MA2101 - Homework Solutions

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August 6, 2023

Prove the following

1. For $x < y$, we have, $x < \frac{x+y}{2} < y$

   Solution : Since $x < y$, we have $\frac{x}{2} < \frac{y}{2}$. Then, we have $\frac{x}{2} + \frac{x}{2} < \frac{y}{2} + \frac{x}{2} \Rightarrow x < \frac{x+y}{2}$ Similarly, we have $\frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} \Rightarrow \frac{x+y}{2} < y$ Hence, $x < \frac{x+y}{2} < y$ $\square$

2. If $x \leq y + z$ for all $z > 0$, then $x \leq y$.

   Solution : Let $x, y \in \mathbb{R}$ such that $x \leq y + z$ for all $z > 0$. We claim that, $x \leq y$. Let us assume to the contrary that, $x > y$. Then, we have $x - y > 0$. Let $\epsilon := x - y$. Also observe that, $x - y \leq z$ for all $z > 0$. Let us set $z = \frac{\epsilon}{2}$. Then, $x - y \leq z \Rightarrow \epsilon \leq \frac{\epsilon}{2} \Rightarrow 1 \leq \frac{1}{2}$. This is a contradiction. Hence, $x \leq y$. This proves our claim. $\square$

3. For $0 < x < y$, we have $0 < x^2 < y^2$ and $0 < \sqrt{2} < \sqrt{y}$, assuming te existence of $\sqrt{x}$ and $\sqrt{y}$. More generally, if x and y are positive, then $x < y$ iff $x^n < y^n$ for all $n \in \mathbb{N}$.

   Solution : Done. Shall type it later.

4. For $0 < x < y$, we have $\sqrt{xy} < \frac{x+y}{2}$.

   Solutions : We claim that the statment is true. Let us assume to the contrary that, $\frac{x+y}{2} < \sqrt{xy}$. Then, we have,

$$\begin{aligned} &\frac{x+y}{2} < \sqrt{xy} \\ \Rightarrow &\left(\frac{x+y}{2}\right)^2 < xy && \text{(Using the result from Problem 2)} \\ \Rightarrow &\left(\frac{x+y}{2}\right)^2 - xy < 0 \\ \Rightarrow &\left(\frac{x-y}{2}\right)^2 < 0 \end{aligned}$$

This is a contradiction since we know that $\alpha^2 \geq 0 \ \forall \alpha \in \mathbb{R}$. This proves our claim. $\square$