# Times Series Assignment5

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# Question 1

## (a)

Consider the following seasonal AR model:

$$(1-\phi B)(1-\Phi B^s)X_t = W_t,$$

which is

$$X_t - \phi W_{t-1} - \Phi X_{t-s} + \phi\Phi X_{t-s-1} = W_t$$

where $W_t \sim WN(0, \sigma^2)$.

Thus, we have

$$\begin{aligned}\gamma_w(h) &= Cov(X_{t+h} - \phi W_{t+h-1} - \Phi X_{t+h-s} + \phi\Phi X_{t+h-s-1},\ X_t - \phi W_{t-1} - \Phi X_{t-s} + \phi\Phi X_{t-s-1}) \\ &= \gamma_x(h) - \phi\gamma_x(h+1) - \Phi\gamma_x(h+s) + \phi\Phi\gamma_x(h+s+1) \\ &\quad - \phi[\gamma_x(h-1) - \phi\gamma(h) - \Phi\gamma_x(h+s-1) + \phi\Phi\gamma_x(h+s)] \\ &\quad - \Phi[\gamma_x(h-s) - \phi\gamma_x(h+1-s) - \Phi\gamma_x(h) + \phi\Phi\gamma_x(h+1)] \\ &\quad + \phi\Phi[\gamma_x(h-s-1) - \phi\gamma_x(h-s) - \Phi\gamma_x(h-1) + \phi\Phi\gamma_x(h)]\end{aligned}$$

Then apply $\gamma_w(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_w(\nu)e^{2\pi i\nu h}d\nu$, $\gamma_x(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(\nu)e^{2\pi i\nu h}d\nu$ and by the uniqueness of Fourier transformation, we can derive that $\frac{g_w(\nu)}{f_x(\nu)} = \frac{\sigma^2}{f_x(\nu)} =$

$$\begin{aligned}&[1+\phi^2+\Phi^2+(\phi\Phi)^2] - 2(\phi+\phi\Phi^2)\cos(2\pi\nu) - 2(\Phi+\phi^2\Phi)\cos(2\pi\nu s) \\ &+ 2\phi\Phi[\cos(2\pi\nu(s+1)) + \cos(2\pi\nu(s-1))].\end{aligned}$$

Therefore, the spectral density of $X_t$ is $f_x(\nu) =$

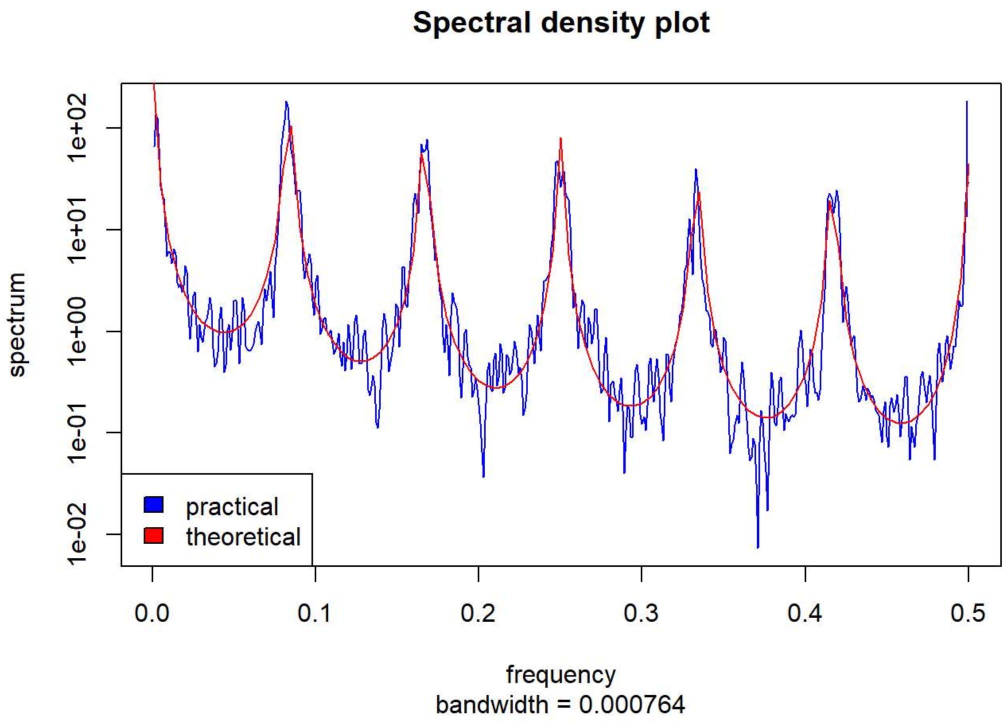
$$\frac{\sigma^2}{1+\phi^2+\Phi^2+(\phi\Phi)^2 - 2(\phi+\phi\Phi^2)\cos(2\pi\nu) - 2(\Phi+\phi^2\Phi)\cos(2\pi\nu s) + 2\phi\Phi[\cos(2\pi\nu(s+1)) + \cos(2\pi\nu(s-1))]}.$$

## (b)

```
library(forecast)
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

```
phi <- rep(0,13)
phi[1] <- 0.5
phi[12]<- 0.9
phi[13] <- -0.45

set.seed(12)

simulated_data <- arima.sim(n = 1000, list(ar = phi))
spec <- spec.pgram(simulated_data, spans = c(2), taper = 0, plot = FALSE)
par(mfrow = c(1,1))
plot(spec, main = "Spectral density plot", col = "blue")

    my_function <- function(sigma_squared, phi,Phi, x) {
      result <- sigma_squared /
        (1 + phi^2 + Phi^2 + (phi*Phi)^2
      - 2*(phi+ phi*Phi^2)*cos(2*pi*x)
      - 2*(Phi + phi^2*Phi)*cos(2*pi*x*12)
      + 2*phi*Phi*(cos(2*pi*x* 13) + cos(2*pi*x*11)))
      return(result)
    }

  curve(my_function(sigma_squared = 1, phi = 0.5, Phi = 0.9, x), from = 0, to = 0.5,
        col = "red", add = TRUE)
  legend("bottomleft", legend = c("practical", "theoretical"),
         fill = c("blue", "red"), cex = 1 )
```




The theoretical and practical results are consistent.

## (c)

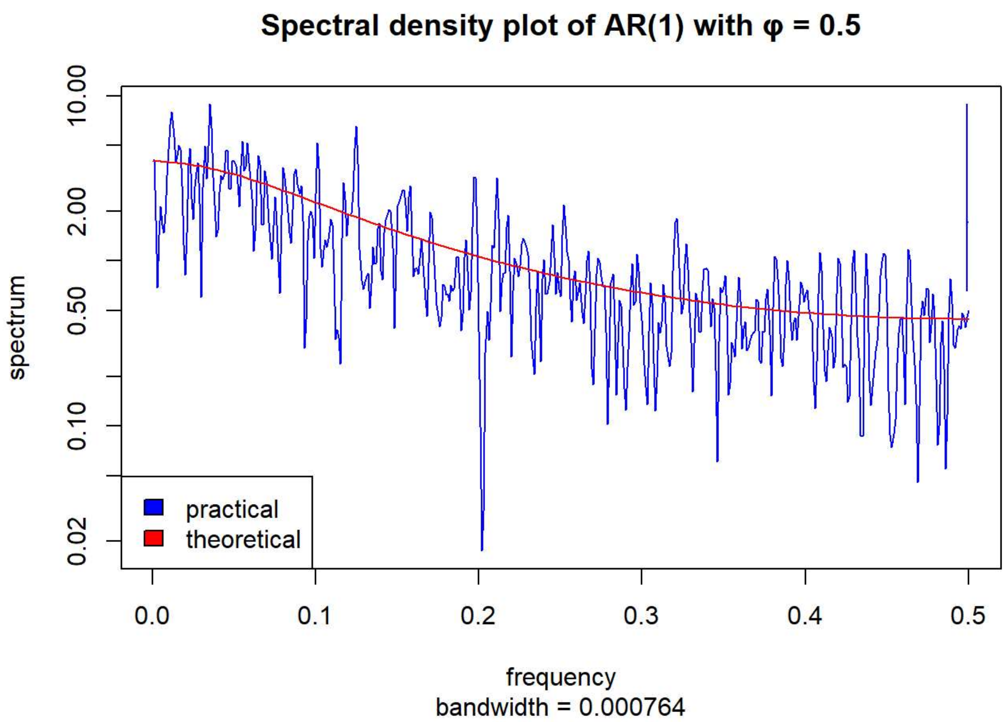
Similarly, for times series $X_t^{(1)}$ which satisfies

$$X_t^{(1)} - 0.5X_{t-1}^{(1)} = W_t,$$

we can derive its spectral density:

$$f_{X_t^{(1)}}(\nu) = \frac{\sigma^2}{1 - \cos(2\pi\nu) + 0.25}.$$

```
set.seed(12)
ts_data <- arima.sim(n = 1000, model = list(order = c(1,0,0), ar = 0.5))
spec <- spec.pgram(ts_data, spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Spectral density plot of AR(1) with φ = 0.5", col = "blue")

my_function1 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(2 * pi * x) + phi^2)
  return(result)
}

curve(my_function1(sigma_squared = 1, phi = 0.5, x), from = 0, to = 0.5
      , col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
       fill = c("blue", "red"), cex = 1 )
```



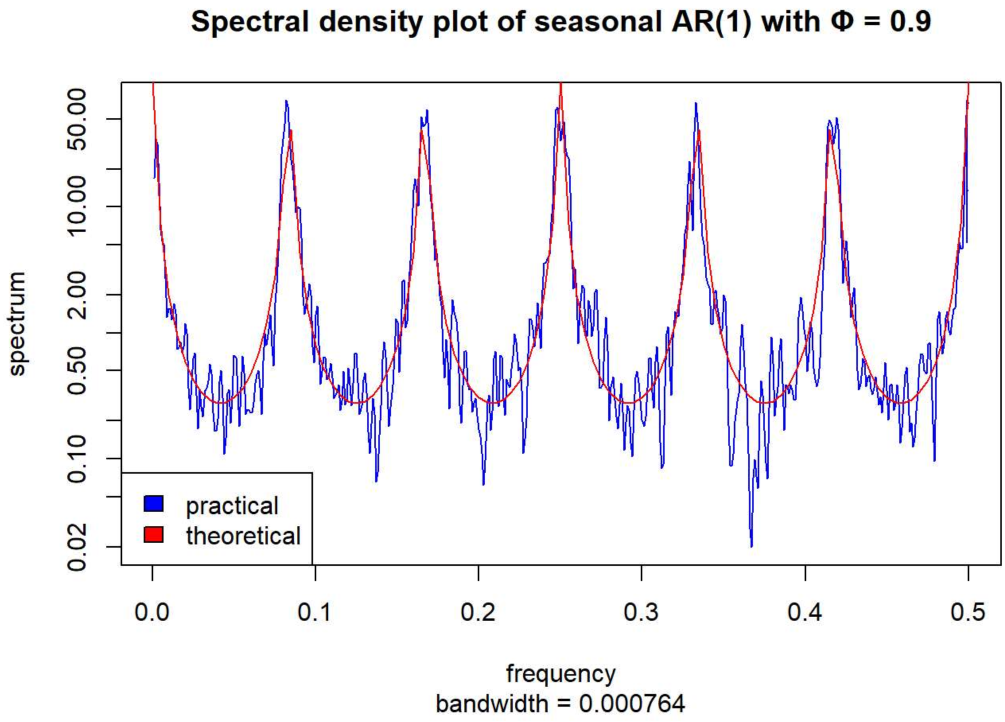

Again by performing the same method, for times series $X_t^{(2)}$ which satisfies

$$X_t^{(2)} - 0.9X_{t-12}^{(2)} = W_t,$$

we can derive its spectral density:

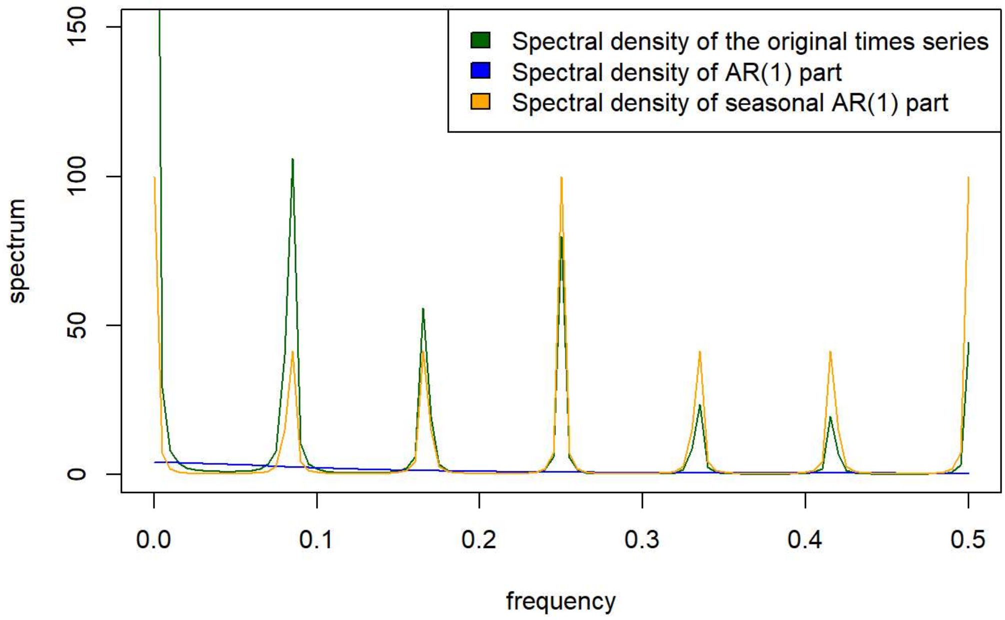
$$f_{X_t^{(2)}}(\nu) = \frac{\sigma^2}{1 - 1.8\cos(24\pi\nu) + 0.81}.$$

```
set.seed(12)
phi <- rep(0,12)
phi[12] <- 0.9
ts_data <- arima.sim(n = 1000, model = list(ar = phi))
spec <- spec.pgram(ts_data, spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Spectral density plot of seasonal AR(1) with Φ = 0.9", col = "blue")

my_function2 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(24 * pi * x) + phi^2)
  return(result)
}

curve(my_function2(sigma_squared = 1, phi = 0.9, x), from = 0, to = 0.5,
      , col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
       fill = c("blue", "red"), cex = 1 )
```




(d)

```
 my_function <- function(sigma_squared, phi,Phi, x) {
      result <- sigma_squared /
        (1 + phi^2 + Phi^2 + (phi*Phi)^2
      - 2*(phi+ phi*Phi^2)*cos(2*pi*x)
      - 2*(Phi + phi^2*Phi)*cos(2*pi*x*12)
      + 2*phi*Phi*(cos(2*pi*x* 13) + cos(2*pi*x*11)))
      return(result)
    }

  curve(my_function(sigma_squared = 1, phi = 0.5, Phi = 0.9, x), from = 0, to = 0.5, ylab = "spectrum", xlab="frequency",
ylim = c(0,150), col = "darkgreen")

  my_function1 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(2 * pi * x) + phi^2)
  return(result)
}

curve(my_function1(sigma_squared = 1, phi = 0.5, x), from = 0, to = 0.5
      , col = "blue", add = TRUE)

my_function2 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(24 * pi * x) + phi^2)
  return(result)
}

curve(my_function2(sigma_squared = 1, phi = 0.9, x), from = 0, to = 0.5,
      , col = "orange", add = TRUE)


legend("topright", legend = c("Spectral density of the original times series", "Spectral density of AR(1) part", "Spectral
density of seasonal AR(1) part"), fill = c("darkgreen","blue","orange"))
```




According to the spectral density plots, we can maintain that the seasonal AR(1) part dominates the seasonality of the original times series.

# Question 2

## (a)

From the inverse Fourier transformation, we have

$$\gamma(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu 0}d\nu = \int_{\frac{1}{6}\leq|\nu|\leq\frac{1}{3}} 5\ d\nu = \frac{5}{3}$$

and

$$\gamma(1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu}d\nu = \int_{\frac{1}{6}\leq|\nu|\leq\frac{1}{3}} 5\cos(2\pi\nu)d\nu = \frac{5}{\pi}[\sin(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})] = 0.$$

## (b)

Since

$$\gamma_y(h) = Cov(X_{t+h} - X_{t+h-12},\ X_t - X_{t-12}) = 2\gamma_x(h) - \gamma_x(h+12) - \gamma_x(h-12),$$

then apply inverse Fourier transformation we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu)e^{2\pi i\nu h}d\nu = \int_{-\frac{1}{2}}^{\frac{1}{2}} [2 - 2\cos(24\pi\nu)]f_x(\nu)e^{2\pi i\nu h}d\nu.$$

Due to the uniqueness of Fourier transformation, we have

$$f_y(\nu) = [2 - 2\cos(24\pi\nu)]f_x(\nu).$$

Consequently, the spectral density of $Y_t$ is

$$f_y(\nu) = \begin{cases} 10 - 10\cos(24\pi\nu), & \frac{1}{6} \le |\nu| \le \frac{1}{3} \\ 0, & else. \end{cases}$$

# Question 3

## (a)

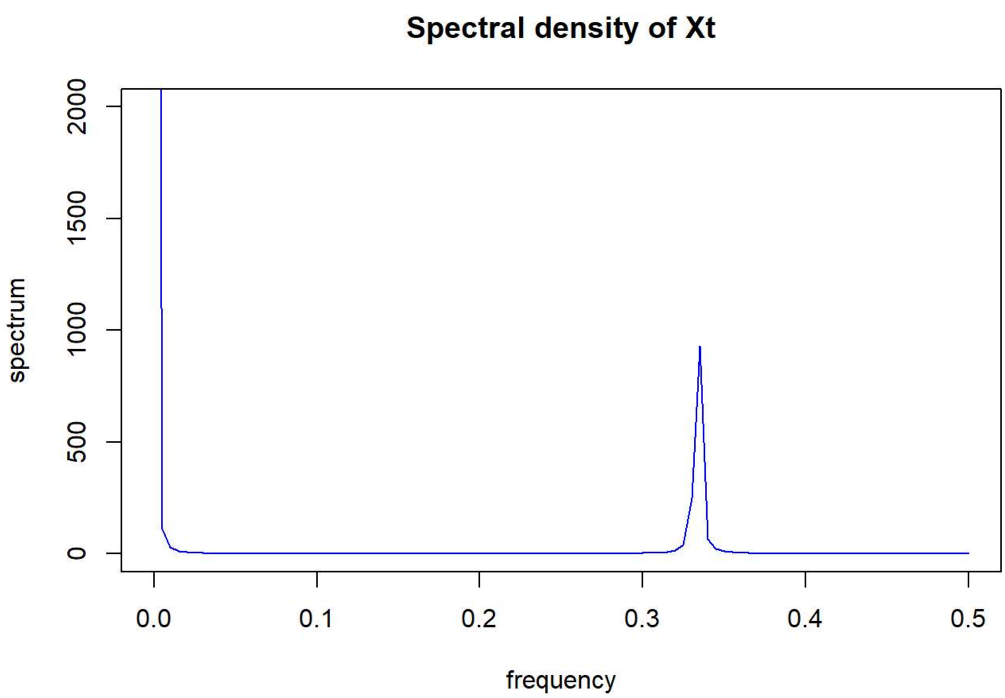
Consider the model $X_t - 0.99X_{t-1} = W_t$, where $W_t \sim WN(0, \sigma^2)$, we have $\gamma_w(h) = Cov(X_{t+h} - 0.99X_{t+h-3},\ X_t - 0.99X_{t-3}) = (1 + 0.99^2)\gamma_x(h) - 0.99[\gamma_x(h+3) + \gamma_x(h-3)]$. Then apply inverse Fourier transformation $\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu h}d\nu$ and notice the uniqueness of Fourier transformation, we will obtain

$$f_w(\nu) = [1 + 0.99^2 - 1.98\cos(6\pi\nu)]f_x(\nu).$$

Thus, the spectral density of $X_t$ is

$$f_x(\nu) = \frac{f_w(\nu)}{1 + 0.99^2 - 1.98\cos(6\pi\nu)} = \frac{1}{1 + 0.99^2 - 1.98\cos(6\pi\nu)}.$$

```
my_function <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.99, x), from = 0, to = 0.5, ylim = c(0,2000)
      , col = "blue", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum")
```

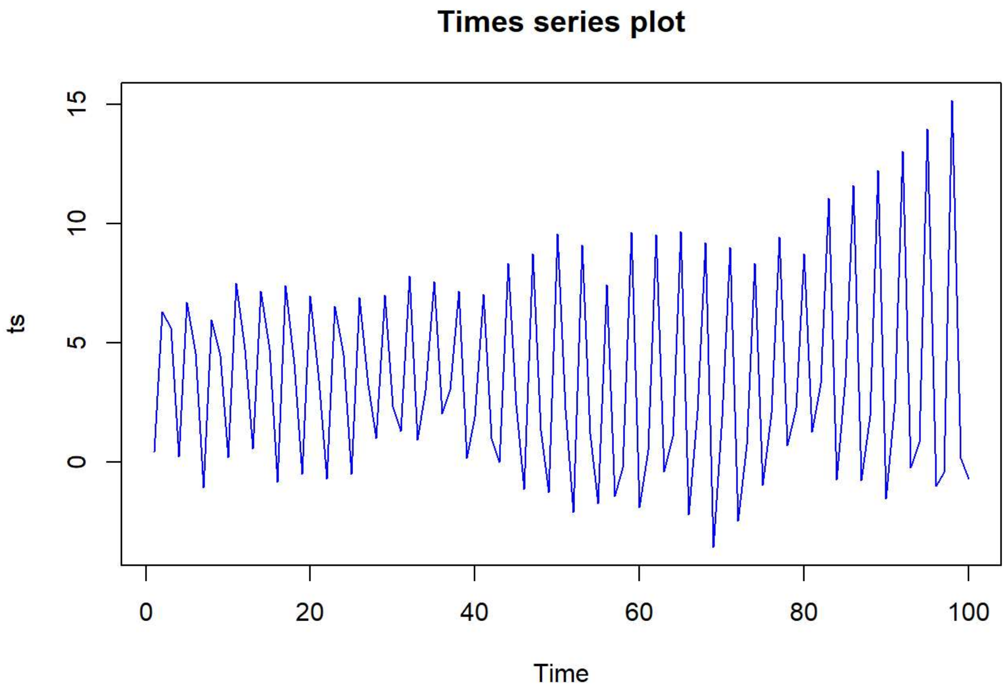



## (b)

Check the spectral density plot and we can figure out that it shows an obvious sudden peak at frequency $\nu = 0.33$ approximately, which means the times series $X_t$ exhibit an oscillation with frequency 0.33, i.e., the period is 3 approximately.

## (c)

```
set.seed(123)
ts <- arima.sim(model = list(ar = c(0,0,0.99)), n = 100)
plot(ts, col = "blue", main = "Times series plot")
```




From the plot we can maintain that the times series $X_t$ oscillates with period 3 approximately, which is consistent with the conclusion of part (b).

## (d)

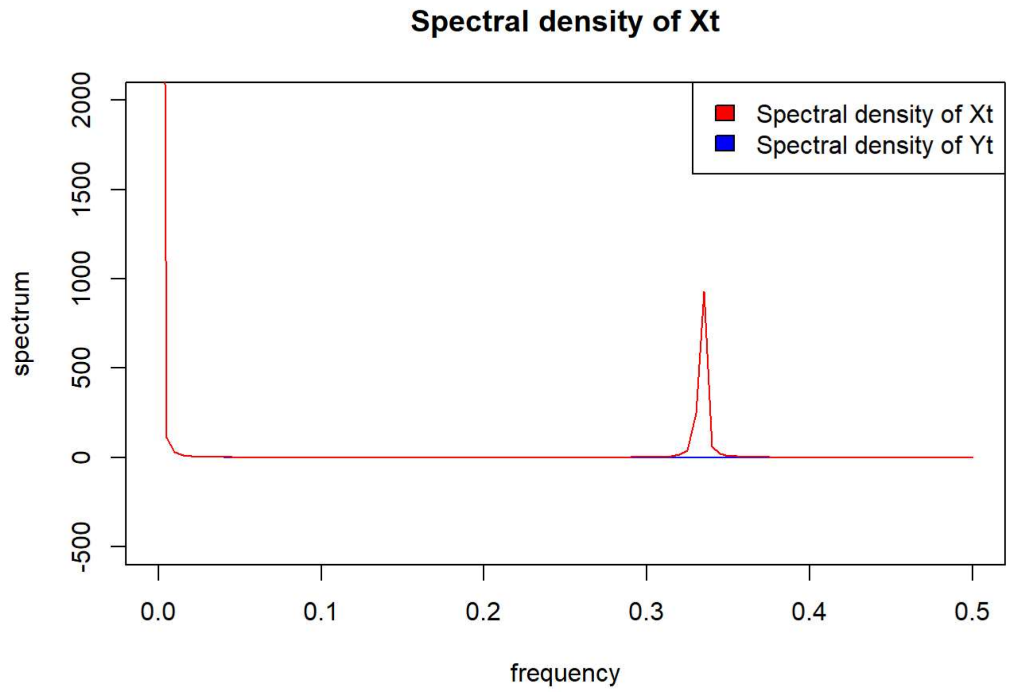
Since $Y_t = \frac{X_{t-1}+X_t+X_{t+1}}{3}$, then we have

$$\gamma_y(h) = \frac{1}{9}[3\gamma_x(h) + 2(\gamma_x(h+1) + \gamma(h-1)) + (\gamma_x(h-2) + \gamma_x(h+2))].$$

Thus, by performing the inverse Fourier transformation and notice the uniqueness of Fourier transformation, we can obtain that

$$f_y(\nu) = [\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)]f_x(\nu) = \frac{\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)}{1 + 0.99^2 - 1.98\cos(6\pi\nu)}.$$

```
 my_function1 <- function(phi, x) {
  result <- (1/3 + 4*cos(2*pi*x)/9 + 2*cos(4*pi*x)/9) / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function1( phi = 0.99, x), from = 0, to = 0.5, ylim = c(-500,2000)
      , col = "blue", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum")

 my_function <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.99, x), from = 0, to = 0.5, ylim = c(0,2000)
      , col = "red", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum", add =TRUE)
legend("topright", legend = c("Spectral density of Xt", "Spectral density of Yt" ),
       fill = c("red", "blue"))
```

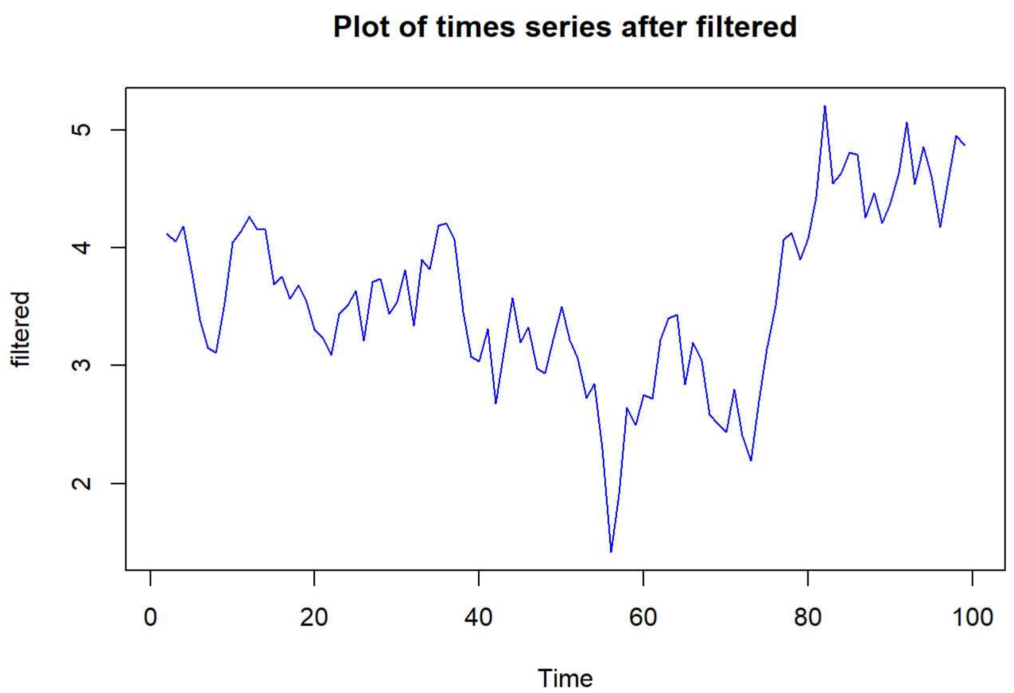



Mathematically, $f_y(\nu) = [\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)]f_x(\nu)$.

Moreover, graphically, the spectral density of $X_t$ exhibits an obvious sudden peak at the frequency $\nu = 0.33$ while the spectral density of $Y_t$ stays constant except at extremely low frequency, which means filtering is effective. Furthermore, their spectral density plots coincide mostly except around frequency $\nu = 0.33$.

## (e)

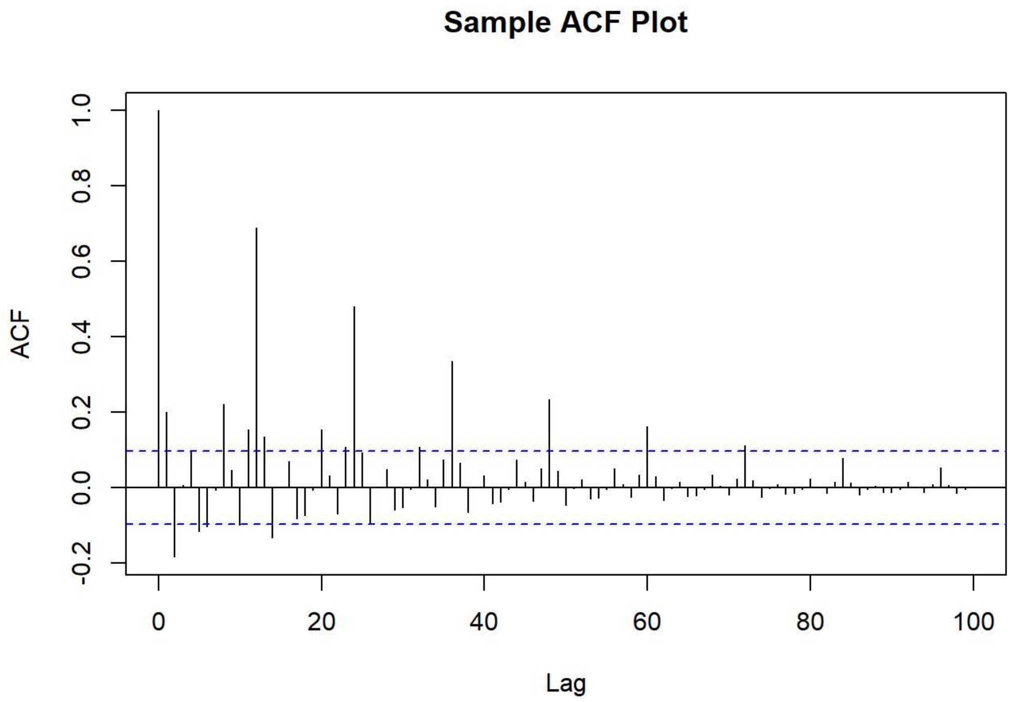
```
set.seed(123)
ts <- arima.sim(model = list(ar = c(0,0,0.99)), n = 100)

filtered <- filter(ts, rep(1/3,3), side = 2)
plot(filtered, col = "blue", main = "Plot of times series after filtered")
```



The filtered times series $Y_t$ does NOT exhibit an obvious oscillatory behavior or say, the frequency is extremely small and the period is extremely large so that we cannot figure out any oscillatory behavior, which rightly support the spectral density plot in part (d).

# Question 4

```
set.seed(123)
startvalue <- rnorm(13)
ts <- rep(0, 413)
ts[1:13] <- startvalue
for (t in 14:413) {
  ts[t] <- 0.5*ts[t-1] + 0.7*ts[t-12] - 0.35*ts[t-13]
}
ts <- as.ts(ts)
acf(ts, main = "Sample ACF Plot", lag.max = 100)
```




Consider the equivalent seasonal ARMA model

$$X_t = 0.5X_{t-1} + 0.7X_{t-12} - 0.35X_{t-13} + W_t$$

where $W_t \sim WN(0, \sigma^2)$.

Notice that for any time $t$, $X_t$ is determined only by $X_{t-1}$, $X_{t-12}$ and $X_{t-13}$, which means $\gamma(h) = 0$ for all $h \notin \{0, k, 12k, 13k\}$ where $k \in \mathbb{Z}$. Moreover, we can obtain

$$\gamma(h) = 0.5\gamma(h-1) + 0.7\gamma(h-12) - 0.35\gamma(h-13).$$

Since $|\phi_{12}| = 0.7$ is the largest one of $|\phi_1|$, $|\phi_{12}|$ and $|\phi_{13}|$, which indicates the seasonal AR part $X_t = 0.7X_{t-12} + W_t$ dominates the times series to some extent, then we can approximately maintain that $\gamma(12h)$ should exhibit an exponentially decreasing behaviour, i.e., $\gamma(12h) \approx e^{0.7h}$.

Consequently, we give some properties of true ACF should behave:

- $\gamma(h) = 0$ for all $h \notin \{0, k, 12k, 13k\}$ where $k \in \mathbb{Z}$.
- $\gamma(12h)$ should exhibit an exponentially decreasing behaviour with rate 0.7.

From the sample ACF plot, we discover that it substantially obey the properties above. Though $\hat{\gamma}(2)$ is beyond $95\%$ confidence interval, we are still able to draw the conclusion that the sample ACF is consistent with the true theoretical ACF.