Assignment 3: CS 663, Fall 2021

Question 5

- If a function $f(x, y)$ is real, prove that its Discrete Fourier transform $F(u, v)$ satisfies $F^*(u, v) = F(-u, -v)$. If $f(x, y)$ is real and even, prove that $F(u, v)$ is also real and even. The function $f(x, y)$ is an even function if $f(x, y) = f(-x, -y)$. [15 points]

Answer:

a) **Given:**

1) $f(x, y)$ is a real function.
2) $F(u, v)$ is the Discrete Fourier Trensform (DTF) of $f(x, y)$.

To Prove That: $F^*(u, v) = F(-u, -v)$.

From the defenition of DFT, we know that,

$$F(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{-j2\pi xu/W_1} e^{-j2\pi yv/W_2}$$
$$\implies F^*(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f^*(x,y) e^{j2\pi xu/W_1} e^{j2\pi yv/W_2}$$

As $f(x, y)$ is real valued, $f^*(x, y) = f(x, y)$. Therefore,

$$F^*(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{j2\pi xu/W_1} e^{j2\pi yv/W_2} \tag{1}$$

Replacing u with $-\mu$ and v with $-\nu$, we get,

$$F^*(-\mu,-\nu) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{-j2\pi x\mu/W_1} e^{-j2\pi y\nu/W_2}$$
$$= F(\mu, \nu)$$

Changing the variables back to u and v, we get,

$$F^*(u, v) = F(-u, -v)$$

b) **Given:**

c) Additionally, $f(x, y) = f(-x, -y)$, i.e., $f(x, y)$ is even.

To Prove That: $F(u,v)$ is also real and even.

From equation 1, we know that,

$$F^*(u,v) = \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{j2\pi xu/W_1} e^{j2\pi yv/W_2}$$

Replacing x with $-x$ and y with $-y$, we get,

$$\begin{aligned} F^*(u,v) &= \sum_{x=1-W_1}^{0} \sum_{y=1-W_2}^{0} f(-x,-y) e^{-j2\pi xu/W_1} e^{-j2\pi yv/W_2} \\ &= \sum_{x=1-W_1}^{0} \sum_{y=1-W_2}^{0} f(x,y) e^{-j2\pi xu/W_1} e^{-j2\pi yv/W_2} \end{aligned}$$

We now use the periodicity of $e^{-j2\pi xu/W_1}$, it's corresponding in y and $f(x,y)$. That is, $e^{-j2\pi(x-W_1)u/W_1} = e^{-j2\pi xu/W_1}$ and $f(x-W_1, y-W_2) = f(x,y)$. Also, $f(0,0) = f(W_1, W_2)$. Thus replacing x by $x - W_1$ and y by $y - W_2$

$$\begin{aligned} F^*(u,v) &= \sum_{x=1}^{W_1} \sum_{y=1}^{W_2} f(x,y) e^{-j2\pi xu/W_1} e^{-j2\pi yv/W_2} \\ &= \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) e^{-j2\pi xu/W_1} e^{-j2\pi yv/W_2} \\ &= F(u,v) \end{aligned}$$

Therefore, $F(u,v)$ is real valued. As $F^*(u,v) = F(u,v)$, and from a), $F^*(u,v) = F(-u,-v)$, we get $F(u,v) = F(-u,-v)$. Therefore, $F(u,v)$ is also even.

Hence Proved.