MATH 220 Practice Finals 2 — October, 2024, Duration: 2.5 hours

*This test has **9 questions** on **19 pages**, for a total of 90 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:	/90								

10 Marks 1. Carefully define or restate each of the following:

(a) A relation on A

(b) A function $f : A \mapsto B$

(c) The limit of a function $f : \mathbb{R} \mapsto \mathbb{R}$ as $x \to$

(d) Euclidean division

(e) The principle of mathematical induction

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

   (a) There exists a prime $p \in \mathbb{Z}$ such that there exists $r, k \in \mathbb{Z}$ such that $0 \leq r \leq p-1$ and $k < p$ and $p \mid kr$.

(b) For all $n \in \mathbb{N}$, for all $p, q \in \mathbb{N}$ such that $k, \ell \in \mathbb{N}$ are the smallest integers such that $n \mid kp$ and $n \mid \ell q$, then if $k = \ell$, $|\{px \in \mathbb{Z} | 0 \leq px \leq kp \in \mathbb{Z}\}| = |\{qy \in \mathbb{Z} | 0 \leq yq \leq \ell q \in \mathbb{Z}\}|$

$\boxed{\text{10 Marks}}$ 3. Let $(\mathbb{Q}, +)$ be the set of rational numbers with addition as the operation and $(\mathbb{Q}\setminus\{0\}, \cdot)$ be the set of rational numbers except 0 with multiplication as the operation. Prove that there does not exist a bijective $f : (\mathbb{Q}, +) \mapsto (\mathbb{Q}\setminus\{0\}, \cdot)$ such that for all $x, y \in (\mathbb{Q}, +)$, $f(x + y) = f(x)f(y)$.

10 Marks

4. Let $S = \{1, 2, 3, \ldots, m \in \mathbb{N}\}$ and $p_i : S \mapsto S$ be a function that swaps some $a, b \in S$ around where $a \neq b$ and fixes everything else, denoted $p_i = (ab)$. Prove that the identity map e can only be written as a product $p_1 p_2 p_3 \ldots p_n$ where n is even by induction on the number of p_i.

10 Marks

5. (a) Let D_n denote the set of rotations by an angle of $\frac{2\pi}{i}$ where $1 \leq n$ and reflections across an axis of symmetry on an $n-$gon, and let $\sim$ be a relation $\sigma \sim \tau$ if and only if $\sigma\tau$ is a rotation where $\sigma, \tau \in D_n$. Prove that R is an equivalence relation.

(b) Let R be the set of rotations and for all $\sigma \in D_n$, let σR denote the set $\{\sigma x | x \in R\}$. Prove that the collection of σR forms a partition on D_n. Furthermore, prove that for all $\sigma \in D_n$, $\sigma R = [\sigma]$ where $[\sigma]$ is the equivalence class of σ under $\sim$.

10 Marks

6. (a) Prove that if $A_1, A_2, A_3 \dots A_n$ and $B_1, B_2, B_3, \dots B_n$ are non-empty sets such that for all $i \leq n \in \mathbb{N}$, $|A_i| \leq |B_i|$, then $|\prod_{i=1}^{n} A_i| \leq |\prod_{i=1}^{n} B_i|$ by constructing an explicit injection $f : \prod_{i=1}^{n} A_i \mapsto \prod_{i=1}^{n} B_i$

(b) Prove that the result no longer holds when there exists B_i such that B_i is empty.

10 Marks

7. Prove or disprove each of the following:

   (a) There exists a bijective function $f : \mathbb{R} \mapsto S_1$ where S_1 is the unit circle in $\mathbb{C}$.

(b) For all A, B, $|A \times B| = |B \times A|$.

(c) A countable union of countable sets is countable.

(d) A denumerable intersection of uncountable sets is countable.

10 Marks

8. Let $B_r(x) \in \mathbb{R}^n$, called an open ball of radius r, be defined as $\{y \in \mathbb{R}^n | \|x-y\| < r\}$ for some $r > 0$, note that $\|x-y\|$ refers to the Euclidean norm $\|x-y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

   (a) Let $p \in \mathbb{R}^n$ and $q \in B_r(p)$. Show that there exists $B_{r_2}(q) \subseteq B_r(p)$.

(b) A set $E \subseteq \mathbb{R}^n$ is open if for all $p \in E$, there exists $B_r(p) \subseteq E$. Prove that E is open if and only if E is a union of open balls.

100 Marks 9. Suppose a set $A \subseteq \mathbb{R}$ is bounded above if there exists $y \in \mathbb{R}$, called an upper bound, such that for all $x \in A, x \leq y$. Let $\sup(A)$ be an upper bound such that if y is an upper bound of A, $\sup(A) \leq y$. Note that in $\mathbb{R}$, if A is bounded above, then there exists $\sup(A) \in \mathbb{R}$.

(a) Suppose A is bounded below if there exists a $y \in \mathbb{R}$, called a lower bound, such that for all $x \in A$, $y \leq x$. Let $\inf A$ be a lower bound such that if y is a lower bound of A, then $y \leq \inf A$. Prove that if $-A = \{-a | a \in A\}$, then $-\sup A = \inf A$.

(b) Let $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ be a non-decreasing sequence, and let $X = \{x_n | n \in \mathbb{N}\}$. Prove that if $(x_n)_{n\in\mathbb{N}}$ is bounded above, then $\lim_{n\to\infty} x_n = \sup X$.

(c) Prove that if $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ is bounded above or bounded below, then $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ has a finite limit.