Assignment 2: CS 663, Fall 2021

Question 2

- In bicubic interpolation, the image intensity value is expressed in the form $v(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x^i y^j$ where a_{ij} are the coefficients of interpolation and (x,y) are spatial coordinates. This uses sixteen nearest neighbors of a point (x,y). Given the intensity values of these 16 neighbors, explain with the help of matrix-based equations, how one can determine the coefficients a_{ij} that determine the function $v(x,y)$? Why do you require 16 neighbors for determining the coefficients? [10 points]

  Answer: Bicubic interpolation

$$v(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x^i y^j =$$
$$1 + a_{10}x + a_{20}x^2 + a_{30}x^3$$
$$+a_{01}y + a_{11}xy + a_{21}x^2y + a_{31}x^3y$$
$$+a_{02}y^2 + a_{12}xy^2 + a_{22}x^2y^2 + a_{32}x^3y^2$$
$$+a_{03}y^3 + a_{13}xy^3 + a_{23}x^2y^3 + a_{33}x^3y^3$$

  The above equation has 16 coefficients (4 choices for i and j), so we need at least 16 linear equations to solve for them. Hence we need 16 neighbouring pixel intensity values:

$$v(x_1,y_1),\ v(x_2,y_1),\ v(x_3,y_1),\ v(x_4,y_1),$$
$$v(x_1,y_2),\ v(x_2,y_2),\ v(x_3,y_2),\ v(x_4,y_2),$$
$$v(x_1,y_3),\ v(x_2,y_3),\ v(x_3,y_3),\ v(x_4,y_3),$$
$$v(x_1,y_4),\ v(x_2,y_4),\ v(x_3,y_4),\ v(x_4,y_4)$$

  The 16 equations can be seen as follows:

$$v(x_1,y_1) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_1^i y_1^j$$
$$v(x_1,y_2) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_1^i y_2^j$$
$$\ldots$$
$$\ldots$$
$$v(x_4,y_4) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_4^i y_4^j$$

  In the above equations, a_{ij} are the variables, $x_m^i y_n^j$ are the coefficients of the variables.

In matrix form:

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 & x_1y_1 & x_1^2y_1 & ... & x_1^2y_1^3 & x_1^3y_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_2 & x_1y_2 & x_1^2y_2 & ... & x_1^2y_2^3 & x_1^3y_2^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_3 & x_1y_3 & x_1^2y_3 & ... & x_1^2y_3^3 & x_1^3y_3^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_4 & x_4^2 & x_4^3 & y_3 & x_4y_3 & x_4^2y_3 & ... & x_4^2y_3^3 & x_4^3y_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 & x_4y_4 & x_4^2y_4 & ... & x_4^2y_4^3 & x_4^3y_4^3 \end{bmatrix}$$

$$x = \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ \vdots \\ a_{23} \\ a_{33} \end{bmatrix}, \quad b = \begin{bmatrix} v(x_1, y_1) \\ v(x_1, y_2) \\ v(x_1, y_3) \\ \vdots \\ v(x_4, y_3) \\ v(x_4, y_4) \end{bmatrix}$$

A is the coefficient matrix for the 16 linear equations and also 16×16, x is the variable vector which we need to solve for and b is just the resultant vector.

$$Ax = b$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 & x_1y_1 & x_1^2y_1 & ... & x_1^2y_1^3 & x_1^3y_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_2 & x_1y_2 & x_1^2y_2 & ... & x_1^2y_2^3 & x_1^3y_2^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_3 & x_1y_3 & x_1^2y_3 & ... & x_1^2y_3^3 & x_1^3y_3^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_4 & x_4^2 & x_4^3 & y_3 & x_4y_3 & x_4^2y_3 & ... & x_4^2y_3^3 & x_4^3y_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 & x_4y_4 & x_4^2y_4 & ... & x_4^2y_4^3 & x_4^3y_4^3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ \vdots \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} v(x_1, y_1) \\ v(x_1, y_2) \\ v(x_1, y_3) \\ \vdots \\ v(x_4, y_3) \\ v(x_4, y_4) \end{bmatrix}$$

$$x = A^{-1}b$$

This way x vector can be estimated from A and b matrices, which gives us the coefficients of interpolation and hence the bicubic interpolation function.