PH3102 - Quantum Mechanics

Assignment 2 Solutions

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Question 1. Consider $\hat{O}$ to be an operator defined by

$$\hat{O} = |\phi\rangle \langle\psi| \,,$$

where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

(a) Give the condition for $\hat{O}$ to be Hermitian.

(b) Calculate $\hat{O}^2$. State the condition for $\hat{O}$ to be a projection operator.

(c) Show that $\hat{O}$ can always be written in the form of $\hat{O} = \lambda P_1 P_2$, where λ is a constant and P_1 and P_2 are projection operators corresponding to the vectors $|\phi\rangle$ and $|\psi\rangle$ respectively.

Solution.

(a) For $\hat{O}$ to be Hermitian, we must have

$$\begin{aligned}
&\hat{O} = \hat{O}^\dagger \\
&\Rightarrow |\phi\rangle \langle\psi| = (|\phi\rangle \langle\psi|)^\dagger \\
&\Rightarrow |\phi\rangle \langle\psi| = |\psi\rangle \langle\phi| \\
&\Rightarrow |\phi\rangle \langle\psi|\psi\rangle = |\psi\rangle \langle\phi|\psi\rangle && \text{(Acting on } |\psi\rangle) \\
&\Rightarrow |\phi\rangle \langle\psi|\psi\rangle = \frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle} |\psi\rangle \\
&\Rightarrow |\phi\rangle = c\,|\psi\rangle
\end{aligned}$$

Where $c = \dfrac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}$. Now we have,

$$\begin{aligned}
&\hat{O} = \hat{O}^\dagger \\
&\Rightarrow |\phi\rangle \langle\psi| = (|\phi\rangle \langle\psi|)^\dagger \\
&\Rightarrow |\phi\rangle \langle\psi| = |\psi\rangle \langle\phi| \\
&\Rightarrow c^* |\psi\rangle \langle\psi| = c\,|\psi\rangle \langle\psi| && (|\phi\rangle = c\,|\psi\rangle) \\
&\Rightarrow c^* = c \\
&\Rightarrow c \in \mathbb{R}
\end{aligned}$$

Hence, for $\hat{O}$ to be Hermitian we must meet the following conditions

$$\boxed{|\phi\rangle = c\,|\psi\rangle \,,\ c \in \mathbb{R}}$$

(b) We first calculate $\hat{O}^2$.

$$\begin{aligned}
&\hat{O}^2 = |\phi\rangle\langle\psi| \cdot |\phi\rangle\langle\psi| = |\phi\rangle \langle\psi|\phi\rangle \langle\psi| = \langle\psi|\phi\rangle\, \hat{O} \\
\Rightarrow &\boxed{\hat{O}^2 = \langle\psi|\phi\rangle\, \hat{O}}
\end{aligned}$$

Hence, for $\hat{O}$ to be a projection operator, we must have $\hat{O}^2 = \hat{O}$. Thus we must have

$$\boxed{\langle\phi|\psi\rangle = 1}$$

(c) We are given that $P_1 = |\phi\rangle\langle\phi|$ and $P_2 = |\psi\rangle\langle\psi|$ Then, we have

$$\begin{aligned}
&P_1P_2 = |\phi\rangle\langle\phi| \cdot |\psi\rangle\langle\psi| \\
\Rightarrow &P_1P_2 = |\phi\rangle \langle\phi|\psi\rangle \langle\psi| \\
\Rightarrow &P_1P_2 = \langle\phi|\psi\rangle |\phi\rangle \langle\psi| \\
\Rightarrow &\frac{P_1P_2}{\langle\phi|\psi\rangle} = \hat{O} && \text{(Assuming that } \langle\phi|\psi\rangle \neq 0) \\
\Rightarrow &\hat{O} = \lambda P_1 P_2
\end{aligned}$$

Where $\lambda = \dfrac{1}{\langle\phi|\psi\rangle}$. Observe that, if $\langle\phi|\psi\rangle = 0$, $P_1P_2 = 0$. Hence, we will not be able to find a lambda such that $\hat{O} = \lambda P_1 P_2$.

Question 2. Consider a real-valued wavefunction $\psi(x)$.

(a) For this $\psi(x)$, show that the expectation value of momentum given by $\langle\hat{p}\rangle$ is zero.

(b) Now show that if $\psi(x)$ has a mean momentum given by $\langle\hat{p}\rangle$, $e^{ip_0x/\hbar}\psi(x)$ has mean momentum $\langle\hat{p}\rangle + p_0$.

Use the Dirac "bra-ket" notation to carry out the computations.

Solution.

(a) We first calculate $\langle\hat{p}\rangle$ as

$$\begin{aligned}
\langle\hat{p}\rangle &= \langle\psi|\hat{p}|\psi\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\psi|x'\rangle \langle x'|\hat{p}|x\rangle \langle x|\psi\rangle \, dx' dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x')\left(-i\hbar\delta(x'-x)\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) dx' dx && (\psi^*(x') = \psi(x')) \\
&= \int_{-\infty}^{\infty} \psi(x)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x) dx \\
&= -i\hbar\int_{-\infty}^{\infty} \psi(x)\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} dx && \text{(i)} \\
&= -i\hbar\left[\psi^2(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(x)\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} dx\right] \\
&= i\hbar\int_{-\infty}^{\infty} \psi(x)\frac{\mathrm{d}\psi(x)}{\mathrm{d}x} dx && \text{(Since } \psi(x) \text{ must vanish at } \pm\infty) \\
&= -\langle\hat{p}\rangle && \text{(using (i))}
\end{aligned}$$

Hence, we have

$$\langle\hat{p}\rangle = -\langle\hat{p}\rangle \Rightarrow \boxed{\langle\hat{p}\rangle = 0}$$

(b) Let $\phi(x) = e^{ip_0x/\hbar}\psi(x)$. Then we have,

$$\begin{aligned}
\langle\phi|\hat{p}|\phi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\phi|x'\rangle \langle x'|\hat{p}|x\rangle \langle x|\phi\rangle\, dx'dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi^*(x')\left(-i\hbar\delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x}\right)\phi(x)dx'dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ip_0x'/\hbar}\psi(x')\left(-i\hbar\delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{ip_0x/\hbar}\psi(x)dx'dx \\
&= \int_{-\infty}^{\infty} e^{-ip_0x/\hbar}\psi(x)\left(-i\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right)e^{ip_0x/\hbar}\psi(x)dx \\
&= -i\hbar\int_{-\infty}^{\infty} e^{-ip_0x/\hbar}\psi(x)\frac{\mathrm{d}}{\mathrm{d}x}e^{ip_0x/\hbar}\psi(x)dx \\
&= -i\hbar\left[\int_{-\infty}^{\infty} e^{-ip_0x/\hbar}\psi(x)e^{ip_0x/\hbar}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}dx + \int_{-\infty}^{\infty} e^{-ip_0x/\hbar}\psi(x)\frac{ip_0}{\hbar}e^{ip_0x/\hbar}\psi(x)dx\right] \\
&= -i\hbar\left[\int_{-\infty}^{\infty}\psi(x)\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}dx + \frac{ip_0}{\hbar}\int_{-\infty}^{\infty}\psi^2(x)dx\right] \\
&= -i\hbar\int_{-\infty}^{\infty}\psi(x)\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}dx + p_0 && \text{(Assuming } \psi(x) \text{ is normalized)} \\
&= \langle\hat{p}\rangle + p_0 && \text{(using (i))}
\end{aligned}$$

Hence we have,

$$\boxed{\langle\phi|\hat{p}|\phi\rangle = \langle\hat{p}\rangle + p_0}$$

.

Question 3. For the simple harmonic oscillator with the time-independent wavefunctions $\psi_n(x)$ satisfying

$$\hat{H}\psi_n(x) = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n(x),$$

consider the superposition at time $t = 0$

$$\psi(x, t = 0) = \sum_{n=0}^{\infty} c_n\psi_n(x).$$

(a) How should the coefficients be chosen so that $\psi(x, 0)$ is an eigenstate of lowering operator $\hat{a}$ with eigenvalue α (a given complex number), i.e.,

$$\hat{a}\psi(x, 0) = \alpha\psi(x, 0).$$

(b) Using the expression for $\hat{a}$, find the explicit form of the wavefunction at $\psi(x, 0)$. Ensure that $\psi(x, 0)$ is correctly normalized.

Note that eigenstates of $\hat{a}$ are referred to as "coherent states".

Solution. Let us define the kets $|n\rangle$ such that,

$$\psi_n(x) = \langle\hat{x}|n\rangle$$

We know that the raising and lowering operators are given by,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

And, we know that

$$
\begin{aligned}
\hat{a}\left|n\right\rangle &= \sqrt{n}\left|n-1\right\rangle \\
\hat{a}^\dagger\left|n\right\rangle &= \sqrt{n+1}\left|n+1\right\rangle
\end{aligned}
$$

(a) We are given that $\hat{a}\left|\psi\right\rangle = \alpha\left|\psi\right\rangle$. Where $\langle\hat{x}|\psi\rangle = \psi(x, t=0)$ For that to hold, we must have,

$$
\begin{aligned}
&\hat{a}\left|\psi\right\rangle = \sum_{n=0}^{\infty} c_n\hat{a}\left|n\right\rangle \\
\Rightarrow&\alpha\left|\psi\right\rangle = \sum_{n=1}^{\infty} c_n\sqrt{n}\left|n-1\right\rangle \\
\Rightarrow&\sum_{n=0}^{\infty}\alpha c_n\left|n\right\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\left|n\right\rangle \\
\Rightarrow&\alpha c_n = \sqrt{n+1}c_{n+1} \\
\Rightarrow&c_{n+1} = \frac{\alpha}{\sqrt{n+1}}c_n \\
\Rightarrow&c_n = \frac{\alpha}{\sqrt{n}}c_{n-1} \\
\Rightarrow&c_n = \frac{\alpha^2}{\sqrt{n(n-1)}}c_{n-2} \\
&\vdots \\
\Rightarrow&\boxed{c_n = \frac{\alpha^n}{\sqrt{n!}}c_0}
\end{aligned}
$$

Then our state $\left|\psi\right\rangle$ becomes,

$$
\boxed{\left|\psi\right\rangle = c_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle}
$$

For $\left|\psi\right\rangle$ to be normalized, we must have,

$$
\begin{aligned}
&\langle\psi|\psi\rangle = 1 \\
\Rightarrow&|c_0|^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^n\cdot\alpha^m}{\sqrt{n!\cdot m!}}\langle n|m\rangle = 1 \\
\Rightarrow&|c_0|^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha^*)^n\cdot\alpha^m}{\sqrt{n!\cdot m!}}\delta nm = 1 \\
\Rightarrow&|c_0|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!} = 1 \\
\Rightarrow&|c_0|^2e^{|\alpha|^2} = 1 \\
\Rightarrow&|c_0|^2 = e^{-|\alpha|^2} \\
\Rightarrow&\boxed{c_0 = \exp\left(i\phi - \frac{|\alpha|^2}{2}\right)}
\end{aligned}
$$

Where $\phi\in\mathbb{R}$ is a constant. Hence finally our state ψ turns out to be,

$$
\boxed{\left|\psi\right\rangle = \exp\left(i\phi - \frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle}
$$

And in position representation we have,

$$\boxed{\psi(x, t=0) = \exp\left(i\phi - \frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,\psi_n(x)}$$

(b) We are given that, $\hat{a}\psi = \alpha\psi$. Then in the position representation we have,

$$\begin{aligned}
\alpha\psi(x) &= \langle x|\hat{a}|\psi\rangle \\
&= \int_0^\infty\int_0^\infty \langle x|x'\rangle\,\langle x'|\hat{a}|x''\rangle\,\langle x''|\psi\rangle\,dx''dx' \\
&= \sqrt{\frac{m\omega}{2\hbar}}\int_0^\infty\int_0^\infty \langle x|x'\rangle\,\langle x'|\hat{x} + \frac{i}{m\omega}\hat{p}|x''\rangle\,\langle x''|\psi\rangle\,dx''dx' \\
&= \sqrt{\frac{m\omega}{2\hbar}}\int_0^\infty\int_0^\infty \delta(x-x')\left(\delta(x'-x'')x'' + \frac{i}{m\omega}\cdot(-i\hbar)\delta(x'-x'')\frac{\mathrm{d}}{\mathrm{d}x''}\right)\psi(x'')dx''dx' \\
&= \sqrt{\frac{m\omega}{2\hbar}}\int_0^\infty \delta(x-x')\left(x' + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x'}\right)\psi(x')dx' \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x)
\end{aligned}$$

Hence, we arrive at a first order differential equation in $\psi(x)$.

$$\begin{aligned}
\sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi &= \alpha\psi(x) \\
\Rightarrow \frac{\mathrm{d}\psi}{\mathrm{d}x} &= \frac{m\omega}{\hbar}\left(\sqrt{\frac{2\hbar}{m\omega}}\alpha - x\right)\psi(x) \\
\Rightarrow \boxed{\frac{\mathrm{d}\psi}{\mathrm{d}x} = \left(\sqrt{\frac{2m\omega}{\hbar}}\alpha - \frac{m\omega x}{\hbar}\right)\psi(x)}
\end{aligned}$$

Observe that, the following guess solution for $\psi(x)$ works

$$\boxed{\psi(x) = C\exp\left(\sqrt{\frac{2m\omega}{\hbar}}\,\alpha x - \frac{m\omega x^2}{2\hbar}\right)}$$

$$\begin{aligned}
\frac{\mathrm{d}\psi}{\mathrm{d}x} &= C\left(\sqrt{\frac{2m\omega}{\hbar}}\,\alpha - \frac{m\omega x}{\hbar}\right)\exp\left(\sqrt{\frac{2m\omega}{\hbar}}\,\alpha x - \frac{m\omega x^2}{2\hbar}\right) \\
&= \left(\sqrt{\frac{2m\omega}{\hbar}}\,\alpha - \frac{m\omega x}{\hbar}\right)\psi(x)
\end{aligned}$$

Now we must ensure that $\psi(x)$ is normalized. Then, we must have,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1\\
\Rightarrow&|C|^2 \int_{-\infty}^{\infty} \exp\left(\sqrt{\frac{2m\omega}{\hbar}} 2\,\mathrm{Re}(\alpha)x - \frac{m\omega x^2}{\hbar}\right) dx = 1\\
\Rightarrow&|C|^2 \int_{-\infty}^{\infty} \exp\left(-\left(\sqrt{\frac{m\omega}{\hbar}} x - \sqrt{2}\,\mathrm{Re}(\alpha)\right)^2 + 2\,\mathrm{Re}(\alpha)^2\right) dx = 1\\
\Rightarrow&|C|^2 \exp\left(2\,\mathrm{Re}(\alpha)^2\right)\left(\frac{\hbar}{m\omega}\right)^{1/2}\sqrt{\pi} = 1\\
\Rightarrow&|C|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-2\,\mathrm{Re}(\alpha)^2\right)\\
\Rightarrow&\boxed{C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(i\theta - \mathrm{Re}(\alpha)^2\right)}
\end{aligned}$$

Where $\theta \in \mathbb{R}$ is a constant. Hence, our normaized state is given by,

$$\boxed{\psi(x, t = 0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(i\theta - \mathrm{Re}(\alpha)^2 + \sqrt{\frac{2m\omega}{\hbar}}\,\alpha x - \frac{m\omega x^2}{2\hbar}\right)}$$