Higher Mathematics

Patrick Ausel

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Contents

Chapter 1

Straight Line

1.1 Gradient Revision and Equation of a Straight Line

The gradient is the line's steepness. The bigger the number, the steeper it is. Its symbol is m.

Its equation is

$$m = \frac{\text{vertical distance}}{\text{horizontal distance}}$$

or

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.1}$$

If the line is vertical (|), then it has an undefined gradient, e.g. when $x = 1$. If it is horizontal (—), then $m = 0$, e.g. when $y = 1$.

From National 5, it should be known that, given a point and gradient, the equation of a line can be written using

$$y - b = m(x - a). \tag{1.2}$$

1.1.1 Example

If $m = 3$ and the line goes through point (-2,3), find the equation of the line.

$$
\begin{aligned}
y - b &= m(x - a) \\
y - 3 &= 3(x + 2) \\
y &= 3(x + 2) + 3 \\
y &= 3x + 6 + 9 \\
y &= 3x + 9
\end{aligned}
$$

Note that the equation can be left in any form, no extra marks are awarded for leaving it in the form $0 = 3x - y + 9$

1.2 $m = \tan\theta$

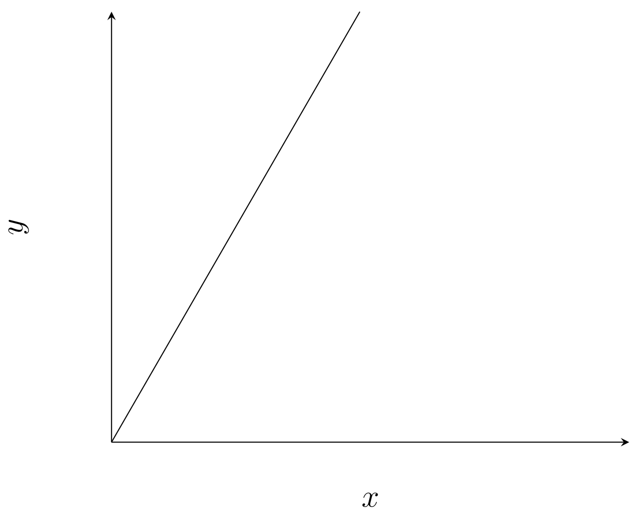


Figure 1.1: Here, θ is 60°.

θ is the (in figure 1.1 an acute) angle made from the x-axis in an anti-clockwise direction (also called in the positive direction of the x-axis). If θ is known, the gradient of the line can be found, and vice versa, by using

$$m = \tan\theta \tag{1.3}$$

1.2.1 Examples

1. A line makes an angle of 120° with the positive direction of the x-axis. Find its gradient.

One could just enter 120° into their calculator and get the right answer, but the following is a method of working it out if the question would appear in a non-calculator paper.

$$\begin{aligned} m &= \tan\theta \\ &= \tan 120^\circ \\ &= -\tan 60 \\ &= -\sqrt{3} \end{aligned}$$

2. A line's gradient is -2. Find the angle it makes in the positive direction of the x-axis.

$$\begin{aligned} \text{If } m &= 2, \\ \tan\theta &= 2 \\ \theta &= \tan^{-1} 2 \\ &= 63.43... \\ \text{Now the proper line where } m &= -2, \\ \theta &= 180 - 63.43... \\ &= 116.56... \\ &\approx 116.6^\circ \end{aligned}$$

1.3 Perpendicular Lines (⊥)

Perpendicular lines sit at right angles to each other. If lines are perpendicular, then

$$m_1 m_2 = -1. \tag{1.4}$$

1.3.1 Examples

If two lines are perpendicular, calculate m_1 if m_2 is

1. 6
2. -7
3. $\frac{2}{7}$

1.

$$\begin{aligned} m_1m_2 &= -1 \\ 3m_1 &= -1 \\ m_1 &= -\frac{1}{3} \end{aligned}$$

2.

$$\begin{aligned} m_1m_2 &= -1 \\ -7m_1 &= -1 \\ m_1 &= \frac{1}{7} \end{aligned}$$

3.

$$\begin{aligned} m_1m_2 &= -1 \\ \frac{2}{7}m_1 &= -1 \\ 2m_1 &= -7 \\ m_1 &= -\frac{7}{2} \end{aligned}$$

As a shortcut, take the recipricol and change the sign.

4. Find the equation of the line perpendicular to $y = -\frac{2}{3}x + 5$ that passes through the point $(1, 7)$.

$$m = -\frac{2}{3} \qquad\qquad m_\perp = \frac{3}{2}$$

Now that the gradient of the perpendicular line has been found, it can be used in the equation.

$$\begin{aligned} y - b &= m(x - a) \\ y - 7 &= \frac{3}{2}(x - 1) \\ y &= \frac{3}{2}x - \frac{3}{2} + 7 \\ y &= \frac{3}{2}x + \frac{11}{2} \end{aligned}$$

1.4 Mid Point Formula

To find the point in the middle of a line, use

$$\text{mid point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{1.5}$$

1.4.1 Examples

1. Find the mid point of $(1, -4)$ and $(7, 8)$.

$$\begin{aligned}\text{mid point} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ &= \left(\frac{1 + 7}{2}, \frac{-4 + 8}{2}\right)\\ &= (4, 2)\end{aligned}$$

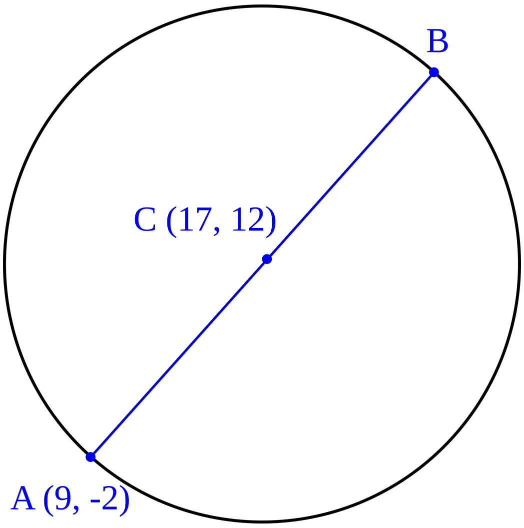


Figure 1.2: Not to scale.

2. A circle has centre point $(17, 12)$. Two points $A(9, -2)$ and B are drawn on its circumference, these are connecting by a diameter (see figure 1.2). Find the coordinates of B.

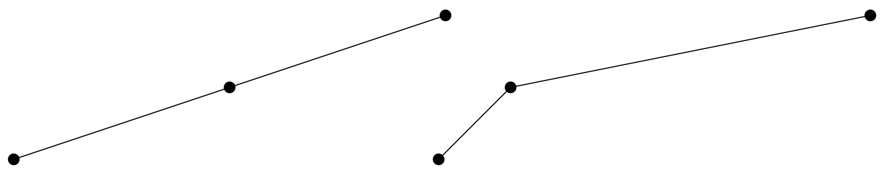
$$\begin{aligned}\text{mid point} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\\ (17, 12) &= \left(\frac{9 + x}{2}, \frac{-2 + y_2}{2}\right)\end{aligned}$$

$$\begin{aligned} \frac{9+x}{2} &= 17 \\ 9+x &= 34 \\ x &= 25 \end{aligned} \qquad \begin{aligned} \frac{-2+y}{2} &= 12 \\ -2+y &= 24 \\ y &= 26 \end{aligned}$$

So the coordinate of B is $(25, 26)$.

1.5 Collinearity

Points that are collinear lie on a straight line.



(a) Collinear. (b) Not collinear.

Figure 1.3: Examples of collinearity.

To test for collinearity of three points A, B, C:

1. find m_{AB},
2. find m_{BC},
3. if $m_{AB} = m_{BC}$, then (because B is a common point) they are collinear.

1.5.1 Examples

1. Show that the points $P(-6,-1), Q(0,2), R(8,6)$ are collinear.

$$m_{PQ} = \frac{2+1}{0+6} = \frac{3}{6} = \frac{1}{2} \qquad m_{QR} = \frac{6-2}{8-0} = \frac{4}{8} = \frac{1}{2}$$

Since $m_{PQ} = m_{QR}$, and Q is a common point, PQR are collinear.

2. If $A\,(1,-1)\,, B\,(-1,k)\,, C\,(5,7)$ are collinear, find k.

$$\begin{aligned} m_{AB} &= m_{BC} \\ \frac{k-(-1)}{-1-1} &= \frac{7-k}{5-(-1)} \\ \frac{k+1}{-2} &= \frac{7-k}{6} \\ 6(k+1) &= -2(7-k) \\ 6k+6 &= -14+2k \\ 4k &= -20 \\ k &= 5 \end{aligned}$$

1.6 Median Lines

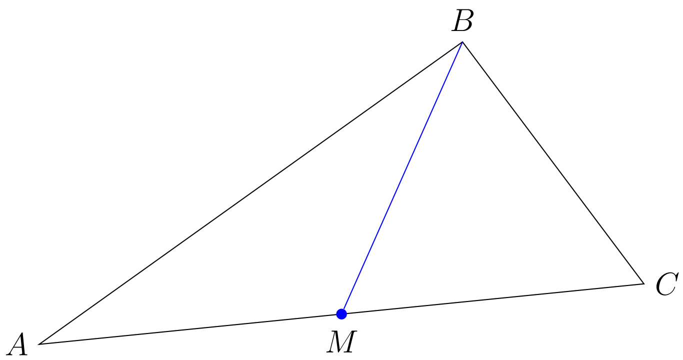


Figure 1.4: Median from B.

The median is a line from a vertex of a triangle to the mid point of the opposite.

To find the median from B in a triangle ABC (such as in figure 1.4):

1. find the mid point of AC (hereafter called M), since the opposite of B is AC,
2. find m_{BM},
3. use $y - b = m(x - a)$ with B or M and m_{BM}.

1.6.1 Example

In $\triangle ABC$, $A(4, -9)$, $B(10, 2)$, and $C(4, -4)$. Find the equation of the median from A.

$$\begin{aligned} \text{mid point} &= \left(\frac{10+4}{2}, \frac{2-4}{2}\right) \\ &= (7, -1) \end{aligned} \qquad \begin{aligned} m_{AM} &= \frac{-1-(-9)}{7-4} \\ &= \frac{8}{3} \end{aligned}$$

$$\begin{aligned} y - b &= m(x - a) \\ y + 9 &= \frac{8}{3}(x - 4) \\ 3y + 27 &= 8(x - 4) \\ 3y &= 8x - 32 - 27 \\ 3y &= 8x - 59 \end{aligned}$$

1.7 Altitude

The altitude is a straight line form a vertex to the other side at right angles.

To find the altitude from B in a triangle ABC (such as in figure 1.5):

1. find m_{AC},
2. find $m_{\perp}$,
3. use $y - b = m(x - a)$ with $m_{\perp}$ and B.

1.7.1 Example

The triangle ABC has vertices $A(3, -5)$, $B(4, 3)$, and $C(-7, 2)$. Find the altitude from A.

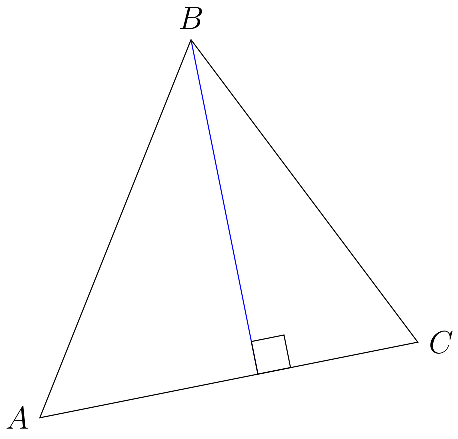


Figure 1.5: Altitude from B.

$$m_{CB} = \frac{3-2}{4-(-7)} = \frac{1}{11} \qquad m_{\perp} = -11$$

$$\begin{aligned} y - b &= m(x-a) \\ y + 5 &= -11(x-3) \\ y &= -11x + 33 - 5 \\ y &= -11x + 28 \end{aligned}$$

1.8 Perpendicular Bisector

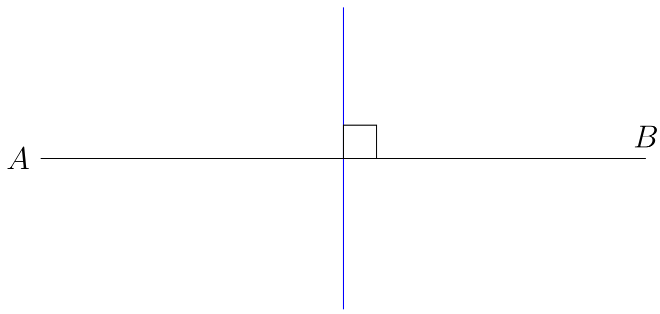


Figure 1.6: The perpendicular bisector of AB is here in blue.

The perpendicular bisector (abbr. PB) is a line that cuts another line in the middle and sits at right angles.

To find the PB of a line AB:

1. find the mid point AB,
2. find m_{AB},
3. find $m_{\perp}$,
4. use $y - b = m(x - a)$ with the mid point AB and $m_{\perp}$.

1.8.1 Example

Two points are $A(-2, 1)$ and $B4, 7)$ are connected by a line. Find the equation of the perpendicular bisector.

$$\begin{aligned} \text{mid point} &= \left(\frac{-2+4}{2}, \frac{1+7}{2}\right) \\ &= (1, 4) \end{aligned} \qquad \begin{aligned} m_{AB} &= \frac{7-1}{4-(-2)} \\ &= 1 \end{aligned} \qquad m_{\perp} = -1$$

$$\begin{aligned} y - b &= m(x - a) \\ y - 4 &= -1(x - 1) \\ y &= -x + 1 + 4 \\ y &= -x + 5 \end{aligned}$$

1.9 Intersection of Lines

The point of intersection (abbr. POI) of two lines can be found through solving each line's equation simultaneously.

The National 5 course most likely focussed on the method of elimination, a better method is to use substitution. However, sometimes, such as in the second example, the equations are so simple that it would be stupid to not eliminate.

1.9.1 Examples

1. Two lines have equations 2x-y+11=0 and x+2y-7 = 0. Find the point of intersection.

$$\begin{aligned} 2x - y + 11 &= 0 \\ y &= 2x + 11 \end{aligned}$$

Sub $y = 2x + 11$ into $x + 2y - 7 = 0$,

$$\begin{aligned} x + 2(2x + 11) - 7 &= 0 \\ x + 4x + 22 - 7 &= 0 \\ 5x + 15 &= 0 \\ 5x &= -15 \\ x &= -3 \end{aligned}$$

$$\begin{aligned} y &= 2x + 11 \\ y &= 2(-3) + 11 \\ &= -6 + 11 \\ &= 5 \end{aligned} \qquad POI = (-3, 5)$$

2. In triangle ABC with points $A(1, 0)$, $B(-4, 3)$, $C(0, -1)$, find the median from A, the altitude from C, and henceforth the POI.

   Median from A:

$$\begin{aligned} \text{mid point} &= \left(\frac{-4 + 0}{2}, \frac{3 + (-1)}{2} \right) \\ &= (-2, 1) \end{aligned} \qquad \begin{aligned} m &= \frac{3 - 0}{-4 - 1} \\ &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} y - b &= m(x - a) \\ y - 1 &= -\frac{1}{3}(x + 2) \\ 3y - 3 &= -(x + 2) \\ 3y &= -x - 2 + 3 \\ 3y &= -x + 1 \end{aligned}$$

   Altitude from C:

$$\begin{aligned} m_{AB} &= \frac{3 - 0}{-4 - 1} \\ &= -\frac{3}{5} \end{aligned} \qquad m_{\perp} = \frac{5}{3}$$

$$
\begin{aligned}
y - b &= m(x - a) \\
y + 1 &= \frac{5}{3}\,(x + 0) \\
3y + 3 &= 5\,(x + 0) \\
3y &= 5x - 3
\end{aligned}
$$

Point of intersection:

$$
\begin{array}{rl}
3y &= -x + 1 \\
- \quad 3y &= 5x - 3 \\ \hline
0 &= -6x + 4 \\
6x &= 4 \\
x &= \dfrac{2}{3}
\end{array}
\qquad\qquad
\begin{aligned}
3y &= -x + 1 \\
3y &= -\frac{2}{3} + 1 \\
9y &= -2 + 3 \\
9y &= 1 \\
y &= \frac{1}{9}
\end{aligned}
$$

$$
POI = \left(\frac{2}{3}, \frac{1}{9}\right)
$$

Chapter 2

Functions and Graphs

2.1 Composite Functions

A function $f(x) = 3x + 9$ will take in any value for x, and substitutes it into the expression $3x + 9$. If another expression would be passed into $f(x)$, such as x-2, then the result would be $3(x - 2) + 9$. A composite function is one where two functions are combined, similar to above.

Suppose $f(x) = 2x$ and $g(x) = x + 1$, then composite functions $f(g(x))$ and $g(f(x))$ can be found as follows:

$$\begin{aligned} f(g(x)) &= 2(x + 1) \\ &= 2x + 2 \end{aligned} \qquad\qquad \begin{aligned} g(f(x)) &= (2x) + 1 \\ &= 2x + 1 \end{aligned}$$

A composite function can get tiring to write out all the time. So it could be written as $h(x) = f(g(x))$.

2.1.1 Example

$f(x) = x^2{+}1$ and $g(x) = \frac{1}{x}\ (x \neq 0)$. Find $h(x) = f(g(x))$ and $h(x) = g(f(x))$.

$$\begin{aligned} h(x) &= f\left(\frac{1}{x}\right) \\ &= \left(\frac{1}{x}\right)^2 - 2 \\ &= \frac{1}{x^2} - 1 \end{aligned} \qquad\qquad \begin{aligned} k(x) &= g\left(x^2 + 1\right) \\ &= \frac{1}{x^2 + 1} \end{aligned}$$

2.2 Domains and Set Notation

A function most often can't take all numbers as inputs. For example, a function of the area of a rectangle might be written as $f(x) = x^2 + 2x - 8$. Since it's a function of real-life area in terms of lengths, the area cannot be negative. After some thinking, it can be concluded that x must be 2 or larger. So the function's domain is any number larger or equal to 2.

Additionally, a real number (aka decimal number) could also be passed in to the function, but maybe that's not what you want. You can also define the domain to only include natural numbers (positive whole numbers).

Applying these two factors, the domain of the function is $\{x \in \mathbb{N} \mid x \geq 2\}$

Set (or rather set-builder) notation is a way of describing a set of numbers. In relation to domains, it details what set of numbers can be an acceptable input of the function. The kinds of set number symbols that exist commonly are

- $\mathbb{N}$, natural numbers. Non-negative integers (so starting with 0, counting up in steps of 1).
- $\mathbb{Z}$, integers. Any number that can be written without a fractional component ("whole" numbers).
- $\mathbb{Q}$, rational numbers. A number which can be written as a fraction with an integer numerator and a non-zero natural denominator[1].
- $\mathbb{R}$, real numbers. Any number that can be represented on a number line.

2.2.1 Example

Find a suitable domain for the function $\frac{1}{x^2-1}$.

$$\{x \in \mathbb{R} \mid x \neq \sqrt{2}\}$$

[1]Negative denominators can exist, but are avoided, as they can be expressed as a negative numerator instead.