MA2102 - Linear Algebra I

Assignment 1 Solutions

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1. (i) Explain why a system of linear equations with more variables than equations always has a solution, whereas a system of such equations with more equations than variables may not have any solution at all.

   (ii) Show that a matrix with more columns than rows (resp. more rows than columns) does not have a left (resp. right) inverse.

2. Compute the determinant of the following matrix:

$$\begin{pmatrix} 2 & -2 & & & & & \\ -1 & 5 & -2 & & & & \\ & -2 & 5 & -2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -2 & 5 & -2 & \\ & & & & -2 & 5 & -1 \\ & & & & & -2 & 2 \end{pmatrix}_{n\times n}$$

3. Let λ be an eigenvalue of an $n \times n$ real matrix A. Show that there exists a positive integer $k \leq n$ such that

$$|\lambda - a_{kk}| \leq \sum_{j=1, j\neq k}^{n} |a_{jk}|$$

   Soltion : First we claim that the eigenvalues of A and A^T.

4. Show that an $n \times n$ real matrix is invertible if and only if its columns span R^n .

5. (i) Let V be the set of all real numbers. Define the binary operation "addition" on V by

$$x \boxplus y = \text{ the maximum of x and y}$$

   for all $x, y \in V$ and define an operation of "scalar multiplication" by

$$\alpha \boxdot y = \alpha x$$

   for all $\alpha \in \mathbb{R}$ and $x \in V$. Is V a vector space over $\mathbb{R}$ under the above operations? Justify your answer!

   Solution : V is not a vector space over $\mathbb{R}$ under the defined operations. Let's assume to the contrary, that V is a vector space. Then, $(V, \boxplus)$ must be an abelian group. Consider the element $3 \in V$. According to the definition of $\boxplus$, $3 \boxplus 2 = 3$. Hence, $2 \in V$ is the identity eement in V. But, we also have $1 \in V$ satisfying, $3 \boxplus 1 = 3$. Hence, 1 is also an indentity element in V. This is a contradiction, since an abelian group must have a unique identity element. This proves our claim. □

   (ii) Let V be the set of all positive real numbers. Define the binary operation "addition" on V by

$$x \boxplus y = xy$$

   for all $x, y \in V$. Define an operation of "scalar multiplication" by

$$\alpha \boxdot x = x^{\alpha}$$

for all $\alpha \in \mathbb{R}$ and $x \in V$. Show that V is a vector space over $\mathbb{R}$. Provide a basis for V.

Solution : V is a vector space over $\mathbb{R}$ under the defined binary operations.

We first show that $(V, \boxplus)$ is an abelian group. Let $x, y, z \in V$ be arbitrary.

Then, if $z := x \boxplus y = xy > 0$. Hence, $z \in R^+ = V$ Hence, V is closed under $\boxplus$.

Also, $x \boxplus (y \boxplus z) = x \boxplus (yz) = x(yz) = (xy)z = (xy) \boxplus z = (x \boxplus y) \boxplus z$ Since multiplication is associative in $\mathbb{R}$. Hence, $\boxplus$ is associative in V.

Consider the element $1 \in V$. Then, we have $1 \boxplus x = 1x = x = x \cdot 1 = x \boxplus 1$. Hence 1 is the identity element in V. Now we show tha uniqueness of the identity element. Let's assume there's another identiy element $\bar{1} \in V$ Then, we have $1 = 1 \boxdot \bar{1} = \bar{1}$. Hence, V has a unique identity element under $\boxplus$.

We know that $\exists$ a unique $x^{-1} \in \mathbb{R}^+ = V$ such that, $x \boxplus x^{-1} = x \cdot x^{-1} = 1$ which is the identity element. Hence, each element in V has a unique inverse under $\boxplus$.

Now, we also have $x \boxplus y = xy = yx = y \boxplus x$ since multiplication is commutative in $\mathbb{R}$. Hence, $\boxplus$ is commutative ion V.

This proves that $(V, \boxplus)$ is an abelian group.

Now let $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$ be arbitrary.

Note that,

(a) $w := \alpha \boxdot u = u^\alpha > 0 \Rightarrow w \in \mathbb{R}^+ = V$

(b) $1 \boxdot v = v^1 = v$

(c) $\alpha \boxdot (\beta \boxdot v) = \alpha \boxdot v^\beta = (v^\beta)^\alpha = v^{\beta\alpha} = v^{\alpha\beta} = (v^\alpha)^\beta = \beta \boxdot (v^\alpha) = \beta \boxdot (\alpha \boxdot v)$

(d) $(\alpha + \beta)v = v^{\alpha+\beta} = v^\alpha \cdot v^\beta = v^\alpha \boxplus v^\beta = \alpha \boxdot v \boxplus \beta \boxdot v$

(e) $\alpha \boxdot (u \boxplus v) = \alpha \boxdot uv = (uv)^\alpha = u^\alpha \cdot v^\alpha = \alpha \boxdot u \boxplus \alpha \boxdot v$

Hence, V is a vector space over $\mathbb{R}$. $\square$

We can take $2 \in V$ as a basis for V since 2^α is a injective continous function from $\mathbb{R} \to \mathbb{R}^+$ Thus, every $y \in V$ will have a unique α such that, $\alpha \boxdot x = x^\alpha = y$.