Convex Optimization Assignment 1

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1 Question 1

If $\mu < 0$, then Ω_μ is empty, which is certainly convex.
If $\mu = 0$, then $\Omega_\mu = \{a\}$ is also convex.
If $\mu > 0$ and $\mu \neq 1$,

$$\begin{aligned}\Omega_\mu &= \{x \in R^n : ||x-a|| \leq \mu||x-b||\} \\ &= \{x \in R^n : ||x-a||^2 \leq \mu||x-b||^2\} \\ &= \{x \in R^n : (1-\mu^2)||x||^2 - 2 < x, a-\mu^2 b > +||a||^2 - \mu^2||b||^2 \leq 0\}\end{aligned}$$

If $0 < \mu < 1$, then

$$\Omega_\mu = \{x \in R^n : ||x - x_0||^2 \leq r\}$$

where

$$x_o = \frac{a-\mu^2 b}{1-\mu^2} \quad and \quad r = |\frac{\mu}{1-\mu^2}| \cdot ||a-b||$$

is a solid ball, which is certainly convex.

If $\mu > 1$, then

$$\Omega_\mu = \{x \in R^n : ||x - x_0||^2 \geq r\}$$

where

$$x_o = \frac{a-\mu^2 b}{1-\mu^2} \quad and \quad r = |\frac{\mu}{1-\mu^2}| \cdot ||a-b||$$

is a complement set of a ball, which is NOT convex.

If $\mu = 1$,

$$\Omega_\mu = \{x \in R^n :< x, a-b > \geq \frac{||a||^2 - ||b||^2}{2}\}$$

is a half-plane, which is convex.
To conclude, when $\mu \in (-\infty, 1]$, then Ω_μ is convex.

2 Question 2

$\forall(y,w)$ belonging to the dual cone of Ω, we have

$$< y,x > \; + wt \geq 0 \quad \forall (x,t) \in \Omega.$$

By generalized Cauchy-schwarz inequality, we can get

$$-||y||_\infty ||x||_1 + wt \geq 0$$

i.e.

$$||y||_\infty ||x||_1 \leq wt \quad \forall ||x|_1 \leq t.$$

Then we take $||x||_1 = t$, yielding

$$||y||_\infty \leq w.$$

Thus, $(y,w) \in \Omega^*$.

Conversely, for any $(y,w) \in \Omega^* = \{(y,w) \in R^n \times R : ||y||_\infty \leq w\}$, we have

$$< (y,w),\; (x,t) > = < y,x > \; + wt \geq -||y||_\infty ||x||_1 + wt \geq 0.$$

Thus, (y,w) belongs to the dual cone of Ω.

To conclude, the dual cone of Ω is $\Omega^* = \{(y,w) \in R^n \times R : ||y||_\infty \leq w\}$.

3 Question 3

$$\begin{aligned} C &= \{x \in R^n : x^T w^* \leq x^T w, \forall w \in \Omega\} \\ &= \{x \in R^n :< x, w^* - w > \leq 0, \forall w \in \Omega\} \end{aligned}$$

$\forall x \in C$ and $\forall \alpha \geq 0$, then we have

$$< \alpha x, w^* - w > \leq 0,\; \forall w \in \Omega$$

which implies C is a cone.

$\forall x_1, x_2 \in C$ and $\forall \lambda \in [0,1]$, we have:

$$< \lambda x_1 + (1-\lambda)x_2, w^* - w > = \lambda < x_1, w^* - w > + (1-\lambda) < x_2, w^* - w > \leq 0.$$

Thus C is convex.

To conclude, C is a convex cone.

4 Question 4

$\forall y, z \in \Omega$, by definition we have $\exists x_1, x_2$ such that

$$f_1(x_1) \le y_1, \cdots, f_m(x_1) \le y_m$$

and

$$f_1(x_2) \le z_1, \cdots, f_m(x_2) \le z_m$$

Then $\forall \lambda \in [0, 1]$, we have

$$f_i(\lambda x_1 + (1-\lambda)x_2) \le \lambda f_i(x_1) + (1-\lambda)f_i(x_2) = \lambda y_i + (1-\lambda)z_i,\ i = 1, 2, ..., m.$$

which means $\lambda y + (1-\lambda) \in \Omega$.

Thus, Ω is convex.

5 Question 5

According to Weiestrass Theorem for Closed Function, we know that f has its attainable lower bound over R^n. Thus, f cannot get to $-\infty$ over R^n.

Therefore, $\forall \alpha > 0$ strictly, let $||x|| \to \infty$, $\alpha||x||^2$ will go to ∞, which implies

$$f(x) + \alpha||x||^2 \to \infty,\ as\ ||x|| \to \infty.$$

Consequently, $f(X) + \alpha||x||^2$ is coercive.

6 Question 6

We will prove it by contradiction.

Assume that $\partial f(x^*)$ contains more than one element in it, i.e. $\partial f(x^*) \supset \{g_1, g_2\}$ where $g_1 \ne g_2$. Then we will yield a contradiction.

Since g is continuous at x^*, then given $0 < \varepsilon < ||g_1 - g_2||$, we can find a δ such that $g(y) \in B(g_1, \varepsilon),\ \forall y \in B(x^*, \delta)$.

Now take $y \in B(x^*, \delta)$, Then since $g_2 \in \partial f(x^*)$, we have

$$f(y) \ge f(x) + \ < g_2,\ y - x^* > .$$

Similarly, since $g(y) \in \partial f(y)$, then we have

$$f(x^*) \ge f(y) + \ < g(y),\ x^* - y > .$$

Add up the inequalities above together, we will get

$$< g(y) - g_2,\ x^* - y > \le 0,\ \forall y \in B(x^*, \delta).$$

However, we take $y \in B(x^*, \delta)$ s.t. $x^* - y = \frac{\delta}{2}(\frac{g_1 - g_2}{||g_1 - g_2||})$, then we will yield the contradiction as follows.

$$
\begin{aligned}
< g(y) - g_2,\ x^* - y > &= < g(y) - g_1 + g_1 - g_2,\ x^* - y > \\
&= < g(y) - g_1,\ \frac{\delta}{2}(\frac{g_1 - g_2}{||g_1\ g_2||}) > \ + \ < g_1 - g_2,\ \frac{\delta}{2}(\frac{g_1 - g_2}{||g_1 - g_2||}) > \\
&= < g(y) - g_1,\ \frac{\delta}{2}(\frac{g_1 - g_2}{||g_1 - g_2||}) > \ + \ \frac{\delta}{2}||g_1 - g_2|| \\
&\geq -\varepsilon \cdot \frac{\delta}{2} + \frac{\delta}{2}||g_1 - g_2|| \\
&= \frac{\delta}{2}(||g_1 - g_2|| - \varepsilon) \\
&> 0
\end{aligned}
$$

Thus, it is a contradiction to the previous statement

$$< g(y) - g_2, x^* - y > \leq 0,\ \forall y \in B(x^*, \delta).$$

Thus, the assumption failed, which means $\partial f(x^*)$ contains only single one element in it. Therefore, f is differentiable at x^*, i.e.

$$\partial f(x^*) = \{\nabla f(x^*)\}.$$

7 Question 7

If Ω is a set with only one element in it, then f is obviously affine over Ω.

If Ω contains more than one element in it, then $\forall x_1, x_2 \in \Omega$ and take $g \in \bigcap\limits_{x \in \Omega} \partial f(x)$, we will have

$$f(x_2) \geq f(x_1) + \ < g, x_2 - x_1 >$$

and

$$f(x_1) \geq f(x_2) + \ < g, x_1 - x_2 > .$$

Thus we have

$$< g, x_2 - x_1 > \geq f(x_2) - f(x_1) \geq < g, x_2 - x_1 > .$$

Consequently,

$$f(x_2) - f(x_1) \ = < g, x_2 - x_1 >, \quad \forall x_1, x_2 \in \Omega.$$

To conclude, f is affine over Ω.

8 Question 8

Denote $h(d) := f^{'}(x^*; d)$, then

$$\partial h(0) = \{g_1 \in R^n : h(d) \geq < g_1, d > \quad \forall d \in R^n\} = \{g_1 \in R^n : f^{'}(x^*; d) \geq g_1 d \quad \forall d \in R^n\}$$

Take arbitrary $g_1 \in \partial h(0)$, we have

$$< g_1, d > \leq \lim_{\alpha \to 0} \frac{f(x^* + \alpha d) - f(x^*)}{\alpha} \quad \forall d \in R^n$$

Let $d = y - x^*$ where y is arbitrary, then we get

$$< g_1, (y - x^*) > \leq \lim_{\alpha \to 0} \frac{f(\alpha y + (1-\alpha)x^*) - f(x^*)}{\alpha} \leq \lim_{\alpha \to 0} \frac{\alpha f(y) - \alpha f(x^*)}{\alpha} = f(y) - f(x^*)$$

i.e.

$$f(y) \geq f(x^*) + < g_1, (y - x^*) > \quad \forall y \in R^n.$$

Thus, $g_1 \in \partial f(x^*)$, which means

$$\partial h(0) \subseteq \partial f(x^*).$$

For the converse direction: $\forall g_2 \in \partial f(x^*)$, we have

$$f(y) \geq f(x^*) + < g_2, (y - x^*) > \quad \forall y \in R^n.$$

Thus, we have

$$h(y) - h(0) = f^{'}(x^*, y) = \lim_{\alpha \to 0} \frac{f(x^* + \alpha y) - f(x^*)}{\alpha} \geq \lim_{\alpha \to 0} \frac{< g, \alpha y >}{\alpha} = < g, y >$$

which means

$$\partial h(0) \supseteq \partial f(x^*).$$

To conclude,

$$\partial h(0) = \partial f(x^*).$$

9 Question 9

Since $y_i \geq 0$ and f_i is convex for $i = 1, 2, ..., m$, then we have

$$\sum_{i=1}^{m} y_i f_i$$

is convex and closed.

Then because supremum keeps closeness and convexity, so $f(x) = \sup_{y=(y_1,...,y_m)\in\Omega} \sum_{i=1}^{m} y_i f_i(x)$ is still convex and closed.

10 Question 10

11 Question 11

11.1 (a)

First we check $R(\Omega)$ is a cone.

$\forall d \in R(\Omega)$, we have:

$$\forall \alpha \geq 0,\ \Omega + \alpha d \subseteq \Omega.$$

Then $\forall k \geq 0$,

$$\forall \alpha \geq 0,\ \Omega + \alpha(kd) \subseteq \Omega.$$

Thus,

$$kd \in R(\Omega),\ \forall k \geq 0.$$

which means $R(\Omega)$ is a cone.

Secondly, we testify it is convex.
$\forall d_1, d_2 \in R(\Omega)$ and $\forall \lambda \in [0,1]$, we have:

$$\forall \alpha \geq 0,\ \Omega + \alpha(\lambda d_1 + (1-\lambda)d_2) = (\Omega + \alpha\lambda d_1) + \alpha(1-\lambda)d_2 \subseteq \Omega + \alpha(1-\lambda)d_2 \subseteq \Omega.$$

Thus, $R(\Omega)$ is convex.

Finally, we check it is closed.
Given a sequence $\{d_n\} \subset R(\Omega)$ and $d_n \to d$, we want to verify d also belongs to $R(\Omega)$.
$\forall x \in \Omega + \alpha d, \exists \omega \in \Omega$ such that

$$x = \omega + \alpha d = \omega + \alpha(d - d_n + d_n) = (\omega + \alpha d_n) + \alpha(d - d_n).$$

Since $d_n \in R(\Omega)$ which implies $\omega + \alpha d_n \in \Omega$, yielding:

$$d(x, \Omega) = \inf_{y \in \Omega}\{d(x,y)\} \leq d(x, \omega + \alpha d_n) = ||\alpha(d - d_n)||.$$

Let $n \to \infty$, then

$$d(x, \Omega) \to 0.$$

which means $x \in \Omega$.
Thus, we obtain

$$\Omega + \alpha d \subseteq \Omega.\ (i.e., d \in R(\Omega))$$

Therefore, $R(\Omega)$ is closed.
To conclude, $R(\Omega)$ is a closed convex cone.

11.2 (b)

If $d \in R(\omega)$, then by definition, we can easily get $\exists \omega \in \Omega$ s.t. $\omega + \alpha d \in \Omega, \forall \alpha \geq 0$.

Conversely, if $\exists \omega_0 \in \Omega$ s.t. $\omega_0 + \alpha d \in \Omega, \forall \alpha \geq 0$, we want to prove $d \in R(\Omega)$.
$\forall \omega_1 \in \Omega$ and given arbitrary $\lambda \in [0,1]$, from the convexity of Ω we can obtain:

$$\lambda(\omega_0 + \alpha d) + (1-\lambda)\omega_1 \in \Omega.$$

Let $\alpha = \frac{\beta}{\lambda}$, then we get

$$\lambda\omega_0 + (1-\lambda)\omega_1 + \beta d \in \Omega$$

Let $\lambda \to 0$, we finally obtain

$$\omega_1 + \beta d \in \Omega,\ \forall \beta \geq 0.$$

Since ω_1 is arbitrarily chosen in Ω, thus $\Omega + d \subseteq \Omega$ which means $d \in R(\Omega)$. So we have completed the proof in the converse direction.

Actually, we can also prove the converse direction by contradiction.

Assume that $d \notin R(\Omega)$, then $\exists \omega_2 \in \Omega, \beta \geq 0$ s.t. $\omega_2 + \beta d \notin \Omega$. Given the premise $\omega_0 + \alpha d \in \Omega,\ \forall \alpha \geq 0$, and by the convexity of Ω, we have:

$$\lambda(\omega_0 + \alpha d) + (1-\lambda)\omega_2 \in \Omega.$$

Let $\alpha = \frac{\beta}{\lambda}$, we get:

$$\lambda\omega_0 + (1-\lambda)\omega_2 + \beta d \in \Omega.$$

Then we let $\lambda \to 0$, yielding:

$$\omega_2 + \beta d \in \Omega$$

which is a contradiction to $\omega_2 + \beta d \notin \Omega$.

Consequently, $d \in R(\Omega)$ is correct.

11.3 (c)

We first verify $R(\cap_{i \in I}\Omega_i) \subseteq \cap_{i \in I} R(\Omega_i)$.

$\forall d \in R(\cap_{i \in I}\Omega_i)$,then

$$\exists \omega \in \cap_{i \in I}\Omega_i\ s.t.\ \omega + \alpha d \in \cap_{i \in I}\Omega_i,\ \forall \alpha \geq 0.$$

Since $\omega \in \Omega_i$ and $\omega + \alpha d \in \Omega_i\ \forall \alpha$, we obtain:

$$d \in R(\Omega_i).$$

which yields $d \in \cap_{i \in I} R(\Omega_i)\ i.e.\ R(\cap_{i \in I}\Omega_i) \subseteq \cap_{i \in I} R(\Omega_i)$.

Now we complete the proof of the converse direction.

Since $\cap_{i \in I}\Omega_i \neq \emptyset$, then we can choose $\omega \in \cap_{i \in I}\Omega_i$. Then $\forall d \in \cap_{i \in I} R(\Omega_i)$, we have:

$$\omega + \alpha d \in \Omega_i,\ \forall i \in I$$

which follows: $\omega + \alpha d \in \cap_{i \in I}\Omega_i$.

Consequently, we get $d \in R(\cap_{i \in I}\Omega_i)\ i.e.\ R(\cap_{i \in I}\Omega_i) \supseteq \cap_{i \in I} R(\Omega_i)$.

To conclude, $R(\cap_{i \in I}\Omega_i) = \cap_{i \in I} R(\Omega_i)$.

11.4 (d)

If there exits $\{\omega_k\}_{k=1}^{\infty} \subset \Omega$ s.t. $\frac{\omega_k}{||\omega_k||} \to d_u$, then we take any ω_k from the sequence.
Since Ω is closed and $\frac{\omega_k}{||\omega_k||} \to d_u$, so we have

$$d_u \in \Omega.$$

Furthermore,

$$\alpha d_u \in \Omega,\ \forall \alpha \geq 0.$$

Then since Ω is convex, we will get

$$\frac{1}{2}\omega_k + \frac{1}{2}\alpha d_u \in \Omega$$

which implies

$$\omega_k + \alpha d_u \in \Omega,\ \forall \alpha \geq 0.$$

Therefore, from the conclusion of (b), we will obtain:

$$d_u \in R(\Omega).$$

For the other direction:

We will prove it by contradiction. Assume there does NOT exist any sequence $\{\omega_k\} \subset \Omega$ s.t. $\frac{\omega_k}{||\omega_k||} \to d_u$, then we can affirm that $d_u \notin \Omega$, because Ω is a closed set.
Then because $d_u \in R(\Omega)$, we can say that

$$\forall \omega \in \Omega\ \forall \alpha \geq 0,\ \omega + \alpha d_u \in \Omega.$$

Now take $\omega = 0 \in \Omega$ and $\alpha = 1$, we will get

$$d_u \in \Omega$$

which is a contradiction to $d_u \notin \Omega$.
Therefore, there must exist a sequence $\{\omega_k\} \in \Omega$ s.t. $\frac{\omega_k}{||\omega_k||} \to d_u$.

To conclude, the unit vector $d_u \in R(\Omega)$ iff there exits a sequence $\{\omega_k\} \subset \Omega$ s.t. $\frac{\omega_k}{||\omega_k||} \to d_u$.

12 Question 12

$\forall x_1, x_2 \in dom(F) = \cap_{i=1}^{m} dom(f_i)$ and $\forall \lambda \in [0, 1]$,

$$g \circ F(\lambda x_1 + (1-\lambda)x_2) = g \begin{bmatrix} f_1(\lambda x_1 + (1-\lambda)x_2) \\ \vdots \\ f_m(\lambda x_1 + (1-\lambda)x_2) \end{bmatrix}$$

$$(because\ g(u) \leq g(v),\ \forall u \leq v, then)$$

$$\leq g \begin{bmatrix} \lambda f_1(x_1) + (1-\lambda)f_1(x_2) \\ \vdots \\ \lambda f_m(x_1) + (1-\lambda)f_m(x_2) \end{bmatrix}$$

$$= g[\lambda F(x_1) + (1-\lambda)F(x_2)]$$

$$(because\ of\ the\ convexity\ of\ g)$$

$$\leq \lambda g(F(x_1)) + (1-\lambda)g(F(x_2))$$

$$= \lambda g \circ F(X_1) + (1-\lambda)g \circ F(x_2)$$

Conspicuously, $g \circ F$ is convex.

13 Question 13

13.1 (a)

Given $d \in R^n$, $\forall \alpha_1, \alpha_2 \in R$ and $\forall \lambda \in [0,1]$, we have:

$$\begin{aligned} \varphi_d(\lambda\alpha_1 + (1-\lambda)\alpha_2) &= f(\bar{x} + (\lambda\alpha_1 + (1-\lambda)\alpha_2)d) \\ &= f(\lambda(\bar{x} + \alpha_1 d) + (1-\lambda)(\bar{x} + \alpha_2 d)) \\ &\leq \lambda f(\bar{x} + \alpha_1 d) + (1-\lambda)f(\bar{x} + \alpha_2 d) \\ &= \lambda\varphi_d(\alpha_1) + (1-\lambda)\varphi_d(\alpha_2) \end{aligned}$$

Therefore, $\varphi_d(\alpha)$ is convex for any $d \in R^n$.

13.2 (b)

If $\bar{x}$ is the minimizer of f, then we check $\bar{\alpha} = 0$ is the minimizer of φ_d for all $d \in R^n$. We will verify it by contradiction.

If for some $d^* \in R^n$, and $\alpha^* \neq 0$ is the minimizer of φ_d, then yielding

$$f(\bar{x} + \alpha^* d^*) \leq f(\bar{x})$$

which is a contradiction to $\bar{x}$ is the minimizer of f. Thus, $\bar{\alpha} = 0$ must be the minimizer of φ_d for all $d \in R^n$.

Then, **for the converse direction:**

If $\bar{\alpha} = 0$ is the minimizer of φ_d for all $d \in R^n$, then we check $\bar{x}$ is the minimizer of f, also by contradiction.

Since f is convex, then the local minimizer of f must be its the global minimizer.

Assume $\bar{x}$ is NOT the minimizer of f, so $\bar{x}$ is not the local minimizer, which means there is some $y \in B(\bar{x}, \varepsilon)$ such that $f(y) \leq f(\bar{x})$. Then let $d = y - \bar{x}$, so by the definition, $\varphi_d(1) < \varphi_d(0)$, so $\bar{\alpha} = 0$ is not the minimizer of φ_d. Contradiction!

Thus, in this case, $\bar{x}$ must be the minimum of f.

To conclude, $\bar{x}$ is the minimizer of f is equivalent to $\bar{\alpha} = 0$ is the minimizer of φ_d for all $d \in R^n$.

14 Question 14

14.1 (a)

If $\bar{x}$ is the interior point of Ω, then there exists a neighbor $B(\bar{x}, \varepsilon)$ such that $B(\bar{x}, \varepsilon) \subset \Omega$. In this neighbor, we can easily find that

$$dist(x; \Omega) = 0, \ \forall x \in B(\bar{x}, \varepsilon) \subset \Omega,$$

which means

$$f_L(x) = f(x), \ \forall x \in B(\bar{x}, \varepsilon).$$

Since $\bar{x}$ is a local minimizer of f over Ω, then $\bar{x}$ is also a local minimizer of f_L over its domain.

When $\bar{x}$ is a boundary point of compact set Ω, the situation becomes a little sophisticated.

By Lipschitz continuity of f with parameter $L_f > 0$, we have:

$$f(y) \geq f(x) - L_f||y - x||, \ \forall x, y \in dom(f).$$

Then as for those $y \in B(\bar{x}, \varepsilon) \cap \Omega^C$, we want to check $f_L(y) \geq f_L(\bar{x})$ in order to prove $\bar{x}$ is the local minimizer of f_L. (For those points in $B(\bar{x}, \varepsilon) \cap \Omega$, we have already verified $f_L(\bar{x})$is the minimum of $f_L(B(\bar{x}, \varepsilon) \cap \Omega)$, as proved in the previous situation.)

Let x becomes the point satisfying

$$dist(y; \Omega) = ||y - x||$$

which means x is the closest point to y among Ω.

So we can obtain:

$$\begin{aligned}
f_L(y) - f_L(\bar{x}) &= f(x) - f(\bar{x}) + L \cdot dist(y; \Omega) \\
&\geq f(x) - L_f||y - x|| - f(\bar{x}) + L \cdot dist(y; \Omega) \\
&= f(x) - f(\bar{x}) + (L - L_f)||y - x|| \\
&\geq f(x) - f(\bar{x}) \\
&\geq 0.
\end{aligned}$$

Thus, $\forall x \in dom(f) = R^n$, we always have $f_L(\bar{x}) \leq f_L(x)$. Therefore, $\bar{x}$ is the local minimizer of f_L.

14.2 (b)

In order to prove $\bar{x}$ is a global minimizer of f over Ω, the only thing we need to check is

$$\bar{x} \in \Omega.$$

If we obtain $\bar{x} \in \Omega$, then since $\bar{x}$ is a global minimizer of f_L over R^n, we will have $\bar{x}$ is also a minimizer of f_L over Ω. Moreover, when constrained in Ω, we have $f_L(x) = f(x)$. Therefore, we can prove that $\bar{x}$ is also a global minimizer of f over Ω.

Now we begin to check $\bar{x} \in \Omega$.

Suppose $\bar{y}$ is the minimizer of f_L over Ω^C, and $\bar{z}$ is the minimizer of f_L over Ω. Then we have:

$$\begin{aligned}
f_L(\bar{y}) - f_L(\bar{z}) &= f(z^*) - f(\bar{z}) + L \cdot dist(\bar{y}; \Omega) \\
&\geq f(z^*) - L_f||\bar{y} - \bar{z}|| - f(\bar{z}) + L \cdot dist(\bar{y}; \Omega) \\
&= f(z^*) - f(\bar{z}) + (L - L_f)||\bar{y} - \bar{x}|| \\
&\geq f(z^*) - f(\bar{z}) \\
&\geq 0.
\end{aligned}$$

where z^* is the closest point to $\bar{y}$ among Ω.

So the global minimizer of f_L over R^n must be in Ω, which is to say $\bar{x} \in \Omega$.

Thus, the proof of "$\bar{x}$ is a global minimizer of f over Ω" has been completed.

15 Question 15

Since

$$E_f = \{(x, \alpha) : \ f(x) < \alpha\}$$

and

$$epi(f) = \{(x, \alpha) : \ f(x) \leq \alpha\}$$

Thus, $E_f \subset epi(f)$ is easily obtained.

Next, we want to prove $epi(f) \subset Cl(E_f)$.

Since $E_f \subset epi(f)$, then the only thing we need to check is

$$\{(x, \alpha) : f(x) = \alpha\} \subset cl(E_f).$$

Given ε, then for any (x, α) satisfying $f(x) = \alpha$, there always exists $(x, \alpha + \varepsilon) \in E_f$ s.t. $d((x, \alpha + \varepsilon), (x, \alpha)) \leq \varepsilon$. Hence, every point in $\{(x, \alpha) : f(x) = \alpha\}$ is an adhesion point of E_f, which is to say

$$\{(x, \alpha) : f(x) = \alpha\} \subset cl(E_f).$$

Thus,

$$epi(f) = (\{(x, \alpha) : f(x) = \alpha\} \cup E_f) \subset cl(E_f).$$

16 Question 16

16.1 (a)

$\forall (y_1, ..., y_m) \in (\prod_{i=1}^{m} \Omega_i)^\circ$, we have

$$< (y_1, ..., y_m), (x_1, ..., x_m) > \leq 0 \quad \forall (x_1, ..., x_n) \in \prod_{i=1}^{m} \Omega_i,$$

which means

$$< y_1, x_1 > + \cdots + < y_m, x_m > \leq 0 \quad \forall (x_1, ..., x_n) \in \prod_{i=1}^{m} \Omega_i.$$

Let all $x_j = 0$ except for x_i, then we get

$$< y_i, x_i > \leq 0 \quad \forall x_i \in \Omega_i.$$

Obviously, $y_i \in \Omega_i^\circ$.
Thus, $(y_1, ..., y_m) \in \prod_{i=1}^{m} \Omega_i^\circ$, which implies

$$(\prod_{i=1}^{m} \Omega_i)^\circ \subseteq \prod_{i=1}^{m} \Omega_i^\circ.$$

For the other direction:
$\forall (y_1, ..., y_m) \in \prod_{i=1}^{m} \Omega_i^\circ$, we have

$$< y_i, x_i > \leq 0 \quad x_i \in \Omega_i$$

Then $\forall (x_1, ..., x_m) \in \prod_{i=1}^{m} \Omega_i$, we can obtain

$$< (y_1, ..., y_m), (x_1, ..., x_m) > = < y_1, x_1 > + \cdots + < y_m, x_m > \leq 0.$$

Thus, $(y_1, ..., y_m) \in (\prod_{i=1}^{m} \Omega_i)^\circ$, which implies

$$(\prod_{i=1}^{m} \Omega_i)^\circ \supseteq \prod_{i=1}^{m} \Omega_i^\circ.$$

To conclude, $(\prod_{i=1}^{m} \Omega_i)^\circ = \prod_{i=1}^{m} \Omega_i^\circ$.

16.2 (b)

$\forall y \in (\bigcup_{i \in I} \Omega_i)^\circ$, we have

$$< y, x > \leq 0 \quad \forall x \in \bigcup_{i \in I} \Omega_i$$

yielding

$$< y, x > \leq 0 \ \forall x \in \Omega_i \quad i \in I.$$

Then we obtain

$$y \in \Omega_i^\circ.$$

Hence,

$$y \in \bigcap_{i \in I} \Omega_i^\circ$$

which means

$$(\bigcup_{i \in I} \Omega_i)^\circ \subseteq \bigcap_{i \in I} \Omega_i^\circ.$$

For the other direction:

$\forall y \in \bigcap_{i \in I} \Omega_i^\circ$, we have

$$y \in \Omega_i^\circ \quad \forall i \in I.$$

Then we get

$$< y, x > \leq 0 \quad \forall x \in \Omega_i \quad \forall i \in I$$

yielding

$$< y, x > \leq 0 \quad \forall x \in \bigcup_{i \in I} \Omega_i$$

which means

$$y \in (\bigcup_{i \in I} \Omega_i)^\circ.$$

Thus,

$$(\bigcup_{i \in I} \Omega_i)^\circ \supseteq \bigcap_{i \in I} \Omega_i^\circ.$$

To conclude, $(\bigcup_{i \in I} \Omega_i)^\circ = \bigcap_{i \in I} \Omega_i^\circ$.

16.3 (c)

$\forall y \in (\Omega_1 + \Omega_2)^\circ$, we have

$$< y, x_1 + x_2 > = < y, x_1 > + < y, x_2 > \leq 0 \quad \forall x_1 \in \Omega_1 \quad \forall x_2 \in \Omega_2.$$

Let $x_1 = 0$, then we get $< y, x_2 > = 0 \quad \forall x_2 \in \Omega_2$. Similarly, let $x_2 = 0$, then we get $< y, x_1 > = 0 \quad \forall x_1 \in \Omega_1$. Thus, $y \in \Omega_1^\circ$ and $y \in \Omega_2^\circ$.

Then we get

$$y \in \Omega_1^\circ \cap \Omega_2^\circ,$$

which is to say

$$(\Omega_1 + \Omega_2)^\circ \subseteq \Omega_1^\circ \cap \Omega_2^\circ.$$

For the other direction:

$\forall y \in \Omega_1^\circ \cap \Omega_2^\circ$, we have

$$< y, x_i > \leq 0 \quad \forall x_i \in \Omega_i \quad i = 1, 2.$$

Consequently, we can obtain:

$$< y, x_1 + x_2 > = < y, x_1 > \ + \ < y, x_2 > \leq 0 \quad \forall x_1 \in \Omega_1 \quad \forall x_2 \in \Omega_2.$$

Thus, $y \in (\Omega_1 + \Omega_2)^\circ$, which implies

$$(\Omega_1 + \Omega_2)^\circ \supseteq \Omega_1^\circ \cap \Omega_2^\circ.$$

To conclude, $(\Omega_1 + \Omega_2)^\circ = \Omega_1^\circ \cap \Omega_2^\circ$.

17 Question 17

Since $int(\Omega^\circ)$ is not empty, then we can choose $y_1, y_2 \in \Omega^\circ$ such that $y_2 \neq \alpha y_1$ where $\alpha \neq 0$ strictly. (Otherwise Ω° is $\{0\}$ or affine, whose interior is empty. Contradiction!)

Let $p = -(y_1 + y_2) \neq 0$, and for any $x \in \Omega$, we have $\frac{x}{||x||} \in \Omega$.

Then we want to check

$$< p, \frac{x}{||x||} > > \ \delta \quad \forall x \in \Omega$$

for some $\delta > 0$ strictly.

First we check $< p, \frac{x}{||x||} > > \ 0$ by contradiction.

Because $y_1, y_2 \in \Omega^\circ$ and $p = -(y_1 + y_2)$, then we must have

$$< p, \frac{x}{||x||} > \geq 0 \quad \forall x \in \Omega.$$

If there exits some $x \in \Omega$ s.t. $< p, \frac{x}{||x||} > = \ 0$ which means $< y_1, \frac{x}{||x||} > = < y_2, \frac{x}{||x||} >= 0$ at the same time. Thus y_1 and y_2 are in the same direction. Obviously it is a contradiction to our choice for y_1 and y_2: $y_2 \neq \alpha y_1$ where $\alpha \neq 0$.

Thus, $< p, \frac{x}{||x||} > > \ 0$ has been checked correct.

Next, we want to show also by contradiction

$$\inf_{x \in \Omega} < p, \frac{x}{||x||} > > 0.$$

Assume $\inf\limits_{x \in \Omega} < p, \frac{x}{||x||} > = 0$, then for any $\varepsilon > 0$, $\exists x_0 \in \Omega$ such that

$$< p, \frac{x_0}{||x_0||} > < \varepsilon.$$

So there must have $0 \geq < y_1, \frac{x_0}{||x_0||} > > -\varepsilon$ and $0 \geq < y_2, \frac{x_0}{||x_0||} > > -\varepsilon$.

Then let $\varepsilon \to 0$, we have

$$< y_1, \frac{x_0}{||x_0||} > = < y_2, \frac{x_0}{||x_0||} > = 0$$

yielding y_1 and y_2 are in the same direction, which is again a contradiction to $y_2 \neq \alpha y_1$ where $\alpha \neq 0$.

Consequently, $\inf\limits_{x \in \Omega} < p, \frac{x}{||x||} > > 0$ strictly.

Then we let $\delta = \inf\limits_{x\in\Omega} < p, \frac{x}{||x||} > > 0$ strictly, we will obtain

$$< p, \frac{x}{||x||} > > \delta \quad \forall x \in \Omega$$

i.e.

$$< p, x > > \delta||x|| \quad \forall x \in \Omega.$$

18 Question 18

18.1 (a)

$\partial f(x) = \{g : f(y) \geq f(x) + g(y-x), \quad \forall y \in R\} = \bigcap\limits_{y\in R} \{g : f(y) \geq f(x) + g(y-x)\}$ is an intersection of closed sets, since intersection keeps the closedness of sets, thus $\partial f(x)$ is still a closed set. Then we turn to its boundedness.

Since for any $g \in \partial f(x)$, we always have

$$f(y) - f(x) \geq g(y-x).$$

Then when $y \geq x$, we get

$$g \leq \frac{f(y)-f(x)}{y-x}.$$

Let $y \to x^+$, then we have

$$g \leq f^{'}(x^+).$$

Similarly, when $y \leq x$, we get

$$g \geq f^{'}(x^-).$$

Therefore, $\partial f(x) \subseteq [f^{'}(x^-), f^{'}(x^+)]$ is a bounded set.

To conclude, $\partial f(x) \subset R$ is a bounded closed set, so it is a compact interval.

However, the proof above of boundedness of $\partial f(x)$ requires the existence of $f^{'}(x^-)$ and $f^{'}(x^+)$ (though they do exist with proof in (b)). Now we give another proof, without knowing the existence of $f^{'}(x^-)$ and $f^{'}(x^+)$:

There is a theorem, saying:

$f : R^n \to \bar{R}$ is proper and convex, $X \subset int(dom\ f)$ is nonempty and compact, then $\bigcup\limits_{x\in X} \partial f(x)$ is nonempty and compact.

In this case, $X = \{x\} \subset int(dom\ f)$ is obviously nonempty and compact, so we can conclude that $\partial f(x)$ is compact.

18.2 (b)

First we need to check that $f^{'}(x;1)$ and $f^{'}(x;-1)$ exist. Since there is a theorem, saying:

$f : R^n \to R$ is proper and convex, $x \in int(dom\ f)$, then $\forall d$ we have $f^{'}(x;d)$ exists.

Since for any $g \in \partial f(x)$, we always have

$$f(y) - f(x) \geq g(y-x).$$

Then when $y \geq x$, we get

$$g \leq \frac{f(y) - f(x)}{y-x}.$$

Let $y = x + \alpha$ where $\alpha > 0$, then let $\alpha \to 0$ we have

$$g \leq \lim_{\alpha \to 0} \frac{f(x+\alpha) - f(x)}{\alpha} = f^{'}(x;1)$$

Similarly, when $y \leq x$, we get

$$g \geq \frac{f(y) - f(x)}{y-x}.$$

Let $y = x - \alpha$ where $\alpha > 0$, then let $\alpha \to 0$ we have

$$g \geq \lim_{\alpha \to 0} \frac{f(x-\alpha) - f(x)}{\alpha} = -f^{'}(x;-1)$$

Thus, we have proved

$$\partial f(x) \subseteq \{g \in R : -f^{'}(x;-1) \leq g \leq f^{'}(x;1)\}.$$

For the other direction:

Because f is convex, then given x fixed we will have

$$h(y) = \frac{f(y) - f(x)}{y-x}$$

is non-decreasing.

Then for y satisfying $y \geq x$, we have

$$g \leq f^{'}(x;1) = \lim_{y_0 \to x} h(y_0) \leq h(y) = \frac{f(y) - f(x)}{y-x}, \quad \forall y \geq x$$

which indicates

$$f(y) \geq f(x) + g(y-x), \quad \forall y \geq x.$$

Similarly for y satisfying $y \leq x$, we have

$$g \geq -f^{'}(x;-1) = \lim_{y_0 \to x} h(y_0) \geq h(y) = \frac{f(y) - f(x)}{y-x}, \quad \forall y \leq x$$

which indicates

$$f(y) \geq f(x) + g(y-x), \quad \forall y \leq x.$$

Therefore, we finally obtain:

$$f(y) \geq f(x) + g(y-x), \quad \forall y \in R$$

yielding $g \in \partial f(x)$ i.e.,

$$\partial f(x) \supseteq \{g \in R : -f^{'}(x;-1) \leq g \leq f^{'}(x;1)\}.$$

To conclude, $\partial f(x) = \{g \in R : -f^{'}(x;-1) \leq g \leq f^{'}(x;1)\}$ has been proved.

19 Question 19

If $g \in \partial f(x)$, then we have

$$f(y) \geq f(x) + < g, y - x > \quad \forall y$$

Certainly we have

$$f(y) \geq f(x) + < g, y - x > -\varepsilon \quad \forall y$$

which means $g \in \partial_\varepsilon f(x)$.

Thus, $\partial f(x) \subseteq \cap_{\varepsilon > 0} \partial_\varepsilon f(x)$ has been proved.

For the next direction: $\forall g \in \cap_{\varepsilon > 0} \partial_\varepsilon f(x)$, we have

$$\varepsilon \geq f(x) + < g, y - x > -f(y) \quad \forall y \in R^n \quad \forall \varepsilon > 0.$$

Then we prove $g \in \partial f(x)$ by contradiction.

Assume $g \notin \partial f(x)$, i.e. $f(x) + < g, y - x > -f(y) > 0$ strictly for some $y \in R^n$. Then we can find ε_0 such that

$$f(x) + < g, y - x > -f(y) > \varepsilon_0.$$

However, let $\varepsilon = \frac{\varepsilon_0}{2}$, we have already known that

$$f(x) + < g, y - x > -f(y) \leq \frac{\varepsilon_0}{2}.$$

Here comes the contradiction!

Therefore, g must belong to $\partial f(x)$, which is to say, $\partial f(x) \supseteq \cap_{\varepsilon > 0} \partial_\varepsilon f(x)$.

To conclude, $\partial f(x) = \cap_{\varepsilon > 0} \partial_\varepsilon f(x)$ has been verified.

20 Question 20

We want to check f^* is lower semi-continuous i.e. $\forall y_0 \in R^m$, $\forall y_n \to y_0$, we always have

$$f^*(y_0) \leq \lim_{n \to \infty} \inf f^*(y_n) = \lim_{n \to \infty} \inf(\inf_{x \in F(y_n)} f(x, y_n)).$$

Let x_n satisfy

$$f(x_n, y_n) = \inf_{x \in F(y_n)} f(x, y_n).$$

Since $x_n \in F(y_n)$ and $F(y_n)$ is compact for all y_n, then we have the sequence x_n belongs to a compact set, which means there must be a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a so called x_0, i.e.,

$$x_{n_k} \to x_0.$$

Then because of the outer semi-continuity of F, then we have $x_0 \in F(y_0)$.

Thus, we get

$$f(x_0, y_0) = \inf_{x \in F(y_0)} f(x, y_0).$$

Since f is lower semi-continuous, we obtain

$$f(x_0, y_0) \leq \lim_{n\to\infty} \inf f(x_n, y_n).$$

Finally, we have

$$\begin{aligned}
f^*(y_0) &= \inf_{x\in F(y_0)} f(x, y_0) \\
&= f(x_0, y_0) \\
&\leq \lim_{n\to\infty} \inf f(x_n, y_n) \\
&= \lim_{n\to\infty} \inf(\inf_{x\in F(y_n)} f(x, y_n)) \\
&= \lim_{n\to\infty} \inf f^*(y_n) \quad \forall y_n \to y_0
\end{aligned}$$

Therefore, $f^*(y)$ is lower semi-continuous.