MATH 220 Practice Midterm — September, 2024, Duration: 50 minutes

This test has ***5 questions*** *on* ***8 pages****, for a total of 50 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks 1. Negate each of the following and prove or disprove the original statement:

(a) For all $x \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that for all $r < q \in \mathbb{Q}$, $r + q < x$.

(b) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $xy < z$ then $x < 0$ or $x^2 + y^2 < z$

10 Marks 2. Let $p \in \mathbb{N}$ and assume $p > 1$. Prove that if there exists $x \in \mathbb{Z}$ with $x \not\equiv 0 \mod p$ such that for all $y \in \mathbb{Z}$, $xy \not\equiv 1 \mod p$, then p is not prime.

10 Marks 3. Let $a, b \in \mathbb{Z}$ where $a, b \neq 0$ and $S = \{ax + by | x, y \in \mathbb{Z}, ax + by > 0\}$. You may not use Bézout's lemma for this section.

(a) Prove that S is non-empty.

(b) Prove that the minimal element $d = as + bt \in S$ for some $s, t \in \mathbb{Z}$ divides both a and b. (Hint: Euclidean division of a by d and b by d)

(c) Prove that if $c \mid a$ and $c \mid b$, then $c \leq d$.

$\boxed{\text{10 Marks}}$ 4. Let $A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{Q}$ and $S = \{c_1a_1 + c_2a_2 + \ldots + c_na_n | c_i \in \mathbb{Z}\}$. Prove by induction that if A has $n \in \mathbb{N}$ elements, then there exists $q \in \mathbb{Q}$ such that $S = \{cq | c \in \mathbb{Z}\}$. (Hint: Show that $S \subseteq \{cq|c \in \mathbb{Z}\}$ and $S \supseteq \{cq|c \in \mathbb{Z}\}$ and find q in your inductive step)

10 Marks 5. Prove or disprove that the sequence $(x_n)_{n\in\mathbb{N}} = \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots(2n)}$ converges. (Hint: Consider an inequality between this sequence and $\frac{1}{\sqrt{3n+1}}$)