MA1202 Notes - Ordinary Differential Equations

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May 5, 2023

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1 First Order ODEs

1.1 Linear First Oder ODEs

1.1.1 Constant Coefficients

This equations are of the form:

$$y' = ay + b$$

In this case, we use an integrating factor $\mu(x) = e^{-ax}$ such that, after multiplying by $\mu(x)$ on both sides, the equation becomes

$$\begin{aligned} &e^{-ax}(y' - ay) = be^{-ax} \\ \Rightarrow &(ye^{-ax})' = -(\frac{be^{-ax}}{a})' \\ \Rightarrow &ye^{-ax} + \frac{be^{-ax}}{a} = C \end{aligned}$$

Hence, the solution to the given differential equation is,

$$y(x) = Ce^{ax} - \frac{b}{a}$$

1.1.2 Variable Coefficients

These equations are of the form

$$y' = a(x)y + b(x)$$

. In this case as well, we use an integrating factor $\mu(x)$ To obtain this integrating factor we must understand the motivation behind it We are looking for a $\mu(x)$ such that,

$$\mu(y' - a(x)y) = (\mu y)'$$

on simplifying we get,

$$\mu y' - a(x)y\mu = \mu y' + \mu' y \Rightarrow \frac{\mu'}{\mu} = -a(x)y \Rightarrow (ln(\mu))' = -a(x)y$$

Let's say we are solving an initial value problem where $y(x_0) = y_0$ Then,

$$\mu(x) = e^{-A(x)}$$

Where

$$A(x) = \int_{x_0}^{x} a(x)dx$$

Using this integrating factor, the solution of the initial value problem is,

$$y(x) = e^{A(x)}[y_0 + \int_{x_0}^{x} e^{-A(x)}b(x)dx]$$

1.2 Exact Differential Equations

1.2.1 Definition

The equation

$$N(t, y(t))y' + M(t, y(t)) = 0$$

is said to be exact if and only if

$$\partial_t N = \partial_y M$$

1.2.2 Poincarre Lemma

The Poincarre lemma states that, there exists a potential function $\psi(t, y(t)) = C$ such that,

$$\partial_y \psi = N$$

$$\partial_t \psi = M$$

Hence, $\psi(t, y(t)) = C$ is the solution of our exact differential equations
Example :
Find all the solutions to the equation

$$[sin(t) + t^2 e^{y(t)} - 1]y'(t) + y(t)cos(t) + 2te^{y(t)}$$

Solution : As we can see here,

$$N = sin(t) + t^2 e^{y(t)} - 1$$

and,

$$M = y(t)cos(t) + 2te^{y(t)}$$

First we check if the given equation is exact.

$$\partial_t N = cos(t) + 2te^{y(t)}$$

$$\partial_y M = cos(t) + 2te^{y(t)}$$

Since $\partial_t N = \partial_y M$, the given equation is exact. Therefore, by Poincarre lemma a function $\psi(t, y(t)) = C_1$ exists such that, $\partial_y \psi = N$ and $\partial_t \psi = M$ Hence,

$$\partial_y \psi = sin(t) + t^2 e^{y(t)} - 1$$

From this if we integrate both sides w.r.t. y, we get,

$$\psi(t, y(t)) = y(t)sin(t) + t^2 e^{y(t)} - y(t) + F(t)$$

also,

$$\partial_t \psi = y(t)cos(t) + 2te^{y(t)} + F'(t) = M = y(t)cos(t) + 2te^{y(t)}$$

therefore, $F'(x) = 0$ or $F(x) = C_2$ Hence,

$$\psi(t, y(t)) = y(t)sin(t) + t^2 e^{y(t)} - y(t) + C_2 = C_1$$

or,

$$y(t)sin(t) + t^2 e^{y(t)} - y(t) = C$$

is the solution of the given differential equation.

1.3 Method of Integrating Factor

Let us consider a first order ODE

$$N(t, y(t))y'(t) + M(t, y(t)) = 0$$

which is not exact then, we try to find an integrating factor such that, multiplying the equation by the integrating factor makes it exact. Let this integrating factor be $\mu(t)$. Multiplying we get,

$$\mu(t)N(t, y(t))y'(t) + \mu(t)M(t, y(t)) = 0$$

Let

$$\tilde{N} = \mu(t)N$$

and,

$$\tilde{M} = \mu(t)M$$

Then, for this new equation to be exact, $\partial_t \tilde{N} = \partial_y \tilde{M}$ which on simplifying yields,

$$\frac{\mu'}{\mu} = \frac{\partial_y M - \partial_t N}{N}$$

If this is a function of t only, then it can be solved just like an exact differential equation.

2 Second Order ODEs

Given functions $p, q, f : (t_1, t_2) \to \mathbb{R}$ a second order ordinary differential equation in the unknown variable $y : (t_1, t_2) \to \mathbb{R}$ is given by

$$y'' + p(t)y' + q(t)y = f(t)$$

Here $p(t)$ and $q(t)$ can be constant or variable. If $f(t) = 0$, it's called a homogeneous equation.

2.1 Linear Dependancy of Solutions

If the functions y_1 and y_2 are two solutions of the equation

$$y'' + a(t)y' + b(t) = 0$$

then $c_1y_1(t) + c_2y_2(t)$ is also a solution $\forall c_1, c_2 \in \mathbb{R}$.

2.1.1 The Wronskian

The Wronskian of two functions y_1 and y_2 is defined as the determinant,

$$W_{y_1y_2} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

If $W_{y_1y_2} = 0$, then y_1 and y_2 are said to be linearly dependant.

2.2 Homogeneous Equations

2.2.1 Constant Coefficients

The equation will be of the form

$$y'' + ay' + by = 0$$

Guessing $y = e^{rt}$ as a solution we obtain the **characteristic polynomial** of the equation to be

$$P(r) = r^2 + ar + b$$

The **characteristic equation** is

$$P(r) = 0$$

let the solutions to $P(r) = 0$ be r_+ and r_-. then,

$$r_\pm = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

Case 1 : $r_+ = r_-$

$$y(t) = (c_0 + c_1t)e^{r_+t}$$

Case 2 : $r_+ \neq r_-$

$$y(t) = c_0e^{r_+t} + c_1e^{r_-t}$$

Variable Coefficients

2.2.2 Reduction of Order Method

If $y_1(t)$ is a solution of $y'' + a(t)y' + b(t)y = 0$ then, we can assume the other solution $y_2(t)$ as

$$y_2(t) = v(t)y_1(t)$$

Substituting this value into the original equation and simplifying leads to a first order equation in w where $w = v'$.

Example

Find the other linearly independent solution of the equation

$$t^2y'' + 2ty' - 2y = 0$$

if one of the solutions is given to be $y_1(t) = t$.

Solution : Let $y_2 = vt$ then, $y_2' = v + v't$ and $y_2'' = 2v' + v''t$ By substituting these values in the original equation we get,

$$
\begin{aligned}
2t^2v' + v''t^3 + 2tv + 2v't^2 - 2vt = 0 \Rightarrow & v''t^3 + 4t^2v' = 0 \\
\Rightarrow & \frac{v''}{v'} = -\frac{4}{t} \\
\Rightarrow & v' = ct^{-4} \\
\Rightarrow & v = c_0t^{-3} + c_1
\end{aligned}
$$

$$\therefore y = \frac{c_0}{t^2} + c_1 t$$

hence, the other fundamental solution is

$$y_2 = \frac{1}{t^2}$$

2.3 Non-homogeneous Equations

These equations are of the form

$$y'' + p(t)y' + q(t)y = f(t)$$

Let $\mathcal{L}$ be an operator such that

$$\mathcal{L}(y) = y'' + +p(t)y' + q(t)y$$

Then, for homogeneous equations $\mathcal{L}(y) = 0$ and for non-homogeneous equations $\mathcal{L}(y) = f(t)$. This $\mathcal{L}$ operator is linear. Hence, if y_1 and y_2 are solutions of the homogeneous equation, then $\mathcal{L}(c_1y_1 + c_2y_2) = c_1\mathcal{L}(y_1) + c_2\mathcal{L}(y_2)$. If y_p is the solution of the equation $\mathcal{L}(y) = f(t)$, then the general solution of this equation is given by

$$y = c_1y_1 + c_2y_2 + y_p$$

If the function f (also known as the source function) can be written as $f(t) = f_1(t) + \cdots + f_n(t)$, and if there exist y_i such that $\mathcal{L}(y_i) = f_i(t)$, then $y_p = y_1 + \cdots + y_n$ satisfies equation $\mathcal{L}(y_p) = f(t)$

2.3.1 Method of Unknown Coefficients

To Solve a non-homogeneous second order ODE using the method of Unknown Coefficients, we follow the following steps :

1. Find the general solution y_h of the homogeneous equation given by $\mathcal{L}(y_h) = 0$
2. If the source function can be decomposed into multiple functions like $f = f_1 + \cdots + f_n$ where $n \geq 1$, then find y_{p_i} as solution of the equation $\mathcal{L}(y_{p_i}) = f_i(t)$ for all $1 \leq i \leq n$.
3. Given the source function f_i, guess the solution y_{p_i} based on the table given below.

$f_i(t)$	y_{p_i}
Ke^{at}	ke^{at}
Kt^m	$k_mt^m + k_{m-1}t^{m-1} + \cdots + k_0$
$Kcos(bt)$ or $Ksin(bt)$	$k_1sin(bt) + k_2cos(bt)$
$Ke^{at}sin(bt)$ or $Ke^{at}cos(bt)$	$e^{at}(k_1sin(bt) + k_2cos(bt))$
Kt^me^{at}	$e^{at}(k_mt^m + k_{m-1}t^{m-1} + \cdots + k_0)$
$Kt^mcos(bt)$ or $Kt^msin(bt)$	$(k_mt^m + \cdots + k_0)(a_1sin(bt) + a_2cos(bt))$

4. If a guess y_{p_i} happens to be a solution of $\mathcal{L}(y_{p_i}) = 0$, try $t^ky_{p_i}$ with $k \in \mathbb{N}$ sufficiently large so that $\mathcal{L}(y_{p_i}) \neq 0$.
5. Impose the equation $\mathcal{L}(y_{p_i}) = f_i(t)$ to find out the unknown coefficients $k_1, \cdots, k_n$ and then compute $y_p = y_1 + \cdots + y_n$.
6. The solution to the original problem is then given by $y = y_h + y_p$.

Example
Solve:

$$y'' - 3y' - 4y = 3e^{4t}$$

Solution: First we obtain the solution of the homogeneous equation as $y_h = c_1e^{4t} + c_2e^{-t}$
The source function $f(t) = 3e^{4t}$ is not decomposable as the sum of simpler functions. So, from the table given above we guess the solution $y_p = ke^{4t}$. But this is already a solution of the homogeneous equation. So we modify our guess to $y_p = kte^{4t}$. Now, we have to determine the value of k. $y_p' = k(e^{4t} + 4te^{4t})$ and $y_p'' = k(8e^{4t} + 16te^{4t})$. Substituting these values in the given equation we get,

$$\begin{aligned}
&k(8e^{4t} + 16te^{4t}) - 3k(e^{4t} + 4te^{4t}) - 4kte^{4t} = 3e^{4t}\\
\Rightarrow &8k + 16kt - 3k - 12kt - 4kt = 3\\
\Rightarrow &5k = 3\\
\Rightarrow &k = \frac{3}{5}
\end{aligned}$$

$$\therefore y_p = \frac{3}{5}te^{4t}$$

And hence, the general solution of the equation can be written as

$$y = c_1e^{4t} + c_2e^{-t} + \frac{3}{5}te^{4t}$$

2.3.2 Method of Variation of parameters

Let y_1 and y_2 be the fundamental solutions of the homogeneous equation $\mathcal{L}(y) = 0$. Then $y_p = u_1y_1 + u_2y_2$ is a particular solution of the non-homogeneous equation $\mathcal{L}(y) = f(t)$ where u_1 and u_2 are defined as,

$$u_1 = -\int \frac{y_2 f(t)}{W_{y_1y_2}}dt$$

$$u_2 = \int \frac{y_1 f(t)}{W_{y_1y_2}}dt$$

Example

$$y'' - 5y' + 6y = 2e^t$$

Solution: The solutions to the homogeneous equations are, $y_1 = e^{3t}$ and $y_2 = e^{2t}$
Also,

$$W_{y_1y_2} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{pmatrix} = 2e^{5t} - 3e^{5t} = -e^{5t}$$

Hence,

$$u_1 = -\int \frac{e^{2t}\cdot 2e^t}{-e^{5t}}dt = 2\int e^{-2t}dt = -e^{-2t}$$

$$u_2 = \int \frac{e^{3t}\cdot 2e^t}{-e^{5t}}dt = -2\int e^{-t}dt = 2e^{-t}$$

Therefore, the particular solution is $y_p = (-e^{-2t})(e^{3t})+(2e^{-t})(e^{2t}) = e^t$ and the general solution of the equation is,

$$y = c_1e^{3t} + c_2e^{2t} + e^t$$

3 Power Series Solutions

Let the $P, Q, R : (x_1, x_2) \to \mathbb{R}$ be functions and a second order differential equation in an unknown variable $y : (x_1, x_2) \to \mathbb{R}$ be defined as

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

Let $x_0 \in (x_1, x_2)$. If $P(x_0) = 0$, we say that x_0 is a **singular point**. If $P(x_0) \neq 0$, we say that x_0 is a **regular point**. If we have x_0 as a regular point, we can substitute $y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ as a solution, and compute the power series coefficients, to arrive at a solution of the differential equation.

3.1 Examples of power series solutions

Let's consider a few examples.
Example 1.

$$y'' + y = 0$$

Solution: For this equation, $P(x) = 1 \forall x \in \mathbb{R}$. So, for convenience we will choose $x_0 = 0$ for the solution. So, let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Substituting the values of y and y'' in $y'' + y = 0$ we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \\
\Rightarrow &\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + a_n]x^n = 0 \\
\Rightarrow &(n+1)(n+1)a_{n+2} + a_n = 0
\end{aligned}$$

For $n = 2k$,

$$\begin{aligned}
&(2k+2)(2k+1)a_{2k+2} + a_{2k} = 0 \\
\Rightarrow &a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+1)} \\
\Rightarrow &a_{2k} = (-1)^k \frac{a_0}{(2k)!}
\end{aligned}$$

A similar result can be obtained for $n = 2k+1$

$$a_{2k+1} = (-1)^k \frac{a_1}{(2k+1)!}$$

Then, the obtained solution can be written as,

$$y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) = a_0 \cos x + a_1 \sin x$$

Example 2.

$$(x^2 + 1)y'' - 4xy' + 6y = 0$$

Solution: Again, like the previous example $P(x) = x^2 + 1 \neq 0 \ \forall x \in \mathbb{R}$. So, again we will choose to expand around $x_0 = 0$ for convenience. In this example we will compute each term in the equation as that is an easier way (for me) of dealing with the summations in power series

$$6y = \sum_{n=0}^{\infty} 6a_n x^n$$

$$4xy' = \sum_{n=1}^{\infty} 4x \cdot n a_n x^{n-1} = \sum_{n=0}^{\infty} 4n a_n x^n$$

$$(x^2+1)y'' = (x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Substituting these values in $(x^2+1)y'' - 4xy' + 6y = 0$ we get,

$$\begin{aligned}
&\sum_{n=0}^{\infty} [n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n = 0]x^n = 0 \\
\Rightarrow &(n(n-1) - 4n + 6)a_n + (n+2)(n+1)a_{n+2} = 0 \\
\Rightarrow &(n-2)(n-3)a_n + (n+2)(n+1)a_{n+2} = 0
\end{aligned}$$

For $n = 0$,

$$6a_0 + 2a_2 = 0 \Rightarrow a_2 = -3a_0$$

For $n = 1$

$$2a_1 + 6a_3 = 0 \Rightarrow a_3 = -\frac{a_1}{3}$$

For $n = 2$

$$0 \cdot a_2 + 12a_4 = 0 \Rightarrow a_4 = 0 \Rightarrow a_{2k} = 0 \ \ \forall k \geq 2$$

For $n = 3$,

$$0 \cdot a_3 + 20a_5 = 0 \Rightarrow a_5 = 0 \Rightarrow a_{2k+1} = 0 \ \ \forall k \geq 2$$

Hence the obtained solution is,

$$y = a_0 + a_1 x - 3a_0 x^2 - \frac{a_1}{3}x^3 = a_0(1 - 3x^2) + a_1\left(x - \frac{x^3}{3}\right)$$

3.2 Cauchy-Euler Equidimensional Equations

These differential equations have the form

$$(x - x_0)^2 y'' + p_0(x - x_0)y' + q_0 y = 0$$

Note that x_0 is not a regular point, but a singular point. To find the solutions to this equation, substitute $y = (x - x_0)^r$ into the equation. After simplifying we get the Euler characteristic polynomial $P(r)$,

$$P(r) = r(r - 1) + p_0 r + q_0 = 0$$

Let the solutions to this quadratic equation in r be r_+ and r_-. Then there can be three cases for which the corresponding solutions of the differential equation have been listed below :
Case 1 They are real and distinct

$$y = c_0(x - x_0)^{r_+} + c_1(x - x_0)^{r_-}$$

Case 2 They are real and identical
Let $r_+ = r_- = r$ then,

$$y = (x - x_0)^r (c_0 + c_1 \ln(x - x_0))$$

Case 3 They are complex conjugates
Let $r_\pm = \alpha \pm i\beta$ then,

$$y = x^\alpha (c_0 \cos(\beta \ln(x - x_0)) + c_1 \sin(\beta \ln(x - x_0)))$$

Example

$$2x^2 y'' + 3xy' - 15y = 0$$

Solution: On substituting $y = x^n$ in the equation we get,

$$\begin{aligned}
&2n(n-1) + 3n - 15 = 0\\
\Rightarrow &2n^2 - 2n + 3n - 15 = 0\\
\Rightarrow &2n^2 + n - 15 = 0\\
\Rightarrow &(2n - 5)(n + 3) = 0\\
\Rightarrow &n = \frac{5}{2}, -3
\end{aligned}$$

Hence, the solution to the equation is,

$$y = c_0 x^{\frac{5}{2}} + c_1 x^{-3}$$

3.3 Equations with Regular-Singular points

Let x_0 be a singular point of the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

and let

$$f(x) = \frac{(x - x_0)Q(x)}{P(x)}$$
$$g(x) = \frac{(x - x_0)^2 R(x)}{P(x)}$$

then, x_0 is called **regular-singular** point if and only if the limits $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ are finite, and both f(x) and g(x) admit convergent Taylor series expansions around x_0.

Remark. Every equation $P(x)y'' + Q(x)y' + R(x)y = 0$ with a **regular-singular** point at x_0 is close to an Euler equation.

Proof. For $x \neq x_0$ dividing both sides by $P(x)$ we get, $y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$ multiplying it by $(x - x_0)^2$ we get,

$$(x - x_0)^2 + (x - x_0)\left[\frac{(x - x_0)Q(x)}{P(x)}\right]y' + \left[\frac{(x - x_0)^2 R(x)}{P(x)}\right]y = 0$$

Since x_0 is a regular-singular point, as $x \to x_0$, the factors $\frac{(x-x_0)Q(x)}{P(x)} \to p_0(say)$ and $\frac{(x-x_0)^2 R(x)}{P(x)} \to q_0(say)$. Then, the equation becomes

$$(x - x_0)^2 y'' + p_0(x - x_0)y' + q_0 y = 0$$

□

Example
Find the regular-singular points of the differential equation

$$(x + 2)^2(x - 1)y'' + 3(x - 1)y' + 2y = 0$$

Solution: We have two singular points, $x_0 = -2$ and $x_1 = 1$.
When $x_0 = -2$,

$$\lim_{x \to -2} \frac{3(x + 2)(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to -2} \frac{3}{x + 2} = \pm\infty$$

Hence, $x_0 = -2$ is not a regular-singular point.
When $x_1 = 1$,

$$\lim_{x \to 1} \frac{3(x - 1)(x - 1)}{(x + 2)^2(x - 1)} = \lim_{x \to 1} \frac{3(x - 1)}{x + 2} = 0$$

$$\lim_{x \to 1} \frac{2(x - 1)^2}{(x + 2)^2(x - 1)} = \lim_{x \to 1} \frac{2(x - 1)}{x + 2} = 0$$

and both $\frac{3(x-1)}{x+2}$ and $\frac{2(x-1)}{x+2}$ have covergent Taylor series around $x_1 = 1$. Hence, $x_1 = 1$ is a regular singular point of the equation.