# HKUST

**MATH1013 Calculus IB**

**Mock Final Examination (Fall 2024)**

1 Dec 2024 (Updated)

*Time limit: 3 hours*

Name: ____________________

Student ID: ____________________

Lecture Section: ____________________

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**Directions:**

- This is a closed book examination. Calculator of any kind is NOT allowed in this examination.
- **DO NOT open the exam until instructed to do so.**
- Turn off all phones and pagers, and remove headphones. All electronic devices should be kept in a bag away from your body.
- Write your name, ID number, and Lecture Section in the space provided above.
- When instructed to open the exam, please check that you have **13** pages of questions including the cover page. This document is updated, with amendments highlighted in red.
- Answer all questions. Show an appropriate amount of work for each short or long problem. If you do not show enough work, you will get only partial credit.
- **Cheating is a serious violation of the HKUST Academic Code.** This is only a mock exam, with no benefit of cheating here: by cheating, you are not lying to anyone but yourself.
- For answer checking/ marking/ feedback of this mock paper set, please either email me via theskillfulnoob2002@gmail.com or Whatsapp/ Signal via (+852) 9035 4789.

**Please read the following statement and sign your signature:**

I have neither given nor received any unauthorized aid during this examination. The answers submitted are my own work.

I understand that sanctions will be imposed, if I am found to have violated the University's regulations governing academic integrity.

Student's Signature:

______________________________

| Question No. | Points | Out of |
|---|---|---|
| **Q. 1-15** | | 45 |
| **Q. 16** | | 7 |
| **Q. 17** | | 8 |
| **Q. 18** | | 9 |
| **Q. 19** | | 13 |
| **Q.20** | | 18 |
| **Total Points** | | **100** |

# Part I: Multiple Choice Questions (45 Points)

**Answer all of the following multiple choice questions.**

- Mark your answers clearly in the Multiple Choice Boxes below.
- Mark only one answer for each MC question. Multiple answers will be treated as incorrect answer.

| **Question** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **Answer** | | | | | | | | |
| **Question** | 9 | 10 | 11 | 12 | 13 | 14 | 15 | **Total** |
| **Answer** | | | | | | | | |

**Internal References**

- Watered down versions of Questions 2-6, 8, 10 and 14 are featured in other streams (1003/ 1012) of this mock paper set.

1. The function $f(x) = \cos^{-1}(e^{2x} - \frac{1}{2})$ is one-to-one and hence has an inverse function. Find the domain of the inverse function of $f$.

   A. $[0, \frac{\pi}{3})$ B. $(0, \frac{\pi}{3}]$ C. $[0, \frac{2\pi}{3})$ D. $[0, \frac{2\pi}{3}]$ E. $(0, \frac{2\pi}{3}]$

2. Find the limit $\lim_{h \to 3} \frac{\sqrt{1+h} - 2}{|h-3|}$ if it exists.

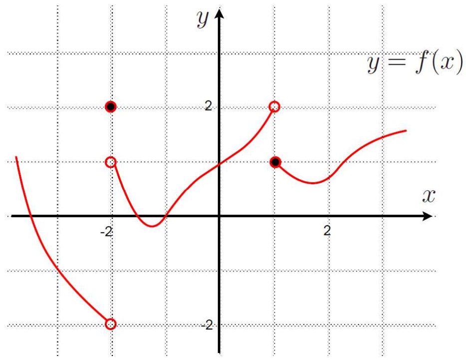
   A. Does not exist B. $-4$ C. $4$ D. $-\frac{1}{4}$. E. $\frac{1}{4}$.

3. How many vertical asymptotes does the following function have?

$$y = \frac{x^2 - 2x + 1}{(x^2 - x)(x^2 - 5x - 6)(x^2 + 4)}$$

   A. 1 B. 2 C. 3 D. 4 E. 5

4. Find the one-sided limit $\lim_{x \to 1^+} f(3f(x) - 5)$, with the graph of $f$ given:




A. Does not exist B. 0 C. 1 D. 2. E. $-2$

5. Let $f(x)$ be a function differentiable everywhere over $x \in \mathbb{R}$ such that $f(2) = 4$ and $f'(2) = 8$. Find the limit $\lim_{h \to 0} \dfrac{(2+h)f(2+h) - 2f(2+h+h^2)}{h}$.

A. 4 B. 8 C. 12 D. 16 E. 24

6. A function $f$ is defined on the interval $(-e, e)$. It is known that

$$f(x) = \begin{cases} 4e^x + mx - 4 & \text{if } -e < x \le 0 \\ \ln(\ln(x+e)) & \text{if } 0 < x < e \end{cases}$$

If $f$ is differentiable, find the value of constant $m$.

A. $\dfrac{1}{e} - 2$ B. $-2$ C. $-4$ D. $\dfrac{1}{e} - 4$ E. $\dfrac{1}{e} + 4$

7. Two functions $f, g$ with continuous derivatives are given. Some function values of these functions are shown in the following table:

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 4 | $-1$ | 0 | 1 |
| 2 | $-1$ | 4 | 1 | 3 |
| 3 | 2 | 0 | 3 | 2 |
| 4 | 0 | 2 | 6 | 4 |

It is also given that $g$ is one-to-one function. Find $\frac{d}{dx}\left(f \circ g^{-1}(x)\right)\Big|_{x=1}$.

A. 4 B. $\frac{4}{3}$ C. 0 D. $\frac{1}{4}$ E. $-1$

8. Define $h(x) = x^{2e^{3x}}$ for $x > 0$. Find $\frac{h'(x)}{h(x)}$.

A. $\frac{2e^{3x}}{x}$ B. $e^{3x}\left(\frac{2}{x} + 6\ln x\right)$ C. $\frac{6e^{3x}}{x}$ D. $e^{3x}\left(\frac{6}{x} + 18\ln x\right)$ E. $18e^{3x}\ln x$

9. To find an approximate root of the equation $x^3 - 1 + 3\cos x = 0$ by Newton's Method, the iteration formula is:

A. $x_{n+1} = x_n - \dfrac{x_n^3 + 1 + 3\cos x_n}{3x_n^2 - 3\sin x_n}$

B. $x_{n+1} = x_n - \dfrac{x_n^3 - 1 + 3\cos x_n}{3x_n^2 + 3\sin x_n}$

C. $x_{n+1} = x_n - \dfrac{x_n^3 - 1 + 3\cos x_n}{3x_n^2}$

D. $x_{n+1} = x_n - \dfrac{x_n^3 - 1 + 3\cos x_n}{3x_n^2 - 3\sin x_n}$

E. $x_{n+1} = x_n + \dfrac{x_n^3 - 1 + 3\cos x_n}{3x_n^2 - 3\sin x_n}$

10. An open-top box is to be made from a square cardboard of side 6 inches by removing a square from each of its corner and folding up the flaps on each side. Find the maximum volume of the box in cubic inches.

    A. 8 B. 16 C. 24 D. 32 E. 48

11. Suppose $g(x)$ is differentiable over $\mathbb{R}$ such that $g'(x) = (x-3)^{2023}$. Which of the following must be true?

    A. $g(x)$ attains a minimum at $x = 3$
    B. $g(x)$ has an inflection point at $x = 3$
    C. $xg(x)$ attains a minimum at $x = 3$
    D. $g(3) = 0$
    E. None of the above

12. Which of the following is an antiderivative of $\dfrac{1}{1+\cos x}$ $(-\pi < x < \pi)$?

    A. $-\dfrac{\sin x}{1+\cos x}+C$ B. $\dfrac{\cos x}{1+\cos x}+C$ C. $\dfrac{\sin x}{1+\cos x}+C$ D. $-\dfrac{\cos x}{1+\cos x}+C$ E. $\dfrac{\sin x\cos x}{1+\cos x}+C$

13. Suppose a continuous function $f$ satisfies the equation

$$\int_0^x f(t)dt + \int_0^{x^3} tf(t)dt = x^4$$

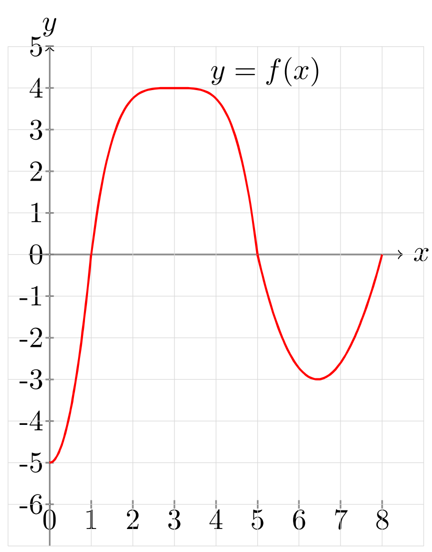
Find the value of $f(1)$.

A. 0 B. 1 C. 2 D. 4 E. 8

14. Evaluate the integral $\int_0^1 \frac{1+2e^{3x}}{1+e^{3x}}dx$ with a suitable substitution.

A. $1+\frac{1}{3}\ln\left(\frac{1+e^3}{2}\right)$ B. $1+\frac{1}{3}\ln\left(1+e^3\right)$ C. $\frac{1}{3}\ln\left(\frac{1+e^3}{2}\right)$ D. $1+\frac{1}{3}\ln\left(\frac{2}{1+e^3}\right)$ E. $1+\ln(1+e^3)$

15. Given the graph of the function $y = f(x)$ as shown below, find $\int_0^8 |f'(x)|dx$.




A. $-5$ B. 5 C. 9 D. 14 E. 19

# Part II: Long Questions (55 Points)

**Answer each of the following 5 long questions. Unless otherwise specified, numerical answers should be either exact or correct to 2 decimal places.**

16. *(7 pts)* Suppose the limit $\lim_{x \to 0} \frac{\sin^2 x + Ax^2}{x^4}$ exists and is equal to $L$. Find the value of

    (a) constant $A$, [4pts]

    (b) $L$. [3pts]

    *(Hint: Consider using L'Hospital's rule for more than once.)*

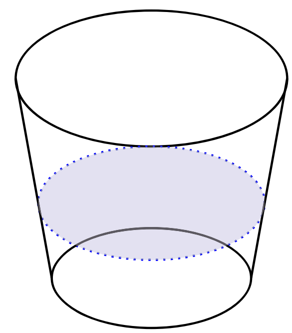
17. *(8 pts)* A container in the form of an inverted right circular frustrum is held vertically. The height, the upper radius and the base radius of the container are $h_1 = 24$ cm, $r_1 = 50$ cm and $r_2 = 40$ cm respectively. Water is now poured into the container until it is full.

(a) Let $h$ cm, $A$ cm$^2$ and $V$ cm$^3$ be the depth of water, wet curved surface area and the volume of water in the frustum container. By considering similar solids or otherwise, prove that

(i) $V = \dfrac{25\pi}{432}\left[(96+h)^3 - 96^3\right]$,

(ii) $A = \dfrac{65\pi}{144}[(96+h)^2 - 96^2]$,

[4pts]



(b) Suppose the rate of pouring is fixed at $\dfrac{dV}{dt} = 60\pi$ cm$^3$ s$^{-1}$.

Describe how the rate of change of the container's wet curved surface area, $\dfrac{dA}{dt}$, varies during the pouring process, giving the end point values. Explain your answer. [4pts]

18. *(9 pts)*

(a) Using a suitable substitution, evaluate $\displaystyle\int \frac{x^3}{9+x^2}dx.$ [5pts]

(b) Denote $f(x) = \dfrac{x^3}{9+x^2}$. A student claims that the Riemann sum

$$S = \lim_{n\to\infty} \frac{4}{n}\left(\frac{1}{9n^3+n} + \frac{8}{9n^3+4n} + \cdots + \frac{n^3}{9n^3+n^3}\right)$$

can be interpreted as

$$\int_0^4 f(x)dx$$

Explain whether the student is correct, by writing $S$ in the form $\lim\limits_{n\to\infty} \Delta x \sum f(x_i)$.
If not, write down the correct integral in $f(x)$. [4pts]

19. *(13 pts)* Let $C$ be the curve $xy = \cos(xy)$, defined for $x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$.

(a) Show that

$$\left(y + x\frac{dy}{dx}\right)(1 + \sin(xy)) = 0.$$

Also briefly explain why $1 + \sin(xy)$ can never be zero for the curve defined. [4pts]

(b) Consider the equation

$$u = \cos u, \quad \text{where } 0 \leq u \leq \frac{\pi}{2} \tag{*}$$

(i) Show that (*) has exactly one root in $\left[0, \frac{\pi}{2}\right]$.

(ii) Hence, show that there exists two tangents to $C$ parallel to the straight line $L : x + y = 0$.

[6pts]

(c) Suppose two tangents to

$$C : xy = \cos(xy) \quad \text{where} \quad x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$

at $x = a < 0$ and $x = b > 0$ are parallel to the straight line $x + y = 0$. It is given that for any point $(x, y)$ lying on $C$, the function $f : y = f(x)$ connecting the two variables is one-to-one. A student claims that:

> Since there are two tangents to $C$ parallel to the straight line $L : x + y = 0$, I can conclude by using mean value theorem that there exists a value $c \in (a, b)$ such that $f"(c) = 0$.

Explain why the student's application of MVT is incorrect.

> Hint: Recall that MVT states for the function $g$ continuous and differentiable over $[a, b]$, there exists some $c \in (a, b)$ such that
> $$g'(c) = \frac{g(b) - g(a)}{b - a}$$

[3pts]

20. *(18 pts)* Let $\Gamma$ be the curve $y = \dfrac{3x^2 - 2x - k}{(x-1)^2}$, where $k$ is a constant.

It is given that $\dfrac{3x^2 - 2x - k}{(x-1)^2} \equiv 3 + \dfrac{4}{x-1} - \dfrac{k-1}{(x-1)^2}$.

(a) State all asymptotes of $\Gamma$. [2pts]

(b) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of $k$. [2pts]

(c) A student claims that $\Gamma$ has a minimum point for all values of $k$. Is the claim correct? Explain your answer.

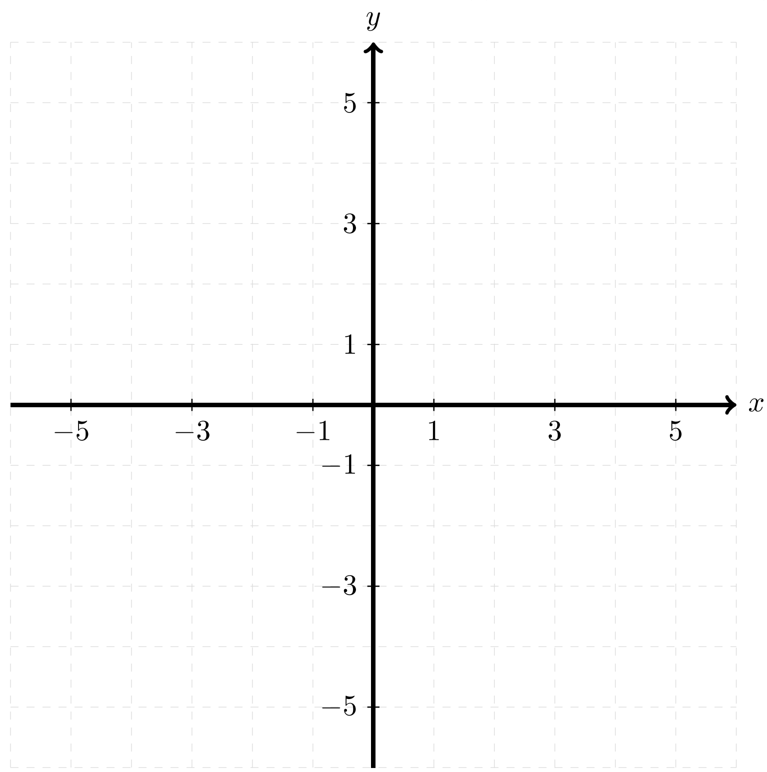
- If yes, find all minimum point(s) of $\Gamma$.
- If no, find all interval(s) of $k$ for which $\Gamma$ has a minimum point.

[5pts]

(d) It is given that the $x$-coordinate of the inflection point of $\Gamma : y = \dfrac{3x^2 - 2x - k}{(x-1)^2}$ is 4.

(i). Show that $k = 5$.
*(Testing for inflection point is not needed.)* [2pts]

(ii). It is given that the $x$-intercepts of $\Gamma$ are $-1$ and $\dfrac{5}{3}$. Sketch the graph of $\Gamma$ for $-6 \leq x \leq 6$, including its intercepts, asymptotes, extreme point(s) and inflection point(s). [4pts]




(iii). Find the area bounded by $\Gamma$, the $x$-axis and the vertical line $x = 4$. [3pts]