# MATH 220 Practice Midterm — September, 2024, Duration: 50 minutes This test has 5 questions on 8 pages, for a total of 50 points.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:					/50

# 10 Marks

- 1. Negate each of the following and prove or disprove the original statement:
  - (a) For all  $x \in \mathbb{R}$ , there exists  $q \in \mathbb{Q}$  such that for all  $r < q \in \mathbb{Q}$ , r + q < x".

**Solution:** The negation is "There exists  $x \in \mathbb{R}$  such that for all  $q \in \mathbb{Q}$ , there exists  $r < q \in \mathbb{Q}$  such that  $r + q \ge x$ .

*Proof.* Let  $x \in \mathbb{R}$ . Pick  $q = \frac{\lfloor x \rfloor}{2} - 1 \in \mathbb{Q}$ . Notice for all  $r < q \in \mathbb{Q}$ ,  $r + q < 2q = \lfloor x \rfloor - 2 \le x - 2 < x$  so the result holds.

(b) For all  $x \in \mathbb{R}$ , there exists  $y \in \mathbb{R}$  such that for all  $z \in \mathbb{R}$ , if xy < z then x < 0 or  $x^2 + y^2 < z$ 

**Solution:** The negation is "There exists  $x \in \mathbb{R}$  such that for all  $y \in \mathbb{R}$ , there exists  $z \in \mathbb{R}$  such that xy < z and  $x \ge 0$  and  $x^2 + y^2 \ge z$ ".

Disproof. Choose x=2 and let  $y\in\mathbb{R}$ . Choose  $z=4+y^2$ . Notice  $x\geq 0$  and  $z\leq 4+y^2=x^2+y^2$ . Now, notice  $y^2-2y+4=(y-1)^2+2\geq 2>0$ , so

$$y^{2} - 2y + 4 > 0$$
$$y^{2} + 4 > 2y$$
$$z = x^{2} + y^{2} > xy$$

and thus xy < z. This proves the negation and thus disproves the original statement.

10 Marks

2. Let  $p \in \mathbb{N}$  and assume p > 1. Prove that if there exists  $x \in \mathbb{Z}$  such that  $x \not\equiv 0 \mod p$  and for all  $y \in \mathbb{Z}$ ,  $xy \not\equiv 1 \mod p$ , then p is not prime.

## Solution:

*Proof.* We prove the contrapositive. Let  $p \in \mathbb{N}$  be prime. Let  $x \in \mathbb{Z}$ .

- If  $x \equiv 0 \mod p$ , then we are done.
- If  $x \not\equiv 0 \mod p$ , we know  $x \equiv r \mod p$  for some  $1 \leq r < p$ . By Bézout's lemma, we know there exists  $\alpha, \beta \in \mathbb{Z}$  such that  $\alpha r + \beta p = \gcd(r, p) = 1$  so  $\alpha r = 1 \beta p$ . It follows that  $\alpha r \equiv 1 \mod p$  so choosing such  $\alpha = y$  would yield  $xy \equiv 1 \mod p$  as required.

10 Marks

- 3. Let  $a, b \in \mathbb{Z}$  where  $a, b \neq 0$  and  $S = \{ax + by | x, y \in \mathbb{Z}, ax + by > 0\}$ . You may not use Bézout's lemma for this section.
  - (a) Prove that S is non-empty.

### **Solution:**

*Proof.* Notice  $a \in S$  so S is non-empty.

(b) Prove that the minimal element  $d = as + bt \in S$  for some  $s, t \in \mathbb{Z}$  divides both a and b. (Hint: Euclidean division of a by d and b by d)

## **Solution:**

Proof. Let S be as stated and let d be the minimal element of S. By Euclidean division, we know there exists  $m, r \in \mathbb{Z}$  such that a = md + r,  $0 \le r < d$ . Notice r = a - md = a - mas - mbt = a(1 - ms) + b(-mt) so  $r \in S \cup \{0\}$ . By assumption, we have d is the minimal element in S, but  $0 \le r < d$  so r = 0. Likewise, there exists  $n, r \in \mathbb{Z}$  such that b = nd + r,  $0 \le r < d$ . It follows that r = a(-ns) + b(1 - nt) so  $r \in S \cup \{0\}$ , and by d is the minimal element, we have r = 0. It follows that a = md and b = nd so  $d \mid a$  and  $d \mid b$ .

(c) Prove that if  $c \mid a$  and  $c \mid b$ , then  $c \leq d$ .

# Solution:

*Proof.* Let S and d be as stated and assume  $c \mid a$  and  $c \mid b$ , so a = ck and  $b = c\ell$  for some  $k, \ell \in \mathbb{Z}$ . Notice  $d = as + bt = c(sk + t\ell)$  so  $c \mid d$ . By assumption, since  $d \in S$ , d > 0, so  $c \le d$  as required.

10 Marks

4. Let  $A = \{a_1, a_2, \dots, a_n\} \subseteq \mathbb{Q}$  and  $S = \{c_1a_1 + c_2a_2 + \dots + c_na_n | a_i \in A, c_i \in \mathbb{Z}\}$ . Prove by induction that if A has  $n \in \mathbb{N}$  elements, then there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$ . (Hint: Show that  $S \subseteq \{cq, c \in \mathbb{Z}\}$  and  $S \supseteq \{cq, c \in \mathbb{Z}\}$  and find q in your inductive step)

### Solution:

*Proof.* Let A and S be as stated. We proceed with mathematical induction on the number of elements in A.

- If A has 1 element a, then,  $S = \{ca, c \in \mathbb{Z}\}$  and since  $a \in \mathbb{Q}$ , the base case holds.
- Assume when  $A = A_k$  has k, that there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$ . Consider  $A = A_{k+1}$  and partition A into  $A \setminus \{a\}$  and  $\{a\}$  for some  $a \in A_{k+1}$ . Notice S consists of all the ways multiples of a can add to multiples of elements in  $A \setminus \{a\}$ , which are precisely elements in  $\{c_1a + c_2q | c_1, c_2 \in \mathbb{Z}\}$ . Since  $a, q \in \mathbb{Q}$ ,  $a = \frac{x_1}{y_1}$  and  $q = \frac{x_2}{y_2}$  where  $y_1, y_2 \neq 0$ . Let  $\frac{\gcd(y_2x_1, y_1x_2)}{y_1y_2} = u$ . We claim that  $\{cu, c \in \mathbb{Z}\} = S$ .
  - Let  $p \in \{cu, c \in \mathbb{Z}\} = S$ . By Bézout's lemma, we know there exists integers  $\alpha, \beta$  such that

$$p = cu = \frac{c(\alpha y_2 x_1 + \beta y_1 x_2)}{y_1 y_2}$$
$$= \frac{c\alpha y_2 x_1 + c\beta y_1 x_2}{y_1 y_2}$$
$$= \frac{c\alpha x_1}{y_1} + \frac{c\beta x_2}{y_2}$$
$$= c\alpha(a) + c\beta(q)$$

so  $p \in \{c_1a + c_2q, c_1, c_2 \in \mathbb{Z}\} = Se, \{cu, c \in \mathbb{Z}\} \subseteq S.$ 

- Let  $p \in S$  so  $p = c_1 a + c_2 q$  for some  $c_1, c_2 \in \mathbb{Z}$ . Then,

$$p = \frac{c_1 x_1}{y_1} + \frac{c_2 x_2}{y_2}$$
$$= \frac{c_1 y_2 x_1 + c_2 y_1 x_2}{y_1 y_2}$$

Notice  $gcd(y_2x_1, y_1x_2) | y_2x_1$  and  $gcd(y_2x_1, y_1x_2) | y_1x_2$ , so

$$y_2 x_1 = \alpha \gcd(y_2 x_1, y_1 x_2)$$

$$y_1x_2 = \beta \gcd(y_2x_1, y_1x_2)$$

where  $\alpha, \beta \in \mathbb{Z}$  and thus

$$p = \frac{c_1 \alpha \gcd(y_2 x_1, y_1 x_2) + c_2 \beta \gcd(y_2 x_1, y_1 x_2)}{y_1 y_2}$$
  
=  $(c_1 \alpha + c_2 \beta) u$ 

so  $p \in \{cu, c \in \mathbb{Z}\}, \{cu, c \in \mathbb{Z}\} \supseteq S$ .

So there exists  $q \in \mathbb{Q}$  such that  $S = \{cq, c \in \mathbb{Z}\}$  as required.

Since the base case and the inductive step hold, by the principle of mathematical induction, the result follows.  $\Box$ 

10 Marks

5. Prove or disprove that the sequence  $(x_n)_{n\in\mathbb{N}} = \frac{1\cdot 3\cdot 5\cdot ...\cdot (2n-1)}{2\cdot 4\cdot 6\cdot ...(2n)}$  converges. (Hint: Consider an inequality between this sequence and  $\frac{1}{\sqrt{3n+1}}$ )

### **Solution:**

*Proof.* We first prove that for all  $n \in \mathbb{N}$ ,  $\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots (2n)} \leq \frac{1}{\sqrt{3n+1}}$ . We proceed with induction.

- For the base case,  $\frac{1}{2} = \frac{1}{\sqrt{4}} = \frac{1}{\sqrt{3+1}}$  so the base case holds.
- Assume that the result holds for n = k. Then, for k + 1, by assumption,

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdot \dots (2n)(2n+2)} \le \frac{(2n+1)}{\sqrt{3n-1}(2n+2)}$$

$$= \sqrt{\frac{(2n+1)^2}{(3n+1)(2n+2)^2}}$$

$$= \sqrt{\frac{(2n+2)^2}{12n^3 + 28n^2 + 20n + 4}}$$

$$\le \sqrt{\frac{(2n+2)^2}{12n^3 + 28n^2 + 19n + 4}}$$

$$= \sqrt{\frac{1}{(3n+4)}}$$

so the inductive step holds

Since the base case and inductive step hold, this proves that  $n \in \mathbb{N}$ ,  $\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \le \frac{1}{\sqrt{3n+1}}$ . Now, let  $\varepsilon > 0$  and choose  $N = \left\lceil \frac{1}{3\varepsilon^2} \right\rceil \in \mathbb{N}$ . Notice, for all n > N,

$$\left| \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} \right| \le \left| \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2N-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2N)} \right|$$

$$\le \left| \frac{1}{\sqrt{3N+1}} \right|$$

$$< \frac{1}{\sqrt{3N}}$$

$$= \frac{1}{\sqrt{3 \left\lceil \frac{1}{3\varepsilon^2} \right\rceil}}$$

$$\le \frac{1}{\sqrt{\frac{3}{3\varepsilon^2}}} = \varepsilon$$

So  $(x_n)_{n\in\mathbb{N}}$  converges as required.