#1.

$$H = B^T (BAB^T)^{-1} B$$

$$\frac{\partial H}{\partial x} = B^T \frac{\partial (BAB^T)^{-1}}{\partial x} B = B^T\left[-(BAB^T)^{-1}\right] \frac{\partial (BAB^T)}{\partial x} (BAB^T)^{-1} B$$

$$= -B^T (BAB^T)^{-1} B \frac{\partial A}{\partial x} B^T (BAB^T)^{-1} B = -H \frac{\partial A}{\partial x} H$$

综上，$\frac{\partial H}{\partial x} = -H \frac{\partial A}{\partial x} H$

#2. (a)

令 $B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$，则 $BX = \begin{bmatrix} X_2 \\ 2X_1 - X_3 \end{bmatrix}$

又 $\because X \sim N(\mu, \Sigma)$，$\therefore BX \sim N(B\mu, B\Sigma B^T)$

即 $BX \sim N\left(\begin{bmatrix} -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 34 \end{bmatrix}\right)$ 即 $\begin{bmatrix} X_2 \\ 2X_1 - X_3 \end{bmatrix} \sim N\left(\begin{bmatrix} -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 34 \end{bmatrix}\right)$

$\because (B\Sigma B^T)_{12} = (B\Sigma B^T)_{21} = 0$

$\therefore X_2$ 与 $2X_1 - X_3$ 互相独立.

(b) 令 $A = \begin{bmatrix} 2 & 0 & -5 \\ 1 & 1 & 0 \end{bmatrix}$，则 $AX = \begin{bmatrix} 2X_1 - 5X_3 \\ X_1 + X_2 \end{bmatrix}$

$\because X \sim N(\mu, \Sigma)$，$\therefore AX \sim N(A\mu, A\Sigma A^T)$

即 $\begin{bmatrix} 2X_1 - 5X_3 \\ X_1 + X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 130 & 25 \\ 25 & 14 \end{bmatrix}\right)$

(c) $\because \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 2 \end{pmatrix}\right)$ $\therefore \begin{pmatrix} X_3 \\ X_2 \\ X_1 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -3 \\ 0 & 9 & 0 \\ -3 & 0 & 5 \end{pmatrix}\right)$

$\therefore X_3 \mid X_2 = -2, X_1 = 1 \sim N\left(0 + [0, -3] \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix}^{-1} \begin{pmatrix} -2+2 \\ 1-3 \end{pmatrix}, 2 - [0, -3] \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -3 \end{bmatrix}\right)$

即 $X_3 \mid X_2 = -2, X_1 = 1 \sim N\left(\frac{6}{5}, \frac{1}{5}\right)$

#3.

$$\sum_{i=1}^{3}(Y_i-\bar{Y})^2=\sum_{i=1}^{3}(Y_i^2-2Y_i\bar{Y}+\bar{Y}^2)=\sum_{i=1}^{3}Y_i^2-3\bar{Y}^2$$

$$\therefore E\sum_{i=1}^{3}(Y_i-\bar{Y})^2=\sum_{i=1}^{3}EY_i^2-3E(\bar{Y}^2)=\sum_{i=1}^{3}\left[Var(Y_i)+(EY_i)^2\right]-\frac{1}{3}E(Y_1+Y_2+Y_3)^2$$

$$=(2+4+1+9+3+16)-\frac{1}{3}E(Y^TAY)$$

$$=35-\frac{1}{3}E(Y^TAY)$$

其中 $A=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$, $\therefore E(Y^TAY)=tr(A\Sigma)+\mu^TA\mu=tr\begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix}+81=87$

$$\therefore E\sum_{i=1}^{3}(Y_i-\bar{Y})^2=35-\frac{1}{3}\times 87=35-29=6$$

#4.

~~当~~ $i<j$时

$$E(Y_i-Y_j)^2=E(Y_i^2+Y_j^2-2Y_iY_j)=EY_i^2+EY_j^2-2EY_iY_j$$

$$=\sigma^2+\mu^2+\sigma^2+\mu^2-2\left[Cov(Y_i,Y_j)+(EY_i)(EY_j)\right]$$

$$=2\sigma^2+2\mu^2-2\mu^2=2\sigma^2$$

$$\therefore E\sum_{i<j}(Y_i-Y_j)^2=\sum_{i<j}E(Y_i-Y_j)^2=\sum_{i<j}2\sigma^2=2\cdot\frac{(1+n-1)(n-1)}{2}\sigma^2=n(n-1)\sigma^2$$

即 $EU=n(n-1)\sigma^2$

若 $k=\frac{1}{n(n-1)}$, 则 $E(kU)=\sigma^2$, 即 $\frac{1}{n(n-1)}U$ 是 σ^2 的无偏估计量.