PH2101 - Waves and Optics

Assignment 2 Solutions

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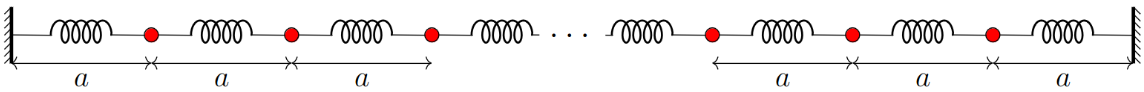
Question 1. Consider a beaded string of N beads each of mass m (approximated as a long chain of spring- mass system as shown in the figure). The beads are uniformly placed on the string and the string has a uniform tension T . The horizontal distance between any two beads in equilibrium is a. The unstretched lengths of the springs are negligible.



(a) Find the equation of the motion of n^{th} bead for the longitudinal mode of vibration

(b) Assuming normal mode vibration, find the normal mode frequency ω_m for m^{th} mode.

(c) Find the amplitudes of the beads in m^{th} mode ($A_n^{(m)}$ using the notation used in the class).

(d) Plot the dispersion relation ω versus k.

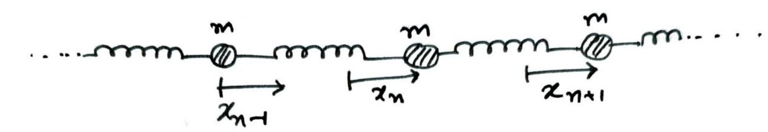
(e) Check whether we have $\omega_{N+2} = \omega_N$?

(f) Check whether we have $A_n^{(N+2)} = A_n^{(N)}$?

(g) Qualitatively plot $A_n^{(1)}$ and $A_n^{(N)}$ for all n.

Solution.

(a) Let the displacement in the n^{th} bead be x_n. Then, for the n^{th} particle we have



$$m\ddot{x}_n = \frac{T}{a}(x_{n-1} - x_n + x_{n+1} - x_n)$$
$$\Rightarrow \ddot{x}_n = -\frac{T}{ma}(2x_n - x_{n+1} - x_{n-1})$$

Let $\omega_0 = \sqrt{\frac{T}{ma}}$. Then, for the n^{th} particle, the equation of motion turns out to be,

$$\boxed{\ddot{x}_n = -\omega_0^2(2x_n - x_{n+1} - x_{n-1})}$$

(b) Let the amplitude of the n^{th} particle and the frequency in the oscillation in a normal mode be be A_n and ω respectively. Then, $x_n = A_n e^{i\omega t}$. So, from the equation of motion we have,

$$\begin{aligned}
\ddot{x}_n &= -\omega_0^2(2x_n - x_{n-1} - x_{n+1}) \\
\Rightarrow -\omega^2 A_n e^{i\omega t} &= -\omega_0^2(2A_n e^{i\omega t} - A_{n-1}e^{i\omega t} - A_{n+1}e^{i\omega t}) \\
\Rightarrow -\omega^2 A_n &= -\omega_0^2(2A_n - A_{n-1} - A_{n+1}) \\
\Rightarrow \frac{2\omega_0^2 - \omega^2}{\omega_0^2} &= \frac{A_{n+1} + A_{n-1}}{A_n} \qquad \text{(i)}
\end{aligned}$$

Observe that, we have a system of N equations, satisfying this relation. We claim that for all $\omega \leq 2\omega_0$, the A_n's that satisfy these N equations, can be written as :

$$A_n = B\cos n\theta + C\sin n\theta$$

where B, C, θ are constants. Now we shall prove this. Observe that

$$\begin{aligned}
\frac{A_{n+1} + A_{n-1}}{A_n} &= \frac{B\cos(n+1)\theta + C\cos(n+1)\theta + B\cos(n-1)\theta + C\cos(n-1)\theta}{B\cos n\theta + C\cos n\theta} \\
&= \frac{2B\cos n\theta\cos\theta + 2C\sin n\theta\cos\theta}{B\cos n\theta + C\sin n\theta} \\
&= 2\cos\theta \qquad \text{(ii)}
\end{aligned}$$

Hence, from equation (i) we have

$$\begin{aligned}
2\cos\theta &= \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \\
\Rightarrow \theta &= \arccos\left(\frac{2\omega_0^2 - \omega^2}{2\omega_0^2}\right) \qquad \text{(iii)}
\end{aligned}$$

Note that, this equation is valid because $\omega \leq 2\omega_0$. Now, $A_0 = B\cos 0 + C\sin 0 = B$. And $A_1 = B\cos\theta + C\sin\theta \Rightarrow A_1 = A_0\cos\theta + C\sin\theta \Rightarrow C = \dfrac{A_1 - A_0\cos\theta}{\sin\theta}$ This uniquely determines $B, C,$ and θ. The claim is satisfied for $n = 0$ and $n = 1$. Now we show inductively that it holds for all n.

Let's say this result holds for all $n \in \{0, \ldots, k\}$. Then, from (ii) we have

$$\begin{aligned}
A_{k+1} &= 2A_k\cos\theta = A_{k-1} \\
&= 2B\cos k\theta\cos\theta + 2C\sin k\theta\cos\theta - B\cos(k-1)\theta - C\sin(k-1)\theta \\
&= B\cos(k+1)\theta + \cancel{B\cos(k-1)\theta} + C\sin(k+1)\theta - \cancel{C\sin(k-1)\theta} - \cancel{B\cos(k-1)\theta} - \cancel{C\sin(k-1)\theta} \\
&= B\cos(k+1)\theta + C\sin(k+1)\theta
\end{aligned}$$

Hence, it holds for $n = k+1$ as well. By the principle of strong induction we can say that the claim holds true for all n. Now, observe that from (iii) we have,

$$\begin{aligned}
\frac{2\omega_0^2 - \omega^2}{2\omega_0^2} &= \cos\theta \\
\Rightarrow \omega^2 &= 2\omega_0^2(1 - \cos\theta) \\
\Rightarrow \omega^2 &= 2\omega_0^2 \cdot 2\sin^2\frac{\theta}{2} \\
\Rightarrow \omega^2 &= 4\omega_0^2\sin^2\frac{\theta}{2} \\
\Rightarrow \omega &= 2\omega_0\sin\frac{\theta}{2} \qquad \text{(iv)}
\end{aligned}$$

Applying the wall boundary conditions we get

$$A_0 = 0 \Rightarrow B\cos 0 + C\sin 0 = 0 \Rightarrow \boxed{B = 0}$$

and

$$A_{N+1} = 0 \Rightarrow C \sin{(N+1)\theta} = 0 \Rightarrow \boxed{\theta = \frac{m\pi}{N+1}}$$

where $m \in \mathbb{N}$. Hence θ can only take these discrete values as determined by m. Hence, m must denote the normal mode. Let ω_m be the normal mode frequency corresponding to the m^{th} normal mode. Then, from (iv) we have,

$$\boxed{\omega_m = 2\omega_0 \sin \frac{m\pi}{2(N+1)}}$$

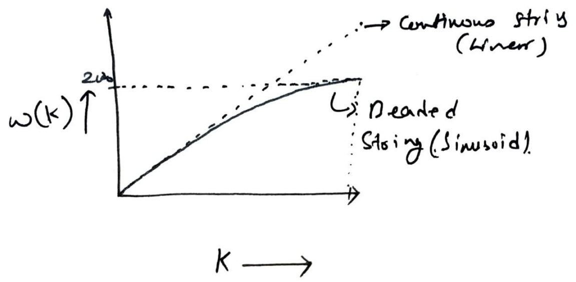
(c) Also, for the m^{th} normal mode the amplitude of the n^{th} particle will be given by,

$$\boxed{A_n^{(m)} = C \sin \frac{nm\pi}{N+1}}$$

(d) In our case of a beaded string, in the m^{th} normal mode, $\lambda = \dfrac{2L}{m\pi}$. Hence, $k = \dfrac{m\pi}{(N+1)a}$ Thus, we have

$$\boxed{\omega = 2\omega_0 \sin \frac{ka}{2}}$$

Hence, the plot for the dispersion relation will be :



(e) From (b) we have,

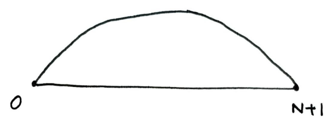
$$\omega_{N+2} = 2\omega_0 \sin \frac{(N+2)\pi}{2(N+1)} = 2\omega_0 \sin \frac{(2N+2-N)\theta}{2(N+1)} = 2\omega_0 \sin\left(\pi - \frac{N\pi}{2(N+1)}\right) = \omega_N$$

Hence, they correspond to the same normal mode.

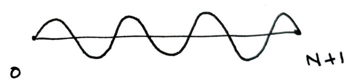
(f) From (c) we have,

$$A_n^{(N+2)} = C \sin \frac{n(N+2)\pi}{N+1} = C \sin \frac{n(2N+2-N)\theta}{N+1} = C \sin\left(2n\pi - \frac{nN\pi}{N+1}\right) = -A_n^{(N)}$$

(g) The plot for $A_n^{(1)}$ looks like,



The plot for $A_n^{(N)}$ looks like,



However, the beads will be on the same sinusoidal enevelope as created by the first normal mode. Only the sign will change, i.e. the first bead will be on the upper envelope, the second bead will be on the bottom envelope, and so on.

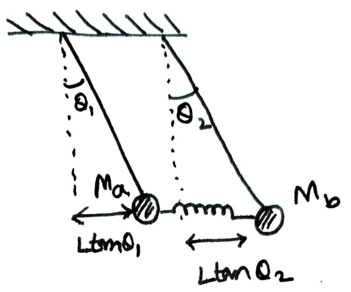
Question 2. Consider two pendulums, a and b, with the same string length L, but with different bob masses, M_a and M_b. They are coupled by a spring of spring constant K which is attached to the bobs. Assuming small angle oscillations,

(a) Find the equations of motion using angles of the pendulums (w.r.t. the vertical) as dynamical variables.

(b) Find the normal modes and the normal frequencies.

(c) For $M_a = M_b = M$ does this reduce to the case considered in class?

Solution. Let the left and right bobs be given small angular displacements θ_1 and θ_2 respectively, according to the diagram below.



(a) Then for the left bob we obtain the equation,

$$\begin{aligned}
&\tau = -M_a g \sin\theta_1 - KL^2 \tan\theta_1 + KL^2 \tan\theta_2 \\
\Rightarrow & I\ddot{\theta}_1 = -M_a g L \theta_1 - KL^2\theta_1 + KL^2\theta_2 \\
\Rightarrow & M_a L^2 \ddot{\theta}_1 = -M_a g L\theta_1 - KL^2(\theta_1 - \theta_2) \\
\Rightarrow & \ddot{\theta}_1 = -\frac{g}{L}\theta_1 - \frac{K}{M_a}(\theta_1 - \theta_2)
\end{aligned}$$

Similarly, for the right bob we will have,

$$
\begin{aligned}
\tau &= -M_b g \sin\theta_2 - KL^2\tan\theta_2 + KL^2\tan\theta_1 \\
\Rightarrow I\ddot{\theta_2} &= -M_b gL\theta_2 - KL^2\theta_2 + KL^2\theta_1 \\
\Rightarrow M_b L^2\ddot{\theta_2} &= -M_b gL\theta_2 - KL^2(\theta_2 - \theta_1) \\
\Rightarrow \ddot{\theta_2} &= -\frac{g}{L}\theta_2 - \frac{K}{M_b}(\theta_2 - \theta_1)
\end{aligned}
$$

Hence the equations of motion are

$$\boxed{\ddot{\theta_1} = -\frac{g}{L}\theta_1 - \frac{K}{M_a}(\theta_1 - \theta_2)}$$

$$\boxed{\ddot{\theta_2} = -\frac{g}{L}\theta_2 - \frac{K}{M_b}(\theta_2 - \theta_1)}$$

(b) Let $\theta_1 = Ae^{i\omega t}$ and $\theta_2 = Be^{i\omega t}$ Then, we have $\ddot{\theta_1} = -\omega^2 Ae^{i\omega t}$ and $\theta_2 = -\omega^2 Be^{i\omega t}$
Putting these values into the equations of motion we get,

$$
\begin{aligned}
-\omega^2 Ae^{i\omega t} &= -\frac{g}{L}Ae^{i\omega t} - \frac{K}{M_a}(Ae^{i\omega t} - Be^{i\omega t}) \\
\Rightarrow \left(\frac{K}{M_a} + \frac{g}{L} - \omega^2\right)A &- \frac{K}{M_a}B = 0
\end{aligned}
$$

$$
\begin{aligned}
-\omega^2 Be^{i\omega t} &= -\frac{g}{L}Be^{i\omega t} - \frac{K}{M_b}(Be^{i\omega t} - Ae^{i\omega t}) \\
\Rightarrow \left(\frac{K}{M_b} + \frac{g}{L} - \omega^2\right)B &- \frac{K}{M_b}A = 0
\end{aligned}
$$

This system of linear equations can be represented as,

$$
\begin{pmatrix} \left(\frac{K}{M_a} + \frac{g}{L} - \omega^2\right) & -\frac{K}{M_a} \\ -\frac{K}{M_b} & \left(\frac{K}{M_b} + \frac{g}{L} - \omega^2\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

For this system to have non-trivial solutions, the determinant of the matrix on the left must be 0. Hence,

$$
\begin{aligned}
\left(\frac{K}{M_a} + \frac{g}{L} - \omega^2\right)\left(\frac{K}{M_b} + \frac{g}{L} - \omega^2\right) &- \frac{K^2}{M_a M_b} = 0 \\
\Rightarrow \left(\frac{g}{L} - \omega^2\right)^2 + \left(\frac{g}{L} - \omega^2\right)\left(\frac{K}{M_a} + \frac{K}{M_b}\right) &= 0 \\
\Rightarrow \left(\frac{g}{L} - \omega^2\right)\left[\left(\frac{g}{L} - \omega^2\right) + \left(\frac{K}{M_a} + \frac{K}{M_b}\right)\right] &= 0 \\
\Rightarrow \omega = \pm\sqrt{\frac{g}{L}} \text{ or, } \omega &= \pm\sqrt{\frac{g}{L} + K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)}
\end{aligned}
$$

We can remove the negative frequencies since they dont give us any new information. Hence we have obtained the normal mode frequencies to be,

$$\boxed{\omega = \sqrt{\frac{g}{L}} \ \ \& \ \ \sqrt{\frac{g}{L} + K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)}}$$

In the normal mode corresposing to $\omega = \sqrt{\frac{g}{L}}$, the system of equations gives us

$$\frac{K}{M_a}A - \frac{K}{M_a}B = 0 \ \ \& \ \ \frac{K}{M_b}B - \frac{K}{M_b}A = 0$$
$$\Rightarrow \boxed{A = B}$$

Hence, in this normal mode, the bobs are oscillating **in phase**.

In the normal mode corresposing to $\omega = \sqrt{\frac{g}{L} + K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)}$, the system of equations gives us

$$-\frac{K}{M_b}A - \frac{K}{M_a}B = 0 \ \ \& \ \ -\frac{K}{M_a}B - \frac{K}{M_b}A = 0$$
$$\Rightarrow \boxed{A = -\frac{M_b}{M_a}B}$$

Hence, in this normal mode, the bobs are oscillating **out of phase**.

(c) If $M_a = M_b = M$, we get the normal mode frequencies to be

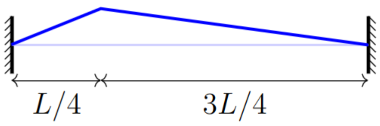
$$\boxed{\omega = \sqrt{\frac{g}{L}} \ \ \& \ \ \sqrt{\frac{g}{L} + \frac{2K}{M}}}$$

This is similar to thenormal modes obtained for the two mass coupled osscilator, which were obtained obtained to be,

$$\boxed{\omega = \sqrt{\frac{K_1}{M}} \ \ \& \ \ \sqrt{\frac{K_1}{M} + \frac{2K_2}{M}}}$$

In our case, $\frac{g}{L}$ coresponds to the $\frac{K_1}{M}$ in the coupled oscillator.

Question 3. Consider the following string, with the given configuration



(a) Find the fourier representation of the string. You should use the sine representation for the string. (not the full representation).

(b) Show that normal modes having nodes at L/4 are absent.

(c) Check numerically that your solution matches with the given shape. You may submit your codes.

Solution.

(a) According to the given configuration,

$$y(x,0) = \begin{cases} \frac{4h}{L}x & \text{for } 0 < x < \frac{L}{4} \\ \\ \frac{4h}{3L}(L-x) & \text{for } \frac{L}{4} < x < L \end{cases}$$

Let the fourier series exapansion of $y(x,0)$ be

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

Then,

$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{8h}{L^2}\int_0^{\frac{L}{4}} x\sin\frac{n\pi x}{L}\mathrm{d}x + \frac{8h}{3L^2}\int_{\frac{L}{4}}^{L}(L-x)\sin\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{8h}{L^2}\left[\int_0^{\frac{L}{4}} x\sin\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{3}\int_{\frac{L}{4}}^{L}(L-x)\sin\frac{n\pi x}{L}\mathrm{d}x\right] \\
&= \frac{8h}{L^2}\left[\int_0^{\frac{L}{4}} x\sin\frac{n\pi x}{L}\mathrm{d}x - \frac{1}{3}\int_{\frac{L}{4}}^{L}x\sin\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{3}\int_{\frac{L}{4}}^{L}L\sin\frac{n\pi x}{L}\mathrm{d}x\right] \\
&= \frac{8h}{L^2}\left[\left(\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{4}} - \frac{1}{3}\left(\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{4}}^{L} - \frac{L^2}{3n\pi}\cos\frac{n\pi x}{L}\Bigg|_{\frac{L}{4}}^{L}\right] \\
&= \frac{8h}{L^2}\left[\left[-\cancel{\frac{L^2}{12n\pi}} + \cancel{\frac{L^2}{3n\pi}} - \cancel{\frac{L^2}{4n\pi}}\right]\cos\frac{n\pi}{4} + \left[-\cancel{\frac{L^2}{3n\pi}} + \cancel{\frac{L^2}{3n\pi}}\right]\cos n\pi + \left(\frac{L}{n\pi}\right)^2\left[1+\frac{1}{3}\right]\sin\frac{n\pi}{4}\right] \\
&= \frac{32h}{3n^2\pi^2}\sin\frac{n\pi}{4}
\end{aligned}
$$

Hence, the fourier representation of the string is

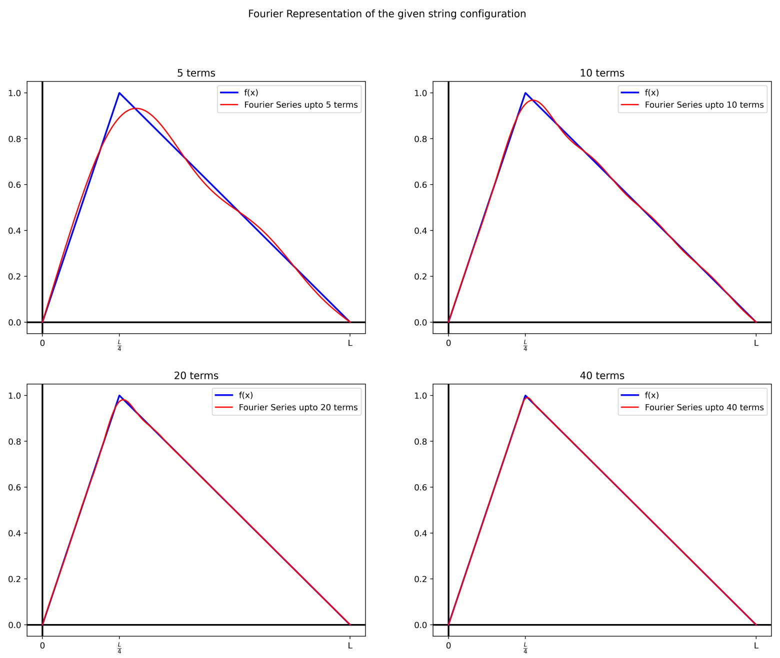
$$
\boxed{y(x,0) = \sum_{n=1}^{\infty}\frac{32h}{3n^2\pi^2}\sin\frac{n\pi}{4}\sin\frac{n\pi x}{L}}
$$

(b) For the normal modes with a node at $x = \frac{L}{4}$ we have, $\sin\frac{n\pi x}{L} = \sin\frac{n\pi}{4} = 0 \Rightarrow n = 4k$ for some $k \in \mathbb{N}$. But, we also have

$$
\boxed{c_{4k} = \frac{32h}{3\cdot 16k^2\cdot\pi^2}\sin\frac{\cancel{4}k\pi}{\cancel{4}} = 0}
$$

Hence, those normal modes are absent in the fourier representation of the string.

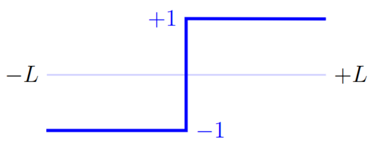
(c) Below is a plot of increasing partial sums of the fourier series expansion of the given function



Here's the code used for generating the above plots :

```python
import matplotlib.pyplot as plt
import numpy as np
import scipy.constants as constants

L = 4
H = 1
x = np.linspace(0, L, 1000)

def c(n):
    return (32*H / (3*n*n*np.pi*np.pi))*np.sin(n*np.pi / 4)
def partial_sums(x, n):
    y = 0
    for i in range(1,n + 1):
        y += c(i)*np.sin((i*constants.pi)*x/L)
    return y
def f(arr):
    y = []
    for x in arr:
        if 0 <= x <= L/4 :
            y.append(4*H*x / L)
        elif L/4 <= x <= L:
            y.append(4*H*(L-x) / (3*L))
    return np.array(y)

fig, ax = plt.subplots(nrows = 2, ncols=2, figsize=(16,12), dpi=400)
fig.suptitle("Fourier Representation of the given string configuration")

n_s = [5, 10, 20, 40]

for i in [0, 1]:
    for j in [0, 1]:
        ax[i, j].set_title(f"{n_s[2*i + j]} terms")
        ax[i, j].axhline(0, color="black", lw=2)
        ax[i, j].axvline(0, color="black", lw=2)
        ax[i, j].set_xticks([0, L/4, L], ["0", r"$\frac{L}{4}$", "L"])
        ax[i, j].plot(x, f(x), "b", label="f(x)", lw=2)
        ax[i, j].plot(x, partial_sums(x, n_s[2*i + j]), "r", label=f"Fourier Series upto {n_s[2*i + j]} terms")
        ax[i, j].legend(loc="upper right")
plt.savefig("fourier_series_1.png", dpi=400)
plt.show()
```

Question 4. Consider the following pattern. Find the Fourier representation of this pattern. Use the complete representation (using sine and cosine). Also, check numerically that your solution matches with the given shape. You may submit your codes.



Solution. Let the fourier series exapansion of $f(x)$ be,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L} \right)$$

Then,

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \mathrm{d}x = 0$$

Also,

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\mathrm{d}x \\
&= -\frac{1}{L}\int_{-L}^{0} \cos\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{L}\int_{0}^{L} \cos\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{1}{L}\int_{0}^{-L} \cos\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{L}\int_{0}^{L} \cos\frac{n\pi x}{L}\mathrm{d}x \\
&= \cancel{-\frac{1}{L}\int_{0}^{L} \cos\frac{n\pi x}{L}}\mathrm{d}x + \cancel{\frac{1}{L}\int_{0}^{L} \cos\frac{n\pi x}{L}}\mathrm{d}x \\
&= 0
\end{aligned}$$

and,

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\
&= -\frac{1}{L}\int_{-L}^{0} \sin\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{L}\int_{0}^{L} \sin\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{1}{L}\int_{0}^{-L} \sin\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{L}\int_{0}^{L} \sin\frac{n\pi x}{L}\mathrm{d}x \\
&= \frac{1}{L}\int_{0}^{L} \sin\frac{n\pi x}{L}\mathrm{d}x + \frac{1}{L}\int_{0}^{L} \sin\frac{n\pi x}{L}\mathrm{d}x \\
&= 2\frac{1}{L}\int_{0}^{L} \sin\frac{n\pi x}{L}\mathrm{d}x \\
&= -\frac{1}{\cancel{L}}\cdot\frac{2\cancel{L}}{n\pi}\cos\frac{n\pi x}{L}\mathrm{d}x\bigg|_0^L \\
&= -\frac{2}{n\pi}(\cos n\pi - 1) \\
&= \begin{cases} \dfrac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

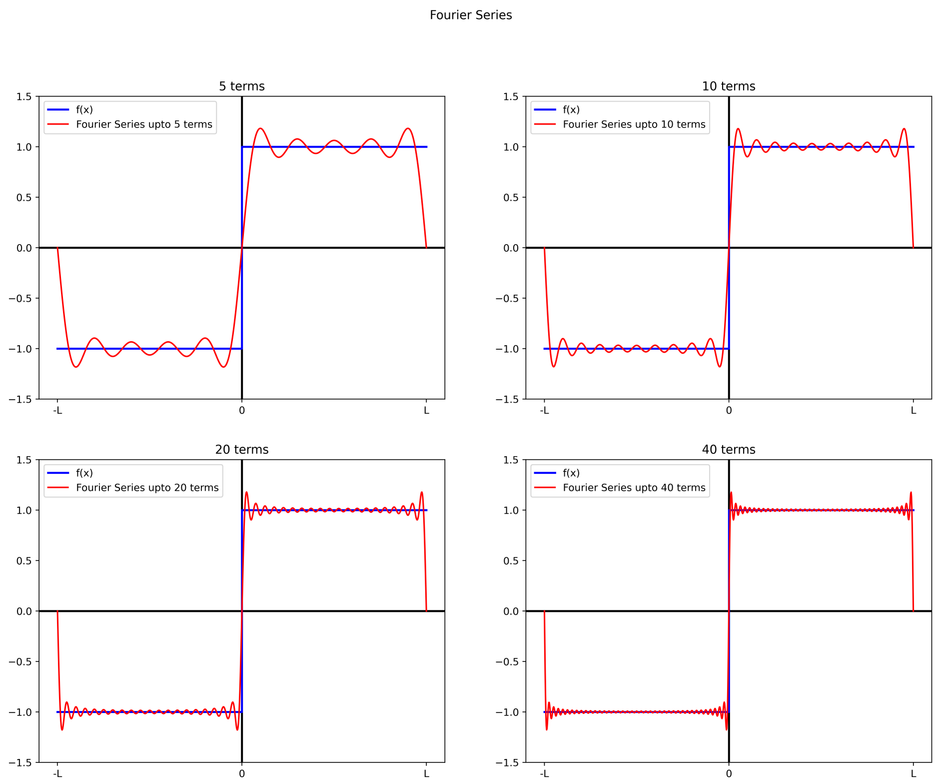
Hence we have

$$f(x) = \sum_{n \text{ is odd}}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{L}$$

Hence, the fourier series exapansion for the given $f(x)$ is

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{L}}$$

Below is a plot of increasing partial sums of the fourier series expansion of the given function



Here's the code used for generating the above plots :

```
import matplotlib.pyplot as plt
import numpy as np
import scipy.constants as constants

L = 1

x = np.linspace(-L, L, 1000)

def b(n):
    return 4/((2*n - 1) * constants.pi)
def partial_sums(x, n):
    y = 0
    for i in range(1,n + 1):
        y += b(i)*np.sin(((2*i - 1)*constants.pi)*x/L)
    return y

fig, ax = plt.subplots(nrows = 2, ncols=2, figsize=(16,12), dpi=400)
fig.suptitle("Fourier Series")

n_s = [5, 10, 20, 40]

for i in [0, 1]:
    for j in [0, 1]:
        ax[i, j].set_title(f"{n_s[2*i + j]} terms")
        ax[i, j].axhline(0, color="black", lw=2)
        ax[i, j].axvline(0, color="black", lw=2)
        ax[i, j].set_ylim(-1.5, 1.5)
        ax[i, j].set_xticks([-L, 0, L], ["-L", "0", "L"])
        ax[i, j].step([-L, 0, L], [-1, -1, 1], "b", label="f(x)", lw=2)
        ax[i, j].plot(x, partial_sums(x, n_s[2*i + j]), "r", label=f"Fourier Series upto {n_s[2*i + j]} terms")
        ax[i, j].legend(loc="upper left")
plt.savefig("fourier_series_2.png", dpi=400)
plt.show()
```