Times Series Assignment5

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Question 1

(a)

Consider the following seasonal AR model:

$$(1-\phi B)(1-\Phi B^s)X_t = W_t,$$

which is

$$X_t - \phi W_{t-1} - \Phi X_{t-s} + \phi\Phi X_{t-s-1} = W_t$$

where $W_t \sim WN(0, \sigma^2)$.

Thus, we have

$$\begin{aligned}
\gamma_w(h) &= Cov(X_{t+h} - \phi W_{t+h-1} - \Phi X_{t+h-s} + \phi\Phi X_{t+h-s-1},\ X_t - \phi W_{t-1} - \Phi X_{t-s} + \phi\Phi X_{t-s-1}) \\
&= \gamma_x(h) - \phi\gamma_x(h+1) - \Phi\gamma_x(h+s) + \phi\Phi\gamma_x(h+s+1) \\
&\quad - \phi[\gamma_x(h-1) - \phi\gamma(h) - \Phi\gamma_x(h+s-1) + \phi\Phi\gamma_x(h+s)] \\
&\quad - \Phi[\gamma_x(h-s) - \phi\gamma_x(h+1-s) - \Phi\gamma_x(h) + \phi\Phi\gamma_x(h+1)] \\
&\quad + \phi\Phi[\gamma_x(h-s-1) - \phi\gamma_x(h-s) - \Phi\gamma_x(h-1) + \phi\Phi\gamma_x(h)]
\end{aligned}$$

Then apply $\gamma_w(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_w(\nu)e^{2\pi i\nu h}d\nu$, $\gamma_x(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_x(\nu)e^{2\pi i\nu h}d\nu$ and by the uniqueness of Fourier transformation, we can derive that $\frac{g_w(\nu)}{f_x(\nu)} = \frac{\sigma^2}{f_x(\nu)} =$

$$\begin{aligned}
&[1+\phi^2+\Phi^2+(\phi\Phi)^2] - 2(\phi+\phi\Phi^2)\cos(2\pi\nu) - 2(\Phi+\phi^2\Phi)\cos(2\pi\nu s) \\
&+ 2\phi\Phi[\cos(2\pi\nu(s+1)) + \cos(2\pi\nu(s-1))].
\end{aligned}$$

Therefore, the spectral density of X_t is $f_x(\nu) =$

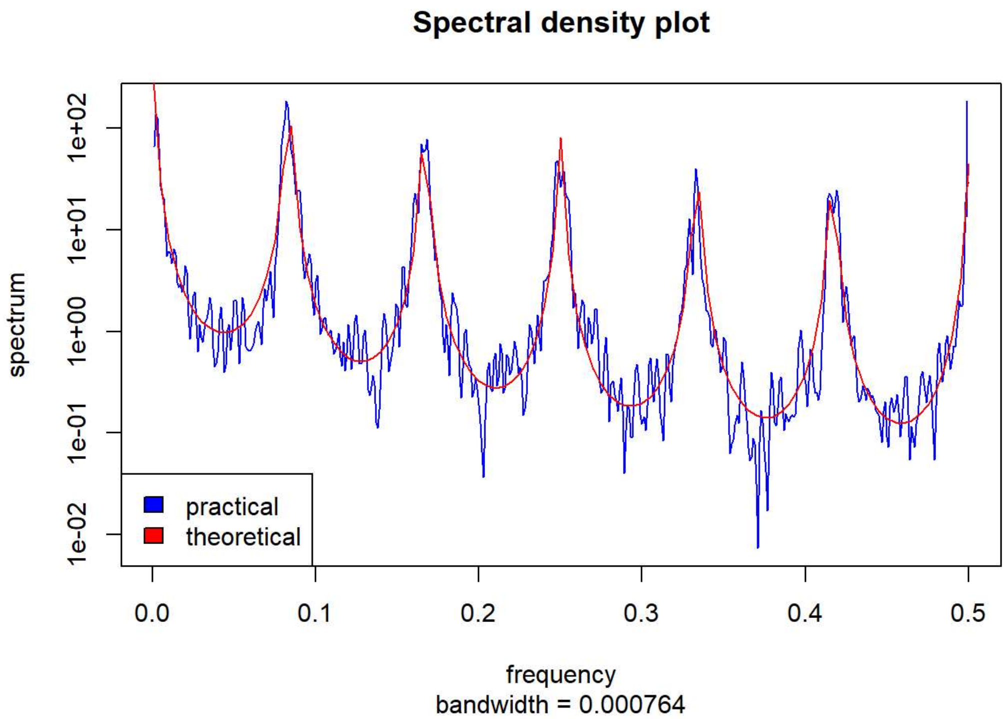
$$\frac{\sigma^2}{1+\phi^2+\Phi^2+(\phi\Phi)^2 - 2(\phi+\phi\Phi^2)\cos(2\pi\nu) - 2(\Phi+\phi^2\Phi)\cos(2\pi\nu s) + 2\phi\Phi[\cos(2\pi\nu(s+1)) + \cos(2\pi\nu(s-1))]}.$$

(b)

```
library(forecast)
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

```
phi <- rep(0,13)
phi[1] <- 0.5
phi[12]<- 0.9
phi[13] <- -0.45

set.seed(12)

simulated_data <- arima.sim(n = 1000, list(ar = phi))
spec <- spec.pgram(simulated_data, spans = c(2), taper = 0, plot = FALSE)
par(mfrow = c(1,1))
plot(spec, main = "Spectral density plot", col = "blue")

    my_function <- function(sigma_squared, phi,Phi, x) {
      result <- sigma_squared /
        (1 + phi^2 + Phi^2 + (phi*Phi)^2
      - 2*(phi+ phi*Phi^2)*cos(2*pi*x)
      - 2*(Phi + phi^2*Phi)*cos(2*pi*x*12)
      + 2*phi*Phi*(cos(2*pi*x* 13) + cos(2*pi*x*11)))
      return(result)
    }

  curve(my_function(sigma_squared = 1, phi = 0.5, Phi = 0.9, x), from = 0, to = 0.5,
        col = "red", add = TRUE)
  legend("bottomleft", legend = c("practical", "theoretical"),
         fill = c("blue", "red"), cex = 1 )
```



The theoretical and practical results are consistent.

(c)

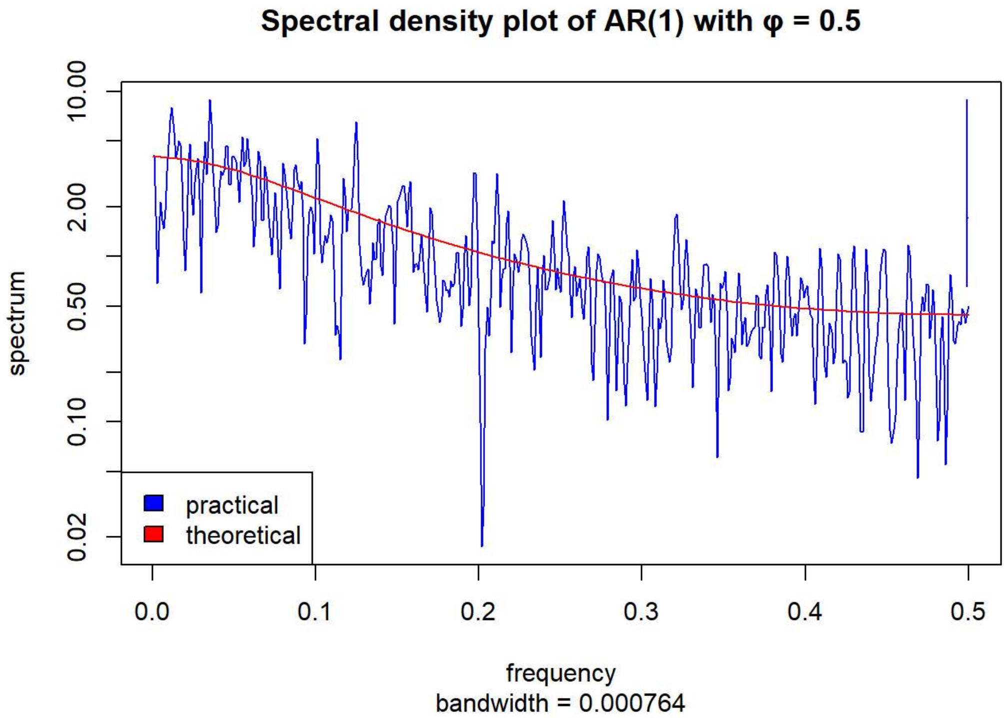
Similarly, for times series $X_t^{(1)}$ which satisfies

$$X_t^{(1)} - 0.5X_{t-1}^{(1)} = W_t,$$

we can derive its spectral density:

$$f_{X_t^{(1)}}(\nu) = \frac{\sigma^2}{1 - \cos(2\pi\nu) + 0.25}.$$

```
set.seed(12)
ts_data <- arima.sim(n = 1000, model = list(order = c(1,0,0), ar = 0.5))
spec <- spec.pgram(ts_data, spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Spectral density plot of AR(1) with φ = 0.5", col = "blue")

my_function1 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(2 * pi * x) + phi^2)
  return(result)
}

curve(my_function1(sigma_squared = 1, phi = 0.5, x), from = 0, to = 0.5
      , col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
       fill = c("blue", "red"), cex = 1 )
```



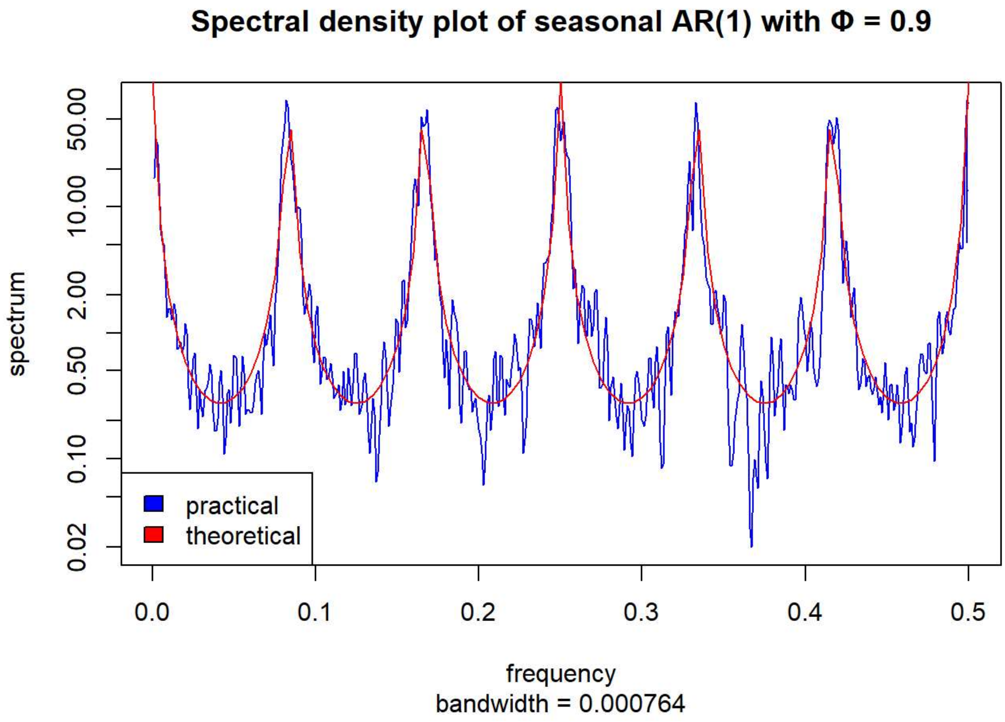
Again by performing the same method, for times series $X_t^{(2)}$ which satisfies

$$X_t^{(2)} - 0.9X_{t-12}^{(2)} = W_t,$$

we can derive its spectral density:

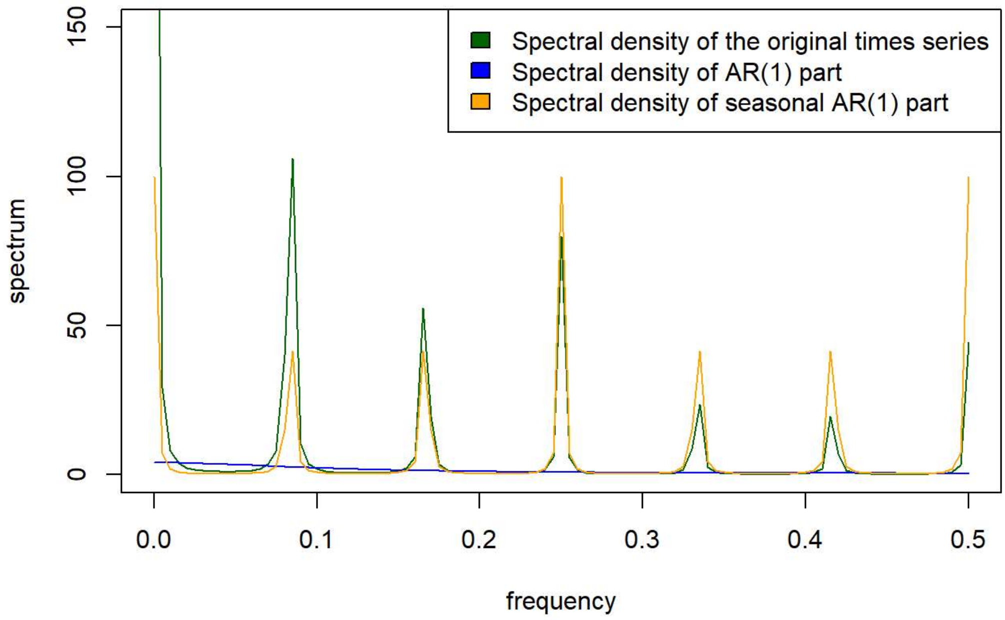
$$f_{X_t^{(2)}}(\nu) = \frac{\sigma^2}{1 - 1.8\cos(24\pi\nu) + 0.81}.$$

```
set.seed(12)
phi <- rep(0,12)
phi[12] <- 0.9
ts_data <- arima.sim(n = 1000, model = list(ar = phi))
spec <- spec.pgram(ts_data, spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Spectral density plot of seasonal AR(1) with Φ = 0.9", col = "blue")

my_function2 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(24 * pi * x) + phi^2)
  return(result)
}

curve(my_function2(sigma_squared = 1, phi = 0.9, x), from = 0, to = 0.5,
      , col = "red", add = TRUE)
legend("bottomleft", legend = c("practical", "theoretical"),
       fill = c("blue", "red"), cex = 1 )
```



(d)

```
 my_function <- function(sigma_squared, phi,Phi, x) {
      result <- sigma_squared /
        (1 + phi^2 + Phi^2 + (phi*Phi)^2
      - 2*(phi+ phi*Phi^2)*cos(2*pi*x)
      - 2*(Phi + phi^2*Phi)*cos(2*pi*x*12)
      + 2*phi*Phi*(cos(2*pi*x* 13) + cos(2*pi*x*11)))
      return(result)
    }

  curve(my_function(sigma_squared = 1, phi = 0.5, Phi = 0.9, x), from = 0, to = 0.5, ylab = "spectrum", xlab="frequency",
ylim = c(0,150), col = "darkgreen")

  my_function1 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(2 * pi * x) + phi^2)
  return(result)
}

curve(my_function1(sigma_squared = 1, phi = 0.5, x), from = 0, to = 0.5
      , col = "blue", add = TRUE)

my_function2 <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(24 * pi * x) + phi^2)
  return(result)
}

curve(my_function2(sigma_squared = 1, phi = 0.9, x), from = 0, to = 0.5,
      , col = "orange", add = TRUE)


legend("topright", legend = c("Spectral density of the original times series", "Spectral density of AR(1) part", "Spectral
density of seasonal AR(1) part"), fill = c("darkgreen","blue","orange"))
```



According to the spectral density plots, we can maintain that the seasonal AR(1) part dominates the seasonality of the original times series.

Question 2

(a)

From the inverse Fourier transformation, we have

$$\gamma(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu 0}d\nu = \int_{\frac{1}{6}\le|\nu|\le\frac{1}{3}} 5\ d\nu = \frac{5}{3}$$

and

$$\gamma(1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu}d\nu = \int_{\frac{1}{6}\le|\nu|\le\frac{1}{3}} 5\cos(2\pi\nu)d\nu = \frac{5}{\pi}[\sin(\frac{2\pi}{3}) - \sin(\frac{\pi}{3})] = 0.$$

(b)

Since

$$\gamma_y(h) = Cov(X_{t+h} - X_{t+h-12},\ X_t - X_{t-12}) = 2\gamma_x(h) - \gamma_x(h+12) - \gamma_x(h-12),$$

then apply inverse Fourier transformation we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\nu)e^{2\pi i\nu h}d\nu = \int_{-\frac{1}{2}}^{\frac{1}{2}} [2 - 2\cos(24\pi\nu)]f_x(\nu)e^{2\pi i\nu h}d\nu.$$

Due to the uniqueness of Fourier transformation, we have

$$f_y(\nu) = [2 - 2\cos(24\pi\nu)]f_x(\nu).$$

Consequently, the spectral density of Y_t is

$$f_y(\nu) = \begin{cases} 10 - 10\cos(24\pi\nu), & \frac{1}{6} \le |\nu| \le \frac{1}{3} \\ 0, & else. \end{cases}$$

Question 3

(a)

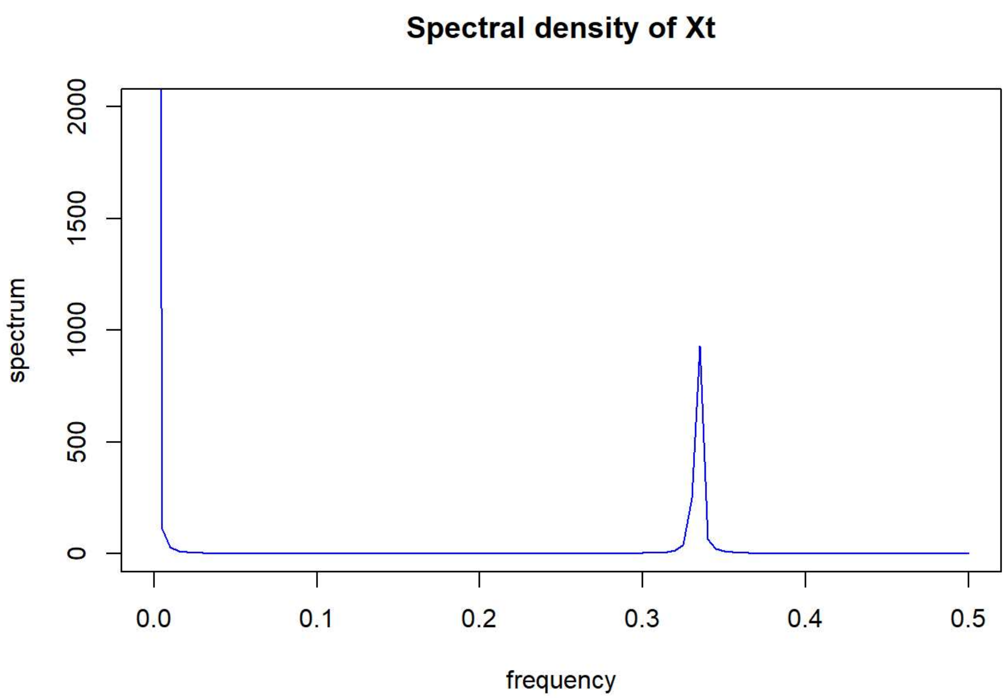
Consider the model $X_t - 0.99X_{t-1} = W_t$, where $W_t \sim WN(0, \sigma^2)$, we have
$\gamma_w(h) = Cov(X_{t+h} - 0.99X_{t+h-3},\ X_t - 0.99X_{t-3}) = (1 + 0.99^2)\gamma_x(h) - 0.99[\gamma_x(h+3) + \gamma_x(h-3)]$. Then apply inverse Fourier transformation $\gamma(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\nu)e^{2\pi i\nu h}d\nu$ and notice the uniqueness of Fourier transformation, we will obtain

$$f_w(\nu) = [1 + 0.99^2 - 1.98\cos(6\pi\nu)]f_x(\nu).$$

Thus, the spectral density of X_t is

$$f_x(\nu) = \frac{f_w(\nu)}{1 + 0.99^2 - 1.98\cos(6\pi\nu)} = \frac{1}{1 + 0.99^2 - 1.98\cos(6\pi\nu)}.$$

```
my_function <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.99, x), from = 0, to = 0.5, ylim = c(0,2000)
      , col = "blue", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum")
```

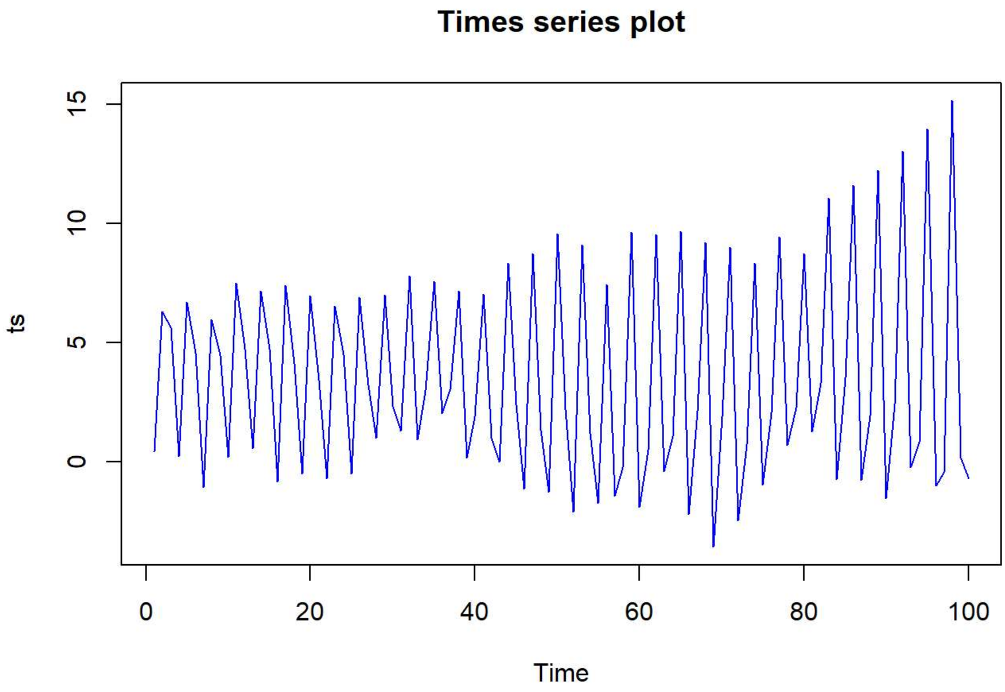


(b)

Check the spectral density plot and we can figure out that it shows an obvious sudden peak at frequency $\nu = 0.33$ approximately, which means the times series X_t exhibit an oscillation with frequency 0.33, i.e., the period is 3 approximately.

(c)

```
set.seed(123)
ts <- arima.sim(model = list(ar = c(0,0,0.99)), n = 100)
plot(ts, col = "blue", main = "Times series plot")
```



From the plot we can maintain that the times series X_t oscillates with period 3 approximately, which is consistent with the conclusion of part (b).

(d)

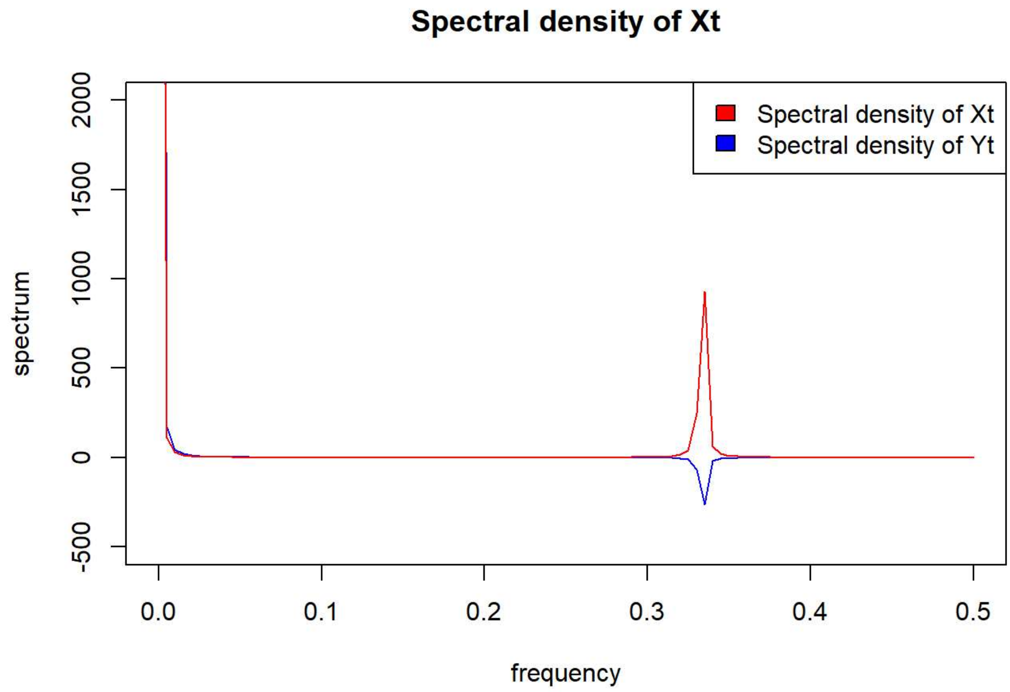
Since $Y_t = \frac{X_{t-1}+X_t+X_{t+1}}{3}$, then we have

$$\gamma_y(h) = \frac{1}{9}[3\gamma_x(h) + 2(\gamma_x(h+1) + \gamma(h-1)) + (\gamma_x(h-2) + \gamma_x(h+2))].$$

Thus, by performing the inverse Fourier transformation and notice the uniqueness of Fourier transformation, we can obtain that

$$f_y(\nu) = [\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)]f_x(\nu) = \frac{\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)}{1 + 0.99^2 - 1.98\cos(6\pi\nu)}.$$

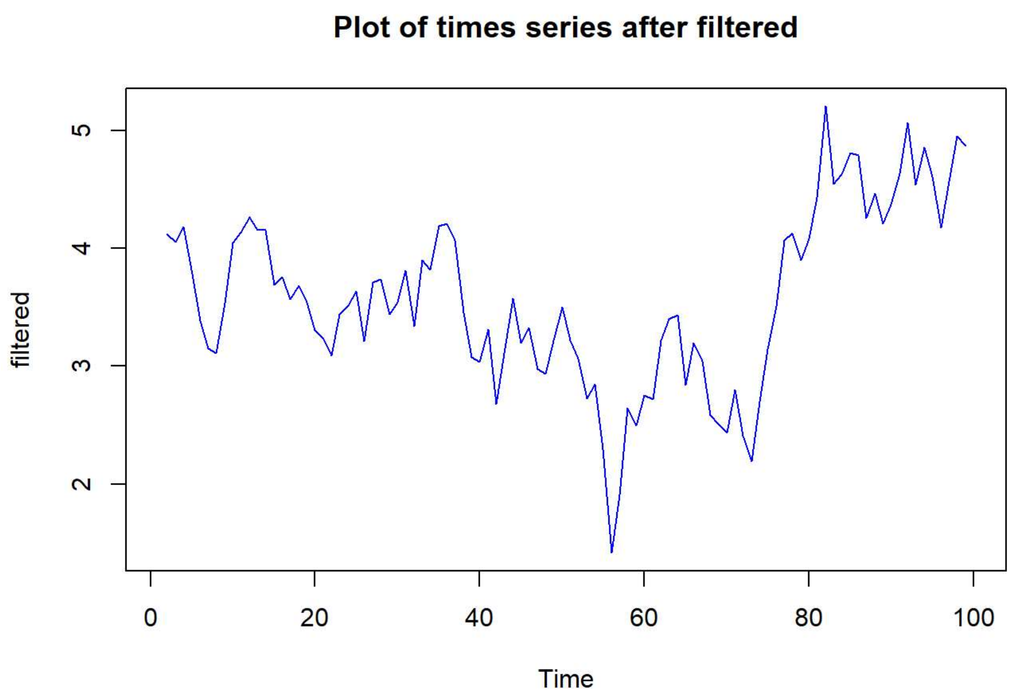
```
 my_function1 <- function(phi, x) {
  result <- (1/3 + 4*cos(2*pi*x)/4 + 2*cos(4*pi*x)/9) / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function1( phi = 0.99, x), from = 0, to = 0.5, ylim = c(-500,2000)
      , col = "blue", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum")

 my_function <- function(sigma_squared, phi, x) {
  result <- sigma_squared / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function(sigma_squared = 1, phi = 0.99, x), from = 0, to = 0.5, ylim = c(0,2000)
      , col = "red", main = "Spectral density of Xt", xlab = "frequency", ylab = "spectrum", add =TRUE)
legend("topright", legend = c("Spectral density of Xt", "Spectral density of Yt" ),
       fill = c("red", "blue"))
```



Mathematically, $f_y(\nu) = [\frac{1}{3} + \frac{4}{9}\cos(2\pi\nu) + \frac{2}{9}\cos(4\pi\nu)]f_x(\nu)$ while graphically, they both exhibit an obvious sudden peak at the same frequency and two spectral density plots coincide mostly except around frequency $\nu = 0.33$.

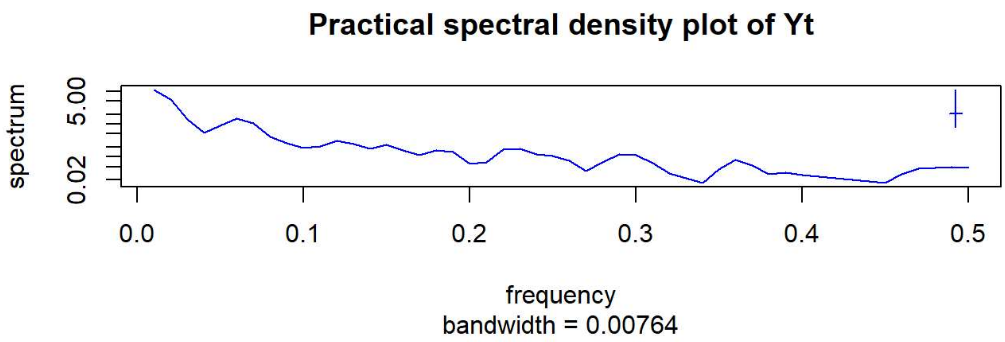
(e)

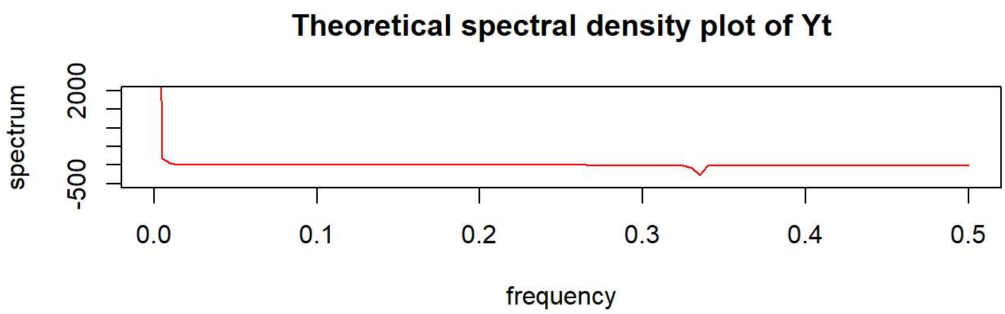
```
set.seed(123)
ts <- arima.sim(model = list(ar = c(0,0,0.99)), n = 100)

filtered <- filter(ts, rep(1/3,3), side = 2)
plot(filtered, col = "blue", main = "Plot of times series after filtered")
```



The filtered times series Y_t does NOT exhibit an oscillatory behavior, which cannot support the spectral density plot in part (d). For detail, we plot both the practical and theoretical spectral density plots of Y_t as follows.

```
par(mfrow = c(2,1))
spec <- spec.pgram(filtered[2:99], spans = c(2), taper = 0, plot = FALSE)
plot(spec, main = "Practical spectral density plot of Yt", col = "blue")

 my_function1 <- function(phi, x) {
  result <- (1/3 + 4*cos(2*pi*x)/4 + 2*cos(4*pi*x)/9) / (1 - 2 * phi * cos(6 * pi * x) + phi^2)
  return(result)
}

curve(my_function1( phi = 0.99, x), from = 0, to = 0.5, ylim = c(-500,2000)
      , col = "red", main = "Theoretical spectral density plot of Yt", xlab = "frequency", ylab = "spectrum")
```

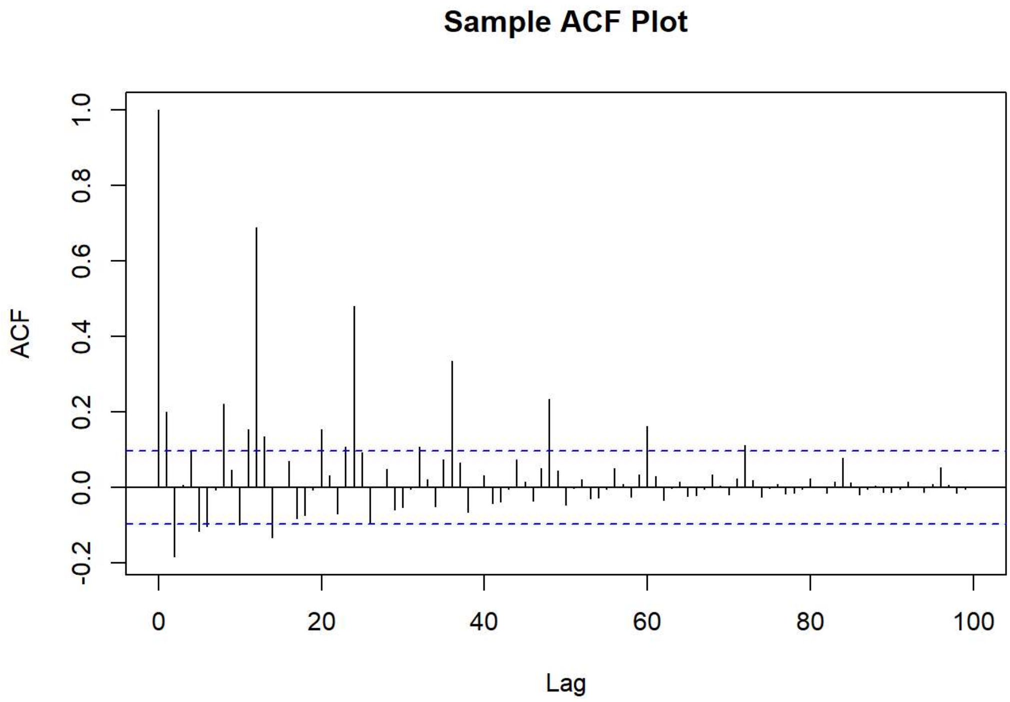




Confidently, we can maintain that the practical results in part (e) are inconsistent with the theoretical results in part (d).

Question 4

```
set.seed(123)
startvalue <- rnorm(13)
ts <- rep(0, 413)
ts[1:13] <- startvalue
for (t in 14:413) {
  ts[t] <- 0.5*ts[t-1] + 0.7*ts[t-12] - 0.35*ts[t-13]
}
ts <- as.ts(ts)
acf(ts, main = "Sample ACF Plot", lag.max = 100)
```



Consider the equivalent seasonal ARMA model

$$X_t = 0.5X_{t-1} + 0.7X_{t-12} - 0.35X_{t-13} + W_t$$

where $W_t \sim WN(0, \sigma^2)$.

Notice that for any time t, X_t is determined only by X_{t-1}, X_{t-12} and X_{t-13}, which means $\gamma(h) = 0$ for all $h \notin \{0, k, 12k, 13k\}$ where $k \in \mathbb{Z}$. Moreover, we can obtain

$$\gamma(h) = 0.5\gamma(h-1) + 0.7\gamma(h-12) - 0.35\gamma(h-13).$$

Since $|\phi_{12}| = 0.7$ is the largest one of $|\phi_1|$, $|\phi_{12}|$ and $|\phi_{13}|$, which indicates the seasonal AR part $X_t = 0.7X_{t-12} + W_t$ dominates the times series to some extent, then we can approximately maintain that $\gamma(12h)$ should exhibit an exponentially decreasing behaviour.

Consequently, we give some properties of true ACF should behave:

- $\gamma(h) = 0$ for all $h \notin \{0, k, 12k, 13k\}$ where $k \in \mathbb{Z}$.
- $\gamma(12h)$ should exhibit an exponential decreasing behaviour.

From the sample ACF plot, we discover that it substantially obey the properties above. Though $\hat{\gamma}(2)$ is beyond 95% confidence interval, we are still able to draw the conclusion that the sample ACF is consistent with the true theoretical ACF.