CS 663 - Fundamentals of Digital Image Processing
Assignment 4

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1 Question 2

We have a unit vector f which is perpendicular to the unit eigen vector e that minimizes:

$$\Sigma_{i=1}^{N}||\mathbf{x_i} - \bar{\mathbf{x}} - (\mathbf{e}.(\mathbf{x_i} - \bar{\mathbf{x}})\mathbf{e}||^2 \tag{1}$$

or in other words, it maximizes $\mathbf{e^t Ce}$, where $\mathbf{C}$ is a $n \times n$ symetric matrix. This means that the eigen value corresponding to $\mathbf{e}$ is the largest (say λ_1)
Now, we have the eigen vector $\mathbf{f}$ perpendicular to $\mathbf{e}$ that maximizes $\mathbf{f^t Cf}$. We need to prove that this eigenvector has the second largest eigen value. Using the concept of Lagrange Multipliers, we can frame this as an optimization problem, where the constraints will be the fact that $\mathbf{f}$ is a unit vector and that the two vectors $\mathbf{e}$ and $\mathbf{f}$ are orthonormal.

$$max_f[\mathbf{f^t Cf} - \alpha_1(\mathbf{f^t f} - 1) - \alpha_2(\mathbf{f^t e})] \tag{2}$$

In order to maximize the expression written in Eqn. (2), we differentiate with respect to $\mathbf{f}$ and set the result to 0. So, we have:

$$\mathbf{Cf} - \alpha_1\mathbf{f} - \alpha_2\mathbf{e} = 0 \tag{3}$$

Premultiplying by $\mathbf{e^t}$ and observing that $\mathbf{e^t}e = 1$ and $\mathbf{C}$ is a symmetric matrix, we get:

$$\begin{aligned}
\mathbf{e^t Cf} - \alpha_1\mathbf{e^t f} - \alpha_2 &= 0 \\
(\mathbf{C}^t e)^t\mathbf{f} - \alpha_1\mathbf{e^t f} - \alpha_2 &= 0 \\
(\mathbf{C}e)^t\mathbf{f} - \alpha_1\mathbf{e^t f} - \alpha_2 &= 0 \\
\lambda_1\mathbf{e^t f} - \alpha_1\mathbf{e^t f} - \alpha_2 &= 0 \\
(\lambda_1 - \alpha_1)\mathbf{e^t f} - \alpha_2 &= 0
\end{aligned} \tag{4}$$

We know that $\mathbf{e^t f} = 0$, so using Eqn. (3), we have:

$$\alpha_2 = 0 \tag{5}$$
$$\mathbf{Cf} = \alpha_1\mathbf{f} \tag{6}$$

So, we get that α_1 is an eigen value of $\mathbf{C}$. Note that we have maximized $\mathbf{f^t Cf}$ which is the same as maximizing the eigen value α_1. But, the maximum eigen value corresponds to the eigen vector $\mathbf{e}$, so we cannot have α_1 to be the maximal eigen value. So, it has to be maximum among all the other eigen values. Note that we have been given that all the non-zero eigen values of $\mathbf{C}$ are distinct, so this eigen value is unique.