ME724 - Essentials of Turbulence
Homework 2

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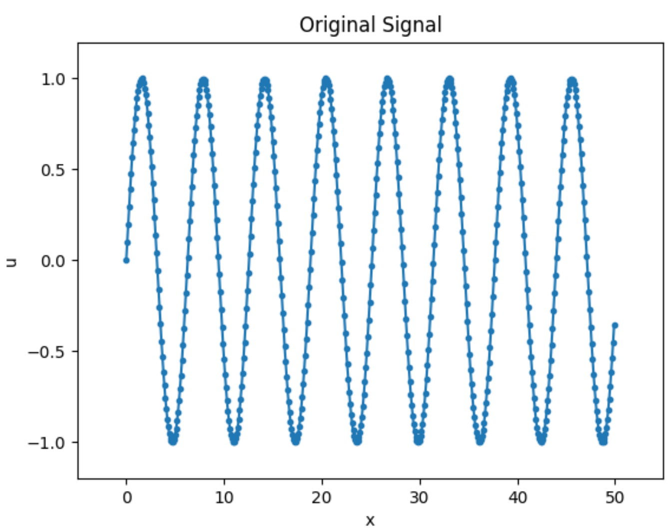
Question 1

The signal is given by -

$$u = sin(x)$$

The signal is plotted below -

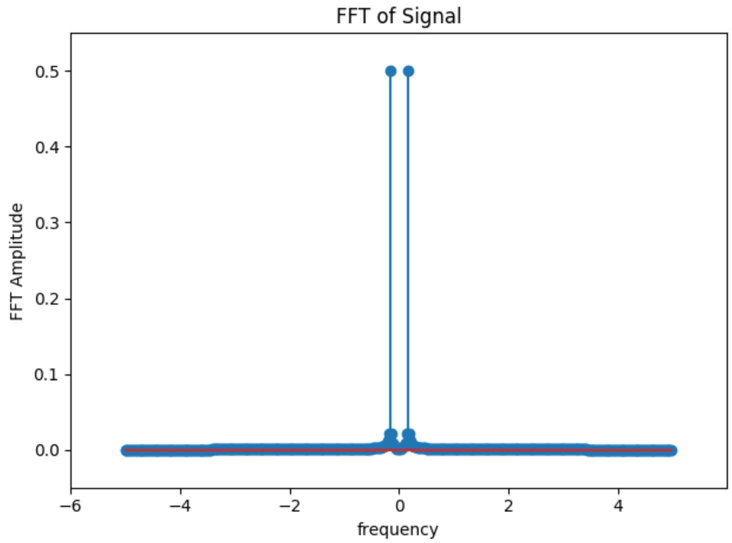


The FFT of the signal is calculated below -

$$\begin{aligned} F(sin(x)) &= \int_{-\infty}^{\infty} \frac{(e^{ix} - e^{-ix})}{2i} e^{-2i\pi f x} dx \\ &= \frac{1}{2i}\left[\int_{-\infty}^{\infty} e^{ix} e^{-2i\pi f x} - \int_{-\infty}^{\infty} e^{-ix} e^{-2i\pi f x}\right] \\ &= \frac{1}{2i}\left[\delta(f - \frac{1}{2\pi}) - \delta(f + \frac{1}{2\pi})\right] \end{aligned}$$

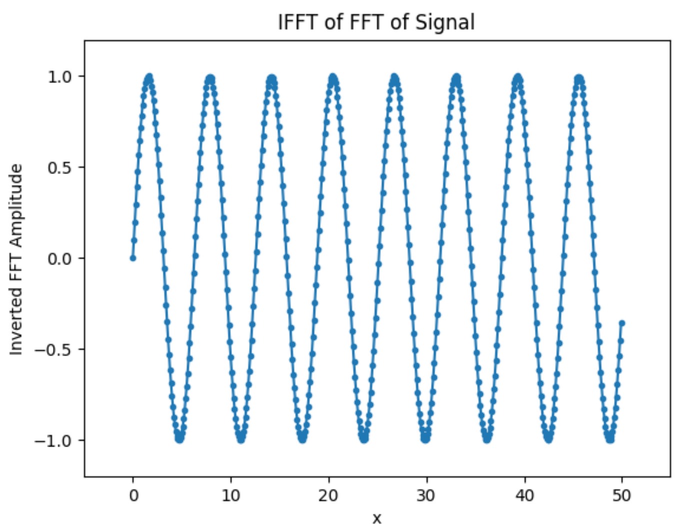
The FFT for the sine wave is purely imaginary (since it is an odd function), so the modulus of the Fourier Transform is plotted.

The FFT is plotted below for the given signal -

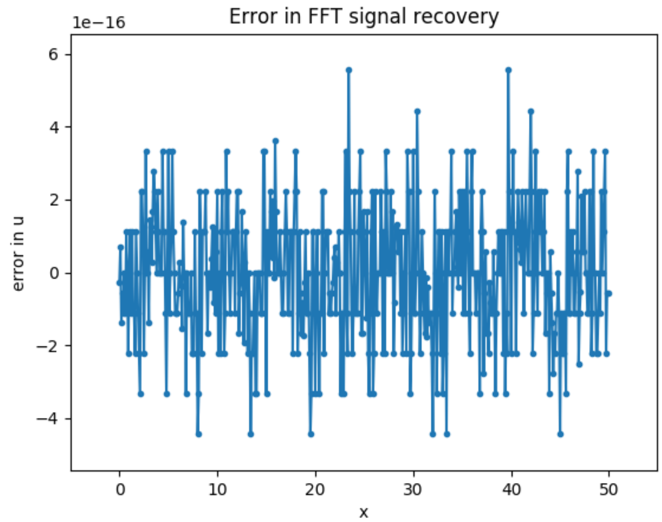


The inverse Fourier transform should ideally give back the original signal.
The Inverse Fourier Transform is calculated and plotted below -

$$
\begin{aligned}
F^{-1}(F(sin(x))) &= \int_{-\infty}^{\infty} \frac{1}{2i}\left[\delta(f-\frac{1}{2\pi}) - \delta(f+\frac{1}{2\pi})\right] e^{2i\pi fx} dx \\
&= \frac{1}{2i}\left[(e^{ix} - e^{-ix})\right] \\
&= sin(x)
\end{aligned}
$$



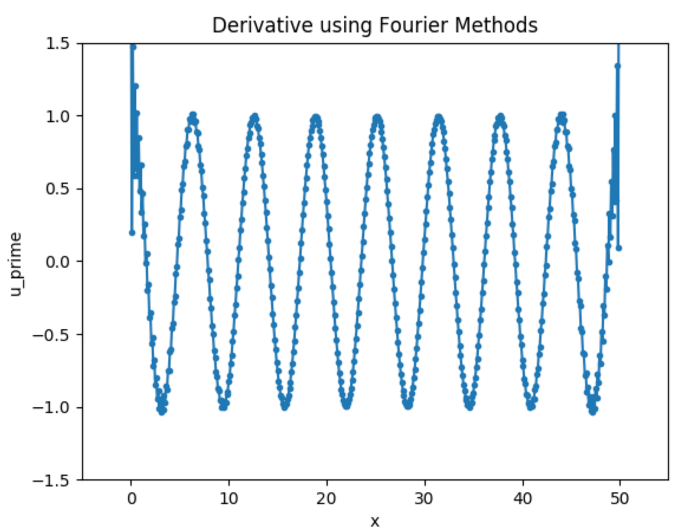
The error in restoring back the signal using Inverse Fast Fourier Transform is plotted below. Note that the errors are of the order of 10^{-16}, which is small enough.



Now, for computing the derivative of the signal using Fourier Method, we have

$$\frac{du}{dx} = F^{-1}(F(\frac{du}{dx})) = F^{-1}(ikF(u))$$

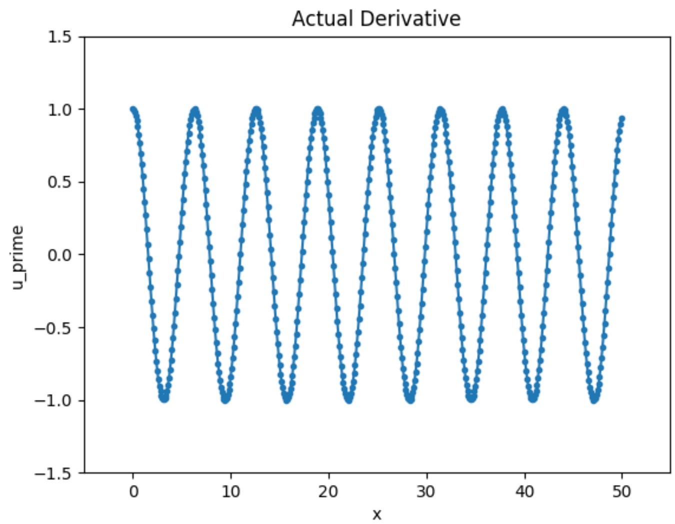
The derivative calculated using this method is plotted below -



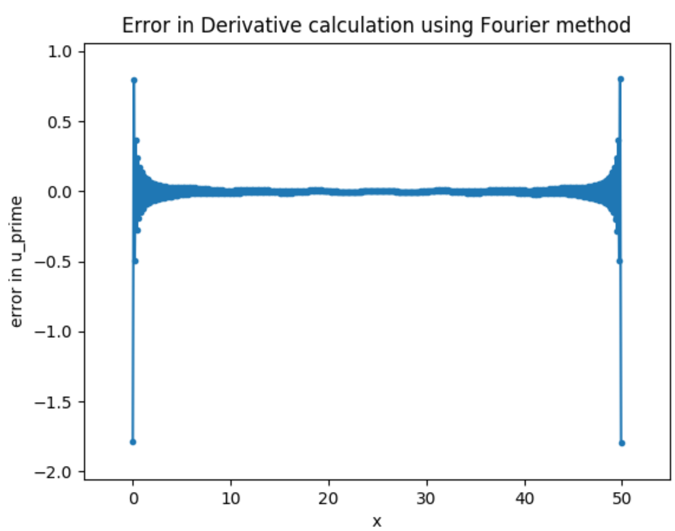
The actual derivative of the signal is given by -

$$\frac{du}{dx} = \frac{d(sin(x))}{dx} = cos(x)$$

The actual derivative of the signal -



The error in derivative calculation using Fourier method is plotted below. Note that except for the boundary points, the error is reasonably low.



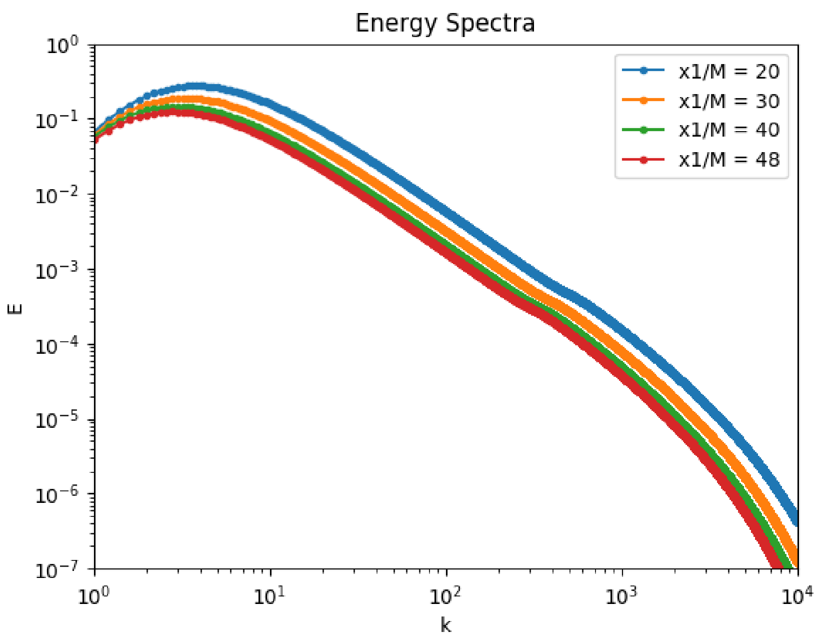
The solution code for the question can be found at this link - **Question 1**

Question 2

The three dimensional turbulent energy spectra is assumed to take the following form -

$$E(\kappa) = c_k \epsilon^{2/3} \kappa^{-5/3} \left[\frac{\kappa \ell}{[(\kappa \ell)^{\alpha_2} + \alpha_1]^{1/\alpha_2}} \right]^{5/3+\alpha_3} e^{-\alpha_4 \kappa \eta} \left[1 + \alpha_5 \left(\frac{1}{\pi} \arctan\left[\alpha_6 \log_{10}(\kappa\eta) + \alpha_7\right] + \frac{1}{2} \right) \right]$$

The energy spectra for the cases x1/M = 20, 30, 40 and 48 has been plotted below. Note that this matches closely with Figure 5 in the paper.



The turbulent kinetic energy can be calculated from the energy spectra in the following way -

$$k = \int_0^{\pi/\Delta} E(\kappa) d\kappa$$

This can be reduced to the following form -

$$\frac{k}{< u_1 >^2} \equiv A \left(\frac{x_1 - x_0}{M} \right)^{-n}$$

The dissipation rate can thus be calculated as the rate of decay of kinetic energy

$$\epsilon^{decay} = -\frac{dk}{dt} = - < u_1 > \frac{dk}{dx_1} = nA \frac{< u_1 >^3}{M} \left(\frac{x_1 - x_0}{M} \right)^{-n-1}$$

The data has been found to fit well to the experimental data for A = 1.80, n = 1.25 and $x_0 = 0$. $< u_1 >$ is taken to be the R.M.S. velocity of the four cases, i.e., 11.2 m/s.

On calculating, the following values are obtained for the turbulent kinetic energy and the dissipation for the four cases -

x_1/M	Turbulent K.E. (k)	$k/<u_1>^2$	K.E. Decay (ϵ^{decay})	Dissipation (ϵ)	Diss. Error (%)
20	5.339	0.0426	24.585	22.8	7.83
30	3.216	0.0256	9.873	9.13	8.14
40	2.245	0.0179	5.168	4.72	9.49
48	1.787	0.0142	3.429	3.41	0.56

The values of the turbulent K.E. closely agree with those plotted in Figure 14 of the paper. Also, the values of dissipation rate match with the actual values tabulated in table 1 (shown in the above table as Dissipation) with maximum error equal to 9.5 %.

Shown below is the plot for the turbulent kinetic energy, which matches to good extent with the plot shown in the paper. Note that the curve has been plotted on log-log scale