EE324: Control Systems Lab
Problem sheet 7

Vinit Awale, 18D070067

March 14, 2021

Contents

1 Problem 1

Given some open loop transfer function we can find the steady state error of the closed loop system as

$$G = \frac{1}{(s+3)(s+4)(s+12)}$$

1.1 Part a

To obtain damping ratio of 0.2 with z=0.01 Now, we need to plot root locus of G*(s+z)/s

So, we can do this in scilab and use the condition for damping ratio as having some specific angle.

If damping ratio, $\zeta = 0.2$ the angle of the line on which the poles must lie will be having angle as- $\tan\theta = \frac{\zeta}{\sqrt{1-\zeta^2}} = 0.16$

Now we can find the intersection of the lines with the root locus.

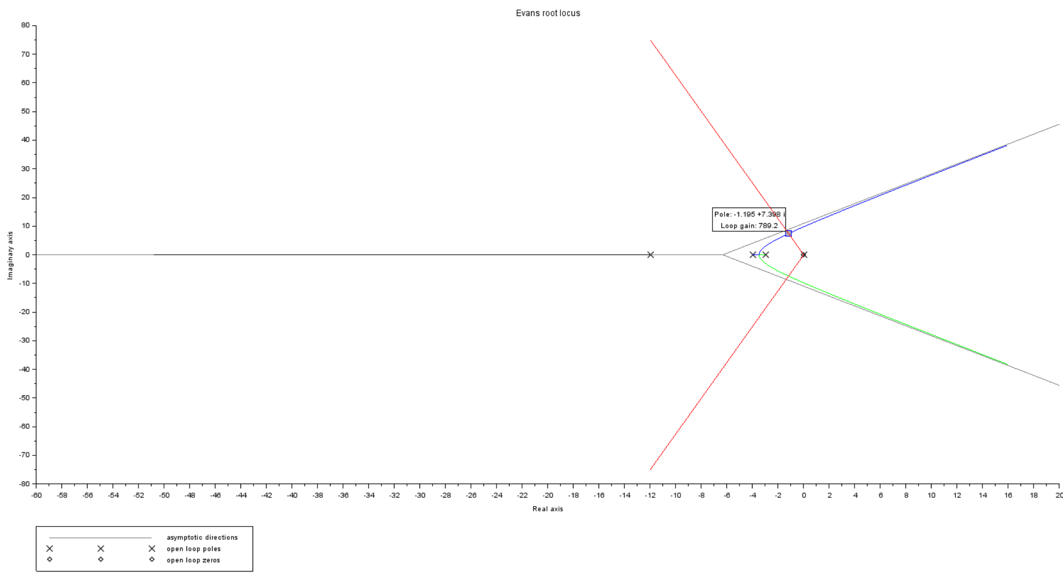
So, from above plot we can say that the intersection point have loop gain=798.2

Thus for K=798.2 this PI controller will be having damping ratio of 0.2

Scilab Code

```
s = poly(0,'s');
G = 1/((s+3)*(s+4)*(s+12));

G = G*(s+0.01)/s
evans(G);

x = -12:0.01:0

y1 = (1/0.16)*x;
y2 = -(1/0.16)*x;
plot(x,y1,'r');
plot(x,y2,'r');
```

Obtained Plot



1.2 Part b

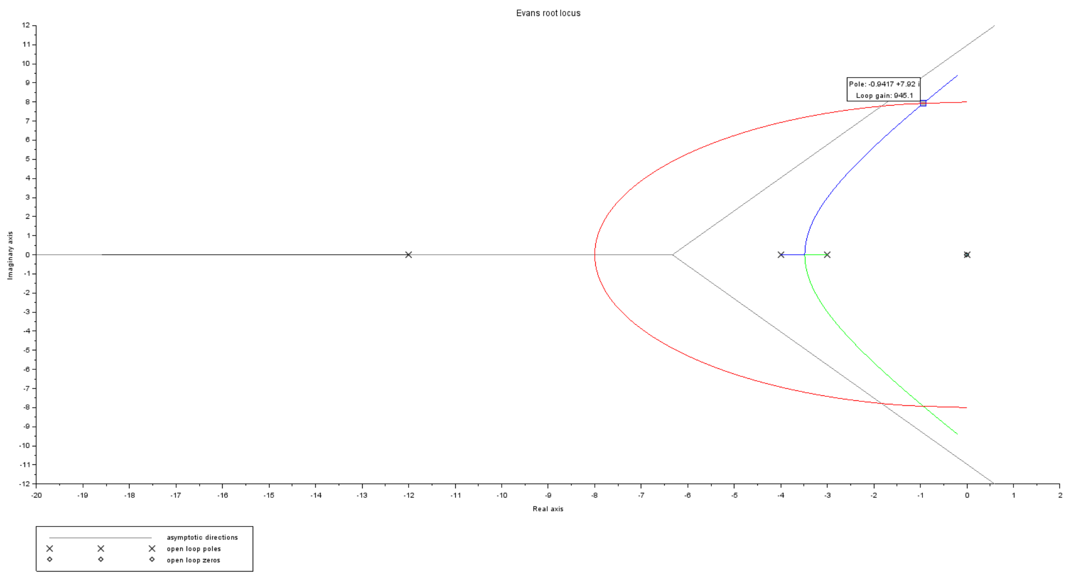
To obtain undamped natural frequency of 8 rad/s

We plot the root locus of $G(s) * \frac{s+0.01}{s}$ and to obtain the gain for undamped natural frequency of 8 rad/s we plot a constant circle of radius 8. The intersection of the two curves gives us the required closed loop pole. We get this pole at the gain of 945.1.

Scilab Code

```
clear
close
clc

s = poly(0,'s');

G = 1/((s+3)*(s+4)*(s+12));
G = G*((s+0.01)/s);
Glin = syslin('c',G);
evans(Glin,1500);

// Plotting the constant omega_n = 8 circle
y = -8:0.1:8;
x = -sqrt(8^2 - y^2);

plot(x,y,'r');
```

Obtained Plot

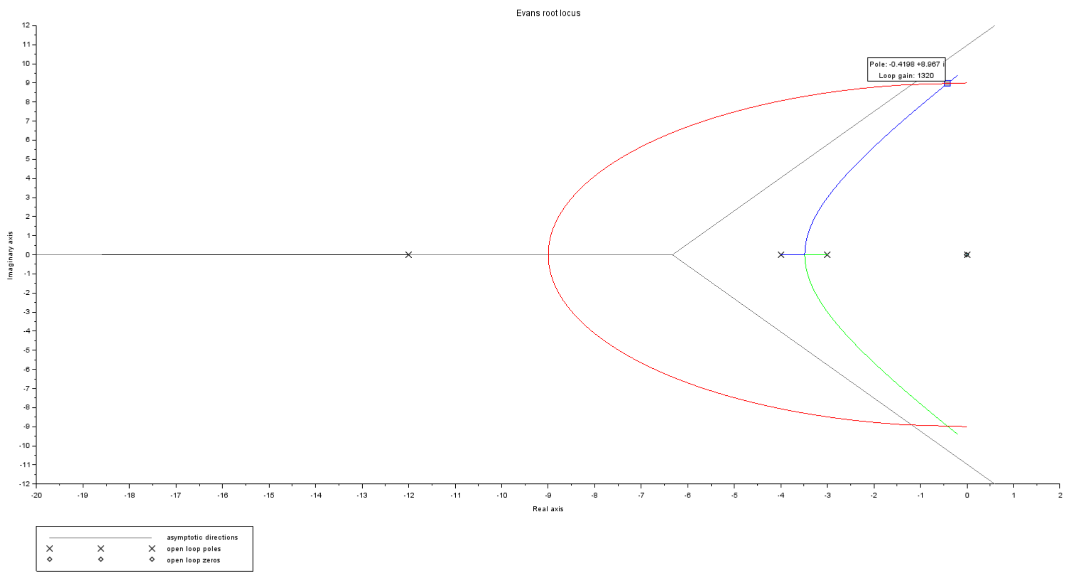


Thus for K=945.1 this PI controller will be have undamped natural frequency of 8 rad/s

To obtain undamped natural frequency of 9 rad/s

We plot the root locus of $G(s) * \frac{s+0.01}{s}$ and to obtain the gain for undamped natural frequency of 9 rad/s we plot a constant circle of radius 9. The intersection of the two curves gives us the required closed loop pole. We get this pole at the gain of 1320.

Obtained Plot



Thus for K=1320 this PI controller will be have undamped natural frequency of 9 rad/s

1.3 Part c

If we vary the value z in [0,3] we get the following root locus. Here we observe that as we increase z, i.e. as the zero goes more towards left, the root locus starts from the same point, i.e. the breakaway point is the same somewhere in between -3 and -4.

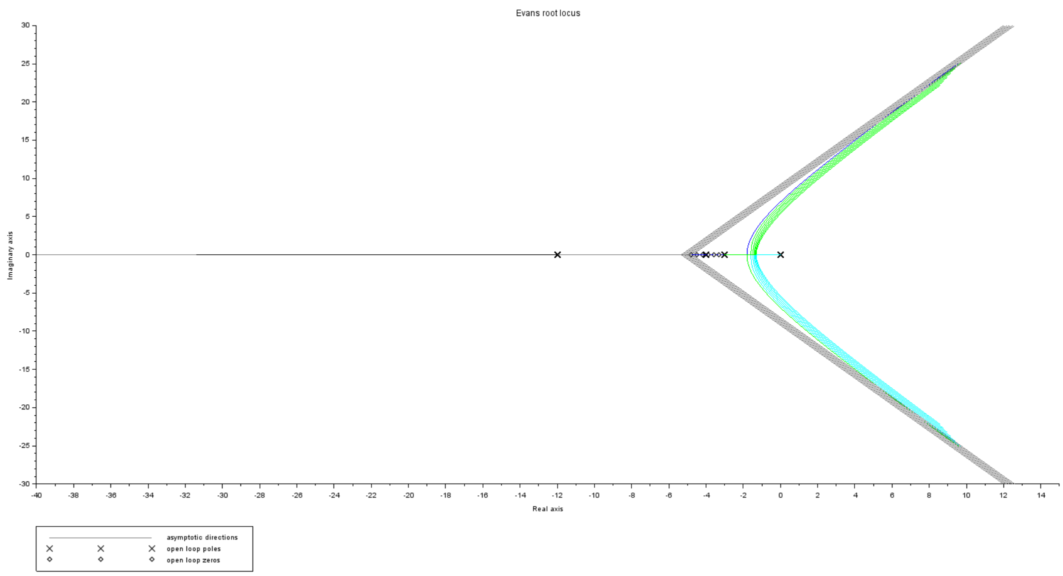
And as we increase z the root locus stays near the x axis for longer time and also the point at which it will cut the imaginary axis is also getting smaller and smaller.

Now if we increase z further i.e. beyond 3, the break away point changes drastically. It comes somewhere in between 0 and -2. Also the breakaway point does not remain constant in this period there is slight movement to the left i.e. towards the origin as we increase z.

Scilab Code

```
s = poly(0,'s');
G = 1/((s+3)*(s+4)*(s+12));

for z = 3:0.3:5
K = G*(s+z)/s
evans(K);
end
```

Obtained Plot



1.4 Part d

So, yes it is possible to have different poles using this PI controller but having the same damping ratio.

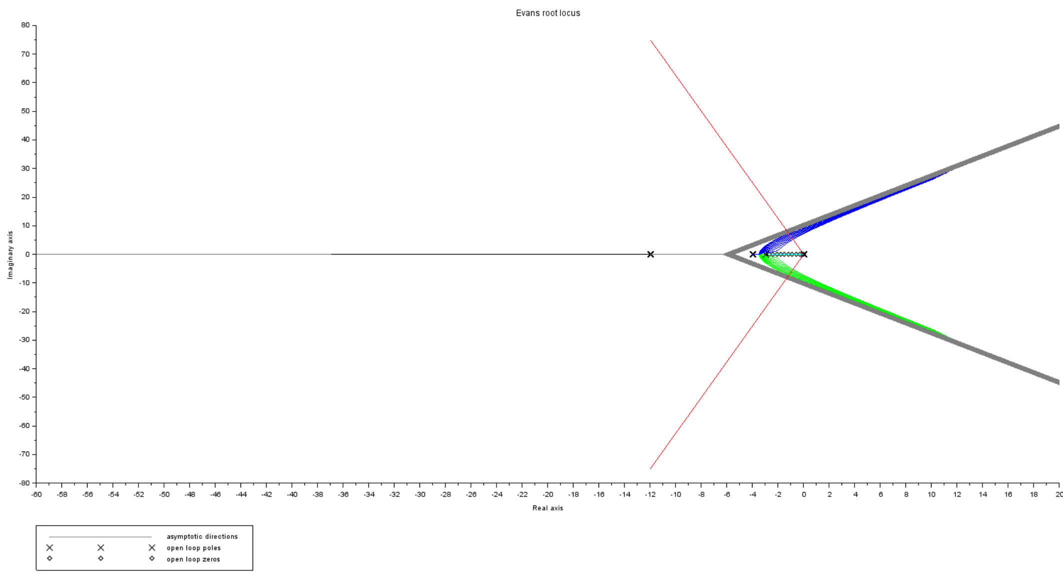
So, below we can see that as we vary z the root locus will also vary and thus we will be getting different points of intersection of the line and the root locus for different z's, as can be seen in the below plot.

So, the points of intersection will actually be the poles, hence proved that we can have different poles with same damping ratio

Scilab Code

```
s = poly(0,'s');
G = 1/((s+3)*(s+4)*(s+12));

for z = 0.01:0.3:3
    K = G*(s+z)/s
    evans(K);
end

x = -12:0.01:0

y1 = (1/0.16)*x;
y2 = -(1/0.16)*x;
plot(x,y1,'r');
plot(x,y2,'r');
```

Obtained Plot



2 Effect of Pole-Zero Location in Lag Compensator

The ratio of zero magnitude to pole magnitude in the lag compensator is 20.

2.1 Problem a

Consider a system with transfer function given by,

$$G(s) = \frac{1}{s^2 + 3s + 2}$$

We have to find gain K such that the closed loop system has 10% overshoot. The closed loop transfer function will be given by,

$$T(s) = \frac{KG(s)}{1 + KG(s)}$$
$$\implies T(s) = \frac{\frac{K}{s^2+3s+2}}{1 + \frac{K}{s^2+3s+2}}$$
$$\implies T(s) = \frac{K}{s^2 + 3s + (2 + K)}$$

Comparing with standard second order transfer function, we can find the natural frequency and the damping ratio as,

$$\omega_n = \sqrt{2 + K} \tag{2.1.1}$$
$$\zeta = \frac{3}{2\sqrt{2 + K}} \tag{2.1.2}$$

For % O.S of 10, we have

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100 = 10$$

Solving this equation for ζ we get it to be equal to,

$$-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = \ln(\frac{10}{100})$$
$$\zeta = 0.591$$

Using this value of ζ and from equation (2.1.2) we get,

$$0.591 = \frac{3}{2\sqrt{2+K}}$$
$$\implies \sqrt{2+K} = \frac{3}{2 \times 0.591} = 2.538$$
$$\implies K = 4.441$$

Answer

K = 4.441

2.2 Problem b

Now, we can find the steady state error as,

$$S.S.E = \lim_{x\to 0} \frac{1}{1+KG(s)} = \frac{1}{1+KG(0)}$$
$$\implies S.S.E = \frac{1}{1+4.441 \times \frac{1}{2}} = 0.310$$

Now, we add the lag compensator. Let the location of the pole of the lag compensator be -0.01. Then we have the location of the zero given by, $20 \times -0.01 = -0.2$. Hence, the transfer function of the of the controller is given by,

$$C(s) = \frac{s+0.2}{s+0.01}$$

Hence, we can find the steady state error as,

$$S.S.E = \lim_{s0} \frac{1}{1+KC(s)G(s)}$$
$$\implies S.S.E = \frac{1}{1+KC(0)G(0)} = \frac{1}{1+4.441 \times (20) \times \frac{1}{2}}$$
$$\implies S.S.E = 0.022$$

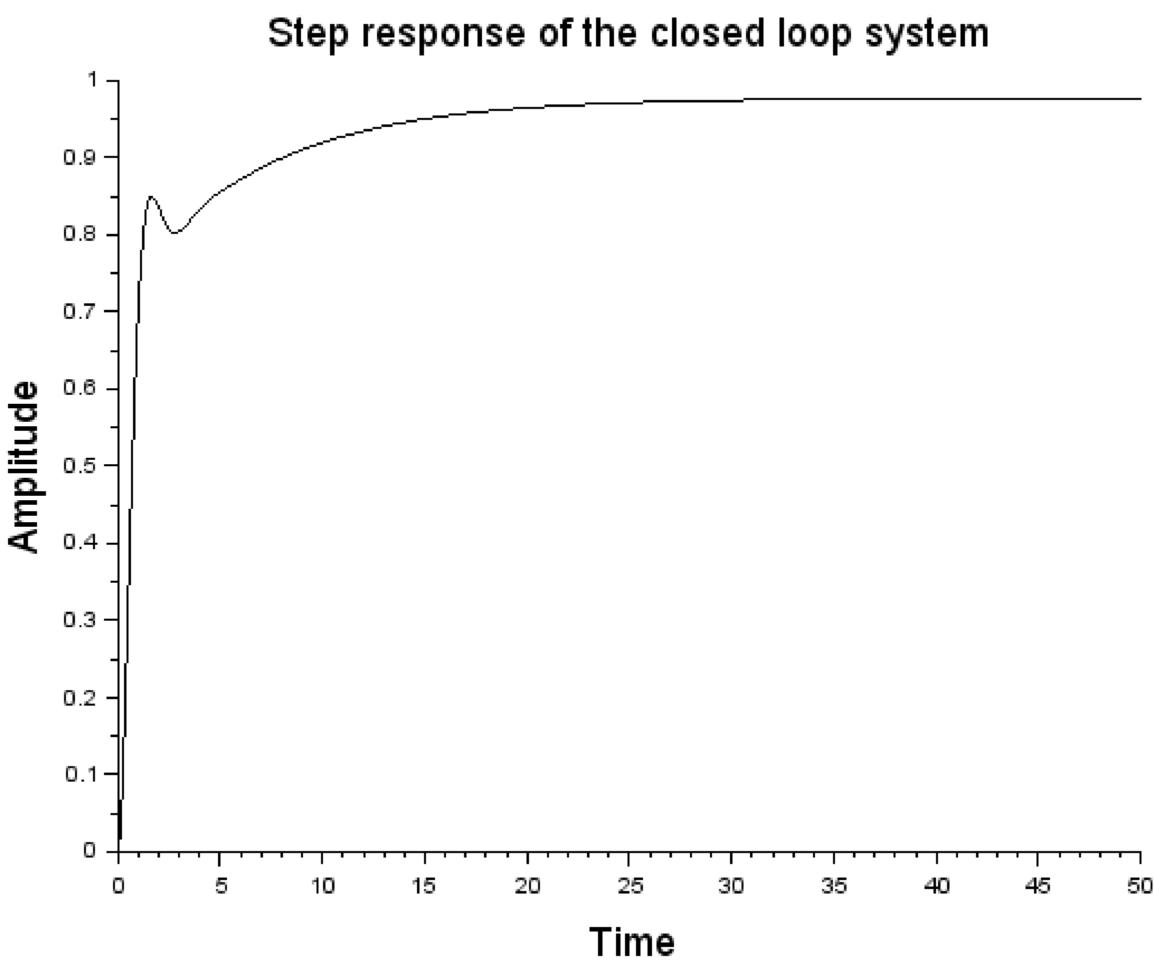
2.3 Problem c

We plot the step response for the following pole zero pairs (we keep ratio of zero magnitude to pole magnitude equal to 20) :

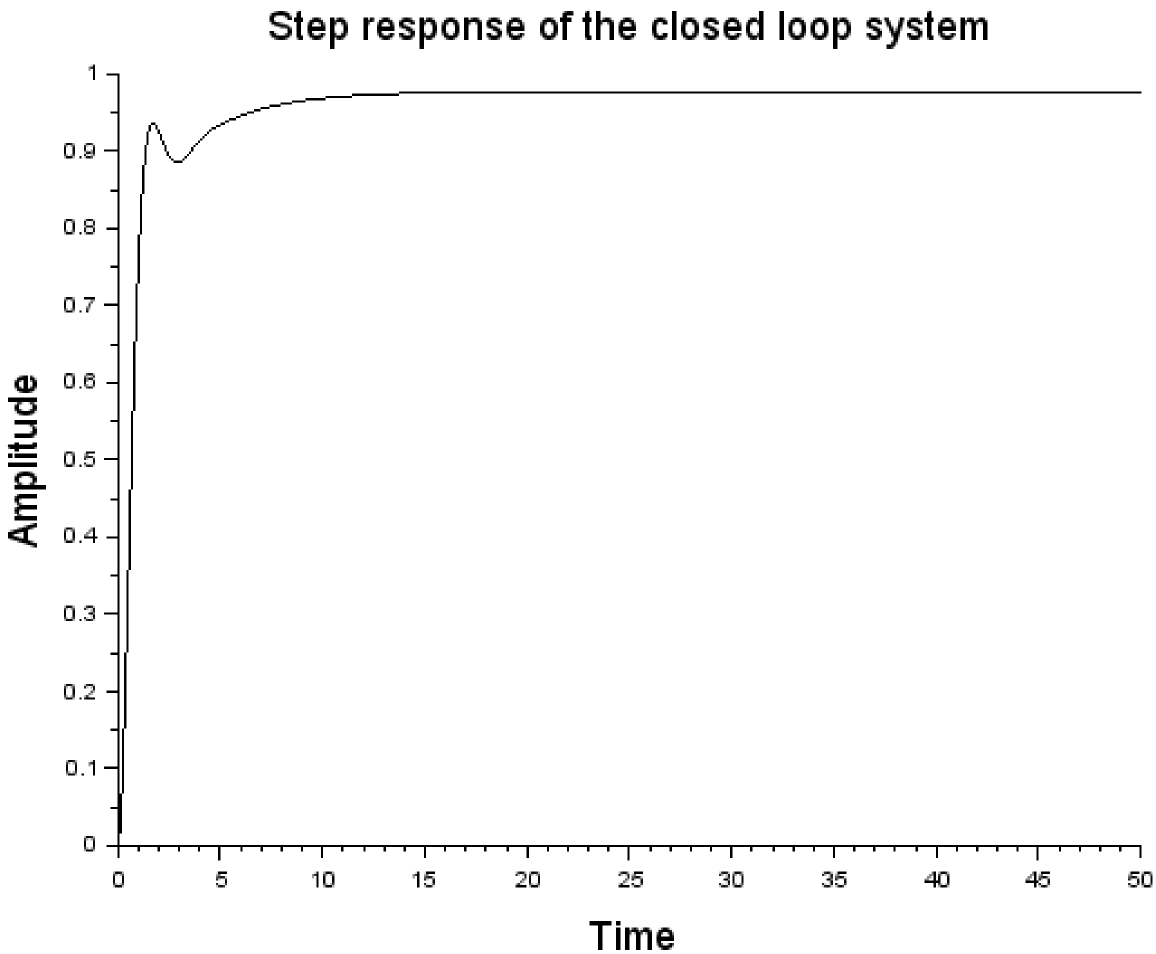
- pole = -0.01 , zero = -0.2
- pole = -0.02 , zero = -0.4
- pole = -0.03 , zero = -0.6
- pole = -0.04 , zero = -0.8
- pole = -0.045 , zero = -0.9

Step response plots

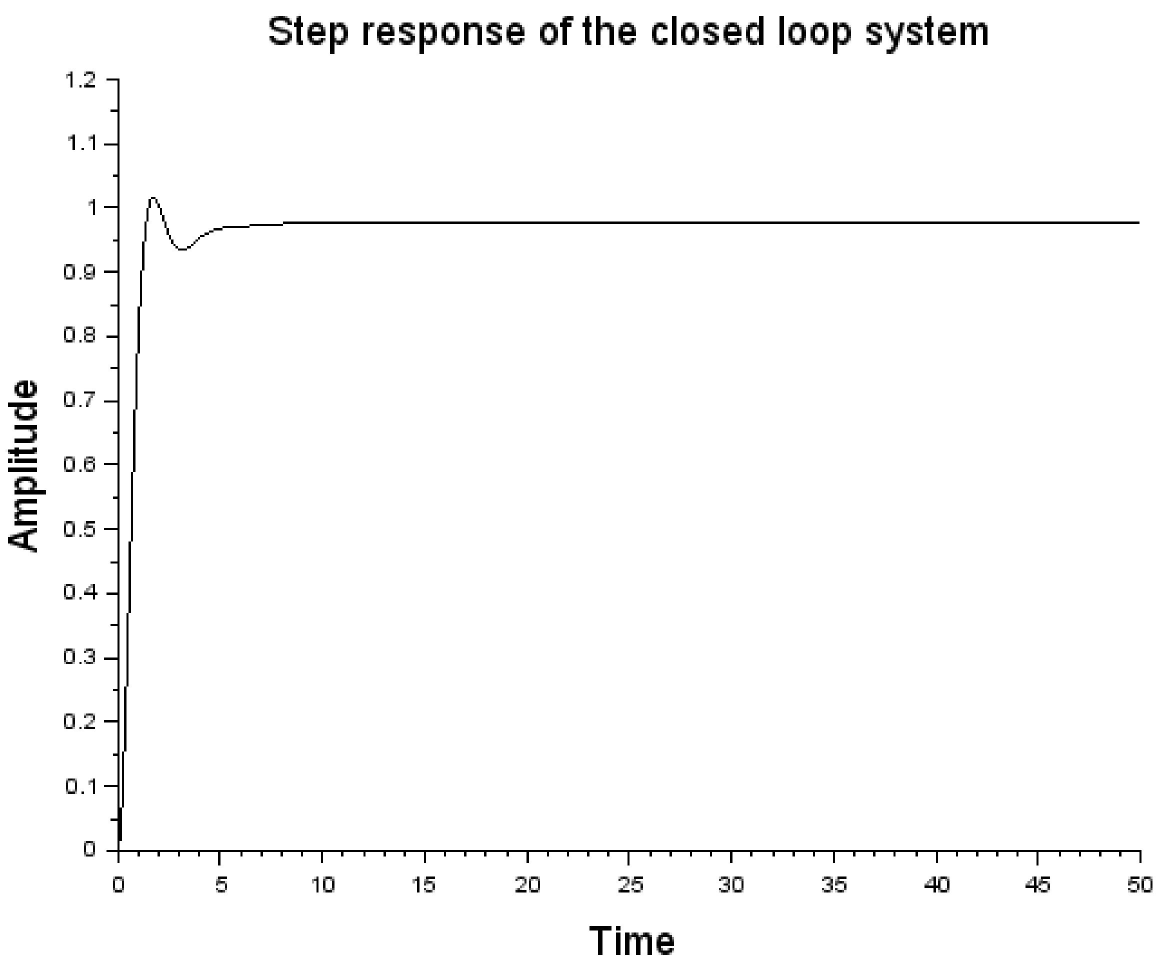
pole = -0.01 , zero = -0.2



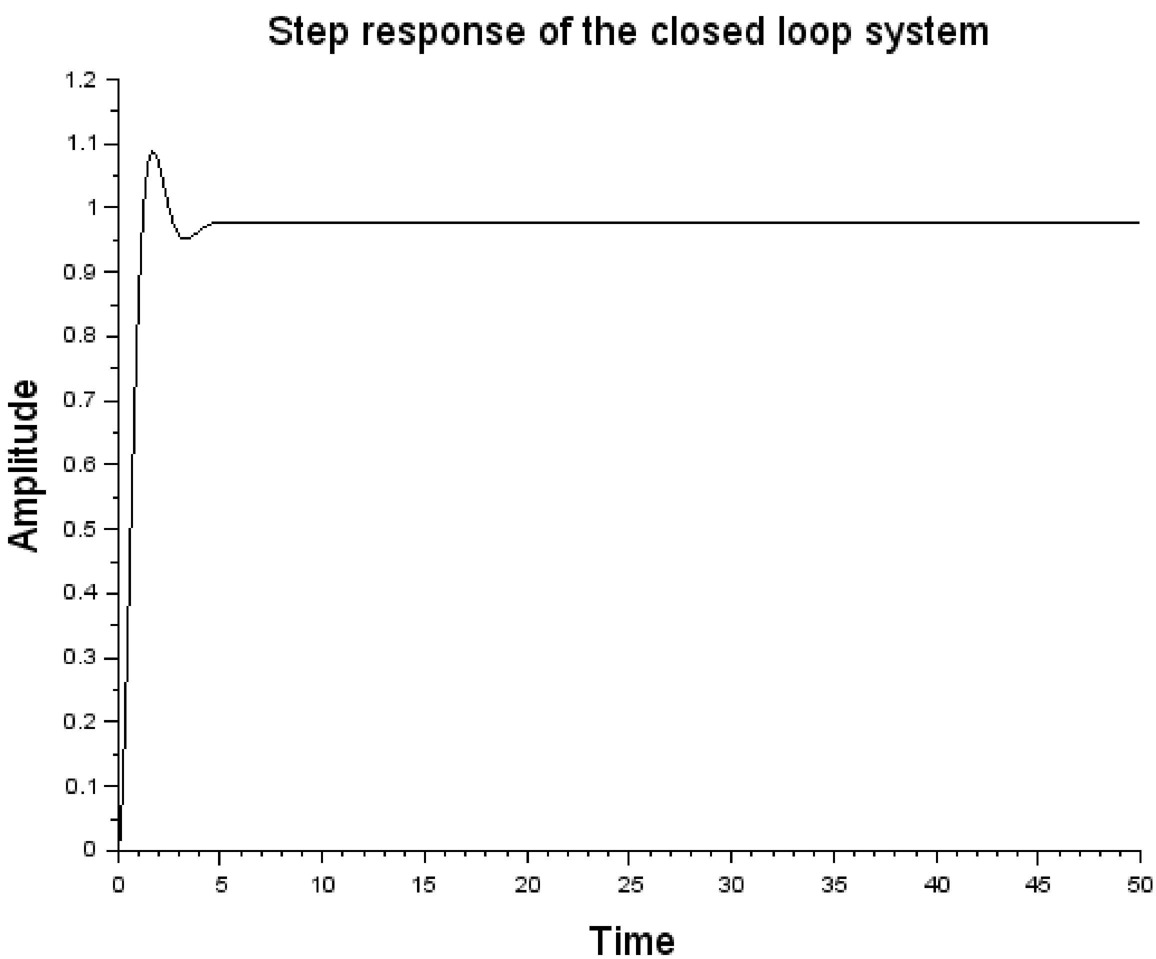
pole = -0.02 , zero = -0.4



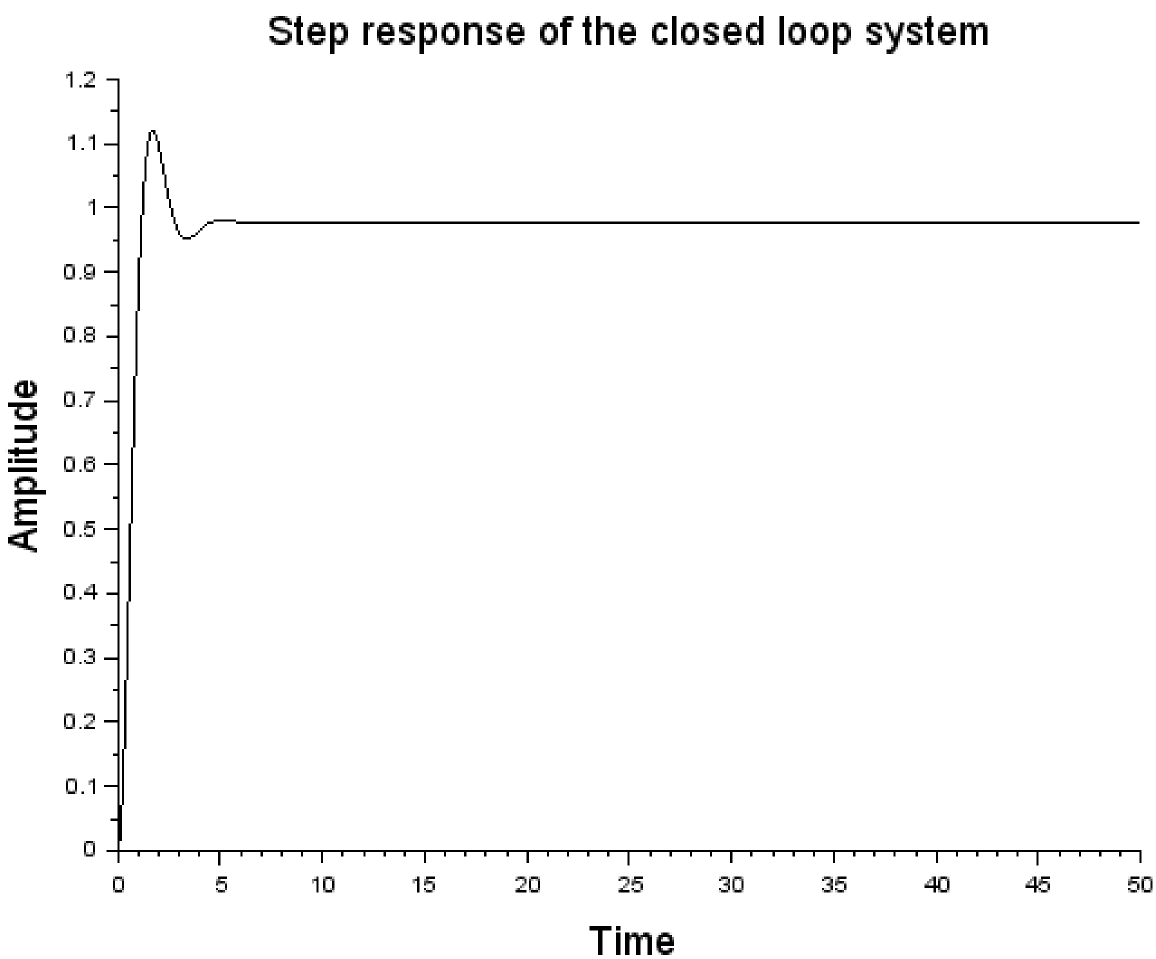
pole = -0.03 , zero = -0.6



pole = -0.04 , zero = -0.8



pole = -0.045 , zero = -0.9



Observation

From these plots we can see that as the pole zero pair moves away from the origin, the steady state is achieved earlier but the percentage overshoot changes from what we had designed the system for.

3 Improving the transient response

3.1 Part a

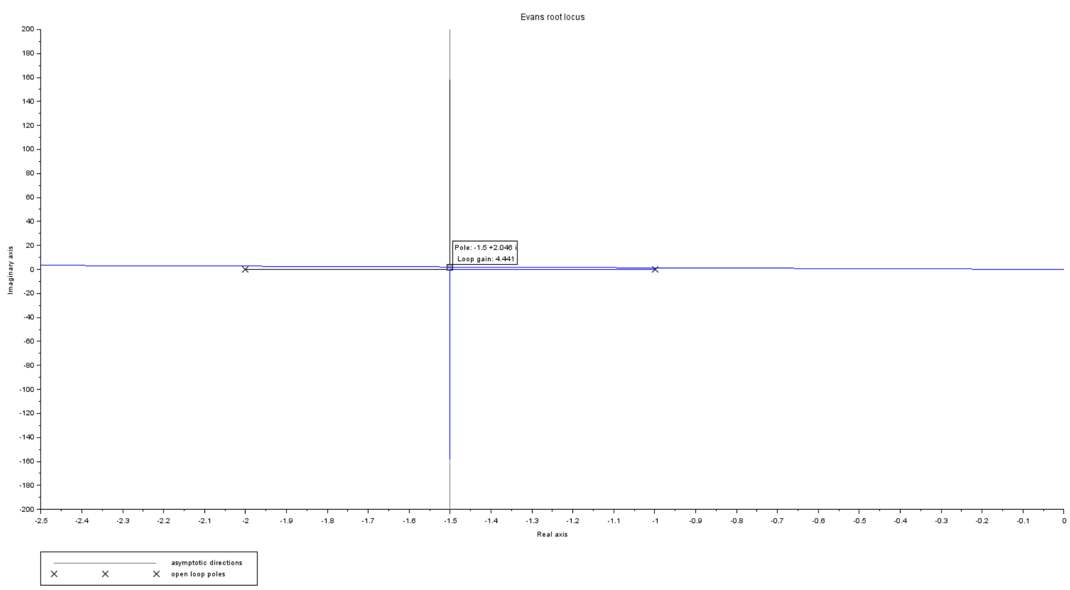
In this part, we design a lead compensator 10% O.S. is equivalent to $\zeta = 0.591$. We search along that damping ratio line for an odd multiple of 180° and find that the dominant,second-order pair of poles is at $-1.5 \pm 2.046i$.

Scilab Code

```
clear
close
clc
```

```
s = poly(0,'s');

G = 1/(s^2 + 3*s + 2);
Glin = syslin('c',G);
evans(Glin,100);

// Finding theta from the damping ratio

zeta = 0.591;
theta = acos(zeta);
x = -10:0.1:0;
y = -tan(theta)*x;
plot(x, y);
```

Obtained Plot



The the settling time of the system is

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.5} = 2.667$$

Now, we proceed to compensate the system. First, we find the location of the compensated system's dominant poles. In order to make the settling time half, we have the new settling time as 1.333. Therefore, the real part of the compensated system's dominant, second-order pole is

$$\sigma = \frac{4}{T_s} = \frac{4}{1.333} = 3$$

To have the same % O.S. the imaginary part can be found out as,

$$\omega_d = 3 \times tan(cos^{-1}(0.591)) = 4.095$$

Arbitrarily assume the zero of the lead compensator to be at -5. Now, we need to find the pole location of the lead compensator. We find the angle contribution of the poles and the zero at the point -3 + 4.095i. contribution due to pole at -2 at -3+ 4.095 i = $180° - tan^{-1}(\frac{4.095}{3-2} = 103.723°$
contribution due to pole at -1 at -3+ 4.095 i = $180° - tan^{-1}(\frac{4.095}{3-1}) = 116.030°$

contribution due to the zero at -5 at -3+4.095 i = $tan^{-1}(\frac{4.096}{5-3} = 63.969°)$

$\therefore$ Total contribution due to the poles = $63.969° - 219.754° = -155.784°$

Hence, the angular contribution required from the compensator zero for the test point to be on the root locus is $-155.784° + 180° = 24.21°$
Hence, we can find the pole location as,

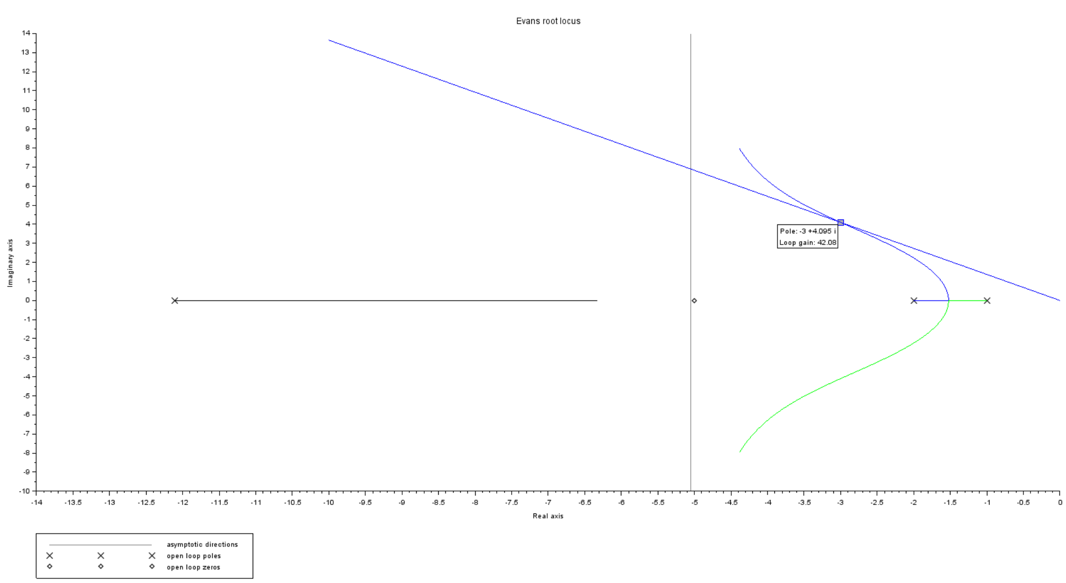
$$\frac{4.095}{p-3} = tan(24.21°)$$
$$\implies z = 12.1053$$

Hence, location of the zero is given by -12.1053. Hence, the total transfer of the compensated system is given by,

$$G_c(s) = \frac{s+5}{(s+12.1053)(s^2+3s+2)}$$

We can check for the intersection of the root locus of the compensated system and the 10% O.S. line as follows.

Obtained plot

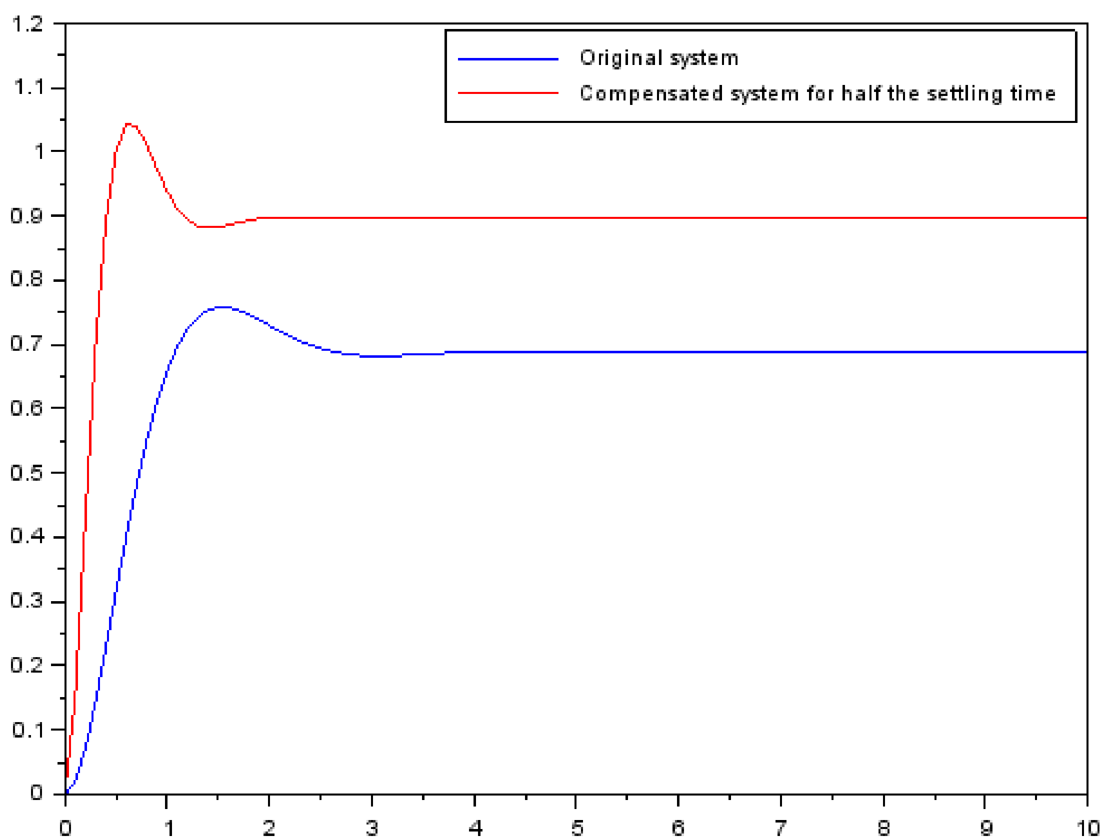


We can see the step response of the original and the compensated system

Scilab Code

```
clear
clc

s = poly(0,'s');

G = 1/(s^2 + 3*s + 2);
k = 4.441;
T = k*G/(1 + k*G);
t = 0:0.1:10;
T1 = syslin('c',T);
ts = csim('step' ,t, T1);
plot(t , ts);

Gc = (s+ 5)/((s^2 + 3*s + 2)*(s+12.1053) );
kc = 42.08;
Tc = kc*Gc/(1+kc*Gc);
T2 = syslin('c',Tc);
ts2 = csim('step' ,t, T2);
plot(t , ts2,'r');
legend('Original system','Compensated system for half the settling time')
```

Obtained Plot



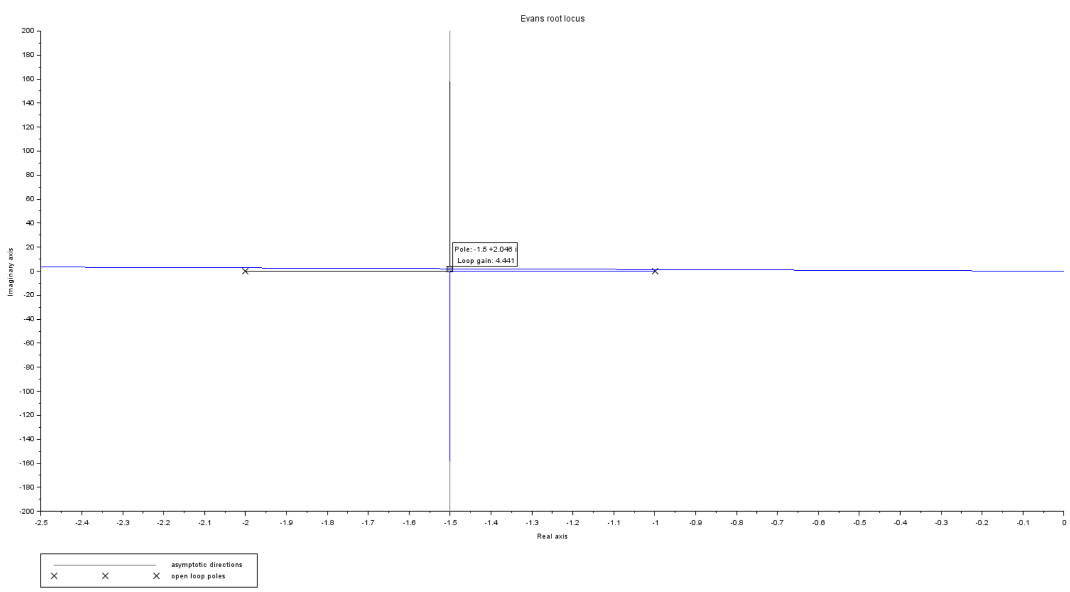
3.2 Part b

In this part we design a PD controller.
10% O.S. is equivalent to $\zeta = 0.591$. We search along that damping ratio line for an odd multiple of 180° and find that the dominant,second-order pair of poles is at $-1.5 \pm 2.046i$.

Scilab Code

```
clear
close
clc

s = poly(0,'s');

G = (s+ 7.923)/(s^2 + 3*s + 2);
Glin = syslin('c',G);
evans(Glin,100);

// Finding theta from the damping ratio

zeta = 0.591;
theta = acos(zeta);
x = -10:0.1:0;
y = -tan(theta)*x;
plot(x, y);
```

Obtained Plot



The settling time of the system is

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{1.5} = 2.667$$

Now, we proceed to compensate the system. First, we find the location of the compensated system's dominant poles. In order to make the settling time half, we have the new settling time as 1.333. Therefore, the real part of the compensated system's dominant, second-order pole is

$$\sigma = \frac{4}{T_s} = \frac{4}{1.333} = 3$$

To have the same % O.S. the imaginary part can be found out as,

$$\omega_d = 3 \times tan(cos^{-1}(0.591)) = 4.095$$

Hence, the location of the closed loop poles for the compensated system is given by $-3 \pm 4.095i$. The angle contribution at these poles from the open loop poles poles is given by.
contribution due to pole at -2 at -3+ 4.095 i = $180° - tan^{-1}(\frac{4.095}{3-2} = 103.723°$
contribution due to pole at -1 at -3+ 4.095 i = $180° - tan^{-1}(\frac{4.095}{3-1}) = 116.030°$
$\therefore$ Total contribution due to the poles = $-219.754°$ Hence, the angular contribution required from the compensator zero for the test point to be on the root locus is $+219.754° - 180° = 39.753°$
Hence, we can find the zero location as,

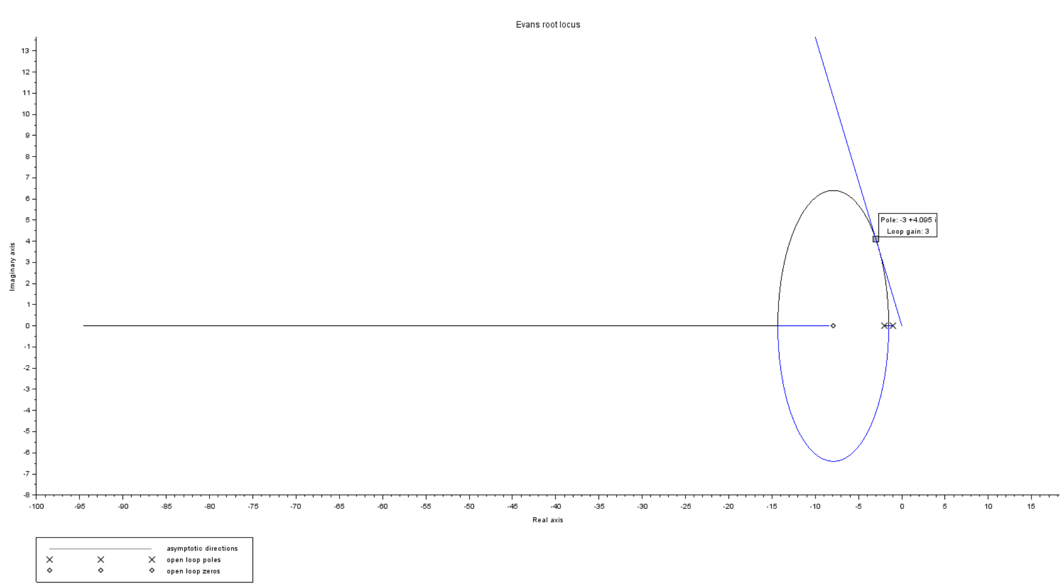
$$\frac{4.095}{z-3} = tan(39.753°)$$
$$\implies z = 7.923$$

Hence, location of the zero is given by -7.923. Hence, the total transfer of the compensated system is given by,

$$G_c(s) = \frac{s+7.923}{s^2+3s+2}$$

We can check for the intersection of the root locus of the compensated system and the 10% O.S. line as follows.

Obtained plot

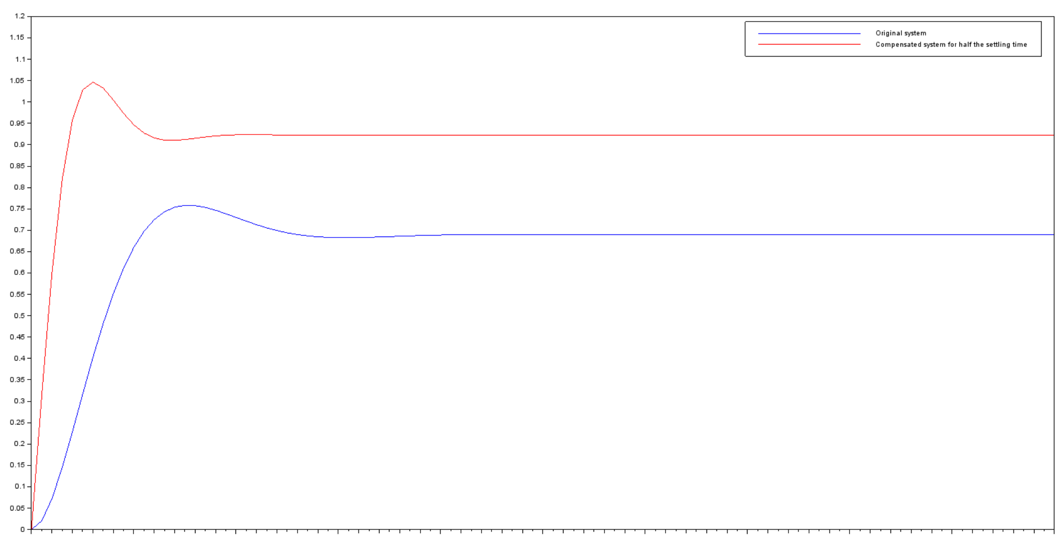


We can see the step response of the original and the compensated system

Scilab Code

```
clear
clc

s = poly(0,'s');

G = 1/(s^2 + 3*s + 2);
k = 4.441;
T = k*G/(1 + k*G);
t = 0:0.1:10;
T1 = syslin('c',T);
ts =  csim('step' ,t, T1);
plot(t , ts);

Gc = (s+7.923)/(s^2 + 3*s +2);
kc = 3;
Tc = kc*Gc/(1+kc*Gc);
T2 = syslin('c',Tc);
ts2 =  csim('step' ,t, T2);
plot(t , ts2,'r');
legend('Original system','Compensated system for half the settling time')
```

Obtained Plot



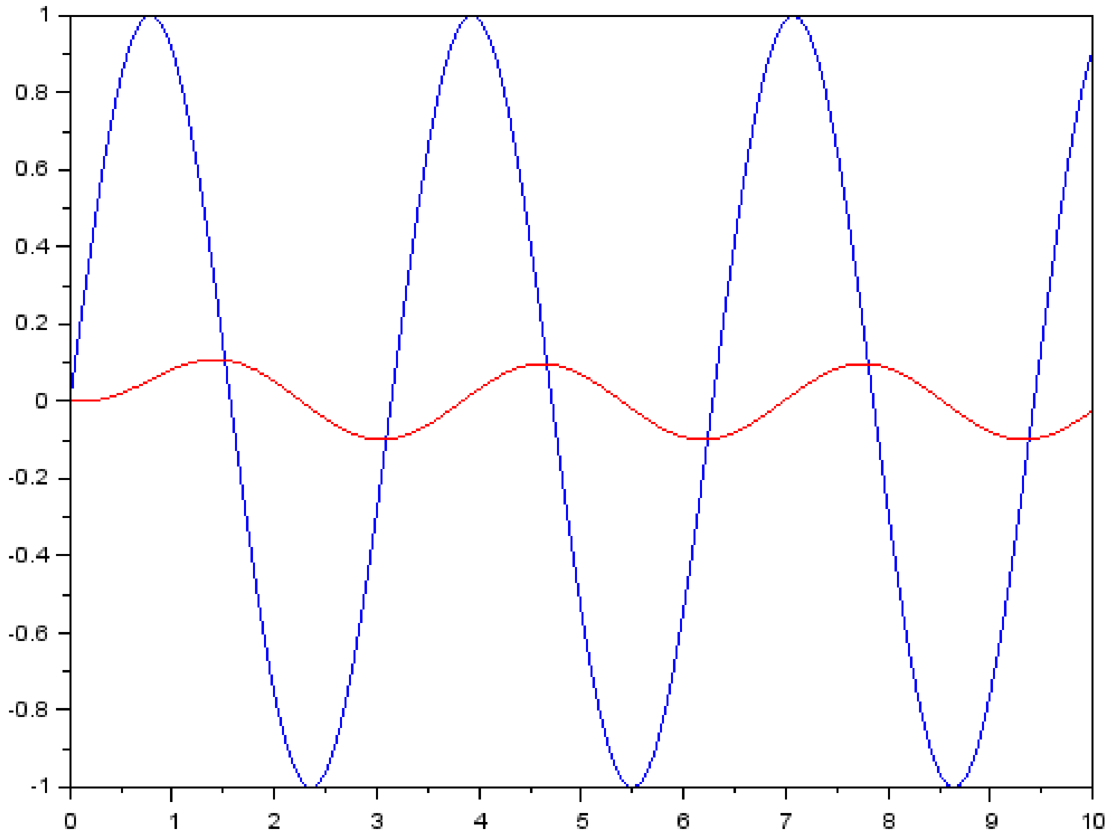
4 Problem 4

For this part consider ω equal to 2,4,6,8,10 rad/s.
The given transfer function is,

$$G(s) = \frac{1}{s^2+5s+6}$$
$$\implies G(j\omega) = \frac{1}{(j\omega)^2+5j\omega+6}$$
$$\implies G(j\omega) = \frac{1}{6-\omega^2+5j\omega}$$
$$\implies |G(j\omega)| = \left|\frac{1}{6-\omega^2+5j\omega}\right|$$
$$\implies |G(j\omega)| = \frac{1}{\sqrt{(6-\omega^2)^2+(5\omega)^2}}$$
$$\angle G(j\omega) = -tan^{-1}\frac{5\omega}{6-\omega^2}$$

- $\omega = 2rad/s$

Obtained plot



Ratio

```
The ratio of amplitudes of output to input is

  0.1099618
```

Also from calculation we have,

$$|G(j\omega)| = 0.0980$$

Phase difference

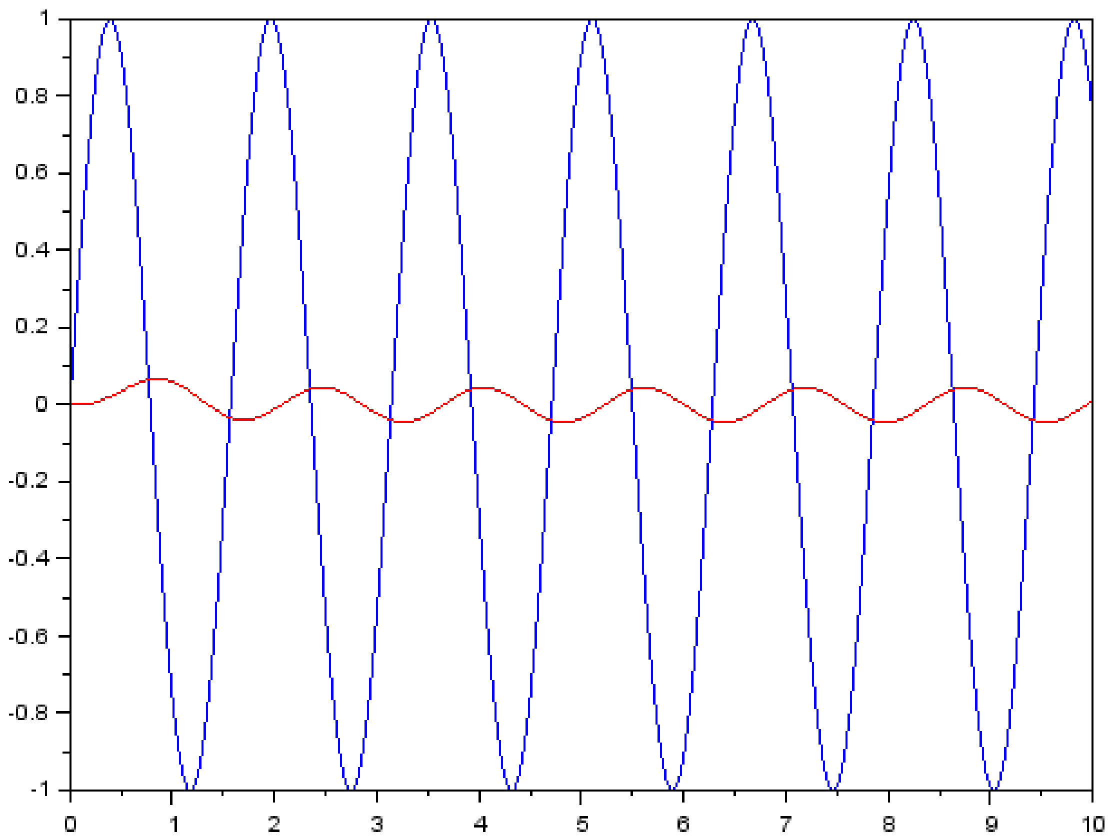
```
The phase difference between input and output is

  1.24
```

Also from calculation we have,

$$\angle G(j\omega) = 1.37$$

- $\omega = 4rad/s$

Obtained plot



Ratio

```
The ratio of amplitudes of output to input is

  0.0462421
```

Also from calculation we have,

$$|G(j\omega)| = 0.0447$$

Phase difference

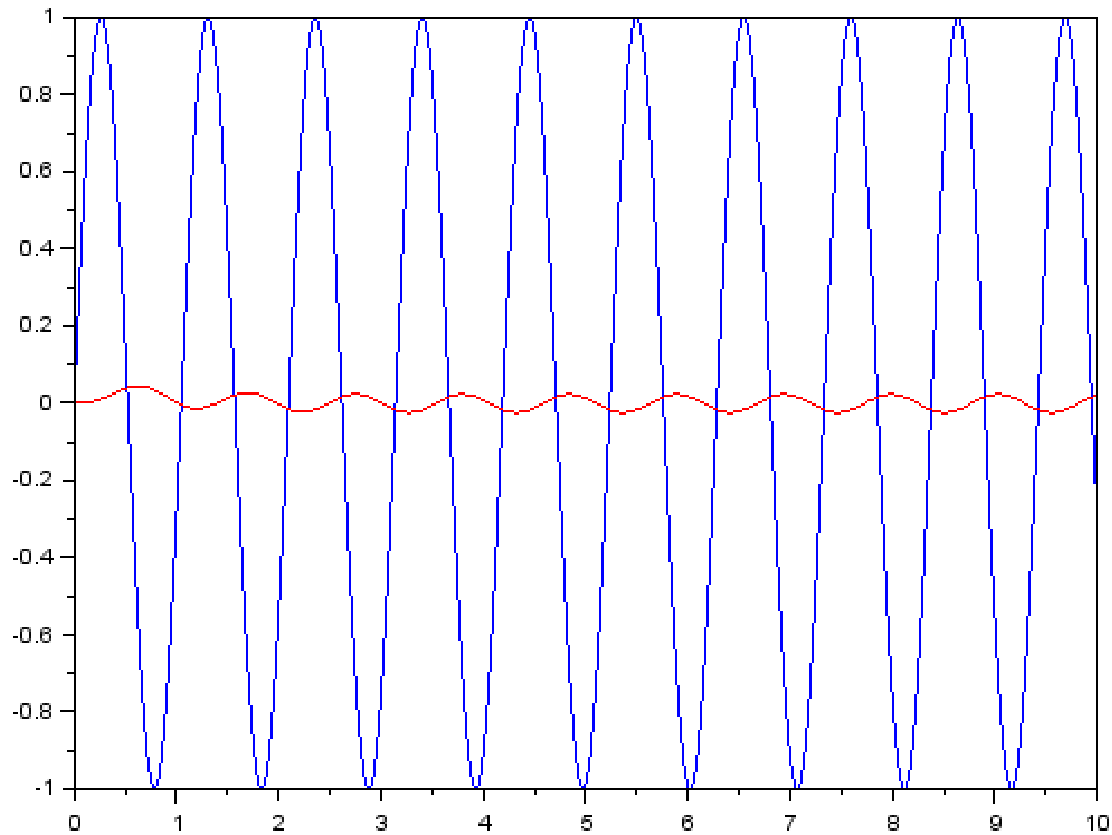
```
The phase difference between input and output is

  1.96
```

Also from calculation we have,

$$\angle G(j\omega) = 2.034$$

- $\omega = 6rad/s$

Obtained plot



Ratio

```
The ratio of amplitudes of output to input is

  0.0457075
```

Also from calculation we have,

$$|G(j\omega)| = 0.0236$$

Phase difference

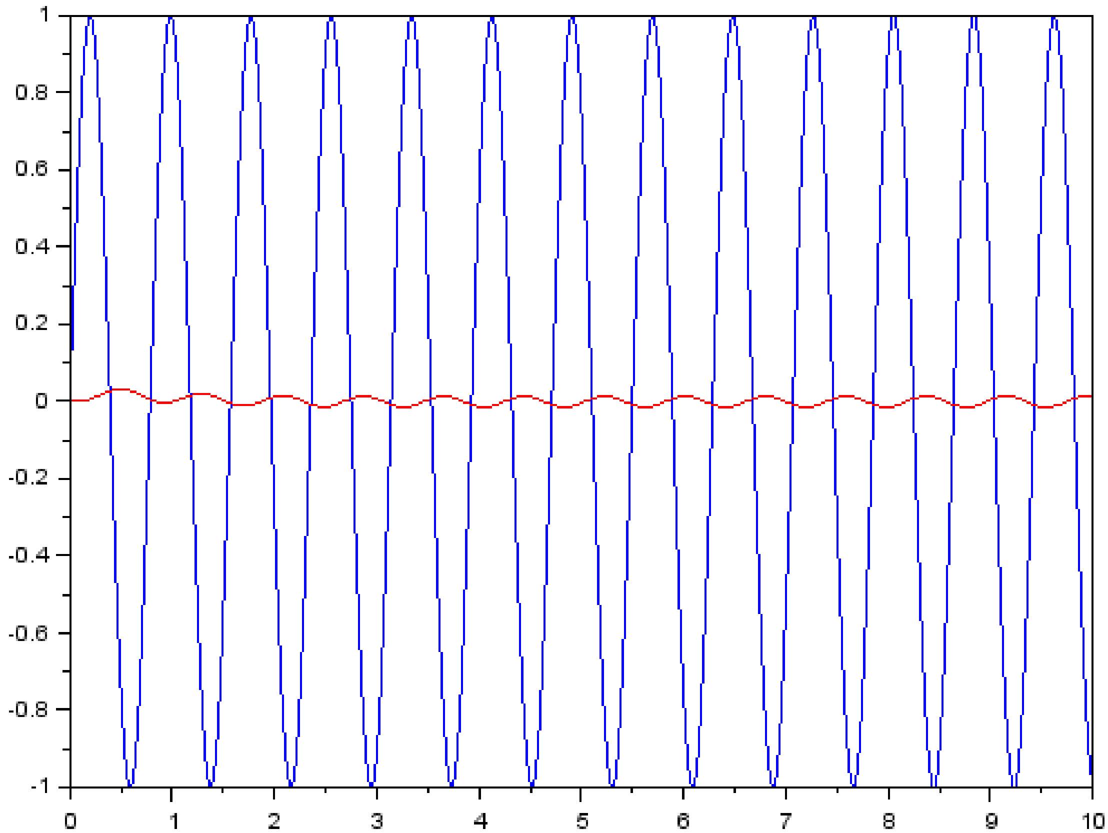
```
The phase difference between input and output is

  2.4
```

Also from calculation we have,

$$\angle G(j\omega) = 2.356$$

- $\omega = 8 rad/s$

Obtained plot



Ratio

```
The ratio of amplitudes of output to input is

  0.0047025
```

Also from calculation we have,

$$|G(j\omega)| = 0.0142$$

Phase difference

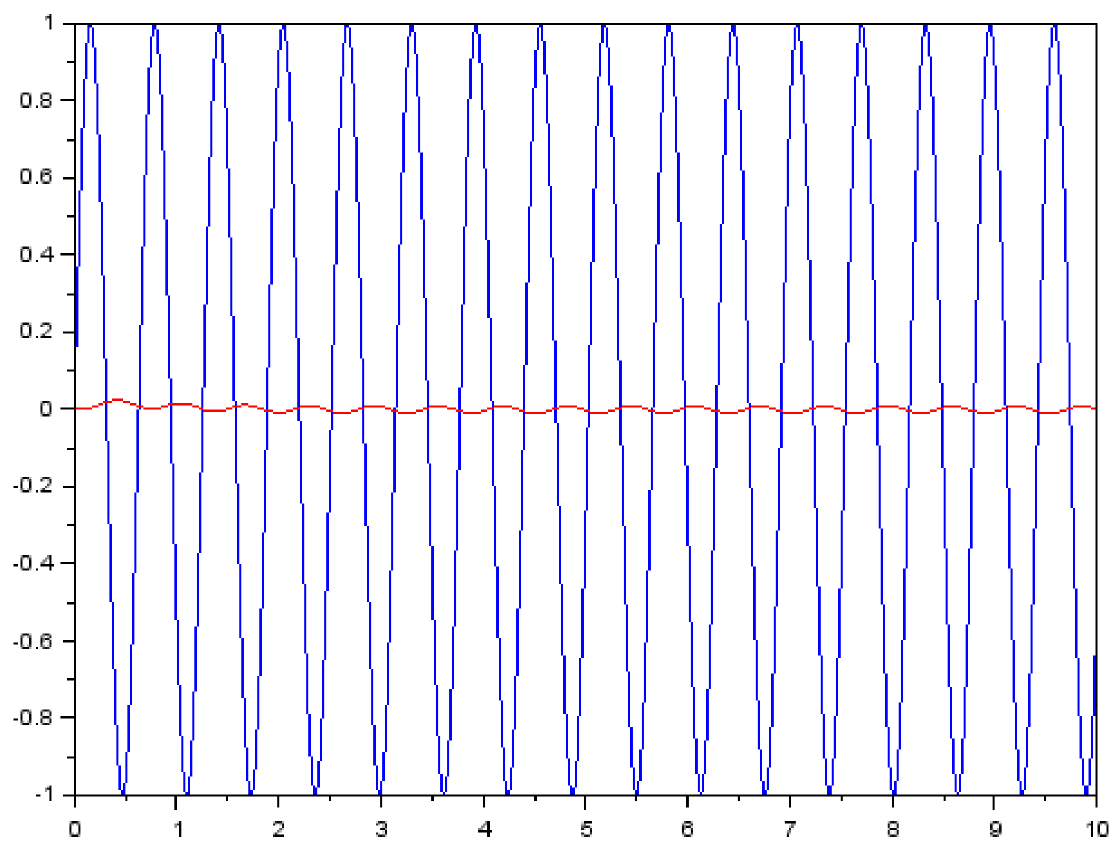
```
The phase difference between input and output is

  2.64
```

Also from calculation we have,

$$\angle G(j\omega) = 2.537$$

- $\omega = 10 rad/s$

Obtained plot



Ratio

```
The ratio of amplitudes of output to input is

  0.0102696
```

Also from calculation we have,

$$|G(j\omega)| = 0.00939$$

Phase difference

```
The phase difference between input and output is

  2.4
```

Also from calculation we have,

$$\angle G(j\omega) = 2.652$$

The Scilab code used for making the above plots

```
clear
close
clc

omega = 10;
t = 30:0.01:31;

sin_omega_t = sin(omega*t);
plot(t , sin_omega_t);


s = poly(0,'s')
G = 1/(s^2 + 5*s + 6);
G1 = syslin('c',G);
gsin =  csim(sin_omega_t ,t, G1);
plot(t , gsin , 'r');

t_max_out = 0;
t_max_in = 0;
max_resp = -20;
max_in = -20;

//for i = 1:1:length(t)
//    if (sin_omega_t(i) > max_in)
//        max_in = sin_omega_t(i);
//        t_max_in = t(i);
//    else
//        break;
//    end
//
//end

//for i = 1:1:length(t)
//    if (gsin(i) > max_resp )
//        max_resp = gsin(i);
//        t_max_out = t(i)
//    else
//        break;
//    end
//end


t_max_out_indx = find(gsin == max(gsin))
```

```
t_max_out = t(t_max_out_indx);

t_max_in_indx = find(sin_omega_t == max(sin_omega_t))
t_max_in = t(t_max_in_indx);

phase = omega * (t_max_out - t_max_in);
disp(gsin(t_max_out_indx),"The ratio of amplitudes of output to input is ");
disp(phase,"The phase difference between input and output is ");
```

4.1 Part b

The units of omega used are rad/s to find the desired relation (between phase difference and angle of $G(j\omega)$

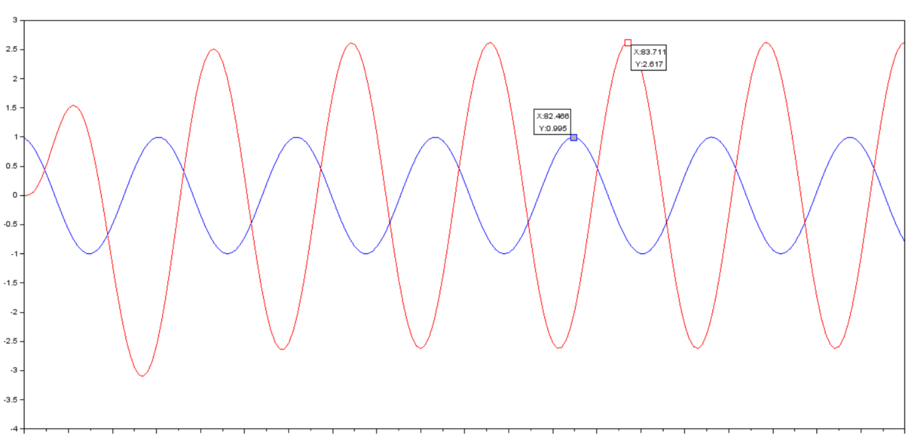
4.2 Part c

For this part consider ω equal to 2,4,6,8,10 rad/s.
The given transfer function is,

$$
\begin{aligned}
G(s) &= \frac{60}{s^3+6s^2+11s+6} \\
\implies G(j\omega) &= \frac{60}{(j\omega)^3+6(j\omega)^2+11(j\omega)+6} \\
\implies G(j\omega) &= \frac{60}{-j\omega^3-6\omega^2+11j\omega+6} \\
\implies |G(j\omega)| &= \left|\frac{60}{6-6\omega^2+j(11\omega-\omega^3)}\right| \\
\implies |G(j\omega)| &= \frac{60}{\sqrt{(6-6\omega^2)^2+(11\omega-\omega^3)^2}} \\
\angle G(j\omega) &= -tan^{-1}\frac{11\omega-\omega^3}{6-6\omega^2}
\end{aligned}
$$

- $\omega = 2rad/s$

  Obtained plot



Ratio

```
 2.6221430

"The ratio of amplitudes of output to input is "
```

Also from calculation we have,

$$|G(j\omega)| = 2.658$$

Phase difference

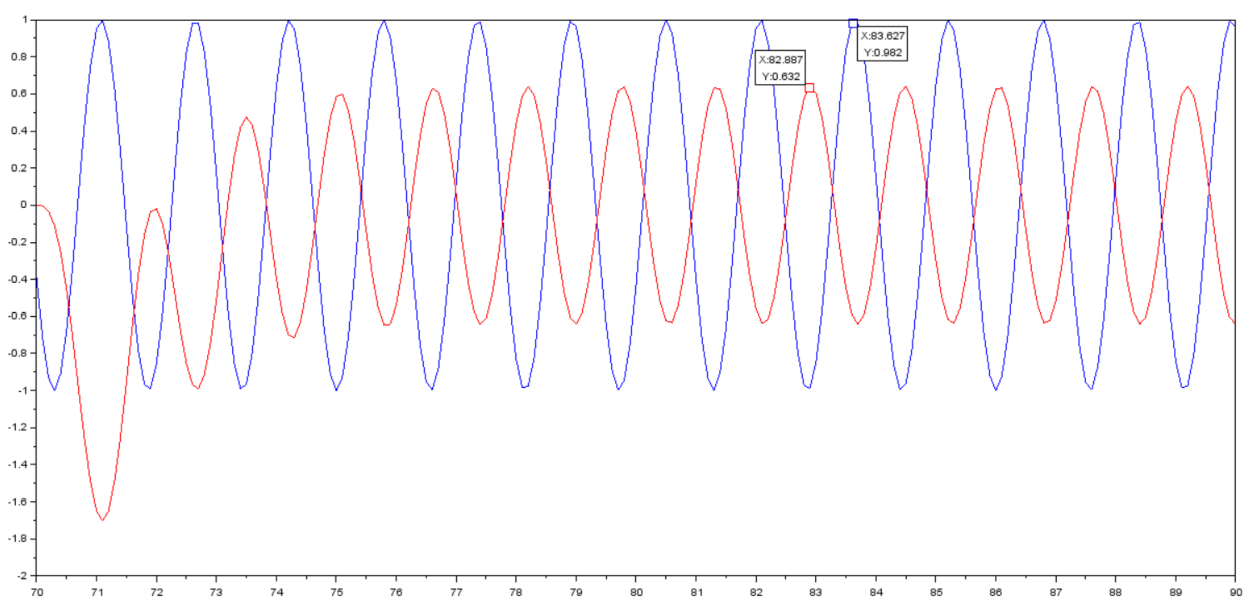
From the plot we can see that the time difference is- 1.245 secs

So, the phase difference is- 1.245*2 = 2.49rad Also from calculation we have,

$$\angle G(j\omega) = 2.478$$

- $\omega = 4 rad/s$

Obtained plot



Ratio

```
0.6421245

"The ratio of amplitudes of output to input is "
```

Also from calculation we have,

$$|G(j\omega)| = 0.656$$

Phase difference

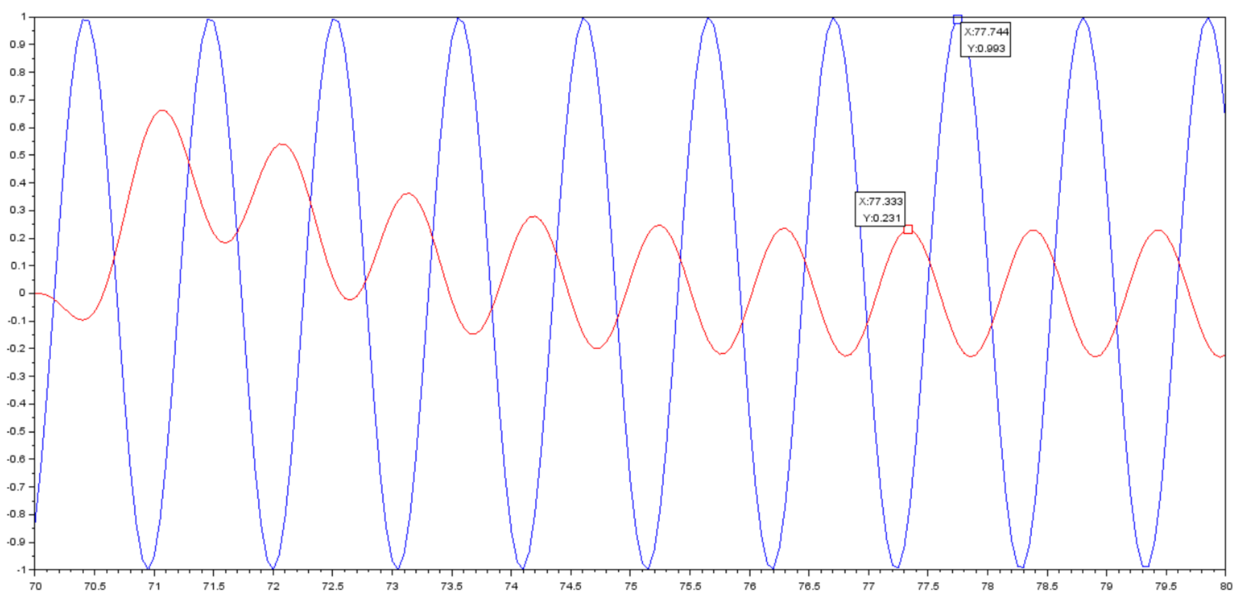
From the plot we can see that the time difference is- 0.74 secs

So, the phase difference is- 0.74*4 = 2.96rad Also from calculation we have,

$$\angle G(j\omega) = 3.06$$

- $\omega = 6rad/s$

Obtained plot



Ratio

From the plot we can see that the ratio is 0.231

Phase difference

From the plot we can see that the time difference is- 0.411 secs
So, the phase difference is- 0.411*6 = 2.466rad Also from calculation we have,

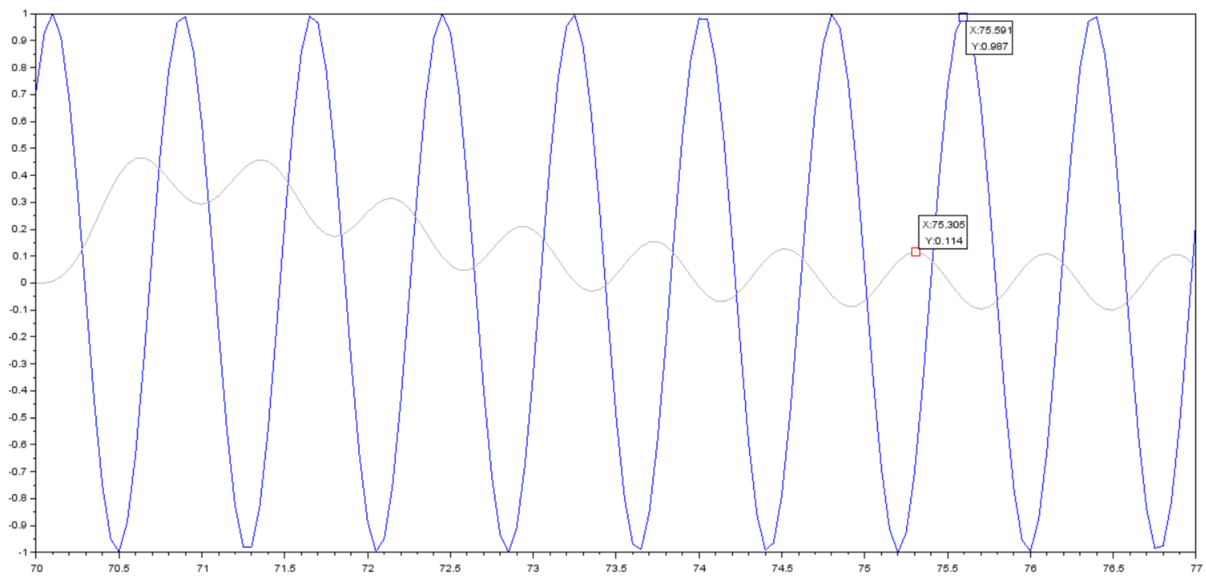
$$|G(j\omega)| = 0.234$$

Also from calculation we have,

$$\angle G(j\omega) = 2.52$$

- $\omega = 8rad/s$

Obtained plot



Ratio

From the plot we can see that the ratio is 0.114

Phase difference

From the plot we can see that the time difference is- 0.286 secs
So, the phase difference is- 0.286*8 = 2.288rad Also from calculation we have,

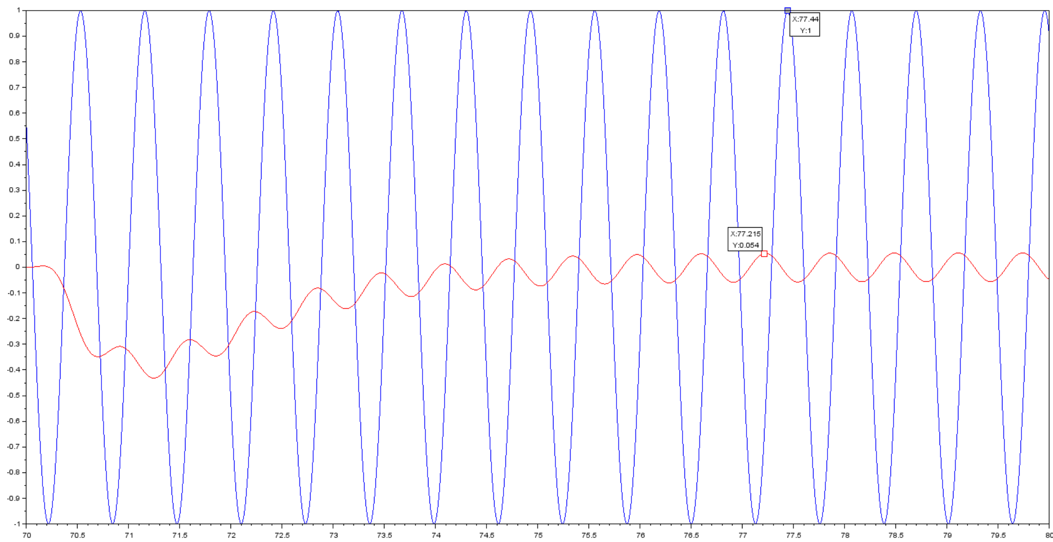
$$|G(j\omega)| = 0.1152$$

Also from calculation we have,

$$\angle G(j\omega) = 2.356$$

- $\omega = 10rad/s$

Obtained plot



Ratio

From the plot we can see that the ratio is 0.054

Phase difference

From the plot we can see that the time difference is- 0.225 secs

So, the phase difference is- 0.225*10 = 2.255rad Also from calculation we have,

$$|G(j\omega)| = 0.056$$

Also from calculation we have,

$$\angle G(j\omega) = 2.159rad$$

The frequency for which the phase difference will be 180 degrees can be calculated as follows-
phase diff $= -tan^{-1}\frac{11\omega-\omega^3}{6-6\omega^2} = 3.14$
Thus, $11\omega = \omega^3$
$\omega = \sqrt{11} = 3.31$
No, the numerator did not play any role in calculation of phase difference.