Times Series Assignment 6

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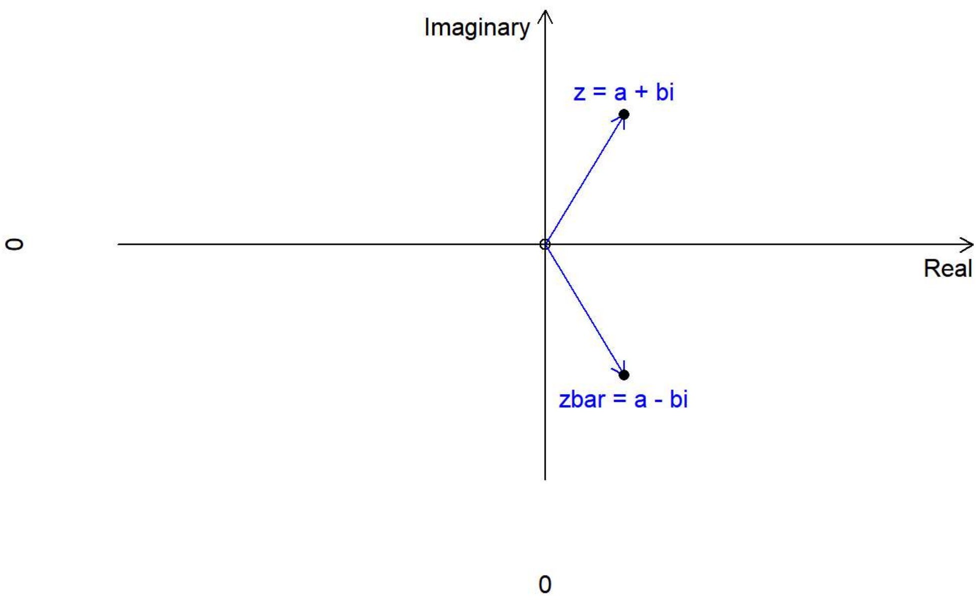
2023-12-12

Question 1

(a)

Let $z = a + bi$ where $a, b \in \mathbb{R}$, followed by $\bar{z} = a - bi$, thus $\bar{z}z = (a - bi)(a + bi) = a^2 + b^2 = |z|^2$.

```
plot(0,0, xlim = c(-5,5), ylim = c(-5,5), axes = FALSE)
abline(h = 0)
abline(v = 0)
arrows(0,0,1,3, col = "blue", length = 0.1)
arrows(0,0,1,-3, col = "blue", length = 0.1)
arrows(0,-5,0,5.4, length = 0.1)
arrows(-5,0, 5.4,0, length = 0.1)
text(5.4, 0, "Real part", pos = 1)
text(0, 5, "Imaginary", pos = 2)
points(1,3, pch = 16)
text(1,3,"z = a + bi", pos = 3, col = "blue")
points(1,-3, pch = 16)
text(1,-3,"zbar = a - bi", pos = 1, col = "blue")
```



(b)

Let $z = re^{i\theta}$ where $r \in \mathbb{R}$ denotes the modules and θ denotes the argument of z, then followed by $\bar{z} = re^{-i\theta}$.Thus, the conjugate of z^j is $\overline{z^j} = \overline{r^j e^{ij\theta}} = r^j \overline{e^{ij\theta}} = r^j e^{-ij\theta} = (re^{-i\theta})^j = \bar{z}^j$.

Consequently, $\overline{z^j} = \bar{z}^j$ has been checked.

(c)

Since $\overline{z^j} = \bar{z}^j$ and $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, thus $p(\bar{z}) = \sum_{j=1}^{k} a_j \bar{z}^j = \sum_{j=1}^{k} \overline{a_j z^j} = \overline{\sum_{j=1}^{k} a_j z^j} = \overline{p(z)}$.

Consequently, $p(z)p(\bar{z}) = |p(z)|^2$.

Question 2

(a)

Denote $\phi(B)X_t = \theta(B)W_t$, then the spectral density of X_t is

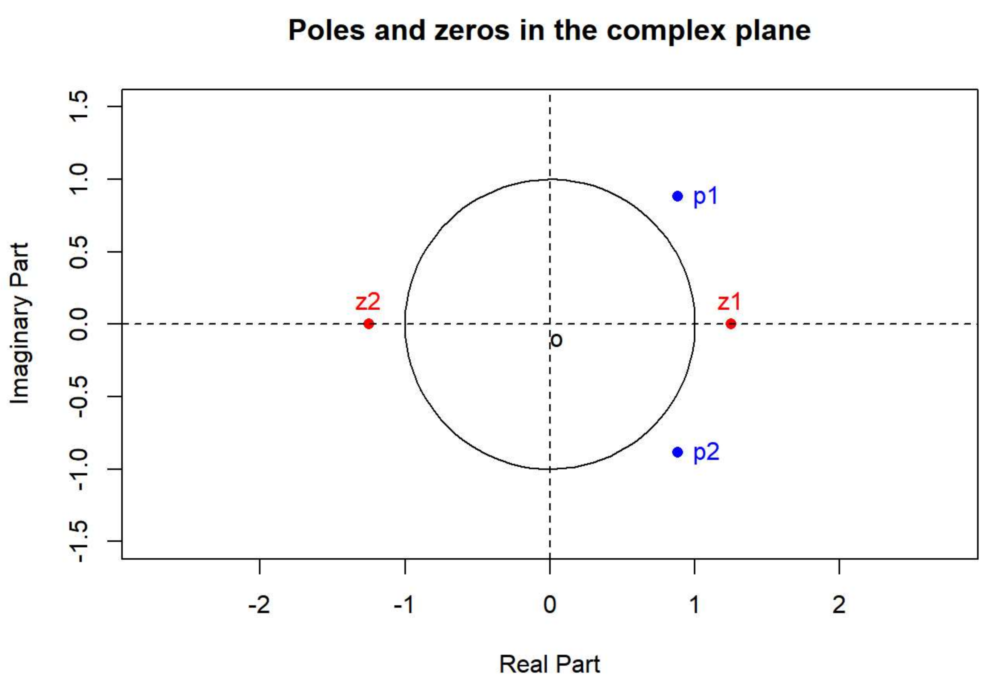
$$f_x(\nu) = \sigma^2 \frac{|\theta(e^{2\pi i\nu})|^2}{|\phi(e^{2\pi i\nu})|^2}.$$

In this case, the spectral density is

$$f_x(\nu) = \frac{|1 - \frac{16}{25}e^{4\pi i\nu}|^2}{|1 - \frac{4\sqrt{2}}{5}e^{2\pi i\nu} + \frac{16}{25}e^{4\pi i\nu}|^2} = \frac{1 + (\frac{4}{5})^4 - \frac{32}{25}\cos(4\pi\nu)}{1 + (\frac{4\sqrt{2}}{5})^2 + (\frac{4}{5})^4 - \frac{8\sqrt{2}}{5}\cos(2\pi\nu) + \frac{32}{25}\cos(4\pi\nu) - \frac{128\sqrt{2}}{125}\cos(6\pi\nu)}.$$

- For zeros: let $\theta(z) = 0$, then we obtain $z_1 = \frac{5}{4}$ and $z_2 = -\frac{5}{4}$.
- For holes: let $\phi(z) = 0$, then we will obtain $p_1 = \frac{5\sqrt{2}}{8}(1+i)$ and $p_2 = \frac{5\sqrt{2}}{8}(1-i)$.

```
theta <- seq(0, 2*pi, length.out = 100)

x <- cos(theta)
y <- sin(theta)

plot(x, y, type = "l", asp = 1, xlim = c(-1.5, 1.5), ylim = c(-1.5, 1.5),
     xlab = "Real Part", ylab = "Imaginary Part", main = "Poles and zeros in the complex plane")


text(-0.1,-0.1,"o", pos = 4)

points(1.25, 0, pch = 16, col = "red")
text(1.25,0,"z1", pos = 3, col = "red")
points(-1.25, 0, pch = 16, col = "red")
text(-1.25,0,"z2", pos = 3, col = "red")

points(5*sqrt(2)/8, 5*sqrt(2)/8, pch = 16, col = "blue")
text(5*sqrt(2)/8, 5*sqrt(2)/8,"p1", pos = 4, col = "blue")
points(5*sqrt(2)/8, -5*sqrt(2)/8, pch = 16, col = "blue")
text(5*sqrt(2)/8, -5*sqrt(2)/8,"p2", pos = 4, col = "blue")

abline(h = 0, col = "black", lwd = 1, lty = 2)
abline(v = 0, col = "black", lwd = 1, lty = 2)
```

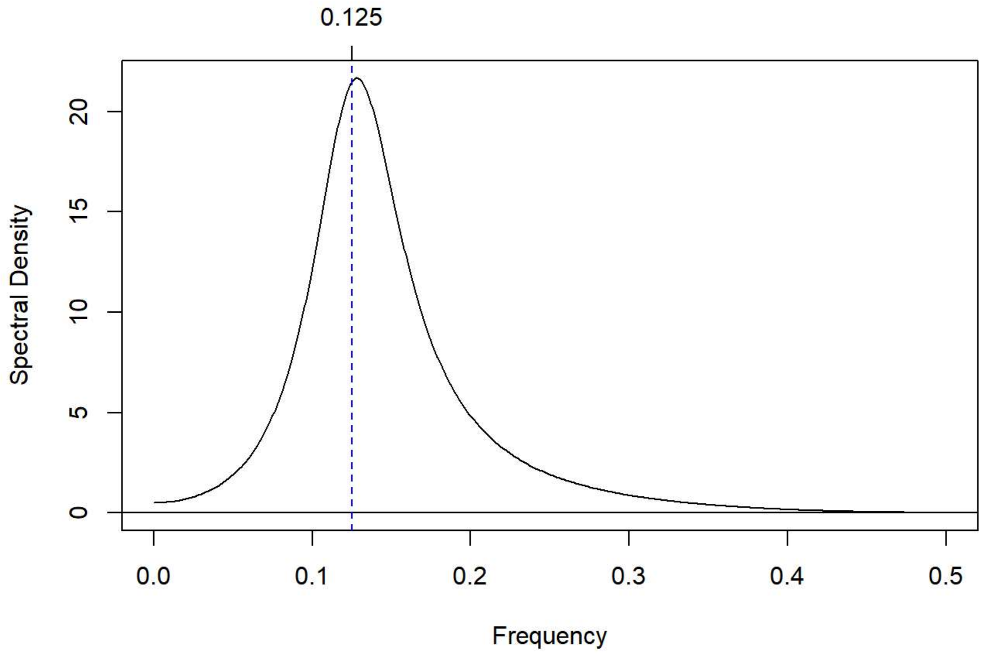


As the argument moves from 0 to π, the closer it is to the zeros, the smaller the spectral density is. The closer it is to the holes, the larger the spectral density is.

In this case, from 0 to $\frac{\pi}{4}$, it goes toward the hole p_1 and away from the zero z_1, thus the spectral density increases from 0 to $\frac{1}{8}$ approximately. Similarly, from $\frac{\pi}{4}$ to π, it goes away from the hole p_1 and toward the zero z_2, thus and the spectral density decreases from $\frac{1}{8}$ to $\frac{1}{2}$ approximately.

Now we use ARMAspec function in R to verify our inference.

```
spectrum <- ARMAspec(model = list(ar = c(4*sqrt(2)/5, -16/25), ma = c(0, -16/25)))
abline(v = 0.125, lty = 2,, col = "blue")
axis(3, at = 0.125, labels = "0.125")
```



We can figure out that the spectral density behaves a unique peak around $\frac{1}{8}$, thus we can maintain that the results are consistent with our inference.

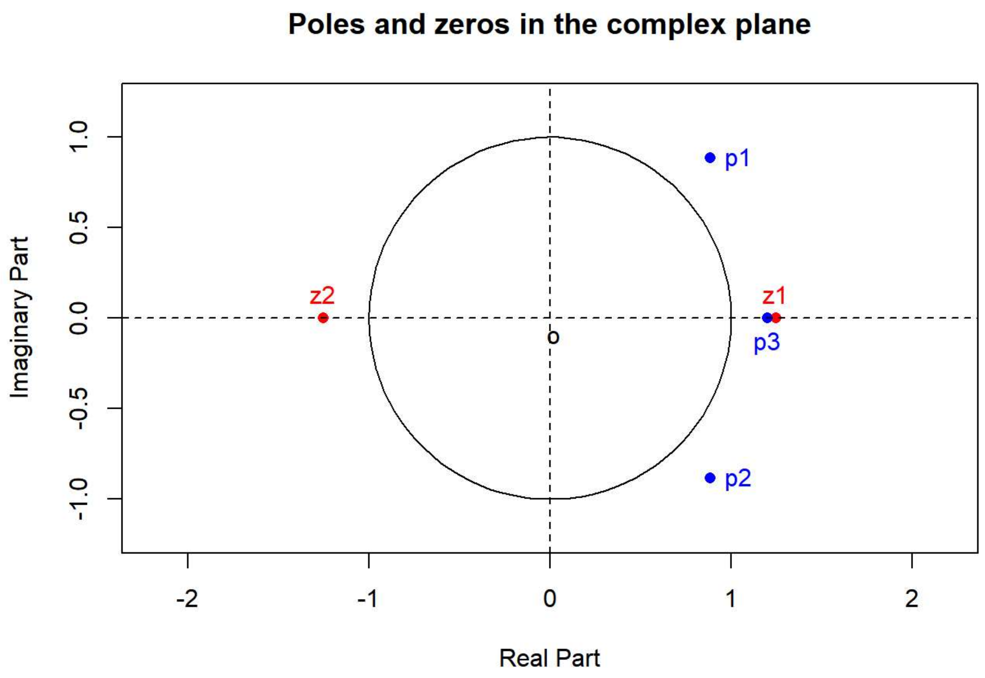
(b)

Since $(1-\frac{5}{6}B)Y_t = X_t$ and $1-\frac{5}{6}z$ has no roots inside the unit circle, thus we have

$$f_y(\nu) = \frac{f_x(\nu)}{|1-\frac{5}{6}e^{2\pi i\nu}|^2} = \frac{1+(\frac{4}{5})^4 - \frac{32}{25}\cos(4\pi\nu)}{(1+(\frac{5}{6})^2 - \frac{5}{3}\cos(2\pi\nu))[1+(\frac{4\sqrt{2}}{5})^2 + (\frac{4}{5})^4 - \frac{8\sqrt{2}}{5}\cos(2\pi\nu) + \frac{32}{25}\cos(4\pi\nu) - \frac{128\sqrt{2}}{125}\cos(6\pi\nu)]}.$$

- For zeros: let $\theta(z) = 0$, then we obtain $z_1 = \frac{5}{4}$ and $z_2 = -\frac{5}{4}$.
- For holes: let $\phi(z) = 0$, then we will obtain $p_1 = \frac{5\sqrt{2}}{8}(1+i)$, $p_2 = \frac{5\sqrt{2}}{8}(1-i)$ and $p_3 = \frac{6}{5}$.

```
theta <- seq(0, 2*pi, length.out = 100)

x <- cos(theta)
y <- sin(theta)

plot(x, y, type = "l", asp = 1, xlim = c(-1.2, 1.2), ylim = c(-1.2, 1.2),
     xlab = "Real Part", ylab = "Imaginary Part", main = "Poles and zeros in the complex plane")


text(-0.1,-0.1,"o", pos = 4)

points(1.25, 0, pch = 16, col = "red")
text(1.25,0,"z1", pos = 3, col = "red")
points(-1.25, 0, pch = 16, col = "red")
text(-1.25,0,"z2", pos = 3, col = "red")

points(5*sqrt(2)/8, 5*sqrt(2)/8, pch = 16, col = "blue")
text(5*sqrt(2)/8, 5*sqrt(2)/8,"p1", pos = 4, col = "blue")
points(5*sqrt(2)/8, -5*sqrt(2)/8, pch = 16, col = "blue")
text(5*sqrt(2)/8, -5*sqrt(2)/8,"p2", pos = 4, col = "blue")
points(6/5, 0, pch = 16, col = "blue")
text(6/5,0,"p3", pos = 1, col = "blue")

abline(h = 0, col = "black", lwd = 1, lty = 2)
abline(v = 0, col = "black", lwd = 1, lty = 2)
```

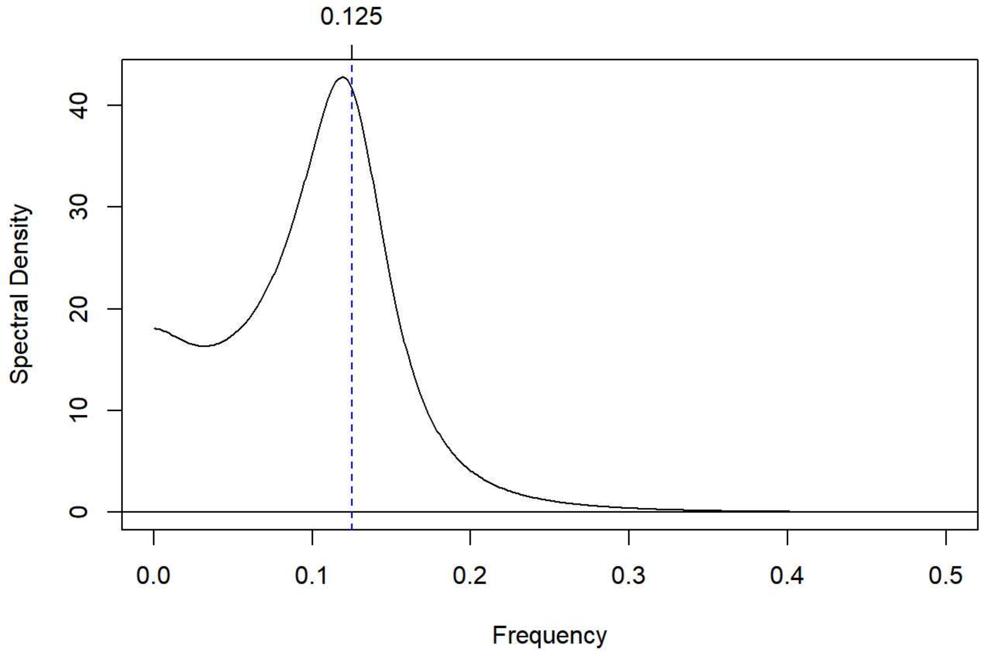


As the argument moves from 0 to π, the closer it is to the zeros, the smaller the spectral density is. The closer it is to the holes, the larger the spectral density is.

In this case, from 0 to $\frac{\pi}{4}$, it goes from the hole p_3 to the hole p_1 and away from the zero z_1, thus the spectral density first decreases and then increases from 0 to $\frac{1}{8}$ approximately. Then, from $\frac{\pi}{4}$ to π, it goes away from the hole p_1 and toward the zero z_2, thus and the spectral density decreases from $\frac{1}{8}$ to $\frac{1}{2}$ approximately.

Now we use ARMAspec function in R to verify our inference.

```
spectrum <- ARMAspec(model = list(ar = c(4*sqrt(2)/5 + 5/6, -16/25 - 2*sqrt(2)/3, 8/15), ma = c(0, -16/25)))
abline(v = 0.125, lty = 2,, col = "blue")
axis(3, at = 0.125, labels = "0.125")
```



We can figure out that the spectral density plot is consistent with our description, which means our inference is correct.

Question 3

Since $\phi(B)X_t = \theta(B)W_t$, and $(1-\frac{5}{6}B)Y_t = X_t$, then we have

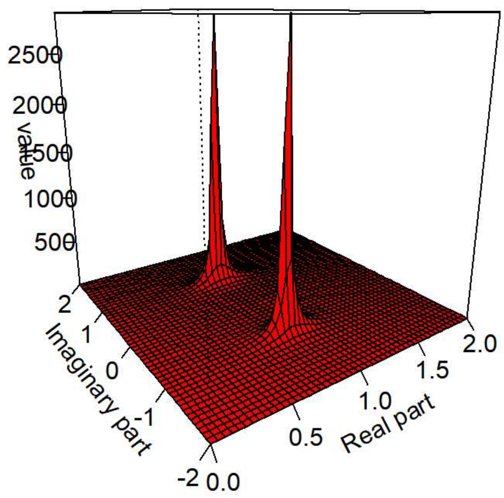
$$(1-\frac{5}{6}B)\phi(B)Y_t = \theta(B)W_t$$

where $\phi(B) = 1 - \frac{4\sqrt{2}}{5}B + (\frac{4}{5})^2B^2$ and $\theta(B) = 1-(\frac{4}{5})^2B^2$.

Thus, we have

$$\psi(z) = \frac{\theta(z)}{\phi(z)(1-\frac{5}{6}z)} = \frac{1-(\frac{4}{5})^2z^2}{(1-\frac{4\sqrt{2}}{5}z+(\frac{4}{5})^2z^2)(1-\frac{5}{6}z)}.$$

```
psi2 <- function(a,b){
  z <- a + 1i*b;
  result <- (1 - (4/5)^2*z^2)/((1 - z*4*sqrt(2)/5 + (4/5)^2*z^2)*(1-z*5/6));
  return((abs(result))^2)

}

x <- seq(0, 2, length.out = 50)
y <- seq(-2, 2, length.out = 50)
value <- outer(x,y, FUN = psi2)

persp(x = x, y = y, z = value,
      xlab = "Real part", ylab = "Imaginary part", zlab = "value", theta = -35, axes = TRUE, box = TRUE, col = "red", ticktyp
e = "detailed")
```

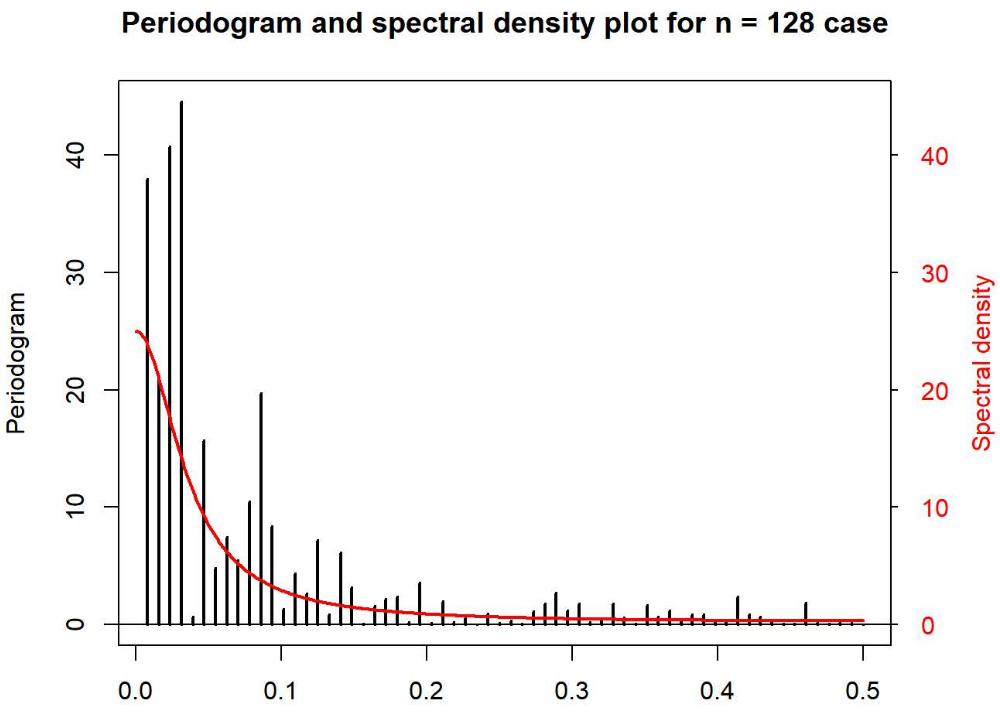


Question 2

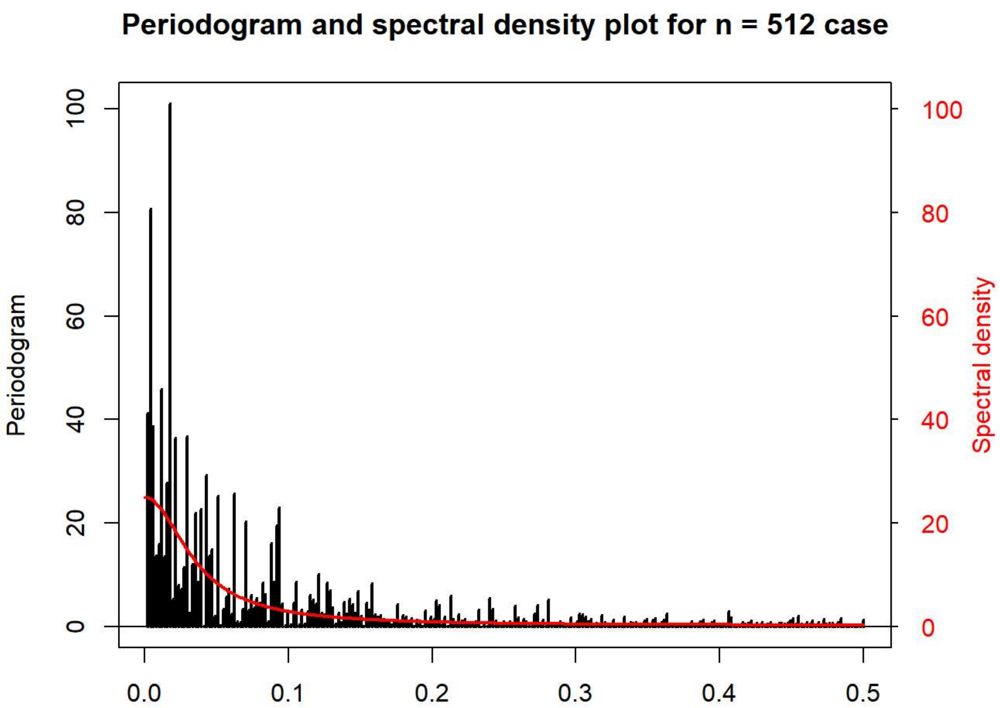
```
set.seed(123)
ts1 <- arima.sim(model = list(ar = 0.8), n = 128)
ts2 <- arima.sim(model = list(ar = 0.8), n = 512)
ts3 <- arima.sim(model = list(ar = 0.8), n = 1024)
ts4 <- arima.sim(model = list(ar = 0.8), n = 2048)
```

(a)

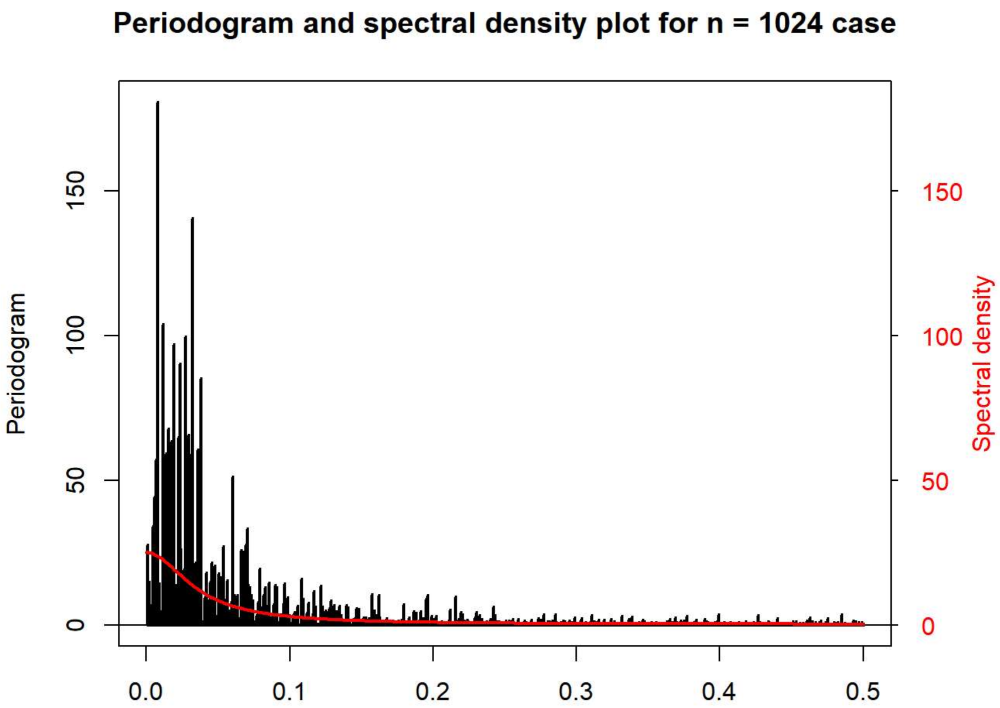
```
library(TSA)
par(mar=c(2,4,4,5))
periodogram(ts1,  main = "Periodogram and spectral density plot for n = 128 case")
abline(h = 0)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F) $spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```



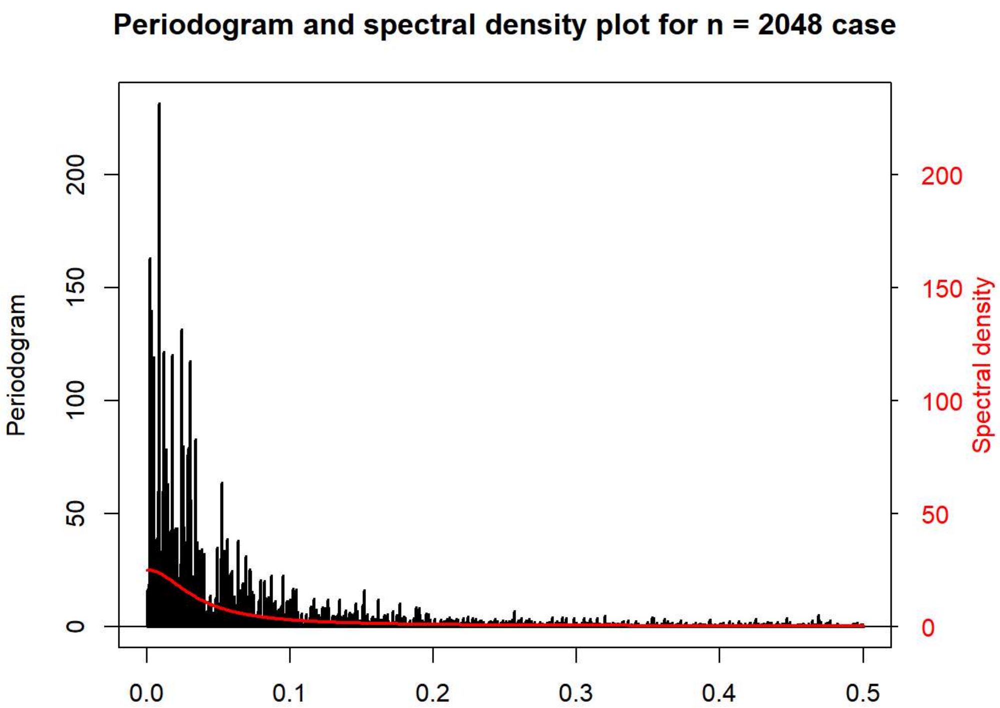

```
library(TSA)
par(mar=c(2,4,4,5))
periodogram(ts2,  main = "Periodogram and spectral density plot for n = 512 case")
abline(h = 0)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F)$spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```




```
library(TSA)
par(mar=c(2,4,4,5))
periodogram(ts3,  main = "Periodogram and spectral density plot for n = 1024 case")
abline(h = 0)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F)$spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```




```
library(TSA)
par(mar=c(2,4,4,5))
periodogram(ts4,  main = "Periodogram and spectral density plot for n = 2048 case")
abline(h = 0)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F) $spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```



(b)

Since $\frac{2}{f(\nu)} I(\hat{\nu}^{(n)}) \xrightarrow{d} \chi^2(2)$, thus we have

$$\Pr(\chi^2(1-\alpha/2, 2) \leq \frac{2}{f(\nu)} I(\hat{\nu}^{(n)}) \leq \chi^2(\alpha/2, 2)) = 1 - \alpha.$$

Consequently, the $100(1-\alpha)\%$ approximate confidence interval is

$$[\frac{2I(\hat{\nu}^{(n)})}{\chi^2(\alpha/2,2)}, \quad \frac{2I(\hat{\nu}^{(n)})}{\chi^2(1-\alpha/2,2)}]$$

where $\chi^2(\beta, 2)$ denotes for the upper-β quantile of $\chi^2(2)$.

```
approx.ci <- function(ts,alpha){
  perd <- periodogram(ts, plot = FALSE)
  index <- which.min(abs(perd$freq - 0.1))
  period.value <- perd$spec[index] # calculate I(\hat{v}^{(n)})
  upper <- 2*period.value / qchisq(alpha/2, df = 2)
  lower <- 2*period.value / qchisq(1- alpha/2, df = 2)
  return(c(lower, upper))
}

approx.ci(ts1, 0.05)
```

```
## [1]  0.3555618 51.8064116
```

```
approx.ci(ts2, 0.05)
```

```
## [1]   0.8456224 123.2096769
```

```
approx.ci(ts3, 0.05)
```

```
## [1]   0.9376798 136.6227239
```

```
approx.ci(ts4, 0.05)
```

```
## [1]   2.246802 327.365655
```

- For n = 128 case, the 95% approximate confidence interval is [0.3555618, 51.8064116].
- For n = 512 case, the 95% approximate confidence interval is [0.8456224, 123.2096769].
- For n = 1024 case, the 95% approximate confidence interval is [0.9376798, 136.6227239].
- For n = 2048 case, the 95% approximate confidence interval is [2.2468017, 327.3656549].

```
psi <- function(z) 1/(1 - 0.8*z)
f <- function(x) abs(psi(exp(2i*pi*x)))^2
f(0.1)
```

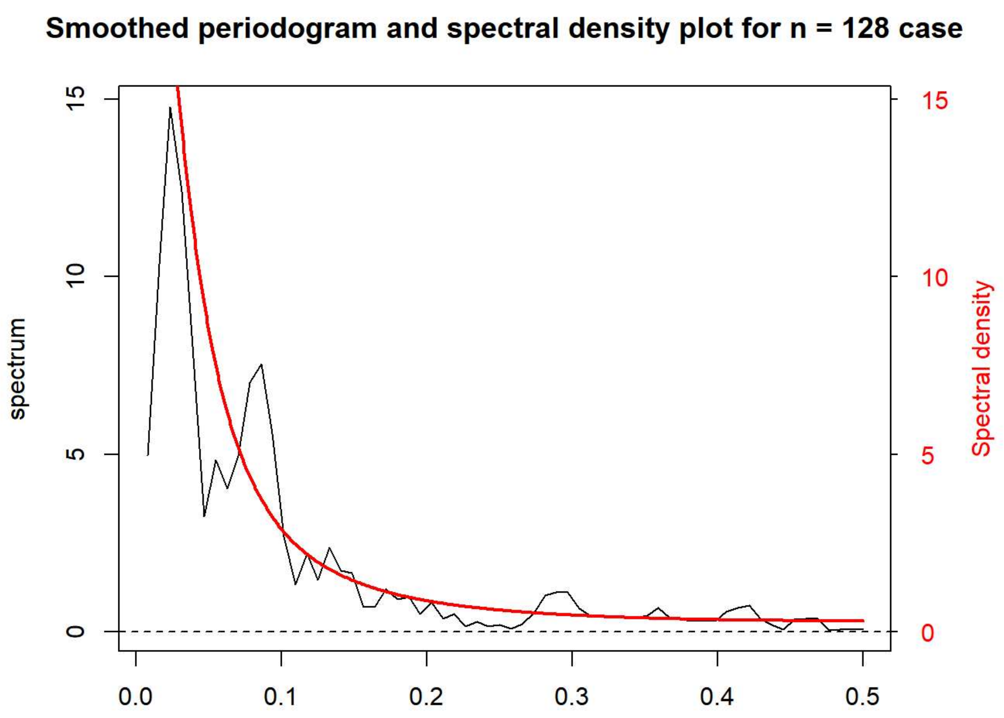
```
## [1] 2.893746
```

Before we evaluate the result, we first calculate the true spectral density $f(0.1)$, which is 2.8937462 with calculation process as shown above.

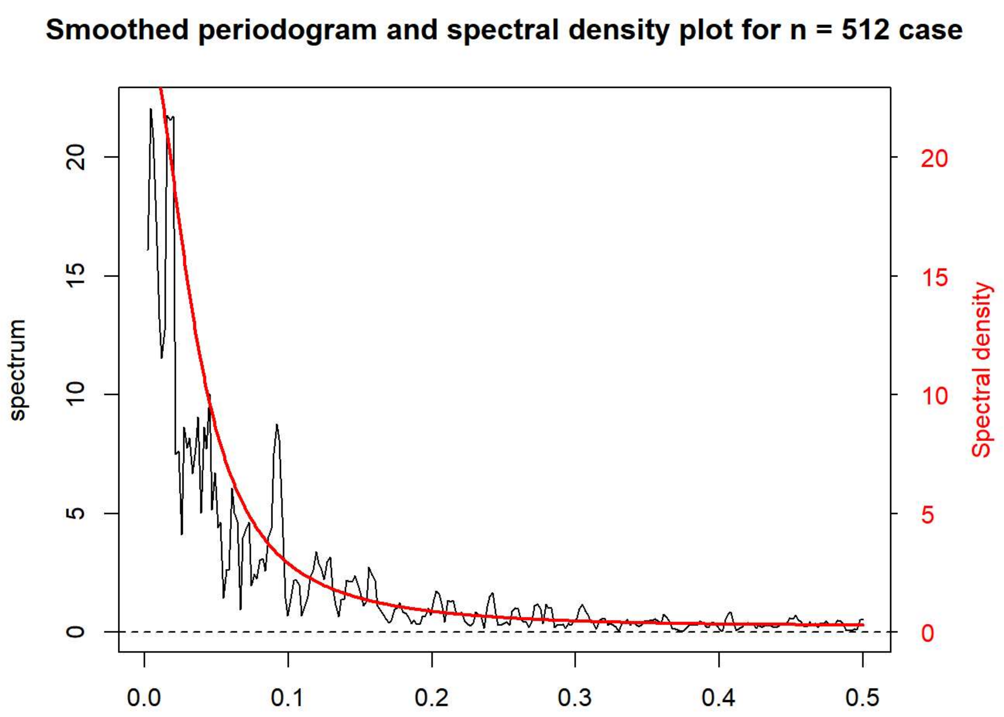
The approximate confidence intervals contain the true value, however, they tend to be quite large. This is consistent with the asymptotic theory, which says that each point of the periodogram has a non-zero asymptotic variance.

Question 5

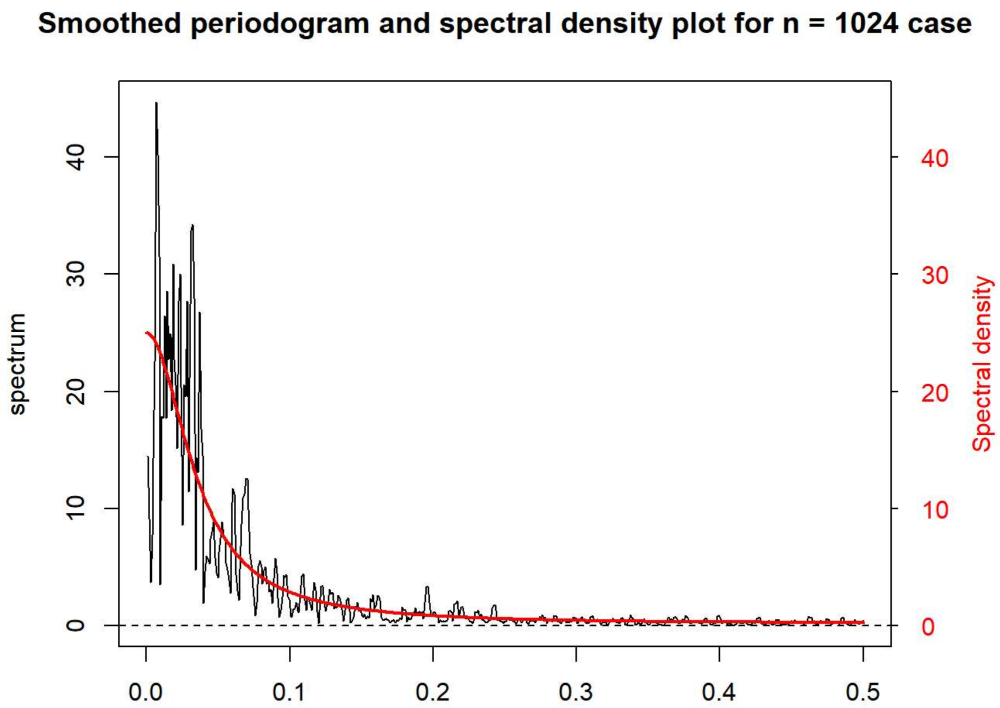
```
par(mar=c(2,4,4,5))
spec.pgram(ts1, kernel = kernel("daniell", floor(sqrt(length(data)))), taper=0, log="no", main = "Smoothed periodogram and sp
ectral density plot for n = 128 case")
abline(h = 0, lty = 2)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F) $spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```




```
par(mar=c(2,4,4,5))
spec.pgram(ts2, kernel = kernel("daniell", floor(sqrt(length(data)))), taper=0, log="no", main = "Smoothed periodogram and sp
ectral density plot for n = 512 case")
abline(h = 0, lty = 2)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F)$spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```




```
par(mar=c(2,4,4,5))
spec.pgram(ts3, kernel = kernel("daniell", floor(sqrt(length(data)))), taper=0, log="no", main = "Smoothed periodogram and sp
ectral density plot for n = 1024 case")
abline(h = 0, lty = 2)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F)$spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```




```
par(mar=c(2,4,4,5))
spec.pgram(ts4, kernel = kernel("daniell", floor(sqrt(length(data)))), taper=0, log="no", main = "Smoothed periodogram and sp
ectral density plot for n = 2048 case")
abline(h = 0, lty = 2)
f=seq(0,.5,by=.001)
lines(f,ARMAspec(model=list(ar=0.8), freq=f,plot=F) $spec,lty='solid', col = "red", lwd = 2)
axis(4,col.axis="red", las=2, cex.axis=1, tck=-.01)
mtext("Spectral density", side = 4, padj = 4, col = "red")
```