

MATH 1003 Mock: Answers and Reminders

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Summary of Testing Areas

Testing Areas

- ① **Function and Limits: (20 Points)**
Q 1-4, 19
- ② **Differentiation: (10 Points)**
Q 5-7, 10
- ③ **Appli. of Differentiation: (50.5 Points)**
Q 8-9, 11-15, 20, 21, 23
- ④ **Integration: (19.5 Points)**
Q 16-18, 22

Section 1

MC Questions (45 Points)

MC Answers

Item	1	2	3	4	5	6
Ans	D	B	B	C	C	D
Item	7	8	9	10	11	12
Ans	A	C	A	A	E	A
Item	13	14	15	16	17	18
Ans	C	B	E	B	E	D

- Questions labelled with * appeared in other advanced streams;
- Questions labelled with @ are watered down versions of those that appeared in other streams.

MCQ 1D/ Function and Limits/ Domain

Question 1

Suppose $f(x) = \frac{x+3}{3x+4}$ and $g(x) = \frac{2x}{4x-1}$.

Give the domain of $f(g(x))$.

- Note that the domain of f excludes $-\frac{4}{3}$ and the domain of g excludes $\frac{1}{4}$: they make the respective denominators zero.
- For the composite function $f(g(x))$, we require

$$4x - 1 \neq 0 \quad \text{and} \quad g(x) \neq -\frac{4}{3}$$

$$\text{i.e. } x \neq \frac{1}{4} \text{ AND } g(x) \neq -\frac{4}{3} \implies x \neq \frac{2}{11}.$$

Reference: MATH 1012 Fall 2015-16 Midterm #9

Question 2

Note that $x - 4$ is a factor on both numerator and denominator of the limit.

- $\lim_{x \rightarrow 4} \frac{x^2 - 3x - 4}{x^2 - 5x + 4} = \lim_{x \rightarrow 4} \frac{(x - 4)(x + 1)}{(x - 4)(x - 1)} = \lim_{x \rightarrow 4} \frac{x + 1}{x - 1}$
- Thus, the required limit is $\frac{4 + 1}{4 - 1} = \frac{5}{3}$.

Reference: MATH 1013 Sample Midterm #2

Question 3

I. True, because $\lim_{x \rightarrow c^+} f(x)g(x) = \lim_{x \rightarrow c^+} f(x) \lim_{x \rightarrow c^+} g(x)$
 $= \lim_{x \rightarrow c^-} f(x) \lim_{x \rightarrow c^-} g(x) = \lim_{x \rightarrow c^-} f(x)g(x)$

II. False. One counterexample:

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}, \quad g(x) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

III. False. Same counterexample as II.

IV. False. One counterexample:

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}, \quad g(x) = 1 \text{ for all real numbers.}$$

Reference: MATH 1003 Fall 2018-19 Midterm #8

Question 4

- Vertical asymptotes may occur when denominator equals zero.
When
 $(x^2 - x)(x^2 - 5x - 6)(x^2 + 4) = (x)(x - 1)(x + 1)(x - 6)(x^2 + 4) = 0$,
 $x = -1$ or 0 or 1 or 6 .
- However, the numerator is $(x - 1)^2$. The factor $x - 1$ cancels out, meaning there is no vertical asymptote at $x = 1$, as the limit at that point is finite.
- Thus, there are only three asymptotes ($x = -1$, $x = 0$, $x = 6$).

Trap: MCQ 4

Factoring Trap of Vertical Asymptotes

Candidates may forget the definition of vertical asymptotes:

- The limit should tends to $\pm\infty$ when $x \rightarrow c$!
- However, when the factor gets cancelled out, the limit when $x \rightarrow c$ would be finite.
- Then $x = c$ is NOT a vertical asymptote!

Reference: MATH 1013 2020-21 Fall Final #3/ MATH 1003 Sample Final #2

Question 5

Rewrite: $u(x) = 3x + \frac{4}{x} + \frac{5}{x^3} \implies u'(x) = 3 - \frac{4}{x^2} - \frac{15}{x^4}$

Reminder: Don't be a fool to apply product rule mindlessly!

- In this case, the denominator is a simple power of x , and can be divided to simplify the expression.
- This question reminds you to be "smart" and think before you do the question :)

Reference: MATH 1003 Fall 2012-13 Final #14

Question 6

- By product rule, $p'(x) = \frac{dp(x)}{dx}$
$$= \frac{d}{dx}(x^2 + 5)^{2024} \cdot (6 - x)^{1203} + (x^2 + 5)^{2024} \cdot \frac{d}{dx}(6 - x)^{1203}$$
$$= 2x(2024)(x^2 + 5)^{2023}(6 - x)^{1203} + (-1)(1203)(x^2 + 5)^{2024}(6 - x)^{1202}$$
- Substituting $x = 0$, the first term becomes 0 as $2x = 2(0) = 0$.
- Thus, the answer is $-1203 \cdot 5^{2024} \cdot 6^{1202}$.

Question 7

$h(x)$ is an obvious candidate for logarithmic differentiation.

- Employing the method, $\ln h(x) = 2x \ln x$.

- Differentiating, $\frac{1}{h(x)} h'(x) = \frac{h'(x)}{h(x)} = 2 \ln x + \frac{2x}{x} = 2 \ln x + 2$

Reference: MATH1013 Fall 2018 Midterm #12

Question 8

Differentiating both sides **with respect to x (why?)**:

- $\frac{x^3}{y^3}$ is a function containing x and y (both change w.r.t. x).

$$\text{So is } x \log_3 y = \frac{x \ln y}{\ln 3}.$$

- Product rule: $\frac{3x^2}{y^3} \frac{dx}{dx} - \frac{3x^3}{y^4} \frac{dy}{dx} + \frac{\ln y}{\ln 3} \frac{dx}{dx} + \frac{x}{y \ln 3} \frac{dy}{dx} = \frac{d(10)}{dx}$

- Grouping terms, $\frac{3x^2}{y^3} + \frac{\ln y}{\ln 3} + \left(\frac{x}{y \ln 3} - \frac{3x^3}{y^4} \right) \frac{dy}{dx} = 0$

Reference: MATH 1003 2018-19 Fall Final #8

Question 9

- Note that $\frac{dy}{dx} = 4(x - 1)^3 - 5$. So, $\frac{dy}{dx}\bigg|_{x=2} = 4(1)^3 - 5 = -1$.
- When $x = 2$, $y = (2 - 1)^4 - 5(2) = -9$
- Tangent equation: $y + 9 = -1(x - 2) \implies y = -x - 7$.

Reference: MATH 1003 2013-14 Fall Final #19

Question 10

- Note that $q(x) = \frac{3(x+1)-3}{x-1} = 3 - \frac{3}{x+1}$.
- $q'(x) = \frac{3}{(x+1)^2}$, $q''(x) = -\frac{6}{(x+1)^3}$
- Therefore, $q''(1) = -\frac{6}{(1+1)^3} = -\frac{3}{4}$.

Reminder: Simplify before differentiate

Similar to Q5, simplifying first can save you time :)

Reference: MATH 1003 Fall 2016-17 Final #5

Question 11

- A. False. Counterexample: $a(x) = (x - 1)^2 + 1$ where $a(1) = 1$ is a critical point.
- B. False. Counterexample: $a(x) = |x|$, where $a(0) = 0$ is a critical value but $a'(0)$ does not exist.
- C. False. Counterexample: $a(x) = x^4$ where $a'(0) = a''(0) = 0$.
- D. False. Counterexample: $a(x) = x^3$ where $(0, a(0))$ is neither minimum or maximum.

Recall that critical point **does NOT** require derivative to exist!

Reference: MATH 1003 Fall 2018-19 Final #12

Question 12

The box has width and length both being $(6 - 2x)$, and height x .

- Objective function is $V(x) = x(6 - 2x)(6 - 2x) = 4x^3 - 24x^2 + 36x$.
- Recall the second derivative test.

To maximize the volume, we require $V'(x) = 0$ and $V''(x) < 0$.

Reference: Libretxts// MATH 1012 2020-21 Fall Final #15

Question 13

Note that

- $R(x) = xP(x) = x(500 - 0.5x) = 500x - 0.5x^2$, and
- $F(x) = x[P(x) - C(x)] = x(400 - x) = 400x - x^2$

- A. False. $R'(x) = 500 - x$ which is > 0 for $x < 500$.
- B. False. As calculated above.
- C. True. $R'(500) = 0$ and $R''(500) = -1 < 0$.
- D. False. $F'(500) = 400 - 2(500) = -300 \neq 0$.

Reference: MATH 1003 Sample Final #11(b)

Question 14

- First, express the volume V in terms of side x : $V = x^3$
- Differentiating **with respect to t** , $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$
- When $V = 512$, $x = \sqrt[3]{512} = 8$.

Therefore, $\frac{dV}{dt} = 3(8)^2\left(\frac{1}{6}\right) = 32$.

Reference: MATH 1003 2012-13 Fall Final #10/ 2013-14 Fall Final #13

Question 15

- $v(T) = 0 \implies -\frac{24}{T+3} + 4 = 0 \implies T = 3$

- Required displacement:

$$s(3) = s(0) + \int_0^3 \left(-\frac{24}{t+3} + 4\right) dt = [-24 \ln |t+3| + 4t]_0^3$$

- It evaluates to $-24 \ln(3+3) + 4(3) + 24 \ln 3 = 12(1 - 2 \ln 2)$

Reference: MATH 1012 Fall 2023-24 Final #15

Question 16

We integrate $g''(x)$ twice to get $g(x)$:

- $g'(x) = \int (e^x + 6)dx = e^x + 6x + C_1$
- $g'(0) = e^0 + 6(0) + C_1 = 2 \implies C_1 = 1$
- $g(x) = \int g'(x)dx = \int (e^x + 6x + 1)dx = e^x + 3x^2 + x + C_2$
- $g(0) = e^0 + 3(0^2) + (0) + C_2 = 5 \implies C_2 = 4$
- $g(1) = e^1 + 3(1^2) + (1) + 4 = e + 8$

Reference: MATH 1003 Fall 2018-19 Final #17

Question 17

- Change the sign to reverse lower/ upper bound:

$$\int_8^0 f(x)dx = - \int_0^8 f(x)dx =$$
$$- \left[\int_0^2 f(x)dx + \int_2^7 f(x)dx + \int_7^8 f(x)dx \right]$$

- Note that region A_2 is under the x -axis and should result in a negative integral.

- So, $\int_0^2 f(x)dx = A_1$, $\int_2^7 f(x)dx = -A_2$ and $\int_7^8 f(x)dx = A_3$

Thus, the required definite integral sums up to $-A_1 + A_2 - A_3$.

Reference: MATH 1003 Fall 2018-19 Final #19

Question 18

- A. True, by the fundamental theorem of calculus.
- B. True. The fact that the derivative $F'(x) = f(x)$ exists implies continuity.
- C. True, by definition of antiderivative.
- D. False.

Reminder: Integration Constant

Don't miss $+C$ in integration in your final exam.

Else... Your grade gets a C.

LQ 19 (a)/ Function and Limits/ One-sided Limit

Question 19(a) - One-sided Limit [3 pts]

- [1 pt: Identifying Sign of Abs. Value]

Note that $x > 3$ for $x \rightarrow 3^+$. So, $3 - x < 0$.

- [1 pt: Cancelling the factor]

$$\text{So, } \lim_{x \rightarrow 3^+} \frac{x(-(x-3))}{x-3} = \lim_{x \rightarrow 3^+} -x$$

- [1 pt: Answer]

The answer is -3 .

Reference: MATH 1013 2017-18 Fall Final #2/ MATH 1003 2012-13 Fall Final #12

LQ 19 (b)/ Function and Limits/ First-principles

Question 19(b) - First Principles [3 pts]

- [1 pt: Factoring out common factor]

$$\lim_{h \rightarrow 0} \frac{e^8(e^{4h} - 1)}{h} = e^8 \lim_{h \rightarrow 0} \frac{(e^{4h} - 1)}{h}$$

- [2 pts: Identifying First Principles + Answer]

$$e^8 \lim_{h \rightarrow 0} \frac{(e^{4h} - 1)}{h} = e^8 \lim_{h \rightarrow 0} \frac{(e^{4h} - e^{4(0)})}{h - 0} = e^8 \frac{d}{dx}(e^{4x}) = 4e^{4x+8}$$

You can of course apply the identity $\lim_{x \rightarrow 0} \frac{e^{kx} - 1}{x} = k$.

However, always be smart to look for shortcuts :)

Reference: MATH 1003 2018-19 Fall Midterm #9

LQ 19 (c)/ Function and Limits/ Rationalization

Question 19(c) - Rationalization [4 pts]

- [2 pts: Successfully apply Rationalization]

$$\begin{aligned} & \lim_{x \rightarrow \infty} \left(2x - \sqrt{4x^2 - 5x + 3} \right) \\ &= \lim_{x \rightarrow \infty} \frac{(2x - \sqrt{4x^2 - 5x + 3})(2x + \sqrt{4x^2 - 5x + 3})}{(2x + \sqrt{4x^2 - 5x + 3})} = \lim_{x \rightarrow \infty} \frac{5x - 3}{2x + \sqrt{4x^2 - 5x + 3}} \end{aligned}$$

- [2 pts: Dividing dominant terms + Answer]

$$\lim_{x \rightarrow \infty} \frac{5x - 3}{2x + \sqrt{4x^2 - 5x + 3}} = \lim_{x \rightarrow \infty} \frac{\frac{5x}{x} - \frac{3}{x}}{\frac{2x}{x} + \sqrt{\frac{4x^2}{x^2} - \frac{5x}{x^2} + \frac{3}{x^2}}}$$

$$\text{The limit evaluates to } \frac{5 - 0}{2 + \sqrt{4 - 0 + 0}} = \frac{5}{4}$$

Reference: MATH 1012 2023-24 MT #5/ MATH 1012 2024-25 MT #8

Question 20(a)(b) - Tangent Equation [5 pts]

- [1 pt: Give $x|_{y=3}$]

$$x^3(3^2) - (3) = 6 \implies 9x^3 = 9 \implies x = 1$$

- [2 pts: Obtain dy/dx from implicit differentiation]

$$\text{Differentiating w.r.t. } x, 3x^2y^2 + 2x^3y \frac{dy}{dx} = \frac{dy}{dx} = 0$$

$$\text{Grouping terms, } \frac{dy}{dx} = \frac{3x^2y^2}{1 - 2x^3y}$$

- [1 pt: Slope at $y = 3$]

$$\left. \frac{dy}{dx} \right|_{(1,3)} = \frac{3(1^2)(3^2)}{1 - 2(1^3)(3)} = -\frac{27}{5}$$

- [1 pt: Equation]

$$\text{Equation: } y - 3 = -\frac{27}{5}(x - 1) \implies 27x + 5y - 42 = 0$$

Question 20(c) - Rate of Change [3 pts]

- [1 pt: Differentiate w.r.t. t]

$$3x^2y^2\frac{dx}{dt} + 2x^3y\frac{dy}{dt} - \frac{dy}{dt} = 0$$

- [2 pt: Substitute Numbers + Answer]

$$3(1^2)(3^2)(-2) + 2(1^3)(3)\frac{dy}{dt} - \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = +\frac{54}{5}\text{units/ sec}$$

Be careful about the **sign (negative: decreasing)** of the rates!

References:

- For (b): MATH 1003 Sample Paper Final #14(c)
- For (c): MATH 1003 Sample Paper Final #11(a)

LQ 21 (a)(b)/ Appli. of Differentiation/ Distances

Question 21(a)(b) - First Eq. of Distance [4 pts]

Draw a figure before stating the equations:

- [2 pts: Express S in terms of t]

The north-south distance at time t is

$520 - at$ (km); the east-west distance is bt .

$$\text{So, } S^2 = (520 - at)^2 + (bt)^2$$

$$S = \sqrt{(520 - at)^2 + (bt)^2}$$

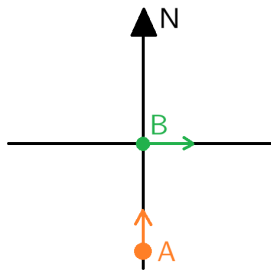
- [2 pts: First equation with $t = 4$]

$$\frac{dS}{dt} = \frac{-2a(520 - at) + 2b(bt)}{2\sqrt{(520 - at)^2 + (bt)^2}}$$

$$\text{At } t = 4, \left. \frac{dS}{dt} \right|_{t=4} = \frac{-2a(520 - 4a) + 2b(4b)}{2\sqrt{(520 - at)^2 + (bt)^2}}$$

$$\text{So, } -2a(520 - 4a) + 2b(4b) = 0$$

$$\text{Therefore, } a^2 + b^2 - 130a = 0$$



Question 21(c) - Solving Speeds [4 pts]

- [2 pts: Constructing another equation]

$$\text{At } t = 13, \left. \frac{dS}{dt} \right|_{t=13} = \frac{-2a(520-13a)+2b(13b)}{2\sqrt{(520-13a)^2+(13b)^2}} = \frac{26(a^2+b^2-40a)}{26\sqrt{(40-a)^2+b^2}} = 60$$

$$\text{Therefore, } \frac{a^2+b^2-40a}{\sqrt{(40-a)^2+b^2}} = 60$$

- [2 pts: Solving + Answer]

$$\text{From (b), } a^2 + b^2 - 40a = 90a.$$

$$\textcircled{1} \frac{90a}{\sqrt{(40-a)^2+b^2}} = 60 \implies \frac{3a}{2} = \sqrt{(40-a)^2+b^2}$$

$$\textcircled{2} \text{ Simplifying, } 5a^2 + 320a - 6400 - 4b^2 = 9a^2 - 200a - 6400 = 0$$

$$\textcircled{3} \text{ } a = 40 \text{ OR } a = -\frac{160}{9} \text{ (rejected). Consequently, } b = 60.$$

This question is inspired from a MATH 1012 homework.

LQ 22 (a)/ Integration/ Simple Functions

Question 22(a) - Simple Integral [3 pts]

For (a), simple and straightforward (directly read from formula sheet).

- Answer: $\frac{1}{6}x^6 - 3 \ln |x| + e^x$ **+C**. [1 pt for each expression]

Adding Constant for Indefinite Integrals

- Don't forget to **+C** for indefinite integrals.
- Otherwise you would get a $C+$ in Calculus...

Reference: MATH 1003 Fall 2016-17 Final #11(a)

LQ 22 (b)/ Integration/ Substitution

Question 22(b) - Integration by Substitution [4 pts]

For (b), apply substitution (the numerator x^2 is relevant to the derivative of the denominator!)

- [2 pts: Give substitution and upper/ lower bounds.]

① Let $u = 5x^3 - 1$. Then $du = 15x^2 dx \implies x^2 dx = \frac{1}{15} du$

② When $x = 1$, $u = 4$. When $x = 2$, $u = 39$.

- [2 pts: Evaluate the integral]

$$\text{So, } \int_1^2 \frac{x^2}{5x^3 - 1} dx = \frac{1}{15} \int_4^{39} \frac{1}{u} du = \frac{1}{15} [\ln |u|]_4^{39} = \frac{1}{15} \ln \left(\frac{39}{4} \right)$$

Reference: MATH 1003 Fall 2018-19 Final #24(b)

Question 22(c) - Area Bounded [5 pts]

We first find when $f(x) > g(x)$ and when $g(x) > f(x)$:

- [2pts: Solving $f(x) = g(x) + 1$ pt: Ranges whichever bigger]

$$f(x) - g(x) = x(x^2 + 5x - 6) = x(x + 6)(x - 1).$$

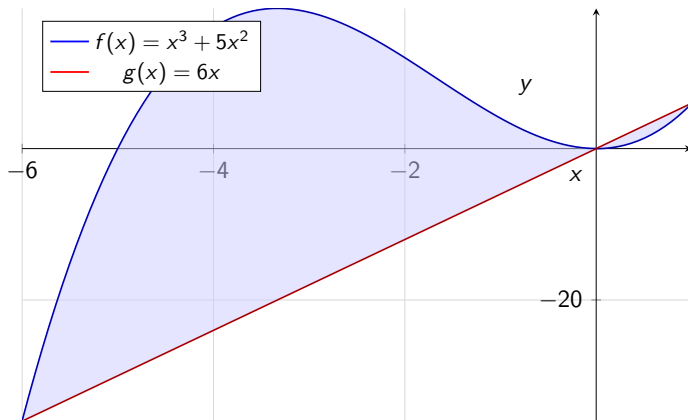
When $f(x) - g(x) = 0$, $x = -6$ OR $x = 0$ OR $x = 1$.

- So, $f(x) > g(x)$ when $x > 1$ or $-6 < x < 0$;
 $f(x) < g(x)$ when $x < -6$ or $0 < x < 1$.

- [2pts: Correct Final Integral]

Required area is $\int_{-6}^0 (f(x) - g(x)) dx + \int_0^1 (g(x) - f(x)) dx$

LQ 22 (c): Figure



References:

MATH 1003 2016-17 Fall Final #12(d)/ 2018-19 Fall Final #25(b)

LQ 23 (a)(b)/ Appli of Differentiation/ Asymptotes

Question 22(a)(b) - Intercept and Asymptotes [3 pts]

- [1 pt: x-intercept (Answers only)]

$$f(x) = 0 \implies x^3 + 5x^2 = x^2(x + 5) = 0 \implies x = -5 \text{ OR } x = 0.$$

So, the x-intercepts are -5 AND 0 .

- [2 pts: Asymptotes (Answers only)]

Directly read from the partial fraction:

- ① The part without fraction: $y = x + 7$ (oblique)
- ② When denominator is 0: $x = 1$ (vertical)

Recall that after **partial fractions/ long div**, you can directly read asymptotes and do easy differentiation!

Question 23(c) - First Derivative [5 pts]

• [3 pts: Intervals of increase/ decrease]

- ① When $f(x) = 0$, $x(x^2 - 3x - 10) = 0 \implies x = 0$ or 5 or -2
- ② $x(x^2 - 3x - 10) > 0$ when $x > 5$ (all factors positive) or $-2 < x < 0$ (one positive).
- ③ As f is undefined at $x = 1$, it is increasing on $(-\infty, -2) \cup (0, 1) \cup (5, \infty)$; and decreasing on $(-2, 0) \cup (1, 5)$

• [2 pts: Minimum/ Maximum]

- ① $x = -2$: f' changes sign from $+$ to $- \implies$ **Maximum** at $(-2, \frac{4}{3})$.
- ② $x = 0$: f' changes sign from $-$ to $+$ \implies **Minimum** at $(0, 0)$.
- ③ $x = 5$: f' changes sign from $-$ to $+$ \implies **Minimum** at $(5, \frac{125}{8})$.

(1 pt for any correct; 2 pts for all correct)

Be careful of the **sign of the denominator** (negative for $x < 1$)!

Question 23(d) - Second Derivative [5 pts]

- [3 pts: Intervals of concave upwards/ downwards]

- ① When $f(x) = 0$, $13x + 5 = 0 \implies x = -\frac{5}{13}$

- ② $2(13x + 5) > 0$ when $x > -\frac{5}{13}$.

- ③ As f is undefined at $x = 1$, f concaves upwards on $(-\frac{5}{13}, 1) \cup (1, \infty)$; and downwards on $(-\infty, -\frac{5}{13})$

- [2 pts: Inflection Points]

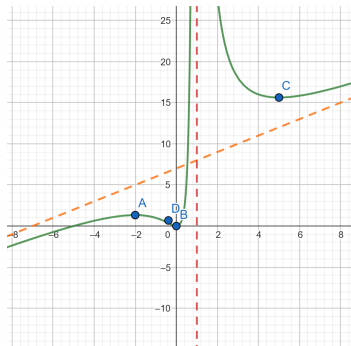
Note that the denominator is always positive.

- ① $x = -\frac{5}{13}$: f' changes sign from $-$ to $+$
 \implies **Inflection** at $(-\frac{5}{13}, \frac{1500}{2197})$.

Question 23(e) - Curve Sketching [4 pts]

Each characteristic is worth 1 pt:

- 1 Two asymptotes
- 2 Maximum (A), Minimums (B, C) and Inflection points (D)
- 3 Overall shape before $x = 1$ (Bending A and B)
- 4 Overall shape after $x = 1$ (a minimum)



Reference: MATH 1003 Fall 2018-19 Final #21