MATH 220 Practice Finals 2 Answers — October, 2024, Duration: 2.5 hours

*This test has **9 questions** on **X pages**, for a total of 90 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:									/90

10 Marks 1. Carefully define or restate each of the following:

(a) An upper bound on a set $A \subseteq X$ where X is ordered

Solution: Let $A \subseteq X$. Then, $y \in X$ is called an upper bound if for all $a \in A$, $a \leq y$.

(b) Bézout's lemma

Solution: Let $a, b \in \mathbb{Z}$. Then, there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$.

(c) A partition on a set X

Solution: $P \subseteq \mathcal{P}(X)$ is a partition on X if

- For all $A \in P$, $A \neq \varnothing$.
- For all $x \in X$, there exists $A \in P$ such that $x \in P$.
- For all $A_1, A_2 \in P$, $A_1 = A_2$ or $A_1 \cap A_2 = \varnothing$.

(d) A bounded sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

Solution: We say $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

(e) The principle of mathematical induction

Solution: Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

   (a) For all $n \in \mathbb{N}$, for all $x \in \mathbb{Z}$, for all $y \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $yk \equiv x \mod n$.

Solution: The negation is "There exists $n \in \mathbb{N}$ such that there exist $x \in \mathbb{N}$ such that there exists $y \in \mathbb{N}$ such that for all $k \in \mathbb{N}, yk \not\equiv x \mod n$". The original statement is false. .

Disproof. Let $n = 4$, $x = 1$, $y = 2$ and let $k \in \mathbb{N}$. Notice for all $k \in \mathbb{N}$, $2k \equiv 2 \mod 4$ or $2k \equiv 0 \mod 4$ so the original statement is false. □

(b) For all people $e \in D_e$ where D_e is the set of all people, there exists a function $f : D_e \mapsto D_e$ such that f maps e to the biological grandmothers of e.

Solution: The negation is "There exists a person $e \in D_e$ such that for all functions $f : D_e \mapsto D_e$, f does not map e to the biological grandmothers of e. The original statement is false.

Disproof. Notice for all $e \in D_e$, e has two biological grandmothers. It follows that if f maps e to their biological grandmothers, then $f(e) = e_1 = e_2$ for some $e_1 \neq e_2$, a contradiction. Hence, there does not exist such f. $\square$

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) If $g \circ f$ is bijective, then f is injective.

Solution 1:

Proof. Let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$ and thus $g \circ f(a_1) = g \circ f(a_2)$. Since $g \circ f$ is injective, it follows that $a_1 = a_2$. $\square$

Solution 2:

Proof. We prove the contrapositive. Assume f is not injective, so there exists $a_1, a_2 \in A$ where $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. Then, $g \circ f(a_1) = g \circ f(a_2)$ so $g \circ f$ is not injective and hence not bijective. $\square$

(b) If $g \circ f$ is bijective, then f is surjective.

Solution:

Disproof. Let $A = \{0\}$, $B = \{0, 1\}$ and $C = \{0\}$, let $f(x) = 0$ and $g(x) = 0$. Notice $g \circ f$ maps 0, the only element in A, to 0, the only element in C, so $g \circ f$ is bijective. However, $f^{-1}(\{1\}) = \varnothing$ so f is not surjective. $\square$

10 Marks 4. Let p be a prime. Prove by induction that for all $n \in \mathbb{Z}$, $p \mid n^p - n$. (Hint: You will need to split into different cases for induction, and use the binomial theorem)

Solution:

Proof. We proceed with a proof by cases.

- For $n \geq 0 \in \mathbb{N}$, we proceed with induction on n.
  - For the base case, it holds trivially when $n = 0$ since $p \mid 0^p - 0 = 0$.
  - Assume that the statement holds for $n = k$. Then, observe that

$$\begin{aligned}(k+1)^p - k - 1 &= k^p + 1 + \sum_{i=1}^{k} \binom{k}{i} k^{k-i} - k - 1 \\ &= k^p - k + ps \\ &= pr + ps \\ &= p(r+s)\end{aligned}$$

    so $p \mid (k+1)^p - k - 1$ and the inductive step holds.

  Since the base case and the inductive step hold, by the principle of mathematical induction, the result holds for all $n \geq 0$.

- For $n < 0$, we proceed with induction on $m = -n$, and we show $p \mid (-m)^p + m$ for all $m \in \mathbb{N}$.
  - For the base case, notice when $m = 1 = -n$, $p = 2$ or $p \neq 2$.
    * If $p = 2$, $p \mid (-1)^2 + 1 = 2$
    * If $p \neq 2$, $p \mid (-1)^p + 1 = 0$

    So the base case holds.
  - Assume that the statement holds for $m = k$. Then,

$$\begin{aligned}(-k-1)^p + k + 1 &= (-k)^p + (-1)^p + \sum_{i=1}^{k} \binom{k}{i} (-k)^{k-i}(-1)^k + k + 1 \\ &= (-k)^p + (-1)^p + k + 1 + ps \\ &= (-1)^p + 1 + pr + ps\end{aligned}$$

    Now $p = 2$ or $p \neq 2$
    * If $p = 2$, $(-1)^p + 1 + pr + ps = 2 + 2(r+s) = 2(1+r+s)$.
    * If $p \neq 2$, $(-1)^p + 1 + pr + ps = -1 + 1 + pr + ps = pr + ps = p(r+s)$.

    In all cases, $p \mid (-k-1)^p + k + 1$ so the inductive step holds.

Since the base case and the inductive step hold, by the principle of mathematical induction, the result holds for all $m > 0$ and hence all $-n > 0$, which is the same as all .

As such, it holds for all $n \in \mathbb{Z}$ as required. $\square$

10 Marks

5. Let $\mathbb{R}[x]$ denote the set of polynomials with real coefficients and define a set $I = \{(x^2+1)p(x) | p(x) \in \mathbb{R}[x]\}$. Let $\sim$ be a relation on $\mathbb{R}[x]$ defined as $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I$.

   (a) Prove $\sim$ is an equivalence relation.

Solution:

Proof. We prove reflexivity, symmetry, and transitivity in turn.

- For reflexivity, let $f(x) \in \mathbb{R}[x]$. Then, $f(x) - f(x) = 0 = (x^2+1) \cdot 0 \in I$ so $f(x) \sim f(x)$.
- For symmetry, let $f(x), g(x) \in \mathbb{R}[x]$ and assume $f(x) \sim g(x)$. Then, $f(x) - g(x) = (x^2+1)p(x)$ for some $p(x) \in \mathbb{R}[x]$ so $g(x) - f(x) = (x^2+1)(-p(x)) \in I$ so $g(x) \sim f(x)$.
- For transitivity, let $f(x), g(x), h(x) \in \mathbb{R}[x]$ and assume $f(x) \sim g(x)$ and $g(x) \sim h(x)$. Then, $f(x) - h(x) = f(x) - g(x) + g(x) - h(x) = (x^2+1)p(x) + (x^2+1)q(x) = (x^2+1)(p(x)+q(x)) \in I$ for some $p(x), q(x) \in \mathbb{R}[x]$ so $f(x) \sim h(x)$.

Hence, $\sim$ is an equivalence relation as required. $\square$

(b) Let $f(x) \in \mathbb{R}[x]$ and $[f(x)]$ be its equivalence class under $\sim$. Prove that $[f(x)]$ must be of the form $\{a + bx + p(x) | p(x) \in I\}$ where $a, b \in \mathbb{R}$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

Solution:

Proof. Let $f(x) \in \mathbb{R}[x]$. By the polynomial division algorithm, we know $f(x) = q(x)(x^2+1)+r(x)$ where $q(x), r(x) \in \mathbb{R}[x]$ and $r(x)$ has degree 0 or 1. By definition, we know $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I$ which implies $f(x) - g(x)$ has the same polynomial remainder $r(x)$, namely, $r(x) = a + bx$ for some $a, b \in \mathbb{R}$ since $r(x)$ has degree 0 or 1. Also notice if $p(x) \in I$, we have $f(x) \sim f(x) + p(x)$ so $r(x) + p(x) \in [f(x)]$ for all $p(x) \in I$ and thus $r(x) + p(x) \in I$ for all $p(x) \in I$. Hence, $[f(x)] = \{a + bx + p(x) | p(x) \in I\}$ as required. $\square$

(c) Prove that if $f(x), g(x) \in \mathbb{R}[x]$ and $f(x)g(x) \in I$, then $f(x) \in I$ or $g(x) \in I$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

Solution:

Proof. We prove the contrapositive. Assume $f(x) \notin I$ and $g(x) \notin I$, so by the polynomial division algorithm $f(x) = q_1(x)p(x)+r_1(x)$ and $g(x) = q_2(x)p(x)+r_2(x)$ where $r_1(x)$ and $r_2(x)$ are of degree 1. Then,

$$\begin{aligned} f(x)g(x) &= (q_1(x)p(x) + r_1(x))(q_2(x)p(x) + r_2(x)) \\ &= q_1(x)p(x)q_2(x)p(x) + r_1(x)q_2(x)p(x) + q_1(x)p(x)r_2(x) + r_1(x)r_2(x) \\ &= k(x)p(x) + (a_1 + b_1x)(a_2 + b_2x) \\ &= k(x)p(x) + a_1a_2 + a_2b_1x + a_1b_2x + b_1b_2x^2 \end{aligned}$$

where one of $a_1, b_1 \in \mathbb{R}$ is non-zero and one of $a_2, b_2 \in \mathbb{R}$ is non-zero.

- If both b_1, b_2 are zero, both a_1, a_2 are non-zero so we have $f(x)g(x) = k(x)p(x)+a_1a_2$. $f(x)g(x)$ cannot be factored by $x^2 + 1$ so $f(x)g(x) \notin I$ as required.
- If b_1, b_2 are both non-zero, then a_1, a_2 are zero so $f(x)g(x) = k(x)p(x)+b_1b_2x^2$ which cannot be factored by $p(x)$ so $f(x)g(x) \notin I$.
- If b_1, a_2 are non-zero but a_1, b_2 are zero, then $f(x)g(x) = k(x)p(x) + a_2b_1x$ so $x^2 + 1$ cannot factor $f(x)g(x)$ and $f(x)g(x) \notin I$. Without loss of generality, the same holds for b_1, a_2 are zero but a_1, b_2 are non-zero.

This proves all cases so $f(x)g(x) \notin I$ and so this proves the contrapositive. □

10 Marks

6. Prove or disprove each of the following:

   (a) If A is a infinite and $P \subseteq \mathcal{P}(A)$ is a finite partition of A, then for all $X \in P$, X is infinite.

Solution:

Disproof. Let $A = \mathbb{N} \cup \{0\}$ and $P = \{\{0\}, \mathbb{N}\}$. Observe that A is infinite and P is a finite partition, but $X = \{0\} \in P$ is finite, so the statement is false. $\square$

(b) If A is a infinite and $P \subseteq \mathcal{P}(A)$ is an infinite partition of A, then for all $X \in P$, X is finite.

Solution:

Disproof. Let $A = \mathbb{N}$ and P be the partition where all composite numbers form their own equivalence classes and the set of primes is one equivalence class. Since there are infinite composite numbers, P is an infinite partition. However, there are also infinite primes so there exists $X \in P$ such that X is infinite. $\square$

10 Marks

7. (a) Prove that if $A_1, A_2, A_3 \dots A_n$ and $B_1, B_2, B_3, \dots B_n$ are non-empty sets such that for all $i \leq n \in \mathbb{N}$, $|A_i| \leq |B_i|$, then $|\prod_{i=1}^n A_i| \leq |\prod_{i=1}^n B_i|$ by constructing an explicit injection $f : \prod_{i=1}^n A_i \mapsto \prod_{i=1}^n B_i$

Solution:

Proof. Since for all non-empty sets $A_1, A_2, A_3 \dots A_n$ and $B_1, B_2, B_3, \dots B_n$, we have $|A_i| \leq |B_i|$, we know that for all A_i there exists an injection $f_i : A_i \mapsto B_i$. Then, I claim $f(a_1, a_2, \dots, a_n) = (f_1(a_1), f_2(a_2), \dots, f_n(a_n))$ would be an injection. This is clear since if

$$\begin{aligned} f(a_1, a_2, \dots, a_n) &= (f_1(a_1), f_2(a_2), \dots, f_n(a_n)) \\ &= (f_1(a_1'), f_2(a_2'), \dots, f_n(a_n')) \\ &= f(a_1', a_2', \dots, a_n') \end{aligned}$$

Then by the injectivity of f_i, we have $a_i = a_i'$ for all $0 \leq i \leq n$, so $(a_1, a_2, \dots, a_n) = (a_1', a_2', \dots, a_n')$. Hence, f is an injection. □

(b) Prove or disprove that the result holds when there exists B_i such that B_i is empty.

Solution:

Proof. Assume for all $A_1, A_2, A_3 \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$, we have $|A_i| \leq |B_i|$, and there exists B_i such that B_i is empty. Then, since $|A_i| \leq |B_i|$, $A_i = \varnothing$. It follows that $\prod_{i=1}^{n} A_i = \varnothing = \prod_{i=1}^{n} B_i$ so $|\prod_{i=1}^{n} A_i| \leq |\prod_{i=1}^{n} B_i|$ as required, $\square$

10 Marks

8. Let $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -2x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$

(a) Recall that $f(x)$ is continuous if for all $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = f(a)$. Prove that $f(x)$ is discontinuous. (Hint: Use density of rationals/irrationals in the reals)

Solution:

Proof. Let $a = 1$ and choose $\varepsilon = 1$. Then, let $\delta > 0$ and choose x to be any irrational number between 1 and $\delta + 1$ so $0 < |x - 1| < \delta$. Then,

$$\begin{aligned} |f(x) - f(1)| &= |f(x) - 2| \\ &= |-2x - 2| \\ &= |-1||2x + 2| \\ &= 2x + 2 \geq 1 = \varepsilon \end{aligned}$$

So f is discontinuous as required. $\square$

(b) We say $f(x)$ is everywhere discontinuous if for all $a \in \mathbb{R}$ there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that $0 < |x-a| < \delta$ and $|f(x)-f(a)| \geq \epsilon$. Prove or disprove that $f(x)$ everywhere discontinuous.

Solution: $f(x)$ is not everywhere discontinuous.

Disproof. I claim $f(x)$ is continuous at $a = 0$. Let $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$ and assume $0 < |x| < \delta$. Then, $|f(x) - f(0)| = \left|\frac{\pm 2\varepsilon}{4}\right| = \frac{\varepsilon}{2} < \varepsilon$ so $f(x)$ is continuous at $a = 0$ and hence not everywhere discontinuous. $\square$

10 Marks 9. Let $f : (0, 1) \mapsto \mathbb{R}$ be defined as $f(x) = \frac{2x-1}{x-x^2}$.

(a) Prove that f is injective.

Solution:

Proof. Let f be as stated and assume $f(x_1) = f(x_2)$. Then,

$$\begin{aligned}
\frac{2x_1 - 1}{x_1 - {x_1}^2} &= \frac{2x_2 - 1}{x_2 - {x_2}^2} \\
(2x_1 - 1)(x_2 - {x_2}^2) &= (2x_2 - 1)(x_1 - {x_1}^2) \\
2x_1(x_2 - {x_2}^2) - (x_2 - {x_2}^2) &= 2x_2(x_1 - {x_1}^2) - (x_1 - {x_1}^2) \\
2x_2{x_1}^2 - 2x_1{x_2}^2 + x_1 - {x_1}^2 - x_2 + {x_2}^2 &= 0 \\
2x_2{x_1}^2 - {x_1}^2 + x_1 - 2x_1{x_2}^2 + {x_2}^2 - x_2 &= 0 \\
(x_1 - x_2)(2x_1x_2 - x_1 + 1) &= 0
\end{aligned}$$

So $x_1 = x_2$ or $2x_1x_2 - x_1 + 1 = 0$. Assume for the sake of contradiction that $x_1 \neq x_2$. Then, $2x_1x_2 - x_1 + 1 = 0$ s0

$$\begin{aligned}
2x_1x_2 - x_1 &= -1 \\
x_1(2x_2 - 1) &= -1 \\
2x_2 - 1 &= \frac{-1}{x_1}
\end{aligned}$$

Now $x_1 > 0$, so $2x_2 - 1 < 0$. Notice LHS < -1 since $0 < x_1 < 1$, but RHS > -1, so LHS $\neq$ RHS, a contradiction. Hence, $x_1 = x_2$ and f is injective as required. $\square$

(b) Prove that f is surjective.

Solution:

Proof. Let $y \in \mathbb{R}$. Then, we solve for $y = \frac{2x-1}{x-x^2}$ where $x \in (0,1)$ so we solve $yx^2 + (y-2)x - 1 = 0$. Either $y = 0$ or $y \neq 0$.

- If $y = 0$, notice if $x = \frac{1}{2}$ then $f(x) = 0$.
- If $y \neq 0$, observe that by the quadratic formula,

$$\begin{aligned} x &= \frac{y-2 \pm \sqrt{(y-2)^2 + 4y}}{2y} \\ &= \frac{y-2 \pm \sqrt{y^2+4}}{2y} \end{aligned}$$

  Now to check $x \in (0,1)$, it suffices to check $0 < y - 2 + \sqrt{y^2+4} < 2y$. Now observe that $\sqrt{(2-y)^2} = \sqrt{y^2 - 4y + 4} < \sqrt{y^2+4} < \sqrt{y^2+4y+4} = \sqrt{(y+2)^2}$ so

$$\begin{aligned} y - 2 + \sqrt{(2-y)^2} < y - 2 + \sqrt{y^2+4} &< y - 2 + \sqrt{(y+2)^2} \\ 0 < y - 2 + \sqrt{y^2+4} &< y - 2 + y + 2 = 2y \end{aligned}$$

  So $x = \frac{y-2+\sqrt{y^2+4}}{2y}$ maps to y as required.

Hence, f is surjective. □

(c) Hence prove that $|(0,1)| = |\mathbb{R}|$.

Solution:

Proof. From above, f is injective and surjective so f is a bijection between $(0,1)$ and $\mathbb{R}$ so $|(0,1)| = |\mathbb{R}|$. □