MATH 220 Practice Midterm 2 — September, 2024, Duration: 50 minutes

*This test has **5 questions** on **9 pages**, for a total of 50 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks 1. Negate each of the following and prove or disprove the original statement:

(a) For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$, such that there exists $z \in \mathbb{Z}$, such that $xy > z$ implies $x + y < z$".

Solution: The negation is "There exists $x \in \mathbb{Z}$ such that for all $y \in \mathbb{Z}$, there exists $z \in \mathbb{Z}$ such that $xy > z$ and $z \leq x + y$.

Proof. Let $x \in \mathbb{Z}$. Choose $y = 0$ and $z = -1$. Notice the hypothesis $xy > z$ is false, so the result holds. □

(b) For all $x \in \mathbb{N}$, for all $y \in \mathbb{N}$ such that $x < y$, there exists $a, b \in \mathbb{Z}$ such that $ax + by < \left|\frac{x}{y}\right|$

Solution: The negation is "There exists $x \in \mathbb{N}$ such that there exists $y \in \mathbb{N}$ such that $x < y$ such that for all $a, b \in \mathbb{Z}$, $ax + by \geq \left|\frac{x}{y}\right|$

Proof. Let $x, y \in \mathbb{N}$. Choose $a, b = -1$ and hence the result holds. $\square$

10 Marks 2. Let $n, k \in \mathbb{N}$. Prove that if for all $m \in \mathbb{Z}$, $m^k \neq n$, then $n^{1/k}$ is irrational.

Solution:

Proof. Let $n, k \in \mathbb{N}$. We prove the contrapositive. Assume $n^{1/k}$ is rational, so $n^{1/k} = \frac{a}{b}$ for some coprime $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Notice $n = \frac{a^k}{b^k}$. Since n is an integer, we know $b = 1$ so $n = a^k$ and thus the results follows. $\square$

10 Marks 3. The Archimedean property of the reals guarantees that for all $x, y \in \mathbb{R}$ where $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

(a) Let $x, y \in \mathbb{R}^+$. Prove that there exists $m \in \mathbb{N}$ such that $(m-1)x \leq y < mx$.

Solution:

Proof. Let $x, y \in \mathbb{R}^+$. By the Archimedean property of the reals, we know there exists $m \in \mathbb{N}$ such that $mx > y$. Now choose m to be the smallest possible m such that $mx > y$. It follows that $(m-1)x \leq y$ as otherwise we would have $(m-1)x = mx$ which is impossible since $x \neq 0$. Thus, $(m-1)x \leq y < mx$ as required. □

(b) Using (a), prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $q \in \mathbb{Q}$ such that $x < q < y$. (Hint: Consider $y - x$ and also notice $1 \in \mathbb{R}^+$)

Solution:

Proof. Let $x, y \in \mathbb{R}^+$. By the Archimedean property of the reals, we know there exists $n \in \mathbb{N}$ such that $n(y-x) > 1$. From part (a), we also know there exists $m \in \mathbb{N}$ such that $(m-1) \leq nx < m$. Hence,

$$\begin{aligned} 1 &< n(y-x) \\ 1 + nx &< ny \\ 1 + (m-1) \leq 1 + nx &< ny \\ m &< ny \\ nx < m &< ny \\ x < \frac{m}{n} &< y \end{aligned}$$

Since $n \in \mathbb{N}$, it follows $\frac{m}{n} \in \mathbb{Q}$ so there exists $q \in \mathbb{Q}$ such that $x < q < y$ as required. □

(c) Assume that $\sqrt{2}$ is irrational. Using $\sqrt{2}$, prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$. You may assume that irrational numbers added to and multiplied by rational numbers are still irrational. (Hint: From part (b), we have that there exists $p, q \in \mathbb{Q}$ such that $x < p < q < y$)

Solution:

Proof. Let $x, y \in \mathbb{R}^+$ and assume that $\sqrt{2}$ is irrational. Notice, $\frac{\sqrt{2}}{2}$ is also irrational and $\frac{\sqrt{2}}{2} < 1$. From part (b), we know there exists $p, q \in \mathbb{Q}$ such that $x < p < q < y$. Choose $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then, $q - p > 0$ since $q > p$, so $\alpha > p$. But also, since $\frac{\sqrt{2}}{2} < 1$, $\alpha < p + (q - p) = q$. Since α is irrational and $x < p < \alpha < y$, we have that for all x, y, there exists $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$ as required. $\square$

10 Marks 4. Let $A = \left\{-1, 2, \frac{1}{2}\right\}$

(a) Prove by induction that if 1 is written as a product $1 = p_1 p_2 p_3 \dots p_n$ where $p_i \in A$, then n is even.

Solution:

Proof. Let A be as stated. We proceed with mathematical induction on the number of $p_i \in A$ that are being multiplied together.

- For the base case, notice $1 \notin A$ so $1 \neq p_1$. The base case holds.
- Assume that if for all i such that $1 \leq i \leq k$, $1 = p_1 p_2 p_3 \dots p_k$ is only possible when k is even. For $k+1$, we proceed with a proof by cases.
  - Assume $p_{k+1} = \frac{1}{p_i}$ for some i such that $1 \leq i \leq k$. Then, $p_i p_{k+1} = 1$ and thus we are left with $p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_k$, which has a length of $k-1$ and thus by our assumption, $k-1$ is even or 1 cannot be written as a product $1 = p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_k$. In both cases, we have $k+1$ is even or 1 cannot be written as a product $1 = p_1 p_2 \dots p_{k+1}$
  - Assume $p_{k+1} \neq \frac{1}{p_i}$ for all i such that $1 \leq i \leq k$. We split this into another proof by cases.
    - If $p_{k+1} = -1$, then we know for all i such that $1 \leq i \leq k$, $p_i \neq -1$ so $p_1 p_2 \dots p_{k+1} < 0$ and hence cannot be 1.
    - If $p_{k+1} \neq -1$, then there exists j where $1 \leq j \leq k$ such that $p_{k+1} = p_j$. There are 2 cases to verify.
      - Assume for all i such that $1 \leq i \leq k$, $p_i \neq 2$. Then, $p_1 p_2 \dots p_{k+1} \leq \frac{1}{2}$ and hence cannot be 1.
      - Assume for all i such that $1 \leq i \leq k$, $p_i \neq \frac{1}{2}$ and thus $p_1 p_2 \dots p_{k+1} \geq 2$ and hence cannot be 1.

    Since we have that this product is equal to 1 in none of the cases, our inductive step still holds.

Since both the base case and the inductive step hold, by the principle of mathematical induction, the result holds for all $n \in \mathbb{N}$. □

(b) Prove or disprove that the same applies for $B = \left\{-1, \pm 2, \pm \frac{1}{2}\right\}$.

Solution:

Disproof. Let B be as stated. Notice $1 = -1 \cdot -2 \cdot \frac{1}{2}$ so $1 = p_1 p_2 p_3$ where $p_1, p_2, p_3 \in B$ but 3 is not even, so the same does not apply for B. $\square$

10 Marks 5. Prove that for all $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{\frac{\pi}{2}+2\pi n} < \delta$, and hence prove that the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right) = L$ does not exist. (Hint: Archimedean property of the reals)

Solution:

Proof. Let $\delta > 0$. Notice $2\pi, \frac{1}{\delta} - \frac{\pi}{2} \in \mathbb{R}$ and by the Archimedean property of the reals, we know there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} 2\pi n &> \frac{1}{\delta} - \frac{\pi}{2} \\ \frac{\pi}{2} + 2\pi n &> \frac{1}{\delta} \end{aligned}$$

so $\frac{1}{\frac{\pi}{2}+2\pi n} < \delta$ as required. Let $\varepsilon = 1$, $\delta > 0$ and choose $x = \frac{1}{\frac{\pi}{2}+2\pi n}$ such that $x < \delta$, and notice $x = |x|$ and $x > 0$ so we have $0 < |x| < \delta$. Observe that $\left|\sin\left(\frac{1}{x}\right)\right| = \left|\sin\left(\frac{\pi}{2} + 2\pi n\right)\right| = 1 = \varepsilon$ so the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right) = L$ does not exist. $\square$