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To cite this article: Denis Sych and Gerd Leuchs 2009 *New J. Phys.* **11** 013006

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## A complete basis of generalized Bell states

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*New Journal of Physics* **11** (2009) 013006 (9pp)

Received 8 September 2008

Published 7 January 2009

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/11/1/013006

**Abstract.** A generalization of the Bell states and Pauli matrices to dimensions which are powers of 2 is considered. A basis of maximally entangled multidimensional bipartite states (MEMBS) is chosen very similar to the standard Bell states and constructed of only symmetric and antisymmetric states. This special basis of MEMBS preserves all basic properties of the standard Bell states. We present a recursive and non-recursive method for the construction of MEMBS and discuss their properties. The antisymmetric MEMBS possess the property of rotationally invariant exclusive correlations which is a generalization of the rotational invariance of the antisymmetric singlet Bell state.

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## 1. Introduction

Entanglement is one of the most astonishing and most exploited effects in the quantum world. Although this astonishing aspect was already noticed in the early stages of quantum theory [1]–[3], it was not until much later that it attracted appreciable attention, with the theoretical and experimental development of quantum information science.

The simplest example of entanglement is represented by the four maximally entangled two-qubit states, or *Bell states*  $|\Phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B \pm |1\rangle_A|1\rangle_B)$ ,  $|\Psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B \pm |1\rangle_A|0\rangle_B)$  [4]. Their properties are well known and can be found in various books on quantum information [5]–[7]. Here we mention only the following:

1. the Bell states form an orthonormal basis in two-qubit Hilbert space, i.e. each pure two-qubit state can be written as a superposition of the Bell states;
2. the Bell states are of only two types—symmetric and antisymmetric with respect to permutation of the subsystems, i.e. if one exchanges indices of subsystems  $A \leftrightarrow B$ , the symmetric Bell states remain the same  $|\Phi^\pm\rangle_{BA} = |\Phi^\pm\rangle_{AB}$ ,  $|\Psi^\pm\rangle_{BA} = |\Psi^\pm\rangle_{AB}$ , and the antisymmetric Bell state obtains a minus sign:  $|\Psi^-\rangle_{BA} = -|\Psi^-\rangle_{AB}$ ; and
3. finally, the antisymmetric state  $|\Psi^-\rangle_{AB}$  delivers perfect rotationally invariant *exclusive correlations*, i.e. the probability of obtaining the same result  $|\nu\rangle$  of arbitrary quantum measurements of each qubit is equal to zero irrespective of the state  $|\nu\rangle$ :  $\text{Tr}_{AB}[(|\nu\rangle_A\langle\nu|_A \otimes |\nu\rangle_B\langle\nu|_B)|\Psi^-\rangle_{AB}\langle\Psi^-|_{AB}] = 0$ . As a consequence, the antisymmetric state is invariant under applying the same unitary transformation  $U$  to each qubit, i.e.  $|\tilde{i}\rangle = U|i\rangle$ :  $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|\tilde{0}\rangle_A|\tilde{1}\rangle_B - |\tilde{1}\rangle_A|\tilde{0}\rangle_B) = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$ .

The Bell states are a rather simple example of entanglement, nevertheless, they are widely used in both theoretical and experimental work. For a number of applications it is useful to consider possible generalizations of the Bell states. A significant amount of work in this direction is concentrated on the investigation of entanglement between several qubits, as this approach is very natural for certain applications such as quantum computation [8]. For other situations (e.g. quantum cryptography), it would be more natural to consider entanglement between two multidimensional systems, or *two-qudit* entanglement [9]–[12].

There are several directions towards the generalization of Bell states and Bell basis to higher dimensions. The most obvious one is to consider a state  $|\phi\rangle_{AB} = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} |k\rangle_A |k\rangle_B$  as a straightforward generalization of the symmetric Bell state  $|\Phi^+\rangle_{AB}$ . As far as all basis maximally entangled multidimensional bipartite states (MEMBS) can be transformed into each other by local unitary transformations, in certain situations it is enough to consider this symmetric state to demonstrate properties of multidimensional entanglement.

In other situations it is important to have a particular property of (anti)symmetry. For example, the antisymmetric singlet Bell state  $|\Psi^-\rangle_{AB}$  can be thought as the state of two particles with individual spin  $1/2$  and zero total spin. An interesting multidimensional generalization of the singlet Bell state is the state  $|\phi\rangle_{AB} = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^j (-1)^{j-m} |m\rangle_A |-m\rangle_B$ , which was mentioned as a rotationally invariant state with zero total spin for  $j = N/2$ , with arbitrary integer  $N$  (if states  $|m\rangle$  are associated with spin projections  $m$ ) [13]. Interestingly, it is either symmetric (for even  $N$ ) or antisymmetric (for odd  $N$ ). This state was further analyzed in the context of the Bell inequality [14]. The property of rotational invariance of the generalized singlet state was also investigated in the context of EPR correlations and hidden-variable theories [15].

An important generalization of the complete Bell basis was constructed for the qudit teleportation scheme [16]. The basis of MEMBS was chosen to be of the form  $|\psi\rangle_{AB}^{mn} = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} e^{i\frac{2\pi}{D}mk} |k\rangle_A |k-n\rangle_B$ . Later this special basis was associated with ‘generalized EPR states’ or ‘generalized Bell states’ and widely used in various quantum information processing schemes with two qudits such as teleportation, dense coding, cryptography.

Another interesting choice of the generalized Bell states was suggested for a set of  $2^n$  qubits—the ‘Bell gems’ [17]. An iterative method for construction of these Bell gems was chosen in such a way that it possesses either subsystem exchange symmetry or antisymmetry.

The property of (anti)symmetry of the Bell states arises in various contexts. For instance, in atomic physics the two natural classes of joint states of two spins  $1/2$  are those with the total spin 0 (represented by the antisymmetric singlet) and the total spin 1 (and its projections 1, 0,  $-1$  represented by the symmetric triplet).

An important aspect of (anti)symmetry is that the symmetric and antisymmetric subspaces are orthogonal to each other. By a single projection of an unknown state to these subspaces one can obtain unambiguous information about the state. As an example, this feature is used in the experimental determination of entanglement with a single measurement [18] by the entanglement concurrence measure [19]. Another interesting example which exploits the property of (anti)symmetry of the Bell states is the adaptive state estimation [20].

The purpose of this paper is to demonstrate a special way of constructing a basis of MEMBS, which naturally preserves all above mentioned basic properties of the standard Bell states. We construct a basis of MEMBS in such a way that it consists only of either symmetric or antisymmetric states. We prove that the antisymmetric MEMBS possess rotationally invariant exclusive correlations. This property is a generalization of rotationally invariant anticorrelations of the singlet Bell state. We also note a close connection between the generalized Pauli matrices and the generalized Bell states.

## 2. Structure of maximally entangled bipartite states

Let us start with a note that a pure state  $|\phi\rangle_{AB}$  of a bipartite system  $A+B$  is maximally entangled if and only if the Schmidt number is equal to the dimension  $D$ , or, equally, the state can be transformed by local operations to the form

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{D}} \sum_{k,l=0}^{D-1} \lambda_{kl} |k\rangle_A |l\rangle_B, \quad (1)$$

where in each row  $k$  and each column  $l$  of the matrix  $\lambda$  there is a single nonzero element  $\lambda_{kl} = e^{i\alpha_{kl}}$  with arbitrary real parameter  $\alpha_{kl}$ . Indeed, tracing a bipartite state (1) over one system we obtain an identity density matrix (normalized to the dimension  $D$ ) of another system:  $\hat{\rho}_A = \text{Tr}_B |\phi\rangle_{AB} \langle \phi|_{AB} = \hat{1}_A/D$ ,  $\hat{\rho}_B = \text{Tr}_A |\phi\rangle_{AB} \langle \phi|_{AB} = \hat{1}_B/D$ . The above mentioned symmetric state  $|\phi\rangle_{AB} = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} |k\rangle_A |k\rangle_B$  is a special case of (1), when  $\lambda_{kl} = \delta_{kl}$ , i.e.  $\lambda$  is a diagonal unit matrix.

Thus we can represent, according to (1), a pure maximally entangled state  $|\phi\rangle_{AB}$  by a  $D \times D$  matrix  $\lambda$ . This matrix representation turns out to be very convenient for the analysis of the properties of entangled states. The symmetry of the state  $|\phi\rangle_{AB}$  with respect to permutation of systems  $A$  and  $B$  is equal to the symmetry of the corresponding matrix  $\lambda$  with respect to its transposition: a symmetric matrix corresponds to a symmetric state, while

an antisymmetric matrix relates to an antisymmetric state. For example, matrix representation  $\{\lambda^{\Phi^+}, \lambda^{\Phi^-}, \lambda^{\Psi^+}, \lambda^{\Psi^-}\}$  of the Bell states  $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$  is the following set of matrices:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \quad (2)$$

In order to form a basis of two-qudit states it is sufficient to form a basis of  $D \times D$  matrices, i.e. to define a set  $\{\lambda\}$  of  $D^2$  linearly independent matrices. The question of forming a basis has no unique solution, because we can fix phases  $\alpha_{kl}$  in (1) quite arbitrarily to satisfy the linear independence of all matrices. Generally speaking, a basis of MEMBS can be associated with a basis of unitary transformations. Here, we mention two particularly useful constructions of unitary bases—nice error bases [21] and ‘shift-and-multiply’ bases [22]. A complete classification of unitary bases for small dimensions was investigated in [23].

Concerning the generalized Bell states, in [16] a basis of MEMBS was chosen of the form

$$|\psi\rangle_{AB}^{mn} = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} e^{i\frac{2\pi}{D}mk} |k\rangle_A |k \ominus n\rangle_B, \quad (3)$$

where  $k \ominus n \equiv (k - n) \bmod D$ . In terms of matrix representation (1), states (3) correspond to a set  $\{\lambda^{mn}\}$  with matrix elements  $\lambda_{kl}^{mn} = e^{i(2\pi/D)mk} \delta_{l, k \ominus n}$ , where upper indices  $m, n$  enumerate states in the basis  $\{|\psi\rangle_{AB}^{mn}\}$  and lower indices  $k, l$  enumerate terms of direct product  $|k\rangle_A |l\rangle_B$  in each state  $|\psi\rangle_{AB}^{mn} = \frac{1}{\sqrt{D}} \sum_{k,l=0}^{D-1} \lambda_{kl}^{mn} |k\rangle_A |l\rangle_B$ . For  $D = 2$  this basis actually leads to the standard Bell states with matrices (2), but in general the scheme (3) produces neither symmetric nor antisymmetric states, for instance in the case  $D = 4$  the scheme (3) produces two antisymmetric states, six symmetric ones, whereas the remaining 8 states are of neither symmetric nor antisymmetric form. Thus the states (3) are not the most straightforward generalization of the Bell states.

Fortunately, among an infinite variety of all possible ways to define a basis of MEMBS we can choose a special one, when MEMBS are of only either symmetric or antisymmetric form, just like Bell states for  $D = 2$ . To show the explicit structure of these MEMBS let us choose the dimension of each qudit equal to power of 2 ( $D = 2^d$ ) and follow the notation:  $\lambda(d)$  is a  $2^d \times 2^d$  matrix representing a pure maximally entangled state according to (1) and  $\{\lambda(d)\}$  is a basis of these matrices. The choice of the dimension  $D = 2^d$  allows us to represent each system as a collection of qubits. However, the following idea with proper modifications can be also exploited in the case of other dimensions.

Note the number of basis states in Hilbert space with the dimension  $D = 2^{d+1}$  is four times greater than with the dimension  $D = 2^d$ . Let  $\{\lambda(d)\}$  be a basis consisting of only symmetric and antisymmetric matrices  $\lambda(d)$ . To construct a four-times-larger basis  $\{\lambda(d+1)\}$ , we associate each matrix  $\lambda(d)$  with four matrices  $\lambda(d+1)$  in the following way:  $\{\lambda^{\Phi^+} \otimes \lambda(d), \lambda^{\Phi^-} \otimes \lambda(d), \lambda^{\Psi^+} \otimes \lambda(d), \lambda^{\Psi^-} \otimes \lambda(d)\}$ , where the symbol  $\otimes$  means the tensor product of the matrices. In other words, matrices  $\lambda(d+1)$  have the block structure

$$\begin{pmatrix} \lambda(d) & \mathcal{O} \\ \mathcal{O} & \lambda(d) \end{pmatrix}, \begin{pmatrix} \lambda(d) & \mathcal{O} \\ \mathcal{O} & -\lambda(d) \end{pmatrix}, \begin{pmatrix} \mathcal{O} & \lambda(d) \\ \lambda(d) & \mathcal{O} \end{pmatrix}, \begin{pmatrix} \mathcal{O} & \lambda(d) \\ -\lambda(d) & \mathcal{O} \end{pmatrix}, \quad (4)$$

where  $\mathcal{O}$  is a  $2^d \times 2^d$  matrix (a matrix of the same size as the matrix  $\lambda(d)$ ), consisting of only zero elements.

As far as each of the subsystems can be thought of as a collection of qubits, one can rewrite this recursive scheme in the explicit nonrecursive form:

$$\lambda(d)_{i_1 i_2 \dots i_d} = \bigotimes_{j=1}^d \lambda(1)_{i_j}, \quad (5)$$

where  $\{\lambda(1)\}$  are the Bell state matrices numbered by a subscript index  $i_j = 1 \dots 4$  and  $\{\lambda(d)\}$  are the generalized Bell state matrices numbered by a composite subscript index  $i_1 i_2 \dots i_d = 1 \dots 4^d$ .

If the matrix  $\lambda(d)$  is symmetric, then the first three matrices in (4) are also symmetric and the last one is antisymmetric, and vice versa, if  $\lambda(d)$  is antisymmetric, then the first three matrices in (4) are antisymmetric and the last is symmetric. Thus all elements in  $\{\lambda(d+1)\}$  are of either symmetric or antisymmetric form. From the construction we see that all matrices  $\{\lambda(d+1)\}$  are linearly independent and form a complete basis of  $2^{d+1} \times 2^{d+1}$  matrices. The basis of entangled states (1) formed by the set  $\{\lambda(d+1)\}$  is a complete orthonormal basis in the  $2^{d+1} \times 2^{d+1}$  Hilbert space.

The total numbers of symmetric ( $N_s$ ) and antisymmetric ( $N_a$ ) states, formed by the scheme (4), are given by the following recurrence relations:

$$N_s(d+1) = 3N_s(d) + N_a(d), \quad N_a(d+1) = 3N_a(d) + N_s(d). \quad (6)$$

We can start the scheme (4) from the smallest case  $d = 0$  ( $D = 2^0 = 1$ ). Here, we have a trivial  $1 \times 1$  matrix, consisting of a single element 1. It is symmetric, thus the initial conditions for (6) are  $N_s(0) = 1$ ,  $N_a(0) = 0$ . Taking it into account, the non-recurrence solution of (6) is

$$N_s = D(D+1)/2, \quad N_a = D(D-1)/2, \quad (7)$$

which are the dimensions of symmetric and antisymmetric subspaces of a bipartite Hilbert space.

One can calculate the numbers  $N_s$  and  $N_a$  in another way. A state is antisymmetric if, in its tensor product representation (5), the number of antisymmetric Bell state matrices is odd, otherwise it is symmetric. Therefore the total number of symmetric states is given by  $N_s = \sum_k C(d, k) 3^{d-k}$ , where we sum over even numbers  $0 \leq k \leq d$ , and  $C(d, k) = d!/(k!(d-k)!)$  is the binomial coefficient. Respectively, the number of antisymmetric states is  $N_a = \sum_k C(d, k) 3^{d-k}$ , where we sum over odd numbers  $1 \leq k \leq d$ . By direct computation one obtains the same values as (7).

In the case  $d = 1$  ( $D = 2^1 = 2$ ) we have the four Bell states (2) obeying (4) with the number of symmetric and antisymmetric states  $N_s = 3$ ,  $N_a = 1$ . The next case  $d = 2$  ( $D = 2^2 = 4$ ) gives us ten symmetric pairs of quarts ( $N_s = 10$ ) and six antisymmetric ( $N_a = 6$ ). In the limit of infinite-dimensional states the number of symmetric states tends to the number of antisymmetric states, or more accurately  $\lim_{D \rightarrow \infty} N_a/N_s = \lim_{D \rightarrow \infty} (D-1)/(D+1) = 1$ .

### 3. Rotational invariance of antisymmetric MEMBS

Properties of the antisymmetric MEMBS, particularly those formed by the scheme (4), are similar to the properties of the antisymmetric Bell state, namely, they deliver *perfect rotationally invariant exclusive correlations*. By exclusive correlations we mean that any two identical measurements on both subsystems never give the same result, i.e. the case of two identical

results is excluded. For the two-dimensional subsystems exclusive correlations simply mean correlations of the orthogonal states, or anticorrelations.

To prove exclusive correlations property of antisymmetric MEMBS, let us calculate the probability  $P_{AB}(\alpha, \beta)$  of measuring a state  $\alpha$  of system  $A$  and a state  $\beta$  of system  $B$ . It is equal to the mean value of a joint projector  $\hat{E}_A(\alpha) \otimes \hat{E}_B(\beta)$ , where  $\hat{E}(\alpha) = |\alpha\rangle\langle\alpha|$ ,  $|\alpha\rangle = \sum_{k=0}^{D-1} \alpha_k |k\rangle$  and  $\hat{E}(\beta) = |\beta\rangle\langle\beta|$ ,  $|\beta\rangle = \sum_{l=0}^{D-1} \beta_l |l\rangle$ :

$$P_{AB}(\alpha, \beta) = \text{Tr}_{AB} \left[ \left( \hat{E}_A(\alpha) \otimes \hat{E}_B(\beta) \right) |\phi\rangle_{AB} \langle\phi|_{AB} \right] = \frac{1}{D} \left| \sum_{k,l=0}^{D-1} \lambda_{kl}^* \alpha_k \beta_l \right|^2. \quad (8)$$

The probability of obtaining the same result  $\alpha = \beta$  in both systems  $A$  and  $B$  is given by (8) rewritten in a form

$$P_{AB}(\alpha, \alpha) = \text{Tr} \left[ \left( \hat{E}_A(\alpha) \otimes \hat{E}_B(\alpha) \right) |\phi\rangle_{AB} \langle\phi|_{AB} \right] = \frac{1}{D} \left| \sum_{k=0}^{D-1} \lambda_{kk}^* \alpha_k^2 + \sum_{k>l=0}^{D-1} \alpha_k \alpha_l (\lambda_{kl}^* + \lambda_{lk}^*) \right|^2, \quad (9)$$

which is constant with respect to  $\alpha$  (namely, it is equal to zero) if and only if  $\lambda_{kl} = -\lambda_{lk}$ , i.e. if  $\lambda$  is an antisymmetric matrix. This means that the results of the same measurement of systems  $A$  and  $B$  never coincide, or, more accurately, the probability (9) of measuring the same result is equal to zero regardless of the measurement. Thus, we have proved *perfect rotationally invariant exclusive correlations* of antisymmetric states.

Here, we note that any antisymmetric state demonstrates rotationally invariant exclusive correlations, not only those formed by the scheme (4). The state must be not necessarily maximally entangled. But the dimension  $D$  of the Hilbert space of each system  $A$  and  $B$  must be even, because there are no antisymmetric matrices with odd rank  $D$ . Indeed, all nonzero elements must be pairwise ( $\lambda_{lk} = -\lambda_{kl}$ ), so the total number of nonzero elements is even, thus the dimension  $D$  must also be even.

The property of rotationally invariant exclusive correlations (i.e. non-coincident results) can be used in various quantum information processing schemes. For qubits, exclusive correlation means that identical measurements on both subsystems always yield orthogonal results. This is a direct consequence of the two-dimensional Hilbert space. However, in higher dimensions this is not as simple as for qubits.

#### 4. Relation to the Pauli matrices

We note that the matrix representation of the Bell states (2) is very similar to the Pauli matrices (we follow the notation where  $\sigma_0 = \hat{1}$ )

$$\{\hat{1}, \vec{\sigma}\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \quad (10)$$

The only antisymmetric Pauli matrix  $\sigma_2$  is multiplied by  $i$  with respect to the antisymmetric Bell matrix  $\lambda^{\Psi^-}$  (2). If in the scheme (4) we replace the matrix  $\lambda^{\Psi^-}$  with the Pauli matrix  $\sigma_2$  and start from  $d = 0$  with  $\lambda(0) = 1$ , we obtain exactly the set of Pauli matrices (10) for  $d = 1$ . In terms of block elements, all matrices  $\lambda(d+1)$  have a block structure very similar to (4):

$$\begin{pmatrix} \lambda(d) & \mathcal{O} \\ \mathcal{O} & \lambda(d) \end{pmatrix}, \begin{pmatrix} \mathcal{O} & \lambda(d) \\ \lambda(d) & \mathcal{O} \end{pmatrix}, \begin{pmatrix} \mathcal{O} & i\lambda(d) \\ -i\lambda(d) & \mathcal{O} \end{pmatrix}, \begin{pmatrix} \lambda(d) & \mathcal{O} \\ \mathcal{O} & -\lambda(d) \end{pmatrix}. \quad (11)$$



All properties of the scheme (11) are the same as those mentioned for the scheme (4), e.g. the number of symmetric and antisymmetric states, the rotational invariance of antisymmetric states, etc.

Matrices formed by the recursive scheme (11) can be rewritten in a non-recursive way similarly to (5):

$$\lambda(d)_{i_1 i_2 \dots i_d} = \bigotimes_{j=1}^d \sigma_{i_j}, \quad (12)$$

where  $\{\sigma_{i_j}\}$  are the standard Pauli matrices numbered by a subscript index  $i_j = 0 \dots 3$  and  $\{\lambda(d)\}$  are the generalized Bell state matrices numbered by a composite subscript index  $i_1 i_2 \dots i_d = 1 \dots 4^d$ .

Here, we can see that these matrices corresponding to the generalized Bell states are similar to the matrices of the generalized Pauli group [5], which appear particularly in quantum error correction [24]. The explanation of this fact is our choice of the matrix dimension. We consider a case when the dimension of each qudit is equal to a power of 2. In other words, the qudit can be represented by a collection of qubits.

The matrices formed by the scheme (12) preserve the following properties of the  $2 \times 2$  Pauli matrices:

1. unitarity ( $\lambda^* = \lambda^{-1}$ );
2. hermiticity ( $\lambda^* = \lambda^T$ );
3. trace zero, except the first unit matrix;
4. all matrices are linearly independent and form a complete basis in space of  $D \times D$  matrices;
5. the determinant of all matrices is equal to 1, except three  $2 \times 2$  Pauli matrices  $\sigma_{1,2,3}$ , whose determinant is equal to  $-1$ ; and
6. multiplication of any two matrices from the set  $\{\lambda(d)\}$  is a matrix which belongs to the same set apart from a certain coefficient ( $\pm i, \pm 1$ ).

The last property can be more precisely described by the structure of commutators  $[\lambda_k, \lambda_l] \equiv \lambda_k \lambda_l - \lambda_l \lambda_k$  and anticommutators  $\{\lambda_k, \lambda_l\} \equiv \lambda_k \lambda_l + \lambda_l \lambda_k$ . For convenience we will use slightly modified expressions instead of the standard definitions, namely, we denote

$$\mathcal{C}_{kl} = [\lambda_k, \lambda_l]/(2i), \quad \mathcal{A}_{kl} = \{\lambda_k, \lambda_l\}/2. \quad (13)$$

Each element  $\mathcal{C}_{kl}$  and  $\mathcal{A}_{kl}$  is equal to either 0 or a matrix  $\lambda_j$  from the set  $\{\lambda\}$  within a coefficient  $\pm 1$ , thus we can simply use 0 or a number  $\pm j$  of matrix  $\lambda_j$  to express elements of matrices  $\mathcal{C}$  and  $\mathcal{A}$ . It is important to follow the rule for state numbering in  $\{\lambda(d+1)\}$ : take elements one after another from  $\{\lambda(d)\}$  and add, according to (11), four elements to  $\{\lambda(d+1)\}$ .

For example, in the case  $d = 1$  (i.e. the standard  $2 \times 2$  Pauli matrices) we have the following matrices  $\mathcal{C}$  and  $\mathcal{A}$ :

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 4 & 0 & -2 \\ 0 & -3 & 2 & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$



For any  $D = 2^d$  we have:

$$\mathcal{C} = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \\ C_2 & C_1 & -A_4 & A_3 \\ C_3 & A_4 & C_1 & -A_2 \\ C_4 & -A_3 & A_2 & C_1 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & -C_4 & C_3 \\ A_3 & C_4 & A_1 & -C_2 \\ A_4 & -C_3 & C_2 & A_1 \end{pmatrix}, \quad (15)$$

where  $C_i$  and  $A_i$  denote the commutator and anticommutator matrices (13) for the one step lower dimension, i.e.  $\mathcal{C}$  and  $\mathcal{A}$  refer to the dimension  $D = 2^d$ , whereas  $C$  and  $A$  refer to the dimension  $D = 2^{d-1}$ .

We can see that the standard  $2 \times 2$  Pauli matrices actually have the same structure of commutators and anticommutators (14) as the generalized ones (15), but the one step lower dimension  $2^0 = 1$  for them is just a trivial  $1 \times 1$  matrix consisting of only one element 1, whose commutator and anticommutator (13) are equal to 0 and 1, respectively. Thus in the case of the Pauli matrices, all commutators  $C_i$  are equal to zero and all anticommutators  $A_i$  are equal to the element number  $i$ .

## 5. Conclusions

We have shown how to generalize Bell states by constructing a special set of MEMBS while preserving the basic properties of the standard Bell states. This can potentially lead to the extension of the previously known schemes of quantum information processing exploiting such properties as (anti)symmetry of the Bell states. Particularly, the antisymmetric MEMBS generalize the rotational invariance of the singlet Bell state and possess perfect rotationally invariant exclusive correlations. We also noticed a close connection between the generalized Bell states and the generalized Pauli matrices.

A possible extension of the obtained results is to consider an arbitrary dimension of the Hilbert space. In our case the dimension is equal to a power of 2, because we are interested in the property of (anti)symmetry of MEMBS, which is uniquely attributed to (anti)symmetry of even-dimensional MEMBS.

In principle, a basis of MEMBS can be constructed in a regular way (similarly to (4) or (5)) for dimensions which are powers of arbitrary numbers. But in this general case the structure of a basis of MEMBS will be irregular and the properties of MEMBS will be incomplete with respect to the standard Bell states. In this sense we carried out the most accurate generalization of the Bell states from qubits to qudits, although with special dimensions.

## Acknowledgments

We thank Reinhard Werner, Nicolas Cerf and Geir Ove Myhr for useful comments, Luis Sanchez-Soto and Andrej Klimov for perspective discussions, and Tim Bartley for help with the preparation of this manuscript. DS acknowledges the Alexander von Humboldt Foundation for a fellowship.

## References

- [1] Schrödinger E 1935 *Proc. Camb. Phil. Soc.* **31** 555
- [2] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

- [3] Bohr N 1935 *Phys. Rev.* **48** 696
- [4] Braunstein S, Mann A and Revzen M 1992 *Phys. Rev. Lett.* **68** 3259
- [5] Nielsen M and Chuang I 2000 *Quantum Computation and Quantum Information* (New York: Cambridge University Press)
- [6] Peres A 2002 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)
- [7] Bouwmeester D, Ekert A and Zeilinger A (ed) 1999 *The Physics of Quantum Information* (Berlin: Springer)
- [8] Briegel H J and Raussendorf R 2001 *Phys. Rev. Lett.* **86** 910
- [9] Bechmann-Pasquinucci H and Tittel W 2000 *Phys. Rev. A* **61** 062308
- [10] Bourennane M, Karlsson A and Björk G 2001 *Phys. Rev. A* **64** 012306
- [11] Cerf N J, Bourennane M, Karlsson A and Gisin N 2002 *Phys. Rev. Lett.* **88** 127902
- [12] Sych D, Grishanin B and Zadkov V 2004 *Phys. Rev. A* **70** 052331
- [13] Landau L and Lifshitz L 1958 *Course of Theoretical Physics* vol 3 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon)
- [14] Mermin N D 1980 *Phys. Rev. D* **22** 356
- [15] Werner R F 1989 *Phys. Rev. A* **40** 4277
- [16] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
- [17] Jaeger G 2004 *Phys. Lett. A* **329** 425
- [18] Walborn S P, Ribeiro P H S, Davidovich L, Mintert F and Buchleitner A 2006 *Nature* **440** 1022
- [19] Wootters W K 1998 *Phys. Rev. Lett.* **80** 2245
- [20] Freyberger M and Happ C 2008 arXiv:0811.2360
- [21] Knill E 1996 arXiv:quant-ph/9608048
- [22] Werner R 2001 *J. Phys. A: Math. Gen.* **34** 7081
- [23] Klappenecker A and Rötteler 2002 *IEEE Trans. Inf. Theory* **48** 2392–9
- [24] Calderbank A R, Rains E M, Shor P M and Sloane N J A 1998 *IEEE Trans. Inf. Theory* **44** 1369–87