MATH 220 Practice Midterm 2 — September, 2024, Duration: 50 minutes

*This test has **5 questions** on **10 pages**, for a total of 50 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks

1. Negate each of the following and prove or disprove the original statement:

   (a) For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$, such that there exists $z \in \mathbb{Z}$, such that $xy > z$ implies $x + y < z$.

(b) For all $x \in \mathbb{N}$, for all $y \in \mathbb{N}$ such that $x < y$, there exists $a, b \in \mathbb{Z}$ such that $ax + by < \left|\frac{x}{y}\right|$

$\boxed{\text{10 Marks}}$ 2. Let $n, k \in \mathbb{N}$. Prove that if for all $m \in \mathbb{Z}$, $m^k \neq n$, then $n^{1/k}$ is irrational.

10 Marks

3. The Archimedean property of the reals guarantees that for all $x, y \in \mathbb{R}$ where $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

   (a) Let $x, y \in \mathbb{R}^+$. Prove that there exists $m \in \mathbb{N}$ such that $(m-1)x \leq y < mx$.

   (b) Using (a), prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $q \in \mathbb{Q}$ such that $x < q < y$. (Hint: Consider $y - x$ and also notice $1 \in \mathbb{R}^+$)

(c) Assume that $\sqrt{2}$ is irrational. Using $\sqrt{2}$, prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$.You may assume that irrational numbers added to and multiplied by rational numbers are still irrational. (Hint: From part (b), we have that there exists $p, q \in \mathbb{Q}$ such that $x < p < q < y$)

10 Marks

4. (a) Let $A = \left\{-1, 2, \frac{1}{2}\right\}$. Prove by induction that if 1 is written as a product $1 = p_1 p_2 p_3 \ldots p_n$ where $p_i \in A$, then n is even.

(b) Prove or disprove that the same applies for $B = \left\{-1, \pm 2, \pm\frac{1}{2}\right\}$.

$\boxed{\text{10 Marks}}$ 5. (a) Prove that for all $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{\frac{\pi}{2}+2\pi n} < \delta$ and likewise, there exists $m \in \mathbb{N}$ such that $\frac{1}{\frac{-\pi}{2}+2\pi m} < \delta$ (Hint: Archimedean property of the reals)

(b) Hence prove that the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = L$ does not exist.