Chapter-5

Random ~~Variable~~ Numbers:

The number generated by certain process that cannot be regenerated are called random no.s.
Eg: Numbers generated by Noise sampling.

Properties of random nos.

1. Uniformity - All the generated numbers in the given range have equal probability of occurance

2. Independence - The numbers in the given range ~~of~~ or the sequence of numbers are independent of each other i.e. they are not auto correlated.

- Pseudo - random numbers

- Computer generated random numbers are called pseudo random numbers.

- These numbers are generated using algorithm and the no. of generation can be updated after certain interval of time. This called pseudo random numbers.

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Random Number Generation Techniques:

1 Linear Congruential Generator (LCG)

→ $z_i = (a z_{i-1} + c) \% M$

↳ a ⇒ Multiplicative factor
c ⇒ Incremental factor
z_{i-1} ⇒ Seed value
M ⇒ Modulus value

Multiplicative ma c = 0
Incremementtal a = 1

Let a = 5, c = 3, z_0 = 11, M = 16

$z_1 = (5 * 11 + 3) \% 16$
$= 58 \% 16$
$= 10$

$U_1 = \frac{10}{16} =$

$z_2 = (5 * 10 + 3) \% 16$
$= 53 \% 16$
$= 5$

$U_2 = \frac{5}{16} =$

$z_3 = (5 * 5 + 3) \% 16$
$= 28 \% 16 = 12$

$U = \frac{\quad}{16}$

$z_4 = (5 * 12 + 3) \% 16$
$= 63 \% 16$
$= 15$

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$$Z_5 = (5*15+3)\%16$$
$$= 14$$

$$Z_6 = (5*14+3)\%16$$
$$= 9$$

$$Z_7 = (5*9+3)\%16$$
$$= 0$$

$$Z_8 = (5*0+3)\%16$$
$$= 3$$

$$Z_9 = (5*3+3)\%16$$
$$= 2$$

$$Z_{10} = (6*2+3)\%16$$
$$= 13$$

$$Z_{11} = (6\times13+3)\%16$$
$$= 4$$

$$Z_{12} = (6\times4+3)\%16$$
$$= 7$$

$$Z_{13} =$$

i	z_i	u_i	e	D_i
0	11			
1	10			
2	5			
3	12			
4	15			
5	14			
6	9			
7	0			
8	3			
9	2			
10	13			
11	4			
12	7			
13	6			
14	1			
15	8			
16	11			

For full period we have a theorem

1. The only positive value that exactly divides c & m is 1.

2. If q is a prime no. ~~then~~ that divides m then q divides a-1.

3. If 4 divides M then 4 divides a-1.

- Mid-Square Method

1. Take starting number (seed) say n digits
2. Square it to get 2n digits, if required add leading zeros.
3. Next random no. is the mid n digits of the squarered number
4. Put Decimal before this number to get next U_i

$z_i = 12$ $n = 2$

i	z_i	z_i^2	u_i
0	12	0144	0.12
1	14	0196	0.14
2	19	0361	0.19
3	36	1296	0.36
4	29	0841	0.29
5	84	7056	0.84
6	05	0025	.05
7	02	0004	.02
8	00	—	.00

Draw back

1. May generate same type of random numbers repeatedly (i.e no randomness)
2. Prediction is easy for next random number as the method is based on mid-square which is not efficient method of generation.
3. In some cases $z_i^2 = 00$, the next number will be zero forever

Mixed Generators

→ For a large period and high density of U_i's on [0,1] we want m to be large. Choice of m is 2^b where b is number of binary digits in a word on the computer for data storage.

eg: for 32-bit compilers, the left most bit is sign bit, b = 31
$\therefore m = 2^{31}$

The largest integer that can be represented is $2^b - 1$

eg: b = 4 bits per word.

$m = 16$
$Z_3 = 12$ from $Z_2 = 5$

Now, $5 * Z_2 + 3 = 28$ which in binary representation is 11100. Since, 4-bit computer can store only four binary digits the leftmost bit is dropped, leaving the binary 1100 which is representation of $Z_3 = 12$.

Thus, $m = 2^b$ appears a good choice for m.
We obtain full period if C is odd and a-1 is divisible by 4, Z_0 can b any value between 0 to m-1

For the choice of a

$a = 2^l + 1$ where l is some positive integer.

Then,

$a z_{i-1} = 2^l z_{i-1} + z_{i-1}$

So, that $a z_{i-1}$ can be obtained by "shifting" binary representation of z_{i-1}

Test for Random Numbers:

→ Random numbers ~~and~~ have desirable properties i.e uniformity and independent. To achieve the desirable properties, number of tests can be performed. Five tests for random numbers are:

1) Frequency Test - Uniformity
2) Run Test
3) Test for auto correlation
4) Gap Test
5) Poker Test

(2–5) - Independence

For uniformity test, the hypothesis are as follows:

$$H_0 : R_i \sim U[0,1]$$
$$H_1 : R_i \nsim U[0,1]$$

The null hypothesis, H_0 reads the numbers generated are uniformly distributed. Failure to reject the hypothesis mean that there is no evidence of non-uniformity on the basis of test.

For independence ~~teste~~ taste, the hypotheses are as follows:

$$H_0 : R_i \sim \text{independent}$$
$$H_1 : R_i \nsim \text{independent}.$$

The null hypothesis H_0 reads the numbers generated are independent. Failure to reject the the hypothesis mean that there is no evidence that the numbers are dependent on the basis of test.

For testing level of significance α is specified which gives the probability to accept or reject.

i.e. $\alpha = P(H_0 \text{ reject} / H_0 \text{ true})$

Commonly α is 0.05

i) Frequency Test

1) K-S Test

The procedure used to perform K-S test are as follows:-

Step 1: Rank the random number from smaller to larger i.e

$$R_{(1)} \leq R_{(2)} \leq \ldots \; R_{(n)}$$

Step 2: Compute

$$D^+ = \max_{1 \leq i \leq N} \left(\frac{i}{N} - R_{(i)} \right)$$

$$D^- = \max_{1 \leq i \leq N} \left(R_{(i)} - \frac{i-1}{N}\right)$$

Step 3 : Compute

$$D = \max(D^+, D^-)$$

Step 4 : For specified level of specificance α and given sample size N, take the table value D_α.

Step 5 : Compare the table value D_α with D. If $D \leq D_\alpha$, the null hypothesis cannot be rejected i.e the numbers are uniformly distributed.

Q Suppose 0.44, 0.81, 0.14, 0.05, 0.93 were generated. Perform K.S test to check the uniformity property for $\alpha = 0.05$, The table value $D_\alpha = 0.565$.

Soln,

Arranging in ascending order

0.05, 0.14, 0.44, 0.81, 0.93

Now for $D_1^+ = \frac{1}{5} - 0.05$

$= 0.15$

$D_2^+ = \frac{2}{5} - 0.14$

$= 0.26$

$D_3^+ = \frac{3}{5} - 0.44$

$= 0.16$

$D_4^+ = \frac{4}{5} - 0.81 = -0.01$

$D_5^+ = \frac{5}{5} - 0.93$

$= 0.07$

So, $D^+_{max} = 0.26$

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Now for $D_1^- = 0.05 - \frac{1-1}{5}$

$= 0.05$

$D_2^- = 0.14 - \frac{1}{5}$

$= -0.06$

$D_3^- = 0.44 - \frac{2}{5}$

$= 0.04$

$D_4^- = 0.81 - \frac{3}{5}$

$= 0.21$

$D_5^- = 0.93 - \frac{4}{5}$

$= 0.13$

$D^-_{max} = 0.21$

Now

$D = max(D^+, D^-)$
$= max(0.26, 0.21)$
$= 0.26$

Since, $D (0.26) < D_\alpha (0.565)$, the null hypothesis cannot be rejected on the basis of this test. i.e the numbers are uniformly distributed.

2) Chi-square

i) Divide the sample into n intervals of equal size (0-0.1, 0.11 - 0.2, ... 0.91 - 1.0)

ii) Count the observed frequency on the given range.

iii) Calculate the estimated frenquency using

$$E(i) = \frac{N}{n}$$, where N => total. no. of observation
n => Number of intervals.

iv) Perform

$$X_0^2 = \frac{\sum_{i=1}^{n} (O_i - E_i)^2}{E_i}$$

v) Generate critical value, $\chi^2_{\alpha, n-1}$ from the table for level of significance α and n-1 degree of freedom.

vi) If $X_0^2 \leq \chi^2_{\alpha, n-1}$, the null hypothesis that the numbers are uniformity distributed cannot be rejected.

chi - Square Test

Q Use the chi-square Test with $\alpha = 0.05$ to test whether the data shown below are uniformly distributed. Let $n = 10$ with equal length. The critical value ~~0.05~~ is $X^2_{0.05,9} = 16.9$

~~0.34~~ ~~0.83~~ ~~0.96~~ ~~0.47~~

~~0.34~~ ~~0.83~~ ~~0.96~~ ~~0.47~~ ~~0.47~~

0.34	0.90	0.25	0.89	0.87	0.44	0.12	0.21	0.46	0.67
0.83	0.76	0.79	0.64	0.70	0.81	0.94	0.74	0.22	0.74
0.96	0.99	0.77	0.67	0.56	0.41	0.52	0.73	0.99	0.02
0.47	0.30	0.17	0.82	0.56	0.05	0.45	0.31	0.78	0.05
0.79	0.71	0.23	0.19	0.82	0.93	0.65	0.37	0.39	0.42
0.99	0.17	0.99	0.46	0.05	0.66	0.10	0.42	0.18	0.49
0.37	0.51	0.54	0.01	0.81	0.28	0.69	0.34	0.75	0.49
0.72	0.43	0.56	0.97	0.30	0.94	0.96	0.58	0.73	0.05
0.06	0.39	0.84	0.24	0.40	0.64	0.40	0.19	0.79	0.62
0.18	0.26	0.97	0.88	0.64	0.47	0.60	0.11	0.29	0.78

	i	O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
0-0.1	1	8	10	4	0.4
0.1-0.2	2	8	10	4	0.4
0.2-0.3	3	10	10	0	0
0.3-0.4	4	9	10	1	0.1
0.4-0.5	5	12	10	4	0.4
0.5-0.6	6	8	10	4	0.4
0.7-0.8	7	10	10	0	0
	8	14	10	16	1.6
	9	10	10	0	0
	10	11	10	1	0.1

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$$= 3.4$$

Since, χ_0^2 (3.4) < $\chi^2_{0.05,9}$ (16.9), the null hypothesis that ~~null hy~~ the random numbers are uniformly distributed cannot be ~~reg~~ rejected on the basis of this test.

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- Independence Test

1) Run Test
- Run up & down
- Run above & below mean
- length of run

→ A run is defined as a succession of similar events preceded and followed by a different event. The length of the run is the number of events that occur in the run. eg: Let us toss a coin 10 times and following sequence

H H H T T H T T H T

Here, Every sequence begins and ends with no event. The first three generation has same sequence i.e HHH. This sequence is the run with run length = 3.

We have two possible concered in run test for sequence of numbers. First concern is number of runs and second concern is length of runs.

① Runs up and Runs down Test
→ An up run is sequence of numbers each of

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which is succeeded by a large number. Similarly, A down run is sequence of numbers succeded by a small numbers. up run is responsented by '+' and down run is by '−'.

If a is the total number of runs in a truly random sequence, the mean and variance of a is given by

$$\mu_a = \frac{2N-1}{3}$$

$$\sigma_a^2 = \frac{16N-29}{90}$$

for $N > 20$, the distribution of a is resonably approximated by a normal distributed $N(\mu_a, \sigma_a^2)$. The approximation can be used to test the independence of numbers. The test statistic is

$$Z_0 = \frac{a - \mu_a}{\sigma_a}$$

Substituting (I) & (II) in (III)

$$Z_0 = \frac{a - \left[\frac{2N-1}{3}\right]}{\sqrt{\frac{16N-29}{90}}}$$

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when $Z_0 \sim N(0,1)$. failure to reject the hypothesis of independence occurs when.
$-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$, where α is level of significance.

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Based on runs up and ~~down~~ runs down, determine whether the following sequence of 40 numbers is such that the hypothesis of independence can be rejected where $\alpha = 0.05$ table value $= 1.96$

+	+	+	−	−	+	−	+	−	−
0.41	0.68	0.89	0.94	0.74	0.91	0.55	0.62	0.36	0.27
0.19	0.72	0.75	0.08	0.54	0.02	0.01	0.36	0.16	0.28
0.18	0.01	0.95	0.69	0.18	0.47	0.23	0.32	0.82	0.53
0.31	0.42	0.73	0.04	0.83	0.45	0.13	0.67	0.63	0.29

⇒ The sequence is as follows:

+ + + − + − − + − − − + + + − + − − + − + − −
+ − − + − + + − − + + − + − − + + −

$a = 26$

Now, $\mu_a = \frac{2N-1}{3}$

$= \frac{2 \times 40 - 1}{3}$

$= \frac{80-1}{3}$

$= \cancel{29.66}\ 26.333$

$\sigma_a^2 = \sqrt{\frac{16N - 29}{90}}$

$= \sqrt{\frac{16 \times 40 - 29}{90}} = \sqrt{6.788}$

$$Z_0 = \frac{26 - \cancel{29.66}\ 26.333}{2.6053}$$

$$= -0.1266$$

$$= -0.13$$

Here, $Z_0\,(0.13) < Z_{\alpha/2}\,(1.96)$, so the null hypothesis that the number are independence cannot be rejected on the basis of this test.

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2. Runs above and below the mean.

- In this case, a run can be achieved by comparing the random numbers by the mean value (0.496). Numbers larger than mean is represented by + and numbers smaller than mean is represented by −. Let n_1 and n_2 be the number of individual observations above and below the mean and be the total no. of runs. The maximum no. of runs i $\leq N = n_1 + n_2$ and the minimum number of runs is one. The mean and variance of b for a truly independent sequence is given by

$$\mu_b = \frac{2n_1 n_2}{N} + \frac{1}{2} \quad \text{—} \; (1)$$

$$\sigma_b^2 = \frac{2n_1 n_2 (2n_1 n_2 - N)}{N^2(N-1)} \quad \text{—} \; (11)$$

For either n_1 or n_2 greater than 20, b is approximately distributed. The test statistics is given by

$$Z_0 = \frac{b - \mu_b}{\sigma_b} \quad \text{—} \; (111)$$

from (1) (11) & (111)

$$Z_0 = \frac{b - \left(\frac{2n_1 n_2}{N} + \frac{1}{2}\right)}{\sqrt{\frac{2n_1 n_2 (2n_1 n_2 - N)}{N^2(N-1)}}}$$

Failure to reject hypothesis occur when $-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$ where α is level of significance.

Previous q^n using run above & below the mean

mean = 0.495

− + + + + + + + − − − + + − + − − − −
− − − + + − − − − − + + − − + − + − − +
+ −

$n_1 = 18$ $\quad$ $b = 17$

$n_2 = 22$

$$\mu_b = \frac{2 \times 18 \times 22}{40} + \frac{1}{2}$$

$$= 19.8 + \frac{1}{2}$$

$$= 20.3$$

$$\sigma_b^2 = \frac{2 \times 18 \times 22\,(2 \times 18 \times 22 - 40)}{40^2\,(40-1)}$$

$$= \frac{792 \times 752}{1639} = \cancel{363.382}\ 9.544$$

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$$Z_0 = \frac{b - \mu_0}{\sqrt{\sigma_0}}$$

$$= \frac{17 - 20.3}{\sqrt{9.544}}$$

$$= -1.068$$

$$= -1.07$$

Here, $Z_0(-1.07) < Z_{\alpha/2}(1.96)$, So

3. length of Runs :-

for runs up & runs down. let Y_i be the number of N numbers. for an independent sequence, the expected value of Y_i is given by

$$E(Y_i) = \frac{2}{(i+3)!}\left[N(i^2+3i-1)-(i^3+3i^2-i-4)\right],\ i \leq n-2$$

$$E(Y_i) = \frac{2}{N!},\ i = N-1$$

For runs above and below mean

$$E(Y_i) = \frac{N w_i}{E(I)},\ N > 20$$

where w_i, the approximate probability that a run has length i, is given by

$$w_i = \left(\frac{n_1}{N}\right)^i\left(\frac{n_2}{N}\right) + \left(\frac{n_1}{N}\right)\left(\frac{n_2}{N}\right)^i,\ N > 20$$

and $E(I)$, the approximate probability that a run has length, i, is given by

$$w_i = \left(\frac{n_1}{N}\right)^i\left(\frac{n_2}{N}\right) + \left(\frac{n_1}{N}\right)\left(\frac{n_2}{N}\right)^2,\ N > 20$$

and $E(I)$, the approximate expected length of run, is given by

$$E(I) = \frac{n_1}{n_2} + \frac{n_2}{n_1}, \quad N > 20$$

The ~~approxim~~ appropriate test is chi-square test with O_i being the observed number of runs of length i. Then, the test is

$$\chi_0^2 = \sum_{i=1}^{L} \frac{[O_i - E(Y_i)]^2}{E(Y_i)}$$

where $L = N-1$ for runs up and runs down and $L = N$ for runs above and below mean. If the null hypothesis is true, then χ_0^2 is approximately distributed with $L-1$ degree of freedom

Q. Given the following sequence of numbers can the hypothesis that the numbers are ~~dif~~ independent be rejected on the basis of length of runs up and down at $\alpha = 0.05$.

0.30	0.48							
0.48	0.86	0.36	0.01	0.54	0.34	0.96	0.06	0.61
0.42	0.83	0.14	0.86	0.89	0.37	0.49	0.60	0.04
0.95	0.27	0.37	0.21	0.90	0.89	0.91	0.79	0.57
0.73	0.47	0.41	0.81	0.96	0.31	0.09	0.06	0.23
0.60	0.84	0.13	0.55	0.11	0.76	0.36	0.25	0.23
		0.70	0.30	0.26	0.38	0.05	0.19	0.73

0.85
0.83
0.99.
0.77
0.72
0.44

~~+ − − + − + − + + − + − + + − + + − + − +~~
~~− − + − + − − + − − + + + − − − + + + −~~
~~− + − + − − − + + − − + − + − − −~~

+ − − + − + − + + − + − + + − + + − +
− + − − + − + − − + − − + + + − − − +
+ − − − + − + − − − + − + − − + − +
+ −

1 2 1 1 1 1 2 1 1 1 2 1 2 1 1 1 1 2 1 1 1 2 1 2 3 3 2 3 1
1 1 3 1 1 1 3 1 1 2 1

~~up = 20 down = 19~~

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$i_1 = 1, \; O_1 = 26$
$i_2 = 2, \; O_2 = 9$
$i_3 = 3, \; O_3 = 5$

Now,

$$E(Y_1) = \frac{2}{(i+3)!_0}\left[N(i^2+3i+1) - (i^3+3i^2-i-4)\right]$$

$$= \frac{2}{(1+3)!_0}\left[60(1+3+1) - (1+3-1-4)\right]$$

$$= \frac{2}{24}\left[60 \times 5 + 1\right]$$

$$= \frac{2 \times 301}{24}$$

$$= \frac{301}{12} = 25.0833$$

Then,

$$E(Y_2) = \frac{2}{(2+3)!_0}\left[N(2^2+3\times2+1) - (2^3+3\times2^2-2-4\right]$$

$$= \frac{2}{5!_0}\left[60 \times 11 - 14\right]$$

$$= \frac{2}{120} \times 646 = 10.7667$$

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$$E(Y_3) = \frac{2}{6!}\left[66(90+9+1) - (27+27-3-4)\right]$$
$$= 3.04$$

$O_1 = 26$
$O_2 = 9$
$O_3 = 5$

$\cancel{\chi_0^2 = 1.56}$

$\chi^2_{0.05,2} = 5.99$

i	O_i	$E(Y_i)$	$O_i - E(Y_i)^2 / E(Y_i)$
1	26	25.08	$0.03 = (26 - 25.08)^2 / 25.08$
2	9	10.77	$0.3 = (9 - 10.77)^2 / 10.77$
3	5	3.04	$1.26 = (5 - 3.04)^2 / 3.04$

~~Since~~

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$$= 1.56$$

Since $\chi_0^2 = 1.56 < \chi^2_{0.05,2} = 5.99$ the null hypothesis that the numbers are independently distributed cannot be rejected on the basis of this test.

Q. Given the same sequence of numbers in above example, can the hypothesis that the numbers are independent be rejected on the basis of length of runs above and below the mean at $\alpha = 0.05$

mean value = 0.495

~~- - - - + - + - + + - + - + + - - + -~~

~~+ - + - - + + + + + + - + - - -~~

- - - - + - + - + + - + - + + - - + - + - + - - -

+ + + + + + + - - + + - - - - + + - - + - + - -

- + + + + - - - - - + -

4 1 1 1 1 2 1 1 1 2 1 2 1 1 1 1 1 2 7 2 2 4 2 2 1 1 1 3

4 5 1 1

i	O_i	$E(Y_i)$	$O_i - E(Y_i)^2/E(Y_i)$
1	17	14.79	$(17-14.79)^2/14.79 = 0.3302$
2	8	7.40	$(8-7.40)^2/7.40 = 0.048$
≥ 3	6	3.71	$(6-3.71)^2/3.71 = 1.4131$

$n_1 = 28$

$n_2 = 32$

$$E(Y_i) = \frac{N w_i}{E(I)}$$

$$w_1 = \left(\frac{n_1}{N}\right)^i \frac{n_2}{N} + \frac{n_1}{N}\left(\frac{n_2}{N}\right)^i$$

$$= \left(\frac{28}{60}\right)^1 \times \frac{32}{60} + \frac{28}{60} \times \left(\frac{32}{60}\right)^1$$

$$= \cancel{0.42667}\ 0.498$$

$$w_2 = \left(\frac{28}{60}\right)^2 \times \frac{32}{60} + \frac{28}{60} \times \left(\frac{32}{60}\right)^2$$

$$= 0.249$$

$$w_3 = \left(\frac{28}{60}\right)^3 \times \frac{32}{60} + \frac{28}{60} \times \left(\frac{32}{60}\right)^3$$

$$= 0.125$$

$$E(I) = \frac{n_1}{n_2} + \frac{n_2}{n_1}$$

$$= \frac{28}{32} + \frac{32}{28}$$

$$= 2.02$$

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$$E(Y_1) = \frac{N \times w_i}{E(I)}$$

$$= \frac{60 \times 0.498}{2.02}$$

$$= 14.7920$$

$$E(Y_2) = \frac{60 \times 0.249}{2.02}$$

$$= 7.40$$

$$E(Y_3) = \frac{60 \times 0.125}{2.02}$$

$$= 3.71$$

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$$= 2.09$$

Since $\chi_0^2 = 2.09 < \chi^2_{0.05, 2} = 5.99$ the null hypothesis the numbers are independently distributed can't be rejected on the basis of this test.

- ## Test for autocorrelation

~~0.12 0.01 0.23 0.28 0.8~~

0.12 0.01 0.23 0.28 0.89 0.31 0.64 0.28 0.83 0.93,
0.99 0.15 0.33 0.35 0.91, 0.41 0.60 0.27 0.75 0.88,
0.68 0.49 0.05 0.43 0.95 0.58 0.19 0.36 0.69 0.87,

The test to be described below requires computation of auto correlation between every m numbers. (m is also known as lag) starting with i^{th} number. Theus, the auto correlation $\rho_{i,m}$ between the following numbers are of interest $R_i, R_{i+m}, R_{i+2m} \ldots R_{i+(M+1)m}$. The value M is the largest integer such that $i+(M+1)m \leq N$, where N is the total number of value in the sequence.

The test statistic is given by

$$Z_0 = \frac{\hat{\rho}_{im}}{\sigma_{\hat{\rho}_{im}}}$$

The formula for $\hat{\rho}_{im}$ is given by ~~$\hat{\rho}_{im} = \frac{1}{M+1}$~~

$$\hat{\rho}_{im} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

and

$$\sigma_{\hat{\rho}_{im}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

After computing Z_0, do not reject the null hypothesis of independence if $-Z_{\alpha/2} \leq Z_0 \leq Z_{\alpha/2}$ where α is level of significance.

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$i = 3$

$m = 5$

$M = 4$

$$\hat{\rho}_{35} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{i+(k+1)m}\right] - 0.25$$

$$= \frac{1}{5}\left[R_3 R_8 + R_8 R_{13} + R_{13} R_{18} + R_{18} R_{23} + R_{23} R_{28}\right] - 0.25$$

$$= \frac{1}{5}\left[0.23 \times 0.28 + 0.28 \times 0.33 + 0.33 \times 0.27 + 0.27 \times 0.05 + 0.05 \times 0.36\right] - 0.25$$

$$= -0.1945$$

$$\sigma_{\hat{\rho}_{35}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

$$= \frac{\sqrt{13 \times 4 + 7}}{12(4+1)}$$

$$= \frac{\sqrt{59}}{60}$$

$$= 0.1280$$

~~Here, the critical value $Z_{0.025} = 1.96$ is greater~~

$$Z_0 = \frac{\hat{\rho}_{im}}{\sigma_{\hat{\rho}_{im}}}$$

$$= \frac{-0.1945}{0.1280}$$

$$= -1.516$$

Here, the critical value $Z_{0.025} = 1.96$ is greater than the computed value. Hence, the null hypothesis that the numbers are autocorrelated cannot be rejected on the basis of this test.