MATH 220 Practice Finals 1 — October, 2024, Duration: 2.5 hours

*This test has **10 questions** on **20 pages**, for a total of 100 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:										/100

10 Marks

1. Carefully define or restate each of the following:

   (a) A rational number $q \in \mathbb{Q}$

   (b) Bézout's lemma

   (c) The Fundamental Theorem of Arithmetic

   (d) A convergent sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

   (e) The principle of mathematical induction

10 Marks

2. Write the negation of each of the following and prove or disprove the original statement.

(a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$.

(b) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$.

$\boxed{\text{10 Marks}}$ 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) For all $U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

(b) For all $U \subseteq B$, $(g \circ f)^{-1}(g(U)) = f^{-1}((U))$

10 Marks

4. Let $n \in \mathbb{N}$ be even and $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$. Let $S = \{k \in \mathbb{Z}/n\mathbb{Z} | 2k \equiv 0 \mod n\} \subseteq \mathbb{Z}/n\mathbb{Z}$. Prove that $|S|$ is odd.

10 Marks

5. Prove that $f : \mathbb{R} \mapsto \mathbb{C} \setminus \{0\}$ where $f(x) = e^{2\pi i x}$ is injective and for all $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$.

(b) Let R be a relation on $\mathbb{R}$ be defined as xRy if and only if $x = y + z$ where $z \in f^{-1}(1)$. Prove that R is an equivalence relation.

(c) Find all equivalence classes under R. Show that the operation $[a] + [b] = [a + b]$ is well defined for $a, b \in \mathbb{R}$.

(d) Let $\mathbb{R}/R$ denotes the set of equivalence classes under R. Find a bijective map $g : \mathbb{R}/R \mapsto \mathrm{Im}(f)$ such that $g([x] + [y]) = g([x])g([y])$. Prove your result.

10 Marks

6. (a) Let X be a set. Prove that any equivalence relation on X forms a partition on X.

(b) Prove that any partition on X corresponds to equivalence classes of an equivalence relation on X.

$\boxed{\text{10 Marks}}$ 7. (a) Prove that if $a \neq 0$, $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$.

(b) Prove that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

$\boxed{\text{10 Marks}}$ 8. Prove or disprove each of the following:

(a) Let $A, B \subseteq C$. If $|A \backslash C| = |B \backslash C|$, then $|A| = |B|$

(b) Let $A_i \in X$. Then, $(\bigcup_{i\in I} A_i)\backslash X = \bigcap_{i\in I}(A_i\backslash X)$.

10 Marks 9. Prove or disprove that the set of decimal numbers $0.a_1a_2a_3\dots$ where $a_1, a_2, a_3 \in \{0, 1\}$ is uncountable.

$\boxed{\text{10 Marks}}$ 10. Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let S_n be the set of bijective functions $f : N \mapsto N$.

(a) Prove by induction that for all $n \in \mathbb{N}$, $|S_n| = n!$.

(b) Prove by induction that for all $n \geq 3 \in \mathbb{N}$, there exists $f, g \in S_n$ such that $f \circ g \neq g \circ f$.