Question 3

- Compressive sensing reconstructions involve estimating a sparse signal $\mathbf{x} \in \mathbb{R}^n, n \gg 2$ from a vector $\mathbf{y} \in \mathbb{R}^m (m \ll n)$ of compressed measurements of the form $\mathbf{y} = \mathbf{\Phi x}$ where $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$ is the measurement matrix (assume there is no noise). Now answer the following questions, from first principles. **Do not merely quote theorems or algorithms.**

  1. If it is known that $\mathbf{x}$ has only 1 non-zero element and that the other elements are zero, can you uniquely estimate $\mathbf{x}$ if $m = 1$? If yes, how? If not, why not? Now further suppose, you knew beforehand the index (but not the value) of the non-zero element of $\mathbf{x}$? Does this help you any further? If yes, how? If not, why not?
  2. If it is known that $\mathbf{x}$ has only 1 non-zero element and that the other elements are zero, can you uniquely estimate $\mathbf{x}$ if $m = 2$? If yes, how? If not, why not?
  3. If it is known that $\mathbf{x}$ has only 2 non-zero elements and that the other elements are zero, can you uniquely estimate $\mathbf{x}$ if $m = 3$? If yes, describe an algorithm that is guaranteed to estimate it accurately. If not, explain why not, and explain whether there are any special instances of $\mathbf{\Phi}$ for which unique estimation is possible?
  4. Repeat part (c) with $m = 4$. [1+2+3+4=10 points]

Answer:

a)

$\mathbf{y} = \Phi\mathbf{x}$, $y \in R^m$, $x \in R^n$, $\Phi \in R^{mxn}$

If m=1, we can calculate $\frac{y}{\Phi_j}$ for each $1 \le j \le n$. But since we do not know at which index the non zero element of x is, we cannot find the unique solution of x but we will have a set of finite possible solutions.

If the index (i) of the non zero element is also known, we can uniquely find x if $\Phi_i \neq 0$. If $\Phi_i = 0$, we cannot.

b)

Let the index of the non zero element of x be j.

$$y_1 = \phi_{1j} x_j$$

$$y_2 = \phi_{2j} x_j$$

$$\frac{y_1}{y_2} = \frac{\phi_{1j}}{\phi_{2j}} \tag{1}$$

We can find the column j in Φ for which the above equation 1 is satisfied. Then we can easily calculate

$$x_j = \frac{y_1}{\phi_{1j}}$$

However, there can be multiple columns in Φ which can satisfy equation-1. So for unique solution, no two columns of Φ should have elements in same ratio.

c)

Let i and j be the indices of the 2 non-zero elements of x.

$$y_1 = \phi_{1i}x_i + \phi_{1j}x_j$$

$$y_2 = \phi_{2i}x_i + \phi_{2j}x_j$$

$$y_3 = \phi_{3i}x_i + \phi_{3j}x_j$$

To uniquely find x, it is necessary that any 4 columns of Φ are linearly independent. But since we only have 3 rows, any 4 columns will be linearly dependent. This is because any 4 vectors in 3D space are always linearly dependent (which can be proved by the argument that- if they were linearly independent then we can take any 3 which will also be linearly independent, and use as basis to express the remaining vector as a linear combination of those 3 and hence the contradiction) Thus it is not possible to uniquely find x for any such Φ

d)

If m=4 and x has 2 non zero elements, we can uniquely find x if any 4 columns of Φ are linearly independent.

$$y_1 = \phi_{1i}x_i + \phi_{1j}x_j \tag{2}$$

$$y_2 = \phi_{2i}x_i + \phi_{2j}x_j \tag{3}$$

$$y_3 = \phi_{3i}x_i + \phi_{3j}x_j \tag{4}$$

$$y_4 = \phi_{4i}x_i + \phi_{4j}x_j \tag{5}$$

To find x, we can do the following,-

- Consider the set of all pairs of columns (i,j) of Φ. loop over the set.
- For the measurements say y_1 and y_2, find the solution $(x_i, x_j)^T$ by taking the inverse of the subset 2x2 matrix.(here solving equations 2 and 3)
- Check if the solution satisfies the equations of the remaining two measurements (say equations 4 and 5 here). If it does, we have found the unique solution and so end loop.