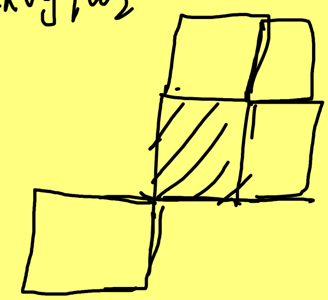
HW-3: Theory Qs

(3)

$$F_d(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y)\, e^{-j2\pi ux/W_1} \; e^{-2\pi vy/W_2}$$

$$f(x,y) = \frac{1}{W_1 W_2} \sum_{u=0}^{W_1-1} \sum_{v=0}^{W_2-1} F_d(u,v)\, e^{j2\pi ux/W_1} \; e^{j2\pi vy/W_2}$$



$\Rightarrow$ f: size: $M_1 \times N_1$, g: size: $M_2 \times N_2$

$$f * g = \sum_{l=0}^{M_1-1} \sum_{m=0}^{N_1-1} f(l,m)\, g(x-l, y-m)$$

pad-zeros to f, g such that their sizes are $M_1+M_2 \times N_1+N_2$

$\Rightarrow$ let $M = M_1+M_2$, $N = N_1+N_2$ | Now $f \xrightarrow{\mathcal{F}} F_d$, $G \xrightarrow{\mathcal{F}} G_d$

$$\Rightarrow \mathrm{IDFT}(F_d \cdot G_d)(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F_d(u,v)\, G_d(u,v)\, e^{j2\pi ux/M} \, e^{j2\pi vy/N}$$

$$\Rightarrow \frac{1}{MN} \sum_{u,v} \sum_{p,q} f(p,q)\, e^{-j2\pi\left(\frac{up}{M} + \frac{vq}{N}\right)} \cdot G_d(u,v)\, e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

$$\Rightarrow \sum_{p,q} f(p,q) \left[\frac{1}{MN} \sum_{u,v} G_d(u,v)\, e^{j2\pi\left(\frac{u(x-p)}{M} + \frac{v(y-q)}{N}\right)} \right]$$

$$\Rightarrow \sum_{p,q} f(p,q)\, g(x-p, y-q) = (f * g)(x,y)$$

original & zero-padded signals have the same convolution.

④ Gradient of a image at pixel (x,y) : $\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} f(x+1,y)-f(x,y) \\ f(x,y+1)-f(x,y) \end{bmatrix}$

magnitude: $\left[\left(f(x+1,y)-f(x,y)\right)^2 + \left(f(x,y+1)-f(x,y)\right)^2 \right]^{1/2}$

$\Rightarrow F_x(u,v) = f(u,v)\left[e^{j2\pi u/M} - 1\right]$

$F_y(u,v) = F(u,v)\left[e^{j2\pi v/M} - 1\right]$

$\mathcal{F}\left[(\text{Gradient magnitude})^2\right] = \left[F_x(u,v) * F_x(u,v) + F_y(u,v) * F_y(u,v)\right]$

It is possible.

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⑤ $F(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y)\, e^{-j2\pi ux/W_1}\, e^{-j2\pi vy/W_2}$

$F^*(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f^*(x,y)\, e^{-j2\pi(-u)x/W_1}\, e^{-j2\pi(-v)y/W_2}$

$\hookrightarrow f(x,y)$

$\Rightarrow F^*(u,v) = F(-u,-v)$

$F(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y)\, e^{-j2\pi ux/W_1}\, e^{-j2\pi vy/W_2}$

$x = -X, \quad y = -Y$

$\Rightarrow F(u,v) = \sum_{X=-(W_1-1)}^{0} \sum_{Y=-(W_2-1)}^{0} f(-X,-Y)\, e^{-j2\pi(-u)X/W_1}\, e^{-j2\pi(-v)Y/W_2}$

$\hookrightarrow f(X,Y)$

periodicity ignores this

$\Rightarrow F(u,v) = F(-u,-v) = F^*(u,v)$

$F(u,v)$ is real & even.

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⑥ $\mathcal{F}(f(t))(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, dt$

$$\Rightarrow \mathcal{F}(\mathcal{F}(f(t)))(v) = \int_{-\infty}^{\infty} \mathcal{F}(f(t))(\mu)\, e^{-j2\pi v\mu}\, d\mu$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\mu t}\, e^{-j2\pi v\mu}\, dt\, d\mu$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j2\pi\mu(t+v)}\, d\mu \right] f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \delta(-t-v)\, f(t)\, dt = f(-v)$$

$$\Rightarrow \mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(-(-t)) = f(t)$$