MATH 220 Practice Finals 2 — October, 2024, Duration: 2.5 hours

*This test has **9 questions** on **18 pages**, for a total of 90 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:									/90

10 Marks 1. Carefully define or restate each of the following:

(a) An upper bound on a set $A \subseteq X$ where X is ordered

(b) Bézout's lemma

(c) A partition on a set X

(d) A bounded sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

(e) The principle of mathematical induction

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

   (a) For all $n \in \mathbb{N}$, for all $x \in \mathbb{N}$, for all $y \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $yk \equiv x \mod n$.

(b) For all people $e \in D_e$ where D_e is the set of all people, there exists a function $f : D_e \mapsto D_e$ such that f maps e to the biological grandmothers of e.

$\boxed{\text{10 Marks}}$ 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) If $g \circ f$ is bijective, then f is injective.

(b) If $g \circ f$ is bijective, then f is surjective.

10 Marks

4. Let p be a prime. Prove by induction that for all $n \in \mathbb{Z}$, $p \mid n^p - n$. (Hint: You will need to split into different cases for induction, and use the binomial theorem)

10 Marks

5. Let $\mathbb{R}[x]$ denote the set of polynomials with real coefficients and define a set $I = \{(x^2+1)p(x) | p(x) \in \mathbb{R}[x]\}$. Let $\sim$ be a relation on $\mathbb{R}[x]$ defined as $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I$.

(a) Prove $\sim$ is an equivalence relation.

(b) Let $f(x) \in \mathbb{R}[x]$ and $[f(x)]$ be its equivalence class under $\sim$. Prove that $[f(x)]$ must be of the form $\{a + bx + p(x) | p(x) \in I\}$ where $a, b \in \mathbb{R}$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

(c) Prove that if $f(x), g(x) \in \mathbb{R}[x]$ and $f(x)g(x) \in I$, then $f(x) \in I$ or $g(x) \in I$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

10 Marks

6. Prove or disprove each of the following:

   (a) If A is a infinite and $P \subseteq \mathcal{P}(A)$ is a finite partition of A, then for all $X \in P$, X is infinite.

(b) If A is a infinite and $P \subseteq \mathcal{P}(A)$ is an infinite partition of A, then for all $X \in P$, X is finite.

10 Marks

7. (a) Prove that if $A_1, A_2, A_3 \dots A_n$ and $B_1, B_2, B_3, \dots B_n$ are non-empty sets such that for all $i \leq n \in \mathbb{N}$, $|A_i| \leq |B_i|$, then $|\prod_{i=1}^{n} A_i| \leq |\prod_{i=1}^{n} B_i|$ by constructing an explicit injection $f : \prod_{i=1}^{n} A_i \mapsto \prod_{i=1}^{n} B_i$

(b) Prove or disprove that the result holds when there exists B_i such that B_i is empty.

10 Marks 8. Let $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -2x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$

(a) Recall that $f(x)$ is continuous if for all $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = f(a)$. Prove that $f(x)$ is discontinuous. (Hint: Use density of rationals/irrationals in the reals)

(b) We say $f(x)$ is everywhere discontinuous if for all $a \in \mathbb{R}$ there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that $0 < |x-a| < \delta$ and $|f(x)-f(a)| \geq \epsilon$. Prove or disprove that $f(x)$ everywhere discontinuous.

10 Marks

9. Let $f : (0,1) \mapsto \mathbb{R}$ be defined as $f(x) = \frac{2x-1}{x-x^2}$.

   (a) Prove that f is injective.

(b) Prove that f is surjective.

(c) Hence prove that $|(0,1)| = |\mathbb{R}|$.