MA2101: Analysis I Lecture Notes

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§1 Algebra of the Real Number System

§1.1 Properties of Addition

The properties of addition(+) in the real number system are:

(A1) $x + y = y + x \; \forall \; x, y \in \mathbb{R}$

(A2) $(x + y) + z = x + (y + z) \; \forall \; x, y, z \in \mathbb{R}$

(A3) $\exists! 0 \in \mathbb{R} \; s.t. \; x + 0 = 0 + x = x \; \forall \; x \in \mathbb{R}$

(A4) $\forall x \in \mathbb{R} \exists! y \in \mathbb{R}$ s.t. $x + y = y + x = 0$

§1.2 Properties of Multiplication

The properties of multiplication($\cdot$) in the real number system are:

(M1) $x \cdot y = y \cdot x \forall x, y \in \mathbb{R}$

(M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z) \forall x, y, z \in \mathbb{R}$

(M3) $\exists! 1 \in \mathbb{R}$ s.t. $x \cdot 1 = x \forall x \in \mathbb{R}$

(M4) $\forall x \in \mathbb{R} \setminus \{0\} \exists! y \in \mathbb{R}$ s.t. $x \cdot y = y \cdot x = 0$

§1.3 Distributive Property

The multiplication operator distributes over addition inn real numbers.

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Since addition and multiplication have these properties in real numbers, $(\mathbb{R}, +, \cdot)$ is a Field.

§1.4 Order in Reals

§1.4.1 Law of Trichotomy

Given two $x, y \in \mathbb{R}$, exaclt one of the following statements is true :

(i) $x = y$

(ii) $x > y$

(iii) $x < y$

§1.4.2 Properties of $<$

(i) If $x < y$ and $y < z$ then $x < z$

(ii) If $x > 0$, $y > 0$ then, $xy > 0$

(iii) If $x < y$ then, $x + z < y + z \ \forall z \in \mathbb{R}$

(iv) $x < y \Rightarrow -x > -y$

(v) If $x < y$ and $z > 0$ then $xz < yz$

(vi) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$

(vii) $x^2 \geq 0 \ \forall x \in \mathbb{R}$

Remark 1.1

Let $x, y \in \mathbb{R}$ such that, $x \leq y$ and $y \leq x$. Then, $x = y$.

Proof 1:

Let's assume to teh contrary that $x \neq y$. Then, by the law of trichotomy, either $x < y$ or $y < x$. Let $y < x$. From $x \leq y$ we have either $x < y$ or $x = y$. By the law of trichotomy, neither of them can be true. Hence, $y \not< x$ Now, let $x < y$. From $y \leq x$ we have either $y < x$ or $y = x$. Again, by the law of trichotomy, neither of them can be true. Hence, $x \not< y$ This is a contradiction. Hence, $x = y$ Q.E.D.

Example 1:

For $x < y$, we have, $x < \frac{x+y}{2} < y$. The point $\frac{x+y}{2}$ is called the midpoint between x and y.

Proof 2:

Since $x < y$, we have $\frac{x}{2} < \frac{y}{2}$. Then, we have $\frac{x}{2} + \frac{x}{2} < \frac{y}{2} + \frac{x}{2} \Rightarrow x < \frac{x+y}{2}$ Similarly, we have $\frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} \Rightarrow \frac{x+y}{2} < y$ Hence, $x < \frac{x+y}{2} < y$ Q.E.D.

Example 2:

If $x \leq y + z$ for all $z > 0$, then $x \leq y$.

Proof 3:

Let $x, y \in \mathbb{R}$ such that $x \leq y + z$ for all $z > 0$. We claim that, $x \leq y$. Let us assume to the contrary that, $x > y$. Then, we have $x - y > 0$. Let $\epsilon := x - y$. Also observe that, $x - y \leq z$ for all $z > 0$. Let us set $z = \frac{\epsilon}{2}$. Then, $x - y \leq z \Rightarrow \epsilon \leq \frac{\epsilon}{2} \Rightarrow 1 \leq \frac{1}{2}$. This is a contradiction. Hence, $x \leq y$. This proves our claim. Q.E.D.

Example 3:

For $0 < x < y$, we have $0 < x^2 < y^2$ and $0 < \sqrt{2} < \sqrt{y}$, assuming te existence of $\sqrt{x}$ and $\sqrt{y}$. More generally, if x and y are positive, then $x < y$ iff $x^n < y^n$ for all $n \in \mathbb{N}$.

Proof 4:

will type up later Q.E.D.

Example 4:
For $0 < x < y$, we have $\sqrt{xy} < \dfrac{x+y}{2}$.

Proof 5:
We claim that the statment is true. Let us assume to the contrary that, $\dfrac{x+y}{2} < \sqrt{xy}$. Then, we have,

$$\begin{aligned}
&\frac{x+y}{2} < \sqrt{xy} \\
\Rightarrow &\left(\frac{x+y}{2}\right)^2 < xy && \text{(Example 1)} \\
\Rightarrow &\left(\frac{x+y}{2}\right)^2 - xy < 0 \\
\Rightarrow &\left(\frac{x-y}{2}\right)^2 < 0
\end{aligned}$$

This is a contradiction since we know that $\alpha^2 \geq 0 \ \forall \alpha \in \mathbb{R}$. This proves our claim. Q.E.D.

§2 Upper and Lower Bounds

Definition 1 (Upper Bound)
Let $A \subset \mathbb{R}$ be nonempty. A number $\alpha \in \mathbb{R}$ is said to be the upper bound of A if $\forall x \in A$, we have $x \leq \alpha$

Geometrically, this means that on the real number line, all the elements of A are to the left of α. If $\alpha \in \mathbb{R}$ is not an upper bound of A, then $\exists x \in A$ s.t. $x > \alpha$

Definition 2 (Lower Bound)
Let $A \subset \mathbb{R}$ be nonempty. A number $\alpha \in \mathbb{R}$ is said to be the lower bound of A if $\forall x \in A$, we have $x \geq \alpha$

Geometrically, this means that on the real number line, all the elements of A are to the right of α. If $\alpha \in \mathbb{R}$ is not an lower bound of A, then $\exists x \in A$ s.t. $x < \alpha$

Example 5:
Consider the set $A := \{1, 2, 3, 4, 5\}$ Then the upper bounds of this A are $5, 6, 1729 \dots$ etc. And the lower bound of A are $1, 0, -1 \dots$ etc. Hence, lower and upper bounds of a set are not unique.

Definition 3 (Bounded above set)
Let $\phi \neq A \subset \mathbb{R}$. Then, A is said to be bounded above, if $\exists \alpha \in \mathbb{R}$ such that α is an upper bound of A.

Definition 4 (Bounded below set)
Let $\phi \neq A \subset \mathbb{R}$. Then, A is said to be bounded below, if $\exists \alpha \in \mathbb{R}$ such that α is an lower bound of A.

Theorem 2.1

Let A be a bounded above set. Let $\alpha \in \mathbb{R}$ be an upper bound. Let $\beta \in \mathbb{R}$ such that $\beta \geq \alpha$. Then β is an upperbound of A.

Proof 6:

Since α is an upper bound, we have $x \leq \alpha \; \forall x \in A$ But, $\alpha \leq \beta$. Then we have, $x \leq \beta \; \forall x \in A$. Hence β is an upper bound of A. Q.E.D.

A similar theorem can be stated and proved analagously for bounded below sets and lower bounds.

Definition 5 (Maximum)

Let $\phi \neq A \subset \mathbb{R}$. $\alpha \in \mathbb{R}$ is said to be the maximum of A if

(i) $\alpha \in \mathbb{R}$

(ii) α is an upper bound of A

The maximum of a set A is denoted by $\max\{A\}$

Theorem 2.2

The maximum of a set is unique.

Proof 7:

Let α and β be two maxima of A s.t. $\alpha \neq \beta$. Then, by the law of trichotomy, either $\alpha > \beta$ or $\beta > \alpha$. If $\alpha > \beta$, then α cannot be an upper bound since, $\beta \in A$. And, if $\beta > \alpha$, then β cannot be an upper bound since, $\alpha \in A$. This is a contradiction. Hence, $\alpha = \beta$. Q.E.D.