MATH 220 Practice Midterm — September, 2024, Duration: 50 minutes

*This test has **5 questions** on **8 pages**, for a total of 50 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks

1. Negate each of the following and prove or disprove the original statement:

   (a) For all $x \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that for all $r < q \in \mathbb{Q}$, $r + q < x$".

Solution: The negation is "There exists $x \in \mathbb{R}$ such that for all $q \in \mathbb{Q}$, there exists $r < q \in \mathbb{Q}$ such that $r + q \geq x$.

Proof. Let $x \in \mathbb{R}$. Pick $q = \frac{\lfloor x \rfloor}{2} - 1 \in \mathbb{Q}$. Notice for all $r < q \in \mathbb{Q}$, $r + q < 2q = \lfloor x \rfloor - 2 \leq x - 2 < x$ so the result holds. □

(b) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $xy < z$ then $x < 0$ or $x^2 + y^2 < z$

Solution: The negation is "There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $xy < z$ and $x \geq 0$ and $x^2 + y^2 \geq z$".

Disproof. Choose $x = 2$ and let $y \in \mathbb{R}$. Choose $z = 4 + y^2$. Notice $x \geq 0$ and $z \leq 4 + y^2 = x^2 + y^2$. Now, notice $y^2 - 2y + 4 = (y-1)^2 + 2 \geq 2 > 0$, so

$$\begin{aligned} y^2 - 2y + 4 &> 0 \\ y^2 + 4 &> 2y \\ z = x^2 + y^2 &> xy \end{aligned}$$

and thus $xy < z$. This proves the negation and thus disproves the original statement. $\square$

10 Marks

2. Let $p \in \mathbb{N}$ and assume $p > 1$. Prove that if there exists $x \in \mathbb{Z}$ such that $x \not\equiv 0 \mod p$ and for all $y \in \mathbb{Z}$, $xy \not\equiv 1 \mod p$, then p is not prime.

Solution:

Proof. We prove the contrapositive. Let $p \in \mathbb{N}$ be prime. Let $x \in \mathbb{Z}$.

- If $x \equiv 0 \mod p$, then we are done.
- If $x \not\equiv 0 \mod p$, we know $x \equiv r \mod p$ for some $1 \leq r < p$. By Bézout's lemma, we know there exists $\alpha, \beta \in \mathbb{Z}$ such that $\alpha r + \beta p = \gcd(r, p) = 1$ so $\alpha r = 1 - \beta p$. It follows that $\alpha r \equiv 1 \mod p$ so choosing such $\alpha = y$ would yield $xy \equiv 1 \mod p$ as required.

□

10 Marks 3. Let $a, b \in \mathbb{Z}$ where $a, b \neq 0$ and $S = \{ax + by | x, y \in \mathbb{Z}, ax + by > 0\}$. You may not use Bézout's lemma for this section.

(a) Prove that S is non-empty.

Solution:

Proof. Notice $a \in S$ so S is non-empty. $\square$

(b) Prove that the minimal element $d = as + bt \in S$ for some $s, t \in \mathbb{Z}$ divides both a and b. (Hint: Euclidean division of a by d and b by d)

Solution:

Proof. Let S be as stated and let d be the minimal element of S. By Euclidean division, we know there exists $m, r \in \mathbb{Z}$ such that $a = md + r$, $0 \leq r < d$. Notice $r = a - md = a - mas - mbt = a(1 - ms) + b(-mt)$ so $r \in S \cup \{0\}$. By assumption, we have d is the minimal element in S, but $0 \leq r < d$ so $r = 0$. Likewise, there exists $n, r \in \mathbb{Z}$ such that $b = nd + r$, $0 \leq r < d$. It follows that $r = a(-ns) + b(1 - nt)$ so $r \in S \cup \{0\}$, and by d is the minimal element, we have $r = 0$. It follows that $a = md$ and $b = nd$ so $d \mid a$ and $d \mid b$. $\square$

(c) Prove that if $c \mid a$ and $c \mid b$, then $c \leq d$.

Solution:

Proof. Let S and d be as stated and assume $c \mid a$ and $c \mid b$, so $a = ck$ and $b = c\ell$ for some $k, \ell \in \mathbb{Z}$. Notice $d = as + bt = c(sk + t\ell)$ so $c \mid d$. By assumption, since $d \in S$, $d > 0$, so $c \leq d$ as required. $\square$

10 Marks

4. Let $A \subseteq \mathbb{Q}$ and $S = \{c_1a_1 + c_2a_2 + c_3a_3 \ldots | a_1, a_2, , a_3 \ldots \in A, c_1, c_2, c_3 \ldots \in \mathbb{Z}\}$. Prove by induction that if A is finite and non-empty, then there exists $q \in \mathbb{Q}$ such that $S = \{cq, c \in \mathbb{Z}\}$. (Hint: Show that $S \subseteq \{cq, c \in \mathbb{Z}\}$ and $S \supseteq \{cq, c \in \mathbb{Z}\}$)

Solution:

Proof. Let A and S be as stated. We proceed with mathematical induction on the number of elements in A.

- If A has 1 element a, then, $S = \{ca, c \in \mathbb{Z}\}$ and since $a \in \mathbb{Q}$, the base case holds.
- Assume there exists $q \in \mathbb{Q}$ such that $S = \{cq, c \in \mathbb{Z}\}$ where $A = A_k$ has k elements. Consider $A = A_{k+1}$ and partition A into $A \setminus \{a\}$ and $\{a\}$ for some $a \in A_{k+1}$. Notice S consists of all the ways multiples of a can add to multiples of elements in $A \setminus \{a\}$, which is precisely elements in $\{c_1a + c_2q, c_1, c_2 \in \mathbb{Z}\}$. Since $a, q \in \mathbb{Q}$, $a = \frac{x_1}{y_1}$ and $q = \frac{x_2}{y_2}$ where $y_1, y_2 \neq 0$. Let $\frac{\gcd(y_2x_1, y_1x_2)}{y_1y_2} = u$. We claim that $\{cu, c \in \mathbb{Z}\} = S$.
  - Let $p \in \{cu, c \in \mathbb{Z}\} = S$. By Bézout's lemma, we know there exists integers α, β such that

$$\begin{aligned} p = cu &= \frac{c(\alpha y_2 x_1 + \beta y_1 x_2)}{y_1 y_2} \\ &= \frac{c\alpha y_2 x_1 + c\beta y_1 x_2}{y_1 y_2} \\ &= \frac{c\alpha x_1}{y_1} + \frac{c\beta x_2}{y_2} \\ &= c\alpha(a) + c\beta(q) \end{aligned}$$

    so $p \in \{c_1a + c_2q, c_1, c_2 \in \mathbb{Z}\} = Se$, $\{cu, c \in \mathbb{Z}\} \subseteq S$.
  - Let $p \in S$ so $p = c_1a + c_2q$ for some $c_1, c_2 \in \mathbb{Z}$. Then,

$$\begin{aligned} p &= \frac{c_1x_1}{y_1} + \frac{c_2x_2}{y_2} \\ &= \frac{c_1y_2x_1 + c_2y_1x_2}{y_1y_2} \end{aligned}$$

    Notice $\gcd(y_2x_1, y_1x_2) \mid y_2x_1$ and $\gcd(y_2x_1, y_1x_2) \mid y_1x_2$, so

$$y_2x_1 = \alpha \gcd(y_2x_1, y_1x_2)$$

$$y_1x_2 = \beta \gcd(y_2x_1, y_1x_2)$$

where $\alpha, \beta \in \mathbb{Z}$ and thus

$$\begin{aligned} p &= \frac{c_1\alpha \gcd(y_2x_1, y_1x_2) + c_2\beta \gcd(y_2x_1, y_1x_2)}{y_1y_2} \\ &= (c_1\alpha + c_2\beta)u \end{aligned}$$

so $p \in \{cu, c \in \mathbb{Z}\}$, $\{cu, c \in \mathbb{Z}\} \supseteq S$.

So there exists $q \in \mathbb{Q}$ such that $\S = \{cq, c \in \mathbb{Z}\}$ as required.

Since the base case and the inductive step hold, it follows that for any finite and non-empty A, there exists $q \in \mathbb{Q}$ such that $S = \{cq, c \in \mathbb{Z}\}$. $\square$

10 Marks

5. Prove or disprove that the sequence $(x_n)_{n\in\mathbb{N}} = \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots(2n)}$ converges. (Hint: Consider an inequality between this sequence and $\frac{1}{\sqrt{3n+1}}$)

Solution:

Proof. We first prove that for all $n \in \mathbb{N}$, $\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots(2n)} \leq \frac{1}{\sqrt{3n+1}}$. We proceed with induction.

- For the base case, $\frac{1}{2} = \frac{1}{\sqrt{4}} = \frac{1}{\sqrt{3+1}}$ so the base case holds.
- Assume that the result holds for $n = k$. Then, for $k+1$, by assumption,

$$\begin{aligned}
\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)(2n+1)}{2\cdot4\cdot6\cdot\ldots(2n)(2n+2)} &\leq \frac{(2n+1)}{\sqrt{3n-1}(2n+2)} \\
&= \sqrt{\frac{(2n+1)^2}{(3n+1)(2n+2)^2}} \\
&= \sqrt{\frac{(2n+2)^2}{12n^3+28n^2+20n+4}} \\
&\leq \sqrt{\frac{(2n+2)^2}{12n^3+28n^2+19n+4}} \\
&= \sqrt{\frac{1}{(3n+4)}}
\end{aligned}$$

so the inductive step holds

Since the base case and inductive step hold, this proves that $n \in \mathbb{N}$, $\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots(2n)} \leq \frac{1}{\sqrt{3n+1}}$. Now, let $\varepsilon > 0$ and choose $N = \left\lceil \frac{1}{3\varepsilon^2} \right\rceil \in \mathbb{N}$. Notice, for all $n > N$,

$$\begin{aligned}
\left|\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots(2n)}\right| &\leq \left|\frac{1\cdot3\cdot5\cdot\ldots\cdot(2N-1)}{2\cdot4\cdot6\cdot\ldots(2N)}\right| \\
&\leq \left|\frac{1}{\sqrt{3N+1}}\right| \\
&< \frac{1}{\sqrt{3N}} \\
&\leq \frac{1}{\sqrt{3\left\lceil\frac{1}{3\varepsilon^2}\right\rceil}} \\
&\leq \frac{1}{\sqrt{\frac{3}{3\varepsilon^2}}} = \varepsilon
\end{aligned}$$

So $(x_n)_{n\in\mathbb{N}}$ converges as required. $\square$