad x=3\end{cases}$ then $f'(x)=\begin{cases}2 & : \quad x\neq 3\\ 0 & : \quad x=3\end{cases}$.

c. If $f(x)=\begin{cases}\frac{2x^2-6x}{x-3} & : \quad x\neq 3\\ 4 & : \quad x=3\end{cases}$ then $f'(x)=\begin{cases}2 & : \quad x\neq 3\\ 6 & : \quad x=3\end{cases}$.

d. If $f(x)=\begin{cases}x^2-4x & : \quad x\geq 3\\ 6-3x & : \quad x<3\end{cases}$ then $f'(x)=\begin{cases}2x-4 & : \quad x\geq 3\\ -3 & : \quad x<3\end{cases}$.

e. If $f(x)=\begin{cases}x^2-6x & : \quad x\geq 2\\ 6-2x & : \quad x<2\end{cases}$ then $f'(x)=\begin{cases}2x-6 & : \quad x\geq 2\\ -2 & : \quad x<2\end{cases}$.

f. If $f(x)=\begin{cases}x^2-6x & : \quad x\geq 2\\ -2x-4 & : \quad x<2\end{cases}$ then $f'(x)=\begin{cases}2x-6 & : \quad x\geq 2\\ -2 & : \quad x<2\end{cases}$.