Assignment 3: CS 663, Fall 2021

Question 6

- If $\mathcal{F}$ is the continuous Fourier operator, prove that $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)$. Hint: Prove that $\mathcal{F}(\mathcal{F}(f(t))) = f(-t)$ and proceed further from there. [15 points]

  Answer: To Prove That: $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)$.

  We know that,

$$\mathcal{F}(f(t)) = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

  Then,

$$\begin{aligned}\mathcal{F}(\mathcal{F}(f(\nu)) = \mathcal{F}(F(t)) &= \int_{-\infty}^{\infty} F(t) e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\nu) e^{-j2\pi t\nu} d\nu \right] e^{-j2\pi\mu t} dt\end{aligned}$$

  Exchanging the integrals using Fubini's Theorem,

$$\begin{aligned}\mathcal{F}(\mathcal{F}(f(\nu)) = \mathcal{F}(F(t)) &= \int_{-\infty}^{\infty} f(\nu) \left[\int_{-\infty}^{\infty} e^{-j2\pi t\nu} e^{-j2\pi\mu t} dt \right] d\nu \\ &= \int_{-\infty}^{\infty} f(\nu) \left[\int_{-\infty}^{\infty} e^{-j2\pi(\nu+\mu)t} dt \right] d\nu \\ &= \int_{-\infty}^{\infty} f(\nu)\delta(\nu + \mu) d\nu \\ &= f(-\mu)\end{aligned}$$

  Therefore, $\mathcal{F}(\mathcal{F}(f(t))) = f(-t)$ *(Replacing μ an ν with t is valid as they are independent variables).*
  Now, $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = \mathcal{F}(\mathcal{F}(f(-t))) = f(-(-t)) = f(t)$.

  Therefore, $\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)$.

  Hence Proved.