

The CIEDE2000 Color-Difference Formula: Implementation Notes, Supplementary Test Data, and Mathematical Observations

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Abstract: This article and the associated data and programs provided with it are intended to assist color engineers and scientists in correctly implementing the recently developed CIEDE2000 color-difference formula. We indicate several potential implementation errors that are not uncovered in tests performed using the original sample data published with the standard. A supplemental set of data is provided for comprehensive testing of implementations. The test data, Microsoft Excel spreadsheets, and MATLAB scripts for evaluating the CIEDE2000 color difference are made available at the first author's website. Finally, we also point out small mathematical discontinuities in the formula. © 2004 Wiley Periodicals, Inc. Col Res Appl, 30, 21–30, 2005; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/col.20070

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INTRODUCTION

The CIEDE2000 formula was published by the CIE in 2001.¹ Developed by members of CIE Technical Committee 1-47, the formula provides an improved procedure for the computation of industrial color differences. The methodology used for developing the formula from experimental color-difference data was described by Luo, Cui, and Rigg² in an article published in this journal. The article also

included the equations for the computation of the CIEDE2000 color difference as an appendix.

The CIEDE2000 formula is considerably more sophisticated and computationally involved than its predecessor color-difference equations for CIELAB³ ΔE_{ab}^* and the CIE94⁴ color-difference ΔE_{94} . Therefore it is important to verify that software implementations for computing color differences based on the new formula are extensively tested prior to their deployment. Toward this end, both the CIEDE2000 publications mentioned in the previous paragraph included an identical set of worked examples for “confirmation of software implementation of CIEDE2000.”¹ However, these worked examples are a small set and do not adequately test the implementation of the CIEDE2000 formula. We discovered this limitation, in our efforts to implement the formula and validate it against publicly available implementations on the World Wide Web. In this process, we found several independent implementations. For the worked examples included in the CIE technical report on CIEDE2000¹ all of these implementations provided results that were in agreement with the published data (up to reasonable numerical precision). However, upon further testing, we discovered that on additional data, the results for the different implementations were different. In fact, several of the implementations distributed on the Internet, including some from reputable sources, were erroneous. Several of our own early implementations of the formula were also in this category of implementations that are erroneous but nevertheless provide correct results over the limited set of worked examples included with the draft standard. We therefore believe that

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correct implementation of the CIEDE2000 formula is non-trivial and our goal in this article is to assist in this process.

This article has three main contributions. First, we provide a comprehensive set of data for testing software implementations of the CIEDE2000 color-difference equation. The data are specifically designed for exercising components in the CIEDE2000 equations that may be erroneously computed and are not exercised by the original worked examples included with the draft standard. We also point out several of the common errors and indicate parts of the test data that help identify these. Second, we provide implementations of the CIEDE2000 color-difference formula in Microsoft Excel and MATLAB.⁵ These implementations agree with the original worked examples included in the CIE Technical Report and with each other on the supplemental data included in this article. We believe these to be correct implementations of the equations. Third, we highlight and characterize discontinuities in the CIEDE2000 color-difference formula that arise from its defining equations.

This article addresses only the computational and mathematical aspects of the CIEDE2000 equations and does not in any way address psychophysical evaluation of the formula, attempt to improve its uniformity, or define its domain of applicability. Some comments of that nature are included in the notes by Kuehni⁶ and Luo *et al.*⁷ published in this journal.

THE CIEDE2000 COLOR-DIFFERENCE FORMULA

In this section, we briefly outline the equations for the computation of the CIEDE2000 color difference. The description follows the presentation of Luo, Cui, and Rigg² but is slightly modified to make it closer to an algorithmic statement. It is included here to relate implementation notes and comments in subsequent sections to the equations presented here.

The CIEDE2000 color-difference formula is based on the CIELAB color space. Given a pair of color values in CIELAB space L_1^*, a_1^*, b_1^* and L_2^*, a_2^*, b_2^* , we denote the CIEDE2000 color difference between them as follows[†]:

$$\Delta E_{00}(L_1^*, a_1^*, b_1^*, L_2^*, a_2^*, b_2^*) = \Delta E_{00}^{12} = \Delta E_{00}. \quad (1)$$

Given two CIELAB color values $\{L_i^*, a_i^*, b_i^*\}_{i=1}^2$ and parametric weighting factors k_L , k_C , and k_H , the process of computation of the color difference is summarized in the following equations, grouped as three main steps.

1. Calculate C'_i , h'_i :

$$C_{i,ab}^* = \sqrt{(a_i^*)^2 + (b_i^*)^2} \quad i=1, 2 \quad (2)$$

$$\bar{C}_{ab}^* = \frac{C_{1,ab}^* + C_{2,ab}^*}{2} \quad (3)$$

$$G = 0.5 \left(1 - \sqrt{\frac{\bar{C}_{ab}^{*7}}{\bar{C}_{ab}^{*7} + 25^7}} \right) \quad (4)$$

$$a_i' = (1+G)a_i^* \quad i=1, 2 \quad (5)$$

$$C_i' = \sqrt{(a_i')^2 + (b_i^*)^2} \quad i=1, 2 \quad (6)$$

$$h_i' = \begin{cases} 0 & b_i^* = a_i' = 0 \\ \tan^{-1}(b_i^*/a_i') & \text{otherwise} \end{cases} \quad i=1, 2 \quad (7)$$

2. Calculate $\Delta L'$, $\Delta C'$, $\Delta H'$:

$$\Delta L' = L_2^* - L_1^* \quad (8)$$

$$\Delta C' = C_2' - C_1' \quad (9)$$

$$\Delta H' = \begin{cases} 0 & C_1' C_2' = 0 \\ h_2' - h_1' & C_1' C_2' \neq 0; |h_2' - h_1'| \leq 180^\circ \\ (h_2' - h_1') - 360 & C_1' C_2' \neq 0; (h_2' - h_1') > 180^\circ \\ (h_2' - h_1') + 360 & C_1' C_2' \neq 0; (h_2' - h_1') < -180^\circ \end{cases} \quad (10)$$

$$\Delta H' = 2 \sqrt{C_1' C_2'} \sin\left(\frac{\Delta h'}{2}\right) \quad (11)$$

3. Calculate CIEDE2000 Color-Difference ΔE_{00} :

$$\bar{L}' = (L_1^* + L_2^*)/2 \quad (12)$$

$$\bar{C}' = (C_1' + C_2')/2 \quad (13)$$

$$\bar{h}' = \begin{cases} \frac{h_1' + h_2'}{2} & |h_1' - h_2'| \leq 180^\circ; C_1' C_2' \neq 0 \\ \frac{h_1' + h_2' + 360^\circ}{2} & |h_1' - h_2'| > 180^\circ; (h_1' + h_2') < 360^\circ; \\ & C_1' C_2' \neq 0 \\ \frac{h_1' + h_2' - 360^\circ}{2} & |h_1' - h_2'| > 180^\circ; (h_1' + h_2') \geq 360^\circ; \\ & C_1' C_2' \neq 0 \\ (h_1' + h_2') & C_1' C_2' = 0 \end{cases} \quad (14)$$

$$T = 1 - 0.17 \cos(\bar{h}' - 30^\circ) + 0.24 \cos(2\bar{h}') \\ + 0.32 \cos(3\bar{h}' + 6^\circ) - 0.20 \cos(4\bar{h}' - 63^\circ) \quad (15)$$

$$\Delta\theta = 30 \exp\left\{-\left[\frac{\bar{h}' - 275^\circ}{25}\right]^2\right\} \quad (16)$$

$$R_C = 2 \sqrt{\frac{\bar{C}'^7}{\bar{C}'^7 + 25^7}} \quad (17)$$

$$S_L = 1 + \frac{0.015(\bar{L}' - 50)^2}{\sqrt{20 + (\bar{L}' - 50)^2}} \quad (18)$$

$$S_C = 1 + 0.045\bar{C}' \quad (19)$$

$$S_H = 1 + 0.015\bar{C}' T \quad (20)$$

$$R_T = -\sin(2\Delta\theta) R_C \quad (21)$$

$$\Delta E_{00}^{12} = \Delta E_{00}(L_1^*, a_1^*, b_1^*, L_2^*, a_2^*, b_2^*) \\ = \sqrt{\left(\frac{\Delta L'}{k_L S_L}\right)^2 + \left(\frac{\Delta C'}{k_C S_C}\right)^2 + \left(\frac{\Delta H'}{k_H S_H}\right)^2 + R_T \left(\frac{\Delta C'}{k_C S_C}\right) \left(\frac{\Delta H'}{k_H S_H}\right)}. \quad (22)$$

[†] The commonly accepted notation is simply ΔE_{00} , however, in certain cases, we will find it useful to use the longer and more explicit versions.

Several notes clarifying the implementation of the above equations are helpful:

1. The definition of the modified hue h'_i in Eq. (7) uses a four-quadrant arctangent. The modified hue h'_i has the geometric interpretation in the two dimensional $a'-b^*$ plane as the angular position of the point (a'_i, b_i^*) measured from the positive a' axis. To emphasize this we use the notation $\tan^{-1}(b_i^*, a'_i)$ as opposed to the more common notation $\tan^{-1}(b_i^*/a'_i)$. The four-quadrant arctangent may be readily obtained using the two argument inverse tangent function available in several programming environments. Typically, these functions return an angular value in radians ranging from $-\pi$ to π . This must be converted to a hue angle in degrees between 0° and 360° by addition of 2π to negative hue angles, followed by a multiplication by $180/\pi$ to convert from radians to degrees. This will provide a result consistent with subsequent equations. For the case where $b_i^* = a'_i = 0$, the modified hue is indeterminate and we have defined it as zero, a fact that we exploit in Eq. (14).
2. The chroma difference $\Delta C'$ in Eq. (9) and the hue difference $\Delta H'$ in Eq. (11) should be computed as signed differences as indicated in the equations and not as absolute values of the differences. Although the equations defining the CIEDE2000 formula^{1,2} are unambiguous in this respect, implementers may be susceptible to making this error.[†] The CIELAB³ ΔE_{ab}^* and the CIE94⁴ ΔE_{94} color differences used only terms in ΔH^2 and ΔC^2 hence required only the absolute value of the hue difference. This we conjecture might lead software implementers to use the absolute values for the hue and chroma difference when they begin with existing implementations of earlier color-difference equations. This implementation error is hard to detect, because the sign of the hue and chroma differences plays a role only in the following cross term:

$$R_T \left(\frac{\Delta C'}{k_C S_C} \right) \left(\frac{\Delta H'}{k_H S_H} \right)$$

in Eq. (22). Its impact is localized in the blue hue region (around 275°), outside of which $\Delta\theta$ and therefore the R_T term is rather small. Therefore the error of ignoring the sign of the hue and chroma difference is not readily diagnosed unless specifically tested for.

3. The computation of the hue angle difference in Eq. (10) is based on the description in the CIE Technical Report.¹ In addition, we use the reasonable interpretation that the hue difference between the origin in the $a'-b^*$ plane and any point in the plane is zero. The article by Luo, Cui, and Rigg² does not fully describe the hue-difference computation. Using the arithmetic difference directly can result in an incorrect sign for the metric hue difference

$\Delta H'$ in Eq. (11). Additional details may also be found in Sève.⁸

4. The computation of the mean hue \bar{h}' used in Eq. (14) is not defined unambiguously in either the CIE Technical Report¹ or the article by Luo, Cui and Rigg.² In most computations of angle, any ambiguities of 360° may be ignored. Due to the $\Delta\theta$ term in Eq. (16), this fact is, however, not true of the CIEDE2000 color-difference equations. A verbatim interpretation of the note in Section 2.6.1 of the CIE Technical Report¹ can lead to mean hue angles over 360° , which conflicts with the sentence at the end of Section 2.7 of the same document. Likewise, a verbatim interpretation of the (slightly different) description for the mean hue computation in the appendix of the paper by Luo, Cui, and Rigg² can lead to negative angles. In this article, the definition of Eq. (14) is therefore selected to provide an angle between 0° and 360° that agrees with the CIE Technical Report¹ up to addition and subtraction of multiples of 360° . Most existing implementations of the CIEDE2000 color-difference formula found on the Internet at the time of this writing were seen to have an error in the computation of average hue. The impact of the error may, however, be quite small in some cases.
5. The defined “boundary cases” for the computation of the mean hue in Eq. (14) are classified differently in the CIE Technical Report¹ and in the article by Luo, Cui, and Rigg.² Their descriptions do not agree for the (rather rare) case when the absolute difference between the modified hue angles is exactly 180° . Our equations and implementations follow the CIE Technical Report¹ to the extent it is unambiguous. In the absence of a clear directive, we have (arbitrarily but within reason) defined the computation of mean hue for the case when $|h'_1 - h'_2| > 180^\circ$ and $(h'_1 + h'_2) = 360^\circ$. Our computation for this case results in a mean hue of 0° instead of the other reasonable choice of 360° .
6. The CIEDE2000 color difference is symmetric. However, the signed differences in lightness, hue, and chroma in Eqs. (8)–(11) are clearly not symmetric. Our notation assumes that the subscript 1 represents the reference color from which these signed differences are computed to the sample 2.

SUPPLEMENTAL TEST DATA FOR CIEDE2000 EQUATIONS

Table 1 provides a set of supplemental test data against which implementations of the CIEDE2000 color-difference formula may be tested. The data are also available electronically at the first author’s website⁵ along with Microsoft Excel and MATLAB implementations of what we believe are correct implementations of the formula.

1. The CIELAB pairs labeled 1–3 test that the true chroma difference ΔH (a signed quantity) is used in computing ΔE_{00} instead of using just the absolute value.
2. The CIELAB pairs labeled 4–6 test that the true hue

[†] Some of the implementations of the CIEDE2000 color-difference formula found on the Internet, were seen to have this error.

TABLE I. CIEDE2000 total color difference test data.

Pair	<i>i</i>	L_i	a_i	b_i	a_i'	C_i'	h_i'	\bar{h}'	G	T	S_L	S_C	S_H	R_T	ΔE_{00}
1	1	50.0000	2.6772	-79.7751	2.6774	79.8200	271.9222	270.9611	0.0001	0.6907	1.0000	4.6578	1.8421	-1.7042	2.0425
	2	50.0000	0.0000	-82.7485	0.0000	82.7485	270.0000								
2	1	50.0000	3.1571	-77.2803	3.1573	77.3448	272.3395	271.1698	0.0001	0.6843	1.0000	4.6021	1.8216	-1.7070	2.8615
	2	50.0000	0.0000	-82.7485	0.0000	82.7485	270.0000								
3	1	50.0000	2.8361	-74.0200	2.8363	74.0743	272.1944	271.0972	0.0001	0.6865	1.0000	4.5285	1.8074	-1.7060	3.4412
	2	50.0000	0.0000	-82.7485	0.0000	82.7485	270.0000								
4	1	50.0000	-1.3802	-84.2814	-1.3803	84.2927	269.0618	269.5309	0.0001	0.7357	1.0000	4.7584	1.9217	-1.6809	1.0000
	2	50.0000	0.0000	-82.7485	0.0000	82.7485	270.0000								
5	1	50.0000	-1.1848	-84.8006	-1.1849	84.8089	269.1995	269.5997	0.0001	0.7335	1.0000	4.7700	1.9218	-1.6822	1.0000
	2	50.0000	0.0000	-82.7485	0.0000	82.7485	270.0000								
6	1	50.0000	-0.9009	-85.5211	-0.9009	85.5258	269.3964	269.6982	0.0001	0.7303	1.0000	4.7862	1.9217	-1.6840	1.0000
	2	50.0000	0.0000	-82.7485	0.0000	82.7485	270.0000								
7	1	50.0000	0.0000	0.0000	0.0000	0.0000	0.0000	126.8697	0.5000	1.2200	1.0000	1.0562	1.0229	0.0000	2.3669
	2	50.0000	-1.0000	2.0000	-1.5000	2.5000	126.8697								
8	1	50.0000	-1.0000	2.0000	-1.5000	2.5000	126.8697	126.8697	0.5000	1.2200	1.0000	1.0562	1.0229	0.0000	2.3669
	2	50.0000	0.0000	0.0000	0.0000	0.0000	0.0000								
9	1	50.0000	2.4900	-0.0010	3.7346	3.7346	359.9847	269.9854	0.4998	0.7212	1.0000	1.1681	1.0404	-0.0022	7.1792
	2	50.0000	-2.4900	0.0009	-3.7346	3.7346	179.9862								
10	1	50.0000	2.4900	-0.0010	3.7346	3.7346	359.9847	269.9847	0.4998	0.7212	1.0000	1.1681	1.0404	-0.0022	7.1792
	2	50.0000	-2.4900	0.0010	-3.7346	3.7346	179.9847								
11	1	50.0000	2.4900	-0.0010	3.7346	3.7346	359.9847	89.9839	0.4998	0.6175	1.0000	1.1681	1.0346	0.0000	7.2195
	2	50.0000	-2.4900	0.0011	-3.7346	3.7346	179.9831								
12	1	50.0000	2.4900	-0.0010	3.7346	3.7346	359.9847	89.9831	0.4998	0.6175	1.0000	1.1681	1.0346	0.0000	7.2195
	2	50.0000	-2.4900	0.0012	-3.7346	3.7346	179.9816								
13	1	50.0000	-0.0010	2.4900	-0.0015	2.4900	90.0345	180.0328	0.4998	0.9779	1.0000	1.1121	1.0365	0.0000	4.8045
	2	50.0000	0.0009	-2.4900	0.0013	2.4900	270.0311								
14	1	50.0000	-0.0010	2.4900	-0.0015	2.4900	90.0345	180.0345	0.4998	0.9779	1.0000	1.1121	1.0365	0.0000	4.8045
	2	50.0000	0.0010	-2.4900	0.0015	2.4900	270.0345								
15	1	50.0000	-0.0010	2.4900	-0.0015	2.4900	90.0345	0.0362	0.4998	1.3197	1.0000	1.1121	1.0493	0.0000	4.7461
	2	50.0000	0.0011	-2.4900	0.0016	2.4900	270.0380								
16	1	50.0000	2.5000	0.0000	3.7496	3.7496	0.0000	315.0000	0.4998	0.8454	1.0000	1.1406	1.0396	-0.0001	4.3065
	2	50.0000	0.0000	-2.5000	0.0000	2.5000	270.0000								
17	1	50.0000	2.5000	0.0000	3.4569	3.4569	0.0000	346.2470	0.3827	1.4453	1.1608	1.9547	1.4599	-0.0003	27.1492
	2	73.0000	25.0000	-18.0000	34.5687	38.9743	332.4939								
18	1	50.0000	2.5000	0.0000	3.4954	3.4954	0.0000	51.7766	0.3981	0.6447	1.0640	1.7498	1.1612	0.0000	22.8977
	2	61.0000	-5.0000	29.0000	-6.9907	29.8307	103.5532								
19	1	50.0000	2.5000	0.0000	3.5514	3.5514	0.0000	272.2362	0.4206	0.6521	1.0251	1.9455	1.2055	-0.8219	31.9030
	2	56.0000	-27.0000	-3.0000	-38.3556	38.4728	184.4723								
20	1	50.0000	2.5000	0.0000	3.5244	3.5244	0.0000	11.9548	0.4098	1.1031	1.0400	1.9120	1.3353	0.0000	19.4535
	2	58.0000	24.0000	15.0000	33.8342	37.0102	23.9095								
21	1	50.0000	2.5000	0.0000	3.7494	3.7494	0.0000	3.5056	0.4997	1.2616	1.0000	1.1923	1.0808	0.0000	1.0000
	2	50.0000	3.1736	0.5854	4.7596	4.7954	7.0113								
22	1	50.0000	2.5000	0.0000	3.7493	3.7493	0.0000	0.0000	0.4997	1.3202	1.0000	1.1956	1.0861	0.0000	1.0000
	2	50.0000	3.2972	0.0000	4.9450	4.9450	0.0000								
23	1	50.0000	2.5000	0.0000	3.7497	3.7497	0.0000	5.8190	0.4999	1.2197	1.0000	1.1486	1.0604	0.0000	1.0000
	2	50.0000	1.8634	0.5757	2.7949	2.8536	11.6380								
24	1	50.0000	2.5000	0.0000	3.7493	3.7493	0.0000	1.9603	0.4997	1.2883	1.0000	1.1946	1.0836	0.0000	1.0000
	2	50.0000	3.2592	0.3350	4.8879	4.8994	3.9206								
25	1	60.2574	-34.0099	36.2677	-34.0678	49.7590	133.2085	132.0835	0.0017	1.3010	1.1427	3.2946	1.9951	0.0000	1.2644
	2	60.4626	-34.1751	39.4387	-34.2333	52.2238	130.9584								
26	1	63.0109	-31.0961	-5.8663	-32.6194	33.1427	190.1951	188.8221	0.0490	0.9402	1.1831	2.4549	1.4560	0.0000	1.2630
	2	62.8187	-29.7946	-4.0864	-31.2542	31.5202	187.4490								
27	1	61.2901	3.7196	-5.3901	5.5668	7.7487	315.9240	310.0313	0.4966	0.6952	1.1586	1.3092	1.0717	-0.0032	1.8731
	2	61.4292	2.2480	-4.9620	3.3644	5.9950	304.1385								
28	1	35.0831	-44.1164	3.7933	-44.3939	44.5557	175.1161	176.4290	0.0063	1.0168	1.2148	2.9105	1.6476	0.0000	1.8645
	2	35.0232	-40.0716	1.5901	-40.3237	40.3550	177.7418								
29	1	22.7233	20.0904	-46.6940	20.1424	50.8532	293.3339	291.3809	0.0026	0.3636	1.4014	3.1597	1.2617	-1.2537	2.0373
	2	23.0331	14.9730	-42.5619	15.0118	45.1317	289.4279								
30	1	36.4612	47.8580	18.3852	47.9197	51.3256	20.9901	21.8781	0.0013	0.9239	1.1943	3.3888	1.7357	0.0000	1.4146
	2	36.2715	50.5065	21.2231	50.5716	54.8444	22.7660								
31	1	90.8027	-2.0831	1.4410	-3.1245	3.4408	155.2410	167.1011	0.4999	1.1546	1.6110	1.1329	1.0511	0.0000	1.4441
	2	91.1528	-1.6435	0.0447	-2.4651	2.4655	178.9612								
32	1	90.9257	-0.5406	-0.9208	-0.8109	1.2270	228.6315	218.4363	0.5000	1.3916	1.5930	1.0620	1.0288	0.0000	1.5381
	2	88.6381	-0.8985	-0.7239	-1.3477	1.5298	208.2412								
33	1	6.7747	-0.2908	-2.4247	-0.4362	2.4636	259.8025	263.0049	0.4999	0.9556	1.6517	1.1057	1.0337	-0.0004	0.6377
	2	5.8714	-0.0985	-2.2286	-0.1477	2.2335	266.2073								
34	1	2.0776	0.0795	-1.1350	0.1192	1.1412	275.9978	268.0910	0.5000	0.7826	1.7246	1.0383	1.0100	0.0000	0.9082
	2	0.9033	-0.0636	-0.5514	-0.0954	0.5596	260.1842								

difference ΔC (a signed quantity) is used in computing ΔE_{00} instead of using just the absolute value.

3. The CIELAB pairs labeled 7–16 test that the arctangent computation for the determination of the hue in Eq. (7) and the computation of mean hue in Eq. (14) are performed correctly.
4. Note that the CIEDE2000 color difference formula is de-

signed to be symmetric with respect to the two samples between which the color difference is computed. In the notation of Eq. (1), we have the following:

$$\begin{aligned}\Delta E_{00}^{12} &= \Delta E_{00}(L_1^*, a_1^*, b_1^*, L_2^*, a_2^*, b_2^*) = \Delta E_{00}^{21} \\ &= \Delta E_{00}(L_2^*, a_2^*, b_2^*, L_1^*, a_1^*, b_1^*).\end{aligned}$$

It is recommended that implementations of the formula should explicitly test this by interchanging the two sets of data between which color differences are computed and verifying that color differences are unchanged in this process. In particular, the use of an absolute value for either but not both of the chroma and hue differences will lead to an asymmetric color difference formula. The use of an absolute value for both will produce a symmetric but incorrect formula.

5. Though the CIEDE2000 color-difference equations are only applicable for small color differences, it is preferable that data for testing software implementations should include at least a few large color differences, because larger differences are not easily confused with variation in numerical precision and can therefore more easily distinguish differences or errors in implementations. The CIELAB pairs labeled 17–20 in the table are included for this reason.
6. The CIELAB pairs labeled 25–34 in the table correspond to data published in CIE technical report¹ and in the article by Luo, Cui and Rigg.²
7. The values given in Table 1 are listed up to four decimal places. Although such precision would rarely be justified in practical applications, in testing computer implementations, even discrepancies in the fourth place should not be ignored because they may be indicative of implementation errors that may manifest themselves as larger errors for a different choice of color samples.

Note that the original set of worked examples included in the CIE Technical Report on CIEDE2000¹ test no features of the equations mentioned in items 1–4. Implementations having one or more of these errors are therefore not identified by the original worked examples but readily identified using the test data provided here.

DISCONTINUITIES IN THE CIEDE2000 COLOR-DIFFERENCE FORMULA

From a quick review of equations defining the CIEDE2000 color-difference formula published in the CIE Technical Report,¹ it appears that all the terms involved are well behaved. One would therefore expect the ΔE_{00} difference to be a continuous and differentiable function of the input CIELAB color pairs. In fact, however, the function has three independent sources of mathematical discontinuities that can be seen by examining the equations presented in this article. In the following, we describe and characterize these discontinuities in order of decreasing discontinuity magnitude.

Discontinuity Due to Mean Hue Computation

A discontinuity arises in the ΔE_{00} difference due to the process used in Eq. (14) for the computation of mean hue \bar{h}' from the modified hue angles h'_1 and h'_2 of the two color samples. As indicated in the equation, the mathematical average is used in situations where the absolute difference

between the modified hue angles is less than or equal to 180° and in situations where the absolute difference between the modified hue angles is greater than 180° , we add/subtract 360° to/from the sum of the angles and divide by 2 to get the mean hue. The latter computation is equivalent to adding/subtracting an angle of 180° to/from the mathematical average. This leads to a discontinuity of 180° for the mean hue angle as elaborated in greater detail below.

Consider two modified hue angles ϕ and $\phi + 180^\circ - \epsilon/2$, where ϵ is a small positive quantity, because the absolute value of difference between these two angles is $180 - \epsilon/2$, which is under 180° , the average hue corresponding to these is given by the following:

$$\begin{aligned}\bar{h}'(\phi, \phi + 180^\circ - \epsilon/2) &= (\phi + \phi + 180^\circ - \epsilon/2)/2 \\ &= \phi + 90^\circ - \epsilon/4. \quad (23)\end{aligned}$$

Next, consider the hue angles ϕ and $\phi + 180^\circ + \epsilon/2$. Because the absolute value of difference between these two angles is $180 + \epsilon/2$, which is over 180° , the mean hue corresponding to these is the following:

$$\begin{aligned}\bar{h}'(\phi, \phi + 180^\circ + \epsilon/2) &= \begin{cases} (\phi + \phi + 180^\circ + \epsilon/2 + 360^\circ)/2 = \phi + 270^\circ + \epsilon/4 & h'_1 + h'_2 < 360^\circ \\ (\phi + \phi + 180^\circ + \epsilon/2 - 360^\circ)/2 = \phi - 90^\circ + \epsilon/4 & h'_1 + h'_2 \geq 360^\circ. \end{cases} \\ & \quad (24)\end{aligned}$$

Subtracting Eq. (24) from Eq. (23), we get the following:

$$\begin{aligned}\bar{h}'(\phi, \phi + 180^\circ + \epsilon/2) - \bar{h}'(\phi, \phi + 180^\circ - \epsilon/2) &= \begin{cases} 180^\circ + \epsilon/2 & h'_1 + h'_2 < 360^\circ \\ -180^\circ + \epsilon/2 & h'_1 + h'_2 \geq 360^\circ. \end{cases} \quad (25)\end{aligned}$$

Thus a small perturbation ϵ in one of the hue angles produces a change of approximately 180° in the mean hue. The mean hue is therefore a discontinuous function.

A geometric illustration of the discontinuity provides clearer insight than the equations presented above. The mean hue \bar{h}' can be geometrically interpreted as the hue angle of the angular bisector of line segments drawn at the modified hue angles h'_1 and h'_2 . This is illustrated in Fig. 1. The two dots in this figure labeled 1 and 2 represent sample colors projected onto the $a'-b^*$ plane. Line segments have been drawn from the origin in the plane to each of the colors' projections. The modified hue angle of each color is the angle that the line segment makes with respect to the a' axis measured in the counterclockwise direction as indicated by the arcs labeled h'_1 and h'_2 . The angular bisector of the two line segments is shown as the cyan line segment with a pointed arrow. The mean hue \bar{h}'_{12} corresponds to the angle that this bisector makes with respect to the a' axis measured in the counterclockwise direction as indicated by the arcs labeled \bar{h}'_{12} .

Using the geometric interpretation of the mean hue, the discontinuity introduced in its computation is readily seen in Fig. 2. In the figure, three sample colors labeled 1, 2, and 3

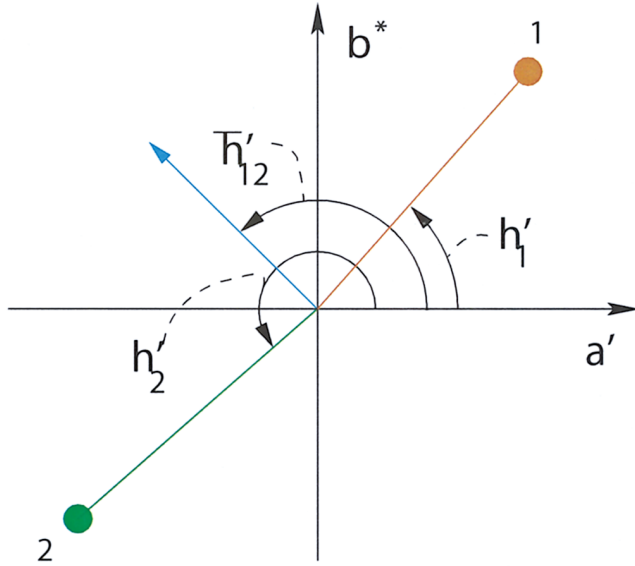


FIG. 1. Geometrical illustration of the computation of mean hue.

are plotted on the $a'-b^*$ plane such that the modified hue angles of 2 and 3 are close to 180° apart from the modified hue angle of 1 with the absolute hue angle difference between 1 and 2 just under 180° and the absolute hue angle difference between 1 and 3 just over 180° . The cyan line segment with the arrow represents the mean hue \bar{h}'_{12} for samples 1 and 2 and the magenta line segment with the arrow represents the mean hue \bar{h}'_{13} for samples 1 and 3. From the figure, it is clear that the small perturbation from 2 to 3 produces a change in mean hue of over 180° .

The discontinuity resulting from the mean hue computation impacts the terms in the CIEDE2000 equations that use

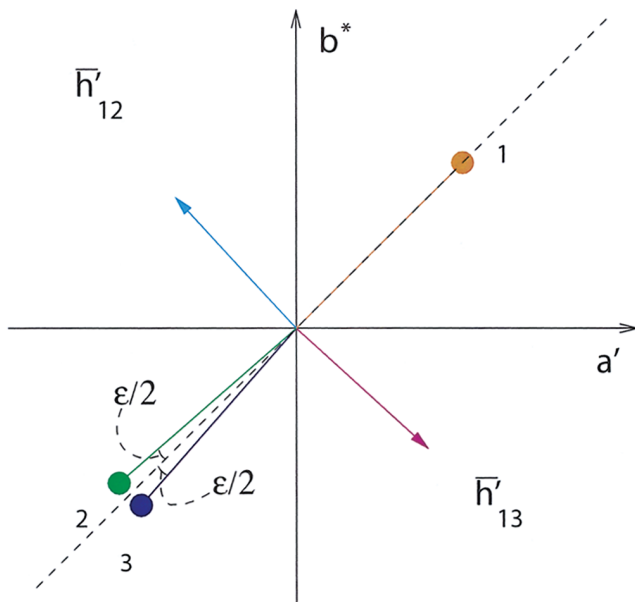


FIG. 2. Geometric illustration of the 180° discontinuity in mean hue computation.

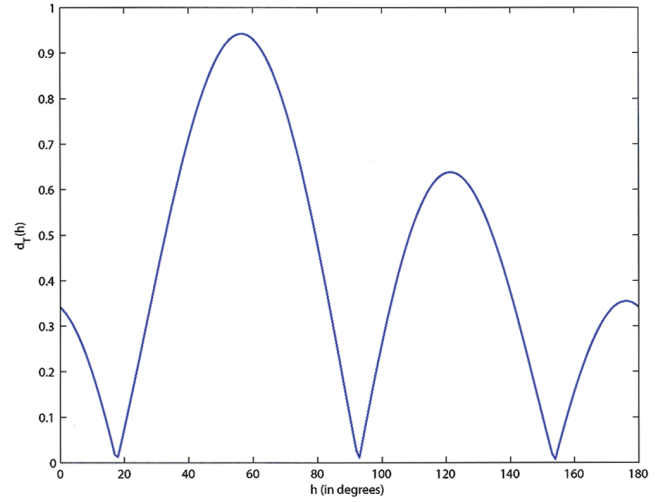


FIG. 3. The magnitude of discontinuity $d_T(h)$ in the term T of the CIEDE2000 equations as a function of mean hue.

this mean hue in further computations. At the lowest level, this occurs in the terms T and $\Delta\theta$ in Eqs. (15) and (16), respectively. If $f(h)$ is any function of the mean hue, the magnitude of the discontinuity introduced in $f(\bar{h})$ can be readily determined as follows:

$$d_f(h) = |f(h) - f(h + 180)| \quad h < 180^\circ. \quad (26)$$

Plots of the discontinuity magnitude in T , $d_T(h)$, and the discontinuity magnitude in $\Delta\theta$, $d_{\Delta\theta}(h)$, are shown in Figs. 3 and 4, respectively.

From the plots in Figs. 3 and 4 it is apparent that the discontinuity in the computation of the mean hue causes a significant discontinuity in the T and $\Delta\theta$ terms of the CIEDE2000 equations. Because these terms are used in combination with other terms in the computation of the final color-difference ΔE_{00} , the magnitude of the discontinuity in ΔE_{00} cannot be directly inferred based on the discontinuity in these terms. The complicated nature of the ΔE_{00} equa-

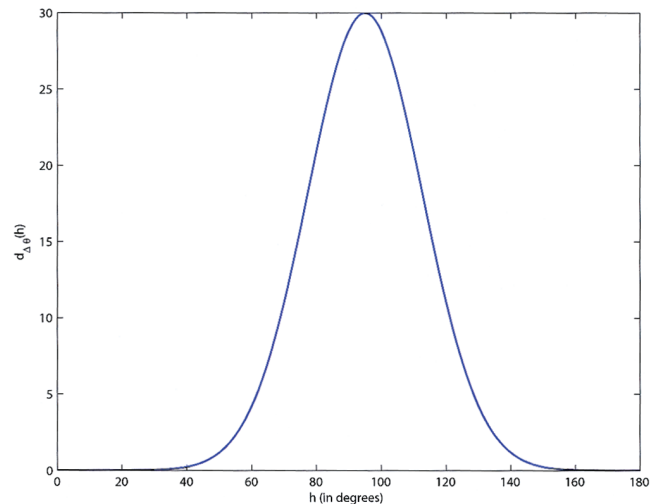


FIG. 4. The magnitude of discontinuity $d_{\Delta\theta}(h)$ in the term $\Delta\theta$ of the CIEDE2000 equations as a function of mean hue.

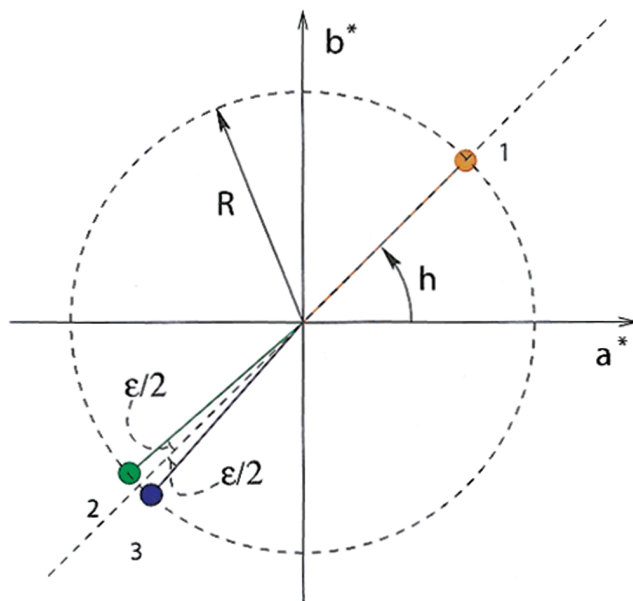


FIG. 5. Color configuration for empirical evaluation of ΔE_{00} discontinuity due to mean hue computation.

tions makes analytical determination of the discontinuity magnitude rather difficult. Instead, we adopt an empirical approach.

We begin by selecting a suitable set of colors in CIELAB space that illustrate the discontinuity while minimizing/eliminating the impact of other discontinuities. From Fig. 2, it is apparent that the discontinuity in the computation of the mean hue arises only for colors that are 180° apart in CIELAB hue angle.[‡] From the CIEDE2000 equations it is also clear that the magnitude of the discontinuity in ΔE_{00} is not a fixed value but will depend on the choice of the pair of color locations. For estimating the magnitude of discontinuity in the ΔE_{00} formula arising due to the mean hue computation, we use the specific configuration shown in the a^*-b^* plane in Fig. 5.[§] As will be illustrated subsequently, the choice of this configuration eliminates any impact of the second discontinuity. The point labeled as 1 represents a reference color is located at CIELAB hue h and has CIELAB chroma R . The points labeled 2 and 3 represent two samples, also having CIELAB chroma R but located at hue angles $h + 180^\circ - \epsilon/2$ and $h + 180^\circ + \epsilon/2$, respectively. For our computations, we use $\epsilon/2 = 10^{-6}$ radians. In general, $\epsilon/2$ should be the smallest possible value whose impact is not masked by the limited precision of computation.

The magnitude of the discontinuity in the CIEDE2000 color-difference formula (due to the discontinuity in mean hue computation) is then estimated as the absolute value of

[‡] Note that colors are under/over/exactly 180° apart in a^*-b^* hue if and only if they are under/over/exactly 180° apart in CIELAB hue.

[§] Because the L^* values of the chosen pairs do not contribute to the discontinuity or influence its magnitude, we fix the L^* values at 50 for our reference and samples (any other value may be equivalently used).

the difference between the CIEDE2000 color-difference ΔE_{00}^{12} between 1 and 2 and the CIEDE2000 color-difference ΔE_{00}^{13} between 1 and 3 as follows:

$$d_{\Delta E}(h) = |\Delta E_{00}^{12} - \Delta E_{00}^{13}|. \quad (27)$$

Figure 6 shows plots of the discontinuity magnitude $d_{\Delta E}(h)$ as a function of the hue h of the reference sample, for values of the chroma R ranging from 0.5 to 2.5 in steps of 0.5. For these and all other numerical/graphical ΔE_{00} values reported in this article, we set the parametric weighting factors to unity (i.e., $k_L = k_C = k_H = 1$). From the figure it can be seen that the magnitude of the discontinuity has local maxima at hue values of roughly 37° , 87° , and 143° , with the highest value at 143° . The magnitude of the discontinuity increases with increase in the chroma value R . For the chosen chroma (R) values of 0.5, 1.0, 1.5, 2.0, and 2.5 the maximum values of the discontinuity magnitude (i.e., its value at 143°) are 0.0119, 0.0465, 0.1025, 0.1786, and 0.2734, respectively. The magnitude of the discontinuity is therefore relatively small, although not negligible.

From a computation of the individual terms involved in Eq. (22), one can also infer that the major contribution to the discontinuity in ΔE_{00} is due to the term $(\Delta H'/(k_H S_H))^2$. A discontinuity is introduced in this term through S_H , which in turn inherits the discontinuity from T , which was discussed earlier.

Discontinuity in Hue-Difference Computation

Similar to the discontinuity in the computation of mean hue, a discontinuity also arises in the computation of hue difference in Eq. (10). The source of the discontinuity is the inherent ambiguity in the sign of the hue difference between two modified hues that are exactly 180° apart. The sign may be arbitrarily chosen as positive or negative. Although we specify the sign as positive in Eq. (10), this does not eliminate the problem of change in sign of the hue difference as one goes from modified hue angles whose arith-

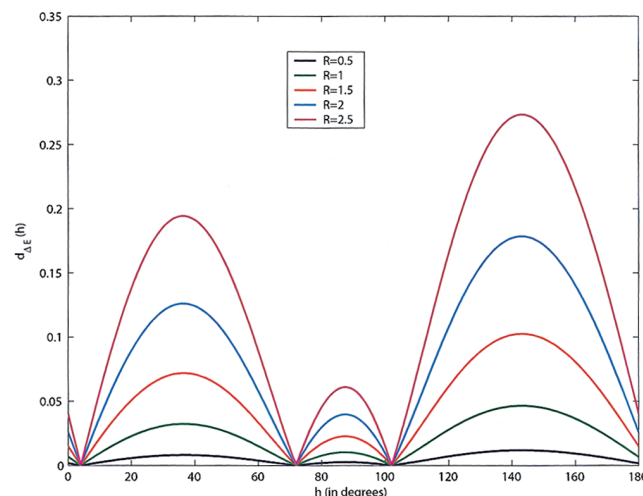


FIG. 6. The magnitude of discontinuity in ΔE_{00} for CIELAB colors as a function of reference color chroma and hue.

metric difference is just under 180° to hue angles whose arithmetic difference is just over 180° . Again, consider three modified hue angles $h'_1 = \phi$, $h'_2 = \phi + 180^\circ - \epsilon/2$, and $h'_3 = \phi + 180^\circ + \epsilon/2$. From Eq. (10) the hue differences between the pairs h'_1, h'_2 and h'_1, h'_3 are as follows:

$$\Delta h'(h'_1, h'_2) = 180^\circ - \epsilon/2 \quad (28)$$

$$\Delta h'(h'_1, h'_3) = (180^\circ + \epsilon/2) - 360^\circ = -180^\circ + \epsilon/2 \quad (29)$$

and

$$\Delta h'(h'_1, h'_2) - \Delta h'(h'_1, h'_3) = 360^\circ - \epsilon$$

illustrating the discontinuity of 360° . This in turn implies a discontinuity of 180° in $\Delta h'/2$, which corresponds to a sign reversal in $\sin(\Delta h'/2)$ and thus in $\Delta H'$ in Eq. (11). Thus the impact of the discontinuity in computation of hue difference is a sign reversal in $\Delta H'$.[¶] The final ΔE_{00} computation in Eq. (22) has only one term that is not invariant to a sign reversal in $\Delta H'$. This is the rotation term:

$$\tau = R_T \left(\frac{\Delta C'}{k_C S_C} \right) \left(\frac{\Delta H'}{k_H S_H} \right). \quad (30)$$

It is clear that this term is zero when $\Delta C' = 0$. Thus the rotation term τ is uniformly zero for color pairs located at the same chroma radius. This is true for the configuration of colors chosen for the illustration of the discontinuity due to mean hue (discussed earlier and presented in Fig. 6). As a result, even though both the discontinuities—due to mean

[¶] The CIE1976 and CIE1994 color-difference formulae can also be expressed in terms of a metric hue difference $\Delta H'$. However, they do not have the discontinuity because they depend only on $|\Delta H'|^2$ and are therefore independent of the sign of $\Delta H'$.

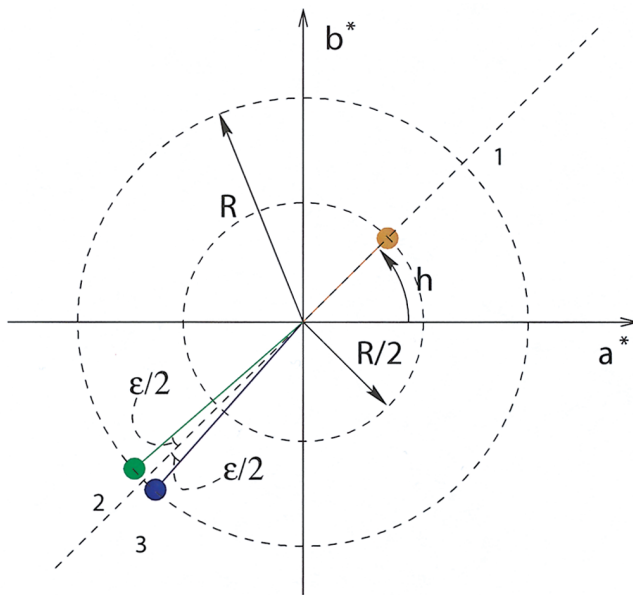


FIG. 7. Color configuration for empirical evaluation of ΔE_{00} discontinuity due to hue-difference computation.

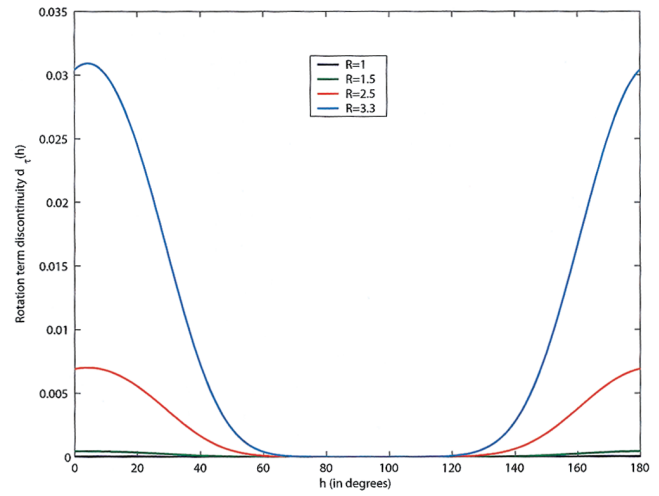


FIG. 8. The magnitude of discontinuity in the rotation term τ for CIELAB colors in the configuration of Fig. 7 as a function of hue for various chroma values R .

hue computation and due to hue-difference computation—occur for colors placed 180° apart in hue, the latter made no contribution in our empirical evaluation of the former that was presented in Fig. 6. One cannot, however, isolate the impact of the discontinuity due to hue-difference computation in a similar fashion. We therefore illustrate the impact of this discontinuity on τ —where it is localized—and on the overall ΔE_{00} , where as we illustrate it is dominated by the discontinuity due to mean hue computation.

From the rotation term in Eq. (30) and the equations for its components, one can readily see that for this term to be nonzero both $\Delta C'$ and $\sqrt{C'_1 C'_2}$ must be nonzero. We therefore select the configuration of colors shown in Fig. 7 in the a^*-b^* plane to empirically evaluate the discontinuity due to hue-difference computation. The reference 1 is located at a hue angle h at chroma radius $R/2$ and the two almost identical samples 2 and 3 are located at a chroma radius of R with hue angles just above and just below $h + 180^\circ$. We estimate the magnitude of discontinuity in τ as follows:

$$d_\tau(h) = |\tau^{12} - \tau^{13}|$$

and the magnitude of the overall discontinuity in ΔE_{00} as in Eq. (27).

Figure 8 shows plots of the discontinuity magnitude $d_\tau(h)$ as a function of the hue h of the reference sample, for values of the chroma R ranging from 1.0 to 3.3. The sign reversal in $\Delta H'$ which causes a sign reversal in τ and therefore contributes to this discontinuity (though it also includes the impact of the discontinuity due to mean hue computation through S_H). From the figure, it can be seen that the discontinuity magnitude is significantly smaller than the discontinuity magnitude in ΔE_{00} due to mean hue computation. The discontinuity magnitude has a unique maximum at approximately 4° and increases with increase in R (chroma of sample). For $R = 3.3$ the maximum value of the discontinuity magnitude in the rotation term τ for this configuration is 0.0309. The location of the maxima for the

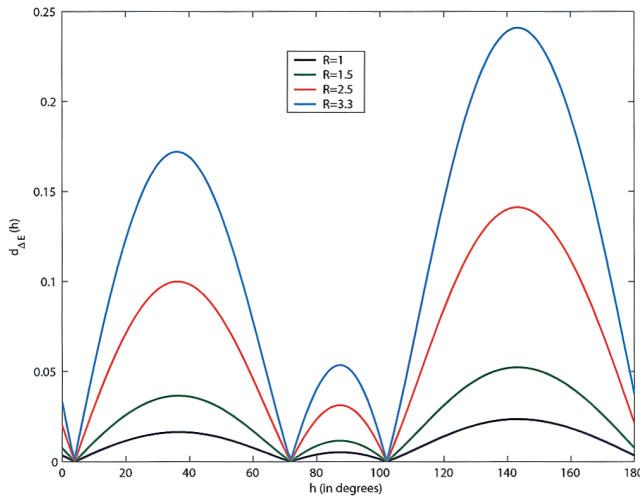


FIG. 9. The magnitude of discontinuity in ΔE_{00} for CIELAB colors in the configuration of Fig. 7 as a function of hue for various chroma values R .

discontinuity is in agreement with the functional form for $\Delta\theta$ in Eq. (16), which indicates that the impact of the rotation term τ will be localized around a mean hue of 275° . The hue locations $h'_1 = 4^\circ$ and $h'_2 = 184^\circ + \epsilon/2$ result in a mean hue of $274^\circ + \epsilon/4$ maximizing $\Delta\theta$, whereas for hue locations of $h'_1 = 4^\circ$ and $h'_2 = 184^\circ - \epsilon/2$ the mean hue is $94 - \epsilon/4$ for which $\Delta\theta$ is negligible. Thus even though the discontinuity due to hue-difference computation reverses the sign of τ , the simultaneous interaction with the mean hue discontinuity actually ensures that at least one of the terms becomes quite small.

Figure 9 shows plots of the overall discontinuity magnitude $d_{\Delta E}(h)$ in ΔE_{00} as a function of the hue h of the reference sample, for colors in the configuration of Fig. 7, for values of the chroma R ranging from 1.0 to 3.3. Upon comparing the plots to those in Fig. 6, one sees a very strong similarity in the shapes of these plots. This indicates that the discontinuity due to mean hue computation is predominant and the impact of the discontinuity due to hue difference computation is negligible. Overall the discontinuity magnitude is smaller than that encountered due to the mean hue computation. Once again the magnitude of discontinuity increases with increase in R .

Discontinuity Due to Hue Rollover at 360°

From a mathematical perspective, there is also another discontinuity in $\Delta\theta$ term in Eq. (16) at a mean hue of $0/360$ degrees because this term is not invariant to shifts of 360° unlike typical trigonometric functions. The break at this discontinuity is, however, extremely small, the change in $\Delta\theta$ is only as follows:

$$30 \left(\exp\left(-\left[\frac{85}{25}\right]^2\right) - \exp\left(-\left[\frac{275}{25}\right]^2\right) \right) \approx 2.8620 \times 10^{-4}.$$

This discontinuity may therefore be disregarded in most practical applications.

For general applications of the ΔE_{00} formula, it is also helpful if we can bound the *maximum discontinuity* magnitude that may be encountered. An empirical evaluation was therefore performed, where for each selected hue angle, we determined the points with the maximal discontinuity in ΔE_{00} that were within a CIELAB ΔE_{ab}^* of 5 units from each other (equivalently, within 5 CIELAB chroma units from each other). In each of the cases tested, the maximal discontinuity occurred for a configuration of colors that was very close to that used in Fig. 5, with $R = 2.5$. Thus the plot corresponding to $R = 2.5$ in Fig. 6 also represents the maximal discontinuity as a function of hue. Globally, for colors that are within 5 CIELAB ΔE_{ab}^* units from each other, the discontinuity in CIEDE2000 color-difference ΔE_{00} is under 0.2734.

The discontinuity magnitude may also be computed for a general configuration, where the reference 1 is located at a hue angle h at chroma radius R_0 and the two almost identical samples 2 and 3 are located at a chroma radius of R_1 with hue angles just above and just below $h + 180^\circ$. Denote the discontinuity magnitude for this configuration as follows:

$$d_{\Delta E}(h, R_0, R_1) = |\Delta E_{00}^{12} - \Delta E_{00}^{13}|. \quad (31)$$

The three-dimensional discontinuity-magnitude function $d_{\Delta E}(h, R_0, R_1)$ can be visualized along two dimensions at a time by fixing the third dimension. The visualization in this form is, however, more suited to interactive demonstration and harder to render in limited 2D views. A MATLAB based GUI has been developed for this purpose⁹ that confirms that the three-dimensional presentation leads to no surprises and the insight gained from the graphs presented in this article is accurate and complete.

IMPLICATIONS OF DISCONTINUITIES IN ΔE_{00}

As outlined in the previous section, the significant discontinuity in the computation of the CIEDE2000 color difference manifests itself only for samples that are 180° apart in hue (i.e., located in opposite quadrants of the a^*-b^* plane). Because the CIEDE2000 formula is applicable primarily for small color differences, both samples will typically be close together. Therefore, the only situation under which they may lie in opposite quadrants is for the case of colors close to gray. These have a low value of chroma and therefore the magnitude of the discontinuity will be small in practical applications. As illustrated in the previous section, if the samples are under $5\Delta E_{ab}^*$ units apart, the discontinuity in CIEDE2000 color-difference ΔE_{00} is under 0.2734, which is small in comparison to color differences encountered in a number of applications, but not negligible. If the samples are $1\Delta E_{ab}^*$ unit apart, the discontinuity magnitude is smaller than 0.0119, which is negligible in most practical situations.

Because of their small magnitude, the discontinuities in the CIEDE2000 color-difference computation may not be a major concern in most industrial applications, where other sources of experimental variation are much larger. How-

ever, the discontinuities do preclude the use of the formula in analysis based on Taylor series approximations^{10,11} and in design techniques using gradient based optimization, that not only require continuity of the function but also continuity of the first derivative. For accommodating these applications, it would be desirable to eliminate the discontinuities all together. Because the source of the main discontinuity is the inherent uncertainty of 180° in computing the “mean” of two angles through the geometric process illustrated in Fig. 2, a suitable modification of this process can eliminate this specific discontinuity.^{||} One option would be to sacrifice the symmetry of the formula and use only the hue angle of the reference sample in the terms T and $\Delta\theta$ instead of the mean.

The discontinuity due to the computation of hue difference and the (extremely small) discontinuity at mean hue of $0/360$ degrees are not eliminated even when the computation of mean hue is modified. To remove the small discontinuity at a mean hue of $0/360$ one may choose a different functional form for $\Delta\theta$ that decays to zero for θ in the neighborhood of 0° and 360° . The discontinuity in the computation of hue difference is inherent in the process of computing differences between angles that are 180° apart, the difference may be assumed to be 180° or -180° . This causes the sign of $\Delta H'$ in Eq. (11) to flip. The discontinuity is, however, eliminated if only the absolute value of $\Delta H'$ is used in subsequent computations. This will require additional changes to the formula to ensure that the rotation term is appropriate.

^{||} From a mathematical standpoint, the discontinuity can be eliminated even if the computation of average hue is unchanged by imposing a constraint of 180° symmetry on terms involving \bar{h}' . This is immediately apparent from Eq. (26). However, such constraints are not really meaningful and therefore likely to hurt the performance of the formula with regard to perceptual uniformity.

CONCLUSION

To validate software implementations of the CIEDE2000 color-difference formula, additional testing is required beyond the worked examples included in the draft standard.¹ In this note, we provided implementation guidelines and supplementary test data to enable this additional testing. We also provide the test data in electronic format and sample implementations in Microsoft Excel and MATLAB at the first author’s website.⁵ We also indicate the presence of three discontinuities in the CIEDE2000 formula and characterize these discontinuities in terms of their magnitudes.

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