Assignment 1: CS 663, Fall 2021

Question 1

August 15, 2021

- A function $f(z)$ is said to be linear if $f(az_1 + z_2) = af(z_1) + f(z_2)$, where z_1, z_2 are two scalar values in the domain of f and a is a scalar constant. This definition extends to the case where $\boldsymbol{z}$ is a vector, i.e. $f(\boldsymbol{z})$ is linear if $f(a\boldsymbol{z_1} + \boldsymbol{z_2}) = af(\boldsymbol{z_1}) + f(\boldsymbol{z_2})$. Now, we have seen the formula for bilinear interpolation which expresses image intensities as a bilinear function of x, y (spatial coordinates) in the form $v(x, y) = ax + by + cxy + d$ where a, b, c, d are scalar constants independent of x, y. Is $v(x, y)$ a linear function of x keeping y constant and vice-versa? Prove or disprove. Is it a linear function of $\boldsymbol{z} \triangleq (x, y)$? Prove or disprove. [15 points]

  Answer: The formula for bilinear interpolation is given as follows

$$v(x, y) = ax + by + cxy + d \tag{1}$$

  - First we check if the bilinear transformation is linear in x when y is kept constant.
    From (1) we can easily see that the bilinear transformation of $\alpha\boldsymbol{z_1} + \boldsymbol{z_2}$ (where $\boldsymbol{z_i} \triangleq (x_i, y_i)$) is given as,

$$\begin{aligned} v(\alpha\boldsymbol{z_1} + \boldsymbol{z_2}) &= v(\alpha x_1 + x_2, \alpha y + y) \qquad \text{...(Since } y \text{ is kept constant)} \\ &= a(\alpha x_1 + x_2) + b(\alpha y + y) + c(\alpha x_1 + x_2)(\alpha y + y) + d \\ &= \alpha a x_1 + a x_2 + \alpha b y + by + \alpha^2 c x_1 y + \alpha c x_1 y + \alpha c x_2 y + c x_2 y + d \end{aligned} \tag{2}$$

    Also,

$$\begin{aligned} v(\boldsymbol{z_1}) &= v(x_1, y) \\ &= ax_1 + by + cx_1 y + d \end{aligned}$$

$$\begin{aligned} v(\boldsymbol{z_2}) &= v(x_2, y) \\ &= ax_2 + by + cx_2 y + d \end{aligned}$$

    Hence,

$$\begin{aligned} \alpha v(\boldsymbol{z_1}) + v(\boldsymbol{z_2}) &= \alpha v(x_1, y) + v(x_2, y) \\ &= \alpha(ax_1 + by + cx_1 y + d) + ax_2 + by + cx_2 y + d \\ &= \alpha a x_1 + \alpha b y + \alpha c x_1 y + \alpha d + a x_2 + by + c x_2 y + d \end{aligned} \tag{3}$$

    From (2) and (3) we can easily see that,

$$\begin{aligned} v(\alpha\boldsymbol{z_1} + \boldsymbol{z_2}) - \alpha v(\boldsymbol{z_1}) + v(\boldsymbol{z_2}) &= \alpha^2 c x_1 y + \alpha c x_2 y - \alpha d \\ &\neq 0 \end{aligned}$$

    Hence, the bilinear transformation $v(.)$ is not linear in x when y is kept constant.

- From symmetry of the equation of bilinear transformation in variables x and y we can easily say that the bilinear transformation is not linear in y when x is kept constant.
- Now, let us check if the bilinear transformation is linear in $\boldsymbol{z}$

$$\begin{aligned} v(\alpha \boldsymbol{z_1} + \boldsymbol{z_2}) &= v(\alpha x_1 + x_2, \alpha y_1 + y_2) \\ &= a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha x_1 + x_2)(\alpha y_1 + y_2) + d \\ &= \alpha a x_1 + a x_2 + \alpha b y_1 + b y_2 + \alpha^2 c x_1 y_1 + \alpha c x_1 y_2 + \alpha c x_2 y_1 + c x_2 y_2 + d \end{aligned} \tag{4}$$

And,

$$\begin{aligned} \alpha v(\boldsymbol{z_1}) + v(\boldsymbol{z_2}) &= \alpha v(x_1, y_1) + v(x_2, y_2) \\ &= \alpha(a x_1 + b y_1 + c x_1 y_1 + d) + a x_2 + b y_2 + c x_2 y_2 + d \\ &= \alpha a x_1 + \alpha b y_1 + \alpha c x_1 y_1 + \alpha d + a x_2 + b y_2 + c x_2 y_2 + d \end{aligned} \tag{5}$$

Using (4) and (5),

$$\begin{aligned} v(\alpha \boldsymbol{z_1} + \boldsymbol{z_2}) - \alpha v(\boldsymbol{z_1}) + v(\boldsymbol{z_2}) &= \alpha^2 c x_1 y_1 + \alpha c x_1 y_2 + \alpha c x_2 y_1 - \alpha c x_1 y_1 - \alpha d \\ &\neq 0 \end{aligned}$$

Hence, by the definition of linearity, bilinear transformation is not linear in $\boldsymbol{z}$.