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HW1 Problem 2

Let SVD of A = USV^T,
Where U and V are orthonormal matrices and S is diagonal.
$A^T A = VSU^T USV^T = VS^2 V^T$ -- (1)

Let the Eigen decomposition of $A^T A$ be represented as: $A^T A = E\Sigma E^{-1}$

Conclusion: the orthonormal matrix V from SVD of A is the eigen vector matrix of Eigen decomposition of $A^T A$ (V = E), and the square of the diagonal matrix from SVD of A is the Eigen value matrix of $A^T A$ ($\Sigma = \mathrm{S}^2$). The proof is shown as follows:

Proof:
Now lets' prove that (1) is also an Eigen decomposition of the square matrix $A^T A$.
In equation (1), multiply on both sides on the right by V:
$A^T AV = VS^2 V^T V = VS^2$ -- (2) (apply the property of orthonormal matrix that $V^T V = 1$)

To continue with eqn (2):
$A^T A[v1\ v2\ v3 \ldots vn] = [v1\ v2\ v3 \ldots vn]S^2 = B$, Let v1, v2, vn be vectors of each column of V, Let Matrix S^2's diagonal be: $s1^2, s2^2, \ldots sn^2$. On the left hand side, the matrix multiplication can be viewed as $A^T A$ dot with v1, v2, … vn. By definition, the dot product of these result in each column of B, call them b1, b2, …bn. In equations:
$A^T A \cdot vi = bi, i\epsilon(1, n)$ -- (3)

Now consider the left hand side, the matrix multiplication can be viewed as the outer product of v1, v2, …vn and each row of matrix S, and the sum of the matrices created by the outer products is exactly B. In equations:
$v1 \cdot s1^2 + v2 \cdot s2^2 + \cdots + vn \cdot sn^2$ --- (4)
Notice that S is a diagonal matrix, which means that the ith term in (4) results in a matrix with all zeros except for the ith column. Thus, the net effect of each term in (4) is to produce a column in matrix B.

Therefore, from (3) and (4), we have bi = $vi \cdot si^2$ ➔
$A^T A \cdot vi = vi \cdot si^2 = si^2 \cdot vi$, -- (6) as si^2 is simply a scalar.

In (6), make the observation that that vi is the eigen vector of matrix $A^T A$, and si^2 is the associating eigen value. The conclusion holds for any i in range (1, n), thus, matrix V is indeed an eigen vectors matrix, and matrix S is indeed an eigen value matrix. Therefore, let's restate the conclusion: (1) is an eigen decomposition of $A^T A$ with V = E and $\Sigma = \mathrm{S}^2$.