

PH3102 - Quantum Mechanics

Assignment 2 Solutions

Debayan Sarkar

22MS002

August 19, 2024

Question 1. Concept of an operator. [5 Marks]

Consider \hat{O} to be an operator defined by

$$\hat{O} = |\phi\rangle \langle\psi|,$$

where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

- (a) Give the condition for \hat{O} to be Hermitian.
- (b) Calculate \hat{O}^2 . State the condition for \hat{O} to be a projection operator.
- (c) Show that \hat{O} can always be written in the form of $\hat{O} = \lambda P_1 P_2$, where λ is a constant and P_1 and P_2 are projection operators corresponding to the vectors $|\phi\rangle$ and $|\psi\rangle$ respectively.

Solution.

- (a) For \hat{O} to be Hermitian, we must have

$$\begin{aligned}\hat{O} &= \hat{O}^\dagger \\ \Rightarrow |\phi\rangle \langle\psi| &= (|\phi\rangle \langle\psi|)^\dagger \\ \Rightarrow |\phi\rangle \langle\psi| &= |\psi\rangle \langle\phi| \\ \Rightarrow |\phi\rangle \langle\psi|\psi\rangle &= |\psi\rangle \langle\phi|\psi\rangle && \text{(Acting on } |\psi\rangle\text{)} \\ \Rightarrow |\phi\rangle \langle\psi|\psi\rangle &= \frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle} |\psi\rangle \\ \Rightarrow |\phi\rangle &= c |\psi\rangle\end{aligned}$$

Where $c = \frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}$. Now we have,

$$\begin{aligned}\hat{O} &= \hat{O}^\dagger \\ \Rightarrow |\phi\rangle \langle\psi| &= (|\phi\rangle \langle\psi|)^\dagger \\ \Rightarrow |\phi\rangle \langle\psi| &= |\psi\rangle \langle\phi| \\ \Rightarrow c^* |\psi\rangle \langle\psi| &= c |\psi\rangle \langle\psi| && (|\phi\rangle = c |\psi\rangle) \\ \Rightarrow c^* &= c \\ \Rightarrow c &\in \mathbb{R}\end{aligned}$$

Hence, for \hat{O} to be Hermitian we must meet the following conditions

$$\boxed{|\phi\rangle = c |\psi\rangle, \quad c \in \mathbb{R}}$$

- (b) We first calculate \hat{O}^2 .

$$\begin{aligned}\hat{O}^2 &= |\phi\rangle \langle\psi| \cdot |\phi\rangle \langle\psi| = |\phi\rangle \langle\psi|\phi\rangle \langle\psi| = \langle\psi|\phi\rangle \hat{O} \\ \Rightarrow \hat{O}^2 &= \langle\psi|\phi\rangle \hat{O}\end{aligned}$$

Hence, for \hat{O} to be a projection operator, we must have $\hat{O}^2 = \hat{O}$. Thus we must have

$$\boxed{\langle \phi | \psi \rangle = 1}$$

(c) We are given that $P_1 = |\phi\rangle\langle\phi|$ and $P_2 = |\psi\rangle\langle\psi|$. Then, we have

$$\begin{aligned} P_1 P_2 &= |\phi\rangle\langle\phi| \cdot |\psi\rangle\langle\psi| \\ \Rightarrow P_1 P_2 &= |\phi\rangle \langle\phi|\psi\rangle \langle\psi| \\ \Rightarrow P_1 P_2 &= \langle\phi|\psi\rangle |\phi\rangle \langle\psi| \\ \Rightarrow \frac{P_1 P_2}{\langle\phi|\psi\rangle} &= \hat{O} && \text{(Assuming that } \langle\phi|\psi\rangle \neq 0) \\ \Rightarrow \hat{O} &= \lambda P_1 P_2 \end{aligned}$$

Where $\lambda = \frac{1}{\langle\phi|\psi\rangle}$. Observe that, if $\langle\phi|\psi\rangle = 0$, $P_1 P_2 = 0$. Hence, we will not be able to find a lambda such that $\hat{O} = \lambda P_1 P_2$.

Question 2. Characteristics of a real wavefunction. [5 Marks]

Consider a real-valued wavefunction $\psi(x)$.

- (a) For this $\psi(x)$, show that the expectation value of momentum given by $\langle \hat{p} \rangle$ is zero.
- (b) Now show that if $\psi(x)$ has a mean momentum given by $\langle \hat{p} \rangle$, $e^{ip_0 x/\hbar} \psi(x)$ has mean momentum $\langle \hat{p} \rangle + p_0$.

Use the Dirac “bra-ket” notation to carry out the computations.

Solution.

(a) We first calculate $\langle \hat{p} \rangle$ as

$$\begin{aligned} \langle \hat{p} \rangle &= \langle \psi | \hat{p} | \psi \rangle \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \psi | x' \rangle \langle x' | \hat{p} | x \rangle \langle x | \psi \rangle dx' dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x') \left(-i\hbar \delta(x' - x) \frac{d}{dx} \right) \psi(x) dx' dx && (\psi^*(x') = \psi(x')) \\ &= \int_{-\infty}^{\infty} \psi(x) \left(-i\hbar \frac{d}{dx} \right) \psi(x) dx \\ &= -i\hbar \int_{-\infty}^{\infty} \psi(x) \frac{d\psi(x)}{dx} dx && \text{(i)} \\ &= -i\hbar \left[\psi^2(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \psi(x) \frac{d\psi(x)}{dx} dx \right] \\ &= i\hbar \int_{-\infty}^{\infty} \psi(x) \frac{d\psi(x)}{dx} dx && \text{(Since } \psi(x) \text{ must vanish at } \pm\infty) \\ &= -\langle \hat{p} \rangle && \text{(using (i))} \end{aligned}$$

Hence, we have

$$\langle \hat{p} \rangle = -\langle \hat{p} \rangle \Rightarrow \boxed{\langle \hat{p} \rangle = 0}$$

(b) Let $\phi(x) = e^{ip_0x/\hbar}\psi(x)$. Then we have,

$$\begin{aligned}
\langle \phi | \hat{p} | \phi \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \phi | x' \rangle \langle x' | \hat{p} | x \rangle \langle x | \phi \rangle dx' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(x') \left(-i\hbar \delta(x - x') \frac{d}{dx} \right) \phi(x) dx' dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ip_0x'/\hbar} \psi^*(x') \left(-i\hbar \delta(x - x') \frac{d}{dx} \right) e^{ip_0x/\hbar} \psi(x) dx' dx \\
&= \int_{-\infty}^{\infty} e^{-ip_0x/\hbar} \psi(x) \left(-i\hbar \frac{d}{dx} \right) e^{ip_0x/\hbar} \psi(x) dx \\
&= -i\hbar \int_{-\infty}^{\infty} e^{-ip_0x/\hbar} \psi(x) \frac{d}{dx} e^{ip_0x/\hbar} \psi(x) dx \\
&= -i\hbar \left[\int_{-\infty}^{\infty} e^{-ip_0x/\hbar} \psi(x) e^{ip_0x/\hbar} \frac{d\psi(x)}{dx} dx + \int_{-\infty}^{\infty} e^{-ip_0x/\hbar} \psi(x) \frac{ip_0}{\hbar} e^{ip_0x/\hbar} \psi(x) dx \right] \\
&= -i\hbar \left[\int_{-\infty}^{\infty} \psi(x) \frac{d\psi(x)}{dx} dx + \frac{ip_0}{\hbar} \int_{-\infty}^{\infty} \psi^2(x) dx \right] \\
&= -i\hbar \int_{-\infty}^{\infty} \psi(x) \frac{d\psi(x)}{dx} dx + p_0 \quad (\text{Assuming } \psi(x) \text{ is normalized}) \\
&= \langle \hat{p} \rangle + p_0 \quad (\text{using (i)})
\end{aligned}$$

Hence we have,

$$\boxed{\langle \phi | \hat{p} | \phi \rangle = \langle \hat{p} \rangle + p_0}$$

Question 3. Coherent States. [5 Marks]

For the simple harmonic oscillator with the time-independent wavefunctions $\psi_n(x)$ satisfying

$$\hat{H}\psi_n(x) = \hbar\omega \left(n + \frac{1}{2} \right) \psi_n(x),$$

consider the superposition at time $t = 0$

$$\psi(x, t = 0) = \sum_{n=0}^{\infty} c_n \psi_n(x).$$

- (a) How should the coefficients be chosen so that $\psi(x, 0)$ is an eigenstate of lowering operator \hat{a} with eigenvalue α (a given complex number), i.e.,

$$\hat{a}\psi(x, 0) = \alpha\psi(x, 0).$$

- (b) Using the expression for \hat{a} , find the explicit form of the wavefunction at $\psi(x, 0)$. Ensure that $\psi(x, 0)$ is correctly normalized.

Note that eigenstates of \hat{a} are referred to as "coherent states".

Solution. Let us define the kets $|n\rangle$ such that,

$$\psi_n(x) = \langle \hat{x} | n \rangle$$

We know that the raising and lowering operators are given by,

$$\begin{aligned}
\hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right) \\
\hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right)
\end{aligned}$$

And, we know that

$$\begin{aligned}\hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \\ \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle\end{aligned}$$

(a) We are given that $\hat{a} |\psi\rangle = \alpha |\psi\rangle$. Where $\langle \hat{x} | \psi \rangle = \psi(x, t=0)$ For that to hold, we must have,

$$\begin{aligned}\hat{a} |\psi\rangle &= \sum_{n=0}^{\infty} c_n \hat{a} |n\rangle \\ \Rightarrow \alpha |\psi\rangle &= \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle \\ \Rightarrow \sum_{n=0}^{\infty} \alpha c_n |n\rangle &= \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle \\ \Rightarrow \alpha c_n &= \sqrt{n+1} c_{n+1} \\ \Rightarrow c_{n+1} &= \frac{\alpha}{\sqrt{n+1}} c_n \\ \Rightarrow c_n &= \frac{\alpha}{\sqrt{n}} c_{n-1} \\ \Rightarrow c_n &= \frac{\alpha^2}{\sqrt{n(n-1)}} c_{n-2} \\ &\vdots \\ \Rightarrow c_n &= \frac{\alpha^n}{\sqrt{n!}} c_0\end{aligned}$$

Then our state $|\psi\rangle$ becomes,

$$|\psi\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

For $|\psi\rangle$ to be normalized, we must have,

$$\begin{aligned}\langle \psi | \psi \rangle &= 1 \\ \Rightarrow |c_0|^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha^*)^n \cdot \alpha^m}{\sqrt{n!} \cdot m!} \langle n | m \rangle &= 1 \\ \Rightarrow |c_0|^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha^*)^n \cdot \alpha^m}{\sqrt{n!} \cdot m!} \delta_{nm} &= 1 \\ \Rightarrow |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} &= 1 \\ \Rightarrow |c_0|^2 e^{|\alpha|^2} &= 1 \\ \Rightarrow |c_0|^2 &= e^{-|\alpha|^2} \\ \Rightarrow c_0 &= \exp\left(i\phi - \frac{|\alpha|^2}{2}\right)\end{aligned}$$

Where $\phi \in \mathbb{R}$ is a constant. Hence finally our state ψ turns out to be,

$$|\psi\rangle = \exp\left(i\phi - \frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

And in position representation we have,

$$\psi(x, t = 0) = \exp\left(i\phi - \frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \psi_n(x)$$

(b) We are given that, $\hat{a}\psi = \alpha\psi$. Then in the position representation we have,

$$\begin{aligned} \alpha\psi(x) &= \langle x|\hat{a}|\psi\rangle \\ &= \int_0^\infty \int_0^\infty \langle x|x'\rangle \langle x'|\hat{a}|x''\rangle \langle x''|\psi\rangle dx'' dx' \\ &= \sqrt{\frac{m\omega}{2\hbar}} \int_0^\infty \int_0^\infty \langle x|x'\rangle \langle x'|\hat{x} + \frac{i}{m\omega}\hat{p}|x''\rangle \langle x''|\psi\rangle dx'' dx' \\ &= \sqrt{\frac{m\omega}{2\hbar}} \int_0^\infty \int_0^\infty \delta(x-x') \left(\delta(x'-x'')x'' + \frac{i}{m\omega} \cdot (-i\hbar)\delta(x'-x'') \frac{d}{dx''} \right) \psi(x'') dx'' dx' \\ &= \sqrt{\frac{m\omega}{2\hbar}} \int_0^\infty \delta(x-x') \left(x' + \frac{\hbar}{m\omega} \frac{d}{dx'} \right) \psi(x') dx' \\ &= \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \psi(x) \end{aligned}$$

Hence, we arrive at a first order differential equation in $\psi(x)$.

$$\begin{aligned} \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \psi &= \alpha\psi(x) \\ \Rightarrow \frac{d\psi}{dx} &= \frac{m\omega}{\hbar} \left(\sqrt{\frac{2\hbar}{m\omega}} \alpha - x \right) \psi(x) \\ \Rightarrow \frac{d\psi}{dx} &= \left(\sqrt{\frac{2m\omega}{\hbar}} \alpha - \frac{m\omega x}{\hbar} \right) \psi(x) \end{aligned}$$

Observe that, the following guess solution for $\psi(x)$ works

$$\psi(x) = C \exp\left(\sqrt{\frac{2m\omega}{\hbar}} \alpha x - \frac{m\omega x^2}{2\hbar}\right)$$

$$\begin{aligned} \frac{d\psi}{dx} &= C \left(\sqrt{\frac{2m\omega}{\hbar}} \alpha - \frac{m\omega x}{\hbar} \right) \exp\left(\sqrt{\frac{2m\omega}{\hbar}} \alpha x - \frac{m\omega x^2}{2\hbar}\right) \\ &= \left(\sqrt{\frac{2m\omega}{\hbar}} \alpha - \frac{m\omega x}{\hbar} \right) \psi(x) \end{aligned}$$

Now we must ensure that $\psi(x)$ is normalized. Then, we must have,

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1 \\
 \Rightarrow & |C|^2 \int_{-\infty}^{\infty} \exp\left(\sqrt{\frac{2m\omega}{\hbar}} 2\operatorname{Re}(\alpha)x - \frac{m\omega x^2}{\hbar}\right) dx = 1 \\
 \Rightarrow & |C|^2 \int_{-\infty}^{\infty} \exp\left(-\left(\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{2}\operatorname{Re}(\alpha)\right)^2 + 2\operatorname{Re}(\alpha)^2\right) dx = 1 \\
 \Rightarrow & |C|^2 \exp(2\operatorname{Re}(\alpha)^2) \left(\frac{\hbar}{m\omega}\right)^{1/2} \sqrt{\pi} = 1 \\
 \Rightarrow & |C|^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp(-2\operatorname{Re}(\alpha)^2) \\
 \Rightarrow & \boxed{C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp(i\theta - \operatorname{Re}(\alpha)^2)}
 \end{aligned}$$

Where $\theta \in \mathbb{R}$ is a constant. Hence, our normalized state is given by,

$$\boxed{\psi(x, t = 0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(i\theta - \operatorname{Re}(\alpha)^2 + \sqrt{\frac{2m\omega}{\hbar}} \alpha x - \frac{m\omega x^2}{2\hbar}\right)}$$