MATH 1003 Mock: Answers and Reminders

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Summary of Testing Areas

Testing Areas

1. **Function and Limits**: (20 Points)
   Q 1-4, 19
2. **Differentiation**: (10 Points)
   Q 5-7, 10
3. **Appli. of Differentiation**: (50.5 Points)
   Q 8-9, 11-15, 20, 21, 23
4. **Integration**: (19.5 Points)
   Q 16-18, 22

Section 1

MC Questions (45 Points)

MC Answers

Item	1	2	3	4	5	6
Ans	D	B	B	C	C	D
Item	7	8	9	10	11	12
Ans	A	C	A	A	E	A
Item	13	14	15	16	17	18
Ans	C	B	E	B	E	D

- Questions labelled with * appeared in other advanced streams;
- Questions labelled with @ are watered down versions of those that appeared in other streams.

MCQ 1D/ Function and Limits/ Domain

Question 1

Suppose $f(x) = \dfrac{x+3}{3x+4}$ and $g(x) = \dfrac{2x}{4x-1}$.

Give the domain of $f(g(x))$.

- Note that the domain of f excludes $-\frac{4}{3}$ and the domain of g excludes $\frac{1}{4}$: they make the respective denominators zero.
- For the composite function $f(g(x))$, we require

$$4x - 1 \neq 0 \quad \text{and} \quad g(x) \neq -\frac{4}{3}$$

i.e. $x \neq \dfrac{1}{4}$ AND $g(x) \neq -\dfrac{4}{3} \implies x \neq \dfrac{2}{11}$.

Reference: MATH 1012 Fall 2015-16 Midterm #9

Question 2

Note that $x - 4$ is a factor on both numerator and denominator of the limit.

- $\lim_{x \to 4} \frac{x^2 - 3x - 4}{x^2 - 5x + 4} = \lim_{x \to 4} \frac{(x-4)(x+1)}{(x-4)(x-1)} = \lim_{x \to 4} \frac{x+1}{x-1}$
- Thus, the required limit is $\frac{4+1}{4-1} = \frac{5}{3}$.

Reference: MATH 1013 Sample Midterm #2

Question 3

I. True, because $\lim_{x \to c^+} f(x)g(x) = \lim_{x \to c^+} f(x) \lim_{x \to c^+} g(x)$
$= \lim_{x \to c^-} f(x) \lim_{x \to c^-} g(x) = \lim_{x \to c^-} f(x)g(x)$

II. False. One counterexample:
$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}, \quad g(x) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

III. False. Same counterexample as II.

IV. False. One counterexample:
$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}, \quad g(x) = 1 \text{ for all real numbers.}$$

Reference: MATH 1003 Fall 2018-19 Midterm #8

Question 4

- Vertical asymptotes may occur when denominator equals zero. When $(x^2-x)(x^2-5x-6)(x^2+4) = (x)(x-1)(x+1)(x-6)(x^2+4) = 0$, $x = -1$ or 0 or 1 or 6.
- However, the numerator is $(x-1)^2$. The factor $x-1$ cancels out, meaning there is no vertical asymptote at $x = 1$, as the limit at that point is finite.
- Thus, there are only three asymptotes ($x = -1$, $x = 0$, $x = 6$).

Factoring Trap of Vertical Asymptotes

Candidates may forget the definition of vertical asymptotes:

- The limit should tends to $\pm\infty$ when $x \to c$!
- However, when the factor gets cancelled out, the limit when $x \to c$ would be finite.
- Then $x = c$ is NOT a vertical asymptote!

Reference: MATH 1013 2020-21 Fall Final #3/ MATH 1003 Sample Final #2

Question 5

Rewrite: $u(x) = 3x + \frac{4}{x} + \frac{5}{x^3} \implies u'(x) = 3 - \frac{4}{x^2} - \frac{15}{x^4}$

Reminder: Don't be a fool to apply product rule mindlessly!

- In this case, the denominator is a simple power of x, and can be divided to simplify the expression.
- This question reminds you to be "smart" and think before you do the question :)

Reference: MATH 1003 Fall 2012-13 Final #14

Question 6

- By product rule, $p'(x) = \frac{dp(x)}{dx}$
  $= \frac{d}{dx}(x^2+5)^{2024} \cdot (6-x)^{1203} + (x^2+5)^{2024} \cdot \frac{d}{dx}(6-x)^{1203}$
  $= 2x(2024)(x^2+5)^{2023}(6-x)^{1203} + (-1)(1203)(x^2+5)^{2024}(6-x)^{1202}$
- Substituting $x = 0$, the first term becomes 0 as $2x = 2(0) = 0$.
- Thus, the answer is $-1203 \cdot 5^{2024} \cdot 6^{1202}$.

Question 7

$h(x)$ is an obvious candidate for logarithmic differentiation.

- Employing the method, $\ln h(x) = 2x \ln x$.
- Differentiating, $\dfrac{1}{h(x)} h'(x) = \dfrac{h'(x)}{h(x)} = 2 \ln x + \dfrac{2x}{x} = 2 \ln x + 2$

Reference: MATH1013 Fall 2018 Midterm #12

Question 8

Differentiating both sides with respect to x *(why?)*:

- $\dfrac{x^3}{y^3}$ is a function containing x and y (both change w.r.t. x).
  So is $x \log_3 y = \dfrac{x \ln y}{\ln 3}$.
- Product rule: $\dfrac{3x^2}{y^3}\dfrac{dx}{dx} - \dfrac{3x^3}{y^4}\dfrac{dy}{dx} + \dfrac{\ln y}{\ln 3}\dfrac{dx}{dx} + \dfrac{x}{y \ln 3}\dfrac{dy}{dx} = \dfrac{d(10)}{dx}$
- Grouping terms, $\dfrac{3x^2}{y^3} + \dfrac{\ln y}{\ln 3} + \left(\dfrac{x}{y \ln 3} - \dfrac{3x^3}{y^4}\right)\dfrac{dy}{dx} = 0$

Reference: MATH 1003 2018-19 Fall Final #8

Question 9

- Note that $\frac{dy}{dx} = 4(x-1)^3 - 5$. So, $\left.\frac{dy}{dx}\right|_{x=2} = 4(1)^3 - 5 = -1$.
- When $x = 2$, $y = (2-1)^4 - 5(2) = -9$
- Tangent equation: $y + 9 = -1(x-2) \implies y = -x - 7$.

Reference: MATH 1003 2013-14 Fall Final #19

Question 10

- Note that $q(x) = \frac{3(x+1)-3}{x-1} = 3 - \frac{3}{x+1}$.
- $q'(x) = \dfrac{3}{(x+1)^2}$, $q''(x) = -\dfrac{6}{(x+1)^3}$
- Therefore, $q''(1) = -\dfrac{6}{(1+1)^3} = -\dfrac{3}{4}$.

Reminder: Simplify before differentiate

Similar to Q5, simplifying first can save you time :)

Reference: MATH 1003 Fall 2016-17 Final #5

Question 11

A. False. Counterexample: $a(x) = (x-1)^2 + 1$ where $a(1) = 1$ is a critical point.

B. False. Counterexample: $a(x) = |x|$, where $a(0) = 0$ is a critical value but $a'(0)$ does not exist.

C. False. Counterexample: $a(x) = x^4$ where $a'(0) = a''(0) = 0$.

D. False. Counterexample: $a(x) = x^3$ where $(0, a(0))$ is neither minimum or maximum.

Recall that critical point does NOT require derivative to exist!

Reference: MATH 1003 Fall 2018-19 Final #12

MCQ 12A*/ Appli. of Differentiation/ Box Optimization

Question 12

The box has width and length both being $(6-2x)$, and height x.

- Objective function is $V(x) = x(6-2x)(6-2x) = 4x^3 - 24x^2 + 36x$.
- Recall the second derivative test.
  To maximize the volume, we require $V'(x) = 0$ and $V''(x) < 0$.

Reference: Libretexts// MATH 1012 2020-21 Fall Final #15

Question 13

Note that

- $R(x) = xP(x) = x(500 - 0.5x) = 500x - 0.5x^2$, and
- $F(x) = x[P(x) - C(x)] = x(400 - x) = 400x - x^2$

A. False. $R'(x) = 500 - x$ which is > 0 for $x < 500$.

B. False. As calculated above.

C. True. $R'(500) = 0$ and $R''(500) = -1 < 0$.

D. False. $F'(500) = 400 - 2(500) = -300 \neq 0$.

Reference: MATH 1003 Sample Final #11(b)

Question 14

- First, express the volume V in terms of side x: $V = x^3$
- Differentiating with respect to t, $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$
- When $V = 512$, $x = \sqrt[3]{512} = 8$.

Therefore, $\frac{dV}{dt} = 3(8)^2(\frac{1}{6}) = 32$.

Reference: MATH 1003 2012-13 Fall Final #10/ 2013-14 Fall Final #13

MCQ 15E*/ Appli. of Differentiation/ s, v, a

Question 15

- $v(T) = 0 \implies -\frac{24}{T+3} + 4 = 0 \implies T = 3$
- Required displacement:
  $$s(3) = s(0) + \int_0^3 (-\frac{24}{t+3} + 4)dt = [-24\ln|t+3| + 4t]_0^3$$
- It evaluates to $-24\ln(3+3) + 4(3) + 24\ln 3 = 12(1 - 2\ln 2)$

Reference: MATH 1012 Fall 2023-24 Final #15

MCQ 16B/ Integration/ Twice Calculation

Question 16

We integrate $g''(x)$ twice to get $g(x)$:

- $g'(x) = \int (e^x + 6)dx = e^x + 6x + C_1$
- $g'(0) = e^0 + 6(0) + C_1 = 2 \implies C_1 = 1$
- $g(x) = \int g'(x)dx = \int (e^x + 6x + 1)dx = e^x + 3x^2 + x + C_2$
- $g(0) = e^0 + 3(0^2) + (0) + C_2 = 5 \implies C_2 = 4$
- $g(1) = e^1 + 3(1^2) + (1) + 4 = e + 8$

Reference: MATH 1003 Fall 2018-19 Final #17

Question 17

- Change the sign to reverse lower/ upper bound:
$$\int_8^0 f(x)dx = -\int_0^8 f(x)dx =$$
$$-\left[\int_0^2 f(x)dx + \int_2^7 f(x)dx + \int_7^8 f(x)dx\right]$$
- Note that region A_2 is under the x-axis and should result in a negative integral.
- So, $\int_0^2 f(x)dx = A_1$, $\int_2^7 f(x)dx = -A_2$ and $\int_7^8 f(x)dx = A_3$

Thus, the required definite integral sums up to $-A_1 + A_2 - A_3$.

Reference: MATH 1003 Fall 2018-19 Final #19

MCQ 18D/ Integration/ FTC Concept

Question 18

A. True, by the fundamental theorem of calculus.

B. True. The fact that the deriative $F'(x) = f(x)$ exists implies continuity.

C. True, by definition of antiderivative.

D. False.

Reminder: Integration Constant

Don't miss $+C$ in integration in your final exam.

Else... Your grade gets a C.

LQ 19 (a)/ Function and Limits/ One-sided Limit

Question 19(a) - One-sided Limit [3 pts]

- [1 pt: Identifying Sign of Abs. Value]
  Note that $x > 3$ for $x \to 3^+$. So, $3 - x < 0$.
- [1 pt: Cancelling the factor]
  So, $\displaystyle \lim_{x \to 3^+} \frac{x(-(3-x))}{x-3} = \lim_{x \to 3^+} x \cdot \lim_{x \to 3^+} \frac{x-3}{x-3} = \lim_{x \to 3^+} x \cdot 1$
- [1 pt: Answer]
  The answer is 3.

Reference: MATH 1013 2017-18 Fall Final #2/ MATH 1003 2012-13 Fall Final #12

LQ 19 (b)/ Function and Limits/ First-principles

Question 19(b) - First Principles [3 pts]

- [1 pt: Factoring out common factor]
  $$\lim_{h\to 0}\frac{e^8(e^{4h}-1)}{h} = e^8 \lim_{h\to 0}\frac{(e^{4h}-1)}{h}$$
- [2 pts: Identifying First Principles + Answer]
  $$e^8 \lim_{h\to 0}\frac{(e^{4h}-1)}{h} = e^8 \lim_{h\to 0}\frac{(e^{4h}-e^{4(0)})}{h-0} = e^8\ \frac{d}{dx}(e^{4x})\Big|_{x=0} = 4e^8$$

You can of course apply the identity $\displaystyle\lim_{x\to 0}\frac{e^{kx}-1}{x} = k$.
However, always be smart to look for shortcuts :)

Reference: MATH 1003 2018-19 Fall Midterm #9

Question 19(c) - Rationalization [4 pts]

- [2 pts: Successfully apply Rationalization]
$$\lim_{x\to\infty}\left(2x-\sqrt{4x^2-5x+3}\right)$$
$$=\lim_{x\to\infty}\frac{\left(2x-\sqrt{4x^2-5x+3}\right)\left(2x+\sqrt{4x^2-5x+3}\right)}{\left(2x+\sqrt{4x^2-5x+3}\right)}=\lim_{x\to\infty}\frac{5x-3}{2x+\sqrt{4x^2-5x+3}}$$
- [2 pts: Dividing dominant terms + Answer]
$$\lim_{x\to\infty}\frac{5x-3}{2x+\sqrt{4x^2-5x+3}}=\lim_{x\to\infty}\frac{\frac{5x}{x}-\frac{3}{x}}{\frac{2x}{x}+\sqrt{\frac{4x^2}{x^2}-\frac{5x}{x^2}+\frac{3}{x^2}}}$$
The limit evaluates to $\dfrac{5-0}{2+\sqrt{4-0+0}}=\dfrac{5}{4}$

Reference: MATH 1012 2023-24 MT #5/ MATH 1012 2024-25 MT #8

Question 20(a)(b) - Tangent Equation [5 pts]

- [1 pt: Give $x|_{y=3}$]
  $x^3(3^2) - (3) = 6 \implies 9x^3 = 9 \implies x = 1$
- [2 pts: Obtain dy/dx from implicit differentiation]
  Differentating w.r.t. x, $3x^2y^2 + 2x^3y\dfrac{dy}{dx} = \dfrac{dy}{dx} = 0$
  Grouping terms, $\dfrac{dy}{dx} = \dfrac{3x^2y^2}{1-2x^3y}$
- [1 pt: Slope at $y = 3$]
  $\left.\dfrac{dy}{dx}\right|_{(1,3)} = \dfrac{3(1^2)(3^2)}{1-2(1^3)(3)} = -\dfrac{27}{5}$
- [1 pt: Equation]
  Equation: $y - 3 = -\frac{27}{5}(x-1) \implies 27x + 5y - 42 = 0$

Question 20(c) - Rate of Change [3 pts]

- [1 pt: Differentiate w.r.t. t]
  $$3x^2y^2\frac{dx}{dt} + 2x^3y\frac{dy}{dt} - \frac{dy}{dt} = 0$$
- [2 pt: Substitute Numbers + Answer]
  $$3(1^2)(3^2)(-2) + 2(1^3)(3)\frac{dy}{dt} - \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = +\frac{54}{5}\text{units/ sec}$$

Be careful about the sign (negative: decreasing) of the rates!

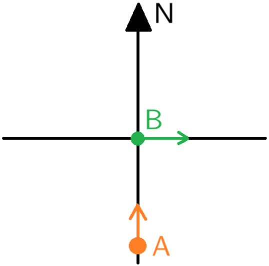
References:

- For (b): MATH 1003 Sample Paper Final #14(c)
- For (c): MATH 1003 Sample Paper Final #11(a)

Question 21(a)(b) - First Eq. of Distance [4 pts]

Draw a figure before stating the equations:

- [2 pts: Express S in terms of t]
  The north-south distance at time t is $520 - at$ (km); the east-west distance is bt.
  So, $S^2 = (520 - at)^2 + (bt)^2$
  $S = \sqrt{(520 - at)^2 + (bt)^2}$
- [2 pts: First equation with $t = 4$]
  $$\frac{dS}{dt} = \frac{-2a(520 - at) + 2b(bt)}{2\sqrt{(520 - at)^2 + (bt)^2}}$$
  At $t = 4$, $\left.\frac{dS}{dt}\right|_{t=4} = \frac{-2a(520-4a)+2b(4b)}{2\sqrt{(520-at)^2+(bt)^2}}$
  So, $-2a(520 - 4a) + 2b(4b) = 0$
  Therefore, $a^2 + b^2 - 130a = 0$



LQ 21 (c)/ Appli. of Differentiation/ Solve Distances

Question 21(c) - Solving Speeds [4 pts]

- [2 pts: Constructing another equation]
  At $t = 13$, $\frac{dS}{dt}\Big|_{t=13} = \frac{-2a(520-13a)+2b(13b)}{2\sqrt{(520-13a)^2+(13b)^2}} = \frac{26(a^2+b^2-40a)}{26\sqrt{(40-a)^2+b^2}} = 60$
  Therefore, $\frac{a^2+b^2-40a}{\sqrt{(40-a)^2+b^2}} = 60$

- [2 pts: Solving + Answer]
  From (b), $a^2 + b^2 - 40a = 90a$.
  1. $\dfrac{90a}{\sqrt{(40-a)^2+b^2}} = 60 \implies \dfrac{3a}{2} = \sqrt{(40-a)^2+b^2}$
  2. Simplifying, $5a^2 + 320a - 6400 - 4b^2 = 9a^2 - 200a - 6400 = 0$
  3. $a = 40$ OR $a = -\frac{160}{9}$ (rejected). Consequently, $b = 60$.

This question is inspired from a MATH 1012 homework.

Question 22(a) - Simple Integral [3 pts]

For (a), simple and straightforward (directly read from formula sheet).

- Answer: $\frac{1}{6}x^6 - 3\ln|x| + e^x \quad +C$. [1 pt for each expression]

Adding Constant for Indefinite Integrals

- Don't forget to $+C$ for indefinite integrals.
- Otherwise you would get a $C+$ in Calculus...

Reference: MATH 1003 Fall 2016-17 Final #11(a)

LQ 22 (b)/ Integration/ Substitution

Question 22(b) - Integration by Substitution [4 pts]

For (b), apply substitution (the numerator x^2 is relevant to the derivative of the denominator!)

- [2 pts: Give substitution and upper/ lower bounds.]
  1. Let $u = 5x^3 - 1$. Then $du = 15x^2 dx \implies x^2 dx = \frac{1}{15} du$
  2. When $x = 1$, $u = 4$. When $x = 2$, $u = 39$.
- [2 pts: Evaluate the integral]

  So, $\displaystyle\int_1^2 \frac{x^2}{5x^3 - 1} dx = \frac{1}{15} \int_4^{39} \frac{1}{u} du = \frac{1}{15} \left[\ln |u|\right]_4^{39} = \frac{1}{15} \ln\left(\frac{39}{4}\right)$

Reference: MATH 1003 Fall 2018-19 Final #24(b)

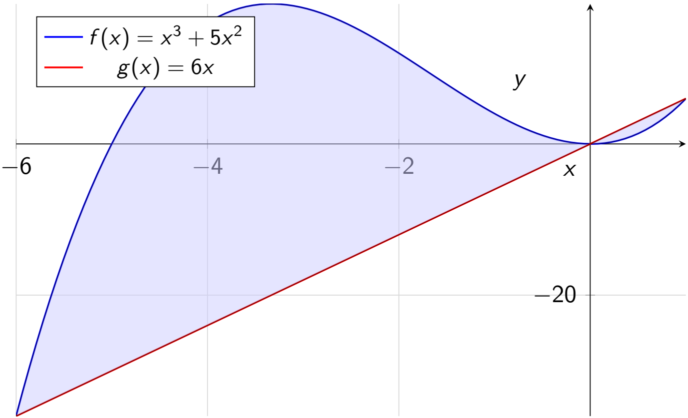
LQ 22 (c)/ Integration/ Area Bounded

Question 22(c) - Area Bounded [5 pts]

We first find when $f(x) > g(x)$ and when $g(x) > f(x)$:

- [2pts: Solving $f(x) = g(x)$ + 1pt: Ranges whichever bigger]
  $f(x) - g(x) = x(x^2 + 5x - 6) = x(x+6)(x-1)$.
  When $f(x) - g(x) = 0$, $x = -6$ OR $x = 0$ OR $x = 1$.
- So, $f(x) > g(x)$ when $x > 1$ or $-6 < x < 0$;
  $f(x) < g(x)$ when $x < -6$ or $0 < x < 1$.
- [2pts: Correct Final Integral]
  Required area is $\int_{-6}^{0} (f(x) - g(x))\, dx + \int_{0}^{1} (g(x) - f(x))\, dx$

LQ 22 (c): Figure



References:
MATH 1003 2016-17 Fall Final #12(d)/ 2018-19 Fall Final #25(b)

Question 22(a)(b) - Intercept and Asymptotes [3 pts]

- [1 pt: x-intercept (Answers only)]
  $f(x) = 0 \implies x^3 + 5x^2 = x^2(x+5) = 0 \implies x = -5$ OR $x = 0$.
  So, the x-intercepts are -5 AND 0.
- [2 pts: Asymptotes (Answers only)]
  Directly read from the partial fraction:
  1. The part without fraction: $y = x + 7$ (oblique)
  2. When denominator is 0: $x = 1$ (vertical)

Recall that after partial fractions/ long div, you can directly read asymptotes and do easy differentiation!

LQ 23 (c)/ Appli. of Differentiation/ First Deriv.

Question 23(c) - First Derivative [5 pts]

- [3 pts: Intervals of increase/ decrease]
  1. When $f(x) = 0$, $x(x^2 - 3x - 10) = 0 \implies x = 0$ or 5 or -2
  2. $x(x^2 - 3x - 10) > 0$ when $x > 5$ (all factors positive) or $-2 < x < 0$ (one positive).
  3. As f is undefined at $x = 1$, it is increasing on $(-\infty, -2) \cup (0, 1) \cup (5, \infty)$; and decreasing on $(-2, 0) \cup (1, 5)$
- [2 pts: Minimum/ Maximum]
  1. $x = -2$: f' changes sign from $+$ to $-$ $\implies$ **Maximum** at $(-2, \frac{4}{3})$.
  2. $x = 0$: f' changes sign from $-$ to $+$ $\implies$ **Minimum** at $(0, 0)$.
  3. $x = 5$: f' changes sign from $-$ to $+$ $\implies$ **Minimum** at $(5, \frac{125}{8})$.

  (1 pt for any correct; 2 pts for all correct)

Be careful of the sign of the denominator (negative for $x < 1$)!

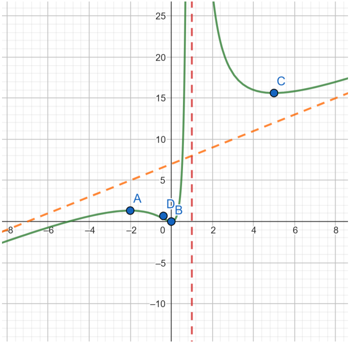
Question 23(d) - Second Derivative [5 pts]

- [3 pts: Intervals of concave upwards/ downwards]
  1. When $f(x) = 0$, $13x + 5 = 0 \implies x = -\frac{5}{13}$
  2. $2(13x + 5) > 0$ when $x > -\frac{5}{13}$.
  3. As f is undefined at $x = 1$, f concaves upwards on $(-\frac{5}{13}, 1) \cup (1, \infty)$; and downwards on $(-\infty, -\frac{5}{13})$
- [2 pts: Inflection Points]
  Note that the denominator is always positive.
  1. $x = -\frac{5}{13}$: f' changes sign from $-$ to $+$
     $\implies$ **Inflection** at $(-\frac{5}{13}, \frac{1500}{2197})$.

Question 23(e) - Curve Sketching [4 pts]

Each characteristic is worth 1 pt:

1. Two asymptotes
2. Maximum (A), Minimums(B, C) and Inflection points (D)
3. Overall shape before $x = 1$ (Bending A and B)
4. Overall shape after $x = 1$ (a minimum)



Reference: MATH 1003 Fall 2018-19 Final #21