

An Algorithm to Calculate Correlated Colour Temperature

A new algorithm to calculate correlated colour temperature is given. This algorithm is based on a rational Chebyshev approximation of the Planckian locus in the CIE 1960 UCS diagram and a bisection procedure. Thus time-consuming search procedures in tables or charts are no longer necessary.

Introduction

The colour temperature of a source is defined to be the temperature of a Planckian radiator emitting radiation of the same chromaticity as the source. When a source does not have the chromaticity of any possible Planckian radiator, a correlated colour temperature may be assigned to such a source, being defined as the temperature of the Planckian radiator whose chromaticity is nearest to that of the source in the CIE 1960 UCS diagram. This temperature is commonly found by computing u and v and then consulting charts such as those published by Kelly¹ showing lines of constant correlated colour temperature, which are by definition normal to the Planckian locus in the CIE 1960 UCS diagram. To make better use of such charts Robertson² developed a method based directly on that graphical method that allows a good interpolation between two adjacent members of a set of isothermperature lines. Unfortunately the method requires a large amount of data to be stored in the computer.

A method of Hohm and Krochmann³ based on curve rectification⁴ is not very useful for automatic computation, because again a large amount of data must be stored in the computer and the coefficients depend on the loci in the chromaticity diagram, i.e., the rectification is not complete.

In this article an algorithm is given that is based on a rational Chebyshev approximation of the Planckian locus and a bisection procedure to find the correlated colour temperature of a source.

Rational Chebyshev Approximation of the Planckian Locus

The radiance L_λ of a Planckian radiator is given by

$$L_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1 \right]^{-1}, \quad (1)$$

where T is the temperature, λ the wavelength, c the speed of light, h Planck's constant, and k_B Boltzmann's constant. The tristimulus values $X(T)$, $Y(T)$, and $Z(T)$ of a Planckian radiator of temperature T are obtained by multiplying the radiance by the CIE spectral tristimulus values $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, and $\bar{z}(\lambda)$ and subsequently integrating over the wavelength:

$$X(T) = k \int_0^\infty \bar{x}(\lambda) L_\lambda(\lambda, T) d\lambda, \quad (2)$$

$$Y(T) = k \int_0^\infty \bar{y}(\lambda) L_\lambda(\lambda, T) d\lambda, \quad (3)$$

$$Z(T) = k \int_0^\infty \bar{z}(\lambda) L_\lambda(\lambda, T) d\lambda. \quad (4)$$

Here k is a normalization constant. The chromaticity coordinates $x(T)$ and $y(T)$ of the Planckian radiator in the CIE 1931 chromaticity diagram are given by

$$x(T) = \frac{X(T)}{X(T) + Y(T) + Z(T)}, \quad (5)$$

$$y(T) = \frac{Y(T)}{X(T) + Y(T) + Z(T)}, \quad (6)$$

and the coordinates $u(T)$ and $v(T)$ in the CIE 1960 UCS diagram by

$$u(T) = \frac{4x(T)}{-2x(T) + 12y(T) + 3}, \quad (7)$$

$$v(T) = \frac{6y(T)}{-2x(T) + 12y(T) + 3}. \quad (8)$$

Combining eqs. (2) to (8) yields

$$u(T) = \frac{4 \int_0^\infty \bar{x}(\lambda) \lambda^{-5} \left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1 \right]^{-1} d\lambda}{\int_0^\infty [\bar{x}(\lambda) + 15\bar{y}(\lambda) + 3\bar{z}(\lambda)] \lambda^{-5} \left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1 \right]^{-1} d\lambda}, \quad (9)$$

$$v(T) = \frac{6 \int_0^\infty \bar{y}(\lambda) \lambda^{-5} \left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1 \right]^{-1} d\lambda}{\int_0^\infty [\bar{x}(\lambda) + 15\bar{y}(\lambda) + 3\bar{z}(\lambda)] \lambda^{-5} \left[\exp\left(\frac{hc}{\lambda k_B T}\right) - 1 \right]^{-1} d\lambda}. \quad (10)$$

Equations (9) and (10) are the parameter representation of the Planckian locus in the CIE 1960 UCS diagram. The integrals are commonly replaced by summation over 1-nm intervals.

The representation of the Planckian locus as given by eqs. (9) and (10) is too complicated to be useful in practical computation of correlated colour temperature. Therefore a rational Chebyshev approximation has been done. This approximation is defined by

$$\min_{T_i \leq T \leq T_f} \max_{T_i \leq T \leq T_f} \left| u(T) - \frac{P(T)}{Q(T)} \right|, \quad (11)$$

$$P(T) = \sum_{k=0}^n a_k T^k, \quad (12)$$

$$Q(T) = \sum_{k=0}^m b_k T^k. \quad (13)$$

That is, one has to find a quotient of two polynomials in T which minimizes the absolute error within an interval (T_i , T_f). Such an approximation is known to be the best possible within the class of rational functions (including simple polynomials!).⁵ The calculation has been done by use of the Remes algorithm (for details see, e.g., ref. 5); the results for an interval (1000 K, 15,000 K) are:

$$\bar{u}(T) = \frac{0.860117757 + 1.54118254 \times 10^{-4} T + 1.28641212 \times 10^{-7} T^2}{1 + 8.42420235 \times 10^{-4} T + 7.08145163 \times 10^{-7} T^2}, \quad (14)$$

$$\bar{v}(T) = \frac{0.317398726 + 4.22806245 \times 10^{-5} T + 4.20481691 \times 10^{-8} T^2}{1 - 2.89741816 \times 10^{-5} T + 1.61456053 \times 10^{-7} T^2}. \quad (15)$$

The errors of the approximation are

$$|u(T) - \bar{u}(T)| < 8 \times 10^{-5} \quad (16)$$

$$|v(T) - \bar{v}(T)| < 9 \times 10^{-5} \quad (17)$$

That is, $\bar{u}(T)$ and $\bar{v}(T)$ may be used instead of $u(T)$ and $v(T)$ in practical calculations because the approximation errors are much smaller than the usual measurement uncertainties.

Computation of Correlated Colour Temperature by Bisection

The lines of constant correlated colour temperature (isotemperature lines) are by definition normal to the Planckian locus in the CIE 1960 UCS diagram, i.e., they are given

by the following formula:

$$\frac{u(T_c) - u_s}{v(T_c) - v_s} = -\frac{dv(T_c)}{du(T_c)}, \quad (18)$$

where u_s and v_s are the chromaticity coordinates of the source of which the correlated colour temperature T_c is to be calculated. Introducing a function

$$F(u_s, v_s, T_c) = \frac{du(T_c)}{dT_c} [u(T_c) - u_s] + \frac{dv(T_c)}{dT_c} [v(T_c) - v_s], \quad (19)$$

eq. (18) can also be written:

$$F(u_s, v_s, T_c) = 0. \quad (20)$$

That is, the correlated colour temperature of a source with chromaticity coordinates u_s and v_s given, is a zero of the function $F(u_s, v_s, T_c)$ as defined in eq. (19). This zero (if present) can be found by a bisection procedure. The algorithm is as follows:

- (a) $T_i = 1000$ K, $T_f = 15000$ K
- (b) If $\text{sgn } F(T_i) = \text{sgn } F(T_f)$ then " T_c undefined!"
- (c) $T_c = 0.5 * (T_i + T_f)$

- (d) If $F(T_c) = 0$ then "done!"
- (e) If $\text{sgn } F(T_c) = \text{sgn } F(T_i)$ then $T_i = T_c$ else $T_f = T_c$
- (f) If $\text{abs}(T_f - T_i) > \epsilon$ then repeat steps c to f else "done!"

where ϵ gives the desired precision.

The algorithm stops after a finite number of steps and is

accurate and fast. The computation of $F(T)$ using the rational approximations $\bar{u}(T)$ and $\bar{v}(T)$ is also fast if Horner's algorithm is used to calculate the values of the polynomials and their derivatives.

The algorithm to calculate the correlated colour temperature was programmed in machine code of the 6502 micro-processor and also in BASIC (the program is available on request from the author). The machine-code program needs less than 1 s to find T_c with an accuracy of 1 K, while the BASIC program needs approximately 10 s for the same task, running on the CBM 8032 computer.

Discussion of Errors

Two kinds of errors must be discussed. The first one is the uncertainty of T_c coming from the approximation errors of the Planckian locus, and the second one is due to measurement uncertainties. The influence of the approximation error can be found as follows. Equation (20) may be written

$$\frac{d}{dT_c} \{[u(T_c) - u_s]^2 + [v(T_c) - v_s]^2\} = 0. \quad (21)$$

Replacing $u(T_c)$ and $v(T_c)$ by the approximation $\bar{u}(T_c)$ and $\bar{v}(T_c)$ causes errors ϵ_u and ϵ_v , respectively. Thus eq. (21) changes to

$$\frac{d}{dT_c} \{[u(T_c) + \epsilon_u - u_s]^2 + [v(T_c) + \epsilon_v - v_s]^2\} = 0. \quad (22)$$

That is, the approximation errors yield a value of T_c belonging to chromaticity coordinates $u_s - \epsilon_u$ and $v_s - \epsilon_v$, rather than u_s and v_s , respectively.

Measurement uncertainties can be considered by adding terms Δu_s and Δv_s to u_s and v_s , respectively, and ΔT_c to T_c . Thus eq. (20) must then be written.

$$\begin{aligned} \frac{du(T_c + \Delta T_c)}{dT_c} [u(T_c + \Delta T_c) - u_s - \epsilon_u - \Delta u_s] \\ + \frac{dv(T_c + \Delta T_c)}{dT_c} [v(T_c + \Delta T_c) - v_s - \epsilon_v \\ - \Delta v_s] = 0. \end{aligned} \quad (23)$$

$$|\Delta T_c| = \frac{\left| \frac{du(T_c)}{dT_c} (\epsilon_u + \Delta u_s) + \frac{dv(T_c)}{dT_c} (\epsilon_v + \Delta v_s) \right|}{\left| \frac{d^2u(T_c)}{dT_c^2} [u(T_c) - u_s] + \frac{d^2v(T_c)}{dT_c^2} [v(T_c) - v_s] + \left[\frac{du(T_c)}{dT_c} \right]^2 + \left[\frac{dv(T_c)}{dT_c} \right]^2 \right|}. \quad (24)$$

Expansion of $u(T_c + \Delta T_c)$ and $v(T_c + \Delta T_c)$ into a Taylor's series up to linear terms only, neglect of all terms containing products of errors, making use of eq. (20), and finally taking absolute values yields eq. (24).

A further estimation of $|\Delta T_c|$ from eq. (24) is:

$$\frac{\Delta T_c}{T_c} \leq M(T_c) [\max(\epsilon_u, \epsilon_v) + \max(|\Delta u_s|, |\Delta v_s|)] \quad (25)$$

with

$$M(T_c) = \frac{\left| \frac{du(T_c)}{dT_c} + \frac{dv(T_c)}{dT_c} \right|}{T_c \left| \left[\frac{du(T_c)}{dT_c} \right]^2 + \left[\frac{dv(T_c)}{dT_c} \right]^2 \right|}. \quad (26)$$

The function $M(T_c)$ varies monotonically between $M(1000 \text{ K}) = 3.87$ and $M(15,000 \text{ K}) = 53.56$, i.e., the relative error of T_c caused by the approximation is less than 0.03

and 0.48% for $T_c = 1000 \text{ K}$ and $T_c = 15,000 \text{ K}$, respectively. The measurement uncertainties are usually at least 10 times greater than the approximation errors, i.e., the relative error of T_c can reach 5% at $T_c = 15,000 \text{ K}$! Note that this error is not due to the method of calculating T_c , but comes from measurement uncertainties (e.g., to get an uncertainty of $\Delta T_c = 200 \text{ K}$ at $T_c = 7000 \text{ K}$ an absolute measurement uncertainty of less than 10^{-3} in u_s and v_s is needed)!

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