CS 663 - Fundamentals of Digital Image Processing
Assignment 4

Gagan Jain - 180100043
Hitesh Kandala - 180070023

November 6, 2020

1 Question 3

We have $\mathbf{P} = \mathbf{A^T A}$ and $\mathbf{Q} = \mathbf{AA^T}$ for matrix $\mathbf{A}$ of size $m \times n$, where $m \leq n$.

1.1 Part a

We have,

$$\begin{aligned}
\mathbf{y^t P y} &= \mathbf{y^t A^T A y} \\
\mathbf{y^t P y} &= (\mathbf{Ay})^\mathbf{T} \mathbf{Ay} \\
\mathbf{y^t P y} &= (\mathbf{Ay}).(\mathbf{Ay}) \\
\mathbf{y^t P y} &\geq 0
\end{aligned} \tag{1}$$

This is because the dot product of a vector with itself is always non negative. Similarly, we have,

$$\begin{aligned}
\mathbf{z^t Q z} &= \mathbf{z^t A A^T z} \\
\mathbf{z^t Q z} &= (\mathbf{A^T z})^\mathbf{T} \mathbf{A^T z} \\
\mathbf{z^t Q z} &= (\mathbf{A^T z}).(\mathbf{A^T z}) \\
\mathbf{z^t Q z} &\geq 0
\end{aligned} \tag{2}$$

Now, we have, for scalar eigen values λ and μ, corresponding to matrices $\mathbf{P}$ and $\mathbf{Q}$ respectively,

$$\mathbf{Py} = \lambda \mathbf{y} \ \& \ \mathbf{Qz} = \mu \mathbf{z} \tag{3}$$

Premultiplying the two equations by $\mathbf{y^t}$ and $\mathbf{z^t}$ respectively,

$$\mathbf{y^t P y} = \lambda \mathbf{y^t y} \ \& \ \mathbf{z^t Q z} = \mu \mathbf{z^t z} \tag{4}$$

The LHS of the two equations are positive [Eq. (1) and Eq. (2)] and for any vector $\mathbf{x}$, we have $\mathbf{z^t z} \geq 0$. So,

$$0 \leq \lambda \ \& \ 0 \leq \mu \tag{5}$$

Hence, the eigen values of the matrices $\mathbf{P}$ and $\mathbf{Q}$ are essentially non-negative.

1.2 Part b

We have $\mathbf{u}$ as an eigen vector of $\mathbf{P}$ with eigen value λ, so, premultiplying by $\mathbf{A}$, we get,

$$\begin{aligned}
\mathbf{Pu} &= \lambda \mathbf{u} \\
\mathbf{A^T A u} &= \lambda \mathbf{u} \\
\mathbf{A A^T A u} &= \lambda \mathbf{Au} \\
\mathbf{QAu} &= \lambda \mathbf{Au}
\end{aligned} \tag{6}$$

So, it is evident that the vector $\mathbf{Au}$ is an eigen vector of the matrix $\mathbf{Q}$ with eigen value λ. Now, for $\mathbf{v}$ as an eigen vector of $\mathbf{Q}$ with eigen value μ, so, premultiplying by $\mathbf{A^T}$, we get,

$$\begin{aligned}
\mathbf{Qv} &= \mu\mathbf{v} \\
\mathbf{AA^Tv} &= \mu\mathbf{v} \\
\mathbf{A^TAA^Tv} &= \mu\mathbf{A^Tv} \\
\mathbf{PA^Tv} &= \mu\mathbf{A^Tv}
\end{aligned} \tag{7}$$

So, it is evident that the vector $\mathbf{A^Tv}$ is an eigen vector of the matrix $\mathbf{P}$ with eigen value μ.
Now, we have $\mathbf{P}$ and $\mathbf{Q}$ to be $n \times n$ and $m \times m$ dimensional respectively. So, $\mathbf{u}$ and $\mathbf{v}$ should be n dimensional and m dimensional vectors respectively.

1.3 Part c

We have $\mathbf{v_i}$ as an eigen vector of $\mathbf{Q}$ and we define,

$$\mathbf{u_i} = \frac{\mathbf{A^Tv_i}}{||\mathbf{A^Tv_i}||_\mathbf{2}} \tag{8}$$

So, we must have,

$$\mathbf{Qv_i} = \lambda_i\mathbf{v_i} \tag{9}$$

Now, we will have,

$$\begin{aligned}
\mathbf{Au_i} &= \mathbf{A}\frac{\mathbf{A^Tv_i}}{||\mathbf{A^Tv_i}||_\mathbf{2}} \\
\mathbf{Au_i} &= \mathbf{Q}\frac{\mathbf{v_i}}{||\mathbf{A^Tv_i}||_\mathbf{2}} \\
\mathbf{Au_i} &= \gamma_i\mathbf{v_i}\ , \gamma_i = \frac{\lambda_i}{||\mathbf{A^Tv_i}||_\mathbf{2}}
\end{aligned} \tag{10}$$

So, we have a real non-negative (since λ_i is non negative) scalar γ_i such that $\mathbf{Au_i} = \gamma_i\mathbf{v_i}$.

1.4 Part d

We define $\mathbf{U} = [\mathbf{v_1}|\mathbf{v_2}|\mathbf{v_3}|....|\mathbf{v_m}]$ and $\mathbf{V} = [\mathbf{u_1}|\mathbf{u_2}|\mathbf{u_3}|....|\mathbf{u_n}]$ and we know that $\mathbf{u_i^Tu_j} = 0$ and $\mathbf{v_i^Tv_j} = 0$ for $i \neq j$. We need to show that $\mathbf{A} = \mathbf{U\Gamma V^T}$, where $\mathbf{\Gamma}$ is a diagonal matrix containing the non-negative values $\gamma_1, \gamma_2, .., \gamma_m$. We also have that the vectors $\mathbf{v_i}$ are orthogonal to each other and also of unit magnitude as they are the eigen values of $\mathbf{Q}$. Using this, we have for $i \neq j$,

$$\begin{aligned}
\mathbf{u_i^tu_j} &= \frac{\mathbf{v_i^tAA^Tv_j}}{||\mathbf{A^Tv_i}||_\mathbf{2}||\mathbf{A^Tv_j}||_\mathbf{2}} \\
\mathbf{u_i^tu_j} &= \frac{\mathbf{v_i^tQv_j}}{||\mathbf{A^Tv_i}||_\mathbf{2}||\mathbf{A^Tv_j}||_\mathbf{2}} \\
\mathbf{u_i^tu_j} &= \mu\frac{\mathbf{v_i^tv_j}}{||\mathbf{A^Tv_i}||_\mathbf{2}||\mathbf{A^Tv_j}||_\mathbf{2}} \\
\mathbf{u_i^tu_j} &= 0
\end{aligned} \tag{11}$$

Now, from part c, we have the result that for real scalar eigen values γ_i, for every $i \in [1, m]$, we have,

$$\mathbf{Au_i} = \gamma_i\mathbf{v_i} \tag{12}$$

So, for a diagonal matrix, Γ such that $\Gamma_{ii} = \gamma_i$ and $\Gamma_{ij} = 0$ for $i \neq 0$, we can write,

$$\mathbf{AV} = \mathbf{U}\Gamma \tag{13}$$

Now, using that $\mathbf{U}$ and $\mathbf{V}$ are orthonormal and of unit magnitude, we post-multiply Eq. (12) by $\mathbf{V^T}$.

$$\begin{aligned}
\mathbf{AVV^T} &= \mathbf{U}\Gamma\mathbf{V^T} \\
\mathbf{A} &= \mathbf{U}\Gamma\mathbf{V^T}
\end{aligned} \tag{14}$$

So, we have the singular value decomposition of the matrix $\mathbf{A}$ in the desired form.