Module 1 - Research

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For my own reference, I will write down the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Of course, $\sigma_1, \sigma_2, \sigma_3$ correspond to $\sigma_x, \sigma_y, \sigma_z$. We also have the group of matrices:

$$SU(2) = \left\{ \begin{bmatrix} z & -w^* \\ w & z^* \end{bmatrix} \middle| z, w \in \mathbb{C} \text{ and } |z|^2 + |w|^2 = 1 \right\}$$

Where we have the claim that:

$$\Sigma_j = e^{it\sigma_j} \in SU(2)$$

for any $t \in \mathbb{R}$ and $j \in \{1,2,3\}$. Finally, we have one more group defined like so:

$$S = \left\{ e^{it(\sigma_j \otimes \sigma_k)} \middle| t \in \mathbb{R} \text{ and } j, k \in \{1,2,3\} \right\}$$

We are looking at some $\mathcal{G} \in S$, and if there exists a $g \in SU(2)$ such that $\mathcal{G} = g \otimes g$.

My initial guess is that in S, if we had $\sigma_j \otimes \sigma_j$, we would be able to find that the $g \in SU(2)$ created with σ_j could be tensor producted with itself to find the matrix in S. I think a good way to start exploring this would be to get all the tensor products I am looking for:

$$\sigma_1 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we need to worry about exponentiating matrices. This we can do quite simply if we note that, for some matrix A, and we have its eigenvalue matrix D and eigenvector matrix V, we can write:

$$A = VDV^{-1}$$

So we can therefore say:

$$A^n = VD^nV^{-1}$$

And as D is simply a diagonal matrix, it will look like this:

$$D^n = \begin{bmatrix} \lambda_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k^n \end{bmatrix}$$

This might seem weird if we are looking for A^n, but note that:

$$e^x = \sum_n \frac{x^n}{n!}$$

So, if we have:

$$e^A = \sum_n \frac{A^n}{n!} = \sum_n \frac{VD^nV^{-1}}{n!} = V \sum_n \begin{bmatrix} \frac{\lambda_1^n}{n!} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\lambda_k^n}{n!} \end{bmatrix} V^{-1} = V \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_k} \end{bmatrix} V^{-1}$$

Nice. To compute $SU(2)$, we can use this, but we must start by finding eigenvectors/values for the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{bmatrix}$$

$$(-\lambda)^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y = x, \quad x = y$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$y = -x, \quad x = -y$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{bmatrix}$$

$$(-\lambda)^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$-iy = x, \quad ix = y$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$-iy = -x, \quad ix = -y$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}}i \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

And I got lazy:

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

I just want to prove to myself that exponentiating this actually results in something in $SU(2)$... So:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} V^{-1}$$

I will computationally find V^{-1} at some point, I just don't want to solve for it properly. Anyway, this means that we can say:

$$e^{it\sigma_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{it(-1)} & 0 \\ 0 & e^{it(1)} \end{bmatrix} V^{-1}$$

Actually exponentiating this with a calculator:

$$e^{it\sigma_1} = \begin{bmatrix} 0.5e^{it} + 0.5e^{-it} & 0.5e^{it} - 0.5e^{-it} \\ 0.5e^{it} - 0.5e^{-it} & 0.5e^{it} + 0.5e^{-it} \end{bmatrix}$$

$$e^{it\sigma_2} = \begin{bmatrix} 0.5e^{it} + 0.5e^{-it} & -0.5i \cdot e^{it} + 0.5i \cdot e^{-it} \\ 0.5i \cdot e^{it} - 0.5i \cdot e^{-it} & 0.5e^{it} + 0.5e^{-it} \end{bmatrix}$$

$$e^{it\sigma_3} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

Just to prove to myself that these make sense in $SU(2)$, for σ_3 you can see that the complex conjugate requirements are met pretty clearly. As for the normalization, we should see:

$$\left|e^{it}\right|^2 = 1$$

Which wolfram alpha says is true. Okay I am convinced.

Now I wanted to look at what happens when we calculate the exponentiations of S. We get the following:

$$e^{it(\sigma_1\otimes\sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & 0.5e^{it}-0.5e^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5e^{it}-0.5e^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_1\otimes\sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5ie^{it}+0.5ie^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5ie^{it}-0.5ie^{-it} & 0 \\ 0 & -0.5ie^{it}+0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_1\otimes\sigma_3)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}+0.5e^{-it} & 0 & -0.5e^{it}+0.5e^{-it} \\ 0.5e^{it}-0.5e^{-it} & 0 & 0.5e^{it}+0.5e^{-it} & 0 \\ 0 & -0.5e^{it}+0.5e^{-it} & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_2\otimes\sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5ie^{it}+0.5ie^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & -0.5ie^{it}+0.5ie^{-it} & 0 \\ 0 & 0.5ie^{it}-0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_2\otimes\sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5e^{it}+0.5e^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ -0.5e^{it}+0.5e^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_2\otimes\sigma_3)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & -0.5ie^{it}+0.5ie^{-it} & 0 \\ 0 & 0.5e^{it}+0.5e^{-it} & 0 & 0.5ie^{it}-0.5ie^{-it} \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0.5e^{it}+0.5e^{-it} & 0 \\ 0 & -0.5ie^{it}+0.5ie^{-it} & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_3\otimes\sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 & 0 \\ 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 & 0 \\ 0 & 0 & 0.5e^{it}+0.5e^{-it} & -0.5e^{it}+0.5e^{-it} \\ 0 & 0 & -0.5e^{it}+0.5e^{-it} & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_3\otimes\sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & -0.5ie^{it}+0.5ie^{-it} & 0 & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 & 0 \\ 0 & 0 & 0.5e^{it}+0.5e^{-it} & 0.5ie^{it}-0.5ie^{-it} \\ 0 & 0 & -0.5ie^{it}+0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_3\otimes\sigma_3)} = \begin{bmatrix} 1.0e^{it} & 0 & 0 & 0 \\ 0 & 1.0e^{-it} & 0 & 0 \\ 0 & 0 & 1.0e^{-it} & 0 \\ 0 & 0 & 0 & 1.0e^{it} \end{bmatrix}$$

Now what happens when instead of taking the tensor product and then exponentiating it, if we do the opposite (what one would expect to yeild the same result):

$$e^{it\sigma_1} \otimes e^{it\sigma_1} =$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}-e^{-it})^2 \\ (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 & 0.25(e^{it}-e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) \\ (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}-e^{-it})^2 & 0.25(e^{it}+e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) \\ 0.25(e^{it}-e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1} \otimes e^{it\sigma_2} =$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) \\ (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 & (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) \\ (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) \\ (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1} \otimes e^{it\sigma_3} =$$

$$\left[\begin{matrix}(0.5e^{it} + 0.5e^{-it})e^{it} & 0 & (0.5e^{it} - 0.5e^{-it})e^{it} & 0\\0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & 0 & (0.5e^{it} - 0.5e^{-it})e^{-it}\\(0.5e^{it} - 0.5e^{-it})e^{it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{it} & 0\\0 & (0.5e^{it} - 0.5e^{-it})e^{-it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it}\end{matrix}\right]$$

$$e^{it\sigma_2} \otimes e^{it\sigma_1} =$$

$$\left[\begin{matrix}0.25(e^{it} + e^{-it})^2 & (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it})\\(0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it})\\(0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 & (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it})\\(0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2\end{matrix}\right]$$

$$e^{it\sigma_2} \otimes e^{it\sigma_2} =$$

$$\left[\begin{matrix}0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25 - ie^{it} + ie^{-it}{}^2\\(0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5ie^{it} - 0.5ie^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it})\\(0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5ie^{it} - 0.5ie^{-it}) & 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it})\\0.25ie^{it} - ie^{-it}{}^2 & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2\end{matrix}\right]$$

$$e^{it\sigma_2} \otimes e^{it\sigma_3} =$$

$$\left[\begin{matrix}(0.5e^{it} + 0.5e^{-it})e^{it} & 0 & (-0.5ie^{it} + 0.5ie^{-it})e^{it} & 0\\0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & 0 & (-0.5ie^{it} + 0.5ie^{-it})e^{-it}\\(0.5ie^{it} - 0.5ie^{-it})e^{it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{it} & 0\\0 & (0.5ie^{it} - 0.5ie^{-it})e^{-it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it}\end{matrix}\right]$$

$$e^{it\sigma_3} \otimes e^{it\sigma_1} =$$

$$\left[\begin{matrix}(0.5e^{it} + 0.5e^{-it})e^{it} & (0.5e^{it} - 0.5e^{-it})e^{it} & 0 & 0\\(0.5e^{it} - 0.5e^{-it})e^{it} & (0.5e^{it} + 0.5e^{-it})e^{it} & 0 & 0\\0 & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & (0.5e^{it} - 0.5e^{-it})e^{-it}\\0 & 0 & (0.5e^{it} - 0.5e^{-it})e^{-it} & (0.5e^{it} + 0.5e^{-it})e^{-it}\end{matrix}\right]$$

$$e^{it\sigma_3} \otimes e^{it\sigma_2} =$$

$$\left[\begin{matrix}(0.5e^{it} + 0.5e^{-it})e^{it} & (-0.5ie^{it} + 0.5ie^{-it})e^{it} & 0 & 0\\(0.5ie^{it} - 0.5ie^{-it})e^{it} & (0.5e^{it} + 0.5e^{-it})e^{it} & 0 & 0\\0 & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & (-0.5ie^{it} + 0.5ie^{-it})e^{-it}\\0 & 0 & (0.5ie^{it} - 0.5ie^{-it})e^{-it} & (0.5e^{it} + 0.5e^{-it})e^{-it}\end{matrix}\right]$$

$$e^{it\sigma_3} \otimes e^{it\sigma_3} =$$

$$\left[\begin{matrix}e^{2.0it} & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & e^{-2.0it}\end{matrix}\right]$$

So that is a mess, but I think we can actually simplify this:

$$(0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it})$$

Which this is a difference of squares:

$$\frac{1}{4}e^{2it} - \frac{1}{4}e^{-2it}$$

$$\frac{1}{4}\left(\cos 2t + i\sin 2t - \cos(-2t) - i\sin(-2t)\right)$$

$$\frac{1}{4}\left(2i\sin 2t\right)$$

Double angle identity:

$$i\sin t\cos t$$

We should also simplify:

$$(-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it})$$

$$-\frac{1}{4}ie^{2it} - \frac{1}{4}i + \frac{1}{4}i + \frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it} - e^{-2it}\right)$$

$$i\left(i\sin t\cos t\right)$$

$$-\sin t\cos t$$

And...

$$(-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}-0.5e^{-it})$$

$$-\frac{1}{4}ie^{2it}+\frac{1}{4}i+\frac{1}{4}i-\frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it}+2-e^{-2it}\right)$$

$$\frac{1}{4}i\left(e^{2it}-e^{-2it}\right)+\frac{1}{2}i$$

$$-\sin t\cos t+\frac{1}{2}i$$

And And...

$$(0.5ie^{it}-0.5ie^{-it})(0.5e^{it}+0.5e^{-it})$$

$$\frac{1}{4}ie^{2it}+\frac{1}{4}i-\frac{1}{4}i-\frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it}-e^{-2it}\right)$$

$$-\sin t\cos t$$

And^3...

$$(0.5ie^{it}-0.5ie^{-it})(0.5e^{it}-0.5e^{-it})$$

$$\frac{1}{4}ie^{2it}-\frac{1}{4}i-\frac{1}{4}i+\frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it}+e^{-2it}\right)-\frac{1}{2}i$$

$$\frac{1}{2}i\cos 2t-\frac{1}{2}i$$

And^4...

$$(-0.5ie^{it}+0.5ie^{-it})(0.5ie^{it}-0.5ie^{-it})$$

$$\frac{1}{4}e^{2it}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}e^{-2it}$$

$$\frac{1}{4}\left(e^{2it}+e^{-2it}\right)-\frac{1}{2}$$

$$\frac{1}{2}\cos 2t-\frac{1}{2}$$

It would also be nice to simplify:

$$(0.5e^{it}+0.5e^{-it})e^{it}$$

$$\frac{1}{2}e^{2it}+\frac{1}{2}$$

And this thing:

$$(0.5e^{it}-0.5e^{-it})e^{it}$$

$$\frac{1}{2}e^{2it}-\frac{1}{2}$$

And this other thing:

$$(0.5e^{it}+0.5e^{-it})e^{-it}$$

$$\frac{1}{2}+\frac{1}{2}e^{-2it}$$

Another thing:

$$(0.5ie^{it}-0.5ie^{-it})e^{-it}$$

$$\frac{1}{2}i-\frac{1}{2}ie^{-2it}$$

Another another thing:

$$(-0.5ie^{it}+0.5ie^{-it})e^{it}$$

$$-\frac{1}{2}ie^{2it}+\frac{1}{2}i$$

Another[3] thing:

$$(-0.5ie^{it}+0.5ie^{-it})e^{-it}$$

$$-\frac{1}{2}i+\frac{1}{2}ie^{-2it}$$

Finally (hopefully):

$$(0.5e^{it}-0.5e^{-it})e^{-it}$$

$$\frac{1}{2}-\frac{1}{2}e^{-2it}$$

$$e^{it\sigma_1}\otimes e^{it\sigma_1}=$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t & i\sin t\cos t & 0.25(e^{it}-e^{-it})^2 \\ i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & 0.25(e^{it}-e^{-it})^2 & i\sin t\cos t \\ i\sin t\cos t & 0.25(e^{it}-e^{-it})^2 & 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t \\ 0.25(e^{it}-e^{-it})^2 & i\sin t\cos t & i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1}\otimes e^{it\sigma_2}=$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t & i\sin t\cos t & -\sin t\cos t+\frac{1}{2}i \\ -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & \frac{1}{2}i\cos 2t-\frac{1}{2}i & i\sin t\cos t \\ i\sin t\cos t & -\sin t\cos t+\frac{1}{2}i & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t \\ \frac{1}{2}i\cos 2t-\frac{1}{2}i & i\sin t\cos t & -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1}\otimes e^{it\sigma_3}=$$

$$\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & \frac{1}{2}e^{2it}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & 0 & \frac{1}{2}-\frac{1}{2}e^{-2it} \\ \frac{1}{2}e^{2it}-\frac{1}{2} & 0 & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}-\frac{1}{2}e^{-2it} & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_2}\otimes e^{it\sigma_1}=$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t & -\sin t\cos t & -\sin t\cos t+\frac{1}{2}i \\ i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t+\frac{1}{2}i & -\sin t\cos t \\ -\sin t\cos t & \frac{1}{2}i\cos 2t-\frac{1}{2}i & 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t \\ \frac{1}{2}i\cos 2t-\frac{1}{2}i & -\sin t\cos t & i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_2}\otimes e^{it\sigma_2}=$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t & -\sin t\cos t & 0.25{-}ie^{it}+ie^{-it^2} \\ -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & \frac{1}{2}\cos 2t-\frac{1}{2} & -\sin t\cos t \\ -\sin t\cos t & \frac{1}{2}\cos 2t-\frac{1}{2} & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t \\ 0.25ie^{it}-ie^{-it^2} & -\sin t\cos t & -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_2}\otimes e^{it\sigma_3}=$$

$$\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & -\frac{1}{2}ie^{2it}+\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & 0 & -\frac{1}{2}i+\frac{1}{2}ie^{-2it} \\ (0.5ie^{it}-0.5ie^{-it})e^{it} & 0 & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}i-\frac{1}{2}ie^{-2it} & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_3}\otimes e^{it\sigma_1}=$$

$$\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & \frac{1}{2}e^{2it}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}e^{2it}-\frac{1}{2} & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & \frac{1}{2}-\frac{1}{2}e^{-2it} \\ 0 & 0 & \frac{1}{2}-\frac{1}{2}e^{-2it} & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_3}\otimes e^{it\sigma_2}=$$

$$\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & -\frac{1}{2}ie^{2it}+\frac{1}{2}i & 0 & 0 \\ (0.5ie^{it}-0.5ie^{-it})e^{it} & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & -\frac{1}{2}i+\frac{1}{2}ie^{-2it} \\ 0 & 0 & \frac{1}{2}i-\frac{1}{2}ie^{-2it} & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_3} \otimes e^{it\sigma_3} =$$

$$\begin{bmatrix} e^{2.0it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2.0it} \end{bmatrix}$$

Cool! That gives us a bunch of matrices that we can compare. The first comparison that jumps out to me is the matrix from S:

$$e^{it(\sigma_3 \otimes \sigma_3)} = \begin{bmatrix} 1.0e^{it} & 0 & 0 & 0 \\ 0 & 1.0e^{-it} & 0 & 0 \\ 0 & 0 & 1.0e^{-it} & 0 \\ 0 & 0 & 0 & 1.0e^{it} \end{bmatrix}$$

And the other double σ_3 matrix from $SU(2)$:

$$e^{it\sigma_3} \otimes e^{it\sigma_3} =$$

$$\begin{bmatrix} e^{2.0it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2.0it} \end{bmatrix}$$

We can pretty clearly see that at $t = 0$, these matrices are equal. Suggesting possibly that for no "time" passing, we can find that:

$$e^{it(\sigma_3 \otimes \sigma_3)} = e^{it\sigma_3} \otimes e^{it\sigma_3}$$