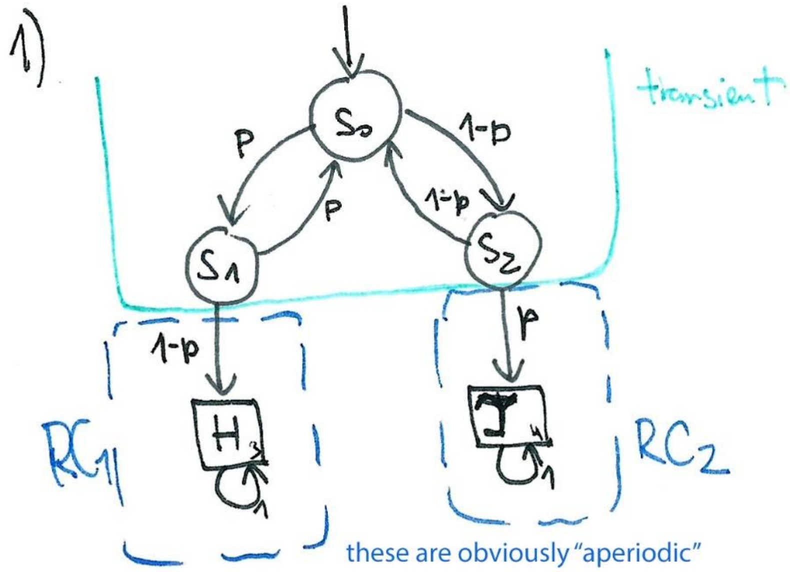
1)




I'm not even computing the unbounded reachability for T = { $\tau$ } because if $s_0$, $s_1$ and $s_2$ are transient, the probability of reaching { H, $\tau$ } is 1, because there are only two states in this set.
Therefore, when the probability of reaching one of them is 0.5 (as shown on this page), the probability of the other one must also be 0.5.

unbounded reachability:

$T = \{\boxed{H}\} \qquad S_0 = \{\boxed{\tau}\}$

$r(H) = 1$

$r(T) = 0$

$r(s_0) = p \cdot r(s_1) + (1-p) \cdot r(s_2)$

$r(s_1) = p \cdot r(s_0) + (1-p) \cdot r(H)$

$r(s_2) = \cancel{p \cdot r(T)} + (1-p) \cdot r(s_0)$

$\quad = (1-p) \cdot r(s_0)$

$r(s_0) = p \cdot r(s_1) + (1-p) \cdot (1-p) \cdot r(s_0)$

$r(s_1) = p \cdot r(s_0) + (1-p)$

$r(s_0) = \boxed{p \cdot p \cdot r(s_0)} + p \cdot (1-p) + \boxed{(1-p)^2 \cdot r(s_0)}$

$r(s_0) \cdot (p^2 + (1-p)^2 - 1) + p(1-p) = 0$

$r(s_0) \cdot (p^2 + \cancel{1} - 2p + p^2 \cancel{-1}) + p(1-p) = 0$

$$\boxed{\begin{aligned} r(s_0) &= \frac{-p(1-p)}{2p^2-2p} \\ &= \frac{p(1-p)}{2p(-1+p)} \\ &= +\frac{1}{2} \end{aligned}}$$ ✓

expected number of tosses until it terminates

let's use a reward model with $\underline{\underline{\rho \equiv 1}}$

we're computing the reachability reward for $T = \{H, \Upsilon\}$

1) Is $\mathbb{P}[F\{H, \Upsilon\}] = 1$?

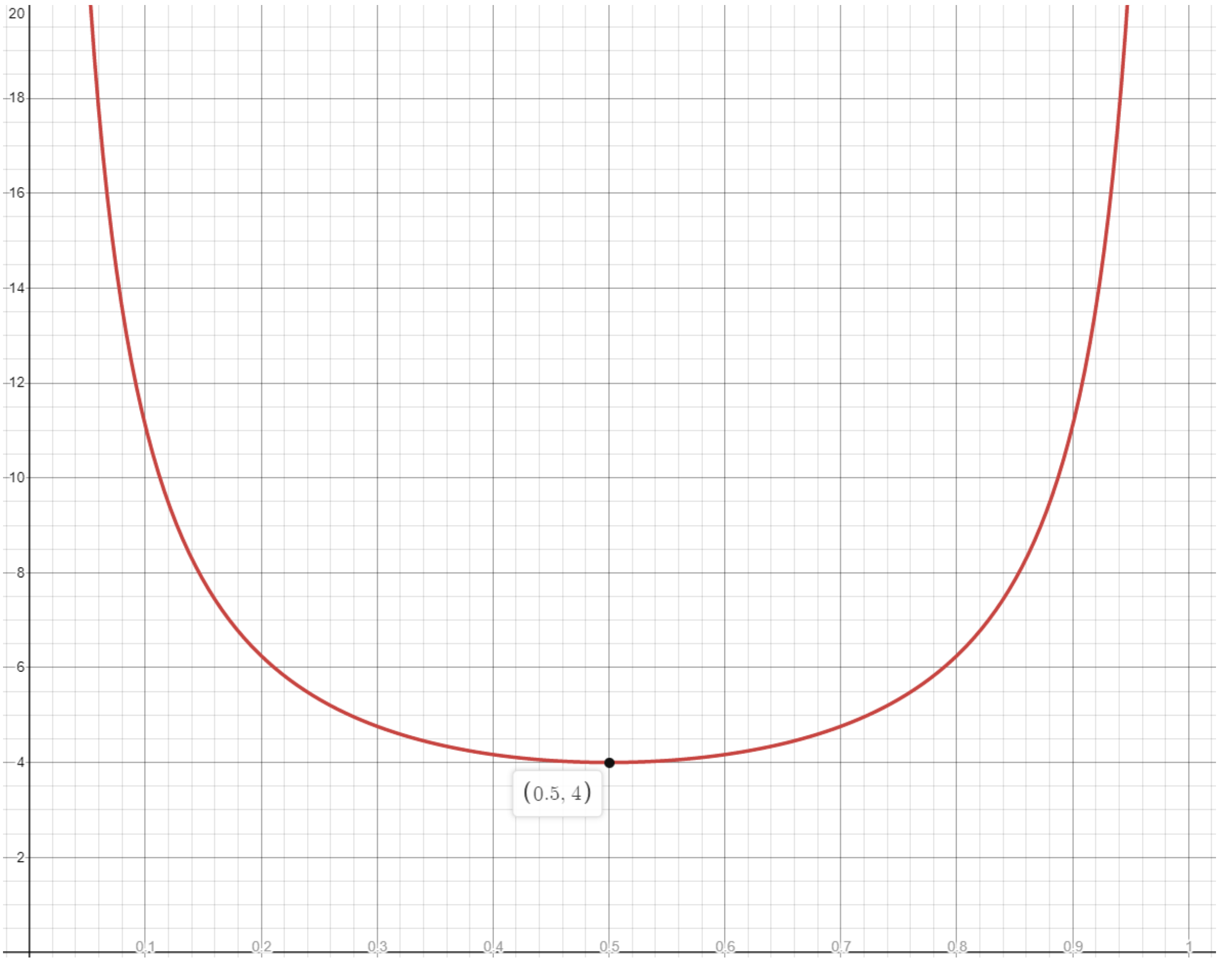
→ It has to be. We could spend a lot of time cycling between $s_0, s_1, s_2$, but in the end, we <u>always</u> reach either $H$ or $\Upsilon$ and when we get there, we cannot leave.

2)
$$e(H) = 0$$
$$e(\Upsilon) = 0$$
$$e(s_0) = 1 + p \cdot e(s_1) + (1-p) \cdot e(s_2)$$
$$e(s_1) = 1 + p \cdot e(s_0) + \cancel{(1-p) \cdot e(H)}$$
$$e(s_2) = 1 + (1-p) \cdot e(s_0) + \cancel{p \cdot e(\Upsilon)}$$

let $e(s_0) = e$

$$e = 1 + p \cdot (1 + p \cdot e) + (1-p) \cdot (1 + (1-p) \cdot e)$$
$$e = 1 + p + p^2 e + (1-p)(1 + e - ep)$$
$$e = 1 \boxed{\cancel{+ p}} + p^2 e + 1 + e - ep \boxed{\cancel{- p}} - pe + ep^2$$
$$\cancel{e} = 2 + 2p^2 e - 2pe + \cancel{e}$$
$$-2 = 2p^2 e - 2pe$$
$$-1 = e(p^2 - p)$$
$$e = -\frac{1}{p^2 - p} = \frac{1}{p - p^2} = \frac{1}{p(1-p)}$$

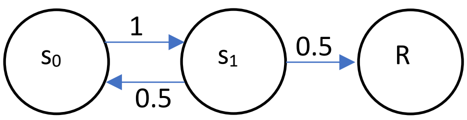
$$\underline{\underline{e(s_0) = \frac{1}{p(1-p)}}}$$




The plot of the resulting formula for the expected number of tosses $t(p) = e(s_0) = \frac{1}{p(1-p)}$ on the interval $[0, 1]$ ($p$ is a probability, so it must be from this interval) show us what we would expect: the "less fair" the coin is, the more tosses we need to end in one of the terminating states. The expected number of tosses until this protocol, which attempts to achieve fairness, could be viewed as a kind of "unfairness" indicator of the used coin.

It may be interesting to evaluate the meaning of the minimum of this function. Again, it comes as no surprise that the minimum occurs for $p = 0.5$: when the coin used to simulate a fair coin is fair itself, the "unfairness" for such coin is minimal.

However, I am mostly wondering about why the *value* of this minimum is *4*. Because we are only concerned about when this protocol *terminates*, we could "squash" the two terminating states into one state $R$ and analyse about the following chain:




I honestly don't know whether it is more painful to do this in Word or in LaTeX/TikZ.

My intuition about this says this, more or less: We always need **one** toss to get to $s_1$. Then, we need at least **one more** toss. However, in that toss, we could get back to $s_0$. Now we need **another** toss to get to $s_1$ again. Finally, because the coin is fair, in an "ideal world far away at the *limit* of everything", after the **next** toss, we would now get to $R$ at last, because at this point, we've "already seen" the side of the tossed coin that got us back to $s_0$, so now we must see the other side that leads to $R$.