Weak Measurement and Weak Values

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This paper explores weak measurements and weak values, broadening our understanding of quantum mechanics beyond the traditional strong measurement framework. The intriguing weak values are dissected and analyzed due to their ability to exist outside of conventional eigenvalue ranges. We also discuss the experimental methodologies that were developed for these nuanced measurements, emphasizing the inherent amplification effect that highlights the promise of weak values for sensing minuscule quantum shifts. The investigation concludes with a look at complex variable measurement in the context of weak values, opening up new avenues for quantum mechanics research.

I. INTRODUCTION

Quantum mechanics, the theoretical foundation underlying the smallest building blocks of the universe, is replete with interpretational challenges and paradoxes that continue to puzzle us. The quantum measurement problem stands central to these intriguing facets of the theory. Our traditional approach, often described within the von Neumann or "strong" measurement framework, although insightful, is not without its limitations. Recognizing these boundaries, we are encouraged to explore beyond them and delve into the domain of weak measurements and weak values.

Yakir Aharonov, David Z. Albert, and Lev Vaidman published a seminal paper in 1988 that introduced the theoretical concept of weak measurements to the physics community [1]. It represented a significant departure from the traditional framework of quantum measurements, also known as "strong" or von Neumann measurements. Aharonov, Albert, and Vaidman's novel approach was intriguing because it involved gentle probing of the quantum system, resulting in minimal disturbance of the system state. The subsequent definition of "weak values," as an outcome of weak measurements, proved to be even more intriguing. Weak values represented quantities that, contrary to popular belief, could exceed the traditional eigenvalue spectrum of quantum observables. This enigmatic property sparked considerable interest, which was fueled further by Simon and Aharonov's initial experimental confirmation in their study [2].

This paper aims to provide a comprehensive examination of weak measurements and weak values, traversing their theoretical foundations, discussing experimental implementations, and reflecting on the unique interpretational questions they raise. We will also investigate the potential of weak measurements for signal amplification and measuring complex quantum variables. However, it is critical to understand the term paper's scope and limitations. As a vast and complex field, quantum mechanics contains facets that go beyond the scope of a single paper. While we aim to provide a thorough understanding of weak measurements and weak values, certain complexities, particularly interpretational debates, may not be fully addressed.

Having established the larger context, we now turn our attention to the domain of quantum measurements. The traditional approach of strong measurements provides a solid foundation but limits our ability to probe the quantum realm. This motivates us to shift our focus to the less intrusive but profoundly intriguing concept of weak measurements and weak values. So we embarked on our exploratory journey into the fascinating landscape of weak measurements.

II. WEAK MEASUREMENT

A weak measurement is a standard measurement with weakened coupling, which is described by the Hamiltonian (1). Unlike the strong measurement, the initial state of the pointer Q is not well-localized. It has a large uncertainty, meaning that the conjugate momentum P is localized around zero with small uncertainty. Although the measurement becomes imprecise, one can improve the precision by a factor $\sqrt{N}$ by performing the weak measurement on an ensemble of N identical systems [3]

$$\hat{H} = g(t)\hat{A} \otimes \hat{P}_d \tag{1}$$

One can treat weak measurement as a generalized quantum strong measurement and consider both the system and the measurement device as quantum systems [4]. To apply weak measurement, first, the quantum system is weakly coupled to the measurement device; then a strong measurement should apply to the measuring device so that the collapsed state of the measurement device can be referred to as the outcome of the weak measurement process [5]. It should be noted that a weak measurement requires a standard deviation of the measurement outcome greater than the difference between the eigenvalues.

The position of the needle q of the measuring device can be described with the wave function:

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$$|\phi\rangle = |\phi_d\rangle = \int_x \phi(q)\,|q\rangle\,dx \tag{2}$$

which is represented in the position basis and $\hat{Q}_d$ is the position operator for the measuring device such that $\hat{Q}_d\,|q\rangle = q\,|q\rangle$. Moreover, $\phi(q)$ is a normal distribution around 0 [5]:

$$\phi(q) = (2\pi\sigma^2)^{-1/4} e^{-q^2/4\sigma^2} \tag{3}$$

Now, suppose $\hat{A}$ is an observable of the system S and has N eigenvectors $|a_j\rangle$ such that $\hat{A}\,|a_j\rangle = a_j\,|a_j\rangle$; therefore, we can define a general state vector in the eigenbasis of $\hat{A}$:

$$|\psi\rangle = \sum_j \alpha_j\,|a_j\rangle \tag{4}$$

Consider Hamiltonian (1) as the interaction Hamiltonian $\hat{H}_{int}$ where the coupling happens between the time interval $[0, T]$ and the strength of the coupling is governed by $g(t)$ which is a sharp time-dependent function used to characterize the time interval of the measurement. Suppose $g(t) = \gamma\delta(t)$ and the following [5]:

$$g(t) = \int_0^T g(t) = 1 \tag{5}$$

Similar to strong measurement, when there is a time evolution via H_{int} on the vector $|\psi\rangle \otimes |\phi(q)\rangle$, one can observe the translation on the position of the needle by applying Heisenberg evolution:

$$\begin{aligned}\hat{Q}_d(T) - \hat{Q}_d(0) &= \int_0^T dt \frac{\partial \hat{Q}_d}{\partial t} \\ &= \int_0^T \frac{1}{\hbar}[\hat{H}, \hat{Q}_d]dt = a_j\end{aligned} \tag{6}$$

The corresponding transformation of the wave function can be described as:

$$e^{-i\hat{H}T/\hbar}\,|\psi\rangle \otimes |\phi\rangle = \sum_j \alpha_j\,|a_j\rangle \otimes |\phi(q - \gamma a_j)\rangle \tag{7}$$

As mentioned previously, since the initial state of the measuring device is not well-localized and has a large uncertainty, in other words, ϕ has a large variance, they overlap each other. This means the higher the variance, the weaker the measurement process [5]. Finally, to observe the result, a strong measurement is applied to the measuring device:

$$P(q = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_j |\alpha_j|^2 e^{\frac{-(x-\gamma a_j)^2}{2\sigma^2}} \tag{8}$$

If σ is much smaller than the difference of the spectrum of $\hat{A}$ or the interaction is sufficiently large: $\gamma \cdot \min(a_i - a_j) \gg \sigma$, then we can visualize $P(q = x)$ as a weighted summation of many Gaussian distributions localized at different centers which are proportional to the eigenvalue of the $\hat{A}$. In this case, weak measurement performs a similar behavior to strong measurement. For the cases where σ is much larger than all $a_j\gamma$, the following approximation works:

$$\begin{aligned}P(q = x) &\approx \frac{1}{\sqrt{2\pi\sigma^2}} \sum_j |\alpha_j|^2 (1 - \frac{(x - \gamma a_j)^2}{2\sigma^2}) \\ &\approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\gamma\langle\hat{A}\rangle)^2/2\sigma^2}\end{aligned} \tag{9}$$

where $\langle\hat{A}\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle$. One measurement is not enough to have information on the exact value due to the large spread caused by $\sigma \gg \gamma\langle\hat{A}\rangle$ [1]. As mentioned before, it is possible to make the measurement $\langle\hat{A}\rangle$ precise by preparing a large number of such systems and making the same measurement on each member of the ensemble.

A. Strong Measurements as a Contrast to Weak Measurements

In a strong measurement scheme, the pointer Q is well localized in the sense that the deviation σ of the pointer in the position basis is less than the eigenvalue spacings of an observable so that one can resolve the outcome of the measurement perfectly. Assume observable A is σ_z for the ease of discussion where $\{|0\rangle, |1\rangle\}$ is the eigenbasis and $\{1, -1\}$ is the corresponding eigenvalues. Assume the state to be measured is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{10}$$

Then, the pointer state (assuming they are initialized to 0-position) and $|\Psi\rangle$ after the interaction Hamiltonian becomes (from Eq.7);

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\phi(-1)\rangle) + |1\rangle \otimes |\phi(1)\rangle) \tag{11}$$

The pointer and the state $|\Psi\rangle$ are entangled with each other. The measurement of the pointer space yields either a pointer centered at x=1 or x=-1 with equal probabilities. The outcome of measurement for observable σ_z is then -1 or +1 respectively. Also, the final state of $|\Psi\rangle$ is $|1\rangle$ or $|0\rangle$ respectively. Therefore, a strong measurement collapsed the state $|\Psi\rangle$ to one of the eigenstates of the observable σ_z. Also, averaging many measurement results of $|\Psi\rangle$ yields the expectation value of observable σ_z given as $\langle\Psi| \sigma_z |\Psi\rangle$ analytically. For this case, it is $\frac{1}{2}$ and guaranteed to be between the eigenvalues $\{-1, 1\}$ for any state $|\Psi\rangle$.

III. WEAK VALUE

In 1988, Aharanov et. al.[1] performed weak measurements with pre and post-selected spin-$\frac{1}{2}$ ensembles. With a particular choice of pre- and post-selected states out of the ensemble, they measured the z-spin component to be 100, lying far above the allowed eigenvalue spectrum $\{-1, 1\}$. They coined this new type of quantum variable, i.e., the outcome of weak measurement with particular pre- and post-selected states, as "weak value". Aharanow et al. [3] introduced the two-state vector formalism of quantum mechanics in the process of developing a time-symmetric quantum mechanics formalism before the weak value experiments. If one conducts a measurement of an observable **A** on a quantum state at time t_1, the outcome is one of the eigenstates of the observable A, say $|\psi\rangle$. $|\psi\rangle$ is the **pre-selected state** since the quantum state is known to be $|\psi\rangle$ at t_1. Then, at a later time t_2, the quantum state becomes;

$$U(t_2) |\psi\rangle = e^{\frac{-iH(t_2-t_1)}{\hbar}} |\psi\rangle \tag{12}$$

At the time t_2, the **post selection process** occurs. The time-evolved system state (possibly with the conjugate pointer state) is projected to a state $|\Phi\rangle$ in the system space. Then, this quantum state between $[t_1, t_2]$ is described by a two-state vector $\langle\Phi| \ |\psi\rangle$.

A single quantum state $|\Psi\rangle$ formalism of quantum mechanics is inherently time-asymmetrical[3]. $|\Psi\rangle$, which is the result of a particular measurement at a certain time t_1, only constraints the future measurement outcomes since after the unitary evolution of $|\Psi\rangle$, any non-orthogonal state to $|\Psi\rangle$ has the possibility of being measured. However, in classical physics, a measurement both in the future or the past uniquely constrains the state in a time-symmetrical manner. The reason Aharanov et al. [3] introduce two-state vector formalism is to address the question that "is the reason of time-asymmetry in quantum mechanics merely due to our use of inherently time-asymmetric concepts like single quantum state $|\Psi\rangle$?". They developed the probability amplitudes formula (ABL formula) for the strong measurements of two-state vectors between the times $[t_1, t_2]$, which are time-symmetrical[3];

$$P(c_n) = \frac{|\langle\Phi| \boldsymbol{P_{C=c_n}} |\Psi\rangle|^2}{\sum_j |\langle\Phi| \boldsymbol{P_{C=c_j}} |\Psi\rangle|^2} \tag{13}$$

Then, they questioned what happens if the measurement of a two-state vector is not strong but a weak one so that the two-state vector is minimally perturbed[3]. Then, they coined the term **weak values** for weak measurement outcomes of two-state vectors. Besides the theoretical motivation of Aharanov et al., weak values posited numerous experimental benefits that we explore in the following section in detail. The first-order weak value of an observable A for a particular two-state vector is given by

$$A_w = \frac{\langle\Phi| A |\Psi\rangle}{\langle\Phi|\Psi\rangle} \tag{14}$$

First, we demonstrate that the Gaussian pointer (weakly coupled to the system in the context of weak measurements) shifts by A_w after post-selection. In other words, the measurement on the system yields the first-order weak value, A_w. More specifically, the position uncertainty of the Gaussian pointer is wide since the measurement is weak, but the center of the Gaussian pointer is A_w. Therefore, one needs to do repeated measurements and averaging to reach the value, A_w. The theoretical account of the process is as follows. The state before the post-selection is the following equation where H is given by Eq.(1).

$$e^{\frac{-iHt}{\hbar}} |\psi\rangle \otimes |\phi\rangle \tag{15}$$

In the post-processing, the composite system-pointer state is going to be measured with the projector[5];

$$\hat{p} = |\Phi\rangle \langle\Phi| \otimes I_p \tag{16}$$

Since the measurement is weak, the first-order expansion of the unitary operator is enough. The final state becomes[5];

$$(|\Phi\rangle \langle\Phi| \otimes I_p)(1 - i\hat{A} \otimes \frac{\hat{P}T}{\hbar})(|\Psi\rangle \otimes |\phi(x)\rangle) \tag{17}$$

$$|\Phi\rangle \otimes \langle\Phi|\Psi\rangle (1 - iA_w \frac{\hat{P}T}{\hbar} |\phi(x)\rangle) \tag{18}$$

$$|\Phi\rangle \otimes \langle\Phi|\Psi\rangle e^{-iA_w \frac{\hat{P}T}{\hbar}} |\phi(x)\rangle \tag{19}$$

$$|\Phi\rangle \otimes \langle\Phi|\Psi\rangle |\phi(x - A_w T)\rangle \tag{20}$$

As apparent from Eq.(20), the center of the Gaussian pointer shifts by weak value A_w times the coupling duration T. If one picks the pre- and post-selected states nearly orthogonal, then $\langle\Phi|\Psi\rangle$ becomes really small, thereby making A_w arbitrarily large as can be seen from Eq.(14). Therefore, one can get big "weak values" far outside of the eigenvalue spectrum.

The first important remark about the previous calculation is that the weak value A_w one gets during the weak measurement is constant with respect to the weak coupling duration T. If one measures the weak value A_w at any instant t of the coupling duration between $[t_1, t_2]$, he measures the shift in the Gaussian pointer as $A_w * (t - t_1)$ and thus observes the same A_w. The second remark is that instead of interpreting weak values as outcomes of weak measurements of some coupling duration T, we can interpret them as a property of the two-state vector quantum states at a particular time[6]. In other words, we can assign a particular two-state vector and weak value to a quantum state at a particular time. In the context of weak measurements with pre- and post-selected states, A_w is a function of two-state vector $\langle\Phi| \ |\Psi\rangle$ where $|\Psi\rangle$ is the pre-selected state at time t_1 and $|\Phi\rangle$ is the post selected state at time t_2. Since the weak value A_w is constant during the whole weak coupling and a function of a two-state vector at a particular time, we can infer that the two-vector quantum state at any time of the coupling duration $[t_1, t_2]$ is the same pre and post-selected two-state vector $\langle\Phi| \ |\Psi\rangle$.

Up to this point, we interpreted the weak values in the context of weak measurements since the latter yields the first. However, we need to address weak values also in the context of stronger measurements since one can also do pre- and post-selected measurements with stronger couplings. In the latter case, the quantum states at a particular time of the strong coupling duration still have a weak value, but they are no longer given by Eq. (14) [6]. The reason is that the stronger couplings change the two-state vectors throughout the coupling duration in contrast to them being constant in weak couplings. Therefore, for a general measurement of a pre-and post-selected quantum system, the actual weak value at a particular time and the pointer reading differ. They agree on the weak interaction regime, but the latter exceeds the former as the interaction strength increases since the interaction Hamiltonian (Eq. 1) has higher order contributions as well. Interestingly, a Gaussian pointer as an exception still shows the weak-value outside of the weak coupling regime [6].

In this section, the theoretical account of weak values is presented with single pre- and post-selected pure quantum states $|\Psi\rangle$,$|\Phi\rangle$. However, the experimental scenarios involve ensembles of pre-selected quantum states that are post-selected after a weak interaction with a pointer. Since the pointer has high uncertainty due to the weakness of coupling, one particle is not enough to determine the true mean(A_w) of the pointer state. Therefore, inherently one has to use ensembles of quantum states to determine A_w after averaging. This brings us to the question that "Are the (anomalous) weak values just a statistical phenomenon rather than a true quantum phenomenon?".

To address this question, one needs to envision an experimental setup where a single measuring device coupled to a pre-and post-selected quantum state once is guaranteed to yield an anomalous weak value with a single measurement. Then, since the averaging procedure with an ensemble is not carried out, the possibility of a weak value being a merely statistical phenomenon rather than a quantum one is eliminated. Such an experimental setup is realized by Rebufello et al. [7]. Their setup involves weak coupling of 7 photon polarization quantum state(a single quantum state) to a measurement device

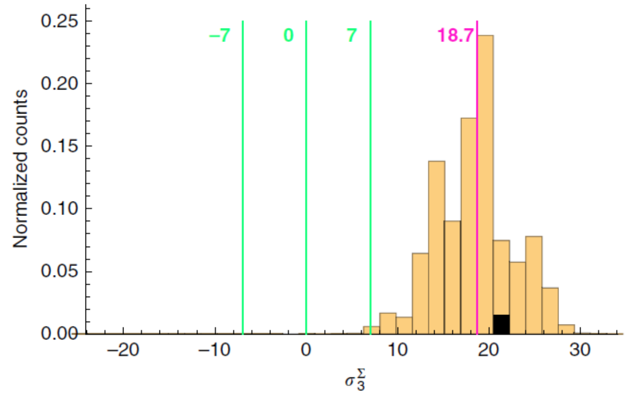

FIG. 1. Green lines represent the eigenvalue spectrum and the red line is a theoretical weak value. The black bar is the first trial.Source:[7]

once. The pre-and post-selection is realized by polarizers. A birefringent material in the middle causes shifts in the transverse direction depending on the polarization. A CCD at the end of the setup measures the transverse displacements of the photons. The observed variable is the polarization of all seven photons, $\sum_{i=1}^{7} S^{(i)}$ where $S = |H\rangle\langle H| - |V\rangle\langle V|$. It has the eigenvalue spectrum [-7,7]. The pointer space is the Gaussian pointer space, p_x introduced in the previous section and this is realized by photons' transverse degree of freedom being a Gaussian.

To ensure that single click measurement yields an anomalous weak value with certainty and provides a good enough estimate for the actual weak value, they picked a smart deviation value σ for Gaussian's(representing photons' transverse degree of freedom) in the position basis. σ being small would mean a strong measurement scenario. Also, for a single click measurement to provide some useful information, σ should not be too large. Their use of a Gaussian pointer enabled them to tweak σ without worrying about changing the weak value of the system since a Gaussian pointer always yields the actual weak value. Their single-click measurement yielded the value 21.4, which lies outside the eigenvalue spectrum $[-7, 7]$ with certainty since the pointer uncertainty is smartly chosen. Hence, they verified the non-statistical nature of weak values and provided a scheme(*robust weak measurements[7]*) where single click measurements provide a good estimate for the actual weak value. Repeated trials are presented as a histogram in Fig.1.

IV. UNDERSTANDING COMPLEX WEAK VALUES

Weak values in quantum physics are distinct from traditional quantum expectation values in two fundamental ways. Firstly, they are not restricted to the eigenvalue spectrum, meaning they can take on exceptionally large or small values that go beyond the bounds set by an operator's eigenvalues. Secondly, weak values A_w can also be complex, contrary to standard quantum expectations, which are confined to real numbers. At first, it might seem peculiar to conceive of a complex number influencing the wave function of a physical entity, such as a needle. This section, therefore, seeks to demystify this phenomenon and provide a lucid comprehension of how such an odd occurrence in quantum physics transpires [5].

Now, think of the scenario where a small perturbation is introduced via operator $\hat{U}(\epsilon) = e^{-i\epsilon\hat{A}}$. As mentioned before, what makes the perturbation weak is to have a small enough ϵ, or one can call it a tiny rotation (if $\hat{A}$ is angular momentum operator) to have a physical understanding. This lets one expand the transformation's Taylor series so that the detection probability after this small perturbation can be expressed as [8]:

$$\begin{aligned} P_\epsilon &= |\langle\Phi|\,\hat{U}(\epsilon)|\,|\Psi\rangle = |\langle\Phi|\,(1 - i\epsilon\hat{A} + ...)\,|\Psi\rangle\,|^2 \\ &= P + 2\epsilon \mathrm{Im}\,\langle\Psi|\Phi\rangle\,\langle\Phi|\,\hat{A}\,|\Psi\rangle + O(\epsilon^2) \end{aligned} \tag{21}$$

where $P = |\langle\Phi|\Psi\rangle|^2$. Dividing the whole expression by P (as long as $|\Phi\rangle$, and $|\Psi\rangle$ are not orthogonal), and using the equation (14) gives the expression [8]:

$$\frac{P_\epsilon}{P} = 1 + 2\epsilon\mathrm{Im}A_w - \epsilon^2[\mathrm{Re}A_w^2 - |A_w|^2] + O(\epsilon^2) \tag{22}$$

where A_w, A_w^2 are first and second-order weak values, respectively.

The entire Taylor Series expansion is defined by complex weak values across all orders. Our operational definition suggests that weak values are the relative modifications to a detection probability caused by a slight intermediate disturbance, leading to an altered detection probability. While we've only shown the expansion to the second order, it's important to note that the full Taylor series expansion for the probability ratio is entirely defined by complex weak values of all orders. There is a linear relationship between the probability adjustment and the first-order weak value when the higher-order terms in the expansion can be ignored. This is known as the weak interaction regime. These terms can be overlooked in two situations [8]:

1. the relative correction $P_\epsilon/P - 1$ is sufficiently small
2. $\epsilon\mathrm{Im}A_w$ is sufficiently large compared to the higher order terms in the sum

Again, consider the Hamiltonian (1) for the interaction and pointer in Eq. (2), then the general state after the post-selection becomes as Eq. (19). Therefore, the state for the pointer after the post-selection becomes $|\alpha\rangle$:

$$|\alpha\rangle = \langle\Psi|\Phi\rangle\, e^{i\hat{A}_w\hat{P}_d}\,|\phi\rangle \tag{23}$$

Once the final state is defined, one can look at the expectation values for an observable, let's say $\hat{M}$, before and after post selections [5]:

$$\langle\hat{M}\rangle_f = \frac{\langle\alpha|\,\hat{M}\,|\alpha\rangle}{\langle\alpha|\alpha\rangle}, \quad \langle\hat{M}\rangle_i = \frac{\langle\phi|\,\hat{M}\,|\phi\rangle}{\langle\phi|\phi\rangle} \tag{24}$$

Since $\langle\hat{A}\rangle_w$ is a complex number, it can be expressed as $a+ib$. This makes the term $e^{-i\langle\hat{A}\rangle_w\hat{P}_d}$ non-unitary due to its complex argument which yields $\langle\alpha|\alpha\rangle = 1 + 2gb\langle\hat{P}\rangle_i$. When $|\alpha\rangle$ is written in terms of $|\phi\rangle$, which gives an expression for $\langle\hat{M}\rangle_f$ in terms of $\langle\hat{M}\rangle_i$ with some correction as below [9]:

$$\begin{aligned} \langle\hat{M}\rangle_f = \langle\hat{M}\rangle_i &+ iga\langle\hat{P}_d\hat{M} - \hat{M}\hat{P}_d\rangle_i \\ &+ gb(\langle\hat{P}_d\hat{M} + \hat{M}\hat{P}_d\rangle_i - 2\langle\hat{P}_d\rangle_i\langle\hat{M}\rangle_i) \end{aligned} \tag{25}$$

Note that the error (or correction part) in the equation has linear dependency on a and b. So one can ask for the observable $\hat{M} = \hat{P}$ or $\hat{M} = \hat{Q}$ what is the correction, which gives [5]:

$$\langle\hat{P}\rangle_f = \langle\hat{P}\rangle_i + 2gb\mathrm{Var}(\hat{P}) \tag{26}$$

$$\langle\hat{Q}\rangle_f = \langle\hat{Q}\rangle_i + ga \tag{27}$$

So given such an interaction Hamiltonian (1), real values of A_w shift the needle's expectation value of $\hat{Q}$ and imaginary values of A_w shift the needle's expectation value of $\hat{P}$.

V. AN EXPERIMENTAL APPROACH

One of the better-known examples of weak value measurements is the canonical optical weak-value amplification device, or COWVAD for short, introduced by Duck, Stevenson, and Sudarshan in 1989. It consists essentially of a birefringent crystal, a laser beam, polarizers for preparing the initial polarization, a linear polarizer for post-selecting the final polarization, and a detector. This device utilizes the weak coupling of polarization to the momentum of the photon to measure the small changes in the polarization of the laser.

Consider the setup shown in 2. An incident laser beam is prepared to be in the initial state of $|i\rangle\,|\psi_i\rangle$, where $|i\rangle$ is the pre-selection polarization and $|\psi_i\rangle$ is the transverse profile of the laser beam. The desired initial polarization state is prepared using a half-wave plate (HWP) and a quarter-wave plate (QWP). The final state of the polarization $|f\rangle$ is achieved by passing the

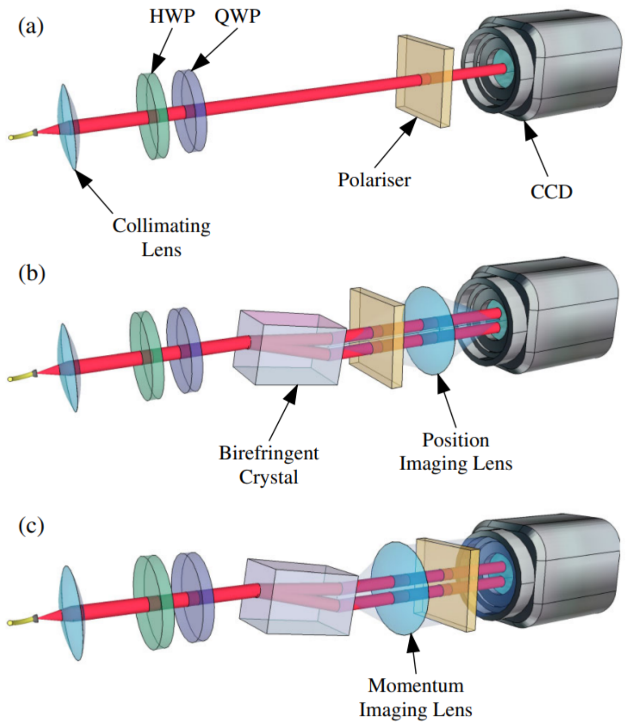

FIG. 2. The visualization of the setup [8]. a) The initial polarization of the laser is adjusted using the half-wave plate (HWP) and the quarter-wave plate (QWP), and the post-selection is done using a final polarizer, after which a detector measures the transverse intensity. b) With a birefringent crystal present, two polarizations follow different paths upon entry. A lens is used to project the transverse position of the beams onto the detector. c) The lens is changed to imaging the far field of the crystal onto the detector as the transverse momentum.

beam through a final linear polarizer. In the absence of any interaction, the probability of measuring a photon is;

$$P = |\langle f|i\rangle|^2 |\langle \psi_f|\psi_i\rangle|^2, \tag{28}$$

where $|\psi_f\rangle$ is the final transverse state on each pixel of the detector. $|\psi_f\rangle$ would be taken as either $|x\rangle$ or $|p\rangle$ for the purposes of the experiment, depending on whether the imaging is taken on the transverse position or transverse momentum, perhaps through the use of a Fourier lens as shown in 2(c). Now, suppose a birefringent crystal is introduced between the wave plates and the final linear polarizer as in 2(b). The crystal separates two polarizations into different paths. The displacement of the beam would depend on the length and the material properties of the crystal. An initial tilt of the crystal is assumed so that the distance covered by both beams would be the same and equal to $\epsilon = \tau v$, where τ is the time it takes for the light to pass through the crystal and v is the speed of light inside the crystal.

As the light passes through the crystal, its time evolution can be shown with $\hat{U}(\tau) = e^{\frac{-i\hat{H}}{\hbar}\tau}$ where the Hamiltonian is,

$$\hat{H} = v\hat{S} \otimes \hat{p}. \tag{29}$$

where $\hat{S} = |H\rangle\langle H| - |V\rangle\langle V|$ is the Stokes polarization operator that takes the values +1 and -1 for horizontal and vertical polarizations respectively, and $\hat{p}$ is the transverse momentum operator that generates transverse displacements in x. Then, after the post-selection, the probability becomes

$$P_\epsilon = |\langle f|\langle\psi_f| e^{-i\epsilon\hat{S}\otimes\hat{p}/\hbar} |i\rangle|\psi_i\rangle|^2 \tag{30}$$

Now, notice that this expression is similar to equation 21 with $|\Psi\rangle = |i\rangle|\psi_i\rangle$ and $\langle\Phi| = \langle f|\langle\psi_f|$. Then, correspondingly $\hat{A} = \frac{\hat{S}\otimes\hat{p}}{\hbar}$. If small values of ϵ are considered, according to equation 22 the following approximation can be made,

$$\frac{P_\epsilon}{P} \approx 1 + \frac{2\tau}{\hbar}\,\mathrm{Im}\,H_w, \tag{31}$$

where H_w is the first weak value of the Hamiltonian. Then, it can be further simplified,

$$\frac{P_\epsilon}{P} - 1 \approx \frac{2\epsilon}{\hbar}[\mathrm{Re}\,S_w\,\mathrm{Im}\,p_w + \mathrm{Im}\,S_w\,\mathrm{Re}\,p_w]. \tag{32}$$

where $S_w = \frac{\langle f|\hat{S}|i\rangle}{\langle f|i\rangle}$ and $p_w = \frac{\langle\psi_f|\hat{p}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}$ are the first order weak values of Stokes polarization and momentum operators respectively. To illustrate, say the initial state is prepared with a polarization state,

$$|i\rangle = \frac{|H\rangle - e^{i\phi}|V\rangle}{\sqrt{2}}, \phi = 0.1, \tag{33}$$

with a Gaussian beam profile such that

$$\langle x|\psi_i\rangle = (2\pi\sigma^2)^{-1/4} e^{-x^2/4\sigma^2}. \tag{34}$$

And let the post-selection states be

$$|f\rangle = \cos\left(\frac{\theta}{2}\right)|H\rangle + \sin\left(\frac{\theta}{2}\right)|V\rangle, \theta = \pi/2 - 0.2. \tag{35}$$

It can be seen that with the chosen values of θ and ϕ, the pre and post-selection states are almost orthogonal and equal to $|\langle f|i\rangle|^2 = 0.012$. $|\psi_f\rangle$ can be chosen to be $|x\rangle$ or $|p\rangle$ with different experimental methods to determine the real and the imaginary parts of the weak values.

For the example of COWVAD, $|\psi_f\rangle = |x\rangle$ is achieved by the configuration shown in 2(b). Then the weak value of transverse momentum p_w becomes,

$$p_w = \frac{\langle x|\,\hat{p}\,|\psi_i\rangle}{\langle x|\psi_i\rangle} = \frac{-i\hbar\partial_x\psi_i(x)}{\psi_i(x)} = \frac{i\hbar x}{\sigma^2}, \tag{36}$$

which is purely imaginary. Then, according to eq. 32,

$$\frac{P_\epsilon}{P} \approx 1 + \frac{x\epsilon}{\sigma^2}\,\mathrm{Re}\,S_w. \tag{37}$$

The post-selection state being the transverse momentum, is achieved by using a Fourier lens before the detection as in 2(c). The resulting weak value for $\hat{p}$ would be,

$$p_w = \frac{\langle p|\,\hat{p}\,|\psi_i\rangle}{\langle p|\psi_i\rangle} = \frac{p\,\langle p|\psi_i\rangle}{\langle p|\psi_i\rangle} = p. \tag{38}$$

Since p_w is purely real, the probability ratio P_ϵ/P becomes,

$$\frac{P_\epsilon}{P} \approx 1 - \frac{2p\epsilon}{\hbar}\,\mathrm{Im}\,S_w, \tag{39}$$

thus isolating $\mathrm{Re}\,S_w$. A similar approach on finding the real and imaginary parts of p_w could be followed.

VI. WEAK VALUE AMPLIFICATION

The intriguing phenomenon of weak value amplification can be explored by leveraging our understanding of weak values and their complex properties. This process, where a small change in a system's state due to a weak interaction leads to a significantly large shift in the outcome of the subsequent measurement, is at the heart of weak value amplification. To detect a small change ϵ amidst background noise, it's crucial that the joint weak value factor in Eq. 10 is sufficiently large. When this factor is large, it amplifies the linear response. A key strategy in this technique is the careful selection of the initial and final states for weak values, p_w and S_w in our case, which can be chosen to yield a high amplification factor [8].

The potential for amplification is one of the most fascinating aspects of weak values. This isn't just a theoretical curiosity, but it also has practical implications in experimental physics, particularly in precision measurements. The amplification effect stems directly from the unique characteristics of weak values. Their complex nature allows them to have large magnitudes, even when the corresponding shifts in the system's state are small. Consider the measurement in Fig. 2(b) for a concrete example in which the centroid is calculated by averaging the positions recorded at each pixel:

$$\int xP_\epsilon(x|\theta)dx \approx \epsilon\mathrm{Re}S_w \tag{40}$$

and for Fig. 2(c) where CCD camera measures the Fourier plane:

$$\int pP_\epsilon(p|\theta)dp \approx \epsilon\frac{\hbar}{2\sigma^2}\mathrm{Im}S_w \tag{41}$$

The amplification process occurs because the factors $\epsilon\mathrm{Re}S_w$ and $\epsilon\frac{\hbar}{2\sigma^2}\mathrm{Im}S_w$ can be increased by making wise polarization post-selection choices [8]. In the previous experimental scenario, the polarization weak value is greater than 1, which is the maximum eigenvalue. This value varies depending on the angle of post-selection used. This allows for the detection of tiny effects that might otherwise be obscured by noise, opening up potential applications in diverse fields, from quantum information science to fundamental physics research [5].

While the amplification of weak values offers intriguing possibilities, it's important to be aware of its inherent challenges. The amplification effect is probabilistic in nature, meaning it doesn't always occur, and when it does, it can be accompanied by significant statistical fluctuations (see Fig. 3). Moreover, the act of measuring a weak value can potentially disturb the system, leading to errors. As the weak value factor increases, the detection probability correspondingly decreases. This implies that the weak interaction approximation, which assumes certain conditions for each pixel, will eventually

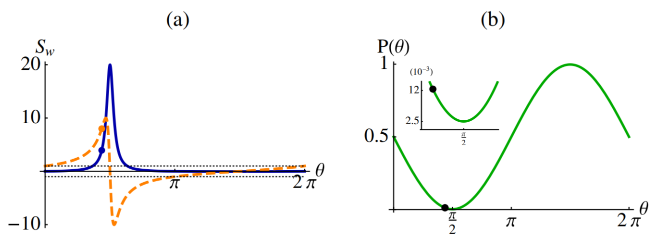

FIG. 3. (a) Real (dashed) and imaginary (solid) parts of polarization S_w, with initial and final states ($|i\rangle$, $|f\rangle$) in Eq. (33) and (35), respectively. The dotted lines represent the eigenvalue limits of ± 1 for reference, and the dots illustrate the final state chosen. (b) The post-selection probability $P(\theta) = |\langle f|i\rangle|^2$ as a function of θ is shown, demonstrating how a large weak value corresponds to a low detection probability. The inset magnifies the region with small probabilities for better visibility, and the dots again represent the final state in Equation (35) [8].

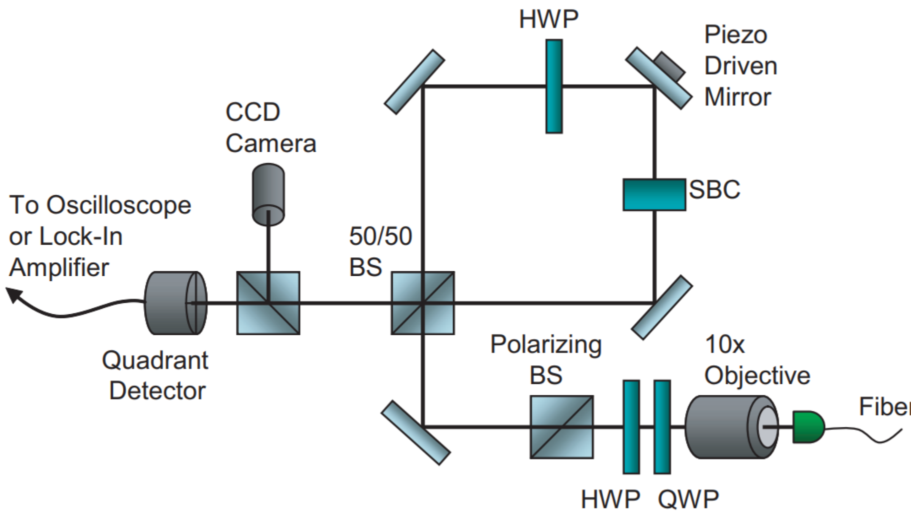

FIG. 4. Experimental setup for beam deflection measurement [10].

cease to hold. At this juncture, it becomes necessary to account for higher-order terms that can disrupt the linear response. Additionally, the diminished detection rate complicates the task of signal detection, necessitating longer collection times to surmount the noise floor. Detailed analysis shows that the signal-to-noise ratio for determining a small interaction parameter within a set time duration remains unchanged as the amplification increases. The signal enhancement achieved by increasing the amplification factors is precisely offset by the uncorrelated shot noise introduced by the reduced detection rate. Furthermore, the measurement scheme may be susceptible to decoherence during the measurement process.

Despite the difficulties, using this amplification technique has two distinct advantages [8]:

1. Because of the post-selection polarizer, the detector only collects a fraction of the total beam power, but it still has a sensitivity comparable to optimal estimation methods. This benefit enables the use of less expensive equipment, while uncollected beam power can be redirected for other purposes.

2. Because of the measurement's weakness, the amplification is resistant to certain types of additional technical noise, such as 1/f noise. This advantage allows the signal to be amplified without amplifying unrelated but common technical noise backgrounds.

When these two advantages are combined, experiments can achieve remarkable precision with relatively simple laboratory equipment.

A. Beam deflection meaurement

Let us now turn our attention to an experimental application that describes the development of a weak value amplification technique for any optical deflection. The weak value measurement in this experiment, in particular, uses the which-path information of a Sagnac interferometer to obtain dramatically improved resolution of an optical beam's deflection.

In Fig. 4, you'll find a schematic representation of a weak value amplification setup utilizing a Sagnac interferometer, a tool well-known in quantum optics for its precision derived from light interference. The operation begins with a light beam that enters the Sagnac interferometer and meets a 50/50 beamsplitter. This component divides the incoming beam into two equal-intensity beams traveling along two distinct paths inside the interferometer, dictated by strategically placed mirrors. These beams move in opposite directions, and after their respective journeys, they reunite at the beamsplitter and exit the interferometer. In a perfectly aligned Sagnac interferometer, all of the light exits through the original input port, making it the "bright port" due to the constructive interference of the two light beams when they meet again at the beamsplitter [10]. The $\pi/2$ phase shifts experienced by the beams at the beamsplitter—once upon initial splitting and again at recombination—facilitate this constructive interference. The other port, which typically stays devoid of light, is known as the "dark port." However, this inherent symmetry of the interferometer can be disrupted by introducing a half-wave plate and a Soleil-Babinet compensator (SBC) into the system. These devices can induce a relative phase shift ϕ between the light paths. By manipulating this phase shift, it's possible to gradually transform the "dark port" into a "bright port," thereby deviating from the original bright-dark configuration of the interferometer [10].

The beam passes through the interferometer, and its spatial shift as it exits the dark port is measured. $\{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$ describes the which-path information of the system. The meter is the transverse position degree of freedom of the beam, denoted by the states $|x\rangle$. The mirror in the interferometer is tilted slightly at the symmetric point. This tilt corresponds to a shift in the beam's transverse momentum. The tilt also breaks the Sagnac interferometer's symmetry, with one propagation direction deflected to the left of the optical axis at the exit of the beamsplitter and the other to the right. In other words, the continuous transverse deflection is linked to the which-path observable. This effect entangles the system with the meter via an impulsive interaction Hamiltonian, resulting in an evolution operator $e^{-ix\hat{A}k}$, where x is the meter's transverse position, k is the mirror's transverse momentum shift, and the system operator $\hat{A} = |\circlearrowright\rangle\langle\circlearrowright| - |\circlearrowleft\rangle\langle\circlearrowleft|$ describes the fact that this momentum-shift is opposite, depending on the propagation direction.
The initial state can be described as [5][10]:

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(e^{-i\phi/2}|\circlearrowright\rangle + e^{i\phi/2}|\circlearrowleft\rangle) \tag{42}$$

and post-selecting the final state as 2 (describing the dark port of the interferometer):

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(i|\circlearrowright + |\circlearrowleft\rangle\rangle) \tag{43}$$

Now, the interaction Hamiltonian becomes:

$$\hat{H}_{int} = gk\hat{A} \otimes \hat{Q} \tag{44}$$

Note that unlike Hamiltonian (1), the wave function of the needle is in momentum representation. One can compute [5]:

$$\hat{A}_w = ik\cot\frac{\phi}{2} \tag{45}$$

$$\langle\psi_f|\psi_i\rangle = \sin^2\frac{\phi}{2} \tag{46}$$

Having an imaginary weak value requires the relation in Eq.(26) giving the shift of $\langle\hat{Q}\rangle$:

$$\langle\hat{Q}\rangle_f = \langle\hat{Q}\rangle_i + 2gk\cot\frac{\phi}{2}\mathrm{Var}(\hat{Q}) \tag{47}$$

When dealing with a notably low value of ϕ, the expected value of $\hat{Q}$ experiences a shift. This behavior exemplifies the sensitivity of quantum states to small changes in parameters, underscoring the delicate balance inherent in quantum systems. However, it's crucial to acknowledge a subsequent challenge: for such small angles, post-selection becomes complex due to $\langle\psi_f|\psi_i\rangle = \sin^2\frac{\phi}{2}$.

VII. DIRECT MEASUREMENT OF QUANTUM COMPLEX VALUES

Some of the complex quantum variables, such as the global phase, may not be accessible through the usual quantum measurement processes as the eigenspectrum of Hermitian operators is necessarily real. However, the weak values are inherently complex, and one can obtain both the imaginary and real parts experimentally as demonstrated with the COWVAD setup throughout this section. Thus, we can employ weak values to measure a complex quantum variable if we write the latter as a simple function of a particular weak value[8]. After the measurement of the weak value, we can obtain the desired complex quantum variable by evaluating the simple function. As an example, we take the direct measurement of a quantum state. In particular, a general state in a specified basis is given by a complex n-dimensional vector where each entry specifies the coefficient of basis elements. One can directly measure these complex coefficients by weak measurements[8]. The conventional method of quantum state detection is *Quantum Tomography*. It involves multiple projective measurements on different basis and a numerical search over the different projective slices to estimate the state[8]. Both measurement and post-processing scheme is too effortful compared to direct measurement with weak values[8]. The latter involves just a measurement of weak values and the simple function evaluation of them to find the complex variable as the post-processing. Assume we want to measure the polarization quantum state of the photon $|i\rangle$. In the polarization basis $\{|H\rangle, |V\rangle\}$, it is

$$|i\rangle = \langle H|i\rangle|H\rangle + \langle V|i\rangle|V\rangle \tag{48}$$

We multiply $|i\rangle$ by a constant c as follows where $|P\rangle$ is the post selection state specifically chosen so that $\langle P|H\rangle = \langle P|V\rangle$[8]:

$$c = \frac{\langle P|H\rangle}{\langle P|i\rangle} = \frac{\langle P|V\rangle}{\langle P|i\rangle} \tag{49}$$

$$c|i\rangle = \frac{\langle P|H\rangle\langle H|i\rangle}{\langle P|i\rangle}|H\rangle + \frac{\langle P|V\rangle\langle V|i\rangle}{\langle V|i\rangle}|V\rangle \tag{50}$$

$$= H_w|H\rangle + V_w|V\rangle \tag{51}$$

$|H\rangle\langle H| = \frac{1}{2}(\mathbb{1} + \hat{S}_z)$ and $|V\rangle\langle V| = \frac{1}{2}(\mathbb{1} - \hat{S}_z)$. If we substitute these to the expressions H_w, V_w,we get $H_w = \frac{1+S_w}{2}$ and $V_w = \frac{1-S_w}{2}$. At the beginning of this section, we discussed how to obtain both real and imaginary parts of S_w. After one measures S_w; H_w, V_w are calculated by

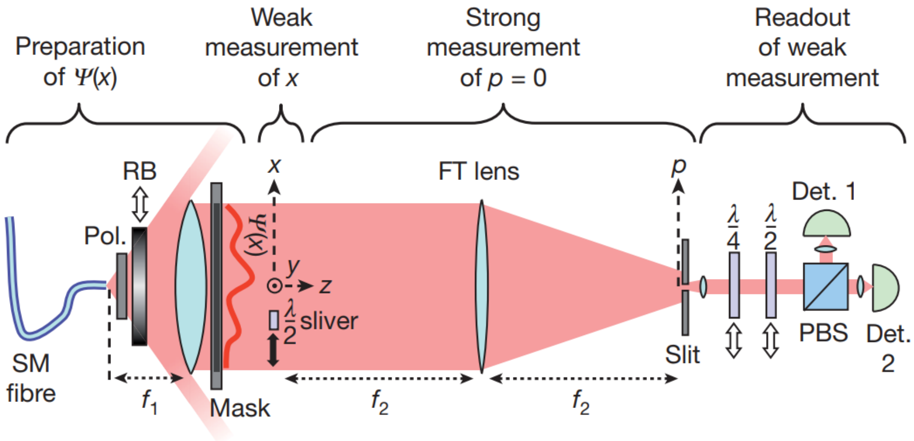

FIG. 5. The process begins with photons having identical wave functions that are transmitted through an optical fiber. The photons exit the fiber and are collimated by a lens. Their nominal initial wave function is then modified to create a variety of test wave functions. A weak measurement is performed by a thin sliver of a half-wave plate, which is followed by an optical Fourier transform (FT) using a second lens. In the Fourier transform plane, photons with a specific momentum (p = 0) are selected. These selected photons are collimated once again and then pass through a wave plate and a polarizing beamsplitter. At each output, the photons are focused onto a detector. The difference in count or signal between the two detectors is related to the real or imaginary part of the wave function, depending on whether they passed through a half-wave plate or a quarter-wave plate, respectively [11].

the given formulas easily, and the normalization yields the state $|i\rangle$ to be measured.

Despite the simplicity of this approach, it has the following drawback. If $\langle P|i\rangle$ is too low, the higher order weak value contributions become comparable to the first order one as can be seen from Eq.22. Therefore, one is no longer in a linear weak interaction regime, and the treatment fails. If the state to be estimated $|i\rangle$ is truly unknown, this might be a problem as the good choice of post-selection state $|P\rangle$ is not apparent.

A. Direct measurement of the quantum wavefunction

This useful property of weak values can be discussed further with another experimental example. Remind that the Heisenberg uncertainty principle states that a precise measurement of X disturbs the particle's wave function forcing a subsequent measurement of P to become random, which makes it impossible to determine a completely unknown wavefunction of a single system.

Let's consider, instead, a measurement of X on an ensemble of particles, Ψ with its Fourier transform $\Phi(p)$ in momentum space. In this case, one must perform a large set of measurements to reconstruct Ψ. The set of measurements helps one to estimate Ψ that is most compatible with the measurement results. This method is generally also known as quantum state tomography. More specifically, the work done in this paper [11] introduces a method to measure Ψ of an ensemble directly, meaning that the average raw signal originating from where the wave function is being probed is simply proportional to its real and imaginary components at that point [11].

Consider the weak measurement of position $A = \pi_x = |x\rangle\langle x|$ followed by a strong measurement $P = p$. In this case, the weak value is described as:

$$\begin{aligned}\langle \pi_x \rangle_w &= \frac{\langle p|x\rangle \langle x|\Psi\rangle}{\langle p|\Psi\rangle} \\ &= \frac{e^{ipx/\hbar}\Psi(x)}{\Phi(p)}\end{aligned} \tag{52}$$

For the case $p = 0$, the result will simply to:

$$\langle \pi_x \rangle_w = k\Psi(x) \tag{53}$$

where $k = 1/\Phi(x)$, a constant. The average result of a weak measurement of π_x is directly linked to the particle's wave function at a particular location, denoted as x. This weak measurement, when methodically performed over varying x coordinates, enables us to derive the complete wave function. For every x coordinate, the noticeable deviations in both the position and momentum of our measurement tool align proportionally with the real and imaginary segments of the wave function, specifically $\mathrm{Re}\Psi(x)$ and $\mathrm{Im}\Psi(x)$. In simpler terms, we can ascertain a single particle's wave function by initially lessening the interference during the X measurement, followed by executing a conventional P measurement [11].

The procedure of the experiment is described in Fig. 5. The transverse position of the photon is weakly measured. This is achieved by coupling it to the photon's own internal degree of freedom, its polarization. By treating the linear polarization angle of the photon as a measurement pointer, we can pinpoint a photon's position, x. At a given x position, where we wish to measure $\pi_x = |x\rangle\langle x|$, we induce a rotation$\alpha$ to the linear polarization of light. Note that if α is set to 90 degrees, we can perfectly discern whether a photon had a specific position x, as we can distinguish between orthogonal polarizations (0 and 90 degrees), meaning that we are applying a strong measurement.

However, we can weaken the measurement by decreasing alpha, resulting in less precise localization of the photon's position, x. The trade-off here is that the disturbance to the wave function of a single photon is also minimized. Following that, we use a Fourier transform lens and a slit to select only photons with momentum p = 0, resulting in a strong measurement of momentum, P. We compute the average value of our weak measurement of pi_x among this subset of photons. The pointer's average rotation (the linear polarization) corresponds to the real part of the weak value, whereas its complementary variable (the rotation in the circular polarization basis) corresponds to the imaginary part of the weak value [11]. For an initial spin polarization, spin 1/2 spin down, the weak value becomes:

$$\langle \pi_x \rangle_w = \frac{1}{\sin\alpha}(\langle s|\,\sigma_x\,|s\rangle + i\,\langle s|\,\sigma_y\,|s\rangle) \tag{54}$$

where σ_x, σ_y are Pauli matrices, and $|s\rangle$ is the final polarization state of the pointer. We can measure the expectation values of the Pauli matrices by sending the photons through a half-wave plate or a quarter-wave plate, respectively. Consequently, we read out ReΨ ImΨ from the imbalance of the detectors in Fig. 5. While this is a general idea, the details can be found in the paper [11].

VIII. THE INTERPRETATION OF TIME SYMMETRY

The study of time symmetry in the context of quantum mechanics, particularly within pre- and post-selected systems, is a compelling field of inquiry. Specifically, this discussion hinges on understanding the differences in manipulating forward and backward-evolving quantum states. To appreciate this, let's first clarify what forward and backward-evolving states are and why they matter in quantum mechanics.

Imagine yourself standing at the bank of a river. The river's current carries water forward - this represents a forward-evolving quantum state. Now, imagine viewing the river through a reversed video, where the water seems to be moving backward - this is akin to a backward-evolving quantum state. The unique aspect of quantum mechanics is that it allows for these both forward and backward-flowing 'currents.' So how does one create a specific quantum state at a certain point in time? For a forward-evolving quantum state, it is similar to releasing a leaf into the river and watching it drift along with the current. If the leaf doesn't land at the desired spot, we can adjust its position. However, for a backward-evolving state, it's like trying to place the leaf at a particular location in the reversed video of the river - it's significantly more challenging because we cannot precisely control where it will land as the water seems to flow in the opposite direction [3].

While quantum mechanics does not inherently favor one direction of time over the other, our ability to manipulate quantum states is affected by what's called the 'arrow of time' due to our memory. Our actions are generally based on past experiences, and we can't predict future events with absolute certainty. This limitation impacts our ability to manipulate backward-evolving states as effectively as forward-evolving ones. Nevertheless, it's crucial to note that these differences apply only to the creation of quantum states. In terms of measurement - the process of 'finding out' what the state is at a particular time - there is no distinction between forward and backward-evolving states [3].

Measurement in quantum mechanics can be likened to checking the time on a clock. Whether the 'hands' of our quantum clock move forward or backward, the time check always yields a result. The same applies to quantum states - regardless of whether they're evolving forward or backward, we learn the outcome only after the measurement is made. However, it's essential to be aware that not all variables in quantum systems can be measured easily. Consider nonlocal variables, those that represent parts of a system separated in space. Measuring these variables is akin to trying to ascertain the condition of two distant points in the river simultaneously [3]. It's certainly possible, but not without challenges and constraints. For instance, while certain variables, such as the Bell operator variable, are measurable, others, including those associated with product state eigenstates, may be less accessible for measurement. These limitations reflect the complexities inherent in the field of quantum mechanics but do not detract from its core principle - the time symmetry in the evolution and measurement of quantum states.

IX. CONTROVERSIES AND OPEN QUESTIONS

Ever since weak values were introduced by Aharonov, Albert, and Vaidman, it has been the center of debate regarding the different methods and results they produce when compared to conventional measurements.

One of these topics of debate is about weak values being a type of quantum measurement that can take on values outside the eigenvalue spectrum of the observable being measured. This has led to a debate about whether weak values are real physical quantities and whether they can be used to gain information about the past. Weak measurements are also contextual, meaning that the outcome of the measurement depends on the pre-selection and post-selection of the quantum system's state [12]. This has led to speculations that weak measurements may be able to violate the laws of causality and interpretations of them.

Another point that critics of weak value measurements argue is the anomalous values and strange effects seen in these measurements can be explained within the framework of standard quantum mechanics without invoking the concept of weak values. Some critics argue that the amplification seen in weak value experiments is only a consequence of the precise alignment of the pre-and post-selected states [12]. Others argue that weak values should be understood purely statistically as weighted averages over possible measurement outcomes. To demonstrate that weak values are merely a statistical phenomenon, Ferry and Combes tried to put forward a classical analog. Their paper is titled "How a coin toss could yield 100?" as a reference to the initial paper on spin-1/2 particles taking weak values of 100. Vaidman[12] criticized that the classical analogs of quantum "weak interaction" and "post-selection" phenomena in their model are not true analogs of the former quantum phenomena and rejected their treatment. Some others argue that the effects observed in weak value experiments can be explained as a form of quantum interference between the different paths the system can take through the pre-selection, interaction, and post-selection stages. A quantum Bayesian treatment of weak values by Qin et al. [13] confirms this view. Their work also involves the refutation of the aforementioned analysis of Ferri and Combes, and they defend the non-existence of any classical analog to weak values. Even though the view that weak value is truly a quantum phenomenon rather than a purely statistical one is more grounded, the weak value experiments still involve ensembles generally. There are further efforts to demonstrate anomalous weak values over a single quantum state to avoid the statistical nature of the ensembles completely. One such successful experiment is introduced in *Weak Values* section. A carefully prepared single quantum state consisting of the sum of polarizations of 7 photons yielded anomalous weak values.

A third matter raises questions about whether weak measurements could change the way quantum mechanics is understood. Weak value measurements bear strange and counter-intuitive properties, which challenge some of the central tenets of the Copenhagen interpretation of quantum mechanics, such as the eigenvalue, eigenstate link, and the collapse of the wave-function upon measurement. Weak values seem to fit more naturally within alternative interpretations, such as the many-worlds interpretation and the consistent histories approach[3].

Despite controversies, weak value measurements are an active area of research. Their peculiar properties, such as contextuality and exceeding the eigenvalue spectrum, puzzle physicists and prompt new investigations. Open questions include the ontological status of weak values and their range of applicability. In addition, there is an ongoing debate about the potential applications of weak values. It is still being disputed if weak value amplification provides a real advantage over traditional measurement techniques.

Some final remarks can be made about the future of weak values. The study of weak value measurements has the potential to shed new light on the foundations of quantum mechanics and its practical applications. The ongoing research in this area promises to deepen our understanding of quantum mechanics and to spur further developments in quantum technology.

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