MA2102 - Linear Algebra I

Assignment 3 Solutions

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Exercise 1. Find the dimension of $\text{span}_{\mathbb{R}} S$ where

$$S := \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}$$

Solution. Let $V := \text{span}_{\mathbb{R}} S$. Observe that,

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, S is not a basis of V. We remove $s_3 := \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ from S in an attempt to construct a basis for V. Let the new set be $S' = S \setminus \{s_3\}$. We claim that, S' is a linearly independent set. Let $x, y \in \mathbb{R}$ be such that,

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, we have

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ x+y \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = y = 0$$

Hence, S' is linearly independent. We already know that, $\text{span}_{\mathbb{R}} S' = V$. Hence, S' is a basis of V. Hence, $\dim V = |S'| = 2$. □

Exercise 2.

(i) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$ And let $V = \{(x, y, z) \in \mathbb{R}^3 : \det(A) = 0\}$. Show that V is a vector space, and find dimension of V.

(ii) Let $V = \{(x_1 - x_2 + x_3,\ x_1 + x_2 - x_3) : (x_1, x_2, x_3) \in \mathbb{R}^3\}$. Show that V is a vector space and find the dimension of V.

Solution.

(i) Observe that, $\det(A) = 0 \Rightarrow (2z - 3y) - (2z - 3x) + (2x - 2y) = 0 \Rightarrow x = y$. Then, any arbitrary $v \in V$ is of the form $v = \begin{pmatrix} x \\ x \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Let $S := \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Then, $v \in \text{span}S$. Hence, $\boxed{V \subseteq \text{span}S}$.

Now, let $v \in \text{span}S$. Then, $v = a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ a \\ b \end{pmatrix}$ Then observe that, for $x = a, y = a$ and $z = b$, $\det(A) = 0$. Hence, $v \in V$. Hence, $\boxed{\text{span}S \subseteq V}$. This implies that $\boxed{V = \text{span}S}$. Hence, V is a vector space.

We claim that S is linearly indepnedent. Let $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, we have

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \alpha = \beta = 0$$

Hence, the set S is linearly independent. Thus S is a basis of V, and $\boxed{\dim V = |S| = 2}$.

(ii) Observe that, any arbitrary $v \in V$ is of the form

$$v = \begin{pmatrix} x_1 - x_2 + x_3 \\ x_1 + x_2 - x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Let $S := \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$. Then, $v \in \text{span}S$. Hence, $\boxed{V \subseteq \text{span}S}$.

Now, let $v \in \text{span}S$. Then, $v = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a - b + c \\ a + b - c \end{pmatrix}$ where $a, b, c \in \mathbb{R}$. Then clearly, for $x_1 = a,\ x_2 = b$ and $x_3 = c$ $v \in V$. Hence, $\boxed{\text{span}S \subseteq V}$. This implies that $\boxed{V = \text{span}S}$

Now, observe that S is not a linearly independent set since, $0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ We remove $s_3 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ from S, to define $S' := S \setminus \{s_3\}$. We claim that S' is linearly independent. Let $\alpha, \beta \in \mathbb{R}$ be such that,

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then,

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha - \beta \\ \alpha + \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \alpha = -\beta \ \ \&\ \ \alpha = \beta \Rightarrow \alpha = \beta = 0$$

Hence, S' is linearly independent. Thus S' is a basis of V and, $\boxed{\dim V = |S'| = 2}$

□

Exercise 3. Let A be an $n \times n$ real matrix such that the sum of the entries of each column of A is λ. Show that λ is an eigenvalue of A.

Solution. Since A is an $n \times n$ matrix, let us define a_{ij} as the the element in the i^{th} row and j^{th} column of A. Then,

$$A^T = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix}$$

Observe that, for A^T we have all the rows summing up, to give λ. Let

$$v := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

We also have ,

$$Av = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{i1} \\ \vdots \\ \sum_{i=1}^{n} a_{in} \end{bmatrix} = \begin{bmatrix} \lambda \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda v$$

Hence, λ is an eigenvalue of A^T. In previous the previous assignment I have shown that the eigenvalue of A and A^T are equal. Hence, λ is an eigenvalues of A. □

Exercise 4. For a prime p, find the cardinality of $\mathrm{GL}_n(\mathbb{F}_p)$.

Solution. For a $n \times n$ matrix to have a non-zero determinant, all the columns of the martix must be linearly independent. Consider the first column. For each entry we have p choices, and there are n entries. Hence, a total of p^n combinations. But, we must exclude the case where all entries are 0, since that would result in matrix whose determinant is 0. Hence, for the first column, we have a total of $p^n - 1$ choices. Now consider constructing the k^{th} column where $k \in [2, \dots, n]$. Again, n entries, with p choices for each entry. This gives us p^n combinations. However, this column, must not be a scalar multiple of any of the previous columns. There are $k-1$ columns, with p scalar multiples of each column (note that this already counts the o vector). This gives us p^{k-1} combinations to exclude. Hence, for the k^{th} column, the number of choices are $p^n - p^{k-1}$.

Hence, total number of all such combinations is $(p^k - 1) \prod_{k=2}^{n} (p^n - p^{k-1}) = \prod_{k=1}^{n} (p^n - p^{k-1}) = \prod_{k=0}^{n-1} (p^n - p^k)$.

Hence,

$$|\mathrm{GL}_n(\mathbb{F}_p)| = \prod_{k=0}^{n-1} (p^n - p^k)$$

□

Exercise 5. Let V be a finite dimensional real vector space, and let $W_1, W_2, \dots, W_n$ be proper subspaces of V. Show that

$$V \neq \bigcup_{i=1}^{n} W_i$$

Solution. First, we remove all W_k that satisfy $W_k \subseteq \bigcup_{\substack{i=1 \\ i \neq k}}^{n} W_i$. Since $\bigcup_{i=1}^{n} W_i = W_k \cup \bigcup_{\substack{i=1 \\ i \neq k}}^{n} W_i = \bigcup_{\substack{i=1 \\ i \neq k}}^{n} W_i$.

Then, we index these subspaces again, using $I := \{1, \dots, n'\}$, as $W'_i \ \forall i \in I$. Clearly, $\bigcup_{i=1}^{n} W_i = \bigcup_{i \in I} W'_i$ We wish to show that $V \neq \bigcup_{i \in I} W'_i$. Let us assume to the contrary, that $V = \bigcup_{i \in I} W'_i$.

Observe that, by the construction of W'_1, $\exists u \in W'_1$ such that $u \notin \bigcup_{i \in I \setminus \{1\}} W'_i$. Note that, this implies that $u \notin W'_i \ \forall i \in I \setminus \{1\}$

Also, $W'_1 \subset V \Rightarrow \exists v \in V$ such that $v \notin W'_1$.

Let us construct a set S as,

$$S := \{v + \alpha u : \alpha \in \mathbb{R} \setminus \{0\}\}$$

Then, clearly, S is an infinite set and $S \subset V$.

Observe that,

$$S \cap W'_1 = \phi$$

Since if not, $\exists \alpha \in \mathbb{R} \setminus \{0\}$ such that $v + \alpha u \in W_1'$ but we also have $u \in W_1' \Rightarrow \alpha u \in W_1'$. Then, $v + \alpha u - \alpha u = v \in W_1'$. This is absurd, since we know that $v \notin W_1'$
Also, observe that

$$|S \cap W_i'| \leq 1 \ \forall i \in I \setminus \{1\}$$

Since, if not, then $\exists i \in I \setminus \{1\}$ such that $v + \alpha_1 u, v + \alpha_2 u \in W_i'$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \neq \alpha_2$. But then, $(v + \alpha_1 u) - (v + \alpha_2 u) = (\alpha_1 - \alpha_2)u \in W_i' \Rightarrow u \in W_i'$ This is absurd, since we know that $u \notin W_i' \ \forall i \in I \setminus \{1\}$.
hence, we have

$$\begin{aligned}
&|S \cap \bigcup_{i \in I} W_i'| \leq n' - 1 \\
\Rightarrow &|S \cap V| \leq n' - 1 \\
\Rightarrow &|S| \leq n' - 1
\end{aligned}$$

But this is a contradiction since we know that S is infinite. Hence, our assumption must be false. i.e., $V \neq \bigcup_{i \in I} W_i'$.Hence,

$$V \neq \bigcup_{i=1}^{n} W_i$$

$\square$