

The Rational Numbers

Math 1 - Chapter 5

Theo Rode

It is time to talk about more numbers! I would say these are new to you, but we have actually already explored these types of numbers.

You ready for it? This is a rational number:

$$1$$

Oh. So it is literally the same as an integer? Why does it have a different name?

Well, 1 is a rational number, but there is more than just 1. There is:

$$\frac{1}{10}$$

Yes. We explored the possibility of fractions being numbers earlier and in fact, they are rational numbers! So then, what does the value $\frac{1}{10}$ represent? Well, we know that multiplying this number by another number “undos” multiplying by 10. And if we multiply this by 1:

$$1 \cdot \frac{1}{10}$$

Well we know that 1 is the multiplicative identity, so this is just:

$$\frac{1}{10}$$

This would indicate that if we multiplied this rational number by 10, we would get 1. Meaning, that this represents a value that is 10 times smaller than 1!

To further convince you of this, consider multiplying this value by 10:

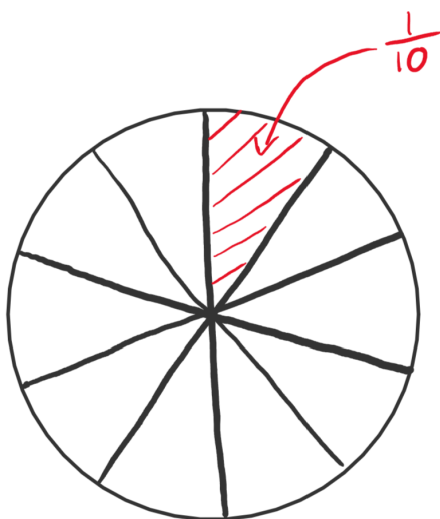
$$\frac{1}{10} \cdot 10$$

Because we know multiplying by the inverse grants us the identity, this must be:

$$\frac{1}{10} \cdot 10 = 1$$

Meaning that it does in fact take 10 groups of $\frac{1}{10}$ to get to the number 1!

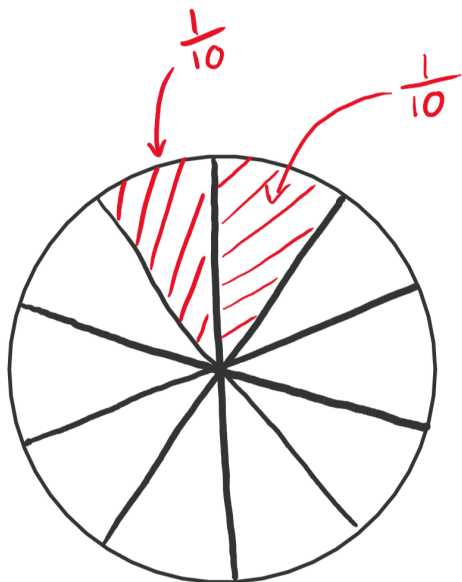
It can sometimes be hard to imagine a number less than 1. What does it mean to have $\frac{1}{10}$ stones? That is even a weird sentence to say ¹ One thing that can help is representing these numbers like so:



¹This is especially because you say $\frac{1}{10}$ as “one-tenth” so the grammar isn’t even right.

I mean the drawing could use some improvement (I am here to talk about math not art), but the idea is what is important. You want to look at splitting the object into 10 parts and, for $\frac{1}{10}$, taking 1 of those parts. You can imagine adding together these “parts” or taking parts of the parts ($\frac{1}{10} \cdot \frac{1}{10}$).

But this begs the question: what if we have more than just 1 part of something? Say we have 2 parts or 3 parts? What does that number look like? Well, it isn’t too difficult to expand our drawing such that we have two parts rather than just 1:



Instead of just highlighting the one part, we highlight two. But what about as a number? Well we know we can use multiplication to represent multiple of something:

$$\frac{1}{10} \cdot 2 = \frac{1}{10} + \frac{1}{10}$$

But that feels hard to write that down every single time. So instead we have come up with a convention for notating this. We say:

$$\frac{1}{10} \cdot 2 = \frac{2}{10}$$

This number, $\frac{2}{10}$, can simply be thought of as $\frac{1}{10} \cdot 2$. But you can also think of it as splitting the number 2 into 10 groups. If you think about it, this is what we were doing with $\frac{1}{10}$. That number represented a part ten times smaller than 1. Take a second to confirm to yourself that it makes sense that something that is ten times smaller than 2 is the same thing as two things that are both ten times smaller than 1. Or that this is true:

$$\frac{1}{10} \cdot 2 = \frac{2}{10}$$

Fractions, properly

So we have gotten to halfway through the second page and I haven’t even fully explained to you the title: Rational Numbers. So let me give you the most general definition of a rational number and we will then explore what it means. I am going to use two variables to represent two integers: a and b (they each can be any integer. So it could be that $a = 5$ and $b = 9$ or $a = -2$ and $b = 5$. Go back to the variables notes if this feels weird to you.) If you remember, we think of integers as the following numbers:

$$\dots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

Where of course we keep counting down in negative numbers and counting up with the positive numbers forever. We define a rational number to be any number that can be created like so:

$$\frac{a}{b}$$

But what does that value mean? Well, much like how we can write $\frac{2}{10} = \frac{1}{10} \cdot 2 = 2 \cdot \frac{1}{10}$ (remember we can reverse multiplication) we can rewrite this in the same way:

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

And we know what these values are! a is just, well, a . It is just some integer. and $\frac{1}{b}$, well that is the multiplicative inverse of b ! (Remember, this means: $b \cdot \frac{1}{b} = 1$)

We can think of multiplying a by the multiplicative inverse of b as splitting a into b parts. So that is all a rational number is. One integer which we have split into a certain number of parts.

Now I am sure that doesn't entirely answer all of your questions about what a fraction is. I am introducing this to you very abstractly and you are in no way expected to entirely understand it right now. But, I do need to explain a few more things to you. For starters, I said that this was a rational number earlier:

$$1$$

But that doesn't look anything like:

$$\frac{a}{b}$$

There is only 1 number and there definitely isn't a horizontal bar... So what's up with that? Well, consider what $\frac{a}{b}$ means. All this is saying is we are splitting a into b parts. Well, what if we were keeping a , well then we would "split" it into 1 part... Or not split it at all. And so if we want to split it into just 1 group, so not split it at all, we can say $b = 1$. And if we just want 1, well 1 split into 1 group is still 1. So we should have the fraction:

$$1 = \frac{1}{1}$$

And now that we understand how to do that, it should be pretty easy to see that if we have the number c , we can create a fraction to represent it like so:

$$c = \frac{c}{1}$$

Adding Fractions

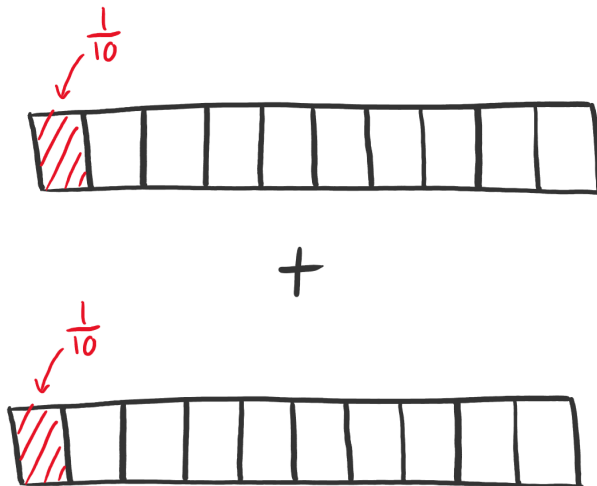
So we have already talked about adding fractions like this:

$$\frac{1}{10} + \frac{1}{10}$$

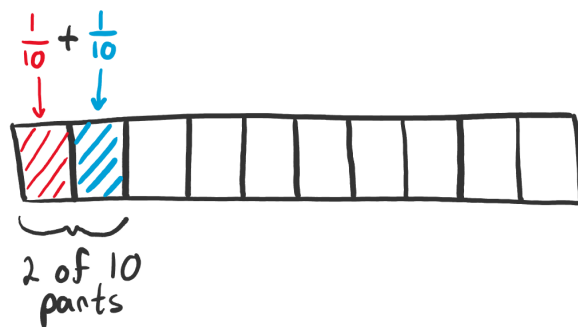
This should be:

$$\frac{2}{10}$$

We can see this through multiplication, but also if we think about it for a little bit. If we were to draw a diagram of this situation:



In order to add the two parts together, we can include them on the same diagram:



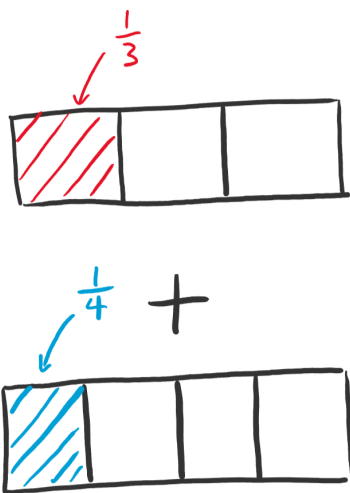
And from this drawing, we can simply count how many of the ten parts are taken up. We find that 2 of them are filled up, meaning that we can say:

$$\frac{1}{10} + \frac{1}{10} = \frac{2}{10}$$

But now say we want to add two different fractions. Say:

$$\frac{1}{3} + \frac{1}{4}$$

How do we add these? If we were to draw a diagram of the situation:



We can't really combine these drawings like we did above, there are different numbers of partitions for the two objects. So then what can we do? This is something you will explore for homework!

Multiplying Fractions

How do we do multiplication with fractions? I mean, we have been using them like we can (and actually doing some multiplication with them), but we haven't really talked about how this works. Of course, we have said the following is true:

$$2 \cdot \frac{1}{10} = \frac{2}{10}$$

And we have also said that $2 = \frac{2}{1}$, so this would suggest that we are saying:

$$\frac{2}{1} \cdot \frac{1}{10} = \frac{2}{10}$$

So here we are multiplying fractions, but it doesn't seem to tell us much about how we would do this for any fraction. We might make the connection that the new fraction is created like so:

$$\frac{2}{1} \cdot \frac{1}{10} = \frac{2 \cdot 1}{1 \cdot 10} = \frac{2}{10}$$

Which would suggest that for any two fractions, we would find:

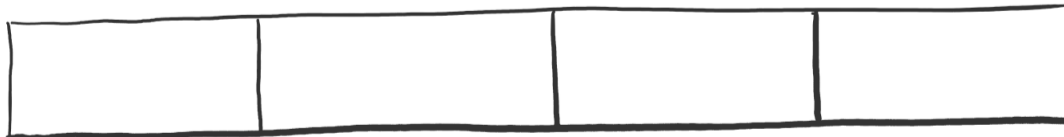
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

But this really isn't motivated by anything? How did we come to this conclusion? What does it mean? Of course I wouldn't have shown you this if it wasn't correct, but I want to actually explain this to you so here that goes.

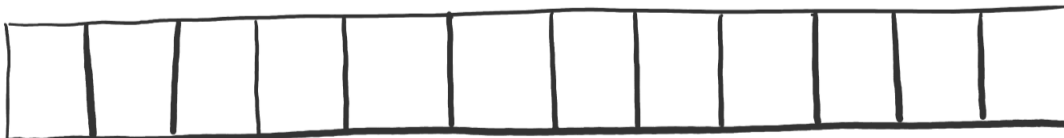
Say we split something into 4 parts. This would give us the fraction: $\frac{1}{4}$. One of 4 parts of the number 1. But now let's say we split that single part into three parts. Now what fraction do we have? Well the one thing we do know is that multiplying by $\frac{1}{3}$ is equivalent to splitting something into 3 parts. So we are looking for the following:

$$\frac{1}{4} \cdot \frac{1}{3} = ???$$

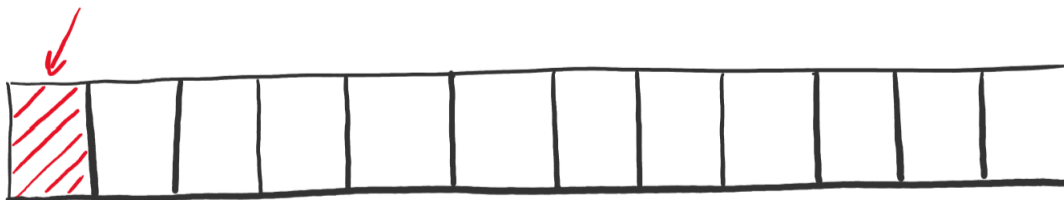
We can try and find an answer by creating another picture! (Pictures solve everything in math) We start by looking at something split into 4 parts:



Taking just one of those 4 parts would be equivalent to $\frac{1}{4}$, just like what we have done before. But now, remember, we want to split each part into 3 more parts. So let's do that:



And... Well, we want just one of those parts:



And simply by counting, we can see that this one part is one-twelfth of the whole. Or the number: $\frac{1}{12}$. This also happens to follow our rule:

$$\frac{1}{4} \cdot \frac{1}{3} = \frac{1 \cdot 1}{4 \cdot 3} = \frac{1}{12}$$

And this makes sense, looking at the way we split the drawing up above, it looks very similar to multiplication. We created 4 groups of 3 things! Just like multiplication! So that hopefully makes sense,