Assignment 2: CS 663, Fall 2021

Question 4

- CProve or disprove: (a) The Laplacian mask with a -4 in the center (see class slides) is a separable filter. (b) The Laplacian mask with a -4 in the center (see class slides) can be implemented entirely using 1D convolutions. [5+5=10 points]

Answer:

(a) By definition, a filter which can be represented in this outer-product form are called separable filters. The Laplacian mask with -4 in the center is given by:

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

For this to be a separable matrix, $\exists$ vectors $u = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ such that,

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \\ &= \begin{bmatrix} v_1u_1 & v_1u_2 & v_1u_3 \\ v_2u_1 & v_2u_2 & v_2u_3 \\ v_3u_1 & v_3u_2 & v_3u_3 \end{bmatrix} \end{aligned} \tag{1}$$

We now observe the following:

(a) $v_1u_1 = 0 \implies$ one of $u_1, v_1 = 0$, but $v_1u_2 = 1$. Therefore, $v_1 \neq 0$ and $u_1 = 0$.

(b) $v_2u_1 = 1 \implies u_1 \neq 0$.

Therefore, we have **contradiction**. Hence, The Laplacian mask with a -4 in the center is **not** a separable filter.

(b) The Laplacian mask with -4 in the center is given by:

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Let the neighbouring pixels of the image I along x axis be $I_x = \begin{bmatrix} I_{x-1,y} & I_{x,y} & I_{x+1,y} \end{bmatrix}$ and along the y axis be $I_y = \begin{bmatrix} I_{x,y-1} \\ I_{x,y} \\ I_{x,y+1} \end{bmatrix}$ We know from 2D convolutions:

$$(L * I)|_{x,y} = I_{x,y-1} + I_{x,y+1} - 4I_{x,y} + I_{x-1,y} + I_{x+1,y} \tag{2}$$

We see that, from 1D convolutions:

$$\begin{bmatrix}1 & -2 & 1\end{bmatrix} * I_x = I_{x-1,y} - 2I_{x,y} + I_{x+1,y} \tag{3}$$

$$\begin{bmatrix}1 & -2 & 1\end{bmatrix} * I_y^T = I_{x,y-1} - 2I_{x,y} + I_{x,y+1} \tag{4}$$

Adding the above two equations:

$$\begin{aligned}\begin{bmatrix}1 & -2 & 1\end{bmatrix} * (I_x + I_y^T) &= I_{x,y-1} + I_{x,y+1} - 4I_{x,y} + I_{x-1,y} + I_{x+1,y} \\ &= \begin{bmatrix}1 & 1 & -4 & 1 & 1\end{bmatrix} * \begin{bmatrix}I_{x,y-1} & I_{x,y+1} & I_{x,y} & I_{x-1,y} & I_{x+1,y}\end{bmatrix} \\ &= L * I|_{x,y}\end{aligned} \tag{5}$$

Therefore, each $I_{x,y}$ of $L * I$ can be found using 1D convolution. Hence, the Laplacian mask with a -4 in the center **can** be implemented entirely using 1D convolutions.