Assignment 3: CS 663, Fall 2021

Question 3

- Prove the convolution theorem for 2D Discrete Fourier transforms. [10 points]

  Answer: Let $f_1(x, y)$ and $f_2(x, y)$ be two discrete functions defined on the same domain (i.e. let both of them be of size $M \times N$). Further,
  Then, the convolution of f_1 and f_2 is defined as:

$$(f_1 * f_2)(x, y) = \sum_{l=0}^{M-1} \sum_{m=0}^{N-1} f_1(l, m) f_2(x - l, y - m)$$

  As $f_1(x, y)$ and $f_2(x, y)$ are periodic with period M along x and N along y,

$$\begin{aligned}(f_1 * f_2)(x + M, y + N) &= \sum_{l=0}^{M-1} \sum_{m=0}^{N-1} f_1(l, m) f_2(x + M - l, y + N - m)\\ &= \sum_{l=0}^{M-1} \sum_{m=0}^{N-1} f_1(l, m) f_2(x - l, y - m)\\ &= (f_1 * f_2)(x, y)\end{aligned}$$

  Hence, $(f_1 * f_2)(x, y)$ is periodic with the same period. Therefore, while computing the convolution, we zero pad f_1 and f_2 to $2M \times 2N$ but consider only the central $M \times N$ output of the convolution. The remaining values outside $M \times N$ are part of the other period of the convolution.

  The 2D Discrete Fourier Transform F, for one period of the signal f is defined as:

$$DFT(f(x, y)) = F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \exp(-2\pi i(\frac{ux}{M} + \frac{vy}{N}))$$

  and the 2D Inverse Discrete Fourier transform F of the signal f is defined as:

$$IDFT(F(u, v)) = f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} (u, v) \exp(2\pi i(\frac{ux}{M} + \frac{vy}{N}))$$

  Now, let $F_1(u, v)$ and $F_2(u, v)$ be the 2D DFT of the signals f_1 and f_2 respectively. Then, the convolution theorem states that:

$$DFT(f_1 * f_2) = F_1 \cdot F_2$$

  We prove this is true by showing that:

$$IDFT(F_1 \cdot F_2) = f_1 * f_2$$

Consider the LHS, by definition of the 2D IDFT we have that:

$$
\begin{aligned}
IDFT(F_1 \cdot F_2) &= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}(F_1(u,v).F_2(u,v))\exp(2\pi i(\frac{ux}{M}+\frac{vy}{N})) \\
&= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\left[\left(\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} f_1(p,q)\exp(-2\pi i(\frac{up}{M}+\frac{vq}{N}))\right)\right].F_2(u,v)\exp(2\pi i(\frac{ux}{M}+\frac{vy}{N})) \\
&\qquad \ldots\text{(by definition of 2D DFT of } f_1)
\end{aligned}
$$

We can write the above equation in a concise form as:

$$
IDFT(F_1 \cdot F_2) = \frac{1}{MN}\sum_{u,v}\left(\sum_{p,q} f_1(p,q)\exp(-2\pi i(\frac{up}{M}+\frac{vq}{N})\right).F_2(u,v)\exp(2\pi i(\frac{ux}{M}+\frac{vy}{N}))
$$

where the limits of the summation can be inferred from the earlier equation. Now, we further simply the equation as follows:

$$
\begin{aligned}
IDFT(F_1 \cdot F_2) &= \frac{1}{MN}\sum_{u,v}\left(\sum_{p,q} f_1(p,q)\exp(-2\pi i(\frac{up}{M}+\frac{vq}{N})\right).F_2(u,v)\exp(2\pi i(\frac{ux}{M}+\frac{vy}{N})) \\
&= \frac{1}{MN}\sum_{p,q}\sum_{u,v} f_1(p,q)\exp(-2\pi i(\frac{up}{M}+\frac{vq}{N}).F_2(u,v)\exp(2\pi i(\frac{ux}{M}+\frac{vy}{N})) \\
&\qquad \ldots\text{(Interchanging order of the two summations)} \\
&= \frac{1}{MN}\sum_{p,q} f_1(p,q)\sum_{u,v}\exp(-2\pi i(\frac{up}{M}+\frac{vq}{N}).F_2(u,v)\exp(2\pi i(\frac{ux}{M}+\frac{vy}{N})) \\
&= \sum_{p,q} f_1(p,q)\left(\frac{1}{MN}\sum_{u,v}F_2(u,v)\exp(2\pi i(\frac{u(x-p)}{M}+\frac{v(y-q)}{N}))\right) \\
&= \sum_{p,q} f_1(p,q)f_2(x-p,y-q) \\
&\qquad \ldots\text{(By Fourier Shift theorem)} \\
&= f_1 * f_2
\end{aligned}
$$

Hence, we have proved the required theorem.