PH2101 - Waves and Optics

Assignment 4 Solutions

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September 21, 2023

Question 1. Consider a one-dimensional oscillator of unit mass and of natural frequency ω_0 (natural frequency: frequency of undamped oscillation). The oscillator experiences a velocity dependent damping with a damping coefficient γ. Initially the oscillator was at rest $x(0) = \dot{x}(0) = 0$. The oscillator is subjected to a force given by $F_0 \cos \omega t$.

(a) Find the complete solution as a function of time.

(b) Check the case when the driving frequency is equal to the natural frequency.

Solution.

(a) The differential equation for the given system will be,

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos \omega t$$

Let $\alpha := \dfrac{\gamma}{m}$. First we solve the homogenous part of this equation by assuming, $x = Ae^{i\omega' t}$. Then, we have

$$\begin{aligned} &-\omega'^2 + i\alpha\omega' + \omega_0^2 = 0 \\ \Rightarrow &\omega'^2 - i\alpha\omega' - \omega_0^2 = 0 \\ \Rightarrow &\omega' = \frac{i\alpha \pm \sqrt{4\omega_0^2 - \alpha^2}}{2} \\ \Rightarrow &\omega' = \frac{i\alpha}{2} \pm \sqrt{\omega_0^2 - \frac{\alpha^2}{4}} \end{aligned}$$

Let $\omega_1 = \sqrt{\omega_0^2 - \dfrac{\alpha^2}{4}}$. Then, the solution to the homogemous solution is,

$$\begin{aligned} x &= Ae^{i(\frac{i\alpha}{2}+\omega_1)t} + Be^{i(\frac{i\alpha}{2}-\omega_1)t} \\ &= e^{-\frac{\alpha t}{2}}(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) \\ &= e^{-\frac{\alpha t}{2}}(C\cos\omega_1 t + D\sin\omega_1 t) \end{aligned}$$

Since the RHS is $\dfrac{F_0}{m}\cos\omega t$, we take a guess that $x(t)$ is of the form $x = Ae^{i\omega t}$. Then, putting this value in the equation we get,

$$\begin{aligned} &A[-\omega^2 - i\alpha\omega + \omega_0^2] = \frac{F_0}{m} \\ \Rightarrow &A = \frac{F_0}{m} \cdot \frac{1}{i\alpha\omega + (\omega_0^2 - \omega^2)} \\ \Rightarrow &A = \frac{F_0}{m} \cdot \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2} e^{i\phi}} \end{aligned}$$

Where $\phi = \arctan\left(\frac{\alpha\omega}{\omega_0^2 - \omega^2}\right)$ Hence, the solution is

$$\boxed{x(t) = e^{-\frac{\alpha t}{2}}\left(C\cos\omega_1 t + D\sin\omega_1 t\right) + A\cos\left(\omega t - \phi\right)}$$

Putting the initial values in the problem we get, $\boxed{C = -A\cos\phi}$ and

$$\begin{aligned}
&-\frac{\alpha}{2}C + \omega_1 D + \omega A\sin\phi = 0\\
\Rightarrow&\omega_1 D = -A\left(\omega\sin\phi + \frac{\alpha}{2}\cos\phi\right)\\
\Rightarrow&\boxed{D = -\frac{A}{\omega_1}\left(\omega\sin\phi + \frac{\alpha}{2}\cos\phi\right)}
\end{aligned}$$

This determines $x(t)$ uniquely.

(b) When driving frequency is equal to the natural frequency, we have $\omega = \omega_0$. Clearly, in this case, the amplitude of oscillation is maximum. The particular solution will be,

$$\boxed{x(t) = \frac{F_0}{m\alpha\omega}e^{i(\omega t - \frac{\pi}{2})}}$$

Since, $\phi = \arctan\frac{\alpha\omega}{\omega_0^2 - \omega^2} = \frac{\pi}{2}$ and $A = \frac{F_0}{m\alpha\omega}$. So, if we want to write the same solution, the values of C and D will be, $\boxed{C = 0}$ and $\boxed{D = -\frac{A\omega}{\omega_1}}$

Question 2. Consider how to solve the steady-state motion of a forced oscillator if the driving force is of the form $F = F_0\sin\omega t$ instead of $F_0\cos\omega t$.

Solution. If it's sin instead of cos, we solve it the exact similar way, except when solving for the particular solution, we put $-i\frac{F_0}{m}$ in the RHS, since we are working with the Sine part of the complex exponential.
Hence, after solving we obtain the particular solution to be $x(t) = A\sin\left(\omega t - \phi\right)$
After that we again proceed similarly to determine the constants C and D from the homogenous part.

Question 3. A block of mass m is connected to a spring, the other end of which is fixed. There is also a viscous damping mechanism. The following observations have been made on this system:

1. If the block is pushed horizontally with a force equal to mg, the static compression of the spring is equal to h.
2. The viscous resistive force is equal to mg if the block moves with a certain known speed u.

(a) For this complete system (including both spring and damper) write the differential equation governing horizontal oscillations of the mass in terms of m, g, h, and u. Answer the following for the case that $u = 3\sqrt{gh}$.

(b) What is the angular frequency of the damped oscillations?

(c) After what time, expressed as a multiple of $\sqrt{h/g}$, is the energy down by a factor $1/e$?

(d) What is the quality factor Q of this oscillator?

(e) This oscillator, initially in its rest position, is suddenly set into motion at $t = 0$ by a bullet of negligible mass but nonnegligible momentum traveling in the positive x direction. Find the value of the phase angle ϕ in the equation $x = Ae^{-\frac{\alpha t}{2}}\cos(\omega t - \phi)$ that describes the subsequent motion, and sketch x versus t for the first few cycles.

(f) If the oscillator is driven with a force $mg\cos\omega t$, where $\omega = \sqrt{2g/h}$, what is the amplitude of the steady-state response?

Solution.

(a) From the given information about the system we can deduce that,

$$k = \frac{mg}{h} \text{ and } \gamma = \frac{mg}{u}$$

Hence, the differential equation turns out to be as in Question 1,

$$\ddot{x} + \frac{g}{u}\dot{x} + \frac{g}{h}x = 0$$

For $u = \sqrt{3gh}$ we get,

$$\ddot{x} + \frac{1}{3}\sqrt{\frac{g}{h}}\dot{x} + \frac{g}{h}x = 0$$

(b) Let $x = Ae^{i\omega' t}$. Then, as previously done for the homogneous part in Question 1,

$$\begin{aligned}\omega' &= \frac{i}{6}\sqrt{\frac{g}{h}} \pm \sqrt{\frac{g}{h} - \frac{g}{36h}} \\ &= \frac{i}{6}\sqrt{\frac{g}{h}} \pm \sqrt{\frac{35g}{36h}}\end{aligned}$$

Let $\omega = \sqrt{\frac{35g}{36h}}$ and $\alpha = \sqrt{\frac{g}{9h}}$. Then, the solution of the equation turns out to be,

$$\boxed{x(t) = e^{-\frac{\alpha}{2}t}C\cos(\omega t - \phi)}$$

Where the angular frequency (ω) is given by,

$$\boxed{\omega = \sqrt{\frac{35g}{36h}}}$$

(c) We have

$$\begin{aligned}\dot{x} &= Ce^{-\frac{\alpha t}{2}}\cos(\omega t - \phi) \\ \Rightarrow \dot{x} &= -Ce^{-\frac{\alpha t}{2}}[\omega\sin(\omega t - \phi) + \frac{\alpha}{2}\cos(\omega t - \phi)] \\ \Rightarrow \dot{x} &= -Ce^{-\frac{\alpha t}{2}}\omega[\sin(\omega t - \phi) + \frac{\alpha}{2\omega}\cos(\omega t - \phi)] \\ \Rightarrow \dot{x} &\sim -Ce^{-\frac{\alpha t}{2}}\omega_0\sin(\omega t - \phi) \qquad (\omega_0 >> \alpha \Rightarrow \omega = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}} \sim \omega_0)\end{aligned}$$

Total energy can be written as,

$$\begin{aligned}E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{mg}{h}x^2 \\ \Rightarrow E &= \frac{mC^2g}{2h}e^{-\alpha t}\end{aligned}$$

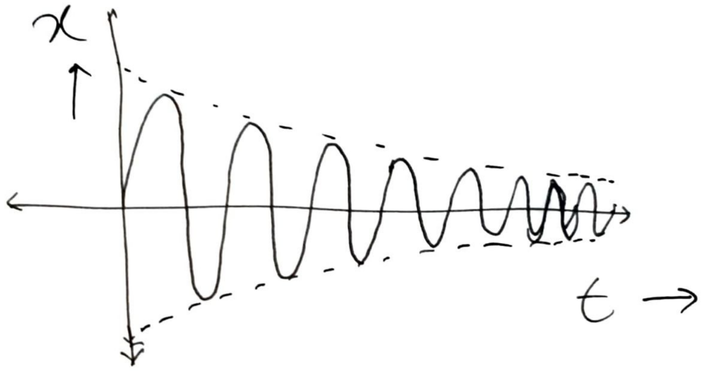
We are given that, $E(t) = \frac{1}{e}E(0) \Rightarrow e^{-\alpha t} = \frac{1}{e} \Rightarrow t = \frac{1}{\alpha} = 3\sqrt{\frac{h}{g}}$

Hence, it's 3 units of $\sqrt{\frac{h}{g}}$.

(d) $Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{g}{h}} \cdot 3\sqrt{\frac{h}{g}} = 3$

(e) According to the given conditions, we have $x = 0$ and $\dot{x} \neq 0$. Then, putting these values we get $\boxed{\phi = \frac{\pi}{2}}$ Hence, the solution is

$$\boxed{x(t) = Ae^{-\frac{\alpha t}{2}} \sin \omega t}$$



(f) The amplitude of the steady state response can be obtained as:

$$
\begin{aligned}
A &= \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \\
\Rightarrow A &= \frac{g}{\sqrt{\frac{g}{9h} \cdot \frac{2g}{h} + (\frac{g}{h} - \frac{2g}{h})^2}} \\
\Rightarrow A &= \frac{h}{\sqrt{1 + \frac{2}{9}}} \\
\Rightarrow &\boxed{A = \frac{3h}{\sqrt{11}}}
\end{aligned}
$$