MATH 220 Practice Finals 1 — October, 2024, Duration: 2.5 hours

*This test has **10 questions** on **19 pages**, for a total of 100 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:										/100

10 Marks

1. Carefully define or restate each of the following:

   (a) A rational number $q \in \mathbb{Q}$

   (b) Bézout's lemma

   (c) The Fundamental Theorem of Arithmetic

   (d) A convergent sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

   (e) The principle of mathematical induction

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

   (a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$.

(b) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$.

$\boxed{\text{10 Marks}}$ 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) For all $U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

(b) For all $U \subseteq B$, $(g \circ f)^{-1}(g(U)) = f^{-1}(U)$

10 Marks

4. Let $n \in \mathbb{N}$ be even and $K = \{0, 1, 2, \ldots, n-1\}$. Let $S = \{k \in K | 2k \equiv 0 \mod n\}$. Prove that $|S|$ is even.

10 Marks 5. (a) Prove that $f : \mathbb{R} \mapsto \mathbb{C} \setminus \{0\}$ where $f(x) = e^{2\pi i x}$ is neither injective nor surjective and for all $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$.

(b) Let R be a relation on $\mathbb{R}$ be defined as xRy if and only if $x = y + z$ where $z \in f^{-1}(\{1\})$. Prove that R is an equivalence relation.

(c) Find all equivalence classes under R. Show that the operation $[a] + [b] = [a + b]$ is well defined for $a, b \in \mathbb{R}$.

$\boxed{\text{10 Marks}}$ 6. (a) Let X be a non-empty set. Prove that any equivalence relation on X forms a partition on X.

(b) Prove that any partition on X corresponds to equivalence classes of an equivalence relation on X.

10 Marks

7. Let $\{x_n\}$ be a sequence and we define a subsequence of $\{x_n\}$ as a sequence obtained from removing terms in $\{x_n\}$.

   (a) Construct an example where $\{x_n\}$ is a sequence and two subsequences of $\{x_n\}$ both converge but converge to different limits, and justify your answer.

(b) Prove that if a sequence $\{x_n\}$ converges to L then every subsequence of $\{x_n\}$ converges to L.

10 Marks 8. Prove or disprove each of the following:

(a) Let $A, B \subseteq C$. If $|C \backslash A| = |C \backslash B|$, then $|A| = |B|$.

(b) Let $A_i \in X$. Then, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$.

10 Marks

9. Let $B_r(\mathbf{x}) \subseteq \mathbb{R}^n$, called an open ball of radius r centered at $\mathbf{x} \in \mathbb{R}^n$, be defined as $\{\mathbf{y} \in \mathbb{R}^n | \|\mathbf{x} - \mathbf{y}\| < r\}$ for some $r > 0 \in \mathbb{R}$, note that $\|\mathbf{x} - \mathbf{y}\|$ refers to the Euclidean norm where $\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ and $x_i, y_i \in \mathbb{R}$.

   (a) Let $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in B_r(p)$. Show that there exists $B_{r_2}(\mathbf{q}) \subseteq B_r(p)$. (Hint: The triangle inequality works for $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\|$ in $\mathbb{R}^n$ too)

(b) A set $E \subseteq \mathbb{R}^n$ is open if for all $\mathbf{p} \in E$, there exists $B_r(\mathbf{p}) \subseteq E$. Prove that E is open if and only if E is a union of open balls.

$\boxed{\text{10 Marks}}$ 10. Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let S_n be the set of bijective functions $f : N \mapsto N$. Prove by induction that for all $n \geq 3 \in \mathbb{N}$, there exists $f, g \in S_n$ such that $f \circ g \neq g \circ f$.