Module 1 - Research

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For my own reference, I will write down the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Of course, $\sigma_1, \sigma_2, \sigma_3$ correspond to $\sigma_x, \sigma_y, \sigma_z$. We also have the group of matrices:

$$SU(2) = \left\{ \begin{bmatrix} z & -w^* \\ w & z^* \end{bmatrix} \middle| z, w \in \mathbb{C} \text{ and } |z|^2 + |w|^2 = 1 \right\}$$

Where we have the claim that:

$$\Sigma_j = e^{it\sigma_j} \in SU(2)$$

for any $t \in \mathbb{R}$ and $j \in \{1, 2, 3\}$. Finally, we have one more group defined like so:

$$S = \left\{ e^{it(\sigma_j \otimes \sigma_k)} \middle| t \in \mathbb{R} \text{ and } j, k \in \{1, 2, 3\} \right\}$$

We are looking at some $\mathcal{G} \in S$, and if there exists a $g \in SU(2)$ such that $\mathcal{G} = g \otimes g$.

My initial guess is that in S, if we had $\sigma_j \otimes \sigma_j$, we would be able to find that the $g \in SU(2)$ created with σ_j could be tensor producted with itself to find the matrix in S. I think a good way to start exploring this would be to get all the tensor products I am looking for:

$$\sigma_1 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we need to worry about exponentiating matrices. This we can do quite simply if we note that, for some matrix A, and we have its eigenvalue matrix D and eigenvector matrix V, we can write:

$$A = VDV^{-1}$$

So we can therefore say:

$$A^n = VD^nV^{-1}$$

And as D is simply a diagonal matrix, it will look like this:

$$D^n = \begin{bmatrix} \lambda_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k^n \end{bmatrix}$$

This might seem weird if we are looking for A^n, but note that:

$$e^x = \sum_n \frac{x^n}{n!}$$

So, if we have:

$$e^A = \sum_n \frac{A^n}{n!} = \sum_n \frac{VD^nV^{-1}}{n!} = V \sum_n \begin{bmatrix} \frac{\lambda_1^n}{n!} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\lambda_k^n}{n!} \end{bmatrix} V^{-1} = V \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_k} \end{bmatrix} V^{-1}$$

Nice. To compute $SU(2)$, we can use this, but we must start by finding eigenvectors/values for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix}$$

$$(-\lambda)^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y = x, \quad x = y$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$y = -x, \quad x = -y$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$