Solutions to Assignment 1.2

Debayan Sarkar, 22MS002 May 22, 2023

Problem 1.

(a) Let x_n be a convergent sequence of reals and there exists $m \in N$ such that $x_n \geq 0\ \forall n \geq m$. Prove that

$$\lim_{n\to\infty} x_n \geq 0$$

Ans : Let's assume to the contrary that,

$$x := \lim_{n\to\infty} x_n < 0$$

Then, for $\epsilon = -\frac{x}{2} > 0\ \exists k \in \mathbb{N}$ such that $K > max\{K(\epsilon), m\}$. Now, $\forall n \geq k$, we have $x_n \geq 0$ and

$$x - \epsilon < x_n < x + \epsilon = x - \frac{x}{2} = \frac{x}{2} < 0 \Rightarrow x_n < 0$$

which is a contradiction. Hence, $x \geq 0$ or,

$$\lim_{n\to\infty} x_n \geq 0$$

□

We are going to use this as lemma 1.a in the following proofs.

(b) Let x_n and y_n be two convergent sequences of reals and $\exists m \in \mathbb{N}$ such that $x_n \geq y_n\ \ \forall n \geq m$. Prove that

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$$

Ans : Let's define

$$x := \lim_{n\to\infty} x_n$$
$$y := \lim_{n\to\infty} y_n$$
$$z_n := x_n - y_n$$

Then we have,

$$z := \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = x - y$$

Also, $z_n \geq 0\ \forall\ n \geq m$. Using lemma 1.a on z_n we know that $z \geq 0 \Rightarrow x - y \geq 0 \Rightarrow x \geq y$ or,

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$$

□

We are going to use this as lemma 1.b for the following proofs.

(c) If x_n is a convergent sequence of Reals, and if $a \leq x_n \leq b\ \forall\ n \in \mathbb{N}$,
Prove that

$$a \leq \lim_{n\to\infty} x_n \leq b$$

Ans : As $x_n \geq a\ \ \forall n \in \mathbb{N}$, Using lemma 1.b on $\{x_n\}$ and $\{a\}$ we get,

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} a = a \tag{1}$$

Similarly, as $b \geq x_n\ \ \forall n \in \mathbb{N}$, Using lemma 1.b on $\{x_n\}$ and $\{b\}$ we get,

$$\lim_{n\to\infty} b = b \geq \lim_{n\to\infty} x_n \tag{2}$$

From (1) and (2) have,

$$a \leq \lim_{n\to\infty} x_n \leq b$$

□

Problem 2. Find the limits of the following sequences using Sandwich Theorem

(i)

$$(2n)^{\frac{1}{n}}$$

Ans : Let

$$x_n := (2n)^{\frac{1}{n}}$$
$$a_n := n^{\frac{1}{n}}$$
$$b_n := 2^{\frac{1}{n}}$$

then, $x_n = a_n \cdot b_n$. Clearly, $a_n > 1 \ \forall \ n \in \mathbb{N}$ Also,

$$\begin{aligned}\left(1+\sqrt{\frac{2}{n}}\right)^n &> 1 + n \cdot \sqrt{\frac{2}{n}} + \frac{n(n-1)}{2} \cdot \frac{2}{n} \\ &> 1 + n - 1 \\ &= n \\ \therefore \left(1+\sqrt{\frac{2}{n}}\right)^n &> n \\ \Rightarrow 1 + \sqrt{\frac{2}{n}} &> n^{\frac{1}{n}}\end{aligned}$$

Hence,

$$1 < n^{\frac{1}{n}} < 1 + \sqrt{\frac{2}{n}} \quad \forall \ n \in \mathbb{N}$$

Also,

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty} 1 + \sqrt{\frac{2}{n}} = 1$$

. Hence using sandwich theorem,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Also trivially $1 < 2^{\frac{1}{n}} < n^{\frac{1}{n}}$. Thus, by sandwich theorem,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 2^{\frac{1}{n}} = 1$$

As $x_n = a_n \cdot b_n$,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = 1$$

or,

$$\lim_{n\to\infty} (2n)^{\frac{1}{n}} = 1$$

□

(ii)

$$\frac{n^2}{n!}$$

Ans : Trivially $\frac{n^2}{n!} > 0$. Also, $n! < n^3 \ \forall n > 5$ we shall prove this by induction.
Base Step : When $n = 6$, $6! = 720 > 216 = 6^3$. i.e. the statement holds.
Induction Step : Let's assume this holds for some $k \in \mathbb{N}$ then,

$$\begin{aligned}k! > k^3 &\Rightarrow (k+1)k! > (k+1)k^3 \\ &\Rightarrow (k+1)! > k^4 + k^3 \\ &\Rightarrow (k+1)! > k^3 + 3k^2 + 3k + 1 \qquad (k \geq 7 \Rightarrow k^4 > 7k^3 = 3k^3 + 3k^3 + k^3 > 3k^2 + 3k + 1) \\ &\Rightarrow (k+1)! > (k+1)^3\end{aligned}$$

thus, the statement also holds for k + 1. By invoking the principle of induction we can say that, $n! > n^3 \ \forall \ n > 5$ Thus, $\forall n > 5$

$$0 < \frac{n^2}{n!} < \frac{n^2}{n^3} = \frac{1}{n}$$

Since,

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$$

By sandwich theorem,

$$\lim_{n\to\infty} \frac{n^2}{n!} = 0$$

□

(iii)

$$\frac{2^n}{n!}$$

Ans : Trivially $\frac{2^n}{n!} > 0$. Also, $(n-1)! > 2^n \ \forall n > 5$ we shall prove this using induction
Base step : When $n = 6$, $(6-1)! = 5! = 120 > 64 = 2^6$
Induction Step : Let's say this holds for some $k \in \mathbb{N}$. then,

$$\begin{aligned}(k-1)! > 2^k &\Rightarrow k(k-1)! > k \cdot 2^k \\ &\Rightarrow k! > 2 \cdot 2^k \\ &\Rightarrow k! > 2^{k+1} \\ &\Rightarrow ((k+1)-1)! > 2^{k+1}\end{aligned}$$

hence, it holds for $(k+1)$ as well. Invoking the principle of mathematical induction we can say that this holds for all $n > 5$.
Then, $\forall n > 5$ we have,

$$0 < \frac{2^n}{n!} < \frac{(n-1)!}{n!} = \frac{1}{n}$$

Since,

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$$

By sandwich theorem,

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0$$

□

(iv)

$$n^{\frac{1}{n^2}}$$

Ans : Clearly,

$$1 < n^{\frac{1}{n^2}} < n^{\frac{1}{n}}$$

Since

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

by sandwich theorem,

$$\lim_{n\to\infty} n^{\frac{1}{n^2}} = 1$$

□

(v)

$$(n!)^{\frac{1}{n^2}}$$

Ans : Clearly,

$$1 < n! < n^n \Rightarrow 1 < (n!)^{\frac{1}{n^2}} < (n^n)^{\frac{1}{n^2}} = n^{\frac{1}{n}}$$

Since

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

by sandwich theorem,

$$\lim_{n\to\infty} (n!)^{\frac{1}{n^2}} = 1$$

□

Problem 3.

(i) Using sandwich theorem prove that

$$\lim_{n\to\infty} (2^n + 3^n)^{\frac{1}{n}} = 3$$

Ans : Clearly,

$$3 = (0 + 3^n)^{\frac{1}{n}} < (2^n + 3^n)^{\frac{1}{n}} < (3^n + 3^n)^{\frac{1}{n}} = 3 \cdot 2^{\frac{1}{n}}$$

Also,

$$\lim_{n\to\infty} 3 = \lim_{n\to\infty} 3 \cdot 2^{\frac{1}{n}} = 3$$

Hence, by sandwich theorem,

$$\lim_{n\to\infty} (2^n + 3^n)^{\frac{1}{n}} = 3$$

$\square$

(ii) Using sandwich theorem prove that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{2k-1}{2k} = 0$$

Ans : Clearly,

$$\prod_{k=1}^{n} \frac{2k-1}{2k} > 0 \tag{1}$$

Note that,

$$\begin{aligned} ln\left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right) &= \sum_{k=1}^{n} ln\left(1 - \frac{1}{2k}\right) \\ &< \sum_{k=1}^{n} -\frac{1}{2k} && (\because ln(1+x) < x \ \ \forall x > -1) \\ &= -\frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} \end{aligned}$$

Hence,

$$\begin{aligned} ln\left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right) &< -\frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} \\ \Rightarrow \prod_{k=1}^{n} \frac{2k-1}{2k} &< e^{-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}} \end{aligned} \tag{2}$$

From (1) and (2) we have,

$$0 < \prod_{k=1}^{n} \frac{2k-1}{2k} < e^{-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}} \ \ \forall \ n \in \mathbb{N}$$

Also, as $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty$,

$$\lim_{n\to\infty} e^{-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}} = 0$$

Thus,

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} e^{-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}} = 0$$

Hence by sandwich theorem,

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{2k-1}{2k} = 0$$

$\square$

Problem 4. Let x_n be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = x$$

(a) If $0 \le x < 1$, then prove that $\lim_{n\to\infty} x_n = 0$

Ans : Let $r \in \mathbb{R}$ such that $0 < x < r < 1$ then, for $\epsilon = r - x > 0$ $\exists k \in \mathbb{N}$ such that $\forall n \ge k$ we have $\frac{x_{n+1}}{x_n} < x + \epsilon = r$. Hence, $\forall n \ge k$,

$$x_{n+1} < rx_n < x < r^2 x_{n-1} < \cdots < r^{n-k+1} x_k$$

let us define $c := \frac{x_k}{r^k}$ then, $\forall k \in \mathbb{N}$, $0 < x_n < cr^n$ since $r < 1$, $\lim_{n\to\infty} cr^n = \lim_{n\to\infty} 0 = 0$. Thus, by sandwich theorem,

$$\lim_{n\to\infty} x_n = 0$$

□

(b) If $x > 1$ prove that $\lim_{n\to\infty} x_n = \infty$

Ans : Let $r \in \mathbb{R}$ such that $x > r > 1$ then, for $\epsilon = x - r > 0$ $\exists k \in \mathbb{N}$ such that $\forall n \ge k$ we have $\frac{x_{n+1}}{x_n} > x - \epsilon = r$. Hence, $\forall n \ge k$,

$$x_{n+1} > rx_n > x > r^2 x_{n-1} > \cdots > r^{n-k+1} x_k$$

let us define $c := \frac{x_k}{r^k}$ then, $\forall k \in \mathbb{N}$, $x_n > cr^n$ since $r > 1$, $\lim_{n\to\infty} cr^n = \infty$. Thus,

$$\lim_{n\to\infty} x_n = \infty$$

□

Problem 5.

(a) Show that the following sequences diverge to ∞

(i)

$$2^n$$

Ans : Let $M > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \log_2 M$ (Archimidean Principle). Then, $\forall n \ge N$, $2^n \ge 2^N > 2^{\log_2 M} = M$. Hence,

$$\lim_{n\to\infty} 2^n = \infty$$

□

(ii)

$$a^n$$

Ans : Let $M > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \log_a M$ (Archimidean Principle). Then, $\forall n \ge N$, $a^n \ge a^N > a^{\log_a M} = M$. Hence,

$$\lim_{n\to\infty} a^n = \infty$$

□

(iii)

$$(n!)^{\frac{1}{n}}$$

Ans : We claim that $n! > n^{\frac{n}{2}} \forall n > 2$. We will prove this using induction. Clearly, $3! = 6 > \sqrt{27} = 3^{\frac{3}{2}}$ thus, the statement holds for $n = 3$. Let's say it's true for $n = k$ then,

$$\begin{aligned} k! > k^{\frac{k}{2}} &\Rightarrow (k!)^2 > k^k \\ &\Rightarrow (k+1)^2 \cdot (k!)^2 > (k+1)^2 \cdot k^k \\ &\Rightarrow ((k+1)!)^2 > (k+1)^2 \cdot k^k \end{aligned} \tag{1}$$

Note that $\forall k > 2$,

$$
\begin{aligned}
(k+1) > 3 > \left(1+\frac{1}{k}\right)^k &\Rightarrow (k+1) > \frac{(k+1)^k}{k^k} \\
&\Rightarrow k^k > (k+1)^{(k-1)} \\
&\Rightarrow (k+1)^2 \cdot k^k > (k+1)^{(k+1)} \qquad (2)
\end{aligned}
$$

Using (2) in (1) we have,

$$((k+1)!)^2 > (k+1)^{(k+1)} \Rightarrow (k+1)! > (k+1)^{\frac{(k+1)}{2}}$$

thus, the statement holds true for $k+1$. Hence, invoking the principle of induction we have

$$n! > n^{\frac{n}{2}} \Rightarrow (n!)^{\frac{1}{n}} > \sqrt{n}$$

Since, $\lim_{n\to\infty} \sqrt{n} = \infty$,

$$\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$$

□

(b) Show that the sequence $x_n = -n^2$ diverges to $-\infty$
Ans : Let $M < 0$ be given. Let $N \in \mathbb{N}$ such that $N > \sqrt{-M}$ (Archimedean property). Then, $\forall n \geq N$ we have,

$$
\begin{aligned}
n^2 \geq N^2 &\Rightarrow n^2 > -M \\
&\Rightarrow -n^2 < M \\
&\Rightarrow x_n < M
\end{aligned}
$$

Hence,

$$\lim_{n\to\infty} x_n = -\infty$$

□

A useful result to show that a sequence diverges to ∞

Lemma. *Let x_n and y_n be two real sequences such that, $x_n \geq y_n$ $\forall n \geq m$ for some $m \in \mathbb{N}$. Then,*

$$\lim_{n\to\infty} y_n = \infty \Rightarrow \lim_{n\to\infty} x_n = \infty$$

Proof. Let $M > 0$ be given. Since $\lim y_n = \infty$, $\exists N_y \in \mathbb{N}$ such that $y_n > M$ $\forall n \geq N_y$. Let us choose $N \in \mathbb{N}$ as $N > max\{N_y, m\}$. Then, $\forall n \geq N$ we have, $x_n \geq y_n > M \Rightarrow x_n > M$. Hence $\lim x_n = \infty$. This proves our claim. □

This result has been used in the solutions of 4.(b) and 5.(a)(iii)

Note that a similar result can be obtained that states, if $x_n \leq y_n$, $\forall n \geq m$, $\lim y_n = -\infty \Rightarrow \lim x_n = -\infty$.

Proof. Easy □