Assignment 2: CS 663, Fall 2021

Question 7

- Prove that the Laplacian operator is rotationally invariant. For this consider a rotation of the coordinate system from (x, y) to $u = x\cos\theta - y\sin\theta, v = x\sin\theta + y\cos\theta$, and show that $f_{xx} + f_{yy} = f_{uu} + f_{vv}$ for any image f. [10 points]

Proof:

$$u = x\cos\theta - y\sin\theta$$
$$v = x\sin\theta + y\cos\theta$$

$$\implies \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(-\sin)\theta - \frac{\partial f}{\partial v}\cos\theta$$

$$\implies \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}(-\sin\theta) + \frac{\partial f}{\partial v}\cos\theta\right) = \frac{\partial}{\partial y}\frac{\partial f}{\partial u}(-\sin\theta) + \frac{\partial}{\partial y}\frac{\partial f}{\partial v}\cos\theta \tag{1}$$

Now, to compute $\frac{\partial}{\partial y}\frac{\partial f}{\partial u}$ and $\frac{\partial}{\partial y}\frac{\partial f}{\partial v}$. For this we make the following assumptions:

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial u} = \frac{\partial}{\partial u}\frac{\partial f}{\partial y}$$
$$\frac{\partial}{\partial y}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\frac{\partial f}{\partial y}$$

We can compute $\frac{\partial}{\partial u}\frac{\partial f}{\partial y}$ and $\frac{\partial}{\partial v}\frac{\partial f}{\partial y}$ as follows

$$\frac{\partial}{\partial u}\frac{\partial f}{\partial y} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}(-\sin\theta) + \frac{\partial f}{\partial v}\cos\theta\right) = \frac{\partial^2 f}{\partial u^2}(-\sin\theta) + \frac{\partial f}{\partial u\partial v}\cos\theta$$
$$\frac{\partial}{\partial v}\frac{\partial f}{\partial y} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}(-\sin\theta) + \frac{\partial f}{\partial v}\cos\theta\right) = \frac{\partial^2 f}{\partial v\partial u}(-\sin\theta) + \frac{\partial^2 f}{\partial v^2}\cos\theta$$

Now, we plug back this into equation (1) to get,

$$\frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial u^2}(-\sin\theta) + \frac{\partial f}{\partial u\partial v}\cos\theta\right)(-\sin\theta) + \left(\frac{\partial^2 f}{\partial v^2}\cos\theta + \frac{\partial f}{\partial v\partial u}(-\sin\theta)\right)\cos\theta$$

$$\implies \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}(\sin^2\theta) + \frac{\partial f}{\partial u\partial v}(-\cos\theta\sin\theta) + \frac{\partial^2 f}{\partial v^2}\cos^2\theta + \frac{\partial f}{\partial v\partial u}(-\sin\theta\cos\theta) \tag{2}$$

Similarly we can find $\frac{\partial^2 f}{\partial x^2}$ to be as follows:

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial u^2}\cos\theta + \frac{\partial f}{\partial u \partial v}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 f}{\partial v^2}\sin\theta + \frac{\partial f}{\partial v \partial u}\cos\theta\right)\sin\theta$$

$$\implies \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}(\cos^2\theta) + \frac{\partial f}{\partial u \partial v}(\sin\theta\cos\theta) + \frac{\partial^2 f}{\partial v^2}\sin^2\theta + \frac{\partial f}{\partial v \partial u}(\cos\theta\sin\theta) \tag{3}$$

Adding equation (2) to equation (3), we get:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2 f}{\partial u^2}\sin^2\theta + \frac{\partial^2 f}{\partial u^2}\cos^2\right) + \left(\frac{\partial^2 f}{\partial v^2}\cos^2\theta + \frac{\partial^2 f}{\partial v^2}\sin^2\right)$$

$$\implies \mathbf{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}}$$

$$\implies \mathbf{f_{xx} + f_{yy} = f_{uu} + f_{vv}}$$

Hence Proved