**MATH 220 Practice Finals 1 — October, 2024, Duration: 2.5 hours**
*This test has **10 questions** on **19 pages**, for a total of 100 points.*

| First Name: | Last Name: |
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| Student Number: | Section: |
| Signature: | |

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10 Marks    1. Carefully define or restate each of the following:

(a) A rational number $q \in \mathbb{Q}$

> **Solution:** $q \in \mathbb{Q}$ if there exists coprime $a, b \in \mathbb{Z}$ where $b \neq 0$ such that $q = \frac{a}{b}$.

(b) Bézout's lemma

> **Solution:** For all $a, b \in \mathbb{Z}$, there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$

(c) The Fundamental Theorem of Arithmetic

> **Solution:** Let $n \in \mathbb{N}$. Then, $n$ can be uniquely factorised into a product of prime powers $p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$ up to order, where $p_i$ are distinct primes and $e_i \in \mathbb{Z}$.

(d) A convergent sequence $(x_n)_{n \in \mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

> **Solution:** $(x_n)_{n \in \mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0 \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \varepsilon$.

(e) The principle of mathematical induction

> **Solution:** Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks    2. Write the negation of each of the following and prove or disprove the original statement.

(a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$.

> **Solution:** The negation is "There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $x + y < z$ and $x - y \le z$". The original statement is false.
>
> *Disproof.* Let $x = 0$ and $y \in \mathbb{R}$. Choose $z = |y| + 1$. Notice $\pm y \le |y| < 1 + |y|$, so $z > x + y$ and $z > x - y$ so the statement is false. $\square$

(b) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$.

> **Solution:** The negation is "For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that there exists $z \in \mathbb{R}$ such that $xy \leq z$". The original statement is false.
>
> *Disproof.* Let $x \in \mathbb{R}$ and choose $y = z = 0$. Then, $xy = 0 = z$ so the statement is false. $\qquad\square$

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) For all $U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

> **Solution:**
>
> *Proof.* Let everything be as stated. We show each inclusion in turn.
>
> - Assume $x \notin f^{-1}(g^{-1}(U))$, so $f(x) \notin g^{-1}(U)$ and $g(f(x)) \notin U$. It follows that $x \notin (g \circ f)^{-1}(U)$, so by contraposition, $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$
> - Assume $x \in f^{-1}(g^{-1}(U))$. Then, $f(x) \in g^{-1}(U)$ and $g(f(x)) \in U$ so $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$.
>
> It follows that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. as required. $\square$

(b) For all $U \subseteq B$, $(g \circ f)^{-1}(g(U)) = f^{-1}(U)$

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**Solution:**

*Disproof.* Let $A = \{1\} = C$, $B = \{0,1\}$, $f(x) = 1$ and $g(x) = 1$. Then, notice for $U = \{0\}$, $f^{-1}((U)) = \varnothing$, but $(g \circ f)^{-1}g(U) = (g \circ f)^{-1}(\{1\}) = \{1\}$ so $(g \circ f)^{-1}(g(U)) \neq f^{-1}((U))$ and thus the statement is false. $\qquad\square$

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10 Marks    4. Let $n \in \mathbb{N}$ be even and $K = \{0, 1, 2, \ldots, n-1\}$. Let $S = \{k \in K | 2k \equiv 0 \mod n\}$. Prove that $|S|$ is even.

> **Solution:**
>
> *Proof.* Assume for the sake of contradiction that $|S|$ is not even, so for there exists an odd number of elements $k \in K$ such that $2k \equiv 0 \mod n$. Consider $K - S$, which has an odd number of elements. Let $a \in K - S$. Since $2a \not\equiv 0 \mod n$, it follows that there exists $a^{-1} \in K - S$ such that $a + a^{-1} \equiv 0 \mod n$, but there are only an odd number of $a$ so there exists $a_0 \in K - S$ such that $a_0 = a_0^{-1}$ which implies $2a_0 \equiv 0 \mod n$, a contradiction. Hence, $|S|$ is even as required. $\square$

10 Marks    5.    (a)   Prove that $f : \mathbb{R} \mapsto \mathbb{C} \setminus \{0\}$ where $f(x) = e^{2\pi i x}$ is neither injective nor surjective and for all $x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$.

> **Solution:**
>
> *Proof.* Notice $0 \neq 1$ but $f(0) = f(1) = 0 \in \mathbb{C}$ so $f$ is not injective. Also notice there does not exist $x \in \mathbb{R}$ such that $f(x) = 0$ since $e^{2\pi i x} \neq 0$ for all $x \in \mathbb{R}$. Hence, $f$ is not surjective. Finally, let $x, y \in \mathbb{R}$. Observe that $f(x + y) = e^{2\pi i (x+y)} = e^{2\pi i x + 2\pi i y} = e^{2\pi i x} e^{2\pi i y} = f(x)f(y)$ so $f(x + y) = f(x)f(y)$ as required.    $\square$

(b) Let $R$ be a relation on $\mathbb{R}$ be defined as $xRy$ if and only if $x = y + z$ where $z \in f^{-1}(\{1\})$. Prove that $R$ is an equivalence relation.

**Solution:**

*Proof.* We prove reflexivity, symmetry, and transitivity in turn.

- Notice $x = x + 0$ and $0 \in f^{-1}(\{1\})$ so $xRx$ as required.
- Assume $xRy$. Then, notice $y = x - z$ and since $f(-z) = e^{-2\pi i z} = \frac{1}{f(z)} = 1$, we have $-z \in f^{-1}(\{1\})$ and so $yRx$.
- Assume $xRy$ and $yRw$. Then, $x = y + z_1$ and $y = w + z_2$ so $x = w + z_2 + z_1$. Now notice $f(z_2 + z_1) = f(z_2)f(z_1) = 1 \cdot 1 = 1$ so $z_2 + z_1 \in f^{-1}(\{1\})$ and thus $xRw$

It follows that $R$ is an equivalence relation. $\qquad\square$

(c) Find all equivalence classes under $R$. Show that the operation $[a] + [b] = [a+b]$ is well defined for $a, b \in \mathbb{R}$.

> **Solution:**
>
> *Proof.* There are infinite equivalence classes. In particular, for all $x \in [0, 1)$, there exists an equivalence class of the form $[x] = \{k + x | k \in \mathbb{Z}\}$. For showing the operation is well defined, we claim that $xRy$ if and only if $f(x) = f(y)$. Assume $x = y + z$ where $z \in f^{-1}(\{1\})$. Then, $f(x) = f(y + z) = f(y)f(z) = f(y)$ so $f(y) = f(x)$. On the other hand, assume $f(x) = f(y)$. Notice for all $k \in \mathbb{Z}$, $f(k) = 1$. Then, $f(x)f(z_1) = f(x)1 = f(y)1 = f(y)f(z_2) = f(y + z_2)$ where $z_1, z_2 \in f^{-1}(\{1\})$ so $x = y + z_2 - z_1$ and since $z_2 - z_1 \in f^{-1}(\{1\})$, $xRy$. Let $a, b \in \mathbb{R}$. Since for all $y \in [a+b]$, $f(y) = f(a+b)$, it suffices to show that for all $x \in [a]+[b]$, $f(x) = f(a+b)$. Notice since $x \in [a] + [b]$, $x = a + z_1 + b + z_2$ where $z_1, z_2 \in f^{-1}(\{1\})$ so $f(x) = f(a + z_1 + b + z_2) = f(a)f(z_1)f(b)f(z_2) = f(a)f(b) = f(a+b)$ so $x \in [a+b]$. Hence, the operation is well defined. $\qquad\square$

$\boxed{\text{10 Marks}}$   6.   (a)   Let $X$ be a non-empty set. Prove that any equivalence relation on $X$ forms a partition on $X$.

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**Solution:**

*Proof.* Let $\sim$ be an equivalence relation on $X$. By reflexivity, we have that for all $x \in X$, $x$ lies in some equivalence class and thus every equivalence class is non-empty. For showing that equivalence classes are disjoint, suppose $x \in [a]$ and $x \in [b]$, so $x \sim a$ and $x \sim b$. By symmetry and transitivity, we have $a \sim x$ and $x \sim b$ so $[a] = [b]$. Thus, equivalence classes are disjoint. It follows that the equivalence classes of an equivalence relation partition a set. $\qquad\square$

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(b) Prove that any partition on $X$ corresponds to equivalence classes of an equivalence relation on $X$.

> **Solution:**
>
> *Proof.* Let $P \subseteq \mathcal{P}(X)$ be a partition of $X$, and define a relation $x \sim y$ if and only if $x, y \in A_i$ where $A_i \in P$.
>
> - Since every $x$ lies in some $A_i \in P$ by the definition of a partition, $x \sim x$.
> - If $x, y \in A_i$, then surely $y, x \in A_i$ so $x \sim y$.
> - Similarly, if $x, y \in A_i$ and $y, z \in A_i$, it follows that $x, z \in A_i$ since every element lies in precisely one $A_i$.
>
> Hence, $\sim$ is an equivalence relation and by our choice of equivalence relation, we have that every equivalence class corresponds precisely to some $A_i \in P$. $\qquad\square$

$\boxed{\text{10 Marks}}$    7. Let $\{x_n\}$ be a sequence and we define a subsequence of $\{x_n\}$ as a sequence obtained from removing terms in $\{x_n\}$.

     (a) Construct an example where $\{x_n\}$ is a sequence and two subsequences of $\{x_n\}$ both converge but converge to different limits, and justify your answer.

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**Solution:** Choose $\{x_n\}$ to be the sequence defined as $x_n = 1$ if $n$ is odd and $-1$ if $n$ is even. Consider the subsequences $\{x_n\}_{\text{odd}}$ and $\{x_n\}_{\text{even}}$. Obviously, both converge since every single term in the sequences are the same. $\{x_n\}_{\text{odd}}$ converges to 1 and $\{x_n\}_{\text{even}}$ converges to $-1$ so they converge to different limits.

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(b) Prove that if a sequence $\{x_n\}$ converges to $L$ then every subsequence of $\{x_n\}$ converges to $L$.

**Solution:**

*Proof.* Assume $\{x_n\}$ converges to $L$. Let $\{y_n\}$ be a subsequence of $\{x_n\}$ and $\varepsilon > 0$. Choose $N_y \in \mathbb{N}$ to be any $N_y$ such that $N_y \geq N_x$ and $x_{N_y} \in \{y_n\}$ where $N_x$ is the choice for $N$ for $\{x_n\}$ by the convergence of $\{x_n\}$. Since $\{y_n\}$ cannot terminate before $y_{N_y}$, and for all $n > N_y$, we have $y_n = x_m$ where $m > N_y$ and so $m > N_y \geq N_x$, it follows that $|y_n - L| = |x_m - L| < \varepsilon$ so $\{y_n\}$ converges to $L$ as required. $\square$

10 Marks    8.  Prove or disprove each of the following:

(a) Let $A, B \subseteq C$. If $|C \backslash A| = |C \backslash B|$, then $|A| = |B|$.

> **Solution:**
>
> *Disproof.* Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \mathbb{N}$. Notice there exists an explicit bijection $f : C \backslash A \mapsto C \backslash B$ where $f(n) = n + 1$, but $|A| = 1 \neq 2 = |B|$ so the statement is false. $\square$

(b) Let $A_i \in X$. Then, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$.

> **Solution:**
>
> *Proof.* Let $A_i \in X$ and $x \in X$. We show each inclusion in turn.
>
> - Let $x \in X \backslash (\bigcup_{i \in I} A_i)$, so $x \notin (\bigcup_{i \in I} A_i)$ and thus $x \notin A_i$ for all $i \in I$. It follows that $x \in A_i \backslash X$ for all $i$ and thus $x \in \bigcap_{i \in I} (X \backslash A_i)$.
>
> - Let $x \in \bigcap_{i \in I} (X \backslash A_i)$ so for all $i \in I$, $x \notin A_i$. It follows that $x \notin \bigcup_{i \in I} A_i$ so $x \in X \backslash (\bigcup_{i \in I} A_i)$.
>
> Hence, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$. $\qquad\square$

10 Marks     9.  Let $B_r(\mathbf{x}) \subseteq \mathbb{R}^n$, called an open ball of radius $r$ centered at $\mathbf{x} \in \mathbb{R}^n$, be defined as $\{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{y}\| < r\}$ for some $r > 0 \in \mathbb{R}$, note that $\|\mathbf{x} - \mathbf{y}\|$ refers to the Euclidean norm where $\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$ and $x_i, y_i \in \mathbb{R}$.

(a) Let $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in B_r(p)$. Show that there exists $B_{r_2}(\mathbf{q}) \subseteq B_r(p)$. (Hint: The triangle inequality works for $\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\|$ in $\mathbb{R}^n$ too)

**Solution:**

*Proof.* Let $\mathbf{q} \in B_r(\mathbf{p})$. Choose $r_2 = r - \|\mathbf{p} - \mathbf{q}\|$ and $q \in B_{r_2}(\mathbf{q})$. Now, $r > \|\mathbf{p} - \mathbf{q}\|$ so $r_2 > 0$. Let $x \in B_{r_2}(\mathbf{q})$. By the triangle inequality, $\|\mathbf{p} - \mathbf{x}\| \le \|\mathbf{p} - \mathbf{q}\| + \|\mathbf{q} - \mathbf{x}\| < \|\mathbf{p} - \mathbf{q}\| + r_2 = r$ so $B_{r_2}(\mathbf{q}) \subseteq B_r(\mathbf{p})$.     $\square$

(b) A set $E \subseteq \mathbb{R}^n$ is open if for all $\mathbf{p} \in E$, there exists $B_r(\mathbf{p}) \subseteq E$. Prove that $E$ is open if and only if $E$ is a union of open balls.

**Solution:**

*Proof.* We prove each direction in turn.

- For one direction, suppose $E$ is open. I claim $E = \bigcup_{\mathbf{x} \in E} B_r(\mathbf{x})$ where $B_r(\mathbf{x})$ is some open ball centered at $\mathbf{x}$ contained in $E$. Notice this yields $\bigcup_{\mathbf{x} \in E} B_r(\mathbf{x}) \subseteq E$. On the other hand, if $\mathbf{x} \in E$, we know $\mathbf{x} \in B_r(\mathbf{x}) \subseteq E$ for some $r > 0$ by $E$ is open, so $\mathbf{x} \in \cup_{\mathbf{x} \in E} B_r(\mathbf{x})$. This proves $E = \cup_{\mathbf{x} \in E} B_r(\mathbf{x})$ so $E$ is a union of open balls.

- For the other direction, suppose $E$ is a union of open balls $\{B_i\}$, and let $\mathbf{x} \in E$. By definition, $x \in B_0$ for some $B_0 \in \{B_i\}$ and from part (a), we know there exists $B_r(\mathbf{x}) \subseteq B_0 \subseteq E$ so $E$ is open as required.

$\square$

10 Marks    10.    Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let $S_n$ be the set of bijective functions $f : N \mapsto N$. Prove by induction that for all $n \geq 3 \in \mathbb{N}$, there exists $f, g \in S_n$ such that $f \circ g \neq g \circ f$.

> **Solution:**
>
> *Proof.* Let $n \geq 3$. We proceed with mathematical induction on $n$.
>
> - For the base case, let $f$ and $g$ in $S_3$ be defined as follows:
>
> $$f(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}, g(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{cases}$$
>
>   Notice $f \circ g(1) = 1$ and $g \circ f(1) = 3$ so $f \circ g \neq g \circ f$.
> - Assume there exists $f, g \in S_k$ such that $f \circ g \neq g \circ f$ for some $k$. For $S_{k+1}$, choose such $f, g \in S_k$ and create $f', g'$ which is the same as $f$ and $g$ except that the $k + 1$th element gets mapped to itself. Notice there exists some $i \in \{1, 2, \ldots k\}$ such that $f' \circ g'(i) = f \circ g(i) \neq g \circ f(i) = g' \circ f'(i)$ so it follows that $f' \circ g' \neq g' \circ f'$ and thus the result holds.
>
> By the principle of mathematical induction, the result holds for all $n \geq 3$.    □