MTH1003 Mathematical Modelling

Project 2 (group): Motion in a potential (on a rollercoaster)

Submission deadline: 12.00 noon on Thursday 6th February.
Poster presentation: 16.30-18.30 on Thursday 13th February.

In this project you will use mathematics and numerical methods coded in Python to study the motion of a rollercoaster cart by treating the cart as a point particle moving in a 1D potential. You will work in groups of five or six people, and each group will make a poster to be submitted in week 4 of term 2 and presented in week 5. The poster should describe the methods used and clearly illustrate the results using plots created in Python.

1 Motion in a potential

Figure 1: Rollercoaster: Getty images

A rollercoaster cart of mass $m = 10^3$kg which we treat as a point particle, travels along a smooth track under a conservative force given by $F(s)$ which is mass times the gravity projected along the track (i.e. $F(s) = -mg\sin(\alpha)$ with α the angle of the track to the horizontal and $g = 10\text{ms}^{-2}$). The potential associated with this force is given by

$$V(s) = -V_0\left(\frac{s}{a}\right)^2\left(\left(\frac{s}{b}\right)^2-1\right)\left(\left(\frac{s}{c}\right)^2-1\right)\exp\left(-2\left(\left(\frac{s}{d}\right)^2-1\right)\right)\ \mathrm{J}, \tag{1}$$

where $V(s)$ is measured in Joules and $a = 100$m, $b = 100$m, $c = 50 + 10G/6$m, $d = 100$m and G as your group number. Here, s denotes the position in metres along the track and V_0 determines the magnitude of the potential (which you will calculate later).

Q1 Numerical derivatives; Convergence.

Find an exact expression for the force $F(s)$. Estimate the maximum value of the function $F(s)/V_0$ to determine the value of V_0 required to give the maximum acceleration in the positive s direction felt by the cart to be $g = 10\mathrm{ms}^{-2}$.

Let the interval size $h = h_0 = 50$m. For a set of points $\{s_1, s_2, ..., s_N\}$ uniformly spaced a distance h_0 apart between -500m and 500m, compute a numerical estimate for the force $F_{num}(s_i)$ for $i = 1, ..., N$ using the Forward Euler finite difference formula. Find the maximum absolute error

$$\epsilon(h) = \max|F_{num}(s_i) - F(s_i)|. \tag{2}$$

Repeat the exercise for $h = h_0 \times 2^{-1}, h_0 \times 2^{-2}, ..., h_0 \times 2^{-6}$. Plot a graph of $\log\epsilon(h)$ versus $\log h$. Does the Forward Euler method converge as predicted by theory?

Q2 Equilibria.

Using one of the root finding methods discussed in lectures, find all the equilibrium positions of the rollercoaster cart. You will need to use different first guesses to find all of the equilibrium positions; you might find it helpful to plot a graph of $F(s)$. By examining the value of $V''(s)$, determine whether each equilibrium position is stable or unstable and calculate the frequency of small-amplitude oscillations about each stable equilibrium position.

Q3 Numerical solution for small amplitude oscillations.

Newton's second law may be written as the pair of equations

$$\frac{ds}{dt} = v, \quad \frac{dv}{dt} = \frac{F(s_n)}{m}. \tag{3}$$

Write a Python code to solve this pair of equations using the Leapfrog time stepping method:

$$s_{n+1} = s_{n-1} + 2hv_n, \tag{4}$$

$$v_{n+1} = v_{n-1} + 2h\frac{F(s)}{m}, \tag{5}$$

with time step h.

You might want to base your code on the code used in lectures for the hard springs example. Use the exact expression for $F(s)$ that you found in part 1.

Compute a numerical solution starting from an initial position at rest close to one of the stable equilibria. You might need to experiment to choose a suitable time step. Discuss whether the particle oscillates as predicted in part 2, i.e. compare the frequency of the oscillations to that predicted in par 2.

Q4 Long term behaviour.

The rollercoaster cart is launched to the positive s from the left-most (largest negative s value) stable equilibrium position with a velocity U. There is some value $U_{crit} > 0$ such that for $0 < U < U_{crit}$ the particle either oscillates or goes to $-\infty$ for large times while for $U > U_{crit}$ the particle's position tends to $+\infty$ for large times. Using the energy equation, along with the values of V at the equilibrium positions found in part 2, determine the value of U_{crit}.

Using the Leapfrog time stepping code from part 3, compute two numerical solutions for $s(t)$ starting from the left-most stable equilibrium position, one with U slightly less than U_{crit}, one with U slightly greater than U_{crit}. Is the behaviour as predicted?

2 Posters

You should prepare a poster presenting your methods and results. **The project will be assessed partly on the results obtained and partly on the presentation quality of the poster.**

Posters should be designed to be printed in A2 size in portrait orientation using the Powerpoint template provided. **Please export the final version as pdf** to reduce the risk of corruption.

Posters should be legible to a reader standing nearby, so dense text or equations is not appropriate. Think about getting the right balance of figures, equations, and text, and make sure you give clear but concise explanations that cover the key points of the method and results. Two-column layout usually works well. There is no need to include Python code on the poster.

Include on the poster your group number and the names of all group members who have contributed.

3 Group working

This module and its assessment are not just about learning new mathematics or applying it in new contexts; they are also about working together in a professional manner, managing and coordinating work within a group of five or six people from diverse backgrounds and abilities, and ensuring a fair division of work between you. For the group work in this project you will need to jointly contribute to preparation of a poster. This should involve you doing some or all of the following:

1. working together in a group on aspects of the problem;
2. doing mathematical calculations;
3. implementing numerical methods in Python;
4. explaining what you have done clearly and concisely;
5. helping others with support and constructive criticism (but do be tactful);
6. designing and/or editing sections, and preparing diagrams for the poster.

We will provide each group with a Wiki on ELE which should be used to document the functioning of the group, and the contribution of individual students. You should use this medium to record notes of key meetings and who was present, tasks with deadlines as assigned within the group, the uploading of draft material (text, figures, scripts, posters), and the commenting on and editing of draft material. The use of the Wiki is part of how we will assess your contribution to the project; see below.

3.1 Dividing up tasks

There are various ways of dividing up the work on this project among group members. One way would be along the lines of the four parts detailed above. Note, however, that there are interdependencies between the different parts, so those working on different parts will need to communicate and coordinate.

Another way is to think about the different types of tasks involved and to try to play to the strengths of the different group members: Who might be a good leader/coordinator/organiser? Who might be good at mathematical derivations? Who might be a good coder? Who might be good with Powerpoint? Who might be good at clear and concise communication and poster layout?

Either way, it will be important to bring all of the pieces of the project together at the end into a coherent piece of work.

Although you'll need to divide up tasks, **it is every group member's responsibility** that the resulting poster is as good as possible in terms of addressing the science and computing tasks and explaining the results to the reader in a manner that is attractive, clear, and 'flows'.

3.2 Working as a group

All members of your group (including you) depend upon each other to engage in the required task and make a full contribution. So do what you say you will do to the best of your ability, and if you are ill or can't attend a group meeting for any reason, then let members of your group know as soon as possible and arrange with them how to rectify the situation. It is a good idea to exchange email addresses and mobile numbers and/or use a social media group facility.

Don't expect your group to countenance clear lack of effort or unreliability on your part.

To encourage all group members to make a full contribution to the project, the mark awarded will depend in part on how your fellow group members rate your contribution. We will explain the mechanics of this rating system towards the end of the project.

Working in a group is not always easy, but is an important learning experience which is valuable for work and life beyond the university, whatever career path you take. If you have any concerns about your group dynamics then do talk with the lecturer or your personal tutor, for general advice and support.

4 Submission

Posters should be submitted in pdf format via ELE. **Only one submission per group is needed.**

The deadline is 12.00 noon on Thursday 6th February. Note that as this is a group project and the presentations happen soon after the submission of the poster, **we are not in a position to allow for either individual or group extensions to the submission deadline.**

5 Poster presentation

We have arranged for each group to give a presentation of their poster on Thursday February 13th 2025 between 4.30 and 6.30. Your exact timeslot (4.30-5.30 or 5.30-6.30) will be arranged nearer the time, but please keep this time free. **All group members should be present, so keep this time free.** Staff will read the poster and ask the group a few questions about it.

6 Assessment

Marks will be awarded as follows:

- 70% Quality of poster: including mathematical accuracy, quality of discussion, quality of figures, overall layout, attractiveness and readability.
- 20% Poster presentation: being able to respond to questions about the poster itself and how the results fit into the wider context, including theory, numerical methods and programming.
- 10% Evidence on the ELE Wiki showing the functioning of the group, including minuting key meetings, uploading and commenting on draft material, and assigning tasks and deadlines within the group.

An overall mark will be given to each group based on the above breakdown. We will also require each student to give information on the contribution of other students within the group, and this will be used to determine individual marks from the group mark, which may be higher or lower. The group ELE Wiki may be used to check on the contribution of students to the group effort. Any student who does not participate in their group, or shows very limited engagement, will gain a correspondingly low mark. We will publish more information on the marking process in due course, on the ELE page.

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