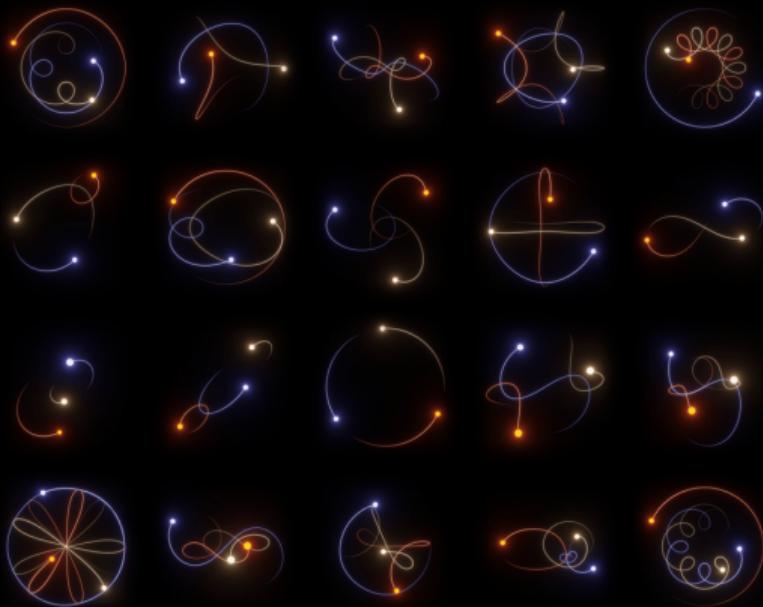


# Sparse Identification of Nonlinear Dynamics (SINDy)

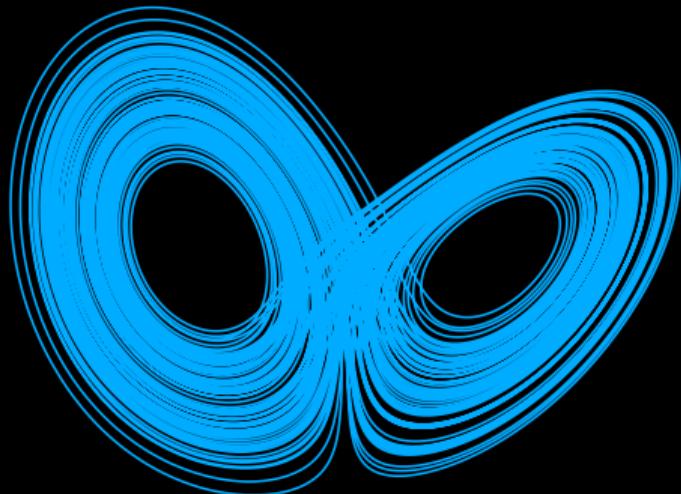
Theo Rode      April 24, 2024

# Dynamical Systems

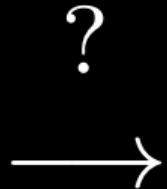
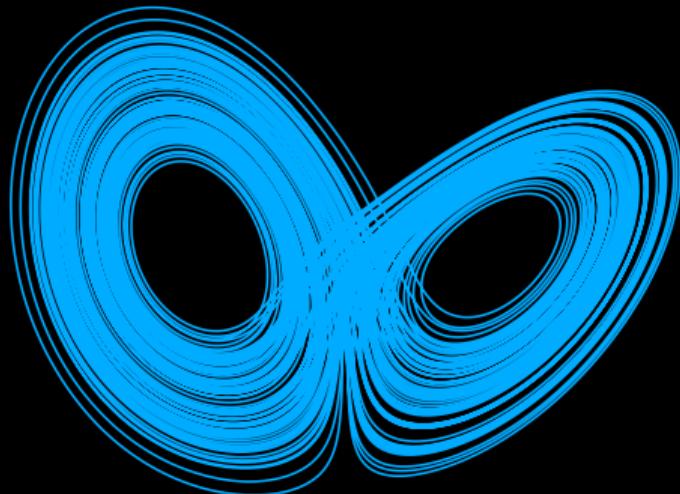


Dynamical System  $\rightarrow$  Behavior

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z\end{aligned}$$



# Data → Dynamical System



$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t))$$

# SINDy - Overview

- (1) Collect data ( $\mathbf{X}$ ) and derivatives ( $\dot{\mathbf{X}}$ )
- (2) Construct function library  $\Theta(\mathbf{X})$
- (3) Solve sparse regression problem  $\dot{\mathbf{X}} = \Theta(\mathbf{X})\boldsymbol{\Xi}$

# Data

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}^T(t_1) \\ \mathbf{x}^T(t_2) \\ \vdots \\ \mathbf{x}^T(t_m) \end{bmatrix} = \begin{bmatrix} x_1(t_1) & x_2(t_1) & \cdots & x_n(t_1) \\ x_1(t_2) & x_2(t_2) & \cdots & x_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t_m) & x_2(t_m) & \cdots & x_n(t_m) \end{bmatrix}$$

# Derivatives

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{x}}^T(t_1) \\ \dot{\mathbf{x}}^T(t_2) \\ \vdots \\ \dot{\mathbf{x}}^T(t_m) \end{bmatrix} = \begin{bmatrix} \dot{x}_1(t_1) & \dot{x}_2(t_1) & \cdots & \dot{x}_n(t_1) \\ \dot{x}_1(t_2) & \dot{x}_2(t_2) & \cdots & \dot{x}_n(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{x}_1(t_m) & \dot{x}_2(t_m) & \cdots & \dot{x}_n(t_m) \end{bmatrix}$$

# Function Library

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & \dots & | & | & \dots \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \dots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \dots \\ | & | & | & | & & | & | & \end{bmatrix}$$

# Function Library

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & \dots & | & | & \dots \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \dots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \dots \\ | & | & | & | & & | & | & \end{bmatrix}$$

$$\mathbf{X}^{P_2} = \begin{bmatrix} x_1^2(t_1) & x_1(t_1)x_2(t_1) & \cdots & x_1(t_1)x_n(t_1) & x_2^2(t_1) & \cdots & x_n^2(t_1) \\ x_1^2(t_2) & x_1(t_2)x_2(t_2) & \cdots & x_1(t_2)x_n(t_2) & x_2^2(t_2) & \cdots & x_n^2(t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1^2(t_m) & x_1(t_m)x_2(t_m) & \cdots & x_1(t_m)x_n(t_m) & x_2^2(t_m) & \cdots & x_n^2(t_m) \end{bmatrix}$$

# Function Library

$$\Theta(\mathbf{X}) = \begin{bmatrix} | & | & | & | & \dots & | & | & \dots \\ 1 & \mathbf{X} & \mathbf{X}^{P_2} & \mathbf{X}^{P_3} & \dots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \dots \\ | & | & | & | & & | & | & \end{bmatrix}$$

$$\sin(\mathbf{X}) = \begin{bmatrix} \sin(x_1(t_1)) & \sin(x_2(t_1)) & \cdots & \sin(x_n(t_1)) \\ \sin(x_1(t_2)) & \sin(x_2(t_2)) & \cdots & \sin(x_n(t_2)) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(x_1(t_m)) & \sin(x_2(t_m)) & \cdots & \sin(x_n(t_m)) \end{bmatrix}$$

# Function Library

$$\dot{x}_i(t) = f_i(\mathbf{x}(t)) = a_1 g_1(\mathbf{x}(t)) + a_2 g_2(\mathbf{x}(t)) + \cdots + a_k g_k(\mathbf{x}(t))$$

# Function Library

$$\dot{x}_i(t) = f_i(\mathbf{x}(t)) = a_1 g_1(\mathbf{x}(t)) + a_2 g_2(\mathbf{x}(t)) + \cdots + a_k g_k(\mathbf{x}(t))$$

$$\Theta(\mathbf{X}) = \begin{bmatrix} g_1(\mathbf{x}(t_1)) & g_2(\mathbf{x}(t_1)) & \cdots & g_k(\mathbf{x}(t_1)) \\ g_1(\mathbf{x}(t_2)) & g_2(\mathbf{x}(t_2)) & \cdots & g_k(\mathbf{x}(t_2)) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(\mathbf{x}(t_m)) & g_2(\mathbf{x}(t_m)) & \cdots & g_k(\mathbf{x}(t_k)) \end{bmatrix}$$

# Sparse Regression

$$\dot{\mathbf{X}} = \Theta(\mathbf{X})\boldsymbol{\Xi}$$

$$\boldsymbol{\Xi} = [\boldsymbol{\xi}_1 \quad \boldsymbol{\xi}_2 \quad \cdots \quad \boldsymbol{\xi}_n]$$

# Sparse Regression

```
function sparse_regression(Theta, Xdot, lambda, kmax)
    Xi = Theta\Xdot # create initial guess for Xi as just regression
    for _ in 1:kmax
        # find coefficients that are less than lambda
        small_coefficients = abs.(Xi) .< lambda
        Xi[small_coefficients] .= 0 # set small coefficients to 0

        # now individually run regression on non-zero coefficients
        for n in 1:(size(Xdot)[2]) # loop through all state dimensions
            nonsmall = .!small_coefficients[:, n]

            Xi[nonsmall, n] = Theta[:, nonsmall]\Xdot[:, n]
        end # for n
    end # for k
    Xi
end # function sparse_regression
```

# Sparse Regression Result

$$\boldsymbol{\Xi} = [\xi_1 \quad \xi_2 \quad \cdots \quad \xi_n]$$

$$\xi_i = \begin{bmatrix} \xi_{i,1} \\ \xi_{i,2} \\ \vdots \\ \xi_{i,k} \end{bmatrix} \implies \dot{x}_i(t) = \xi_{i,1}g_1(\mathbf{x}(t)) + \xi_{i,2}g_2(\mathbf{x}(t)) + \cdots + \xi_{i,k}g_k(\mathbf{x}(t))$$

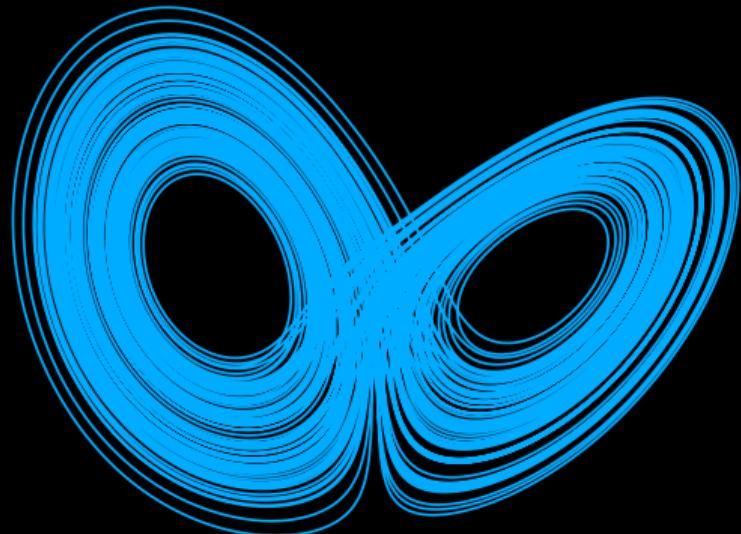
Most  $\xi_{i,j} = 0$ .

# Lorenz System

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$



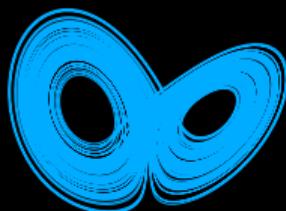
# Found Lorenz Systems

Actual



$$\begin{aligned}\dot{x} &= 10(y - x) \\ \dot{y} &= x(28 - z) - y \\ \dot{z} &= xy - (8/3)z\end{aligned}$$

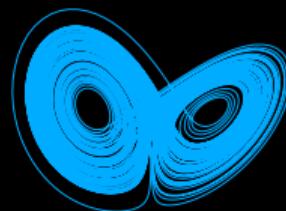
$\eta = 0.01$



$$\begin{aligned}\dot{x} &= 9.976y - 9.947x \\ \dot{y} &= 27.041x - 0.976xz - 0.669y \\ \dot{z} &= 0.996xy - 2.650z\end{aligned}$$

$$E_{\text{der}} = 0.00136$$

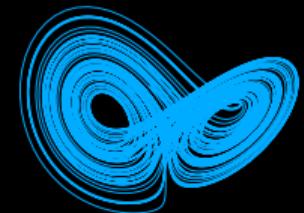
$\eta = 0.1$



$$\begin{aligned}\dot{x} &= 9.838y - 9.806x \\ \dot{y} &= 25.912x - 0.946xz - 0.433y \\ \dot{z} &= 0.982xy - 2.613z\end{aligned}$$

$$E_{\text{der}} = 0.00349$$

$\eta = 1$



$$\begin{aligned}\dot{x} &= 7.144y - 7.004x \\ \dot{y} &= 14.982x - 0.635xz + 1.537y \\ \dot{z} &= 0.839xy - 2.151z - 2.153\end{aligned}$$

$$E_{\text{der}} = 0.10119$$

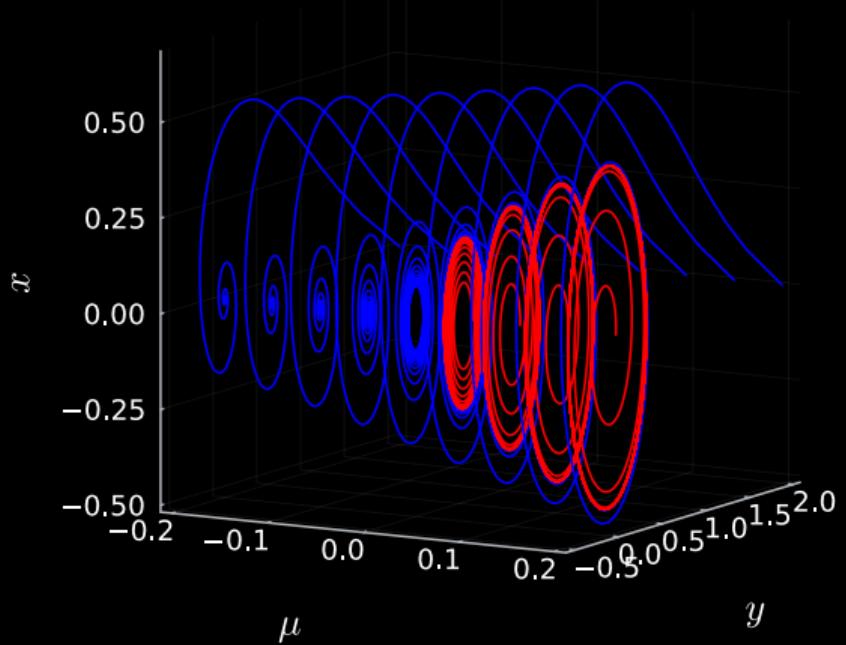
# SINDy and Bifurcations

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}; \mu) \\ \dot{\mu} &= 0\end{aligned}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{\mu_0} \\ \mathbf{X}_{\mu_1} \\ \vdots \\ \mathbf{X}_{\mu_h} \end{bmatrix}$$

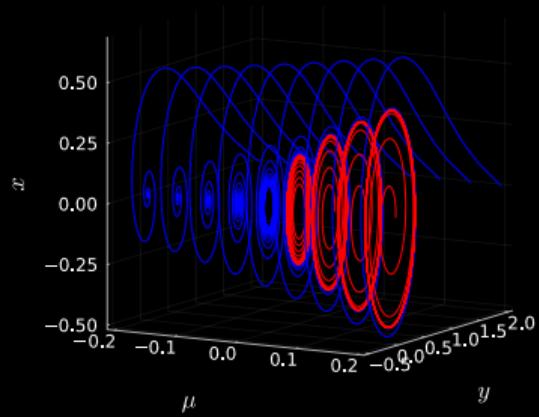
# The Hopf Normal Form

$$\begin{aligned}\dot{x} &= \mu x - \omega y - Ax(x^2 + y^2) \\ \dot{y} &= \omega x + \mu y - Ay(x^2 + y^2)\end{aligned}$$



# Found Hopf Bifurcation

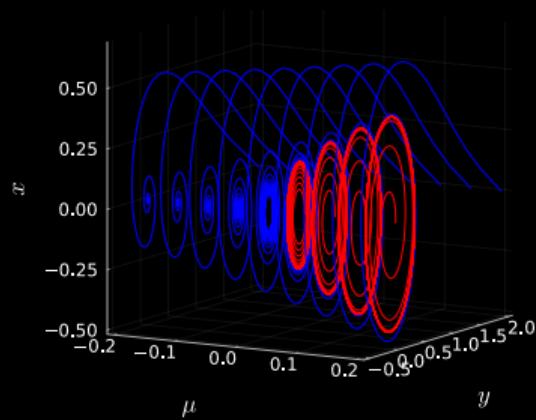
Actual



$$\dot{x} = \mu x - y - x(x^2 + y^2)$$

$$\dot{y} = x + \mu y - y(x^2 + y^2)$$

$\eta = 0.005$



$$\dot{x} = -0.957y + 0.908\mu x - 0.888x^3 - 1.303xy^2$$

$$+ 0.769\mu xy^2 - 0.723\mu^2 x^2 y$$

$$\dot{y} = 0.966x + 0.961\mu y - 0.927yx^2 - 0.962y^3$$

$$E_{\text{der}} = 2.682$$

# Concluding Thoughts

- SINDy demonstrates performance in qualitative dynamics recovery
- Concerns over stability with noise and parameters (how do we choose good basis/parameters?)
- Active learning?

# Citations

- [1] Steven L. Bruton, Joshua L. Proctor, and J. Nathan Kutz. “Discovering governing equations from data by sparse identification of nonlinear dynamical systems”. In: *Proceedings of the national academy of sciences* 113.15 (2016), pp. 3932–3937. DOI:  
<https://doi.org/10.1073/pnas.1517384113>.
- [2] Perosello. *20 examples of periodic solutions to the three-body problem*. [Online; accessed April 23, 2024]. 2023. URL:  
[https://en.wikipedia.org/wiki/Three-body\\_problem#/media/File:5\\_4\\_800\\_36\\_downscaled.gif](https://en.wikipedia.org/wiki/Three-body_problem#/media/File:5_4_800_36_downscaled.gif).