HW-4: Theory questions

① $A: m \times n$, real $\quad AA^T: m \times m$, real, symmetric, psd

$A^TA: n \times n$, real, symmetric, psd

$v \to m \times 1$

$v^T A A^T v = (A^T v)^T (A^T v) \geq 0$

$v^T A^T A v = (Av)^T (Av) \geq 0$

$\Rightarrow$ All eigen values of AA^T & A^TA are real and positive (showing they are real needs more work)

$AA^T v = \lambda v \Rightarrow A^TA(A^Tv) = \lambda (A^T v)$
$A^TA v = \lambda v \Rightarrow AA^T(Av) = \lambda (Av)$
} eigen values of AA^T & A^TA are same.

$\Rightarrow$ By spectral theorem, AA^T & A^TA are diagonalizable

so, $AA^T = U D_m U^T$ with $UU^T = I_m$
$A^TA = V D_n V^T$ with $VV^T = I_n$

as they have the same eigen values
distinct eigen values $\leq \min(m,n)$

Now, choose D_m or D_n accordingly which of m, n is minimum
the diagonal elements give us the eigen values (of A^TA or AA^T).

$\Rightarrow$ take the positive square root of them & Construct $S: m \times n$ matrix by inserting the positive square roots in the principal diagonal of S.

Important step: U can arrange the square root of eigen values in any order but column vectors in U, V should be of corresponding eigen vectors.

* There would be a confusion if $m \neq n$, but the additional eigen vectors in U or V correspond to '0' eigen value. (those extras can be in any order)

Then, $A = USV^T$

The student got $A \neq USV^T$ because he/she forgot to consider the order of eigen vectors (column vectors) in U & V.

The order of singular values in S determine the order of column vectors in U & V.

Hence, to rectify this error, we have re-arrange the column vectors in U & V to match the singular values in S.

Not Correct

(2) $f^t C f, \quad f^t f = 1, \; f^t e = 0$

$$\nabla\left[f^t C f - \lambda_1 (f^t f - 1) - \lambda_2 (f^t e)\right] = 0$$

$$\nabla(f^t e) = \begin{bmatrix} \frac{\partial}{\partial f_1} \sum f_i e_i \\ \frac{\partial}{\partial f_2} \sum f_i e_i \\ \vdots \\ \frac{\partial}{\partial f_n} \sum f_i e_i \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = e$$

$$\Rightarrow 2Cf - 2\lambda_1 f - \lambda_2 e = 0$$

$$\Rightarrow \quad Cf = \lambda_1 f + \frac{\lambda_2 e}{2}$$

$$\Rightarrow (Cf)^T e = \lambda_1 f^t e + \frac{\lambda_2}{2} e^t e$$

$\Rightarrow f^T C e = \frac{\lambda_2}{2}$ $\quad$ [$e^t e = 1$, C is symmetric]

$\Rightarrow f^T \lambda^* e = \frac{\lambda_2}{2}$ $\quad$ [λ^* is largest eigen value]

$\Rightarrow \lambda_2 = 0 \quad \Rightarrow \quad Cf = \lambda_1 f \Rightarrow f$ is an eigen vector of C

Now, $f^t C f = \lambda_1$,

Can't choose $\lambda_1 = \lambda^*$ as space of $(C - \lambda^* I)v = 0$ has unit dimension.

(because non-zero eigen values of C are distinct $\Rightarrow$ algebraic multiplicity of $\lambda^* = 1$ & geometric multiplicity $\leq$ algebraic multiplicity)

Hence $\lambda_1 = \lambda^{**}$ (second highest eigen value)

(3) $g_1 = f_1 + h_2 * f_2$

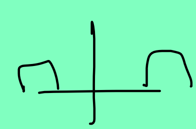
$g_2 = f_2 + h_1 * f_1$

$\Rightarrow \left.\begin{array}{l} G_1 = F_1 + H_2 \cdot F_2 \\ G_2 = F_2 + H_1 F_1 \end{array}\right\} \Rightarrow \quad G_1 - H_2 \cdot G_2 = F_1 (1 - H_1 H_2)$

$$\Rightarrow F_1 = \frac{G_1 - H_2 G_2}{1 - H_1 H_2}$$

Similarly $F_2 = \dfrac{G_2 - H_1 G_1}{1 - H_1 H_2}$. Hence $f_1 = \mathcal{F}^{-1}\left\{\dfrac{G_1 - H_2 G_2}{1 - H_1 H_2}\right\}$

$$f_2 = \mathcal{F}^{-1}\left\{\frac{G_2 - H_1 G_1}{1 - H_1 H_2}\right\}$$



In the above formulas,

F_1, F_2 are undefined when $H_1 H_2 = 1$ which could give incorrect results for f_1 & f_2.