Assignment 2: CS 663, Fall 2021

Question 5

- Suppose I convolve an image f with a mean-filter of size $(2a+1) \times (2a+1)$ where $a > 0$ is an integer to produce a result f_1. Suppose I convolve the resultant image f_1 with the same mean filter once again to produce an image f_2, and so on until you get image f_K in the Kth iteration. Can you express f_K as a convolution of f with some kernel. If not, why not? If yes, with what kernel? Justify. [10 points]

  Answer: Let M be the $(2a+1) \times (2a+1)$ mean filter. Then,

$$
\begin{aligned}
M &= \frac{1}{(2a+1)^2} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}_{(2a+1)\times(2a+1)} \\
&= \frac{1}{2a+1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} * \frac{1}{2a+1} \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \end{bmatrix}
\end{aligned} \tag{1}
$$

  Now, $f_1 = M * f$, $f_2 = M * f_1 = M * (M * f) = (M * M) * f$ and so on...(Since convolution is associative.). Therefore, by induction:

$$
f_K = (M * M * \dots * M)|_K * f \tag{2}
$$

  Let $M_K = (M * M * \dots * M)|_K$. $M_1 = M$.

$$
M_2 = \frac{1}{(2a+1)^4} \begin{bmatrix} 1 & 2 & \dots & a+1 & \dots & 2 & 1 \\ 2 & 4 & \dots & 2(a+1) & \dots & 4 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a+1 & 2(a+1) & \dots & (a+1)^2 & \dots & 2(a+1) & a+1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 4 & \dots & 2(a+1) & \dots & 4 & 2 \\ 1 & 2 & \dots & a+1 & \dots & 2 & 1 \end{bmatrix}_{(2a+3)\times(2a+3)} \tag{3}
$$

  From multinomial theorem on $(x^{-a} + x^{-a+1} + ... + 1 + ... + x^{a-1} + x^{a})^K$, we can show that:

$$
M_K(i,j) = \frac{\binom{K}{i}_{a+1} \binom{K}{j}_{a+1}}{(2a+1)^{2K}} \tag{4}
$$