Cancellation Properties on Natural Numbers

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Cancellation property for Addition

Proposition 1 : $1 + b = 1 + a \Rightarrow b = a \ \forall a, b \in \mathbb{N}$
Proof : Every $n \in \mathbb{N}$ has a unique successor in $\mathbb{N}$. Hence, every successor must also have a unique predecessor. So, if the successors are equal, the predecessors must also be equal.
Claim : We claim that,

$$n + a = n + b \Rightarrow a = b \ \forall n, a, b \in \mathbb{N}$$

Proof : We prove this using the principle of mathematical induction on n.
Base Step : For $n = 1$, the statement becomes

$$1 + a = 1 + b \Rightarrow a = b$$

which is true from Proposition 1. So, this statement holds for $n = 1$
Induction Step : Let's say that this holds for $n = k$ where $k \in \mathbb{N}$. We wish to prove, that this holds for $n = k + 1$.
Let's assume that $(k + 1) + a = (k + 1) + b$

$$\begin{aligned}
(k+1) + a &= k + (1 + a) && \text{(Associativity of addition on } \mathbb{N}) \\
&= k + (a + 1) && \text{(Commutativity of addition on } \mathbb{N}) \\
&= (k + a) + 1 && \text{(Associativity of addition on } \mathbb{N})
\end{aligned}$$

$$\therefore (k + 1) + a = (k + a) + 1 \tag{2}$$

Similarly,

$$\begin{aligned}
(k+1) + b &= k + (1 + b) && \text{(Associativity of addition on } \mathbb{N}) \\
&= k + (b + 1) && \text{(Commutativity of addition on } \mathbb{N}) \\
&= (k + b) + 1 && \text{(Associativity of addition on } \mathbb{N})
\end{aligned}$$

$$\therefore (k + 1) + b = (k + b) + 1 \tag{1}$$

Substituting the values from (1) and (2) in our assumption we get,

$$\begin{aligned}
&(k + a) + 1 = (k + b) + 1 \\
&\Rightarrow k + a = k + b && \text{(From Proposition 1 as } (k + a), (k + b) \in \mathbb{N}) \\
&\Rightarrow a = b && \text{(Because the statement holds for } n = k)
\end{aligned}$$

Hence,

$$(k + 1) + a = (k + 1) + b \Rightarrow a = b$$

The statement holds true for $n = k + 1$ whenever it holds true for $n = k$. By invoking the principle of mathematical induction we can say, that it holds true for every $n \in \mathbb{N}$. This proves our claim

Cancellation property for Multiplication

Claim : We claim that

$$n \cdot a = n \cdot b \Rightarrow a = b \;\; \forall a, b, n \in \mathbb{N}$$

Proof : We will prove this by contradiction. Let $n, a, b \in \mathbb{N}$ such that $n \cdot a = n \cdot b$ and $a \neq b$ Then, there are two possible cases :
Case 1 : $a = b + k$ where $k \in \mathbb{N}$ Multiplying n on both sides we get,

$$\begin{aligned} & n \cdot a = n \cdot (b + k) \\ \Rightarrow & n \cdot a = n \cdot b + n \cdot k && \text{(Distributive Property)} \\ \Rightarrow & n \cdot a \neq n \cdot b \end{aligned}$$

which contradicts our assumption.
Case 2 : $b = a + k$ where $k \in \mathbb{N}$ Multiplying n on both sides we get,

$$\begin{aligned} & n \cdot b = n \cdot (a + k) \\ \Rightarrow & n \cdot b = n \cdot a + n \cdot k && \text{(Distributive Property)} \\ \Rightarrow & n \cdot b \neq n \cdot a \end{aligned}$$

which also contradicts our assumption.

Hence our assumption must be wrong, and $a = b$. This proves our claim.