Question 4

- Consider compressive measurements of the form $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{v}$ for sensing matrix $\boldsymbol{A}$, signal vector $\boldsymbol{x}$, noise vector $\boldsymbol{v}$ and measurement vector $\boldsymbol{y}$. Consider the problem P1 done in class: Minimize $\|\boldsymbol{x}\|_1$ w.r.t. $\boldsymbol{x}$ such that $\|\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}\|_2 \leq e$. Also consider the problem Q1: Minimize $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}\|_2$ w.r.t. $\boldsymbol{x}$ subject to the constraint $\|\boldsymbol{x}\|_1 \leq t$. Prove that if $\boldsymbol{x}$ is a unique minimizer of P1 for some value $e \geq 0$, then there exists some value $t \geq 0$ for which $\boldsymbol{x}$ is also a unique minimizer of Q1. Note that $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_2$ stand for the L1 and L2 norms of the vector $\boldsymbol{x}$ respectively. [15 points] (Hint: Consider $t = \|\boldsymbol{x}\|_1$ and now consider another vector $\boldsymbol{z}$ with L1 norm less than or equal to t).

Answer:

P1: $\underset{x}{minimize}\ \mathbf{x}_1$; s.t. $\mathbf{y}\text{-}\mathbf{A}\mathbf{x}_2 \leq e$

Q1: $\underset{x}{minimize}\ \mathbf{y}\text{-}\mathbf{A}\mathbf{x}_2$; s.t. $\mathbf{x}_1 \leq t$

Now, consider $\mathbf{x}$ to be the unique minimizer for problem P1 for some $e \geq 0$

Let $t = \mathbf{x}_1$

Consider vector $\mathbf{z} \in R^n$, $\mathbf{z} \neq \mathbf{x}$ such that $\mathbf{z}_1 \leq t$

Now since $\mathbf{z}_1 \leq \mathbf{x}_1 = t$ and x is unique minimizer of P1, $\mathbf{z}$ should not satisty the constraint of P1. Thus,

$$\mathbf{y}\text{-}\mathbf{A}\mathbf{z}_2 > e \geq \mathbf{y}\text{-}\mathbf{A}\mathbf{x}_2 \tag{1}$$

Above equation implies that $\mathbf{x}$ is the unique minimizer of Q1 problem.

Hence it is proved that, if $\mathbf{x}$ is a unique minimizer of P1 for some $e \geq 0$ then there exists $t \geq 0$ for which $\mathbf{x}$ is also the unique minimizer of Q1