MA2101: Analysis I Lecture Notes

Instructor: Rajib Dutta

Debayan Sarkar [*] 22MS002
Sabarno Saha [†] 22MS037
Piyush Kumar Singh [‡] 22MS027

August 13, 2023

Contents

§1 Algebra of the Real Number System

§1.1 Properties of Addition

The properties of addition($+$) in the real number system are:

(A1) $x + y = y + x \ \forall \ x, y \in \mathbb{R}$

(A2) $(x + y) + z = x + (y + z) \ \forall \ x, y, z \in \mathbb{R}$

(A3) $\exists!0 \in \mathbb{R} \ s.t. \ x + 0 = 0 + x = x \ \forall \ x \in \mathbb{R}$

(A4) $\forall x \in \mathbb{R} \exists! y \in \mathbb{R}$ s.t. $x + y = y + x = 0$

§1.2 Properties of Multiplication

The properties of multiplication($\cdot$) in the real number system are:

(M1) $x \cdot y = y \cdot x \forall x, y \in \mathbb{R}$

(M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \forall x, y, z \in \mathbb{R}$

(M3) $\exists!1 \in \mathbb{R}$ s.t. $x \cdot 1 = x \forall x \in \mathbb{R}$

(M4) $\forall x \in \mathbb{R} \setminus \{0\} \exists! y \in \mathbb{R}$ s.t. $x \cdot y = y \cdot x = 0$

---

[*] TheSillyCoder.github.io
[†] TheInvisibleFoe.github.io
[‡] iamPiyushKrSingh.github.io

§1.3 Distributive Property

The multiplication operator distributes over addition inn real numbers.

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Since addition and multiplication have these properties in real numbers, $(\mathbb{R}, +, \cdot)$ is a Field.

§1.4 Order in Reals

§1.4.1 Law of Trichotomy

Given two $x, y \in \mathbb{R}$, exactly one of the following statements is true :

(i) $x = y$

(ii) $x > y$

(iii) $x < y$

§1.4.2 Properties of $<$

(i) If $x < y$ and $y < z$ then $x < z$

(ii) If $x > 0$, $y > 0$ then, $xy > 0$

(iii) If $x < y$ then, $x + z < y + z \; \forall z \in \mathbb{R}$

(iv) $x < y \Rightarrow -x > -y$

(v) If $x < y$ and $z > 0$ then $xz < yz$

(vi) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$

(vii) $x^2 \geq 0 \; \forall x \in \mathbb{R}$

Remark 1.1

Let $x, y \in \mathbb{R}$ such that, $x \leq y$ and $y \leq x$. Then, $x = y$.

Proof:

Let's assume to teh contrary that $x \neq y$. Then, by the law of trichotomy, either $x < y$ or $y < x$. Let $y < x$. From $x \leq y$ we have either $x < y$ or $x = y$. By the law of trichotomy, neither of them can be true. Hence, $y \not< x$ Now, let $x < y$. From $y \leq x$ we have either $y < x$ or $y = x$. Again, by the law of trichotomy, neither of them can be true. Hence, $x \not< y$ This is a contradiction. Hence, $x = y$ Q.E.D.

Example 1:

For $x < y$, we have, $x < \frac{x+y}{2} < y$. The point $\frac{x+y}{2}$ is called the midpoint between x and y.

Proof:

Since $x < y$, we have $\frac{x}{2} < \frac{y}{2}$. Then, we have $\frac{x}{2} + \frac{x}{2} < \frac{y}{2} + \frac{x}{2} \Rightarrow x < \frac{x+y}{2}$ Similarly, we have $\frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} \Rightarrow \frac{x+y}{2} < y$ Hence, $x < \frac{x+y}{2} < y$ Q.E.D.

Example 2:

If $x \leq y + z$ for all $z > 0$, then $x \leq y$.

Proof:

Let $x, y \in \mathbb{R}$ such that $x \leq y + z$ for all $z > 0$. We claim that, $x \leq y$. Let us assume to the contrary that, $x > y$. Then, we have $x - y > 0$. Let $\epsilon := x - y$. Also observe that, $x - y \leq z$ for all $z > 0$. Let us set $z = \frac{\epsilon}{2}$. Then, $x - y \leq z \Rightarrow \epsilon \leq \frac{\epsilon}{2} \Rightarrow 1 \leq \frac{1}{2}$. This is a contradiction. Hence, $x \leq y$. This proves our claim. Q.E.D.

Example 3:

For $0 < x < y$, we have $0 < x^2 < y^2$ and $0 < \sqrt{2} < \sqrt{y}$, assuming te existence of $\sqrt{x}$ and $\sqrt{y}$. More generally, if x and y are positive, then $x < y$ iff $x^n < y^n$ for all $n \in \mathbb{N}$.

Proof:

will type up later Q.E.D.

Example 4:

For $0 < x < y$, we have $\sqrt{xy} < \frac{x+y}{2}$.

Proof:

We claim that the statment is true. Let us assume to the contrary that, $\frac{x+y}{2} < \sqrt{xy}$. Then, we have,

$$\begin{aligned} & \frac{x+y}{2} < \sqrt{xy} \\ \Rightarrow & \left(\frac{x+y}{2}\right)^2 < xy && \text{(Example 1)} \\ \Rightarrow & \left(\frac{x+y}{2}\right)^2 - xy < 0 \\ \Rightarrow & \left(\frac{x-y}{2}\right)^2 < 0 \end{aligned}$$

This is a contradiction since we know that $\alpha^2 \geq 0 \ \forall \alpha \in \mathbb{R}$. This proves our claim. Q.E.D.

§2 Upper and Lower Bounds

Definition 1 (Upper Bound)

Let $A \subset \mathbb{R}$ be nonempty. A number $\alpha \in \mathbb{R}$ is said to be the upper bound of A if $\forall x \in A$, we have $x \leq \alpha$

Geometrically, this means that on the real number line, all the elements of A are to the left of α. If $\alpha \in \mathbb{R}$ is not an upper bound of A, then $\exists x \in A$ s.t. $x > \alpha$

Definition 2 (Lower Bound)

Let $A \subset \mathbb{R}$ be nonempty. A number $\alpha \in \mathbb{R}$ is said to be the lower bound of A if $\forall x \in A$, we have $x \geq \alpha$

Geometrically, this means that on the real number line, all the elements of A are to the right of α. If $\alpha \in \mathbb{R}$ is not an lower bound of A, then $\exists x \in A$ s.t. $x < \alpha$

Example 5:
Consider the set $A := \{1, 2, 3, 4, 5\}$ *Then the upper bounds of this* A *are* $5, 6, 1729 \ldots$ *etc. And the lower bound of* A *are* $1, 0, -1 \ldots$ *etc. Hence, lower and upper bounds of a set are not unique.*

Definition 3 (Bounded above set)
Let $\phi \neq A \subset \mathbb{R}$. Then, A is said to be bounded above, if $\exists \alpha \in \mathbb{R}$ such that α is an upper bound of A.

Definition 4 (Bounded below set)
Let $\phi \neq A \subset \mathbb{R}$. Then, A is said to be bounded below, if $\exists \alpha \in \mathbb{R}$ such that α is an lower bound of A.

Theorem 2.1
Let A be a bounded above set. Let $\alpha \in \mathbb{R}$ be an upper bound. Let $\beta \in \mathbb{R}$ such that $\beta \geq \alpha$. Then β is an upperbound of A.

Proof:
Since α is an upper bound, we have $x \leq \alpha \; \forall x \in A$ But, $\alpha \leq \beta$. Then we have, $x \leq \beta \; \forall x \in A$. Hence β is an upper bound of A. Q.E.D.

A similar theorem can be stated and proved analagously for bounded below sets and lower bounds.

Definition 5 (Maximum)
Let $\phi \neq A \subset \mathbb{R}$. $\alpha \in \mathbb{R}$ is said to be the maximum of A if

(i) $\alpha \in \mathbb{R}$

(ii) α is an upper bound of A

The maximum of a set A is denoted by $\max\{A\}$

Theorem 2.2
The maximum of a set is unique.

Proof:
Let α and β be two maxima of A s.t. $\alpha \neq \beta$. Then, by the law of trichotomy, either $\alpha > \beta$ or $\beta > \alpha$. If $\alpha > \beta$, then α cannot be an upper bound since, $\beta \in A$. And, if $\beta > \alpha$, then β cannot be an upper bound since, $\alpha \in A$. This is a contradiction. Hence, $\alpha = \beta$. Q.E.D.

Remark 2.1

A bounded above set need not have a maximum. Consider set $S := (0, 1)$. S clearly has upper bounds for instance 1, 2, etc. However it does not have a maximum, since none of the upper bounds are in the set S itself. Some more of such remarks could be

1. $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$
2. $\{1 - \frac{1}{2^n} : n \in \mathbb{N}\}$

Definition 6 (Least Upper Bound)

Let A be a non-empty subset of $\mathbb{R}$. An upper bound α is said to be the least upper bound of A if $\beta < \alpha \Rightarrow \beta$ is not an upper bound of A.

We denote the least upper bound of A as $\text{lub}A$

Definition 7 (Greatest Lower Bound)

Given a non-empty bounded below set A, a real number β is said to be the greatest lower bound of A if

(i) β is a lower bound of A

(ii) if $\beta' > \beta$ then, $\exists x \in A$ such that $\beta \leq x \leq \beta'$

We denote the Greatest lower bound of A as $\text{glb}A$

Theorem 2.3

The least upper bound(or greatest lower bound) of a bounded above(or below) set is unique.

Definition 8 (Completeness Axiom)

Every bounded above(or below) subset of $\mathbb{R}$ has a least upper bound(or greatest lower bound).

Remark 2.2

This is an important property in $\mathbb{R}$. $\mathbb{Q}$ does not have this property. For instance, consider the set $A = \{x \in \mathbb{Q} : 0 < x^2 < 2\}$. Clearly, $\text{lub}A = \sqrt{2} \notin \mathbb{Q}$

Theorem 2.4 (Archimedean Property)
(AP1) $\mathbb{N}$ is not bounded above in $\mathbb{R}$
(AP2) Let $x > 0$, $y \in \mathbb{R}$, then $\exists n \in \mathbb{N}$ such that $nx > y$

Proof:
First, we prove AP1 by contradiction. Let us assume to the contrary that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Then, by the Completeness Axiom, $\exists \alpha \in \mathbb{R}$ such that $\alpha = \text{lub}\mathbb{N}$. Then, by definition of lub, $\alpha - 1$ is not an upper bound of $\mathbb{N}$. Hence, $\exists n \in \mathbb{N}$ such that $\alpha - 1 < n < \alpha$. Then we have $n + 1 > \alpha$. This is a contradiction since $n + 1 \in \mathbb{N}$ and α is an upper bound of $\mathbb{N}$. Hence, $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Now, we show that AP1 $\Rightarrow$ AP2. However, that trivially follows, since if we have $x, y \in \mathbb{R}$ with $x > 0$, then from $AP1$ we have that $\frac{y}{x}$ cannot be an upper bound of $\mathbb{R}$. Hence, $\exists n \in \mathbb{N}$ s.t. $n > \frac{y}{x}$, i.e. $nx > y$.

The proof for AP2 $\Rightarrow$ AP1 is left as an exercise. Q.E.D.

Definition 9 (Bounded Sets)
Let A be a non-empty subset of $\mathbb{R}$. Then, A is said to be bounded if A is bounded above and bounded below.

Theorem 2.5 (Greatest Integer Function)
Let $x \in \mathbb{R}$. Then, $\exists$ a unique $m \in \mathbb{Z}$ s.t. $m \leq x < m + 1$ This m is denoted as $\lfloor x \rfloor$. We define the **greatest integer function** $f : \mathbb{R} \to \mathbb{Z}$ as, $f(x) = \lfloor x \rfloor$

Proof:
Let $S := \{k \in \mathbb{Z} : k \leq x\}$. S is bounded above as x is an upper bound from set definition. We now aim to show that S is non-empty. Let us suppose to the contrary that S is empty then we have $\forall \quad k \in \mathbb{Z}, \quad k > x$. Thus k is a lower bound of $\mathbb{Z}$ which is a constradiction. Thus S is non empty. We know S is bounded above and non-empty. Then by the completeness axiom, there exists a least upper bound. Let $m = \sup(S)$. We now show that $m \leq x < m + 1$. Clearly $m \leq x$ as x is any upper bound for S. For proving the right inequality we use contradiction. Let us suppose to the contrary that $x \geq m + 1$. Since $m \in \mathbb{Z}$, then $m + 1 \in \mathbb{Z}$. Thus $m + 1 \in S$. Thus we have $m + 1 = \sup(S)$, which is a contradiction. Q.E.D.

Claim: m is unique.

Proof:
Suppose we have $m \neq n$ and $m = \lfloor x \rfloor = n$. WLOG $n > m$. we have,$m \leq x < m+1$ and $n \leq x < m+1$. Since $n > m$ and $n, m \in \mathbb{Z}$, then $n \geq m+1$. Now $m \leq x < m+1 \leq n \quad \Rightarrow x < n$ which is a contradiction of the law of trichotomy of order as we assumed $x \geq n$. Q.E.D.

Theorem 2.6 (Density of Rational Numbers)
$\forall \quad a, b, \in \mathbb{R}$ and $a < b$. Then $\exists \quad q \in \mathbb{Q}$ s.t. $a < q < b$.

Proof:
Since $q \in \mathbb{Q}$, we can write $q = \frac{m}{n}$ where $m, n \in \mathbb{Z} \quad, n \neq = 0$.Thus we have $a < \frac{m}{n} < b \quad \Rightarrow na < m < nb$
Using the Archimedean Property, we choose n_0 s.t. $n_0 > \frac{1}{b - a}$. Now we choose $m = \lfloor n_0 a + 1 \rfloor$. Thus, we have $n_0 a < m \leq n_0 a+1$. We thus have $m \leq n_0 a+1$. We know from the AP inequality $n_0(b-a) > 1$, we get $m < n_0 a + n_0(b - a) = n_0 b$. Thus we have $n_0 a < m < n_0 b$ Q.E.D.

Definition 10 (Irrational Numbers)
If $\alpha \in \mathbb{R}$ and $\alpha \notin \mathbb{Q}$, we say that α is an irrational number.

Theorem 2.7 (Density of Irrational Numbers)
$\forall \;\; a, b, \in \mathbb{R}$ and $a < b$. Then $\exists \;\; \alpha \in \mathbb{R} \backslash \mathbb{Q}$ s.t. $a < \alpha < b$.