

PH2101 - Waves and Optics

Assignment 4 Solutions

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22MS002

September 19, 2023

Question 1. Consider a one-dimensional oscillator of unit mass and of natural frequency ω_0 (natural frequency: frequency of undamped oscillation). The oscillator experiences a velocity dependent damping with a damping coefficient γ . Initially the oscillator was at rest $x(0) = \dot{x}(0) = 0$. The oscillator is subjected to a force given by $F_0 \cos \omega t$.

- (a) Find the complete solution as a function of time.
- (b) Check the case when the driving frequency is equal to the natural frequency.

Solution.

- (a) The differential equation for the given system will be,

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

Let $\alpha := \frac{\gamma}{m}$. First we solve the homogenous part of this equation by assuming, $x = Ae^{i\omega' t}$. Then, we have

$$\begin{aligned} -\omega'^2 + i\alpha\omega' + \omega_0^2 &= 0 \\ \Rightarrow \omega'^2 - i\alpha\omega' - \omega_0^2 &= 0 \\ \Rightarrow \omega' &= \frac{i\alpha \pm \sqrt{4\omega_0^2 - \alpha^2}}{2} \\ \Rightarrow \omega' &= \frac{i\alpha}{2} \pm \sqrt{\omega_0^2 - \frac{\alpha^2}{4}} \end{aligned}$$

Let $\omega_1 = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}$. Then, the solution to the homogenous solution is,

$$\begin{aligned} x &= Ae^{i(\frac{i\alpha}{2} + \omega_1)t} + Be^{i(\frac{i\alpha}{2} - \omega_1)t} \\ &= e^{-\frac{\alpha t}{2}} (Ae^{i\omega_1 t} + Be^{-i\omega_1 t}) \\ &= e^{-\frac{\alpha t}{2}} (C \cos \omega_1 t + D \sin \omega_1 t) \end{aligned}$$

Since the RHS is $\frac{F_0}{m} \cos \omega t$, we take a guess that $x(t)$ is of the form $x = Ae^{i\omega t}$. Then, putting this value in the equation we get,

$$\begin{aligned} A[-\omega^2 - i\alpha\omega + \omega_0^2] &= \frac{F_0}{m} \\ \Rightarrow A &= \frac{F_0}{m} \cdot \frac{1}{i\alpha\omega + (\omega_0^2 - \omega^2)} \\ \Rightarrow A &= \frac{F_0}{m} \cdot \frac{1}{\sqrt{\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2} e^{i\phi}} \end{aligned}$$

Where $\phi = \arctan\left(\frac{\alpha\omega}{\omega_0^2 - \omega^2}\right)$ Hence, the solution is

$$x(t) = e^{-\frac{\alpha t}{2}} (C \cos \omega_1 t + D \sin \omega_1 t) + A \cos(\omega t - \phi)$$

Putting the initial values in the problem we get, $C = -A \cos \phi$ and

$$\begin{aligned} -\frac{\alpha}{2}C + \omega_1 D + \omega A \sin \phi &= 0 \\ \Rightarrow \omega_1 D &= -A \left(\omega \sin \phi + \frac{\alpha}{2} \cos \phi \right) \\ \Rightarrow D &= -\frac{A}{\omega_1} \left(\omega \sin \phi + \frac{\alpha}{2} \cos \phi \right) \end{aligned}$$

This determines $x(t)$ uniquely.

- (b) When driving frequency is equal to the natural frequency, we have $\omega = \omega_0$. Clearly, in this case, the amplitude of oscillation is maximum. The particular solution will be,

$$x(t) = \frac{F_0}{m\alpha\omega} e^{i(\omega t - \frac{\pi}{2})}$$

Since, $\phi = \arctan \frac{\alpha\omega}{\omega_0^2 - \omega^2} = \frac{\pi}{2}$ and $A = \frac{F_0}{m\alpha\omega}$. So, if we want to write the same solution, the

values of C and D will be, $C = 0$ and $D = -\frac{A\omega}{\omega_1}$

Question 2. Consider how to solve the steady-state motion of a forced oscillator if the driving force is of the form $F = F_0 \sin \omega t$ instead of $F_0 \cos \omega t$.

Solution. If it's sin instead of cos, we solve it the exact similar way, except when solving for the particular solution, we put $-i\frac{F_0}{m}$ in the RHS, since we are working with the Sine part of the complex exponential.

Hence, after solving we obtain the particular solution to be $x(t) = A \sin(\omega t - \phi)$

After that we again proceed similarly to determine the constants C and D from the homogenous part.

Question 3. A block of mass m is connected to a spring, the other end of which is fixed. There is also a viscous damping mechanism. The following observations have been made on this system:

1. If the block is pushed horizontally with a force equal to mg , the static compression of the spring is equal to h .
 2. The viscous resistive force is equal to mg if the block moves with a certain known speed u .
- (a) For this complete system (including both spring and damper) write the differential equation governing horizontal oscillations of the mass in terms of m , g , h , and u . Answer the following for the case that $u = 3\sqrt{gh}$.
 - (b) What is the angular frequency of the damped oscillations?
 - (c) After what time, expressed as a multiple of $\sqrt{h/g}$, is the energy down by a factor $1/e$?
 - (d) What is the quality factor Q of this oscillator?
 - (e) This oscillator, initially in its rest position, is suddenly set into motion at $t = 0$ by a bullet of negligible mass but nonnegligible momentum traveling in the positive x direction. Find the value of the phase angle ϕ in the equation $x = Ae^{-\frac{\alpha t}{2}} \cos(\omega t - \phi)$ that describes the subsequent motion, and sketch x versus t for the first few cycles.

- (f) If the oscillator is driven with a force $mg \cos \omega t$, where $\omega = \sqrt{2g/h}$, what is the amplitude of the steady-state response?

Solution.

- (a) From the given information about the system we can deduce that,

$$k = \frac{mg}{h} \text{ and } \gamma = \frac{mg}{u}$$

Hence, the differential equation turns out to be as in Question 1,

$$\ddot{x} + \frac{g}{u} \dot{x} + \frac{g}{h} x = 0$$

For $u = \sqrt{3gh}$ we get,

$$\ddot{x} + \frac{1}{3} \sqrt{\frac{g}{h}} \dot{x} + \frac{g}{h} x = 0$$

- (b) Let $x = Ae^{i\omega' t}$. Then, as previously done for the homogenous part in Question 1,

$$\begin{aligned} \omega' &= \frac{i}{6} \sqrt{\frac{g}{h}} \pm \sqrt{\frac{g}{h} - \frac{g}{36h}} \\ &= \frac{i}{6} \sqrt{\frac{g}{h}} \pm \sqrt{\frac{35g}{36h}} \end{aligned}$$

Let $\omega = \sqrt{\frac{35g}{36h}}$ and $\alpha = \sqrt{\frac{g}{9h}}$. Then, the solution of the equation turns out to be,

$$x(t) = e^{-\frac{\alpha}{2}t} C \cos(\omega t - \phi)$$

Where the angular frequency (ω) is given by,

$$\omega = \sqrt{\frac{35g}{36h}}$$

- (c) We have

$$\begin{aligned} \dot{x} &= C e^{-\frac{\alpha t}{2}} \cos(\omega t - \phi) \\ \Rightarrow \dot{x} &= -C e^{-\frac{\alpha t}{2}} \left[\omega \sin(\omega t - \phi) + \frac{\alpha}{2} \cos(\omega t - \phi) \right] \\ \Rightarrow \dot{x} &= -C e^{-\frac{\alpha t}{2}} \omega \left[\sin(\omega t - \phi) + \frac{\alpha}{2\omega} \cos(\omega t - \phi) \right] \\ \Rightarrow \dot{x} &\sim -C e^{-\frac{\alpha t}{2}} \omega_0 \sin(\omega t - \phi) \quad (\omega_0 \gg \alpha \Rightarrow \omega = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}} \sim \omega_0) \end{aligned}$$

Total energy can be written as,

$$\begin{aligned} E &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{mg}{h} x^2 \\ \Rightarrow E &= \frac{mA^2 g}{2h} e^{-\alpha t} \end{aligned}$$

We are given that, $E(t) = \frac{1}{e} E(0) \Rightarrow e^{-\alpha t} = \frac{1}{e} \Rightarrow t = \frac{1}{\alpha} = 3 \sqrt{\frac{h}{g}}$

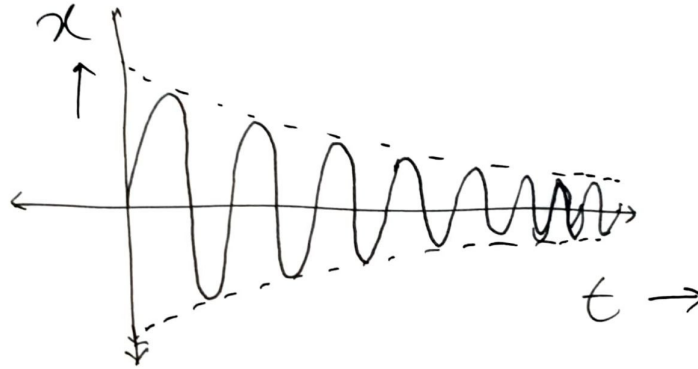
Hence, it's 3 units of $\sqrt{\frac{h}{g}}$.

$$(d) Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{g}{h}} \cdot 3 \sqrt{\frac{h}{g}} = 3$$

- (e) According to the given conditions, we have $x = 0$ and $\dot{x} \neq 0$. Then, putting these values we get

$\phi = \frac{\pi}{2}$ Hence, the solution is

$$x(t) = Ae^{-\frac{\alpha t}{2}} \sin \omega t$$



- (f) The amplitude of the steady state response can be obtained as:

$$\begin{aligned}
 A &= \frac{F_0}{m} \frac{1}{\sqrt{\alpha^2 \omega^2 + (\omega_0^2 - \omega^2)}} \\
 \Rightarrow A &= \frac{g}{\sqrt{\frac{g}{9h} \cdot \frac{2g}{h} + \left(\frac{g}{h} - \frac{2g}{h}\right)^2}} \\
 \Rightarrow A &= \frac{h}{\sqrt{1 + \frac{2}{9}}} \\
 \Rightarrow A &= \frac{3h}{\sqrt{11}}
 \end{aligned}$$