Assignment 2: CS 663, Fall 2021

Question 6

- Consider a 1D ramp image of the form $I(x) = cx + d$ where c, d are scalar coefficients. Derive an expression for the image J which results when I is filtered by a zero-mean Gaussian with standard deviation σ. Derive an expression for the image that results when I is treated with a bilateral filter of parameters σ_s, σ_r. (Hint: in both cases, you get back the same image.) Ignore any border issues, i.e. assume the image had infinite extent. [10 points]

  Notation: In the following answer, $*$ denotes convolution operation.
  Answer: To find the filtered output (J) of the image (I) with a zero-mean Gaussian filter, $F_G(x)$, we can use the following equation:

$$J(x) = I(x) * F_G(x)$$
$$J(x) = I(x) * \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$
$$J(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right)(cx - ct + d)dt$$

  We can split the above integral as follows:

$$J(x) = \frac{1}{\sqrt{2\pi}\sigma}\left((cx+d)\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right)dt + c\int_{-\infty}^{\infty} t\exp\left(-\frac{t^2}{2\sigma^2}\right)dt\right)$$

  We can easily see that the second integral in the above equation is an odd function and hence it will integrate out to be zero. Hence, the above equation can be simplified as follows:

$$J(x) = \frac{1}{\sqrt{2\pi}\sigma}(cx+d)\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right)dt$$

  This is nothing but the integral of the Gaussian function with zero mean and standard deviation σ. It evaluates to $\sqrt{2\pi}\sigma$.Hence, we can write $J(x)$ as follows:

$$\mathbf{J(x) = cx + d}$$
$$\implies \mathbf{J(x) = I(x)}$$

  Hence, we can see that in the case of Gaussian filtering, the output image is the same as the input image.

  Now, let us find the output image when the given image, $I(x)$ is filtered by a bilateral filter, $F_{bilateral}(x)$ with parameters σ_s, σ_r.

By the definition of Bilateral Filter we have,

$$J(x) = \int_{-\infty}^{\infty} I(t) \times \left(\frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(-\frac{x^2}{2\sigma_s^2} \right) \frac{1}{\sqrt{2\pi}\sigma_r} \exp\left(-\frac{I^2(x)}{2\sigma_r^2} \right) \right) dt$$

$$J(x) = \frac{1}{\sqrt{2\pi}\sigma_s} \frac{1}{\sqrt{2\pi}\sigma_r} \int_{-\infty}^{\infty} (ct+d) \exp\left(-\frac{(x-t)^2}{2\sigma_s^2} \right) \exp\left(-\frac{(I(x)-I(t))^2}{2\sigma_r^2} \right) dt$$

$$J(x) = \frac{1}{\sqrt{2\pi}\sigma_s} \frac{1}{\sqrt{2\pi}\sigma_r} \int_{-\infty}^{\infty} (ct+d) \exp\left(-\frac{(x-t)^2}{2\sigma_s^2} \right) \exp\left(-\frac{c^2(x-t)^2}{2\sigma_r^2} \right) dt$$

$$\implies J(x) = \frac{1}{2\pi\sigma_s\sigma_r} \int_{-\infty}^{\infty} I(t) \exp\left(-\frac{(x-t)^2}{2\sigma_s^2} + -\frac{c^2(x-t)^2}{2\sigma_r^2} \right) dt$$

$$\implies J(x) = \frac{1}{2\pi\sigma_s\sigma_r} \int_{-\infty}^{\infty} I(t) \exp\left(-(x-t)^2 \left(\frac{1}{2\sigma_s^2} + \frac{c^2}{2\sigma_r^2} \right) \right) dt$$

Since the above equation resembles the convolution of an image with a Gaussian filter, we can directly derive the following expression by our commputation in previous section:

$$J(x) = \frac{1}{2\pi\sigma_s\sigma_r} \times \sqrt{2\pi} \left(\frac{2\sigma_s\sigma_r}{\sqrt{2\sigma_r^2 + 2c^2\sigma_s^2}} \right) \times I(x)$$

$$\implies \mathbf{J}(\mathbf{x}) = \frac{\mathbf{1}}{\sqrt{\pi(\boldsymbol{\sigma}_{\mathbf{r}}^{\mathbf{2}} + \mathbf{c}^{\mathbf{2}}\boldsymbol{\sigma}_{\mathbf{s}}^{\mathbf{2}})}} \times \mathbf{I}(\mathbf{x})$$

Hence, we can see that in the case of bilateral filter, the output image is same as the input image with a scaling factor.