

**MATH 220 Practice Midterm — September, 2024, Duration: 50 minutes***This test has **5 questions** on **8 pages**, for a total of 50 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks

1. Negate each of the following and prove or disprove the original statement:

(a) For all  $x \in \mathbb{R}$ , there exists  $q \in \mathbb{Q}$  such that for all  $r < q \in \mathbb{Q}$ ,  $r + q < x$ .

**Solution:** The negation is "There exists  $x \in \mathbb{R}$  such that for all  $q \in \mathbb{Q}$ , there exists  $r < q \in \mathbb{Q}$  such that  $r + q \geq x$ ."

*Proof.* Let  $x \in \mathbb{R}$ . Pick  $q = \frac{\lfloor x \rfloor}{2} - 1 \in \mathbb{Q}$ . Notice for all  $r < q \in \mathbb{Q}$ ,  $r + q < 2q = \lfloor x \rfloor - 2 \leq x - 2 < x$  so the result holds.  $\square$

- (b) For all  $x \in \mathbb{R}$ , there exists  $y \in \mathbb{R}$  such that for all  $z \in \mathbb{R}$ , if  $xy < z$  then  $x < 0$  or  $x^2 + y^2 < z$

**Solution:** The negation is "There exists  $x \in \mathbb{R}$  such that for all  $y \in \mathbb{R}$ , there exists  $z \in \mathbb{R}$  such that  $xy < z$  and  $x \geq 0$  and  $x^2 + y^2 \geq z$ ".

*Disproof.* Choose  $x = 2$  and let  $y \in \mathbb{R}$ . Choose  $z = 4 + y^2$ . Notice  $x \geq 0$  and  $z \leq 4 + y^2 = x^2 + y^2$ . Now, notice  $y^2 - 2y + 4 = (y - 1)^2 + 2 \geq 2 > 0$ , so

$$y^2 - 2y + 4 > 0$$

$$y^2 + 4 > 2y$$

$$z = x^2 + y^2 > xy$$

and thus  $xy < z$ . This proves the negation and thus disproves the original statement.  $\square$

10 Marks

2. Let  $p \in \mathbb{N}$  and assume  $p > 1$ . Prove that if there exists  $x \in \mathbb{Z}$  such that  $x \not\equiv 0 \pmod{p}$  and for all  $y \in \mathbb{Z}$ ,  $xy \not\equiv 1 \pmod{p}$ , then  $p$  is not prime.

**Solution:**

*Proof.* We prove the contrapositive. Let  $p \in \mathbb{N}$  be prime. Let  $x \in \mathbb{Z}$ .

- If  $x \equiv 0 \pmod{p}$ , then we are done.
- If  $x \not\equiv 0 \pmod{p}$ , we know  $x \equiv r \pmod{p}$  for some  $1 \leq r < p$ . By Bézout's lemma, we know there exists  $\alpha, \beta \in \mathbb{Z}$  such that  $\alpha r + \beta p = \gcd(r, p) = 1$  so  $\alpha r = 1 - \beta p$ . It follows that  $\alpha r \equiv 1 \pmod{p}$  so choosing such  $\alpha = y$  would yield  $xy \equiv 1 \pmod{p}$  as required.

□

10 Marks

3. Let  $a, b \in \mathbb{Z}$  where  $a, b \neq 0$  and  $S = \{ax + by \mid x, y \in \mathbb{Z}, ax + by > 0\}$ . You may not use Bézout's lemma for this section.

- (a) Prove that  $S$  is non-empty.

**Solution:**

*Proof.* Notice  $a \in S$  so  $S$  is non-empty. □

- (b) Prove that the minimal element  $d = as + bt \in S$  for some  $s, t \in \mathbb{Z}$  divides both  $a$  and  $b$ . (Hint: Euclidean division of  $a$  by  $d$  and  $b$  by  $d$ )

**Solution:**

*Proof.* Let  $S$  be as stated and let  $d$  be the minimal element of  $S$ . By Euclidean division, we know there exists  $m, r \in \mathbb{Z}$  such that  $a = md + r$ ,  $0 \leq r < d$ . Notice  $r = a - md = a - mas - mbt = a(1 - ms) + b(-mt)$  so  $r \in S \cup \{0\}$ . By assumption, we have  $d$  is the minimal element in  $S$ , but  $0 \leq r < d$  so  $r = 0$ . Likewise, there exists  $n, r \in \mathbb{Z}$  such that  $b = nd + r$ ,  $0 \leq r < d$ . It follows that  $r = a(-ns) + b(1 - nt)$  so  $r \in S \cup \{0\}$ , and by  $d$  is the minimal element, we have  $r = 0$ . It follows that  $a = md$  and  $b = nd$  so  $d \mid a$  and  $d \mid b$ . □

(c) Prove that if  $c \mid a$  and  $c \mid b$ , then  $c \leq d$ .

**Solution:**

*Proof.* Let  $S$  and  $d$  be as stated and assume  $c \mid a$  and  $c \mid b$ , so  $a = ck$  and  $b = c\ell$  for some  $k, \ell \in \mathbb{Z}$ . Notice  $d = as + bt = c(sk + t\ell)$  so  $c \mid d$ . By assumption, since  $d \in S$ ,  $d > 0$ , so  $c \leq d$  as required.  $\square$

10 Marks

4. Let  $A \subseteq \mathbb{Q}$  and  $S = \{c_1a_1 + c_2a_2 + c_3a_3 \dots \mid a_1, a_2, \dots, a_3 \dots \in A, c_1, c_2, c_3 \dots \in \mathbb{Z}\}$ . Prove by induction on the elements of  $A$  that if  $A$  is finite and non-empty, then there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$ . (Hint: Show that  $S \subseteq \{cq, c \in \mathbb{Z}\}$  and  $S \supseteq \{cq, c \in \mathbb{Z}\}$  and find  $u$ )

**Solution:**

*Proof.* Let  $A$  and  $S$  be as stated. We proceed with mathematical induction on the number of elements in  $A$ .

- If  $A$  has 1 element  $a$ , then,  $S = \{ca, c \in \mathbb{Z}\}$  and since  $a \in \mathbb{Q}$ , the base case holds.
- Assume there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$  where  $A = A_k$  has  $k$  elements. Consider  $A = A_{k+1}$  and partition  $A$  into  $A \setminus \{a\}$  and  $\{a\}$  for some  $a \in A_{k+1}$ . Notice  $S$  consists of all the ways multiples of  $a$  can add to multiples of elements in  $A \setminus \{a\}$ , which is precisely elements in  $\{c_1a + c_2q, c_1, c_2 \in \mathbb{Z}\}$ . Since  $a, q \in \mathbb{Q}$ ,  $a = \frac{x_1}{y_1}$  and  $q = \frac{x_2}{y_2}$  where  $y_1, y_2 \neq 0$ . Let  $\frac{\gcd(y_2x_1, y_1x_2)}{y_1y_2} = u$ . We claim that  $\{cu, c \in \mathbb{Z}\} = S$ .
  - Let  $p \in \{cu, c \in \mathbb{Z}\} = S$ . By Bézout's lemma, we know there exists integers  $\alpha, \beta$  such that

$$\begin{aligned} p = cu &= \frac{c(\alpha y_2 x_1 + \beta y_1 x_2)}{y_1 y_2} \\ &= \frac{c\alpha y_2 x_1 + c\beta y_1 x_2}{y_1 y_2} \\ &= \frac{c\alpha x_1}{y_1} + \frac{c\beta x_2}{y_2} \\ &= c\alpha(a) + c\beta(q) \end{aligned}$$

so  $p \in \{c_1a + c_2q, c_1, c_2 \in \mathbb{Z}\} = Se, \{cu, c \in \mathbb{Z}\} \subseteq S$ .

- Let  $p \in S$  so  $p = c_1a + c_2q$  for some  $c_1, c_2 \in \mathbb{Z}$ . Then,

$$\begin{aligned} p &= \frac{c_1 x_1}{y_1} + \frac{c_2 x_2}{y_2} \\ &= \frac{c_1 y_2 x_1 + c_2 y_1 x_2}{y_1 y_2} \end{aligned}$$

Notice  $\gcd(y_2 x_1, y_1 x_2) \mid y_2 x_1$  and  $\gcd(y_2 x_1, y_1 x_2) \mid y_1 x_2$ , so

$$y_2 x_1 = \alpha \gcd(y_2 x_1, y_1 x_2)$$

$$y_1 x_2 = \beta \gcd(y_2 x_1, y_1 x_2)$$

where  $\alpha, \beta \in \mathbb{Z}$  and thus

$$\begin{aligned} p &= \frac{c_1 \alpha \gcd(y_2 x_1, y_1 x_2) + c_2 \beta \gcd(y_2 x_1, y_1 x_2)}{y_1 y_2} \\ &= (c_1 \alpha + c_2 \beta) u \end{aligned}$$

so  $p \in \{cu, c \in \mathbb{Z}\}$ ,  $\{cu, c \in \mathbb{Z}\} \supseteq S$ .

So there exists  $q \in \mathbb{Q}$  such that  $\S = \{cq, c \in \mathbb{Z}\}$  as required.

Since the base case and the inductive step hold, it follows that for any finite and non-empty  $A$ , there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$ .  $\square$



10 Marks

5. Prove or disprove that the sequence  $(x_n)_{n \in \mathbb{N}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$  converges. (Hint: Consider an inequality between this sequence and  $\frac{1}{\sqrt{3n+1}}$ )

**Solution:**

*Proof.* We first prove that for all  $n \in \mathbb{N}$ ,  $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \leq \frac{1}{\sqrt{3n+1}}$ . We proceed with induction.

- For the base case,  $\frac{1}{2} = \frac{1}{\sqrt{4}} = \frac{1}{\sqrt{3+1}}$  so the base case holds.
- Assume that the result holds for  $n = k$ . Then, for  $k + 1$ , by assumption,

$$\begin{aligned} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)} &\leq \frac{(2n+1)}{\sqrt{3n-1}(2n+2)} \\ &= \sqrt{\frac{(2n+1)^2}{(3n+1)(2n+2)^2}} \\ &= \sqrt{\frac{(2n+2)^2}{12n^3 + 28n^2 + 20n + 4}} \\ &\leq \sqrt{\frac{(2n+2)^2}{12n^3 + 28n^2 + 19n + 4}} \\ &= \sqrt{\frac{1}{(3n+4)}} \end{aligned}$$

so the inductive step holds

Since the base case and inductive step hold, this proves that  $n \in \mathbb{N}$ ,  $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \leq \frac{1}{\sqrt{3n+1}}$ . Now, let  $\varepsilon > 0$  and choose  $N = \lceil \frac{1}{3\varepsilon^2} \rceil \in \mathbb{N}$ . Notice, for all  $n > N$ ,

$$\begin{aligned} \left| \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right| &\leq \left| \frac{1 \cdot 3 \cdot 5 \cdots (2N-1)}{2 \cdot 4 \cdot 6 \cdots (2N)} \right| \\ &\leq \left| \frac{1}{\sqrt{3N+1}} \right| \\ &< \frac{1}{\sqrt{3N}} \\ &= \frac{1}{\sqrt{3 \lceil \frac{1}{3\varepsilon^2} \rceil}} \\ &\leq \frac{1}{\sqrt{\frac{3}{3\varepsilon^2}}} = \varepsilon \end{aligned}$$

So  $(x_n)_{n \in \mathbb{N}}$  converges as required. □