MATH 1012 Mock: Answers and Reminders

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This document is prepared for people who obtained a MATH1012 Mock.

Summary of Testing Areas

Testing Areas

1. **Function and Limits**: (18 Points)
   Q1-6
2. **Differentiation**: (32 Points)
   Q7-12, 21
3. **Appli. of Differentiation**: (28 Points)
   Q14-15, 20, 22
4. **Integration**: (22 Points)
   Q13, 16-18, 19

Section 1

MC Questions (54 Points)

MC Answers

Item	1	2	3	4	5	6
Ans	C	A	A	E	E	C
Item	7	8	9	10	11	12
Ans	D	B	D	C	B	C
Item	13	14	15	16	17	18
Ans	A	E	A	B	B	A

Question 1

- Note that $e^{2x} > 0$ for all x. On the other hand, $\cos x \leq 1$ for all x.
- So, the range of $f(x)$ is $(\cos^{-1}(0), \cos^{-1}(1)]$.
- Domain of the inverse of f: $(\cos^{-1}(0), \cos^{-1}(1)] = [0, \frac{\pi}{2})$

Remarks: Closed/ Open Intervals
Pay attention to closed/ open intervals. It is a usual area of trap in the final exam.

Reference: MATH 1013 2021-22 Fall Midterm #2

Question 2

Apply rationalization to transform the expression to
$\lim_{h\to 3} \frac{\sqrt{1+h}-2}{|h+3|} = \lim_{h\to 3} \frac{\sqrt{1+h}-2}{|h+3|} \cdot \frac{\sqrt{1+h}+2}{\sqrt{1+h}+2} = \lim_{h\to 3} \frac{h-3}{|h-3|(\sqrt{1+h}+2)}$

- Left limit
$$= \lim_{h\to 3^-} \frac{h-3}{-(h-3)(\sqrt{1+h}+2)} = \lim_{h\to 3^-} \frac{1}{-(\sqrt{1+h}+2)} = -\frac{1}{4}$$
- Right limit
$$= \lim_{h\to 3^+} \frac{h-3}{(h-3)(\sqrt{1+h}+2)} = \lim_{h\to 3^+} \frac{1}{\sqrt{1+h}+2} = \frac{1}{4}$$
- As $\lim_{h\to 3^-} \frac{\sqrt{1+h}-2}{|h+3|} \neq \lim_{h\to 3^+} \frac{\sqrt{1+h}-2}{|h+3|}$, the limit does not exist.

Trap: MCQ 2

Limit Trap for Absolute Values

- Be careful! $|h-3| = -(h-3)$ for $h < 3$.
- Recall that limits only exist when left hand limit $(-)$ equals to the right hand limit $(+)$.
- Absolute values are often culprits of wrong answers in limit questions.

Reference: MATH 1012 Fall 2024-25 Fall Midterm $\#7$

Question 3

Observe that $x^2 - 9$ and $x^2 + 5x + 6$ both have the common factor $x + 3$, and that the term inside the sine function goes to 0.

- $\lim_{x \to -3} \dfrac{x^2 - 9}{x^2 + 5x + 6} \cdot \lim_{x \to -3} \dfrac{x^2 + 5x + 6}{\sin(x^2 + 5x + 6)} =$
  $\lim_{x \to -3} \dfrac{(x-3)(x+3)}{(x+2)(x+3)} \cdot \lim_{y \to 0} \dfrac{y}{\sin y}$
- Cancelling the common factor and using sine identity, the limit is

$$\lim_{x \to -3} \frac{(x-3)}{(x+2)} \cdot 1 = 6$$

Alternative Solution: MCQ 3

Alternative (L' Hospital's Rule):
Both numerator and denominator evaluates to 0. Apply L'Hospital's rule:

$$\lim_{x\to -3} \frac{x^2-9}{\sin(x^2+5x+6)}$$

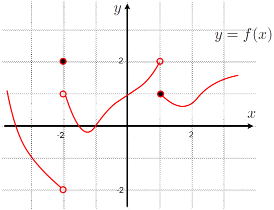
$$= \lim_{x\to -3} \frac{2x}{\cos(x^2+5x+6)\cdot(2x+5)} = \frac{2(-3)}{\cos(0)\cdot -1} = 6$$

Reference: MATH 1013 2019-20 Fall Final #2

Question 4

"Limit graph" involves these steps:

- Putting the limit inside:
  $\lim_{x \to 1^+} f(3f(x) - 5) = f(\lim_{x \to 1^+} 3f(x) - 5)$
- Substitution:
  $f(\lim_{x \to 1^+} 3f(x) - 5) = f(\lim_{y \to 1^-} 3y - 5)$
- The limit evaluates to
  $f(\lim_{u \to -2^-} u) = \lim_{u \to -2^-} f(u) = -2$



Reference: MATH 1012 Fall 2023 Midterm Q7

Question 5

We want $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} g(x)$.

- Obviously, Choices A, C and D do not fit in this criterion.
- For B, $\lim\limits_{x\to-\infty} -e^{-x} = \lim\limits_{y\to\infty} -e^{y} = -\infty$, and $\lim\limits_{x\to-\infty} e^{-x} = +\infty$
  So Choice B is also not correct.
- Choice E is the answer, as $\lim\limits_{x\to-\infty} -e^{2x} = 0 = \lim\limits_{x\to-\infty} e^{2x}$.

Reference: MATH 1013 2020-21 Fall Final #12

MCQ 6C/ Function and limits/ Asymptotes

Question 6

- Vertical asymptotes may occur when denominator equals zero. When $(x^2-x)(x^2-5x-6)(x^2+4) = (x)(x-1)(x+1)(x-6)(x^2+4) = 0$, $x = -1$ or 0 or 1 or 6.
- However, the numerator is $(x-1)^2$. The factor $x-1$ cancels out, meaning there is no vertical asymptote at $x = 1$, as the limit at that point is finite.
- Thus, there are only three asymptotes ($x = -1$, $x = 0$, $x = 6$).

Trap: MCQ 6

Factoring Trap of Vertical Asymptotes

Candidates may forget the definition of vertical asymptotes:

- The limit should tends to $\pm\infty$ when $x \to c$!
- However, when the factor gets cancelled out, the limit when $x \to c$ would be finite.
- Then $x = c$ is NOT a vertical asymptote!

Reference: MATH 1013 2020-21 Fall Final #3/ MATH 1003 Sample Final #2

Question 7

- L.H. derivative: $4e^x + m$ (the function is clearly continuous).
- R.H. derivative: $\frac{d(\ln(\ln(x+e)))}{dx} = \frac{d(\ln(\ln(x+e)))}{d(\ln(x+e))} \cdot \frac{d(\ln(x+e))}{d(x+e)} \cdot \frac{d(x+e)}{dx}$
- At $x = 0$, differentiability requires
  $4e^0 + m = \frac{1}{\ln(0+e)} \cdot \frac{1}{0+e} \implies m = \frac{1}{e} - 4$

Reference: MATH 1013 2019-20 Fall Final #6

Question 8

- Identify the limit as $\dfrac{d}{dx}(xf(x))$ with first principles.
- Applying product rule gives
  $\left[xf'(x) + f(x)\right]\Big|_{x=2} = f(2) + 2f'(2) = 4 + 2(8) = 20.$

RECALL FIRST PRINCIPLES!!!!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Question 9

- By product rule, $p'(x) = \frac{dp(x)}{dx}$
  $= \frac{d}{dx}(x^2+5)^{2024} \cdot (6-x)^{1203} + (x^2+5)^{2024} \cdot \frac{d}{dx}(6-x)^{1203}$
  $= 2x(2024)(x^2+5)^{2023}(6-x)^{1203} + (-1)(1203)(x^2+5)^{2024}(6-x)^{1202}$
- Substituting $x = 0$, the first term becomes 0 as $2x = 2(0) = 0$.
- Thus, the answer is $-1203 \cdot 5^{2024} \cdot 6^{1202}$.

Question 10
Find the derivative $(f^{-1})'(-2)$.

With $f(0) = -2$, $(f^{-1})'(-2) = \frac{1}{f'(f^{-1}(-2))} = \frac{1}{f'(0)} = \frac{1}{4}$.

Recall Inverse Function Theorem:
If f is differentiable and has an inverse f^{-1}, then:

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}.$$

Reference: MATH 1013 Sample Final #8

Question 11

$h(x)$ is an obvious candidate for logarithmic differentiation.

- Employing the method, $\ln h(x) = 2e^{3x} \ln x$.
- Differentiating,
$$\frac{1}{h(x)}h'(x) = \frac{h'(x)}{h(x)} = \frac{2e^{3x}}{x} + 3(2e^{3x})(\ln x) = e^{3x}\left(\frac{2}{x} + 6\ln x\right)$$

Reference: MATH1013 Fall 2018 Midterm #12

Question 12

The denominator $\to 0$. For the limit to exist, it must be of $0/0$ form.

- So, $\lim_{x\to 0}(5-3f(x)) = 0 \implies \lim_{x\to 0} f(x) = \frac{5}{3}$
  As $f(x)$ is continuous, $f(0) = \frac{5}{3}$
- Applying L'hospital's Rule, $\lim_{x\to 0}\frac{5-3f(x)}{x} = \lim_{x\to 0}\frac{-3f'(x)}{1} = 4$
  As $f(x)$ is differentiable, $f'(0) = -\frac{4}{3}$
- So, the linear approximation at $x = 0$ is $y = \frac{5}{3} - \frac{4}{3}x$.

Reminder: MCQ 12

Reminder: Determining Form of Limit

- Sometimes, you need to deduce the form of limit from its existence.
- This is a common technique in HKALE.
- This question is an expansion of a question in your last year's PP.

Reference: HKUST MATH 1012 2023-24 Fall Final #6

MCQ 13A/ Integration/ Approximation

Question 13

- $f(6) - f(0) \approx \Delta x \left(f'(2) + f'(4) + f'(6)\right)$ where $\Delta x = \dfrac{6-0}{3} = 2$
- The value is $2(2 + 0 + 6) = 16$.

Mid-point rule: $2(5 - 1 + 3) = 14$; left-endpoint rule: $2(4 + 2 + 0) = 12$.

Reference: MATH 1003 2023-24 Fall Final #22(a)

Reminder: MCQ 13

Riemann Sum Approximation Rules

To approximate the definite integral $\int_a^b f(x)\,dx$, we use the following rules:

- $\Delta x = \frac{b-a}{n}$ is the width of each subinterval.
- The choice of sample points determines the approximation type.

Rule	Formula	Key Feature
Left	$\sum_{i=1}^{n} f(x_{i-1})\Delta x$	**Left endpoints** of subintervals.
Right	$\sum_{i=1}^{n} f(x_i)\Delta x$	**Right endpoints** of subintervals.
Midpoint	$\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$	**Midpoints** of subintervals

Question 14

- $v(T) = 0 \implies -\frac{24}{T+3} + 4 = 0 \implies T = 3$
- Required displacement:
  $$s(3) = s(0) + \int_0^3 \left(-\frac{24}{t+3} + 4\right)dt = [-24 \ln|t+3| + 4t]_0^3$$
- It evaluates to $-24\ln(3+3) + 4(3) + 24\ln 3 = 12(1 - 2\ln 2)$

Reference: MATH 1012 Fall 2023-24 Final #15

Question 15

The box has width and length both being $(6-2x)$, and height x.

- Objective function is $V(x) = x(6-2x)(6-2x) = 4x^3 - 24x^2 + 36x$.
- Recall the second derivative test.
  To maximize the volume, we require $V'(x) = 0$ and $V''(x) < 0$.

Reference: Libretexts// MATH 1012 2020-21 Fall Final #15

MCQ 16B/ Integration/ Riemann Sum

Question 16

Note that the given limit is a Riemann sum:

- The term $\frac{1}{5n-k}$ suggests using the function $f(x) = \frac{1}{x}$.
- The summation range, $k = 1$ to $k = 2n$, represents $2n$ subintervals.
- The denominator $5n - k$ suggests reindexing k. Set $x_k = 5n - k$:

  As k ranges from 1 to $2n$, x_k ranges from $5n - 2n = 3n$ to $5n - 1$.
- Then $\Delta x = \dfrac{\text{Range of } x}{\#\text{Intervals}} = \dfrac{5n - 3n}{2n} = 1$.

So, we write the Riemann integral as $\displaystyle\int_{3n}^{5n} \frac{1}{x} dx = [\ln|x|]_{3n}^{5n} = \ln\left(\frac{5}{3}\right)$.

Reminder: MCQ 16

Recall the Riemann Integral properties:

Reminder: Riemann Integral as Infinite Sum Limit

- Regardless of whichever rule you use (from MCQ 13), the limits you end up are the same (i.e. converge) for Riemann integrals.
- In exams, you need to write Δx, and identify the lower and upper limits (a and b).
- A common place of mis-concept is confusion over the integral width (or just a constant multiple).

Reference: MATH 1012 Fall 2023-24 Final #18

MCQ 17B/ Integration/ FTC

Question 17

- First fundamental theorem of calculus: $f'(x) = e^{x^2}(x^3 - 7x - 6)$.
- Setting $f'(x) = 0 \implies x^3 - 7x - 6 = (x+1)(x+2)(x-3) = 0$ $x = -1$ or $x = -2$ or $x = 3$.
- Use signs test: $f'(x) < 0$ for $x < -2$, $f'(x) > 0$ for $x \in (-2, -1)$, $f'(x) < 0$ for $x \in (-1, 3)$ and $f'(x) > 0$ for all $x > 3$.

Only ONE local maximum at $x = -1$.

Question 18

Note that $\int_0^1 \frac{1+2e^{3x}}{1+e^{3x}}dx = \int_0^1 \frac{1+e^{3x}}{1+e^{3x}}dx + \int_0^1 \frac{e^{3x}}{1+e^{3x}}dx.$

- The first integral is simply $\int_0^1 dx = [x]_0^1 = 1$
- For the second integral, apply substitution $u = 1 + e^{3x}$:
  $du = 3e^{3x}dx$, when $x = 1$, $u = 1 + e^3$; when $x = 0$, $u = 2$.
- Then, the second integral
  $$\int_0^1 \frac{e^{3x}}{1+e^{3x}}dx = \int_2^{1+e^3} \frac{1}{3}\frac{du}{u} = \frac{1}{3}[\ln u]_2^{1+e^3} = \frac{1}{3}\ln(\frac{1+e^3}{2})$$
- Summing up, the integral evaluates to $1 + \frac{1}{3}\ln(\frac{1+e^3}{2})$.

Reminder: Q18

Reminder: When to substitute? When to break?

- If you see similar terms in numerator/ denominator, factor/ divide them out. You will likely benefit from it!
- Recall differentiating a constant returns 0. For substitution:
  1. It is common to write $d(e^{ax} + k) = ae^{ax}dx$.
  2. Another common substitution is using $d(x^a + b) = ax^{a-1}dx$.
- Obtain the constant a for substitution by dividing it elsewhere.

It is also a common (careless) error that $e^0 = 1$, NOT zero.

Reference: MATH 1012 2015-16 Fall Final #9

LQ 19 (a)(b)/ Integration/ Calculations

Question 19(a) - Basic Values [3 pts]
Question 19(b) - Absolute Value [3 pts]

For (a), simple and straightforward (directly read from formula sheet).

- Answer: $\frac{1}{2}\tan 2x - 5\ln|x| + 7x \quad +C$. [1 pt for each expression]

For (b),

- Note that $e^x - 1 < 0$ when $x < 0$.
- $\int_{-1}^{1} |e^x - 1| dx = \int_0^1 (e^x - 1) dx + \int_{-1}^{0} (1 - e^x) dx$ [1 pt]
- Answer: $[e^x - x]_0^1 + [x - e^x]_{-1}^{0} = e + \frac{1}{e} - 2$ [2 pts]

Trap: Q19(a)(b)

Adding Constant for Indefinite Integrals

- Don't forget to $+C$ for indefinite integrals.
- Otherwise you would get a $C+$ in Calculus...

Integration Bound Trap for Absolute values

- You should switch the sign of the integral within the absolute value!
- If you computed $\int_{-1}^{1}(e^x - 1)dx = e - \frac{1}{e} - 2$, you likely have stepped into this trap!

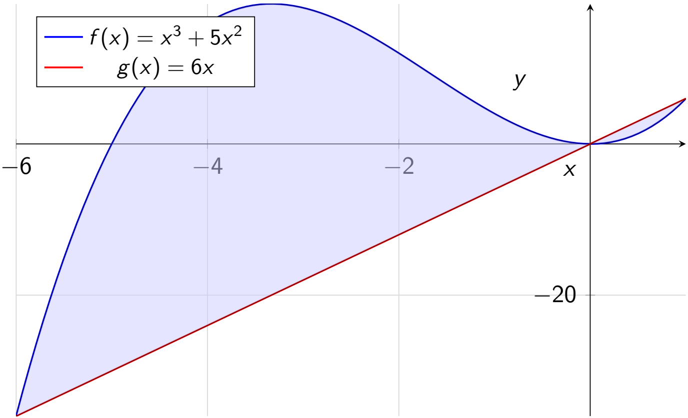
Reference: MATH 1012 Fall 2020 Final #16

Question 19(c) - Area Bounded [4 pts]

We first find when $f(x) > g(x)$ and when $g(x) > f(x)$:

- [2pts: Solving $f(x) = g(x)$ correctly]
  $f(x) - g(x) = x(x^2 + 5x - 6) = x(x+6)(x-1)$.
  When $f(x) - g(x) = 0$, $x = -6$ OR $x = 0$ OR $x = 1$.
- So, $f(x) > g(x)$ when $x > 1$ or $-6 < x < 0$;
  $f(x) < g(x)$ when $x < -6$ or $0 < x < 1$.
- [2pts: Correct Final Integral]
  Required area is $\displaystyle\int_{-6}^{0} (f(x) - g(x))\, dx + \int_{0}^{1} (g(x) - f(x))\, dx$

LQ 19 (c): Figure



References:
MATH 1003 2016-17 Fall Final #12(d)/ 2018-19 Fall Final #25(b)

Question 20(a) - Proving Similar Areas [4 pts]

- [1pt: Surface Area Formula]
  The original cone has curved surface area $A_0 = \pi\sqrt{(r_0)^2 + (h_0)^2} \cdot r = 1040\pi$.
- [2pts: Applying Similar solids correctly (power 2)]
  Applying similar area, $\dfrac{A}{A_0} = \left(\dfrac{h}{h_0}\right)^2$
- [1pt: Followthrough] $A = 1040\pi \cdot \dfrac{h^2}{48^2} = \dfrac{65}{144}h^2$

Question 20(b) - Rate of Change [4 pts]

- [2pts : Differentiate the formula w.r.t. t]
  Differentiating, $\dfrac{dA}{dt} = \dfrac{65}{72}\pi h \dfrac{dh}{dt}$
- [2pts : Correct Substitution + Answer]
  Required rate of change: $\dfrac{dA}{dt} = \dfrac{65}{72}\pi(36)\dfrac{1}{\pi} = \dfrac{65}{2}$ (cm^2 s^{-1})

References:
HKDSE 2017 M2 #6; MATH 1013 2019-2020 Fall Final #12

LQ 21 (a)/ Appli. of Differentiation/ Implicit Diff.

Question 21(a) - Implicit Differentiation [4 pts]

- [2 pts: Implicitly differentiate each sides w.r.t. x]
  $y + x\dfrac{dy}{dx} = -y\sin(xy) - x\sin(xy)\dfrac{dy}{dx}$
  So, $y(1+\sin(xy)) + x\dfrac{dy}{dx}(1+\sin(xy)) = 0$. The result follows.
- [2 pts: Considering Counterargument]
  For $1+\sin(xy) = 0$, $\sin(xy) = -1$, $xy \neq 0$
  But, $\cos(xy) = \sqrt{1-(-1)^2} = 0 = xy$, contradicting with the requirement $xy \neq 0$.
- So, $y + x\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = -\dfrac{y}{x}$

Question 21(b)(i) - Intermediate Value Theorem [3 pts]

- [1 pt: mention $\cos u$ is decreasing]
  Note that $\dfrac{d}{du}(\cos u - u) = -\sin u - 1 < 0$ for all $0 < u < \dfrac{\pi}{2}$.
  So, $\cos u$ is decreasing throughout $0 < u < \dfrac{\pi}{2}$.
- [2 pts: Applying IVT]
  Noting that $0 - \cos(0) = -1 < 0$ and $\frac{\pi}{2} - \cos(\frac{\pi}{2}) = \frac{\pi}{2} > 0$,
  The equation (*) has exactly one root in $\left[0, \dfrac{\pi}{2}\right]$.

Question 21(b)(ii) - Tangent Slope [4 pts]

- [1 pt: Matching Slope]

  The line L has slope -1. So, $\dfrac{dy}{dx} = -1 = -\dfrac{y}{x} \implies x = y$.

- [3 pts: Applying IVT]

  Substituting $x = y$ into the curve C, we have $x^2 = \cos(x^2)$.

  1. By (i), we know for $0 < x^2 < \frac{\pi}{2}$, there is exactly one value of x satisfying $x^2 = \cos(x^2)$.
     Denoting this value by $x = u$. Then, $u^2 = \cos(u^2)$.
  2. Also note that $x = -u$ is the only value $\in (-\dfrac{\pi}{2}, 0)$ satisfying $x^2 = \cos(x^2)$.

  Thus, there exists two tangents to C parallel to the straight line $L : x + y = 0$.

Question 21(c) - Mean Value Theorem [3 pts]

- [2 pts: point out discontinuity]
  Note that $f(x)$ is NOT defined at $x = 0$. Thus, $f(x)$ is NOT continuous over $[a, b]$.
  Alternative answer:
  Note that $f'(x) = \frac{dy}{dx} = -\frac{y}{x}$ is NOT defined at $x = 0$. Thus, $f(x)$ is NOT differentiable over $[a, b]$.
- [1pt: followthrough]
  As the prerequisites of MVT are not satisfied, the student's application of MVT is incorrect.

LQ 22 (a)(b)/ Appli of Differentiation/ Asymptotes

Question 22(a) - Asymptotes [2 pts]
Question 22(b) - Derivative [2 pts]

- [2 pts: Asymptotes (Answers only)]
  Note that $\dfrac{3x^2-2x-5}{(x-1)^2} \equiv 3+\dfrac{4}{x-1}-\dfrac{4}{(x-1)^2}$.
  Horizontal asymptote: $y=3$; Vertical asymptote: $x=1$
- [2 pts: Derivatives (Answers only)]
  $\dfrac{dy}{dx}=-\dfrac{4}{(x-1)^2}+\dfrac{8}{(x-1)^3}$; $\qquad \dfrac{d^2y}{dx^2}=\dfrac{8}{(x-1)^3}-\dfrac{24}{(x-1)^4}$

You may find this unexpectedly shorter than product/ chain rule :)
This question highlights the power of using partial fractions/ long div!

Question 22(c) - Maximum/ Minimum Points [3 pts]

- [1 pt: Set first derivative zero]
  $$\frac{dy}{dx} = -\frac{4}{(x-1)^2} + \frac{8}{(x-1)^3} = \frac{-4x+12}{(x-1)^3} = 0 \implies x = 3$$
- [1 pt: Testing (**Second Derivative**/ Signs)]
  $$\left.\frac{d^2y}{dx^2}\right|_{x=3} = \frac{8}{(3-1)^3} - \frac{24}{(3-1)^4} = -\frac{1}{2} < 0$$
- [1 pt: Answer]

  Thus, Γ has a maximum point at $(3, \dfrac{3(3^2)-2(3)-5}{(3-1)^2}) = (3, 4)$.

Question 22(d) - Inflection Points [3 pts]

- [1 pt: Set second derivative zero]
  $$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3} - \frac{24}{(x-1)^4} = \frac{8x-32}{(x-1)^4} = 0 \implies x = 4$$
- [1 pt: Testing (**Signs**)]

  For $x > 4$, $\dfrac{d^2y}{dx^2} > 0$; For $x < 4$, $\dfrac{d^2y}{dx^2} < 0$.

  Therefore, $\frac{d^2y}{dx^2}$ changes sign through $x = 4$.
- [1 pt: Answer]

  Thus, Γ has an inflection point at $(4, \dfrac{3(4^2) - 2(4) - 5}{(4-1)^2}) = (4, \dfrac{35}{9})$.

Question 22(e) - Curve Sketching [4 pts]

Each characteristic is worth 1 pt:

1. Two asymptotes
2. x-, y- intercepts (points C, D, E)
3. Maximum and Inflection points (points B, A)
4. Overall shape