

Times Series Assignment5

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Question 1

(a)

Consider the following seasonal AR model:

$$(1 - \phi B)(1 - \Phi B^s)X_t = W_t,$$

which is

$$X_t - \phi W_{t-1} - \Phi X_{t-s} + \phi \Phi X_{t-s-1} = W_t$$

where $W_t \sim WN(0, \sigma^2)$.

Thus, we have

$$\begin{aligned}\gamma_w(h) &= Cov(X_{t+h} - \phi W_{t+h-1} - \Phi X_{t+h-s} + \phi \Phi X_{t+h-s-1}, X_t - \phi W_{t-1} - \Phi X_{t-s} + \phi \Phi X_{t-s-1}) \\ &= \gamma_x(h) - \phi \gamma_x(h+1) - \Phi \gamma_x(h+s) + \phi \Phi \gamma_x(h+s+1) \\ &\quad - \phi [\gamma_x(h-1) - \phi \gamma_x(h) - \Phi \gamma_x(h+s-1) + \phi \Phi \gamma_x(h+s)] \\ &\quad - \Phi [\gamma_x(h-s) - \phi \gamma_x(h+1-s) - \Phi \gamma_x(h) + \phi \Phi \gamma_x(h+1)] \\ &\quad + \phi \Phi [\gamma_x(h-s-1) - \phi \gamma_x(h-s) - \Phi \gamma_x(h-1) + \phi \Phi \gamma_x(h)]\end{aligned}$$

Then apply $\gamma_w(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_w(\nu) e^{2\pi i \nu h} d\nu$, $\gamma_x(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(\nu) e^{2\pi i \nu h} d\nu$ and by the uniqueness of Fourier transformation, we can derive that $\frac{g_w(\nu)}{f_x(\nu)} = \frac{\sigma^2}{f_x(\nu)} =$

$$\begin{aligned}&[1 + \phi^2 + \Phi^2 + (\phi \Phi)^2] - 2(\phi + \phi \Phi^2) \cos(2\pi \nu) - 2(\Phi + \phi^2 \Phi) \cos(2\pi \nu s) \\ &+ 2\phi \Phi [\cos(2\pi \nu(s+1)) + \cos(2\pi \nu(s-1))].\end{aligned}$$

Therefore, the spectral density of X_t is $f_x(\nu) =$

$$\frac{\sigma^2}{1 + \phi^2 + \Phi^2 + (\phi \Phi)^2 - 2(\phi + \phi \Phi^2) \cos(2\pi \nu) - 2(\Phi + \phi^2 \Phi) \cos(2\pi \nu s) + 2\phi \Phi [\cos(2\pi \nu(s+1)) + \cos(2\pi \nu(s-1))]}.$$

(b)

```
library(forecast)
```

```
## Registered S3 method overwritten by 'quantmod':  
## method from  
## as.zoo.data.frame zoo
```

```

phi <- rep(0,13)
phi[1] <- 0.5
phi[12]<- 0.9
phi[13] <- -0.45

set.seed(12)

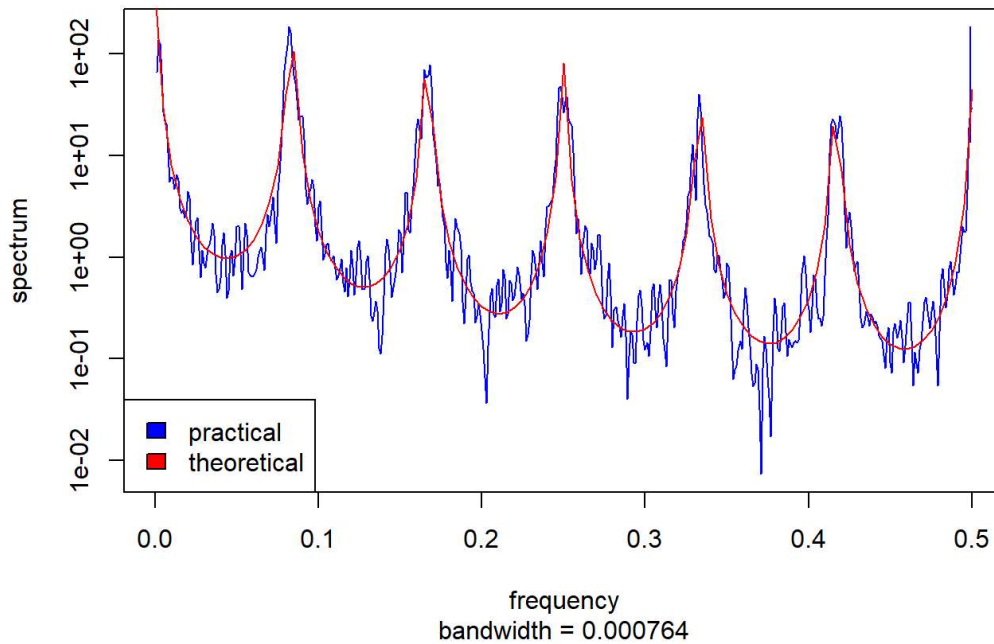
simulated_data <- arima.sim(n = 1000, list(ar = phi))
spec <- spec.pgram(simulated_data, spans = c(2), taper = 0, plot = FALSE)
par(mfrow = c(1,1))
plot(spec, main = "Spectral density plot", col = "blue")

my_function <- function(sigma_squared, phi,Phi, x) {
  result <- sigma_squared /
    (1 + phi^2 + Phi^2 + (phi*Phi)^2
    - 2*(phi+ phi*Phi^2)*cos(2*pi*x)
    - 2*(Phi + phi^2*Phi)*cos(2*pi*x*12)
    + 2*phi*Phi*(cos(2*pi*x* 13) + cos(2*pi*x*11)))
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.5, Phi = 0.9, x), from = 0, to = 0.5,
      col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
      fill = c("blue", "red"), cex = 1 )

```

Spectral density plot



The theoretical and practical results are consistent.

(c)

Similarly, for times series $X_t^{(1)}$ which satisfies

$$X_t^{(1)} - 0.5X_{t-1}^{(1)} = W_t,$$

we can derive its spectral density:

$$f_{X_t^{(1)}}(\nu) = \frac{\sigma^2}{1 - \cos(2\pi\nu) + 0.25}.$$

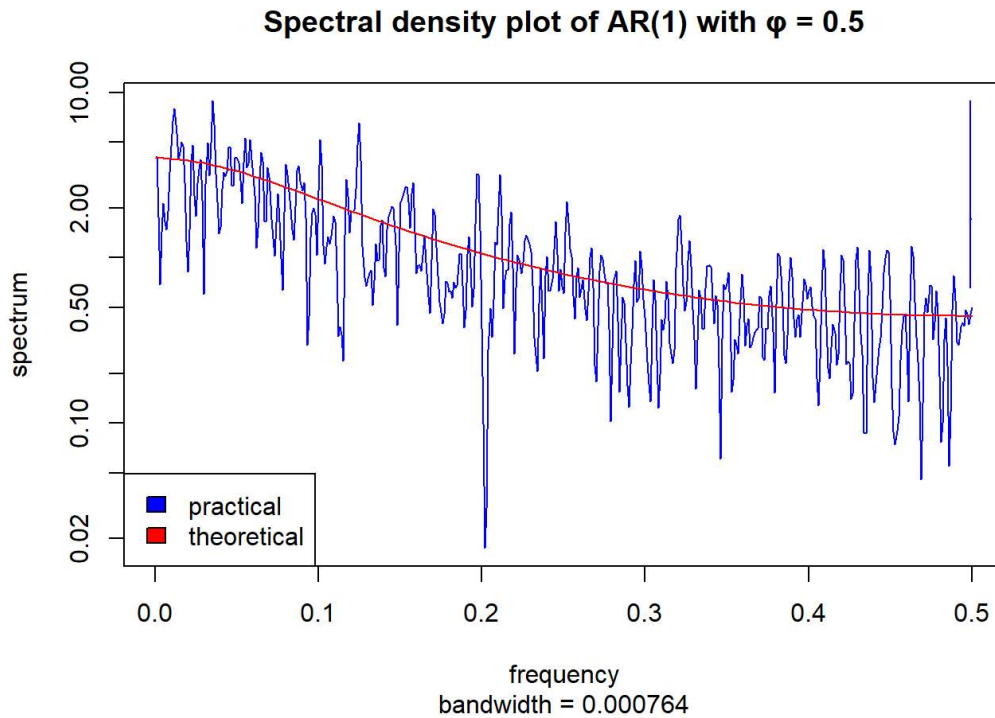
```

set.seed(12)
ts_data <- arima.sim(n = 1000, model = list(order = c(1,0,0), ar = 0.5))
spec <- spec.pgram(ts_data, spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Spectral density plot of AR(1) with  $\phi = 0.5$ ", col = "blue")

my_function1 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(2 * pi * x) + phi^2)
  return(result)
}

curve(my_function1(sigma_squared = 1, phi = 0.5, x), from = 0, to = 0.5
      , col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
      fill = c("blue", "red"), cex = 1 )

```



Again by performing the same method, for times series $X_t^{(2)}$ which satisfies

$$X_t^{(2)} - 0.9X_{t-12}^{(2)} = W_t,$$

we can derive its spectral density:

$$f_{X_t^{(2)}}(\nu) = \frac{\sigma^2}{1 - 1.8 \cos(24\pi\nu) + 0.81}.$$

```

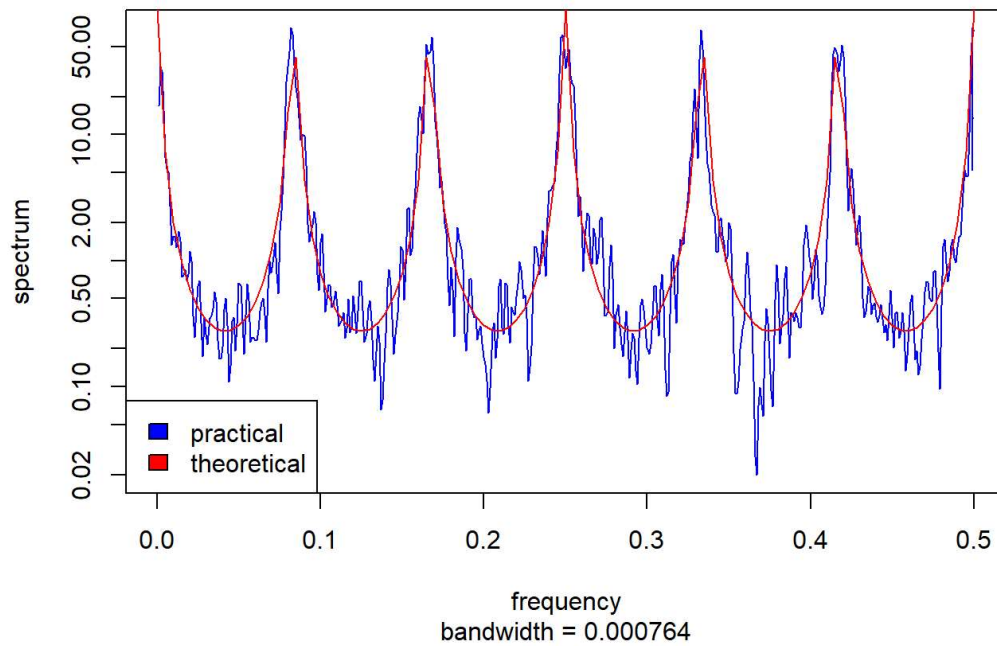
set.seed(12)
phi <- rep(0,12)
phi[12] <- 0.9
ts_data <- arima.sim(n = 1000, model = list(ar = phi))
spec <- spec.pgram(ts_data, spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Spectral density plot of seasonal AR(1) with  $\Phi = 0.9$ ", col = "blue")

my_function2 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(24 * pi * x) + phi^2)
  return(result)
}

curve(my_function2(sigma_squared = 1, phi = 0.9, x), from = 0, to = 0.5,
      , col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
      fill = c("blue", "red"), cex = 1 )

```

Spectral density plot of seasonal AR(1) with $\Phi = 0.9$



(d)

```
my_function <- function(sigma_squared, phi, Phi, x) {
  result <- sigma_squared /
    (1 + phi^2 + Phi^2 + (phi*Phi)^2
     - 2*(phi+ phi*Phi^2)*cos(2*pi*x)
     - 2*(Phi + phi^2*Phi)*cos(2*pi*x*12)
     + 2*phi*Phi*(cos(2*pi*x* 13) + cos(2*pi*x*11)))
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.5, Phi = 0.9, x), from = 0, to = 0.5, ylab = "spectrum", xlab="frequency",
ylim = c(0,150), col = "darkgreen")

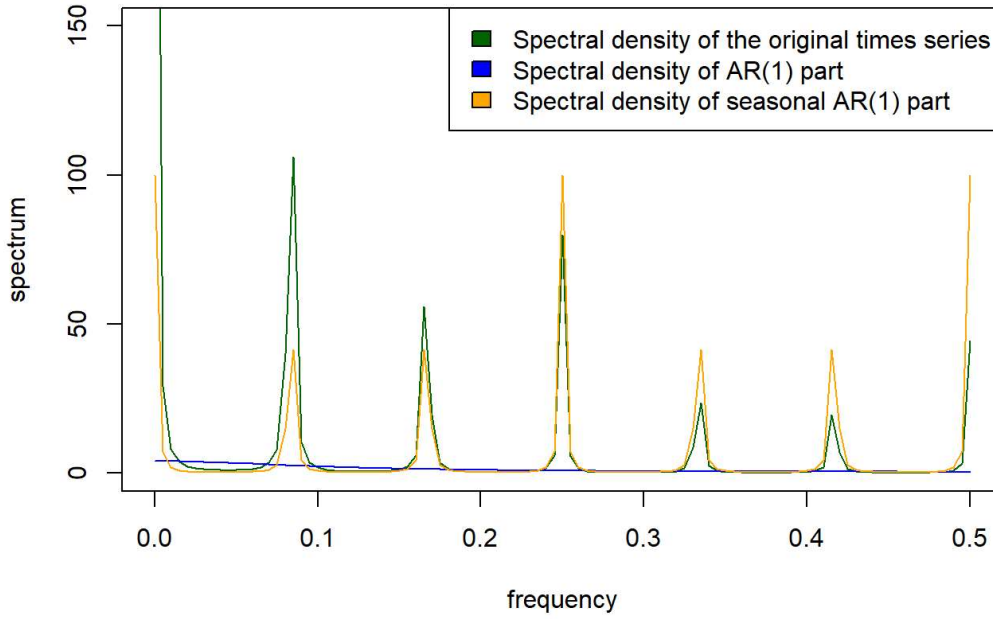
my_function1 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(2 * pi * x) + phi^2)
  return(result)
}

curve(my_function1(sigma_squared = 1, phi = 0.5, x), from = 0, to = 0.5
, col = "blue", add = TRUE)

my_function2 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(24 * pi * x) + phi^2)
  return(result)
}

curve(my_function2(sigma_squared = 1, phi = 0.9, x), from = 0, to = 0.5,
, col = "orange", add = TRUE)

legend("topright", legend = c("Spectral density of the original times series", "Spectral density of AR(1) part", "Spectral
density of seasonal AR(1) part"), fill = c("darkgreen","blue","orange"))
```



According to the spectral density plots, we can maintain that the seasonal AR(1) part dominates the seasonality of the original times series.

Question 2

(a)

From the inverse Fourier transformation, we have

$$\gamma(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu) e^{2\pi i \nu 0} d\nu = \int_{\frac{1}{6} \leq |\nu| \leq \frac{1}{3}} 5 d\nu = \frac{5}{3}$$

and

$$\gamma(1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu) e^{2\pi i \nu} d\nu = \int_{\frac{1}{6} \leq |\nu| \leq \frac{1}{3}} 5 \cos(2\pi \nu) d\nu = \frac{5}{\pi} \left[\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right) \right] = 0.$$

(b)

Since

$$\gamma_y(h) = \text{Cov}(X_{t+h} - X_{t+h-12}, X_t - X_{t-12}) = 2\gamma_x(h) - \gamma_x(h+12) - \gamma_x(h-12),$$

then apply inverse Fourier transformation we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu) e^{2\pi i \nu h} d\nu = \int_{-\frac{1}{2}}^{\frac{1}{2}} [2 - 2 \cos(24\pi \nu)] f_x(\nu) e^{2\pi i \nu h} d\nu.$$

Due to the uniqueness of Fourier transformation, we have

$$f_y(\nu) = [2 - 2 \cos(24\pi \nu)] f_x(\nu).$$

Consequently, the spectral density of Y_t is

$$f_y(\nu) = \begin{cases} 10 - 10 \cos(24\pi\nu), & \frac{1}{6} \leq |\nu| \leq \frac{1}{3} \\ 0, & \text{else.} \end{cases}$$

Question 3

(a)

Consider the model $X_t - 0.99X_{t-1} = W_t$, where $W_t \sim WN(0, \sigma^2)$, we have

$\gamma_w(h) = \text{Cov}(X_{t+h} - 0.99X_{t+h-3}, X_t - 0.99X_{t-3}) = (1 + 0.99^2)\gamma_x(h) - 0.99[\gamma_x(h+3) + \gamma_x(h-3)]$. Then apply inverse Fourier transformation $\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu h} d\nu$ and notice the uniqueness of Fourier transformation, we will obtain

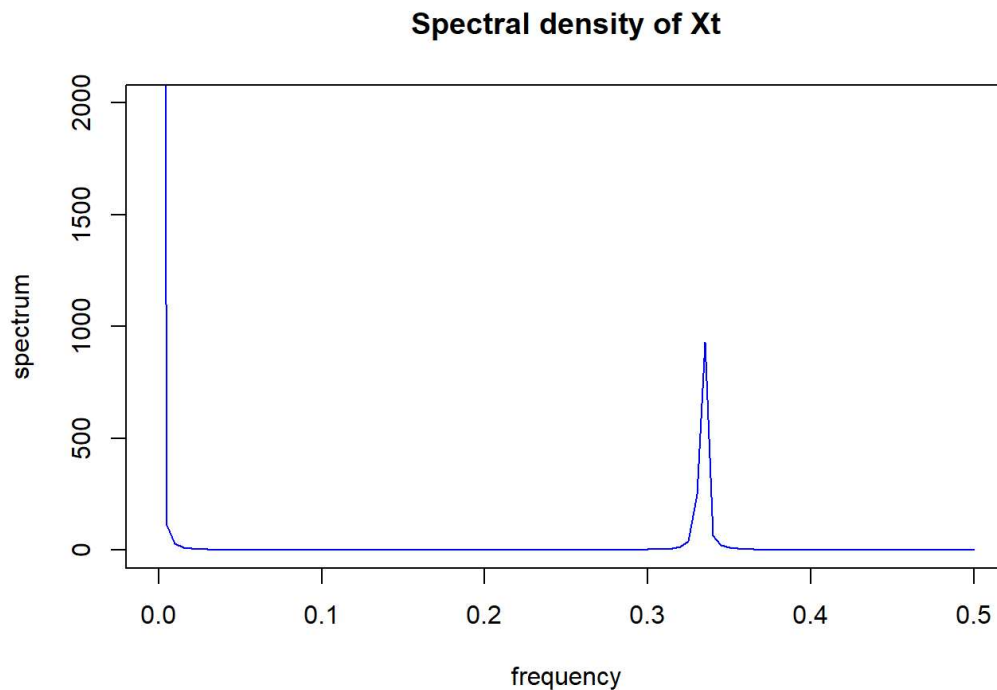
$$f_w(\nu) = [1 + 0.99^2 - 1.98 \cos(6\pi\nu)]f_x(\nu).$$

Thus, the spectral density of X_t is

$$f_x(\nu) = \frac{f_w(\nu)}{1 + 0.99^2 - 1.98 \cos(6\pi\nu)} = \frac{1}{1 + 0.99^2 - 1.98 \cos(6\pi\nu)}.$$

```
my_function <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.99, x), from = 0, to = 0.5, ylim = c(0, 2000),
      col = "blue", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum")
```

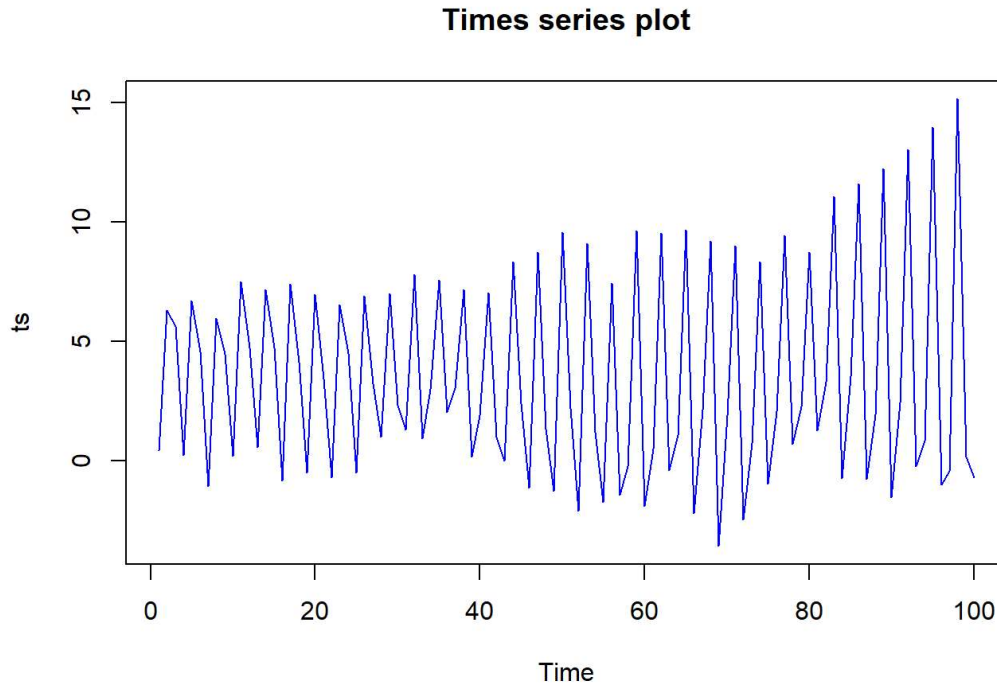


(b)

Check the spectral density plot and we can figure out that it shows an obvious sudden peak at frequency $\nu = 0.33$ approximately, which means the times series X_t exhibit an oscillation with frequency 0.33, i.e., the period is 3 approximately.

(c)

```
set.seed(123)
ts <- arima.sim(model = list(ar = c(0, 0, 0.99)), n = 100)
plot(ts, col = "blue", main = "Times series plot")
```



From the plot we can maintain that the times series X_t oscillates with period 3 approximately, which is consistent with the conclusion of part (b).

(d)

Since $Y_t = \frac{X_{t-1} + X_t + X_{t+1}}{3}$, then we have

$$\gamma_y(h) = \frac{1}{9} [3\gamma_x(h) + 2(\gamma_x(h+1) + \gamma(h-1)) + (\gamma_x(h-2) + \gamma_x(h+2))].$$

Thus, by performing the inverse Fourier transformation and notice the uniqueness of Fourier transformation, we can obtain that

$$f_y(\nu) = \left[\frac{1}{3} + \frac{4}{9} \cos(2\pi\nu) + \frac{2}{9} \cos(4\pi\nu) \right] f_x(\nu) = \frac{\frac{1}{3} + \frac{4}{9} \cos(2\pi\nu) + \frac{2}{9} \cos(4\pi\nu)}{1 + 0.99^2 - 1.98 \cos(6\pi\nu)}.$$

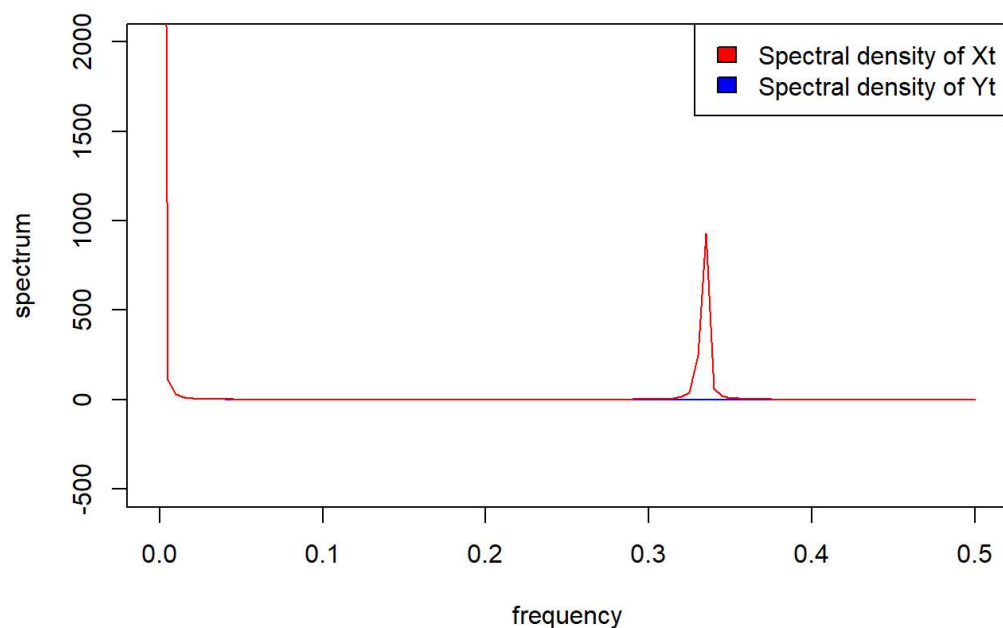
```
my_function1 <- function(phi, x) {
  result <- (1/3 + 4*cos(2*pi*x)/9 + 2*cos(4*pi*x)/9) / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function1( phi = 0.99, x), from = 0, to = 0.5, ylim = c(-500,2000)
      , col = "blue", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum")

my_function <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.99, x), from = 0, to = 0.5, ylim = c(0,2000)
      , col = "red", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum", add =TRUE)
legend("topright", legend = c("Spectral density of Xt", "Spectral density of Yt" ),
      fill = c("red", "blue"))
```

Spectral density of X_t



Mathematically, $f_y(\nu) = [\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)]f_x(\nu)$.

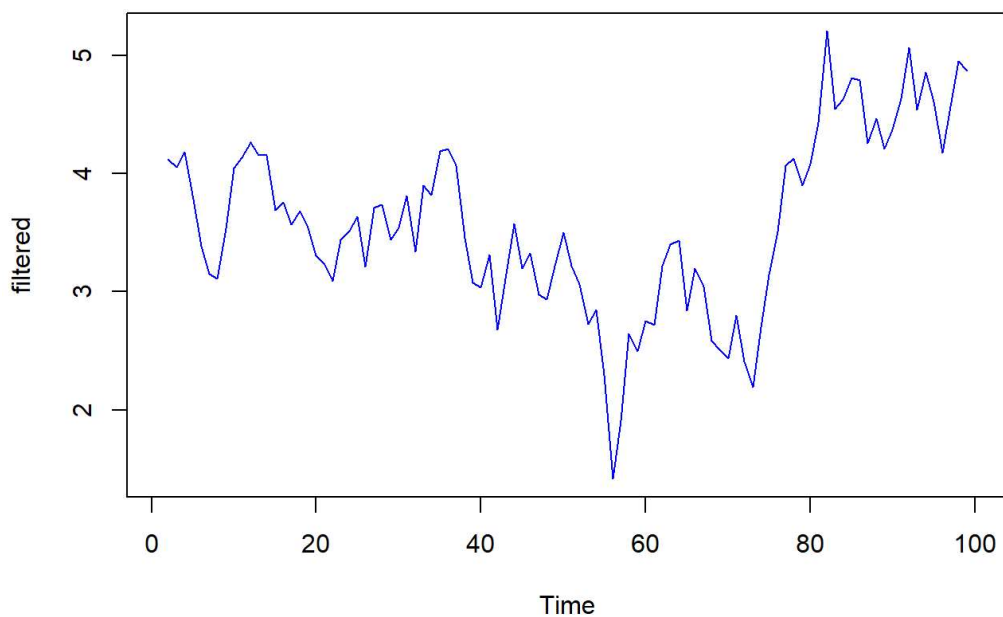
Moreover, graphically, the spectral density of X_t exhibits an obvious sudden peak at the frequency $\nu = 0.33$ while the spectral density of Y_t stays constant except at extremely low frequency, which means filtering is effective. Furthermore, their spectral density plots coincide mostly except around frequency $\nu = 0.33$.

(e)

```
set.seed(123)
ts <- arima.sim(model = list(ar = c(0, 0, 0.99)), n = 100)

filtered <- filter(ts, rep(1/3, 3), side = 2)
plot(filtered, col = "blue", main = "Plot of times series after filtered")
```

Plot of times series after filtered



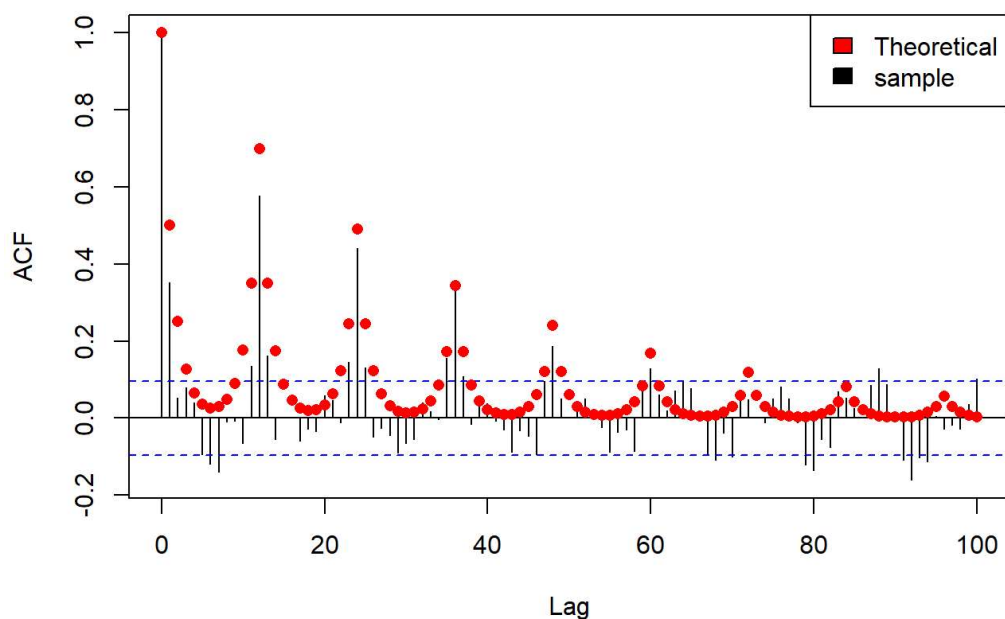
The filtered times series Y_t does NOT exhibit an obvious oscillatory behavior or say, the frequency is extremely small and the period is extremely large so that we cannot figure out any oscillatory behavior, which rightly support the spectral density plot in part (d).

Question 4

```
set.seed(123)
startvalue <- rnorm(13)
ts <- rep(0, 413)
ts[1:13] <- startvalue
set.seed(12)
W <- rnorm(500)
for (t in 14:413) {
  ts[t] <- 0.5*ts[t-1] + 0.7*ts[t-12] - 0.35*ts[t-13] + W[t]
}
ts <- as.ts(ts)
acf(ts, main = "Sample ACF Plot", lag.max = 100)

phi <- rep(0, 13)
phi[1] <- 0.5
phi[12] <- 0.7
phi[13] <- -0.35
true_acf <- ARMAacf(ar = phi, lag.max = 100, pacf = FALSE)
points(0:100, true_acf, col = "red", pch = 16)
legend("topright", legend = c("Theoretical", "sample"), fill = c("red", "black"))
```

Sample ACF Plot



Consider the equivalent seasonal ARMA model

$$X_t = 0.5X_{t-1} + 0.7X_{t-12} - 0.35X_{t-13} + W_t$$

where $W_t \sim WN(0, \sigma^2)$.

Notice that the theoretical acf is almost positive while the sample acf is not. However, the trend of them is consistent.