MATH 220 Practice Finals 2 — October, 2024, Duration: 2.5 hours

*This test has **10 questions** on **20 pages**, for a total of 100 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:										/100

10 Marks

1. Carefully define or restate each of the following:

   (a) An upper bound on a set $A \subseteq X$ where X is ordered

   (b) Bézout's lemma

   (c) A partition on a set X

   (d) A bounded sequence $(x_n)_{n \in \mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

   (e) The principle of mathematical induction

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

   (a) Let A be the set of rational numbers with odd denominators. Then, for all $x \in A$, for all $y \in A$, $x + y \in A$ or $xy \in A$.

(b) For all people $e \in D_e$ where D_e is the set of all people, there exists a function $f : D_e \mapsto D_e$ such that f maps e to the biological grandmothers of e.

$\boxed{\text{10 Marks}}$ 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) If $g \circ f$ is bijective, then f is injective.

(b) If $g \circ f$ is bijective, then f is surjective.

$\boxed{\text{10 Marks}}$ 4. (a) Prove by induction that if A is finite, then $|\mathcal{P}(A)| = 2^{|A|}$.

(b) Using the binomial theorem and without induction, prove the same statement again.

10 Marks

5. Let $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$ denote the equivalence classes of the set of integers under mod n. For all $[a] \in \mathbb{Z}/n\mathbb{Z}$, let $\#a$ denote the smallest natural number such that $\#a[a] = [0]$. and define an equivalence relation $\sim$ on $\mathbb{Z}/n\mathbb{Z}$ where $[a] \sim [b]$ if and only if $\#a = \#b$. Prove that $\sim$ is a relationship.

(b) Let $n > 1$. Find the equivalence classes and determine how many equivalence classes there are.

(c) Prove that for all $[a] \in \mathbb{Z}/n\mathbb{Z}$, $\#a \mid n$.

10 Marks 6. Prove that if $x > 0$ and $y \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $nx > y$.

10 Marks 7. Prove or disprove each of the following:

(a) If A is a infinite and $P \subseteq \mathcal{P}(A)$ is a finite partition of A, then for all $X \in P$, X is infinite.

(b) If A is a infinite and $P \subseteq \mathcal{P}(A)$ is an infinite partition of A, then for all $X \in P$, X is finite.

10 Marks

8. (a) Prove that if $A_1, A_2, A_3 \dots A_n$ and $B_1, B_2, B_3, \dots B_n$ are non-empty sets such that for all $i \leq n \in \mathbb{N}$, $|A_i| \leq |B_i|$, then $|\prod_{i=1}^{n} A_i| \leq |\prod_{i=1}^{n} B_i|$ by constructing an explicit injection $f : \prod_{i=1}^{n} A_i \mapsto \prod_{i=1}^{n} B_i$

(b) Prove that the result no longer holds when there exists B_i such that B_i is empty.

10 Marks

9. Let $B_r(x) \in \mathbb{R}^n$, called an open ball of radius r, be defined as $\{y \in \mathbb{R}^n | \|x - y\| < r\}$ for some $r > 0$, note that $\|x-y\|$ refers to the Euclidean norm $\|x-y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

   (a) Let $p \in \mathbb{R}^n$ and $q \in B_r(p)$. Show that there exists $B_{r_2}(q) \subseteq B_r(p)$.

(b) A set $E \subseteq \mathbb{R}^n$ is open if for all $p \in E$, there exists $B_r(p) \subseteq E$. Prove that E is open if and only if E is a union of open balls.

10 Marks 10. Let $f : (0,1) \mapsto \mathbb{R}$ be defined as $f(x) = \frac{2x-1}{x-x^2}$.

(a) Prove that f is injective.

(b) Prove that f is surjective.

(c) Hence prove that $|(0,1)| = |\mathbb{R}|$.

(d) Prove that f is continuous over $(0, 1)$.