Question 2

- We will prove why the value of the coherence between $m \times n$ measurement matrix $\boldsymbol{\Phi}$ (with all rows normalized to unit magnitude) and $n \times n$ orthonormal representation matrix $\boldsymbol{\Psi}$ must lie within the range $[1, \sqrt{n}]$ (both 1 and $\sqrt{n}$ inclusive). Recall that the coherence is given by the formula $\mu(\boldsymbol{\Phi}, \boldsymbol{\Psi}) = \sqrt{n} \max_{\substack{i \in \{0,1,\ldots,m-1\} \\ j \in \{0,1,\ldots,n-1\}}} |\boldsymbol{\Phi}^i \boldsymbol{\Psi}_j|$. Proving the upper bound should be very easy for you. To prove the lower bound, proceed as follows. Consider a unit vector $\boldsymbol{g} \in \mathbb{R}^n$. We know that it can be expressed as $\boldsymbol{g} = \sum_{k=1}^{n} \alpha_k^i \boldsymbol{\Psi_k}$ as $\boldsymbol{\Psi}$ is an orthonormal basis. Now prove that $\mu(\boldsymbol{g}, \boldsymbol{\Psi}) = \sqrt{n} \max_{i \in \{0,1,\ldots,n-1\}} \frac{|\alpha_i|}{\sum_{j=1}^{n} \alpha_j^2}$. Exploiting the fact that $\boldsymbol{g}$ is a unit vector, prove that the minimal value of coherence is attained when $\boldsymbol{g} = \sqrt{1/n} \sum_{k=1}^{n} \boldsymbol{\Psi_k}$ and that hence the minimal value of coherence is 1. [10 points]

Answer:
First we prove the upper bound. For this first consider two vector $v_i, v_j \in \mathbb{R}^{n \times 1}$ such that both the vectors have a unit norm (in 2-norm), i.e. $||v_i||_2 = 1$ and $||v_j||_2 = 1$. Now, we know that for any such two vectors a function given by the following equation is a valid inner product.

$$f(v_i, v_j) = v_i^T v_j = \langle v_i, v_j \rangle$$

Hence, using Cauchy-Schwarz inequality, we know that the upper bound on the innner product between any such two vectors is given by,

$$|\langle v_i, v_j \rangle| \leq |\sqrt{\langle v_i, v_i \rangle \langle v_j, v_j \rangle}|$$

Now, since we know that

$$\langle v_i, v_i \rangle = ||v_i||_2^2 = 1$$

and

$$\langle v_j, v_j \rangle = ||v_j||_2^2 = 1$$

we can show that the upper bound on the inner product is

$$|\langle v_i, v_j \rangle| \leq |\sqrt{1}| = 1 \tag{1}$$

Now, we know that the rows of the measurement matrix ($\boldsymbol{\Phi}^i \in \mathbb{R}^{1 \times n}$) are unit normalized i.e

$$||\boldsymbol{\Phi}^i||_2 = 1 \qquad \forall i \in \{0, 1, \ldots, m-1\} \tag{2}$$

Also the matrix $\boldsymbol{\Psi}$ is orthonormal, i.e. its column vectors ($\boldsymbol{\Psi} \in \mathbb{R}^{n \times 1}$ are mutually orthogonal and unit normalized, i.e.

$$||\boldsymbol{\psi}_j||_2 = 1 \qquad \forall j \in \{0, 1, \ldots, n-1\} \tag{3}$$

Since, $\mathbf{\Phi}^i$ is a row vector and $\mathbf{\Psi}_j$ is a column vector, we can apply Cauchy-Schwarz inequality on them considering $\mathbf{\Phi}^i\mathbf{\Psi}_j$ as the inner product. Hence, using Cauchy-Schwarz on the row-vector of the measurement matrix and the columns of the matrix $\mathbf{\Psi}$, we know that the upper bound on the inner product between any two vectors is given by,

$$\begin{aligned}
&|\mathbf{\Phi}^i, \mathbf{\Psi}_j| \leq 1 \qquad \forall i \in \{0, 1, \ldots, m-1\} \qquad \forall j \in \{0, 1, \ldots, n-1\} \\
\implies &\sqrt{n} \max_{\substack{i\in\{0,1,\ldots,m-1\} \\ j\in\{0,1,\ldots,n-1\}}} |\mathbf{\Phi}^i\mathbf{\Psi}_j| \leq \sqrt{n} \quad \ldots \quad \text{(Using (1), (2) and (3))}
\end{aligned}$$

$$\implies \mu(\mathbf{\Phi}, \mathbf{\Psi}) \leq \sqrt{n} \tag{4}$$

Now, let us prove the lower bound of the given inequality.
As $\mathbf{\Psi}$ is an orthonormal basis we can represent the i^{th} row of the measurement matrix as,

$$(\mathbf{\Phi}^i)^T = \sum_{k=1}^{n} \alpha_k^i \mathbf{\Psi}_k \qquad \forall i \in \{0, 1, \ldots, m-1\}$$

The transpose is taken as $\mathbf{\Phi}^i$ is a row vector and $\mathbf{\Psi}_k$ is a column vector. Hence, we can write the above equation as,

$$\mathbf{\Phi}^i = \sum_{k=1}^{n} \alpha_k^i \mathbf{\Psi}_k^T$$

Also, since $\mathbf{\Phi}^i$ is unit normalized, we know that,

$$\sum_{k=1}^{n} (\alpha_k^i)^2 = 1 \tag{5}$$

Hence, we can write the coherence between $\mathbf{\Phi}^i$ and $\mathbf{\Psi}_k$ as,

$$\begin{aligned}
\mu(\mathbf{\Phi}, \mathbf{\Psi}) &= \sqrt{n} \max_{\substack{i\in\{0,1,\ldots,m-1\} \\ j\in\{0,1,\ldots,n-1\}}} \left|\mathbf{\Phi}^i\mathbf{\Psi}_j\right| \\
&= \sqrt{n} \max_{\substack{i\in\{0,1,\ldots,m-1\} \\ j\in\{0,1,\ldots,n-1\}}} \left|(\sum_{k=1}^{n} \alpha_k^i \mathbf{\Psi}_k^T)\mathbf{\Psi}_j\right| \\
&= \sqrt{n} \max_{\substack{i\in\{0,1,\ldots,m-1\} \\ j\in\{0,1,\ldots,n-1\}}} \left|\alpha_j^i\right|
\end{aligned}$$

Now, since we want to find the lower bound on $\mu(\mathbf{\Phi}^i\mathbf{\Psi}_j)$ and from equation (5) we know the constraints that each column of the measurement matrix should follow, we can claim that the coherence will be minimized when all the coefficients α_j^i of the i^{th} row of the measurement matrix are equal.

Hence, the coefficients of that i^{th} row of the measurement matrix are given by,

$$\alpha_j^i = \pm\frac{1}{\sqrt{n}} \qquad \forall j \in \{0, 1, \ldots, n-1\}$$

The minimized value of the coherence in this case can be given by,

$$\mu_{min}(\mathbf{\Phi}, \mathbf{\Psi}) = \sqrt{n} \left|\pm\frac{1}{\sqrt{n}}\right|$$

$$\implies \mu_{min}(\mathbf{\Phi}, \mathbf{\Psi}) = 1 \tag{6}$$

Hence, using equation (5) and (6) we can write,

$$1 \leq \mu(\mathbf{\Phi}, \mathbf{\Psi}) \leq \sqrt{n}$$

Hence proved