Assignment 3: CS 663, Fall 2021

Question 4

- You can use the Fourier transform to compute the Laplacian of an image. But can you use the Fourier transform to compute the gradient magnitude at every pixel in an image? If yes, explain how you will do it. If not, explain why this is not possible. [10 points]

  Answer: We can use Fourier transform to compute the gradient magnitude at every pixel in an image.

  Let $f(x, y)$ be the image intensity function of size $M \times M$ and its Fourier transform is $F(u, v)$,

  $$\text{Gradient: } \nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} f(x+1, y) - f(x, y) \\ f(x, y+1) - f(x, y) \end{bmatrix}$$

  $$\text{Gradient magnitude: } \|\nabla f\| = \left[(f(x+1, y) - f(x, y))^2 + (f(x, y+1) - f(x, y))^2\right]^{1/2}$$

  $$\|\nabla f\| = \left[f_x^2 + f_y^2\right]^{1/2}$$

  The idea is that we can get f_x and f_y using Fourier transform:

  $$F_x(u, v) = \mathfrak{F}(f_x) = F(u, v)[e^{j2\pi u/M} - 1] \quad (\because \text{Fourier shift theorem})$$

  $$\implies f_x = \mathfrak{F}^{-1}(F_x(u, v)) = \mathfrak{F}^{-1}(F(u, v)[e^{j2\pi u/M} - 1])$$

  $$F_y(u, v) = \mathfrak{F}(f_y) = F(u, v)[e^{j2\pi v/M} - 1] \quad (\because \text{Fourier shift theorem})$$

  $$\implies f_y = \mathfrak{F}^{-1}(F_y(u, v)) = \mathfrak{F}^{-1}(F(u, v)[e^{j2\pi v/M} - 1])$$

  which can be used to get gradient magnitude:

  $$\mathfrak{F}((\|\nabla f\|)^2) = \mathfrak{F}(f_x^2) + \mathfrak{F}(f_y^2)$$

  $$\mathfrak{F}((\|\nabla f\|)^2) = F_x(u, v) * F_x(u, v) + F_y(u, v) * F_y(u, v) \quad (\because \text{Fourier convolution theorem})$$

  $$\|\nabla f\| = \left[\mathfrak{F}^{-1}(F_x(u, v) * F_x(u, v) + F_y(u, v) * F_y(u, v))\right]^{1/2}$$

  $F_x(u, v)$ and $F_y(u, v)$ can be determined from $F(u, v)$, hence the gradient magnitude can be determined using Fourier transform.