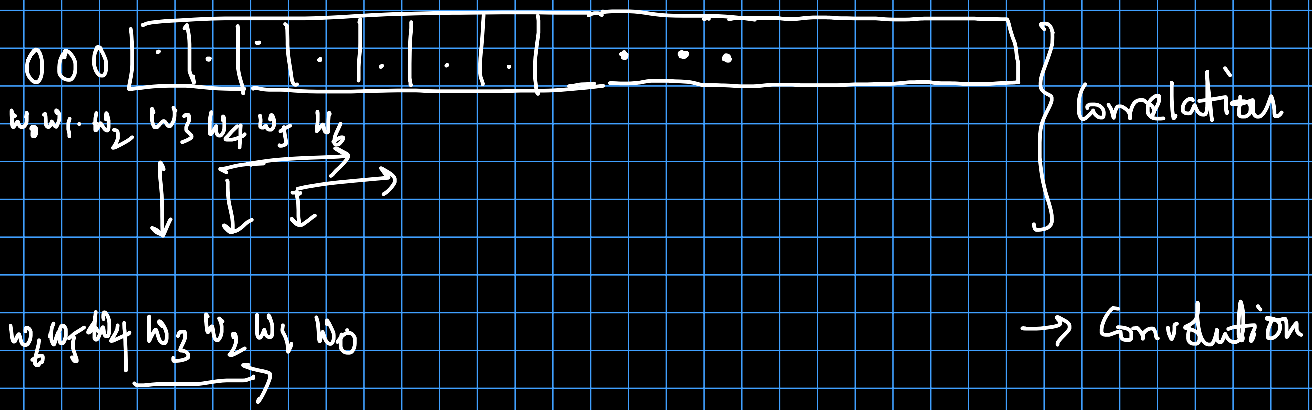
hw2 - theory questions

①



f : $1 \times N$

A : $N \times N$

fA : $1 \times N$ ⓕ

∴

$\boxed{g = fA}$

output 1D image

$$A = \begin{bmatrix} w_3 & w_4 & w_5 & w_6 & 0 & \cdots & 0 \\ w_2 & w_3 & w_4 & w_5 & w_6 & \cdots & 0 \\ w_1 & w_2 & w_3 & w_4 & w_5 & \cdots & 0 \\ w_0 & w_1 & w_2 & w_3 & w_4 & \cdots & \vdots \\ 0 & w_0 & w_1 & w_2 & w_3 & \cdots & \vdots \\ 0 & 0 & w_0 & w_1 & w_2 & \cdots & \vdots \\ 0 & 0 & 0 & w_0 & w_1 & \cdots & \vdots \\ 0 & 0 & 0 & 0 & w_0 & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & w_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & w_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & w_4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & w_3 \end{bmatrix}_{N \times N}$$

← properties?

← Applications?

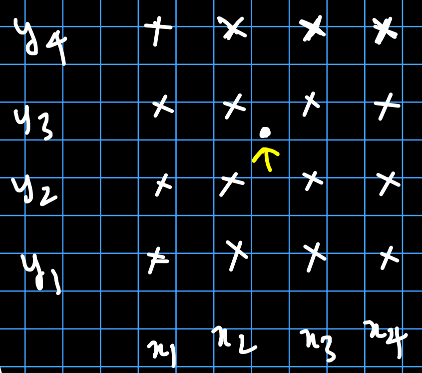
Q: What are the properties & Applications of the matrix A?

② $v(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij}\, x^{i} y^{j}$

$$\overset{\text{coefficient vector}}{c}\qquad A c = \underset{\text{nearest values}}{n}$$

16 coefficients (4×4)

so 16 eqⁿs required



$$\underset{16\times16}{\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 & x_1y_1 & x_1^2y_1 & x_1^3y_1 & y_1^2 & \cdots & x_1^3y_1^3 \\ 1 & x_1 & x_1^2 & x_1^3 & y_2 & x_1y_2 & x_1^2y_2 & x_1^3y_2 & y_2^2 & \cdots & x_1^3y_2^3 \\ '' & '' & '' & '' & y_3 & x_1y_3 & x_1^2y_3 & x_1^3y_3 & y_3^2 & \cdots & x_1^3y_3^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 & x_1y_4 & x_1^2y_4 & x_1^3y_4 & y_4^2 & \cdots & x_4^3y_4^3 \end{bmatrix}} \underset{16\times1}{\begin{bmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{11} \\ \vdots \\ a_{23} \\ a_{33} \end{bmatrix}} = \underset{1\times16}{\begin{bmatrix} v(x_1,y_1) \\ v(x_1,y_2) \\ v(x_1,y_3) \\ \vdots \\ v(x_4,y_3) \\ v(x_4,y_4) \end{bmatrix}}$$

$\Rightarrow Ac = n \Rightarrow c = A^{-1} \cdot n$

if A^{-1} doesnot exist, we need more than 16 values but...

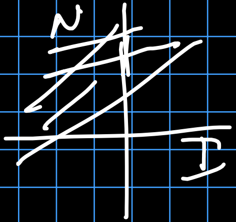
Q: Does A^{-1} always exist?

* * *

③ $R(x,y) = I(x,y) + N(x,y)$ $\qquad$ $N(x,y) \sim \mathcal{N}(0,\sigma)$

$\{N(x,y) \mid x,y \in \Omega\}$ are independent

$p_I(i)$, $p_N(n)$ $\searrow$ $\swarrow$ PDFs $\qquad$ $I, N \rightarrow$ independent



$\Rightarrow p_{IN}(k) = p_I(i)\, p_N(n)$

$$\mathbb{P}(I+N \le k) = \underset{\substack{\downarrow \\ n}}{\int_{-\infty}^{\infty}} \underset{\substack{\downarrow \\ i}}{\int_{-\infty}^{k-n}} \underset{\sim}{p_I(i)}\, p_N(n)\, di\, dn$$

$\Rightarrow p_{I+N}(k) = \frac{\partial}{\partial k} \mathbb{P}(I+N \le k)$

$$\Rightarrow p_{I+N}(k) = \int_{-\infty}^{\infty} p_I(k-n)\, p_N(n)\, dn = p_I(k) * p_N(k)$$

④

(a) $L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix}$

L is not a separable filter.

$a \cdot x = 0 , \quad a \cdot y = 1 , \quad b \cdot x = 1$

$a \cdot y = 1 \Downarrow a \neq 0$

$x = 0 \leftarrow (a \cdot x = 0,\ a \neq 0)$

$x = 0 \rightarrow b \cdot x = 1$ Contradiction!

(b) $\therefore (x+y) * h = (x * b) + (y * h)$ ✓

$$\sum_{t=-\infty}^{\infty} (x+y)(t)\, h(z-t) = \sum_{t=-\infty}^{\infty} x(t)\, h(z-t) + y(t)\, h(z-t)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

Laplacian mask can be implemented using only 1D convolutions.

⑤ $f_1 = f * M,\ f_2 = f_1 * M \Rightarrow f_2 = f*(M*M)$ → Associativity of Convolution

$\Rightarrow f_K = f*(\underbrace{M*M* \cdots *M}_{k \text{ times}})$

$$M = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 144 & 204 & 144 \\ 204 & 289 & 204 \\ 144 & 204 & 144 \end{bmatrix}$$

1^2 1^2

1 2^2 3^2

2 3 5^2

5 7^2

7

10 12^2 17^2

17

17 29^2

.24

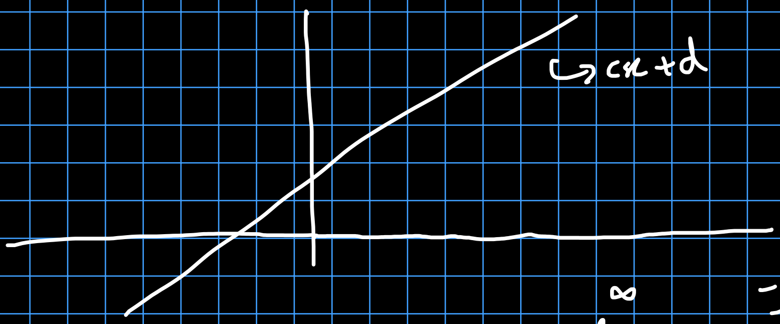
$$M*M = \frac{1}{81}\begin{bmatrix} 4 & 6 & 4 \\ 6 & 9 & 6 \\ 4 & 6 & 4 \end{bmatrix}$$

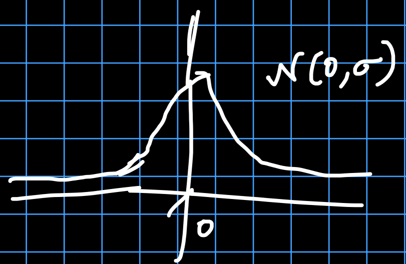
$M^5 =$

$$M*M*M = \frac{1}{3^8}\begin{bmatrix} 25 & 35 & 25 \\ 35 & 49 & 35 \\ 25 & 35 & 25 \end{bmatrix}$$

---

⑥

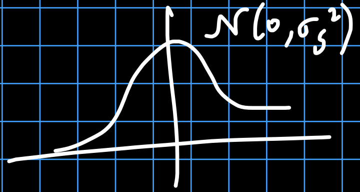


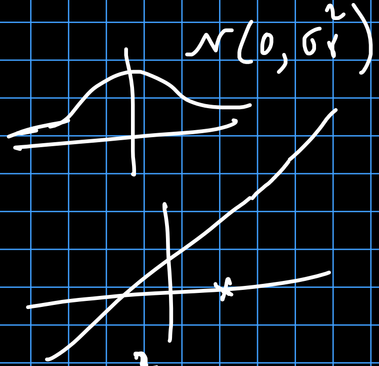


$\Rightarrow g_1(x) = I(x) * f_G(x) \Rightarrow \displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-t^2}{2\sigma}} \cdot (cx - ct + d)\, dt$ odd fn

$= \displaystyle\int_{-\infty}^{\infty} f_G(t)\, I(x-t)\, dt \Rightarrow (cx+d)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma} dt - c\underbrace{\int_{-\infty}^{\infty} \overbrace{f_G(t)\cdot t}^{e}\, dt}_{0}$

=) $g_1(x) = cx+d$





2) $$g_2(x) = \int_{-\infty}^{\infty} I(t)\, f_{G_s}(x-t)\; f_{G_r}(I(x)-I(t))\, dt$$

$$= \int_{-\infty}^{\infty} (ct+d) \left(\frac{1}{\sqrt{2\pi\sigma_s^2}}\, e^{-\frac{(x-t)^2}{2\sigma_s}}\right) \left(\frac{1}{\sqrt{2\pi\sigma_r^2}}\, e^{-\frac{c^2(x-t)^2}{2\sigma_r}}\right) dt$$

$$= \int_{-\infty}^{\infty} I(t)\, \frac{1}{2\pi.\sigma_s\sigma_r}\, e^{-\frac{(x-t)^2}{2}\left[\frac{1}{\sigma_s}+\frac{c^2}{\sigma_r}\right]} dt$$

$$= \frac{1}{2\pi\sigma_s\cancel{\sigma_r}} \times \sqrt{2\pi} \times \frac{\cancel{\sigma_s\sigma_r} \times I(x)}{\sigma_r+\sigma_s c^2} = \frac{I(x)}{\sqrt{2\pi}\,(\sigma_r+c^2\sigma_s)}$$

Didn't get back original image for bilateral filter.

---

⑦