MATH 220 Torture Finals — October, 2024, Duration: 2.5 hours

*This test has **8 questions** on **24 pages**, for a total of 100 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8
Points:								
Total:	/100							

10 Marks

1. Carefully define or restate each of the following:

   (a) A relation on A

   (b) A function $f : A \mapsto B$

   (c) The limit of a function $f : \mathbb{R} \mapsto \mathbb{R}$ as $x \to a$

   (d) Euclidean division

   (e) The principle of mathematical induction

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

   (a) There exists a prime $p \in \mathbb{Z}$ such that there exists $r, k \in \mathbb{Z}$ such that $0 \leq r \leq p-1$ and $k < p$ and $p \mid kr$.

(b) For all $n \in \mathbb{N}$, for all $p, q \in \mathbb{N}$ such that $k, \ell \in \mathbb{N}$ are the smallest integers such that $n \mid kp$ and $n \mid \ell q$, then if $k = \ell$, $|\{px \in \mathbb{Z} | 0 \leq px \leq kp \in \mathbb{Z}\}| = |\{qy \in \mathbb{Z} | 0 \leq yq \leq \ell q \in \mathbb{Z}\}|$

10 Marks 3. Prove that there does not exist a bijective $f : \mathbb{Q} \mapsto \mathbb{Q} \setminus \{0\}$ such that for all $x, y \in \mathbb{Q}$, $f(x+y) = f(x)f(y)$.

10 Marks

4. Let $S = \{1, 2, 3, \ldots, m \in \mathbb{N}\}$ and $p_i : S \mapsto S$ be a function that swaps some $a, b \in S$ around where $a \neq b$ and fixes everything else, denoted $p_i = (ab)$. Prove that the identity map e can only be written as a product $p_1 p_2 p_3 \ldots p_n$ where n is even by induction on the number of p_i.

$\boxed{\text{15 Marks}}$ 5. (a) Let D_n denote the set of rotations by an angle of $\frac{2\pi}{i}$ where $1 \leq n$ and reflections across an axis of symmetry on an $n-$gon, and let $\sim$ be a relation $\sigma \sim \tau$ if and only if $\sigma\tau$ is a rotation where $\sigma, \tau \in D_n$. Prove that $\sim$ is an equivalence relation.

(b) Let $R \subseteq D_n$ be the set of rotations by an angle of $\frac{2\pi}{i}$ where $1 \leq n$, and for all $\sigma \in D_n$, let σR denote the set $\{\sigma x | x \in R\}$. Prove that the collection of σR forms a partition on D_n.

(c) Prove that for all $\sigma \in D_n$, $\sigma R = [\sigma]$ where $[\sigma]$ is the equivalence class of σ under $\sim$.

10 Marks

6. Let m, n be coprime integers with $m < n$. Prove that for all $a, b \in \mathbb{Z}$, there exists $x \in \mathbb{Z}$ such that $x \equiv a \mod m$ and $x \equiv b \mod m$. (Hint: Contradiction and pigeonhole principle)

15 Marks 7. Prove or disprove each of the following:

(a) There exists a bijective function $f : \mathbb{R} \mapsto S_1$ where S_1 is the unit circle in $\mathbb{C}$.

(b) There exists a function $f : \mathbb{R} \mapsto S_1$ where S_1 is the unit circle in $\mathbb{C}$ such that for all $x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$.

(c) A countable union of countable sets is countable.

(d) A denumerable intersection of uncountable sets is countable.

20 Marks

8. (a) Let $A \subseteq \mathbb{R}$. Prove that if $-A = \{-a | a \in A\}$, then $-\sup A = \inf A$.

(b) Let $\{x_n\} : \mathbb{N} \mapsto \mathbb{R}$ be a non-decreasing sequence. Prove that if $\{x_n\}$ is bounded above, then $\lim_{n\to\infty} x_n = \sup\{x_n\}$.

(c) Prove that if $\{x_n\}$ is a non-increasing sequence, and if $\{x_n\}$ is bounded below, then $\lim_{n\to\infty} x_n = \inf \{x_n\}$

(d) Prove that if $\{x_n\} : \mathbb{N} \mapsto \mathbb{R}$ is bounded, then $\{x_n\}$ has a finite limit.

(e) Let $\{x_n\} : \mathbb{N} \mapsto \mathbb{R}$ be a sequence, and we define $\{a_n\}$ to be a subsequence of $\{x_n\}$ if we can obtain $\{a_n\}$ from removing some elements from $\{x_n\}$. Let $n \in \mathbb{N}$ be a peak if for all $m \geq n$, $x_m \geq x_n$. Prove that if $\{x_n\}$ has infinite peaks then $\{x_n\}$ has a monotone (non-decreasing or non-increasing) subsequence.

(f) Prove that if $\{x_n\}$ has finite peaks then $\{x_n\}$ has a monotone subsequence.

(g) Prove that if $\{x_n\}$ is bounded then $\{x_n\}$ has a convergent subsequence.

(h) We denote $\{b_n\} : \mathbb{N} \mapsto \mathbb{R}^n$ to be a sequence of vectors in $\mathbb{R}^n$ and we say a sequence $\{b_n\} : \mathbb{N} \mapsto \mathbb{R}^n$ is bounded if there exists $M > 0 \in \mathbb{R}$ such that for all $i \in \mathbb{N}$, $\|b_i\| \leq M$ where $\|b_i\|$ denotes the Euclidean norm of b_i. Prove that if $\{b_n\}$ which can be represented as $(\{b_{n1}\}, \{b_{n2}\}, \ldots, \{b_{nn}\})$ is bounded, then for all $j \in \mathbb{N}$, $\{b_{nj}\}$ is bounded.

(i) We say $\{b_n\} : \mathbb{N} \mapsto \mathbb{R}^n$ converges to $L \in \mathbb{R}^n$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\|x_n - L\| < \varepsilon$ where $\|x_n - L\|$ denote the Euclidean norm. Prove that if $\{b_n\}$ has a convergent subsequence, then there exists a countable $K \subseteq I$ where I is the index set of $\{b_n\}$

(j) Using the results from parts (g),(h),(i), prove that every bounded sequence $\{b_n\} : \mathbb{N} \mapsto \mathbb{R}^n$ has a convergent subsequence.