ESSENTIALS OF TURBULENCE - Assignment 3
Film Notes for Vorticity - Ascher H. Shapiro
Summarized by Gagan Jain, Roll No. 180100043

1 Introduction

The vorticity is defined as curl of velocity vector: $\omega = \bigtriangledown \times \mathbf{V}$. So, in a manner, the whole fluid space can be thought of as being threaded by vortex lines, which represent the local spin axis of fluid particle at that point. For rigid body rotation, the vorticity is twice the angular velocity. A vorticity meter is used to measure the vorticity of a flow with the help of four vanes and a pointer arrow.

It is related to the moment of momentum of a small spherical fluid particle about its own COM. Several dynamical theorems relate the changes in vorticity to the moments of the forces on that fluid particle.

A misleading association of vorticity with rotation leads to the thinking that a flow has to be curved for vorticity to be present. But, one can also have vorticity present in a straight channel flow, where one could see a drawn fluid cross deform. Also, it is even possible to have no vorticity associated in a flow with curved streamlines. Flow in a sink-vortex tank will make the vorticity meter move in a circular fashion, but won't make it rotate.

2 Crocco's Theorem

For steady, incompressible, inviscid fluid with conservative body forces acting, *Crocco's theorem* has the following form where p_0 is the stagnation pressure and the rest of the terms have usual meanings.

$$\mathbf{V} \times \omega = \frac{1}{\rho} \bigtriangledown p_0; p_0 = p + \frac{1}{2}\rho V^2 + \rho U$$

For a two dimensional flow in the paper plane, the vorticity vector is normal to it and hence, by Crocco's theorem, the gradient of stagnation pressure lies in the plane of paper and is normal to the velocity and the vorticity vectors, which means that every streamline has a constant stagnation pressure and varies between streamlines only if the vorticity is present.

So, the uniformity of angular momentum ($\mathbf{Vr} = \mathit{const}$), is the condition for, (1) stagnation pressure to be constant throughout, and (2) the free surface to be a hyperboloid of revolution. However, the high-velocity gradients and strain rates produce large viscous forces near the axis which reduce the hyperboloidal depression to a deep dimple having a bottom. A flow without rotation may also contain small regions where the vorticity is very large.

3 Fluid Circulation

The Fluid Circulation T is defined as the line integral of the velocity $\mathbf{V}$ around any closed curve C. Mathematically,

$$T = \oint \mathbf{V}.\mathbf{dr} = \iint \bigtriangledown \times \mathbf{V}.d\mathbf{A} = \iint \omega.d\mathbf{A}$$

A definite circulation around C implies that the fluid inside must have vorticity. In a case where the vorticity is very large only in a thin thread of fluid, lumping all the vorticity into a concentrated vortex line is a reasonable simplification. Every horizontal circuit therefore has a circulation equal to twice the product of the angular velocity and the area bounded by the circuit. The air circulation across a airplane wing is what provides it a lift. However, in this case, the vorticity is distributed and not concentrated.

4 Kelvin's Theorem

The Kelvin's theorem relates the time rate of circulation T with the torques produced by the forces acting in the fluid:

$$\frac{DT_c}{Dt} = -\oint \frac{dp}{\rho} + \oint \mathbf{G}.d\mathbf{r} + \oint \frac{\mu}{\rho} \bigtriangledown^2 \mathbf{V}.d\mathbf{r}$$

The three terms on the RHS represent torques due to pressure forces, body forces, viscous forces, respectively.

5 Viscous Torques

The viscous torques change the vorticity of the fluid particle and are thus responsible for the circulation on a bounding circuit. When a fluid flows around a sharp edge, viscous and pressure forces in the boundary layer lead to a separated flow and the fluid forms a concentrated vortex.

6 Body-Force Torques

Centrally directed body forces like gravity are irrotational. Important rotational body forces include: (1) *Coriolis forces*, ($-2\Omega \times \mathbf{V}$), in rotating reference frames, and (2) *Lorentz forces*, ($\mathbf{J} \times \mathbf{B}$) which cause vorticity in oceans, atmosphere (e.g. hurricanes) and magneto-hydrodynamic flows.

7 Pressure Torques

Incompressible flows have no pressure torque acting. If the *isochors* (contours of constant density) are parallel to the *isobars* (lines of constant pressure - a situation describes as *barotropic*, which means that ρ is a function of p alone, then no net torque acts about the COM. If that is not the case, then a torque acts about the COM to change the circulation. The circulation arising in natural convection systems are driven by the pressure-density term in Kelvin's theorem.

8 Origin of Irrotational Flow

Consider an airfoil starting to move suddenly through a fluid initially at rest. The viscous friction, together with the no-slip condition at the solid surface, makes lift generation possible. The viscous effects leads to shedding of "starting vortex" and thus to a circulation. To balance that, a circulation equal and opposite to that should be present on the curve surrounding the vortex. The stopping of airfoil leads to shedding of a stopping vortex and a similar phenomenon occurs.

9 Helmholtz's Vortex-Laws

In absence of all torque-producing factors, the Helmholtz's laws give the following geometric interpretation to fluids -

- Vortex lines either form closed loops or end at a fluid boundary, and the circulation is the same for every contour enclosing the vortex line. Eg. Smoke ring
- A fluid line coinciding with a vortex line at some instant of time will do so always. Eg. A wing of finite span
- On a vortex line, the ratio of the vorticity to the product of the fluid density with the length of the line ($\omega/\rho l$) remains constant with respect to time. Eg. water over a hump

Turbulent flows are full of vorticity and the mutually-induced velocities of these vortex lines cause some of them to lengthen which produce finer-grained turbulence.

10 Secondary Flows

A curved channel flow illustrates this well where the upstream flow has a only horizontal component of vorticity, which results in a drift to the outer side of the bend because the velocity at the inside of the bend is greater than at the outside. The vortex line now has another component along the flow which leads to the secondary flow. These flows increase the frictional losses.