MA1201 - Assignment 2 Solutions

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1. Prove that the collection of all finite subsets of $\mathbb{N}$ is countable.

   Solution : Let N_n denote the set of all subsets of $\mathbb{N}$ with n elements. Then consider the function $f : N_n \to \mathbb{N}$ defined as,
   $$f(S) = \sum_{n \in S} 10^n \ \ \forall S \in N_n$$
   f can be shown to be injective. Since f is an injection from N_k to $\mathbb{N}$ which is a countable set, N_k is also countable. The set of all finite subsets of $\mathbb{N}$,
   $$X := \bigcup_{n \in \mathbb{N}} N_n$$
   is the countable union of countable sets, and hence is countable. $\square$

   Note : The set X must not be confused with $\mathcal{P}(\mathbb{N})$. Even though both of them may seem similar as n grows larger and larger, it's easy to see that $\mathbb{N} \in \mathcal{P}(\mathbb{N})$ but $\mathbb{N} \notin X$. Hence, $X \neq \mathcal{P}(\mathbb{N})$

2. Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

   Solution : Let us define a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as
   $$f((x, y)) = 2^x 3^y \ \ \forall (x, y) \in \mathbb{N} \times \mathbb{N}$$
   Let $(x_1, y_1), (x_2, y_2) \in \mathbb{N} \times \mathbb{N}$ such that, $f((x_1, y_1)) = f((x_2, y_2))$. Then,
   $$\begin{aligned} f((x_1, y_1)) = f((x_2, y_2)) &\Rightarrow 2^{x_1} 3^{y_1} = 2^{x_2} 3^{y_2} \\ &\Rightarrow 2^{x_1 - x_2} = 3^{y_2 - y_1} \\ &\Rightarrow x_1 = x_2 \ \ \& \ \ y_1 = y_2 \qquad \text{(Since 2 and 3 are prime numbers)} \end{aligned}$$
   Hence, f is injective. Since, f is an injection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, and $\mathbb{N}$ is countable, $\mathbb{N} \times \mathbb{N}$ is also countable. $\square$

3. Show that the set $F = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$ is countable.

   Solution : Let us define $f : \mathbb{Q} \times \mathbb{Q} \to F$ as,
   $$f((x, y)) := x + y\sqrt{2} \ \ \forall (x, y) \in \mathbb{Q} \times \mathbb{Q}$$
   Now we claim that f is a bijection. Let $(a_1, b_1), (a_2, b_2) \in \mathbb{Q} \times \mathbb{Q}$ such that $f((a_1, b_1)) = f((a_2, b_2))$. Then,
   $$\begin{aligned} f((a_1, b_1)) = f((a_2, b_2)) &\Rightarrow a_1 + b_1\sqrt{2} = a_2 + b_2\sqrt{2} \\ &\Rightarrow (a_1 - a_2) = (b_2 - b_1)\sqrt{2} \\ &\Rightarrow a_1 = a_2 \ \ \& \ \ b_1 = b_2 \qquad (\mathbb{Q} \text{ is closed under addition}) \\ &\Rightarrow (a_1, b_1) = (a_2, b_2) \end{aligned}$$
   Hence, f is injective. Now, let $z \in F$ be arbitrary. Then, $z = p + q\sqrt{2}$ for some $p, q \in \mathbb{Q}$. Then, $f((p, q)) = p + q\sqrt{2} = z$. Hence f is onto. Thus, f is bijective. Hence, $\mathbb{Q} \times \mathbb{Q} \sim F$. Since $\mathbb{Q} \times \mathbb{Q}$ is countable, F is also countable. $\square$

4. Prove that the set of all polynomials of degree ≤ 3 with integer coefficients is countable.

   Solution : Let S be the set of all polynomials of degree ≤ 3 with integer coefficients. Let us define $f : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to S$ as,
   $$f((a, b, c, d)) := P(x) \ \ \forall (a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

Where $P(x) = ax^3+bx^2+cx+d \ \forall x \in \mathbb{R}$. We claim that f is a bijection. Let $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ such that $f((a_1, b_1, c_1, d_1)) = f((a_2, b_2, c_2, d_2)) \Rightarrow a_1x^3 + b_1x^2 + c_1x + d_1 = a_2x^3 + b_2x^2 + c_2x + d_2 \ \forall x \in \mathbb{R}$. Since the polynomials are equal for all values of x, the coefficients must be equal i.e. $(a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$. Hence, $f((a_1, b_1, c_1, d_1)) = f((a_2, b_2, c_2, d_2)) \Rightarrow (a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$ thus, f is injective. Now, let $P(x) \in S$ defined as $P(x) = ax^3 + bx^2 + cx + d \forall x \in \mathbb{R}$ be arbitrary. Then let $z = (a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then, clearly, $f(z) = P(x)$. Thus, f is surjective. Hence, f is bijective. Thus, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \sim S$. Since $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ is countable, S is also countable. □

5. Prove that the sets $(0, \infty)$ and $\mathbb{R}$ are equipotent.

   Solution : Let us define $f : (0, \infty) \to \mathbb{R}$ as,

$$f(x) := \ln(x) \quad \forall x \in (0, \infty)$$

   Now, we claim that f is a bijection.

   Let $x_1, x_2 \in (0, \infty)$ such that $f(x_1) = f(x_2)$.

$$\begin{aligned} f(x_1) = f(x_2) \Rightarrow & \ln(x_1) = \ln(x_2) \\ \Rightarrow & e^{\ln(x_1)} = e^{\ln(x_2)} \\ \Rightarrow & x_1 = x_2 \end{aligned}$$

   Hence, f is injective. Let $y \in \mathbb{R}$ be arbitrary. Then let us define $x := e^y \in (0, \infty)$. Then, $\ln(x) = \ln(e^y) = y$. Hence, f is surjective. Thus, f is a bijection. This proves our claim. Since f is a bijection, the sets $(0, \infty)$ and $\mathbb{R}$ are equipotent. □

6. Prove that if A and B are countable then $A \times B$ is countable. In general for every $n \in \mathbb{N}$ if $A_1, A_2, \ldots, A_n$ are countable then $A_1 \times A_2 \times \cdots \times A_n$ is countable.

7. Prove or disprove that $A_1 \times A_2 \times \ldots$ is countable, where A_i is a countable set.

8. Prove or disprove that countable union of countable sets in countable.

9. Prove that the set $A = \{a \in \mathbb{R} : \exists a^4 + pa^3 + qa^2 + ra + s = 0\}$ is countable.

   Solution : We know that $\mathbb{Z}^4$ is countable. Let $\mathbb{Z}^4 = \{z_1, z_2, \ldots\}$. Then, for some $n \in \mathbb{N}$ let us construct A_n as,

$$A_n = \{a \in \mathbb{R} : pa^3 + qa^2 + ra + s = 0, (p, q, r, s) = z_n \in \mathbb{Z}^4\}$$

   Since, A_n can have atmost 4 elements, A_n is finite and hence countable for all $n \in \mathbb{N}$. Also,

$$A = \bigcup_{n \in \mathbb{N}} A_n$$

   Hence, A is also countable since the countable union of countable sets is countable. □

10. Let A be a finite set. Prove that the set of all sequences of elements in A of finite length is countable.

    Solution : Each finite sequence of elements in A of length $n \in \mathbb{N}$ can be written as an element of A^n. Since A is countable, A^n is also countable $\forall n \in \mathbb{N}$ Hence, the set of all sequences of elements in A of finite length,

$$X := \bigcup_{n \in \mathbb{N}} A^n$$

    is the countable union of countable sets and hence, is countable. □

11. Prove that if X is countable and $f : X \to Y$ is a surjective function then Y is countable.

    Solution : Look at the solution of 23(c).

12. Let A be an infinite set. If there is an infinite sequence in which each element of A appears at least once, then show that A is countable.

    Solution : Since there is an infinite sequence in which each element of A appears at least once, $\exists f : \mathbb{N} \to A$ such that f is surjective. Since $\mathbb{N}$ is countable, A must also be countable. □

13. Prove that the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

Solution : Let S be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We claim that S is uncountable. Let us assume to the contrary that, S is countable. Then, the set S can be written as

$$S = \{f_1, f_2, \dots\}$$

Now, let us construct $f : \mathbb{N} \to \mathbb{N}$ as,

$$f(n) = \begin{cases} 1 & \text{if } f_n(n) = 1 \\ 2 & \text{if } f_n(n) \neq 1 \end{cases}$$

Then, $f(n) \neq f_n(n)\ \ \forall n \in \mathbb{N}$, i.e. $f \neq f_n\ \ \forall n \in \mathbb{N}$ Hence, $f \notin S$. This is a contradiction since, S is the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. Hence, the set S is uncountable. □

Note : This approach of constructing an element by making all the diagonal elements unequal to prove the uncountability of a set is known as Cantor's diagonal argument. This approach will be used multiple times throughout the solutions.

14. Prove that the set of all decreasing functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.

Solution : Let the set of all decreasing functions from $\mathbb{N}$ to $\mathbb{N}$ be A and, let $f \in A$ be arbitrary. As a consequence of well-ordering principle, $\exists m, n \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$ with $k \geq n$, $f(k) = m$ i.e. the function becomes a constant function after n. We can uniquely represent this function as an ordered tuple $(f(1), f(2), f(3), \dots f(n)) \in \mathbb{N}^n$. Let A_n be the collection of all decreasing functions that become constant after some $n \in \mathbb{N}$. Then, $A_n \subseteq \mathbb{N}^n \Rightarrow A_n$ is countable. $\forall n \in \mathbb{N}$

Then, the collection of all decreasing functions from $\mathbb{N}$ to $\mathbb{N}$

$$A = \bigcup_{n \in \mathbb{N}} A_n$$

is the countable of union of countable sets and hence countable. □

15. Let X and Y be two nonempty finite sets. Then, is the set of all functions from X to Y (i) finite (ii) countably infinite (iii) uncountable? Give justifications

Solution : Let's assume that the set X has m elements and the set Y has n elements. Then, For each element in X, there are n possible elements it can map to. Hence, the total number of functions will be n^m. Hence, the set of all functions from X to Y is finite. □

16. Prove that a set is infinite iff it is bijective with a proper subset of itself.

Solution : Let's assume S is an infinite set. Let $A \subset S$ be countable. We know such a subset exists because, every infinite set has a countable subset. Then, the set A can be written as $A = \{a_1, a_2, \dots\}$. Let us split thus sets into two sets A_1 and A_2, as $A_1 = \{a_1, a_3, \dots\}$ and $A_2 = \{a_2, a_4, \dots\}$. Now, let us consider the function $f : S \to S \setminus A_1$ defined as,

$$f(x) = \begin{cases} x & \text{if } x \in S \setminus A \\ a_{2n} & \text{if } x = a_n \in A \end{cases}$$

f can be shown to be a bijection and hence, $S \sim S \setminus A_1 \subset S$

Now, let us assume that S is equipotent to a proper subset of itself. We shall show that S is ininite. Let's assume to the contrary that, S is finite. Then, Let S have $n \in \mathbb{N}$ elements. Then any proper subset of S has $m \in \mathbb{N}$ elements with $m < n$. Then, it is impossible to find a bijection between them since, for a bijection to exist between two finite sets, the number of elements in the sets must be equal. But, this is a contradiction. Hence, S must be inifinte. □

17. Prove that if $A \cap B = \phi$ then $\mathcal{P}(A \cup B) \sim \mathcal{P}(A) \times \mathcal{P}(B)$

Solution : Consider the function $f : \mathcal{P}(A \cup B) \to \mathcal{P}(A) \times \mathcal{P}(B)$ defined as,

$$f(S) = (\{x : x \in S \cap A\}, \{y : y \in S \cap B\})\ \ \forall S \in \mathcal{P}(A \cup B)$$

Let $S_1, S_2 \in \mathcal{P}(A \cup B)$ such that $S_1 \neq S_2$. WLOG we can assume that, $\exists x \in S_1$ such that $x \notin S_2$

Case 1 $x \in A$

$x \in A \Rightarrow x \in S_1 \cap A$ and $x \notin S_2 \cap A$ Hence, $\{a : a \in S_1 \cap A\} \neq \{b : b \in S_2 \cap A\} \Rightarrow f(S_1) \neq f(S_2)$

Case 2 $x \in B$

$x \in B \Rightarrow x \in S_1 \cap B$ and $x \notin S_2 \cap B$ Hence, $\{a : a \in S_1 \cap B\} \neq \{b : b \in S_2 \cap B\} \Rightarrow f(S_1) \neq f(S_2)$

Hence, f is injective. Now, let $(S_A, S_B) \in \mathcal{P}(A) \times \mathcal{P}(B)$ be arbitrary. Tthen, $S_A \subseteq A$ and $S_B \subseteq B$. Let $S = S_A \cup S_B$. Then, $S \cap A = S_A \cup S_B \cap A = S_A \cup \phi = S_A($ since $A \cap B = \phi)$. Similarly $S \cap B = S_B$. Then,

$$\begin{aligned} f(S) &= (\{x : x \in S \cap A\}, \{y : y \in S \cap B\}) \\ &= (\{x : x \in S_A\}, \{y : y \in S_B\}) \\ &= (S_A, S_B) \end{aligned}$$

Hence, f is onto. Thus f is a bijection and thus, $\mathcal{P}(A \cup B) \sim \mathcal{P}(A) \times \mathcal{P}(B)$ □

18. Is the set of all infinite sequences of 0's and 1's finite, countably infinite or uncountable? Give justification.

Solution : Let S be the set of all infinite sequences of 0's and 1's. We claim that S is uncountable. Let us assume to the contrary that, S is countable. Then, the set S can be written as,

$$S = \{a_1, a_2, a_3, \dots\}$$

Also, i^{th} element in S can be indexed as well like this,

$$a_i = a_{i1}a_{i2}a_{i3}\dots \text{ where } a_{ij} \in \{0, 1\} \ \forall i, j \in \mathbb{N}$$

Now, let us construct a_0 as ,

$$a_{0n} = \begin{cases} 1 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} \neq 0 \end{cases} \forall n \in \mathbb{N}$$

Clearly $a_0 \neq a_n \ \forall n \in \mathbb{N}$. Hence, $a_0 \notin S$. But this is a contradiction since S is the set of all infinite sequences of 0's and 1's. Hence, our assumption is wrong and the set S is uncountable. □

19. Give an example of two sets A and B such that $B \subset A$ and B is bijective with A but $B \neq A$.

Solution : Let $A := \mathbb{N}$ and $B := \{2n : n \in \mathbb{N}\}$. Then, clearly $B \subset A$ and $B \neq A$ but the function $f : A \to B$ defined as $f(a) = 2a \ \forall a \in A$ is bijective i.e. $A \sim B$.

20. Prove that if there exists an injective function from $(0, 1)$ to a set A then the set A is uncountable.

Solution : Let $f : (0, 1) \to A$ be injective. Since, $(0, 1)$ is uncountable, A is uncountable.

Now we shall prove that $(0, 1)$ is uncountable. Let us assume to the contrary that $(0, 1$ is countable. Let $X = (0, 1)$. Then, the set X can be written as, $X = \{x_1, x_2, \dots\}$. Also, since $x_i \in (0, 1)$, each x_i can be written as,

$$x_i = 0.x_{i1}x_{i2}\dots$$

where x_{ij} refers to the j^{th} digit after the decimal point in the decimal expansion of x_i. Now, let us construct x_0 as,

$$x_{0n} = \begin{cases} 1 & \text{if } x_{nn} \neq 1 \\ 2 & \text{if } x_{nn} = 1 \end{cases} \forall n \in \mathbb{N}$$

Then, $\forall n \in \mathbb{N}$ we have, $x_{0n} \neq x_{nn} \Rightarrow x_0 \in x_n \Rightarrow x \notin B$. This is a contradiction. Hence, B cannot be countable. Hence the set $(0, 1)$ is uncountable.

21. Suppose that $A \subseteq B$ then prove that
    (a) B is finite $\implies$ A is finite.
    (b) A is infinite $\implies$ B is infinite.
    (c) B is countable $\implies$ A is countable.
    (d) A is uncountable $\implies$ B is uncountable.

22. Suppose $f : A \to B$ is injective then prove that
    (a) B is finite $\implies$ A is finite.
    (b) A is infinite $\implies$ B is infinite.
    (c) B is countable $\implies$ A is countable.
    (d) A is uncountable $\implies$ B is uncountable.

23. Suppose $f : A \to B$ is surjective then prove that

    (a) A is finite $\implies$ B is finite.
    (b) B is infinite $\implies$ A is infinite.
    (c) A is countable $\implies$ B is countable.
    (d) B is uncountable $\implies$ A is uncountable.

24. Show that the sets $[0, 1]$ and $(0, 1)$ are equipotent.

    Solution : Let us define $f : (0, 1) \to (0, 1]$, $g : [0, 1) \to [0, 1]$ and $h : (0, 1] \to [0, 1)$ as,

$$f(x) = \begin{cases} \frac{1}{n-1} & \text{,if } \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ x & \text{,if } x \neq \frac{1}{n} \forall n \in \mathbb{N} \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{n-1} & \text{,if } \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ x & \text{,if } x \neq \frac{1}{n} \forall n \in \mathbb{N} \end{cases}$$

$$h(x) = 1 - x \;\; \forall x \in (0, 1]$$

Let $x, y \in (0, 1)$ such that $x \neq y$.

Case 1 $x = \frac{1}{m}$ and $y = \frac{1}{n}$ for some $m, n \in \mathbb{N}$

$$\begin{aligned} x \neq y \Rightarrow & \frac{1}{m} \neq \frac{1}{n} \\ \Rightarrow & m \neq n \\ \Rightarrow & m - 1 \neq n - 1 \\ \Rightarrow & \frac{1}{m-1} \neq \frac{1}{n-1} \\ \Rightarrow & f(x) \neq f(y) \end{aligned}$$

Case 2 $x \neq \frac{1}{m}, y \neq \frac{1}{n} \;\; \forall m, n \in \mathbb{N}$

$$f(x) = x \;\; \& \;\; f(y) = y \text{ and since, } x \neq y \text{ we have } f(x) \neq f(y)$$

Case 3 $x = \frac{1}{m}$ for some $m \in \mathbb{N}$ and $y \neq \frac{1}{n} \;\; \forall n \in \mathbb{N}$

$$f(x) = \frac{1}{n-1} \;\; \& \;\; f(y) = y \neq \frac{1}{n-1} \Rightarrow f(x) \neq f(y)$$

Hence f is one-one. Now, let $z \in (0, 1]$ be arbitrary

Case 1 $z = \frac{1}{n}$ for some $n \in \mathbb{N}$

Let us define $x := \frac{1}{n+1}$ then , $f(x) = \frac{1}{n} = z$

Case 2 $z \neq \frac{1}{n} \;\; \forall n \in \mathbb{N}$

Let us set $x = z$. Then we have $f(x) = x = z$

Hence f is surjective. Thus, f is a bijection.

Similarly, g can be shown to be a bijection. Also, h is clearly a bijection And thus, the function $F : (0, 1) \to [0, 1]$ defined as,

$$F(x) = g \circ h \circ f(x) \;\; \forall x \in (0, 1)$$

is a bijection. Hence, $(0, 1) \sim [0, 1]$ □

25. Show that the sets $[0, 1]$ and $\mathcal{P}(\mathbb{N})$ are equipotent.

    Solution : Consider the function $f : \mathcal{P}(\mathbb{N}) \to [0, 1]$ defined as,

$$f(S) = \sum_{n \in S} 2^{-n} \;\; \forall n \in \mathbb{N}$$

Since the binary representation of every real number is unique, f can be shown to be a bijection. Hence, $[0, 1] \sim \mathcal{P}(\mathbb{N})$. □

26. Are the sets $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}(\mathbb{N})$ equipotent? Give justification.

    Solution : We know from questions 24 and 25, $\mathcal{P}(\mathbb{N}) \sim [0, 1] \sim (0, 1)$. Also consider the function $f : (0, 1) \to \mathbb{R}$ defined as $f(x) = \tan(\sin^{-1}(2x - 1))$. It can be shown to be a bijection. Hence, $(0, 1) \sim \mathbb{R}$. Hence, $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$. Now, by Cantor's theorem we know that $P(\mathbb{R}) \not\sim \mathbb{R}$. Hence, $\mathcal{P}(\mathbb{N}) \not\sim \mathcal{P}(\mathbb{R})$ □