Time Series Assignment 2

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1

(a)

When $\phi = 0$, $X_t = W_t$ is trivial.

When $|\phi| < 1$ and $\phi \neq 0$, $$

$$
\begin{aligned}
X_t - \phi X_{t-1} &= c\phi^t + \sum_{j=0}^{+\infty} \phi^j W_{t-j} - \phi(c\phi^{t-1} + \sum_{j=0}^{+\infty} \phi^j W_{t-1-j}) \\
&= \sum_{j=0}^{+\infty} \phi^j W_{t-j} - \sum_{j=0}^{+\infty} \phi^{j+1} W_{t-1-j} \\
Let \quad k = j+1, & \\
&= \sum_{j=0}^{+\infty} \phi^j W_{t-j} - \sum_{k=1}^{+\infty} \phi^k W_{t-k} \\
&= W_t
\end{aligned}
$$

$$

which is to say,

$$X_t - \phi X_{t-1} = W_t$$

Thus, $X_t = c\phi^t + \sum_{j=0}^{+\infty} \phi^j W_{t-j}$ is a solution to the difference equation $X_t - \phi X_{t-1} = W_t$ with $|\phi| < 1$ for every real number c.

(b)

$$\mathbb{E}(X_t) = \mathbb{E}(c\phi^t + \sum_{j=0}^{+\infty} \phi^j W_{t-j}) = \mathbb{E}(c\phi^t) = c\phi^t.$$

The expectation depends on t if $c \neq 0$, thus $X_t = c\phi^t + \sum_{j=0}^{+\infty} \phi^j W_{t-j}$ is non-stationary when $c \neq 0$.

2

(a)

Because Y_t is stationary, its characteristic function $\phi(z)$ has no roots with length equaling 1. Since the polynomial is exchangebable, i.e.,

$$(1-B)^k \phi(B) = \phi(B)(1-B)^K,$$

Thus the characteristic function of $(1-B)^k Y_t$ is still $\phi(z)$, which means it also has no roots with length equaling 1, hence $(1-B)^k Y_t$ is stationary for all $k \geq 1$.

(b)

From (a) we have already verified that $(1-B)^k Y_t$ is stationary for all $k \geq 1$, then to check whether $(1-B)^k X_t$ (where $X_t = \beta_0 + \beta_1 t + \cdots + \beta^q t^q + Y_t$) is stationary is necessarily equivalent to check $(1-B)^k(\beta_0 + \beta_1 t + \cdots + \beta^q t^q)$ is stationary.

- When $k < q$, notice that $(1-B)^k \beta^q t^q = c\beta^q t^{q-k}$ with c is a non-zero constant. Thus, the expectation $\mathbb{E}[(1-B)^k X_t]$ still depends on time t, so it cannot be stationary.
- when $k \geq q$, notice that $(1-B)^k t^n = c_1$ for all $n \leq k$ where c_1 is a constant, which indicates that $(1-B)^k(\beta_0 + \beta_1 t + \cdots + \beta^q t^q)$ where c_2 is a constant. Thus, $(1-B)^k(\beta_0 + \beta_1 t + \cdots + \beta^q t^q)$ is stationary. Therefore, $(1-B)^k X_t$ is stationary.

To conclude, $(1-B)^k X_t$ is stationary when $k < q$ and is non-stationary when $k \geq q$.

3

(a)

We first prove a proposition:

$$\cos(\alpha+\beta)\cos(\alpha)+\sin(\alpha+\beta)\sin(\alpha)=\cos(\beta).$$

$$\begin{aligned} L.H.S. &= [\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)]\cos(\alpha)+[\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)]\sin(\alpha) \\ &= \cos^2(\alpha)\cos(\beta)+\sin^2(\alpha)\cos(\beta) \\ &= \cos(\beta) \\ &= R.H.S. \end{aligned}$$

The proposition has been verified, then we calculate the autocovariance.

$$\begin{aligned} & Cov(X_{t+h}, X_t) \\ &= Cov(U_1\cos(2\pi w_1(t+h))+V_1\sin(2\pi w_1(t+h))+U_2\cos(2\pi w_2(t+h))+V_2\sin(2\pi w_2(t+h)), \\ & U_1\cos(2\pi w_1 t)+V_1\sin(2\pi w_1 t)+U_2\cos(2\pi w_2 t)+V_2\sin(2\pi w_2 t)) \\ &= \sigma^2(\cos(2\pi w_1(t+h))\cos(2\pi w_1 t)+\sin(2\pi w_1(t+h))\sin(2\pi w_1 t)+ \\ & \cos(2\pi w_2(t+h))\cos(2\pi w_2 t)+\sin(2\pi w_2(t+h))\sin(2\pi w_2 t)) \\ By\ the\ proposition\ proved, & \\ &= \sigma^2[\cos(2\pi w_1 h)+\cos(2\pi w_2 h)]. \end{aligned}$$

Hence, autocovariance does not depend on t.

Additionally, $\mathbb{E}X_t = 0$ also does not depend on t.

Therefore, we can conclude that X_t is weakly stationary.

(b)

Since X_t is weakly stationary and as we have already calculated the autocovariance above, thus we have

$$\gamma(h) = Cov(X_{t+h}, X_t) = \sigma^2[\cos(2\pi w_1 h)+\cos(2\pi w_2 h)].$$

Therefore,

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\cos(2\pi w_1 h)+\cos(2\pi w_2 h)}{2}.$$

4

(a)

$$

$$\begin{aligned} Cov(X_{t+h}, X_t) &= Cov(W_{t+h}+\frac{5}{2}W_{t+h-1}-\frac{3}{2}W_{t+h-2}, W_t+\frac{5}{2}W_{t-1}-\frac{3}{2}W_{t-2}) \\ &= \begin{cases} \frac{19}{2}, & when\ h=0 \\ -\frac{5}{4}, & when\ h=1 \\ -\frac{5}{4}, & when\ h=-1 \\ -\frac{3}{2}, & when\ h=2 \\ -\frac{3}{2}, & when\ h=-2 \\ 0, & else. \end{cases} \end{aligned}$$

$$

(b)

$$
\begin{aligned}
Cov(X_{t+h}, X_t) &= Cov(\tilde{W}_{t+h} - \frac{1}{6}\tilde{W}_{t+h-1} - \frac{1}{6}\tilde{W}_{t+h-2}, \tilde{W}_t - \frac{1}{6}\tilde{W}_{t-1} - \frac{1}{6}\tilde{W}_{t-2}) \\
&= \begin{cases} \frac{19}{2}, & when\ h = 0 \\ -\frac{5}{4}, & when\ h = 1 \\ -\frac{5}{4}, & when\ h = -1 \\ -\frac{3}{2}, & when\ h = 2 \\ -\frac{3}{2}, & when\ h = -2 \\ 0, & else. \end{cases}
\end{aligned}
$$

We can discover that the autocovariance result in (b) is the same as that in (a).

(c)

For the first time series, $\theta(z) = 1 + \frac{5}{2}z - \frac{3}{2}z^2$, whose roots are 2 and $-\frac{1}{3}$. Since it has a root $-\frac{1}{3}$ in the unit ball of complex plane, thus it is not invertible.

For the second time series, $\theta(z) = 1 - \frac{1}{6}z - \frac{1}{6}z^2$, whose roots are 3 and 2. Since there does not exist any root in the unit ball of the complex plane, hence it is invertible.

To conclude, MA models in (b) is invertible.

5

Since Y_t is stationary, then for convenience we define $\gamma(h) \triangleq Cov(Y_{t+h}, Y_t)$.

(a)

$$
\begin{aligned}
Cov(X_{t+h}, X_t) &= Cov(Y_{t+h} - 0.4Y_{t+h-1}, Y_t - 0.4Y_{t-1}) \\
&= Cov(Y_{t+h}, Y_t) - 0.4Cov(Y_{t+h-1}, Y - t) - 0.4Cov(Y_{t+h}, Y_{t-1}) + 0.16Cov(Y_{t+h-1}, Y_{t-1}) \\
&= \gamma(h) - 0.4\gamma(h-1) - 0.4\gamma(h+1) + 0.16\gamma(h) \\
&= 1.16\gamma(h) - 0.4\gamma(h-1) - 0.4\gamma(h+1).
\end{aligned}
$$

Since $X_t = Y_t - 0.4Y_{t-1}$ is also stationary, then we define

$$
\gamma_x(h) \triangleq Cov(X_{t+h}, X_t) = 1.16\gamma(h) - 0.4\gamma(h-1) - 0.4\gamma(h+1).
$$

Similarly, for $Z_t = Y_t - 2.5Y_{t-1}$, we have

$$
\begin{aligned}
Cov(Z_{t+h}, Z_t) &= Cov(Y_{t+h} - 2.5Y_{t+h-1}, Y_t - 2.5Y_{t-1}) \\
&= Cov(Y_{t+h}, Y_t) - 2.5Cov(Y_{t+h-1}, Y - t) - 2.5Cov(Y_{t+h}, Y_{t-1}) + 6.25Cov(Y_{t+h-1}, Y_{t-1}) \\
&= \gamma(h) - 2.5\gamma(h-1) - 2.5\gamma(h+1) + 6.25\gamma(h) \\
&= 7.25\gamma(h) - 2.5\gamma(h-1) - 2.5\gamma(h+1).
\end{aligned}
$$

And since Z_t is also stationary, thus we are also able to define

$$
\gamma_z(h) \triangleq Cov(Z_{t+h}, Z_t) = 7.25\gamma(h) - 2.5\gamma(h-1) - 2.5\gamma(h+1).
$$

(b)

The autocorrelation function of X_t is

$$
\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \frac{1.16\gamma(h) - 0.4\gamma(h-1) - 0.4\gamma(h+1)}{1.16\gamma(0) - 0.8\gamma(1)}.
$$

Similarly, the autocorrelation function of Z_t is

$$\rho_z(h) = \frac{\gamma_z(h)}{\gamma_z(0)} = \frac{7.25\gamma(h) - 2.5\gamma(h-1) - 2.5\gamma(h+1)}{7.25\gamma(0) - 5\gamma(1)}.$$

Since

$$\frac{1.16}{7.25} = \frac{0.4}{2.5},$$

then we can check that

$$\rho_x(h) = \rho_z(h).$$

Therefore, the autocorrelation functions of X_t and Z_t are the same.

6

(a)

- $\phi(z) = 1 + 0.81z^2$ with roots $\frac{10}{9}i$ and $-\frac{10}{9}i$ where i is the imaginary unit.
- $\theta(z) = 1 + \frac{1}{3}z$ with root -3.
- $p = 2$ and $q = 1$, i.e., $ARMA(2,1)$.
- Since there does not exist any root of $\phi(z)$ in the unit ball of complex plane, then it is causal.
- Since there does not exist any root of $\theta(z)$ in the unit ball of complex plane, then it is invertible.

(b)

Since

$$X_t - X_{t-1} = W_t - 0.5W_{t-1} - 0.5W_{t-2},$$

which is

$$(1-B)X_t = \frac{1}{2}(1-B)(2+B)W_t.$$

Thus,

$$X_t = \frac{1}{2}(2+B)W_t.$$

- $\phi(z) = 1$ with no roots.
- $\theta(z) = 1 + \frac{1}{2}z$ with root -2.
- $p = 0$ and $q = 1$, i.e., $ARMA(0,1)$.
- Since there does not exist any root of $\phi(z)$ in the unit ball of complex plane, then it is causal.
- Since there does not exist any root of $\theta(z)$ in the unit ball of complex plane, then it is invertible.

(c)

- $\phi(z) = 1 - 3z$ with root $\frac{1}{3}$
- $\theta(z) = 1 + 2z - 8z^2$ with root $\frac{1}{2}$ and $-\frac{1}{4}$.
- $p = 1$ and $q = 2$, i.e., $ARMA(1,2)$.
- Since there exists a root $\frac{1}{3}$ of $\phi(z)$ in the unit ball of complex plane, then it is NOT causal.
- Since there exist roots $\frac{1}{2}$ and $-\frac{1}{4}$ of $\theta(z)$ in the unit ball of complex plane, then it is NOT invertible.

(d)

- $\phi(z) = 1 - 2z + 2z^2$ with roots $\frac{1+i}{2}$ and $\frac{1-i}{2}$, where i is the imaginary unit.
- $\theta(z) = 1 - \frac{8}{9}z$ with root $\frac{9}{8}$.
- $p = 2$ and $q = 1$, i.e., $ARMA(2,1)$.
- Since there exist roots $\frac{1+i}{2}$ and $\frac{1-i}{2}$ of $\phi(z)$ in the unit ball of complex plane, then it is NOT causal.
- Since there does not exist any root of $\theta(z)$ in the unit ball of complex plane, then it is invertible.

(e)

- $\phi(z) = 1 - 4z^2$ with roots $\frac{1}{2}$ and $-\frac{1}{2}$.
- $\theta(z) = 1 - z + \frac{1}{2}z^2$ with no roots.

- $p = 2$ and $q = 2$, i.e., $ARMA(2, 2)$.
- Since there exist roots $\frac{1}{2}$ and $-\frac{1}{2}$ of $\phi(z)$ in the unit ball of complex plane, then it is NOT causal.
- Since there does not exist any root of $\theta(z)$ in the unit ball of complex plane, then it is invertible.

(f)

- $\phi(z) = 1 - \frac{9}{4}z - \frac{9}{4}z^2$ with roots $\frac{1}{3}$ and $-\frac{4}{3}$.
- $\theta(z) = 1$ with no roots.
- $p = 2$ and $q = 0$, i.e., $ARMA(2, 0)$.
- Since there exists a root $\frac{1}{3}$ of $\phi(z)$ in the unit ball of complex plane, then it is NOT causal.
- Since there does not exist any root of $\theta(z)$ in the unit ball of complex plane, then it is invertible.

(g)

Since

$$X_t - \frac{9}{4}X_{t-1} - \frac{9}{4}X_{t-2} = W_t - 3W_{t-1} + \frac{1}{9}W_{t-2} - \frac{1}{3}W_{t-2},$$

which is

$$(1 - 3B)(1 + \frac{3}{4}B)X_t = (1 - 3B)(1 + \frac{1}{9}B^2)W_t.$$

Thus,

$$(1 + \frac{3}{4}B)X_t = (1 + \frac{1}{9}B^2)W_t.$$

- $\phi(z) = 1 + \frac{3}{4}z$ with root $-\frac{4}{3}$.
- $\theta(z) = 1 + \frac{1}{9}z^2$ with root $3i$ and $-3i$ where i is the imaginary unit.
- $p = 1$ and $q = 2$, i.e., $ARMA(1, 2)$.
- Since there does not exist any root of $\phi(z)$ in the unit ball of complex plane, then it is causal.
- Since there does not exist any root of $\theta(z)$ in the unit ball of complex plane, then it is invertible.

7

Causal models are (a), (b) and (g).

Based on the formula

$$\psi_j = \theta_j + \phi_1\psi_{j-1} + \cdots + \phi_p\psi_{j-p},$$

we can start the calculation.

(a)

Given $\phi_2 = -0.81$ and $\theta_1 = \frac{1}{3}$, we have:

- $\psi_0 = 1$.
- $\psi_1 = \theta_1 + \phi_1\psi_0 = \theta_1 = \frac{1}{3}$.
- $\psi_2 = \theta_2 + \phi_2\psi_0 + \phi_1\psi_1 = \phi_2\psi_0 = -0.81$.
- $\psi_3 = \theta_3 + \phi_2\psi_1 + \phi_1\psi_2 = \phi_2\psi_1 = -0.27$.
- $\psi_4 = \theta_4 + \phi_2\psi_2 + \phi_1\psi_3 = \phi_2\psi_2 = (-0.81)^2 = 0.6561$.

(b)

Given $\phi_1 = 1$, $\theta_1 = -\frac{1}{2}$ and $\theta_2 = -\frac{1}{2}$, we have:

- $\psi_0 = 1$.
- $\psi_1 = \theta_1 + \phi_1\psi_0 = \frac{1}{2}$.
- $\psi_2 = \theta_2 + \phi_1\psi_1 = 0$.
- $\psi_3 = \theta_3 + \phi_1\psi_2 = 0$.
- $\psi_4 = \theta_4 + \phi_1\psi_3 = 0$.

(g)

Given $\phi_1 = -\frac{3}{4}$ and $\theta_2 = \frac{1}{9}$, we have:

- $\psi_0 = 1.$
- $\psi_1 = \theta_1 + \phi_1\psi_0 = -\frac{3}{4}.$
- $\psi_2 = \theta_2 + \phi_1\psi_1 = \frac{1}{9} + \frac{9}{16} = \frac{97}{144}.$
- $\psi_3 = \theta_3 + \phi_2\psi_1 + \phi_1\psi_2 = \phi_1\psi_2 = -\frac{3}{4} \times \frac{97}{144} = -\frac{97}{192}.$
- $\psi_4 = \phi_1\psi_3 = -\frac{3}{4} \times (-\frac{97}{192}) = \frac{97}{256}.$

8

Given the formula

$$\gamma(k) = \mathbb{E}(X_{t+k}, X_t) = \begin{cases} \sum_{j=1}^{p} \phi_j \gamma(k-j) + \sigma^2 \sum_{j=k}^{q} \theta_j \psi_{j-k}, & k \leq q \\ \sum_{j=1}^{p} \phi_j \gamma(k-j), & k < q \end{cases}$$

For times series in (a)

Given $\phi_2 = -0.81, \theta_1 = \frac{1}{3}$ and $\{\psi_i\}$ sequence, then according to the formula above, we can derive that:

$$\begin{cases} \gamma(0) + 0.81\gamma(2) = \psi_0 + \theta_1\psi_1 = \frac{10}{9} \\ \gamma(1) + 0.81\gamma(1) = \theta_1\psi_0 = \frac{1}{3} \\ \gamma(2) + 0.81\gamma(0) = 0 \\ \gamma(n) + 0.81\gamma(n-2) = 0, \quad n \geq 2 \end{cases}$$

$$\gamma(k) = \begin{cases} (-0.81)^n \gamma(0), & k = 2n \\ (-0.81)^n \gamma(1), & k = 2n+1 \end{cases}$$

where $\gamma(1) = \frac{1}{3 \times 1.81}$ and $\gamma(0) = \frac{10}{9 \times (1 - 0.81^2)}$.

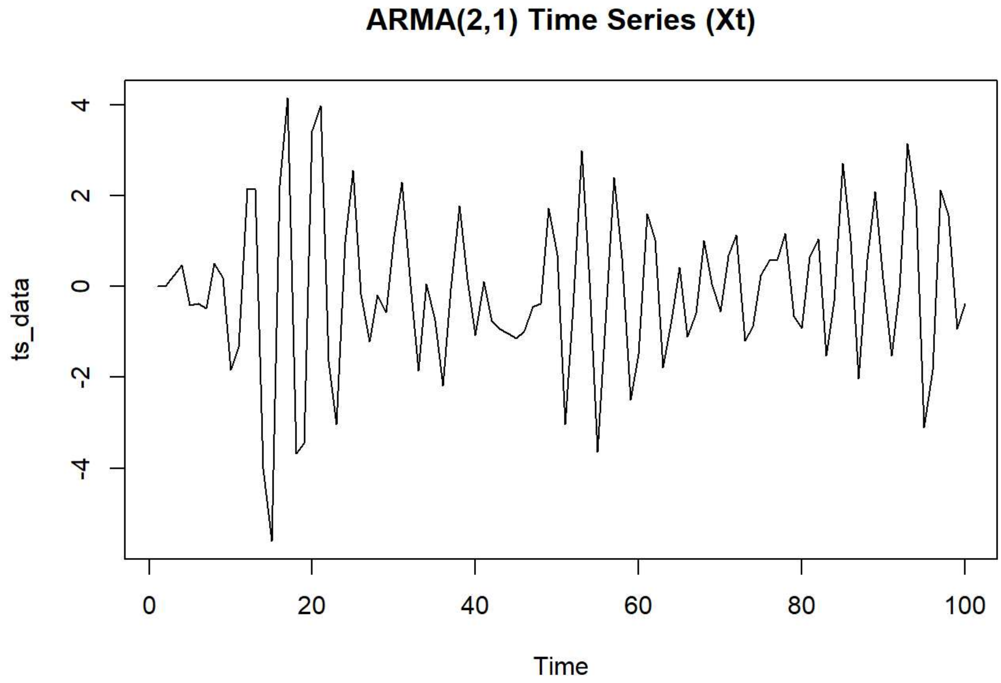
Therefore, the ACF is

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \begin{cases} (-0.81)^n, & k = 2n \\ (-0.81)^n \times \frac{\gamma(1)}{\gamma(0)} = (-0.81)^n \times 0.057, & k = 2n+1 \end{cases}$$

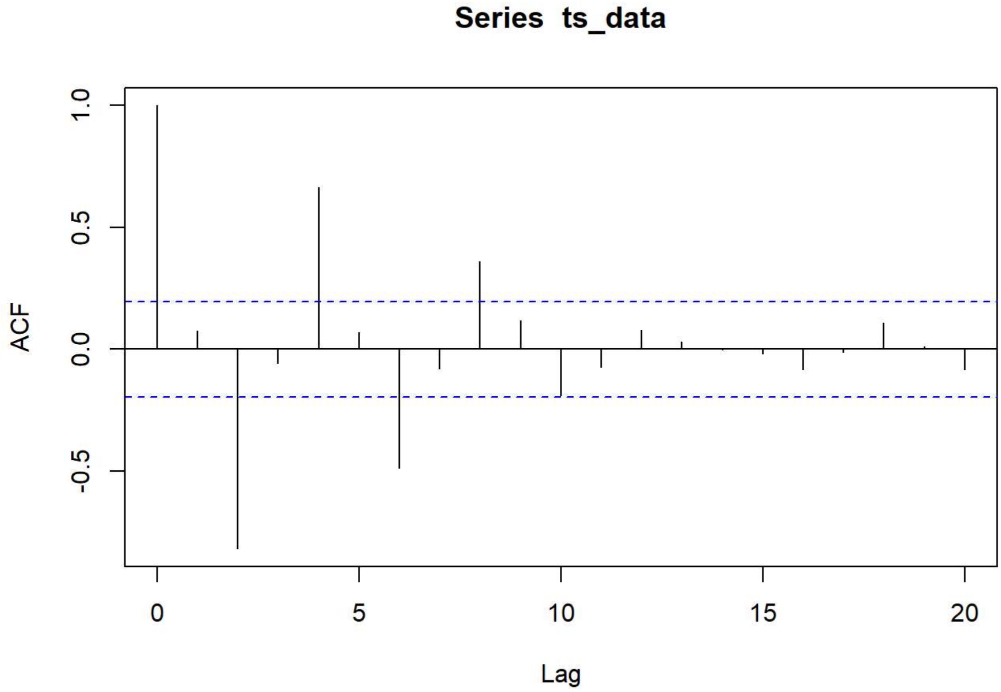
```
library(forecast)
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

```
# 设置随机数种子以获得可重复的结果
set.seed(99)

# 生成随机白噪声项Wt
n <- 100  # 时间序列的长度
white_noise <- rnorm(n)

# 创建Xt项（ARMA(2,1)模型）
Xt <- numeric(n)
for (t in 3:n) {
  Xt[t] <- white_noise[t] - 0.81 * Xt[t - 2] + 1/3 * white_noise[t - 1]
}

# 创建时间序列对象
ts_data <- ts(Xt)

# 可视化时间序列
plot(ts_data, main = "ARMA(2,1) Time Series (Xt)")
```




```
result <- acf(ts_data)
```




```
result
```

```
##
## Autocorrelations of series 'ts_data', by lag
##
##      0      1      2      3      4      5      6      7      8      9     10
##  1.000  0.076 -0.819 -0.059  0.666  0.069 -0.490 -0.079  0.360  0.117 -0.190
##     11     12     13     14     15     16     17     18     19     20
## -0.075  0.077  0.031 -0.003 -0.019 -0.083 -0.012  0.108  0.011 -0.083
```

We can discover that the result is consistent with our theoretical values.

For time series in (b)

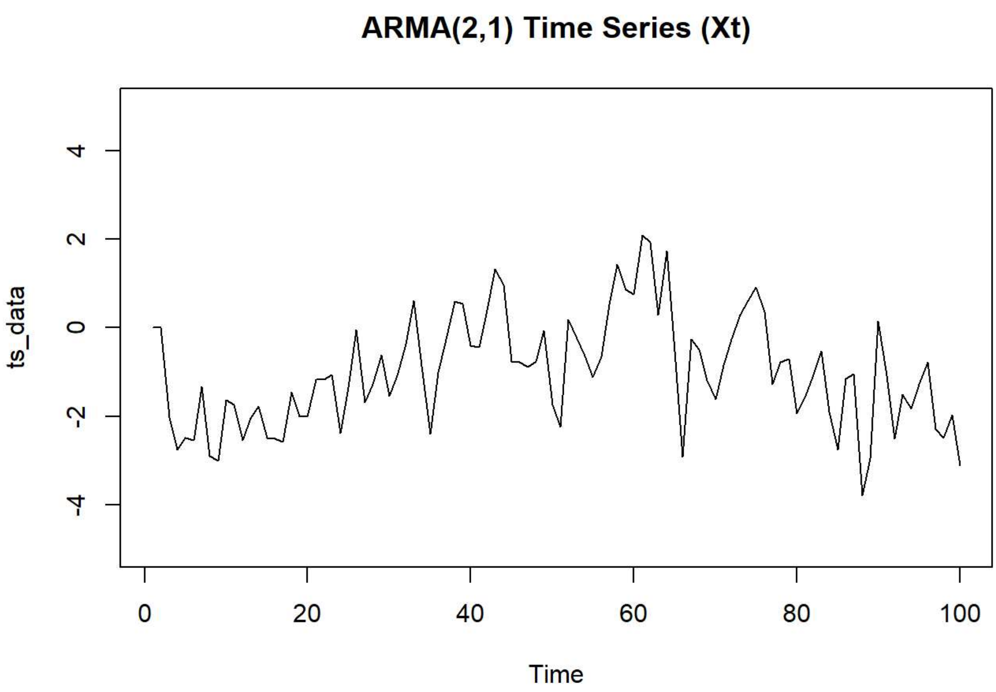
We have already derive that $\psi_0 = 1$, $\psi_1 = \frac{1}{2}$ and $\psi_j = 0, \forall j \neq 1, 2$. Thus,

$$X_t = W_t + \frac{1}{2} W_{t-1}.$$

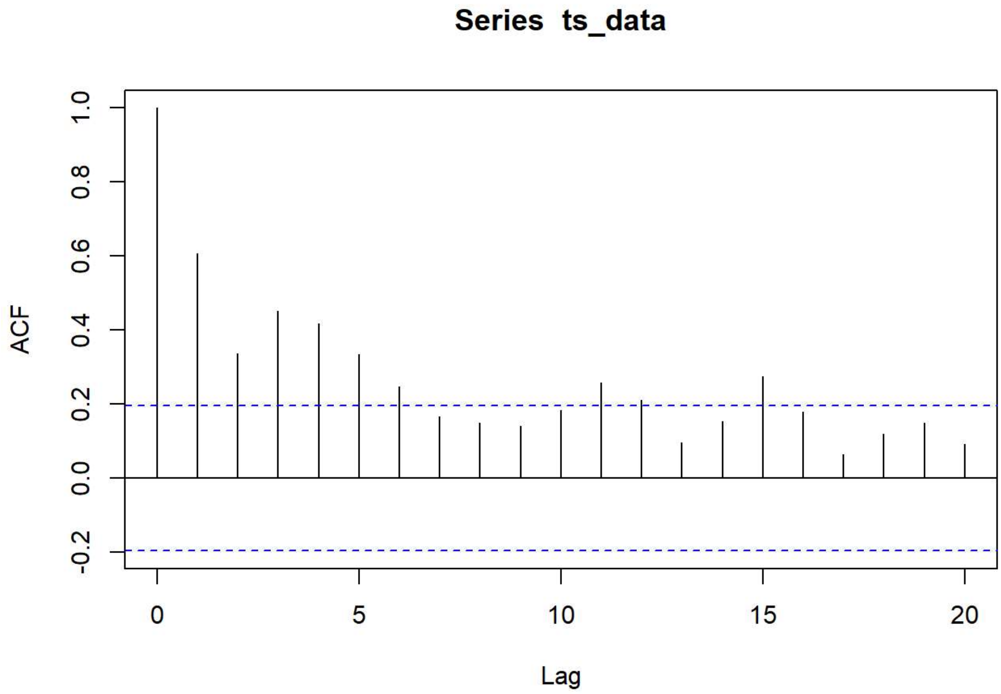
Then by definition, we can calculate that

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1, & h = 0 \\ \frac{1/2}{1+(1/2)^2} = \frac{2}{5}, & h = 1 \; or - 1 \\ 0, & else \end{cases}$$

```
library(forecast)

set.seed(198)

# 生成随机白噪声项Wt
n <- 100  # 时间序列的长度
white_noise <- rnorm(n)

# 创建Xt项（ARMA(2,1)模型）
Xt <- numeric(n)
for (t in 3:n) {
  Xt[t] <- white_noise[t] + Xt[t-1] - 0.5*white_noise[t - 2]
  - 0.5 * white_noise[t - 1]

}

ts_data <- ts(Xt)

plot(ts_data,ylim = c(-5,5), main = "ARMA(2,1) Time Series (Xt)")
```




```
result <- acf(ts_data)
```




```
result
```

```
##
## Autocorrelations of series 'ts_data', by lag
##
##     0     1     2     3     4     5     6     7     8     9    10    11    12
## 1.000 0.607 0.336 0.452 0.418 0.333 0.247 0.167 0.150 0.141 0.184 0.257 0.210
##    13    14    15    16    17    18    19    20
## 0.096 0.154 0.276 0.180 0.064 0.120 0.149 0.092
```

From the result, we can find out that it is consistent with our theoretical value.

For time series in (g)

We can derive a system of equations as follows: $$

$$\begin{cases} \gamma(0) = -\frac{3}{4}\gamma(1) + 1 + \frac{1}{9} \times \frac{97}{144} \\ \gamma(1) = -\frac{3}{4}\gamma(0) + \frac{1}{9} \times (-\frac{3}{4}) \\ \gamma(2) = -\frac{3}{4}\gamma(1) + \frac{1}{9} \\ \gamma(3) = -\frac{3}{4}\gamma(2) \\ \vdots \\ \gamma(n) = -\frac{3}{4}\gamma(n-1) \\ \vdots \end{cases}$$

$$

Then we can figure out the result as follows:

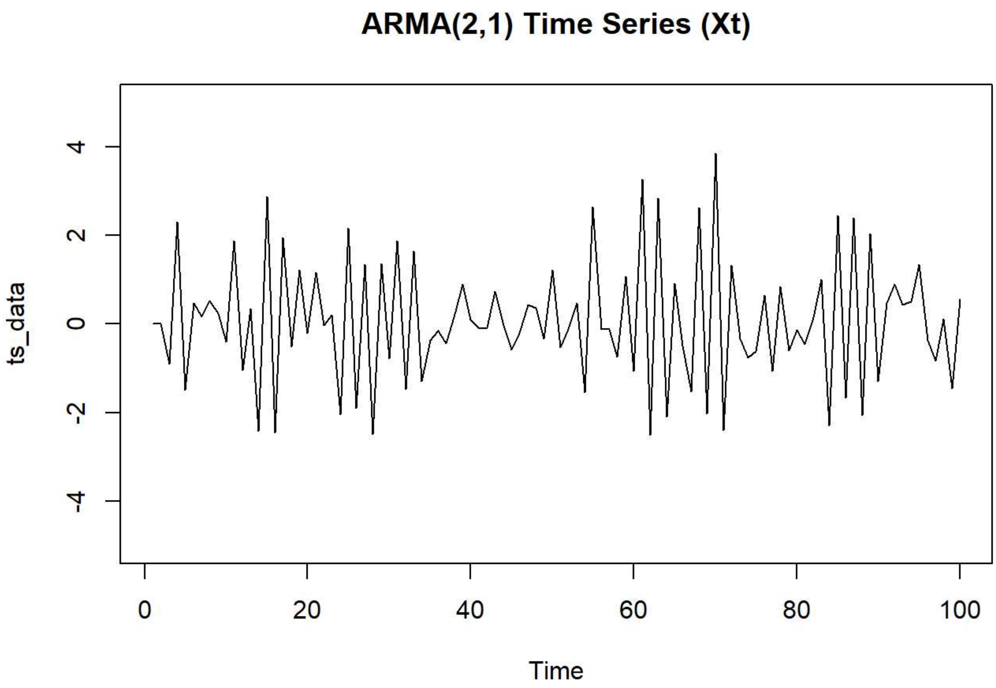
$\gamma(0) = \frac{1474}{567}, \gamma(1) = -\frac{1537}{756}, \gamma(2) = \frac{1649}{1008}, \gamma(3) = -\frac{3}{4}\gamma(2)$ and so on.

Thus, $$(h) =

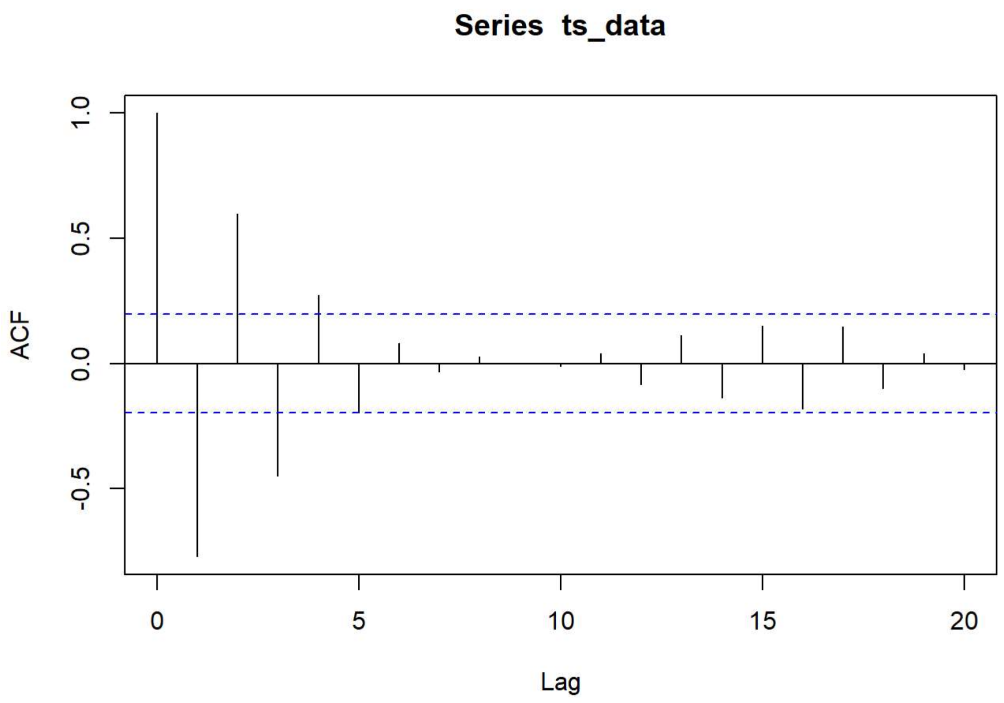
$$\begin{cases} 1, & h = 0 \\ -\frac{4611}{5896}, & h = 1 \\ 0.6293, & h = 2 \\ 0.6293 \times (-\frac{3}{4})^{h-2}, & h \geq 3 \end{cases}$$

$$

```
library(forecast)
 # (b)
# 设置随机数种子以获得可重复的结果
set.seed(1)

# 生成随机白噪声项Wt
n <- 100  # 时间序列的长度
white_noise <- rnorm(n)

# 创建Xt项（ARMA(2,1)模型）
Xt <- numeric(n)
for (t in 3:n) {
  Xt[t] <- white_noise[t] -0.75*Xt[t-1] + 1/9*white_noise[t - 2]

}

ts_data <- ts(Xt)


plot(ts_data,ylim = c(-5,5), main = "ARMA(2,1) Time Series (Xt)")
```




```
result <- acf(ts_data)
```




```
result
```

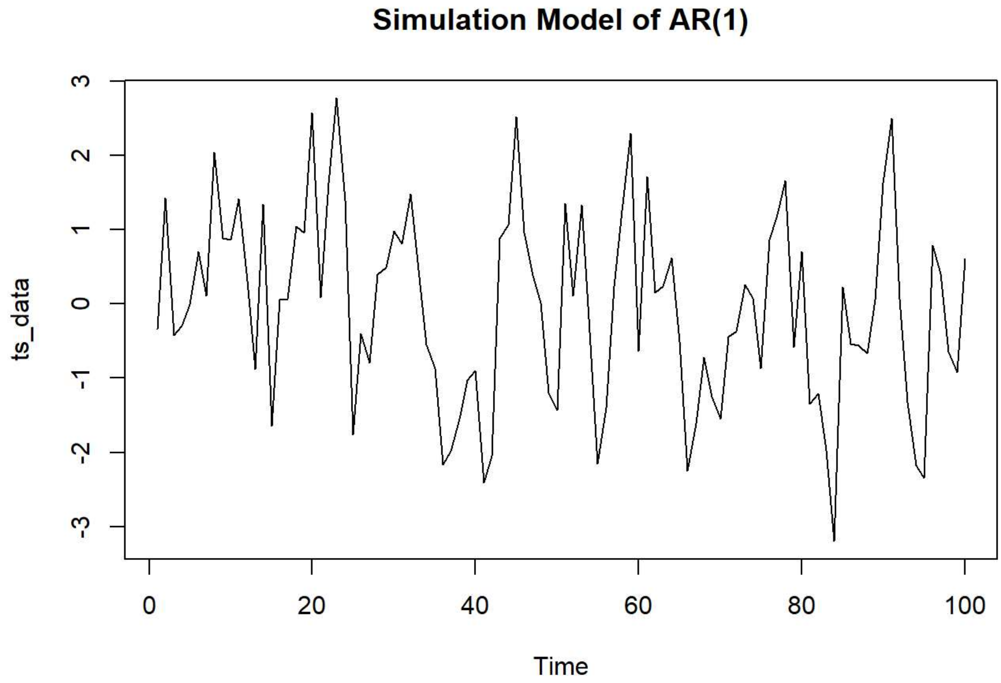
```
##
## Autocorrelations of series 'ts_data', by lag
##
##      0      1      2      3      4      5      6      7      8      9     10
##  1.000 -0.771  0.598 -0.448  0.274 -0.197  0.079 -0.034  0.028 -0.001 -0.009
##     11     12     13     14     15     16     17     18     19     20
##  0.039 -0.083  0.111 -0.136  0.148 -0.180  0.147 -0.100  0.040 -0.023
```

From the figure, we can discover that the result is consistent with our theoretical value.

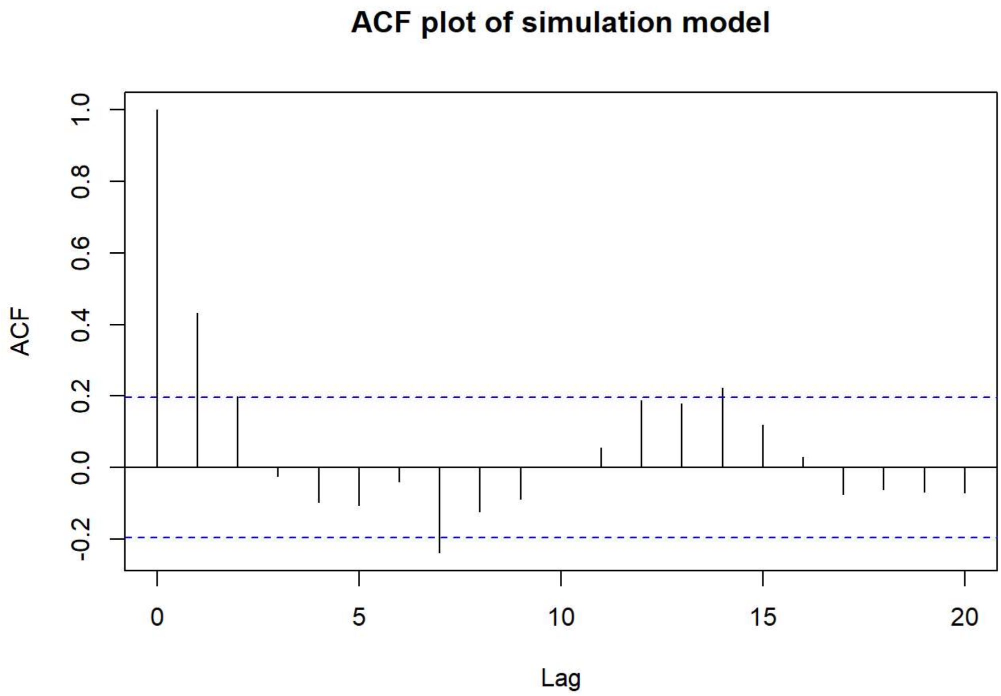
9

(a)

```
set.seed(2)
n <- 100
X0 <- rnorm(1,0,sqrt(4/3))
W <- rnorm(n)
Xt <- numeric(n)
Xt[1] <- 1/2*X0 + W[1]
for (i in 2:100) {
  Xt[i] <- 1/2*Xt[i-1] + W[i]

}

ts_data <- ts(Xt)
plot(ts_data, main = "Simulation Model of AR(1)")
```



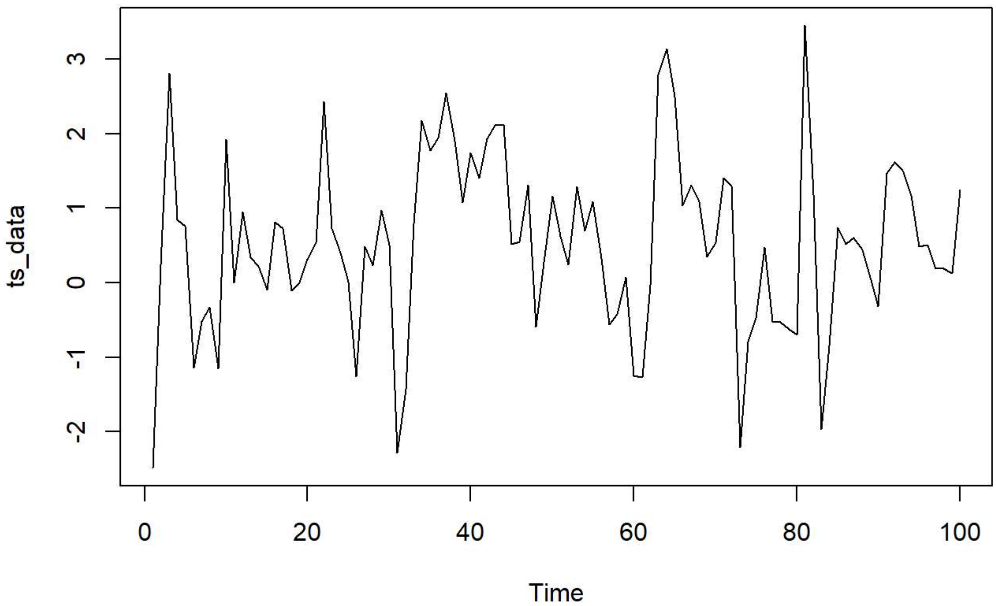

```
acf(ts_data,main = "ACF plot of simulation model")
```



We can discover from the time plot that the mean is zero and from the ACF plot that the acf decays exponentially, which means the parameter ϕ of the model $AR(1)$ is less than 1, indicating that there is no root, with its length equaling 1, of its characteristic function $\phi(z) = 1 - \phi z$. Consequently, it is simulated from a stationary process $AR(1)$.

(b)

```
set.seed(1)
N <- 1000
W <- rt(N,5)
Xt <- numeric(N)
Xt[1] <- W[1]
for (i in 2:N) {
  Xt[i] <- 0.5*Xt[i-1] + W[i]
}

Zt <- Xt[901:1000]
ts_data <- ts(Zt)
plot(ts_data)
```



Since t-distribution is more heavy-tailed than normal distribution, which means the white noise tends to have more extreme values. Thus, there will be more transient states in the data. Here we simulate a long sequence of data can effectively help us capture more recurrent state and filter transient state caused by the heavy-tailed white noise.

Additionally, simulating a long sequence of values can help to reduce initial bias. Since the initial values are usually generated by seed value, this can introduce bias. Thus, simulating a long sequence of values can decrease the potential initial bias.

10

(a)

```
data <- read.csv("C:/Users/Lenovo/Desktop/yahoo(1).csv")
data <- data[-c(1,2),]
colnames(data) <- c('time','yahoo')
data$yahoo <- as.numeric(data$yahoo)

ts <- ts(data$yahoo)

library(zoo)
```

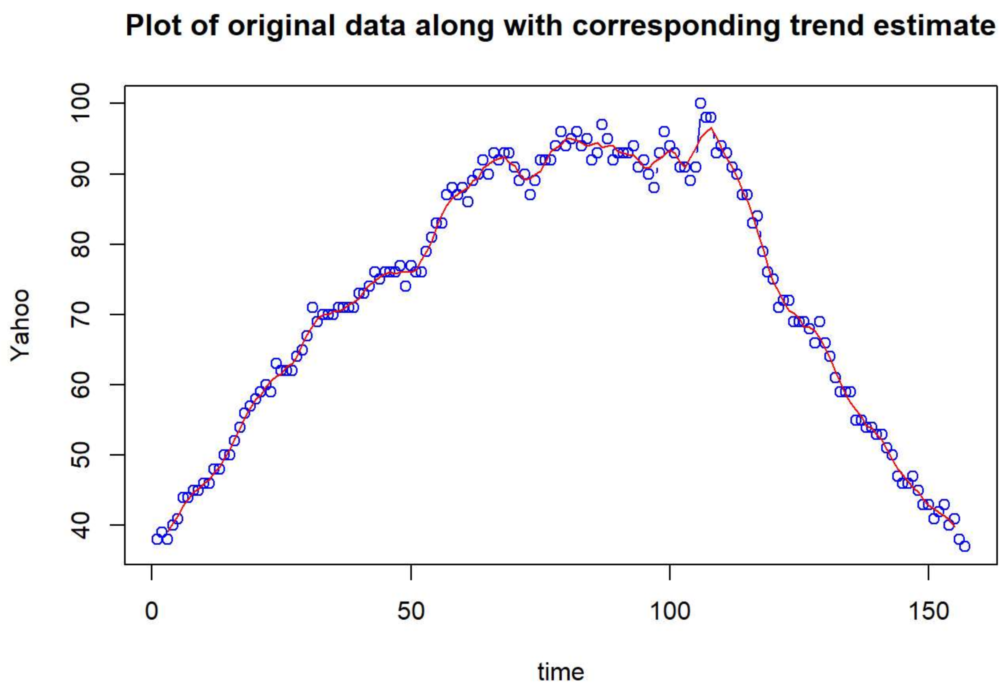
```
##
## 载入程辑包：'zoo'
```

```
## The following objects are masked from 'package:base':
##
##     as.Date, as.Date.numeric
```

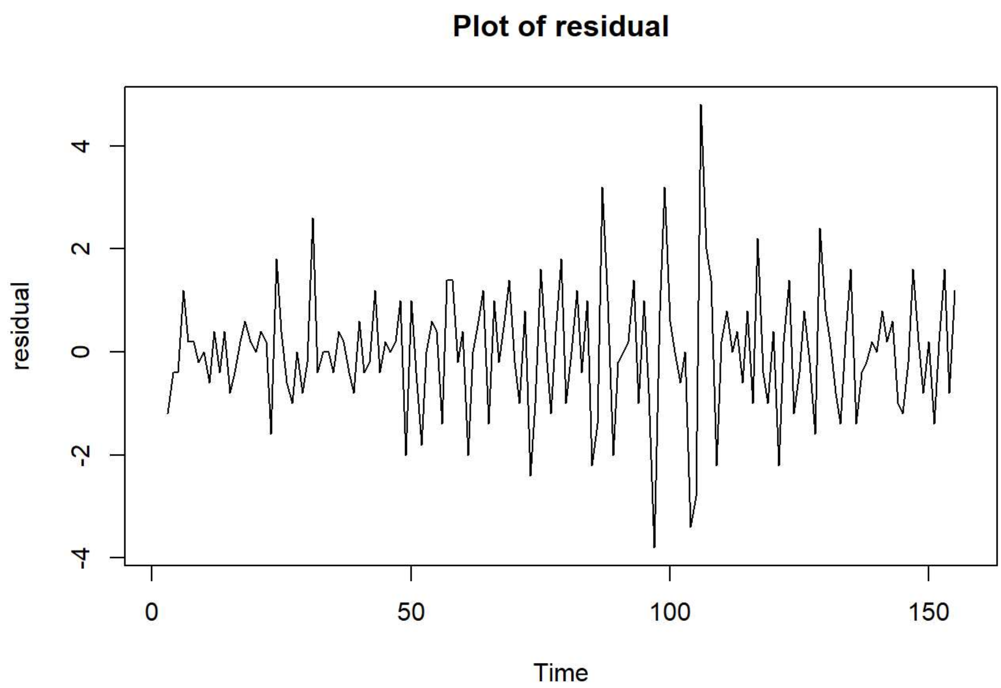
```
smoothed_data <- rollmean(ts, k = 5, fill = NA)
```

We set the parameter $q = 5$ because if it is too small, then the variance will become large whereas the bias is relatively small; on the other hand, if it is too large, then the bias will become large though the variance tends to be small.

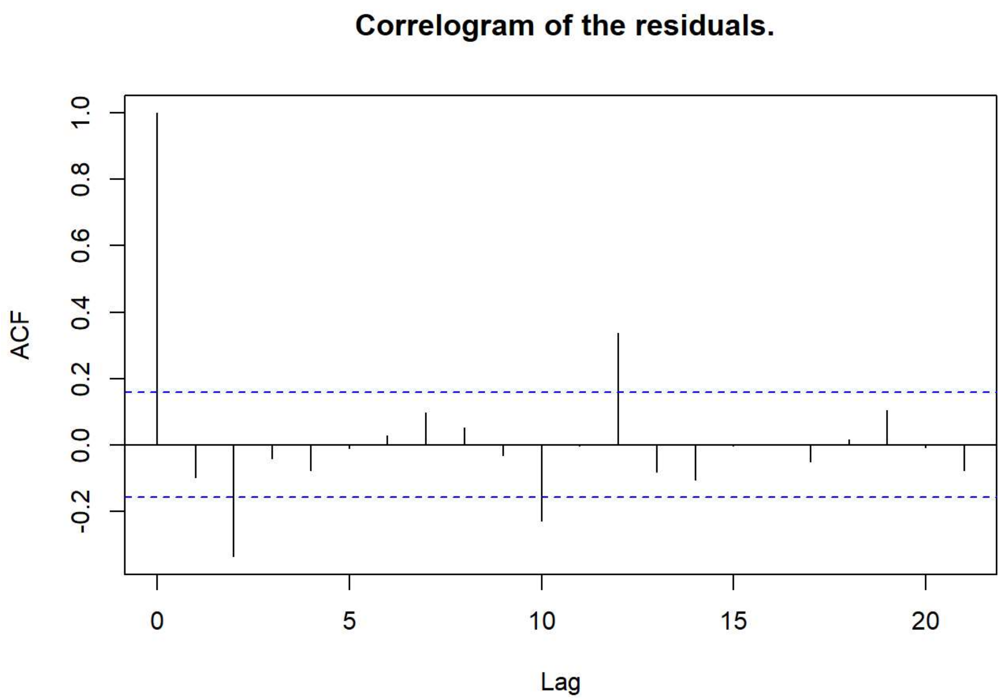
```
plot(ts, type = "b", col = "blue", xlab = "time", ylab = "Yahoo",main = "Plot of original data along with corresponding tre
nd estimate")
lines(smoothed_data, col = "red")
```




```
residual <- ts - smoothed_data
plot(residual,main = "Plot of residual")
```



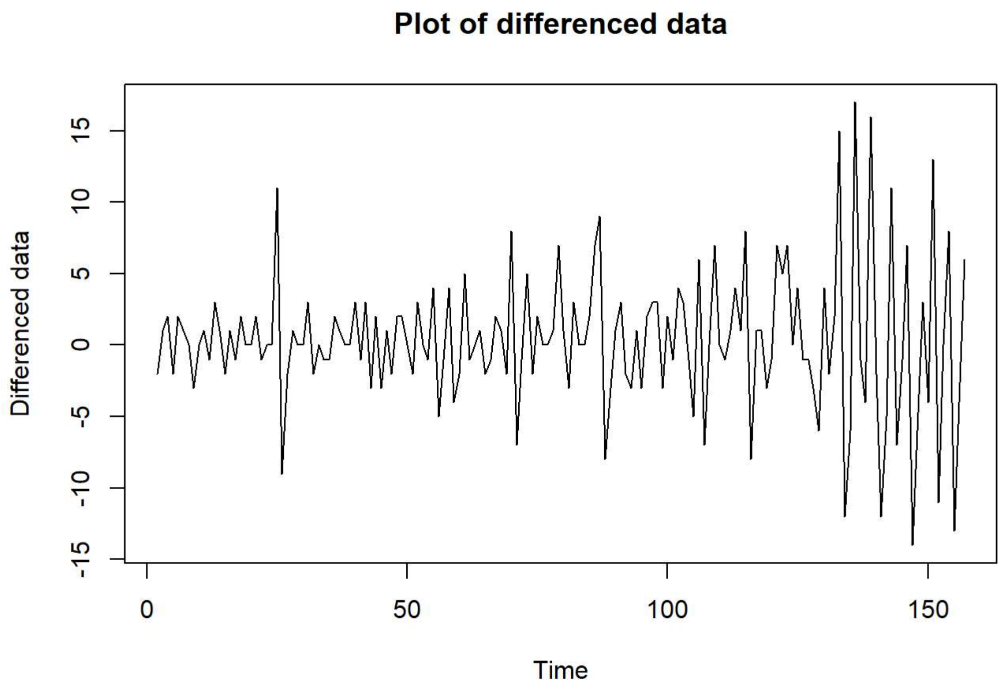

```
acf(residual[c(-1,-2,-156,-157)], main = " Correlogram of the residuals.")
```



From the time plot and the ACF plot of residual, we can discover that the residual tends to be a white noise, which implies the trend estimate of the original data is appropriate.

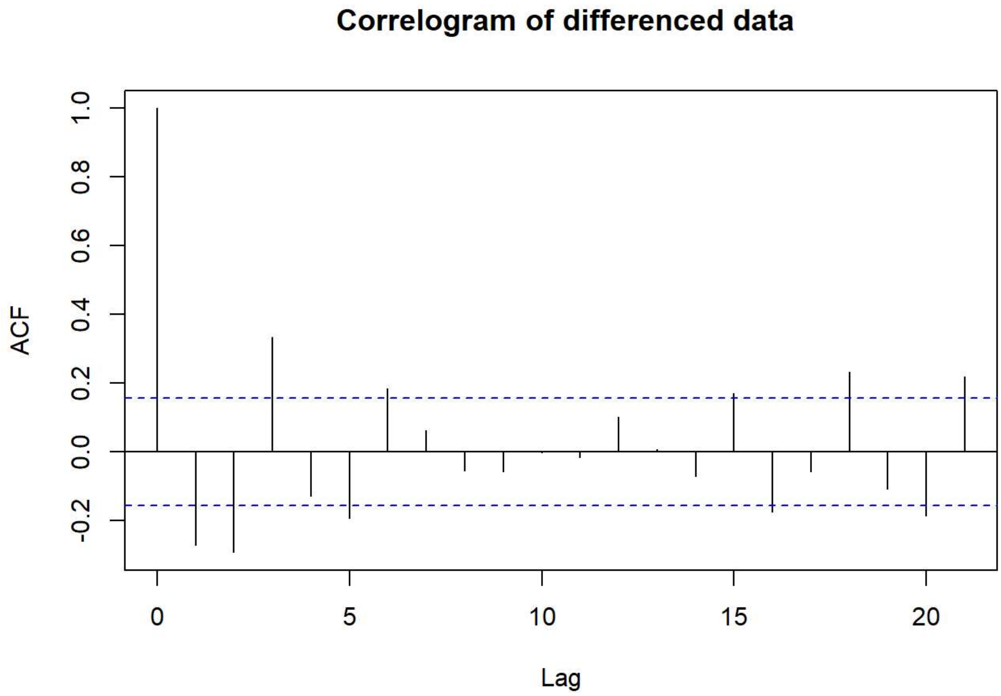
(b)

```
data2 <- read.csv("C:/Users/Lenovo/Desktop/chipotle(1).csv")
colnames(data2) <- c('time','chipolte')
ts <- ts(as.numeric(data2$chipolte[c(-1,-2)]))
dif <- diff(ts)
plot(dif, ylab = "Differenced data", main = "Plot of differenced data")
```



From the time plot of the differenced data, we can figure out that there is no trend in the differenced data since the data values fluctuate around zero.

```
acf(dif, main = " Correlogram of differenced data")
```



According to the ACF plot of the residual, we can draw to the conclusion that the residual tends to be a white noise since most ACF stay nearby zero and thus we can consider it as a white noise. Consequently, in this practice, differencing is able to remove the trend away from the original data.