HKUST

MATH1012 Calculus IA

Mock Final Examination (Fall 2024) Name: ____________________

1 Dec 2024 (Updated) Student ID: ____________________

Time limit: 3 hours Lecture Section: ____________________

Directions:

- This is a closed book examination. Calculator of any kind is NOT allowed in this examination.
- **DO NOT open the exam until instructed to do so.**
- Turn off all phones and pagers, and remove headphones. All electronic devices should be kept in a bag away from your body.
- Write your name, ID number, and Lecture Section in the space provided above.
- When instructed to open the exam, please check that you have **14** pages of questions including the cover page. This document is updated, with amendments highlighted in red.
- Answer all questions. Show an appropriate amount of work for each short or long problem. If you do not show enough work, you will get only partial credit.
- **Cheating is a serious violation of the HKUST Academic Code.** This is only a mock exam, with no benefit of cheating here: by cheating, you are not lying to anyone but yourself.
- For answer checking/ marking/ feedback of this mock paper set, please either email me via `theskillfulnoob2002@gmail.com` or Whatsapp/ Signal via `(+852) 9035 4789`.

Please read the following statement and sign your signature:

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

`Student's Signature:`

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Question No.	**Points**	**Out of**
Q. 1-18		54
Q. 19		10
Q. 20		8
Q. 21		14
Q. 22		14
Total Points		**100**

Part I: Multiple Choice Questions (54 Points)

Answer all of the following multiple choice questions.

- Mark your answers clearly in the Multiple Choice Boxes below.
- Mark only one answer for each MC question. Multiple answers will be treated as incorrect answer.

Question	1	2	3	4	5	6	7	8	9
Answer									
Question	10	11	12	13	14	15	16	17	18
Answer									

References

- All MCs except 5, 9, 13 are direct/ partial PP/ HW references.
- MCs 1-4, 6, 7, 11, 14, 15, 18 appeared in other mock variants.

1. The function $f(x) = \cos^{-1}(e^{2x})$ is one-to-one and hence has an inverse function.
   Find the domain of the inverse function of f.

   A. $[0, \pi)$ B. $(0, \pi]$ C. $[0, \frac{\pi}{2})$ D. $[0, \frac{\pi}{2}]$ E. $(0, \frac{\pi}{2}]$

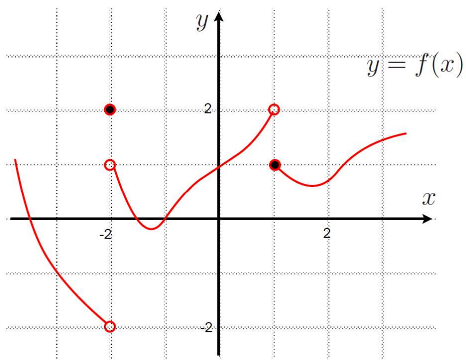
2. Find the limit $\lim_{h \to 3} \frac{\sqrt{1+h} - 2}{|h-3|}$ if it exists.

   A. Does not exist B. -4 C. 4 D. $-\frac{1}{4}$. E. $\frac{1}{4}$.

3. Find the limit $\lim_{x \to -3} \dfrac{x^2 - 9}{\sin(x^2 + 5x + 6)}$ if it exists.

   A. 6 B. 0 C. -1 D. -5. E. Does not exist

4. Find the one-sided limit $\lim_{x \to 1^+} f(3f(x) - 5)$, with the graph of f given:



   A. Does not exist B. 0 C. 1 D. 2. E. -2

5. In applying the Squeeze Theorem to evaluate

$$\lim_{x \to -\infty} e^{2x} \cos(e^{-x}),$$

   we consider that $f(x) \le e^{2x} \cos(e^{-x}) \le g(x)$. Which of the following combinations of $f(x)$ and $g(x)$ would result in a correct deduction?

   A. $f(x) = -1$, $g(x) = 1$

   B. $f(x) = -e^{-x}$, $g(x) = e^{-x}$

   C. $f(x) = -e^{2x}$, $g(x) = 1$

   D. $f(x) = -e^{2x}$, $g(x) = e^{2x} + e^{-x}$

   E. $f(x) = -e^{2x}$, $g(x) = e^{2x}$

6. How many vertical asymptotes does the following function have?

$$y = \frac{x^2 - 2x + 1}{(x^2 - x)(x^2 - 5x - 6)(x^2 + 4)}$$

A. 1 B. 2 C. 3 D. 4 E. 5

7. A function f is defined on the interval $(-e, e)$. It is known that

$$f(x) = \begin{cases} 4e^x + mx - 4 & \text{if } -e < x \leq 0 \\ \ln(\ln(x + e)) & \text{if } 0 < x < e \end{cases}$$

If f is differentiable, find the value of constant m.

A. $\frac{1}{e} - 2$ B. -2 C. -4 D. $\frac{1}{e} - 4$ E. $\frac{1}{e} + 4$

8. Let $f(x)$ be a function differentiable everywhere over $x \in \mathbb{R}$ such that $f(2) = 4$ and $f'(2) = 8$. Find the limit $\lim_{h \to 0} \frac{(2 + h)f(2 + h) - 2f(2)}{h}$.

A. 16 B. 20 C. 24 D. 28 E. 32

9. Let $p(x) = (x^2+5)^{2024} \cdot (6-x)^{1203}$. Find the value of $p'(0)$.

   A. $1203 \cdot 5^{2023} \cdot 6^{1203}$

   B. $-2024 \cdot 5^{2023} \cdot 6^{1203}$

   C. $2024 \cdot 5^{2024} \cdot 6^{1202}$

   D. $-1203 \cdot 5^{2024} \cdot 6^{1202}$

   E. $1203 \cdot 5^{2024} \cdot 6^{1202}$

10. The increasing function $f(x) = x^3 + 4x + 2$ has an inverse function f^{-1}.
    Find the derivative $(f^{-1})'(2)$.

    A. $-\dfrac{3}{4}$ B. $-\dfrac{1}{3}$ C. $\dfrac{1}{4}$ D. $\dfrac{1}{3}$ E. $\dfrac{4}{3}$

11. Define $h(x) = x^{2e^{3x}}$ for $x > 0$. Find $\dfrac{h'(x)}{h(x)}$.

    A. $\dfrac{2e^{3x}}{x}$ B. $e^{3x}\left(\dfrac{2}{x} + 6\ln x\right)$ C. $\dfrac{6e^{3x}}{x}$ D. $e^{3x}\left(\dfrac{6}{x} + 18\ln x\right)$ E. $18e^{3x}\ln x$

12. Suppose $f(x)$ is differentiable everywhere such that $\displaystyle\lim_{x\to 0} \frac{5-3f(x)}{x} = 4$.
    By considering L'Hospital's rule, find the linear approximation of f at $x = 0$.

    A. $\dfrac{5}{3} + \dfrac{4}{3}x$ B. $5 - 4x$ C. $\dfrac{5}{3} - \dfrac{4}{3}x$ D. $\dfrac{5}{3} - \dfrac{2}{3}x$ E. $-\dfrac{5}{3} + 5x$

13. Suppose $f(x)$ is a continuous function. Given the following tables of values $f'(x)$ as x varies:

x	0	1	2	3	4	5	6
$f'(x)$	4	5	2	-1	0	3	6

Approximate $f(6) - f(0)$ by the right-endpoint Riemann sum on three subintervals of equal length.

A. 16 B. 14 C. 12 D. 10 E. 8

14. A particle is traveling along the s-axis with velocity function

$$v(t) = -\frac{24}{t+3} + 4$$

for $t \geq 0$ (in seconds). It is at displacement $s(0) = 0$ initially. Find its displacement $s(t)$ when it is at rest at time $t = T$.

A. $-12\ln 2$ B. $12(\ln 2 - 1)$ C. $-24\ln 2$ D. $12(1 - 2\ln 6)$ E. $12(1 - 2\ln 2)$

15. An open-top box is to be made from a square cardboard of side 6 inches by removing a square from each of its corner and folding up the flaps on each side. To maximize the volume of the box $V(x)$ (in cubic inches) where x inches is the side of the squares to remove, what is the objective function $V(x)$, and the conditions on $V'(x)$ and $V''(x)$?

A. $V(x) = x(6-2x)^2$; $V'(x) = 0$ and $V''(x) < 0$
B. $V(x) = x(6-2x)^2$; $V'(x) = 0$ and $V''(x) > 0$
C. $V(x) = x(6-x)^2$; $V'(x) = 0$ and $V''(x) = 0$
D. $V(x) = x(6-x)^2$; $V'(x) = 0$ and $V''(x) < 0$
E. $V(x) = x(6-x)^2$; $V'(x) = 0$ and $V''(x) > 0$

16. Evaluate the limit $\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{5n-k}$ by consider a suitable definite integral.

A. $\ln\left(\frac{3}{5}\right)$ B. $\ln\left(\frac{5}{3}\right)$ C. $\ln(2)$ D. 0 E. $\frac{1}{2}\ln\left(\frac{5}{3}\right)$

17. Find the number of points at which f attains local maximum if

$$f(x)=\int_0^x e^{y^2}(y^3-7y-6)dy$$

for all x.

A. 0 B. 1 C. 2 D. 3 E. 4

18. Using the substitution $u=1+e^{3x}$, evaluate $\int_0^1 \frac{1+2e^{3x}}{1+e^{3x}}dx$.

(Hint: Separate the integral into two!)

A. $1+\frac{1}{3}\ln\left(\frac{1+e^3}{2}\right)$ B. $1+\frac{1}{3}\ln\left(1+e^3\right)$ C. $\frac{1}{3}\ln\left(\frac{1+e^3}{2}\right)$ D. $1+\frac{1}{3}\ln\left(\frac{2}{1+e^3}\right)$ E. $1+\ln(1+e^3)$

Part II: Long Questions (46 Points)

Answer each of the following 4 long questions. Unless otherwise specified, numerical answers should be either exact or correct to 2 decimal places.

19. *(10 pts)* This question concerns integrals.

(a) Find $\int \sec^2(2x) - 5x^{-1} + 7dx$. [3pts]

(b) Evaluate $\int_{-1}^{1} |e^x - 1|dx$. [3pts]

(c) Express the area of the finite region bounded by the graphs of $f(x) = x^3 + 5x^2$ and $g(x) = 6x$ as one/ multiple definite integral(s).
You need not compute the integral(s), just set it/ them up. [4pts]

20. *(8 pts)* A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 48 cm and 20 cm respectively. Water is now poured into the container.

   (a) Let A cm^2 be the wet curved surface area of the container and h cm be the depth of water in the container. Prove that $A = \dfrac{65}{144}\pi h^2$. [4pts]

   (b) The depth of water in the container increases at a constant rate of $\frac{1}{\pi}$ cm/s . Find the rate of change of the wet curved surface area of the container when the height of the water surface in the container is 36 cm. [4pts]

21. *(14 pts)* Let C be the curve $xy = \cos(xy)$, defined for $x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$.

(a) Show that

$$\left(y + x\frac{dy}{dx}\right)(1 + \sin(xy)) = 0.$$

Also briefly explain why $1 + \sin(xy)$ can never be zero for the curve defined. [4pts]

(b) Consider the equation

$$u = \cos u, \quad \text{where } 0 \leq u \leq \frac{\pi}{2} \qquad (*)$$

(i) Show that (*) has exactly one root in $\left[0, \frac{\pi}{2}\right]$.

(ii) Hence, show that there exists two tangents to C parallel to the straight line $L : x + y = 0$.

[7pts]

(c) Suppose two tangents to

$$C : xy = \cos(xy) \quad \text{where} \quad x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$

at $x = a < 0$ and $x = b > 0$ are parallel to the straight line $x + y = 0$. It is given that for any point (x, y) lying on C, the function $f : y = f(x)$ connecting the two variables is one-to-one. A student claims that:

> Since there are two tangents to C parallel to the straight line $L : x + y = 0$, I can conclude by using mean value theorem that there exists a value $c \in (a, b)$ such that $f"(c) = 0$.

Explain why the student's application of MVT is incorrect.

> Hint: Recall that MVT states for the function g continuous and differentiable over $[a, b]$, there exists some $c \in (a, b)$ such that
> $$g'(c) = \frac{g(b) - g(a)}{b - a}$$

[3pts]

22. *(14 pts)* Let Γ be the curve $y = \dfrac{3x^2 - 2x - 5}{(x-1)^2}$.

(a) State all asymptotes of Γ. [2pts]

(b) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [2pts]

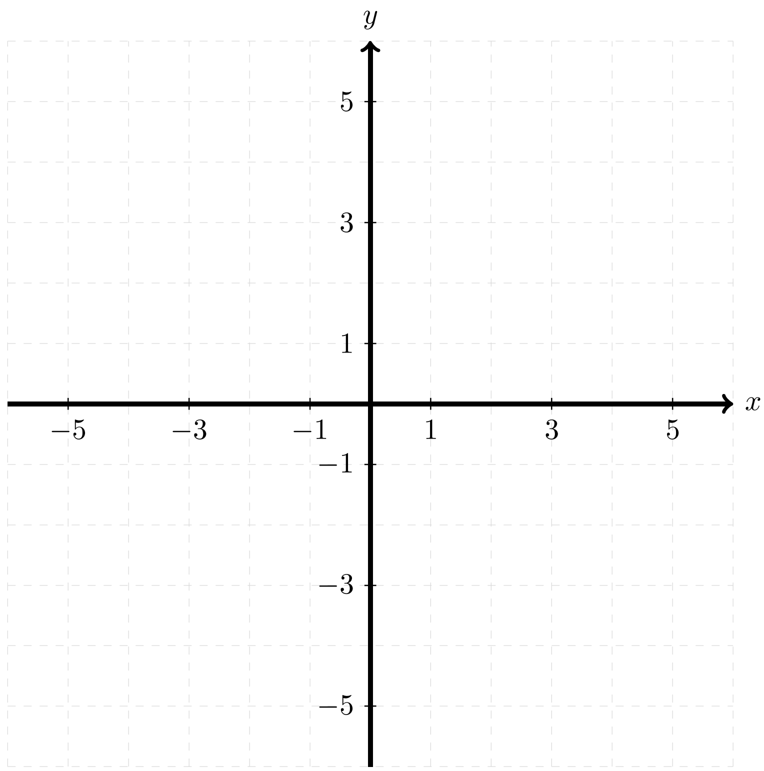
(c) Find the minimum/ maximum point(s) of Γ. [3pts]

(d) Find the inflection point(s) of Γ. [3pts]

(e) It is given the x-intercepts of

$$\Gamma : y = \frac{3x^2 - 2x - 5}{(x-1)^2}$$

are -1 and $\frac{5}{3}$. Sketch the graph of Γ for $-6 \le x \le 6$, including its intercepts, asymptotes, extreme point(s) and inflection point(s). [4pts]



Math1012 Exam Formula Sheet

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$
$$\sin(A-B) = \sin A\cos B - \sin B\cos A$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\sin A\cos B = \frac{1}{2}\Big[\sin(A-B) + \sin(A+B)\Big]$$
$$\cos A\cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big]$$
$$\sin A\sin B = \frac{1}{2}\Big[\cos(A-B) - \cos(A+B)\Big]$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\frac{d(\text{constant})}{dx} = 0 \qquad \frac{dx^p}{dx} = px^{p-1}$$
$$\frac{d\ln x}{dx} = \frac{1}{x} \qquad \frac{de^x}{dx} = e^x$$
$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\cos x}{dx} = -\sin x$$
$$\frac{d\tan x}{dx} = \sec^2 x, \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\big[f(x)g(x)\big]' = f(x)g'(x) + g(x)f(x)$$
$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$
$$\big[f(g(x))\big]' = f'(g(x))g'(x)$$