

**MATH 220 Practice Midterm — September, 2024, Duration: 50 minutes***This test has **5 questions** on **8 pages**, for a total of 50 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks
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1. Negate each of the following and prove or disprove the original statement:

(a) For all  $x \in \mathbb{R}$ , there exists  $q \in \mathbb{Q}$  such that for all  $r < q \in \mathbb{Q}$ ,  $r + q < x$ .

- (b) For all  $x \in \mathbb{R}$ , there exists  $y \in \mathbb{R}$  such that for all  $z \in \mathbb{R}$ , if  $xy < z$  then  $x < 0$  or  $x^2 + y^2 < z$

10 Marks
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2. Let  $p \in \mathbb{N}$  and assume  $p > 1$ . Prove that if there exists  $x \in \mathbb{Z}$  such that  $x \not\equiv 0 \pmod{p}$  and for all  $y \in \mathbb{Z}$ ,  $xy \not\equiv 1 \pmod{p}$ , then  $p$  is not prime.

10 Marks
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3. Let  $a, b \in \mathbb{Z}$  where  $a, b \neq 0$  and  $S = \{ax + by \mid x, y \in \mathbb{Z}, ax + by > 0\}$ . You may not use Bézout's lemma for this section.

(a) Prove that  $S$  is non-empty.

- (b) Prove that the minimal element  $d = as + bt \in S$  for some  $s, t \in \mathbb{Z}$  divides both  $a$  and  $b$ . (Hint: Euclidean division of  $a$  by  $d$  and  $b$  by  $d$ )

(c) Prove that if  $c \mid a$  and  $c \mid b$ , then  $c \leq d$ .

10 Marks
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4. Let  $A = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{Q}$  and  $S = \{c_1 a_1 + c_2 a_2 + \dots + c_n a_n \mid a_i \in A, c_i \in \mathbb{Z}\}$ . Prove by induction that if  $A$  has  $n \in \mathbb{N}$  elements, then there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$ . (Hint: Show that  $S \subseteq \{cq, c \in \mathbb{Z}\}$  and  $S \supseteq \{cq, c \in \mathbb{Z}\}$  and find  $q$  in your inductive step)

10 Marks
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5. Prove or disprove that the sequence  $(x_n)_{n \in \mathbb{N}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$  converges. (Hint: Consider an inequality between this sequence and  $\frac{1}{\sqrt{3n+1}}$ )