

# MA1201 - Assignment 2 Solutions

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1. Prove that the collection of all finite subsets of  $\mathbb{N}$  is countable.

**Solution :** Let  $N_n$  denote the set of all subsets of  $\mathbb{N}$  with  $n$  elements. Then consider the function  $f : N_n \rightarrow \mathbb{N}$  defined as,

$$f(S) = \sum_{n \in S} 10^n \quad \forall S \in N_n$$

$f$  can be shown to be injective. Since  $f$  is an injection from  $N_k$  to  $\mathbb{N}$  which is a countable set,  $N_k$  is also countable. The set of all finite subsets of  $\mathbb{N}$ ,

$$X := \bigcup_{n \in \mathbb{N}} N_n$$

is the countable union of countable sets, and hence is countable.  $\square$

**Note :** The set  $X$  must not be confused with  $\mathcal{P}(\mathbb{N})$ . Even though both of them may seem similar as  $n$  grows larger and larger, it's easy to see that  $\mathbb{N} \in \mathcal{P}(\mathbb{N})$  but  $\mathbb{N} \notin X$ . Hence,  $X \neq \mathcal{P}(\mathbb{N})$

2. Prove that  $\mathbb{N} \times \mathbb{N}$  is countable.

**Solution :** Let us define a function  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  as

$$f((x, y)) = 2^x 3^y \quad \forall (x, y) \in \mathbb{N} \times \mathbb{N}$$

Let  $(x_1, y_1), (x_2, y_2) \in \mathbb{N} \times \mathbb{N}$  such that,  $f((x_1, y_1)) = f((x_2, y_2))$ . Then,

$$\begin{aligned} f((x_1, y_1)) = f((x_2, y_2)) &\Rightarrow 2^{x_1} 3^{y_1} = 2^{x_2} 3^{y_2} \\ &\Rightarrow 2^{x_1 - x_2} = 3^{y_2 - y_1} \\ &\Rightarrow x_1 = x_2 \quad \& \quad y_1 = y_2 \end{aligned} \quad \text{(Since 2 and 3 are prime numbers)}$$

Hence,  $f$  is injective. Since,  $f$  is an injection from  $\mathbb{N} \times \mathbb{N}$  to  $\mathbb{N}$ , and  $\mathbb{N}$  is countable,  $\mathbb{N} \times \mathbb{N}$  is also countable.  $\square$

3. Show that the set  $F = \{p + q\sqrt{2} : p, q \in \mathbb{Q}\}$  is countable.

**Solution :** Let us define  $f : \mathbb{Q} \times \mathbb{Q} \rightarrow F$  as,

$$f((x, y)) := x + y\sqrt{2} \quad \forall (x, y) \in \mathbb{Q} \times \mathbb{Q}$$

Now we claim that  $f$  is a bijection. Let  $(a_1, b_1), (a_2, b_2) \in \mathbb{Q} \times \mathbb{Q}$  such that  $f((a_1, b_1)) = f((a_2, b_2))$ . Then,

$$\begin{aligned} f((a_1, b_1)) = f((a_2, b_2)) &\Rightarrow a_1 + b_1\sqrt{2} = a_2 + b_2\sqrt{2} \\ &\Rightarrow (a_1 - a_2) = (b_2 - b_1)\sqrt{2} \\ &\Rightarrow a_1 = a_2 \quad \& \quad b_1 = b_2 \end{aligned} \quad \text{(\mathbb{Q} is closed under addition)}$$

$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

Hence,  $f$  is injective. Now, let  $z \in F$  be arbitrary. Then,  $z = p + q\sqrt{2}$  for some  $p, q \in \mathbb{Q}$ . Then,  $f((p, q)) = p + q\sqrt{2} = z$ . Hence  $f$  is onto. Thus,  $f$  is bijective. Hence,  $\mathbb{Q} \times \mathbb{Q} \sim F$ . Since  $\mathbb{Q} \times \mathbb{Q}$  is countable,  $F$  is also countable.  $\square$

4. Prove that the set of all polynomials of degree  $\leq 3$  with integer coefficients is countable.

**Solution :** Let  $S$  be the set of all polynomials of degree  $\leq 3$  with integer coefficients. Let us define  $f : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \rightarrow S$  as,

$$f((a, b, c, d)) := P(x) \quad \forall (a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

Where  $P(x) = ax^3 + bx^2 + cx + d \quad \forall x \in \mathbb{R}$ . We claim that  $f$  is a bijection. Let  $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  such that  $f((a_1, b_1, c_1, d_1)) = f((a_2, b_2, c_2, d_2)) \Rightarrow a_1x^3 + b_1x^2 + c_1x + d_1 = a_2x^3 + b_2x^2 + c_2x + d_2 \quad \forall x \in \mathbb{R}$ . Since the polynomials are equal for all values of  $x$ , the coefficients must be equal i.e.  $(a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$ . Hence,  $f((a_1, b_1, c_1, d_1)) = f((a_2, b_2, c_2, d_2)) \Rightarrow (a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2)$  thus,  $f$  is injective. Now, let  $P(x) \in S$  defined as  $P(x) = ax^3 + bx^2 + cx + d \quad \forall x \in \mathbb{R}$  be arbitrary. Then let  $z = (a, b, c, d) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ . Then, clearly,  $f(z) = P(x)$ . Thus,  $f$  is surjective. Hence,  $f$  is bijective. Thus,  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \sim S$ . Since  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$  is countable,  $S$  is also countable.  $\square$

5. Prove that the sets  $(0, \infty)$  and  $\mathbb{R}$  are equipotent.

**Solution :** Let us define  $f : (0, \infty) \rightarrow \mathbb{R}$  as,

$$f(x) := \ln(x) \quad \forall x \in (0, \infty)$$

Now, we claim that  $f$  is a bijection.

Let  $x_1, x_2 \in (0, \infty)$  such that  $f(x_1) = f(x_2)$ .

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow \ln(x_1) = \ln(x_2) \\ &\Rightarrow e^{\ln(x_1)} = e^{\ln(x_2)} \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

Hence,  $f$  is injective. Let  $y \in \mathbb{R}$  be arbitrary. Then let us define  $x := e^y \in (0, \infty)$ . Then,  $\ln(x) = \ln(e^y) = y$ . Hence,  $f$  is surjective. Thus,  $f$  is a bijection. This proves our claim. Since  $f$  is a bijection, the sets  $(0, \infty)$  and  $\mathbb{R}$  are equipotent.  $\square$

6. Prove that if  $A$  and  $B$  are countable then  $A \times B$  is countable. In general for every  $n \in \mathbb{N}$  if  $A_1, A_2, \dots, A_n$  are countable then  $A_1 \times A_2 \times \dots \times A_n$  is countable.
7. Prove or disprove that  $A_1 \times A_2 \times \dots$  is countable, where  $A_i$  is a countable set.
8. Prove or disprove that countable union of countable sets is countable.
9. Prove that the set  $A = \{a \in \mathbb{R} : \exists a^4 + pa^3 + qa^2 + ra + s = 0\}$  is countable.

**Solution :** We know that  $\mathbb{Z}^4$  is countable. Let  $\mathbb{Z}^4 = \{z_1, z_2, \dots\}$ . Then, for some  $n \in \mathbb{N}$  let us construct  $A_n$  as,

$$A_n = \{a \in \mathbb{R} : pa^3 + qa^2 + ra + s = 0, (p, q, r, s) = z_n \in \mathbb{Z}^4\}$$

Since,  $A_n$  can have atmost 4 elements,  $A_n$  is finite and hence countable for all  $n \in \mathbb{N}$ . Also,

$$A = \bigcup_{n \in \mathbb{N}} A_n$$

Hence,  $A$  is also countable since the countable union of countable sets is countable.  $\square$

10. Let  $A$  be a finite set. Prove that the set of all sequences of elements in  $A$  of finite length is countable.

**Solution :** Each finite sequence of elements in  $A$  of length  $n \in \mathbb{N}$  can be written as an element of  $A^n$ . Since  $A$  is countable,  $A^n$  is also countable  $\forall n \in \mathbb{N}$ . Hence, the set of all sequences of elements in  $A$  of finite length,

$$X := \bigcup_{n \in \mathbb{N}} A^n$$

is the countable union of countable sets and hence, is countable.  $\square$

11. Prove that if  $X$  is countable and  $f : X \rightarrow Y$  is a surjective function then  $Y$  is countable.

**Solution :** Look at the solution of 23(c).

12. Let  $A$  be an infinite set. If there is an infinite sequence in which each element of  $A$  appears at least once, then show that  $A$  is countable.

**Solution :** Since there is an infinite sequence in which each element of  $A$  appears at least once,  $\exists f : \mathbb{N} \rightarrow A$  such that  $f$  is surjective. Since  $\mathbb{N}$  is countable,  $A$  must also be countable.  $\square$

13. Prove that the set of all functions from  $\mathbb{N}$  to  $\mathbb{N}$  is uncountable.

**Solution :** Let  $S$  be the set of all functions from  $\mathbb{N}$  to  $\mathbb{N}$ . We claim that  $S$  is uncountable. Let us assume to the contrary that,  $S$  is countable. Then, the set  $S$  can be written as

$$S = \{f_1, f_2, \dots\}$$

Now, let us construct  $f : \mathbb{N} \rightarrow \mathbb{N}$  as,

$$f(n) = \begin{cases} 1 & \text{if } f_n(n) = 1 \\ 2 & \text{if } f_n(n) \neq 1 \end{cases}$$

Then,  $f(n) \neq f_n(n) \forall n \in \mathbb{N}$ , i.e.  $f \neq f_n \forall n \in \mathbb{N}$ . Hence,  $f \notin S$ . This is a contradiction since,  $S$  is the set of all functions from  $\mathbb{N}$  to  $\mathbb{N}$ . Hence, the set  $S$  is uncountable.  $\square$

**Note :** This approach of constructing an element by making all the diagonal elements unequal to prove the uncountability of a set is known as Cantor's diagonal argument. This approach will be used multiple times throughout the solutions.

14. Prove that the set of all decreasing functions from  $\mathbb{N}$  to  $\mathbb{N}$  is countable.

**Solution :** Let the set of all decreasing functions from  $\mathbb{N}$  to  $\mathbb{N}$  be  $A$  and, let  $f \in A$  be arbitrary. As a consequence of well-ordering principle,  $\exists m, n \in \mathbb{N}$  such that  $\forall k \in \mathbb{N}$  with  $k \geq n$ ,  $f(k) = m$  i.e. the function becomes a constant function after  $n$ . We can uniquely represent this function as an ordered tuple  $(f(1), f(2), f(3), \dots, f(n)) \in \mathbb{N}^n$ . Let  $A_n$  be the collection of all decreasing functions that become constant after some  $n \in \mathbb{N}$ . Then,  $A_n \subseteq \mathbb{N}^n \Rightarrow A_n$  is countable.  $\forall n \in \mathbb{N}$

Then, the collection of all decreasing functions from  $\mathbb{N}$  to  $\mathbb{N}$

$$A = \bigcup_{n \in \mathbb{N}} A_n$$

is the countable of union of countable sets and hence countable.  $\square$

15. Let  $X$  and  $Y$  be two nonempty finite sets. Then, is the set of all functions from  $X$  to  $Y$  (i) finite (ii) countably infinite (iii) uncountable? Give justifications

**Solution :** Let's assume that the set  $X$  has  $m$  elements and the set  $Y$  has  $n$  elements. Then, For each element in  $X$ , there are  $n$  possible elements it can map to. Hence, the total number of functions will be  $n^m$ . Hence, the set of all functions from  $X$  to  $Y$  is finite.  $\square$

16. Prove that a set is infinite iff it is bijective with a proper subset of itself.

**Solution :** Let's assume  $S$  is an infinite set. Let  $A \subset S$  be countable. We know such a subset exists because, every infinite set has a countable subset. Then, the set  $A$  can be written as  $A = \{a_1, a_2, \dots\}$ . Let us split this set into two sets  $A_1$  and  $A_2$ , as  $A_1 = \{a_1, a_3, \dots\}$  and  $A_2 = \{a_2, a_4, \dots\}$ . Now, let us consider the function  $f : S \rightarrow S \setminus A_1$  defined as,

$$f(x) = \begin{cases} x & \text{if } x \in S \setminus A \\ a_{2n} & \text{if } x = a_n \in A \end{cases}$$

$f$  can be shown to be a bijection and hence,  $S \sim S \setminus A_1 \subset S$

Now, let us assume that  $S$  is equipotent to a proper subset of itself. We shall show that  $S$  is infinite. Let's assume to the contrary that,  $S$  is finite. Then, Let  $S$  have  $n \in \mathbb{N}$  elements. Then any proper subset of  $S$  has  $m \in \mathbb{N}$  elements with  $m < n$ . Then, it is impossible to find a bijection between them since, for a bijection to exist between two finite sets, the number of elements in the sets must be equal. But, this is a contradiction. Hence,  $S$  must be infinite.  $\square$

17. Prove that if  $A \cap B = \emptyset$  then  $\mathcal{P}(A \cup B) \sim \mathcal{P}(A) \times \mathcal{P}(B)$

**Solution :** Consider the function  $f : \mathcal{P}(A \cup B) \rightarrow \mathcal{P}(A) \times \mathcal{P}(B)$  defined as,

$$f(S) = (\{x : x \in S \cap A\}, \{y : y \in S \cap B\}) \quad \forall S \in \mathcal{P}(A \cup B)$$

Let  $S_1, S_2 \in \mathcal{P}(A \cup B)$  such that  $S_1 \neq S_2$ . WLOG we can assume that,  $\exists x \in S_1$  such that  $x \notin S_2$

**Case 1**  $x \in A$

$x \in A \Rightarrow x \in S_1 \cap A$  and  $x \notin S_2 \cap A$  Hence,  $\{a : a \in S_1 \cap A\} \neq \{b : b \in S_2 \cap A\} \Rightarrow f(S_1) \neq f(S_2)$

**Case 2**  $x \in B$

$x \in B \Rightarrow x \in S_1 \cap B$  and  $x \notin S_2 \cap B$  Hence,  $\{a : a \in S_1 \cap B\} \neq \{b : b \in S_2 \cap B\} \Rightarrow f(S_1) \neq f(S_2)$

Hence,  $f$  is injective. Now, let  $(S_A, S_B) \in \mathcal{P}(A) \times \mathcal{P}(B)$  be arbitrary. Then,  $S_A \subseteq A$  and  $S_B \subseteq B$ . Let  $S = S_A \cup S_B$ . Then,  $S \cap A = S_A \cup S_B \cap A = S_A \cup \phi = S_A$  (since  $A \cap B = \phi$ ). Similarly  $S \cap B = S_B$ . Then,

$$\begin{aligned} f(S) &= (\{x : x \in S \cap A\}, \{y : y \in S \cap B\}) \\ &= (\{x : x \in S_A\}, \{y : y \in S_B\}) \\ &= (S_A, S_B) \end{aligned}$$

Hence,  $f$  is onto. Thus  $f$  is a bijection and thus,  $\mathcal{P}(A \cup B) \sim \mathcal{P}(A) \times \mathcal{P}(B)$   $\square$

18. Is the set of all infinite sequences of 0's and 1's finite, countably infinite or uncountable? Give justification.

**Solution :** Let  $S$  be the set of all infinite sequences of 0's and 1's. We claim that  $S$  is uncountable. Let us assume to the contrary that,  $S$  is countable. Then, the set  $S$  can be written as,

$$S = \{a_1, a_2, a_3, \dots\}$$

Also,  $i^{th}$  element in  $S$  can be indexed as well like this,

$$a_i = a_{i1}a_{i2}a_{i3} \dots \text{ where } a_{ij} \in \{0, 1\} \quad \forall i, j \in \mathbb{N}$$

Now, let us construct  $a_0$  as ,

$$a_{0n} = \begin{cases} 1 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} \neq 0 \end{cases} \quad \forall n \in \mathbb{N}$$

Clearly  $a_0 \neq a_n \quad \forall n \in \mathbb{N}$ . Hence,  $a_0 \notin S$ . But this is a contradiction since  $S$  is the set of all infinite sequences of 0's and 1's. Hence, our assumption is wrong and the set  $S$  is uncountable.  $\square$

19. Give an example of two sets  $A$  and  $B$  such that  $B \subset A$  and  $B$  is bijective with  $A$  but  $B \neq A$ .

**Solution :** Let  $A := \mathbb{N}$  and  $B := \{2n : n \in \mathbb{N}\}$ . Then, clearly  $B \subset A$  and  $B \neq A$  but the function  $f : A \rightarrow B$  defined as  $f(a) = 2a \quad \forall a \in A$  is bijective i.e.  $A \sim B$ .

20. Prove that if there exists an injective function from  $(0, 1)$  to a set  $A$  then the set  $A$  is uncountable.

**Solution :** Let  $f : (0, 1) \rightarrow A$  be injective. Since,  $(0, 1)$  is uncountable,  $A$  is uncountable.

Now we shall prove that  $(0, 1)$  is uncountable. Let us assume to the contrary that  $(0, 1)$  is countable. Let  $X = (0, 1)$ . Then, the set  $X$  can be written as,  $X = \{x_1, x_2, \dots\}$ . Also, since  $x_i \in (0, 1)$ , each  $x_i$  can be written as,

$$x_i = 0.x_{i1}x_{i2} \dots$$

where  $x_{ij}$  refers to the  $j^{th}$  digit after the decimal point in the decimal expansion of  $x_i$ . Now, let us construct  $x_0$  as,

$$x_{0n} = \begin{cases} 1 & \text{if } x_{nn} \neq 1 \\ 2 & \text{if } x_{nn} = 1 \end{cases} \quad \forall n \in \mathbb{N}$$

Then,  $\forall n \in \mathbb{N}$  we have,  $x_{0n} \neq x_{nn} \Rightarrow x_0 \in x_n \Rightarrow x_0 \notin B$ . This is a contradiction. Hence,  $B$  cannot be countable. Hence the set  $(0, 1)$  is uncountable.

21. Suppose that  $A \subseteq B$  then prove that

- (a)  $B$  is finite  $\implies A$  is finite.
- (b)  $A$  is infinite  $\implies B$  is infinite.
- (c)  $B$  is countable  $\implies A$  is countable.
- (d)  $A$  is uncountable  $\implies B$  is uncountable.

22. Suppose  $f : A \rightarrow B$  is injective then prove that

- (a)  $B$  is finite  $\implies A$  is finite.
- (b)  $A$  is infinite  $\implies B$  is infinite.
- (c)  $B$  is countable  $\implies A$  is countable.
- (d)  $A$  is uncountable  $\implies B$  is uncountable.

23. Suppose  $f : A \rightarrow B$  is surjective then prove that

- (a)  $A$  is finite  $\implies B$  is finite.
- (b)  $B$  is infinite  $\implies A$  is infinite.
- (c)  $A$  is countable  $\implies B$  is countable.
- (d)  $B$  is uncountable  $\implies A$  is uncountable.

24. Show that the sets  $[0, 1]$  and  $(0, 1)$  are equipotent.

**Solution :** Let us define  $f : (0, 1) \rightarrow (0, 1]$ ,  $g : [0, 1) \rightarrow [0, 1]$  and  $h : (0, 1] \rightarrow [0, 1)$  as,

$$f(x) = \begin{cases} \frac{1}{n-1} & , \text{if } \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ x & , \text{if } x \neq \frac{1}{n} \forall n \in \mathbb{N} \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{n-1} & , \text{if } \exists n \in \mathbb{N} \text{ such that } x = \frac{1}{n} \\ x & , \text{if } x \neq \frac{1}{n} \forall n \in \mathbb{N} \end{cases}$$

$$h(x) = 1 - x \quad \forall x \in (0, 1]$$

Let  $x, y \in (0, 1)$  such that  $x \neq y$ .

**Case 1**  $x = \frac{1}{m}$  and  $y = \frac{1}{n}$  for some  $m, n \in \mathbb{N}$

$$\begin{aligned} x \neq y &\Rightarrow \frac{1}{m} \neq \frac{1}{n} \\ &\Rightarrow m \neq n \\ &\Rightarrow m - 1 \neq n - 1 \\ &\Rightarrow \frac{1}{m-1} \neq \frac{1}{n-1} \\ &\Rightarrow f(x) \neq f(y) \end{aligned}$$

**Case 2**  $x \neq \frac{1}{m}, y \neq \frac{1}{n} \quad \forall m, n \in \mathbb{N}$

$f(x) = x$  &  $f(y) = y$  and since,  $x \neq y$  we have  $f(x) \neq f(y)$

**Case 3**  $x = \frac{1}{m}$  for some  $m \in \mathbb{N}$  and  $y \neq \frac{1}{n} \quad \forall n \in \mathbb{N}$

$$f(x) = \frac{1}{n-1} \quad \& \quad f(y) = y \neq \frac{1}{n-1} \Rightarrow f(x) \neq f(y)$$

Hence  $f$  is one-one. Now, let  $z \in (0, 1]$  be arbitrary

**Case 1**  $z = \frac{1}{n}$  for some  $n \in \mathbb{N}$

Let us define  $x := \frac{1}{n+1}$  then,  $f(x) = \frac{1}{n} = z$

**Case 2**  $z \neq \frac{1}{n} \quad \forall n \in \mathbb{N}$

Let us set  $x = z$ . Then we have  $f(x) = x = z$

Hence  $f$  is surjective. Thus,  $f$  is a bijection.

Similarly,  $g$  can be shown to be a bijection. Also,  $h$  is clearly a bijection And thus, the function  $F : (0, 1) \rightarrow [0, 1]$  defined as,

$$F(x) = g \circ h \circ f(x) \quad \forall x \in (0, 1)$$

is a bijection. Hence,  $(0, 1) \sim [0, 1]$  □

25. Show that the sets  $[0, 1]$  and  $\mathcal{P}(\mathbb{N})$  are equipotent.

**Solution :** Consider the function  $f : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$  defined as,

$$f(S) = \sum_{n \in S} 2^{-n} \quad \forall S \in \mathcal{P}(\mathbb{N})$$

Since the binary representation of every real number is unique,  $f$  can be shown to be a bijection. Hence,  $[0, 1] \sim \mathcal{P}(\mathbb{N})$ . □

26. Are the sets  $\mathcal{P}(\mathbb{R})$  and  $\mathcal{P}(\mathbb{N})$  equipotent? Give justification.

**Solution :** We know from questions 24 and 25,  $\mathcal{P}(\mathbb{N}) \sim [0, 1] \sim (0, 1)$ . Also consider the function  $f : (0, 1) \rightarrow \mathbb{R}$  defined as  $f(x) = \tan(\sin^{-1}(2x-1))$ . It can be shown to be a bijection. Hence,  $(0, 1) \sim \mathbb{R}$ . Hence,  $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$ . Now, by Cantor's theorem we know that  $\mathcal{P}(\mathbb{R}) \not\sim \mathbb{R}$ . Hence,  $\mathcal{P}(\mathbb{N}) \not\sim \mathcal{P}(\mathbb{R})$  □