Module 1 - Research

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For my own reference, I will write down the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Of course, $\sigma_1, \sigma_2, \sigma_3$ correspond to $\sigma_x, \sigma_y, \sigma_z$. We also have the group of matrices:

$$SU(2) = \left\{ \begin{bmatrix} z & -w^* \\ w & z^* \end{bmatrix} \middle| z, w \in \mathbb{C} \text{ and } |z|^2 + |w|^2 = 1 \right\}$$

Where we have the claim that:

$$\Sigma_j = e^{it\sigma_j} \in SU(2)$$

for any $t \in \mathbb{R}$ and $j \in \{1, 2, 3\}$. Finally, we have one more group defined like so:

$$S = \left\{ e^{it(\sigma_j \otimes \sigma_k)} \middle| t \in \mathbb{R} \text{ and } j, k \in \{1, 2, 3\} \right\}$$

We are looking at some $\mathcal{G} \in S$, and if there exists a $g \in SU(2)$ such that $\mathcal{G} = g \otimes g$.

My initial guess is that in S, if we had $\sigma_j \otimes \sigma_j$, we would be able to find that the $g \in SU(2)$ created with σ_j could be tensor producted with itself to find the matrix in S. I think a good way to start exploring this would be to get all the tensor products I am looking for:

$$\sigma_1 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_1 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_2 \otimes \sigma_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$\sigma_3 \otimes \sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we need to worry about exponentiating matrices. This we can do quite simply if we note that, for some matrix A, and we have its eigenvalue matrix D and eigenvector matrix V, we can write:

$$A = VDV^{-1}$$

So we can therefore say:

$$A^n = VD^nV^{-1}$$

And as D is simply a diagonal matrix, it will look like this:

$$D^n = \begin{bmatrix} \lambda_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k^n \end{bmatrix}$$

This might seem weird if we are looking for A^n, but note that:

$$e^x = \sum_n \frac{x^n}{n!}$$

So, if we have:

$$e^A = \sum_n \frac{A^n}{n!} = \sum_n \frac{VD^nV^{-1}}{n!} = V \sum_n \begin{bmatrix} \frac{\lambda_1^n}{n!} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\lambda_k^n}{n!} \end{bmatrix} V^{-1} = V \begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_k} \end{bmatrix} V^{-1}$$

Nice. To compute $SU(2)$, we can use this, but we must start by finding eigenvectors/values for the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}$$

$$(-\lambda)^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y = x, \quad x = y$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$y = -x, \quad x = -y$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 - \lambda & -i \\ i & 0 - \lambda \end{bmatrix}$$

$$(-\lambda)^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$-iy = x, \quad ix = y$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} i \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$-iy = -x, \quad ix = -y$$

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} i \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}} i & -\frac{1}{\sqrt{2}} i \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

And I got lazy:

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

I just want to prove to myself that exponentiating this actually results in something in $SU(2)$... So:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} V^{-1}$$

I will computationally find V^{-1} at some point, I just don't want to solve for it properly. Anyway, this means that we can say:

$$e^{it\sigma_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{it(-1)} & 0 \\ 0 & e^{it(1)} \end{bmatrix} V^{-1}$$

Actually exponentiating this with a calculator:

$$e^{it\sigma_1} = \begin{bmatrix} 0.5e^{it} + 0.5e^{-it} & 0.5e^{it} - 0.5e^{-it} \\ 0.5e^{it} - 0.5e^{-it} & 0.5e^{it} + 0.5e^{-it} \end{bmatrix}$$

$$e^{it\sigma_2} = \begin{bmatrix} 0.5e^{it} + 0.5e^{-it} & -0.5i \cdot e^{it} + 0.5i \cdot e^{-it} \\ 0.5i \cdot e^{it} - 0.5i \cdot e^{-it} & 0.5e^{it} + 0.5e^{-it} \end{bmatrix}$$

$$e^{it\sigma_3} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

Just to prove to myself that these make sense in $SU(2)$, for σ_3 you can see that the complex conjugate requirements are met pretty clearly. As for the normalization, we should see:

$$\left|e^{it}\right|^2 = 1$$

Which wolfram alpha says is true. Okay I am convinced.

Now I wanted to look at what happens when we calculate the exponentiations of S. We get the following:

$$e^{it(\sigma_1\otimes\sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & 0.5e^{it}-0.5e^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5e^{it}-0.5e^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_1\otimes\sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5ie^{it}+0.5ie^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5ie^{it}-0.5ie^{-it} & 0 \\ 0 & -0.5ie^{it}+0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_1\otimes\sigma_3)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}+0.5e^{-it} & 0 & -0.5e^{it}+0.5e^{-it} \\ 0.5e^{it}-0.5e^{-it} & 0 & 0.5e^{it}+0.5e^{-it} & 0 \\ 0 & -0.5e^{it}+0.5e^{-it} & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_2\otimes\sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5ie^{it}+0.5ie^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & -0.5ie^{it}+0.5ie^{-it} & 0 \\ 0 & 0.5ie^{it}-0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_2\otimes\sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5e^{it}+0.5e^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ -0.5e^{it}+0.5e^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_2\otimes\sigma_3)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & -0.5ie^{it}+0.5ie^{-it} & 0 \\ 0 & 0.5e^{it}+0.5e^{-it} & 0 & 0.5ie^{it}-0.5ie^{-it} \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0.5e^{it}+0.5e^{-it} & 0 \\ 0 & -0.5ie^{it}+0.5ie^{-it} & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_3\otimes\sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 & 0 \\ 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 & 0 \\ 0 & 0 & 0.5e^{it}+0.5e^{-it} & -0.5e^{it}+0.5e^{-it} \\ 0 & 0 & -0.5e^{it}+0.5e^{-it} & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_3\otimes\sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & -0.5ie^{it}+0.5ie^{-it} & 0 & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 & 0 \\ 0 & 0 & 0.5e^{it}+0.5e^{-it} & 0.5ie^{it}-0.5ie^{-it} \\ 0 & 0 & -0.5ie^{it}+0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

$$e^{it(\sigma_3\otimes\sigma_3)} = \begin{bmatrix} 1.0e^{it} & 0 & 0 & 0 \\ 0 & 1.0e^{-it} & 0 & 0 \\ 0 & 0 & 1.0e^{-it} & 0 \\ 0 & 0 & 0 & 1.0e^{it} \end{bmatrix}$$

Now what happens when instead of taking the tensor product and then exponentiating it, if we do the opposite (what one would expect to yeild the same result):

$$e^{it\sigma_1} \otimes e^{it\sigma_1} =$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}-e^{-it})^2 \\ (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 & 0.25(e^{it}-e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) \\ (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}-e^{-it})^2 & 0.25(e^{it}+e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) \\ 0.25(e^{it}-e^{-it})^2 & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1} \otimes e^{it\sigma_2} =$$

$$\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) \\ (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 & (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) \\ (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 & (-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) \\ (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}-0.5e^{-it}) & (0.5e^{it}-0.5e^{-it})(0.5e^{it}+0.5e^{-it}) & (0.5ie^{it}-0.5ie^{-it})(0.5e^{it}+0.5e^{-it}) & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1} \otimes e^{it\sigma_3} =$$

$$\begin{bmatrix} (0.5e^{it} + 0.5e^{-it})e^{it} & 0 & (0.5e^{it} - 0.5e^{-it})e^{it} & 0 \\ 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & 0 & (0.5e^{it} - 0.5e^{-it})e^{-it} \\ (0.5e^{it} - 0.5e^{-it})e^{it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{it} & 0 \\ 0 & (0.5e^{it} - 0.5e^{-it})e^{-it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} \end{bmatrix}$$

$$e^{it\sigma_2} \otimes e^{it\sigma_1} =$$

$$\begin{bmatrix} 0.25(e^{it} + e^{-it})^2 & (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) \\ (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) \\ (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 & (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) \\ (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} - 0.5e^{-it}) & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_2} \otimes e^{it\sigma_2} =$$

$$\begin{bmatrix} 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25{-}ie^{it} + ie^{-it^2} \\ (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5ie^{it} - 0.5ie^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) \\ (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (-0.5ie^{it} + 0.5ie^{-it})(0.5ie^{it} - 0.5ie^{-it}) & 0.25(e^{it} + e^{-it})^2 & (-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) \\ 0.25ie^{it} - ie^{-it^2} & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & (0.5ie^{it} - 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it}) & 0.25(e^{it} + e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_2} \otimes e^{it\sigma_3} =$$

$$\begin{bmatrix} (0.5e^{it} + 0.5e^{-it})e^{it} & 0 & (-0.5ie^{it} + 0.5ie^{-it})e^{it} & 0 \\ 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & 0 & (-0.5ie^{it} + 0.5ie^{-it})e^{-it} \\ (0.5ie^{it} - 0.5ie^{-it})e^{it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{it} & 0 \\ 0 & (0.5ie^{it} - 0.5ie^{-it})e^{-it} & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} \end{bmatrix}$$

$$e^{it\sigma_3} \otimes e^{it\sigma_1} =$$

$$\begin{bmatrix} (0.5e^{it} + 0.5e^{-it})e^{it} & (0.5e^{it} - 0.5e^{-it})e^{it} & 0 & 0 \\ (0.5e^{it} - 0.5e^{-it})e^{it} & (0.5e^{it} + 0.5e^{-it})e^{it} & 0 & 0 \\ 0 & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & (0.5e^{it} - 0.5e^{-it})e^{-it} \\ 0 & 0 & (0.5e^{it} - 0.5e^{-it})e^{-it} & (0.5e^{it} + 0.5e^{-it})e^{-it} \end{bmatrix}$$

$$e^{it\sigma_3} \otimes e^{it\sigma_2} =$$

$$\begin{bmatrix} (0.5e^{it} + 0.5e^{-it})e^{it} & (-0.5ie^{it} + 0.5ie^{-it})e^{it} & 0 & 0 \\ (0.5ie^{it} - 0.5ie^{-it})e^{it} & (0.5e^{it} + 0.5e^{-it})e^{it} & 0 & 0 \\ 0 & 0 & (0.5e^{it} + 0.5e^{-it})e^{-it} & (-0.5ie^{it} + 0.5ie^{-it})e^{-it} \\ 0 & 0 & (0.5ie^{it} - 0.5ie^{-it})e^{-it} & (0.5e^{it} + 0.5e^{-it})e^{-it} \end{bmatrix}$$

$$e^{it\sigma_3} \otimes e^{it\sigma_3} =$$

$$\begin{bmatrix} e^{2.0it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2.0it} \end{bmatrix}$$

So that is a mess, but I think we can actually simplify this:

$$(0.5e^{it} - 0.5e^{-it})(0.5e^{it} + 0.5e^{-it})$$

Which this is a difference of squares:

$$\frac{1}{4}e^{2it} - \frac{1}{4}e^{-2it}$$

$$\frac{1}{4}\left(\cos 2t + i\sin 2t - \cos(-2t) - i\sin(-2t)\right)$$

$$\frac{1}{4}\left(2i\sin 2t\right)$$

Double angle identity:

$$i\sin t\cos t$$

We should also simplify:

$$(-0.5ie^{it} + 0.5ie^{-it})(0.5e^{it} + 0.5e^{-it})$$

$$-\frac{1}{4}ie^{2it} - \frac{1}{4}i + \frac{1}{4}i + \frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it} - e^{-2it}\right)$$

$$i\left(i\sin t\cos t\right)$$

$$-\sin t\cos t$$

And...

$$(-0.5ie^{it}+0.5ie^{-it})(0.5e^{it}-0.5e^{-it})$$

$$-\frac{1}{4}ie^{2it}+\frac{1}{4}i+\frac{1}{4}i-\frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it}+2-e^{-2it}\right)$$

$$\frac{1}{4}i\left(e^{2it}-e^{-2it}\right)+\frac{1}{2}i$$

$$-\sin t\cos t+\frac{1}{2}i$$

And And...

$$(0.5ie^{it}-0.5ie^{-it})(0.5e^{it}+0.5e^{-it})$$

$$\frac{1}{4}ie^{2it}+\frac{1}{4}i-\frac{1}{4}i-\frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it}-e^{-2it}\right)$$

$$-\sin t\cos t$$

And^3...

$$(0.5ie^{it}-0.5ie^{-it})(0.5e^{it}-0.5e^{-it})$$

$$\frac{1}{4}ie^{2it}-\frac{1}{4}i-\frac{1}{4}i+\frac{1}{4}ie^{-2it}$$

$$\frac{1}{4}i\left(e^{2it}+e^{-2it}\right)-\frac{1}{2}i$$

$$\frac{1}{2}i\cos 2t-\frac{1}{2}i$$

And^4...

$$(-0.5ie^{it}+0.5ie^{-it})(0.5ie^{it}-0.5ie^{-it})$$

$$\frac{1}{4}e^{2it}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}e^{-2it}$$

$$\frac{1}{4}\left(e^{2it}+e^{-2it}\right)-\frac{1}{2}$$

$$\frac{1}{2}\cos 2t-\frac{1}{2}$$

It would also be nice to simplify:

$$(0.5e^{it}+0.5e^{-it})e^{it}$$

$$\frac{1}{2}e^{2it}+\frac{1}{2}$$

And this thing:

$$(0.5e^{it}-0.5e^{-it})e^{it}$$

$$\frac{1}{2}e^{2it}-\frac{1}{2}$$

And this other thing:

$$(0.5e^{it}+0.5e^{-it})e^{-it}$$

$$\frac{1}{2}+\frac{1}{2}e^{-2it}$$

Another thing:

$$(0.5ie^{it}-0.5ie^{-it})e^{-it}$$

$$\frac{1}{2}i-\frac{1}{2}ie^{-2it}$$

Another another thing:

$$(-0.5ie^{it}+0.5ie^{-it})e^{it}$$

$$-\frac{1}{2}ie^{2it}+\frac{1}{2}i$$

Another[3] thing:

$$(-0.5ie^{it}+0.5ie^{-it})e^{-it}$$

$$-\frac{1}{2}i+\frac{1}{2}ie^{-2it}$$

Finally (hopefully):

$$(0.5e^{it}-0.5e^{-it})e^{-it}$$

$$\frac{1}{2}-\frac{1}{2}e^{-2it}$$

$$e^{it\sigma_1}\otimes e^{it\sigma_1}=\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t & i\sin t\cos t & 0.25(e^{it}-e^{-it})^2 \\ i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & 0.25(e^{it}-e^{-it})^2 & i\sin t\cos t \\ i\sin t\cos t & 0.25(e^{it}-e^{-it})^2 & 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t \\ 0.25(e^{it}-e^{-it})^2 & i\sin t\cos t & i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1}\otimes e^{it\sigma_2}=\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t & i\sin t\cos t & -\sin t\cos t+\frac{1}{2}i \\ -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & \frac{1}{2}i\cos 2t-\frac{1}{2}i & i\sin t\cos t \\ i\sin t\cos t & -\sin t\cos t+\frac{1}{2}i & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t \\ \frac{1}{2}i\cos 2t-\frac{1}{2}i & i\sin t\cos t & -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_1}\otimes e^{it\sigma_3}=\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & \frac{1}{2}e^{2it}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & 0 & \frac{1}{2}-\frac{1}{2}e^{-2it} \\ \frac{1}{2}e^{2it}-\frac{1}{2} & 0 & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}-\frac{1}{2}e^{-2it} & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_2}\otimes e^{it\sigma_1}=\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t & -\sin t\cos t & -\sin t\cos t+\frac{1}{2}i \\ i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t+\frac{1}{2}i & -\sin t\cos t \\ -\sin t\cos t & \frac{1}{2}i\cos 2t-\frac{1}{2}i & 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t \\ \frac{1}{2}i\cos 2t-\frac{1}{2}i & -\sin t\cos t & i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_2}\otimes e^{it\sigma_2}=\begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t & -\sin t\cos t & 0.25{-}ie^{it}+ie^{-it^2} \\ -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & \frac{1}{2}\cos 2t-\frac{1}{2} & -\sin t\cos t \\ -\sin t\cos t & \frac{1}{2}\cos 2t-\frac{1}{2} & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t \\ 0.25ie^{it}-ie^{-it^2} & -\sin t\cos t & -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it\sigma_2}\otimes e^{it\sigma_3}=\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & -\frac{1}{2}ie^{2it}+\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & 0 & -\frac{1}{2}i+\frac{1}{2}ie^{-2it} \\ (0.5ie^{it}-0.5ie^{-it})e^{it} & 0 & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}i-\frac{1}{2}ie^{-2it} & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_3}\otimes e^{it\sigma_1}=\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & \frac{1}{2}e^{2it}-\frac{1}{2} & 0 & 0 \\ \frac{1}{2}e^{2it}-\frac{1}{2} & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & \frac{1}{2}-\frac{1}{2}e^{-2it} \\ 0 & 0 & \frac{1}{2}-\frac{1}{2}e^{-2it} & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_3}\otimes e^{it\sigma_2}=\begin{bmatrix} \frac{1}{2}e^{2it}+\frac{1}{2} & -\frac{1}{2}ie^{2it}+\frac{1}{2}i & 0 & 0 \\ (0.5ie^{it}-0.5ie^{-it})e^{it} & \frac{1}{2}e^{2it}+\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}+\frac{1}{2}e^{-2it} & -\frac{1}{2}i+\frac{1}{2}ie^{-2it} \\ 0 & 0 & \frac{1}{2}i-\frac{1}{2}ie^{-2it} & \frac{1}{2}+\frac{1}{2}e^{-2it} \end{bmatrix}$$

$$e^{it\sigma_3}\otimes e^{it\sigma_3}=\begin{bmatrix} e^{2.0it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2.0it} \end{bmatrix}$$

Cool! That gives us a bunch of matrices that we can compare. The first comparison that jumps out to me is the matrix from S:

$$e^{it(\sigma_3\otimes\sigma_3)}=\begin{bmatrix} 1.0e^{it} & 0 & 0 & 0 \\ 0 & 1.0e^{-it} & 0 & 0 \\ 0 & 0 & 1.0e^{-it} & 0 \\ 0 & 0 & 0 & 1.0e^{it} \end{bmatrix}$$

And the other double σ_3 matrix from $SU(2)$:

$$e^{it\sigma_3} \otimes e^{it\sigma_3} = \begin{bmatrix} e^{2.0it} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2.0it} \end{bmatrix}$$

We can pretty clearly see that at $t = 0$, these matrices are equal. Suggesting possibly that for no "time" passing, we can find that:

$$e^{it(\sigma_3 \otimes \sigma_3)} = e^{it\sigma_3} \otimes e^{it\sigma_3}$$

Looking at the just σ_1 matrices:

$$e^{it\sigma_1} \otimes e^{it\sigma_1} = \begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t & i\sin t\cos t & 0.25(e^{it}-e^{-it})^2 \\ i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & 0.25(e^{it}-e^{-it})^2 & i\sin t\cos t \\ i\sin t\cos t & 0.25(e^{it}-e^{-it})^2 & 0.25(e^{it}+e^{-it})^2 & i\sin t\cos t \\ 0.25(e^{it}-e^{-it})^2 & i\sin t\cos t & i\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it(\sigma_1 \otimes \sigma_1)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & 0.5e^{it}-0.5e^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5e^{it}-0.5e^{-it} & 0 \\ 0 & 0.5e^{it}-0.5e^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5e^{it}-0.5e^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

Again, for $t = 0$ we find that these matrices are equivalent.
Comparing all other pairs:

$$e^{it\sigma_1} \otimes e^{it\sigma_2} = \begin{bmatrix} 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t & i\sin t\cos t & -\sin t\cos t + \frac{1}{2}i \\ -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 & \frac{1}{2}i\cos 2t - \frac{1}{2}i & i\sin t\cos t \\ i\sin t\cos t & -\sin t\cos t + \frac{1}{2}i & 0.25(e^{it}+e^{-it})^2 & -\sin t\cos t \\ \frac{1}{2}i\cos 2t - \frac{1}{2}i & i\sin t\cos t & -\sin t\cos t & 0.25(e^{it}+e^{-it})^2 \end{bmatrix}$$

$$e^{it(\sigma_1 \otimes \sigma_2)} = \begin{bmatrix} 0.5e^{it}+0.5e^{-it} & 0 & 0 & -0.5ie^{it}+0.5ie^{-it} \\ 0 & 0.5e^{it}+0.5e^{-it} & 0.5ie^{it}-0.5ie^{-it} & 0 \\ 0 & -0.5ie^{it}+0.5ie^{-it} & 0.5e^{it}+0.5e^{-it} & 0 \\ 0.5ie^{it}-0.5ie^{-it} & 0 & 0 & 0.5e^{it}+0.5e^{-it} \end{bmatrix}$$

Now I used a calculator (symbolic engine) for the rest of them and for every single one works in the same way ($t = 0$ makes them equal). Another very interesting thing, though, is that they also all simplify to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It appears we get the same result, the matrices being equal, with a $t = 2\pi$. This also results in the same 4x4 identity. This would lead one to suspect that this pattern should exist with the period of trig functions (2π) which seems to be the case when testing other numbers ($k \cdot 2\pi$).

Another interesting find is when $t = \pi$. With this, we don't get equivalent results, however we do get the consistent result of:

$$e^{it\sigma_j} \otimes e^{it\sigma_k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{it(\sigma_j \otimes \sigma_k)} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Which seems to suggest that these two results are "out of phase" by half a period. Which is interesting. More exploration is probably necessary.