MATH 220 Practice Finals 1 — October, 2024, Duration: 2.5 hours

*This test has **10 questions** on **20 pages**, for a total of 100 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:										/100

10 Marks

1. Carefully define or restate each of the following:

(a) A rational number $q \in \mathbb{Q}$

Solution: $q \in \mathbb{Q}$ if there exists coprime $a, b \in \mathbb{Z}$ where $b \neq 0$ such that $q = \frac{a}{b}$.

(b) Bézout's lemma

Solution: For all $a, b \in \mathbb{Z}$, there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$

(c) The Fundamental Theorem of Arithmetic

Solution: Let $n \in \mathbb{N}$. Then, n can be uniquely factorised into a product of prime powers ${p_1}^{e_1}{p_2}^{e_2} \ldots {p_n}^{e_n}$ up to order, where p_i are distinct primes and $e_i \in \mathbb{Z}$.

(d) A convergent sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

Solution: $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0 \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \varepsilon$.

(e) The principle of mathematical induction

Solution: Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks 2. Write the negation of each of the following and prove or disprove the original statement.

(a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$.

Solution: The negation is "There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $x + y < z$ and $x - y \leq z$". The original statement is false.

Disproof. Let $x = 0$ and $y \in \mathbb{R}$. Choose $z = |y| + 1$. Notice $\pm y \leq |y| < 1 + |y|$, so $z > x + y$ and $z > x - y$ so the statement is false. $\square$

(b) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$.

Solution: The negation is "For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that there exists $z \in \mathbb{R}$ such that $xy \leq z$". The original statement is false.

Disproof. Let $x \in \mathbb{R}$ and choose $y = z = 0$. Then, $xy = 0 = z$ so the statement is false. □

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) For all $U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

Solution:

Proof. Let everything be as stated. We show each inclusion in turn.

- Assume $x \notin f^{-1}(g^{-1}(U))$, so $f(x) \notin g^{-1}(U)$ and $g(f(x)) \notin U$. It follows that $x \notin (g \circ f)^{-1}(U)$, so by contraposition, $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$
- Assume $x \in f^{-1}(g^{-1}(U))$. Then, $f(x) \in g^{-1}(U)$ and $g(f(x)) \in U$ so $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$.

It follows that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. as required. $\square$

(b) For all $U \subseteq B$, $(g \circ f)^{-1}(g(U)) = f^{-1}((U))$

Solution:

Disproof. Let $A = \{1\} = C$, $B = \{0, 1\}$, $f(x) = 1$ and $g(x) = 1$. Then, notice for $U = \{0\}$, $f^{-1}((U)) = \varnothing$, but $(g \circ f)^{-1}g(U) = (g \circ f)^{-1}(\{1\}) = \{1\}$ so $(g \circ f)^{-1}(g(U)) \neq f^{-1}((U))$ and thus the statement is false. $\square$

10 Marks

4. Let $n \in \mathbb{N}$ be even and $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$. Let $S = \{k \in \mathbb{Z}/n\mathbb{Z} | 2k \equiv 0 \mod n\}$. Prove that $|S|$ is even.

Solution:

Proof. Assume for the sake of contradiction that $|S|$ is not even, so for there exists an odd number of elements $k \in \mathbb{Z}/n\mathbb{Z}$ such that $2k \equiv 0 \mod n$. Consider $\mathbb{Z}/n\mathbb{Z} - S$, which has an odd number of elements. Let $a \in \mathbb{Z}/n\mathbb{Z} - S$. Since $2a \not\equiv 0 \mod n$, it follows that there exists $a^{-1} \in \mathbb{Z}/n\mathbb{Z} - S$ such that $a + a^{-1} \equiv 0 \mod n$, but there are only an odd number of a so there exists $a_0 \in \mathbb{Z}/n\mathbb{Z} - S$ such that $a_0 = {a_0}^{-1}$ which implies $2a_0 \equiv 0 \mod n$, a contradiction. Hence, $|S|$ is even as required. $\square$

10 Marks

5. (a) Prove that $f : \mathbb{R} \mapsto \mathbb{C} \setminus \{0\}$ where $f(x) = e^{2\pi i x}$ is neither injective nor surjective and for all $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$.

Solution:

Proof. Notice $0 \neq 1$ but $f(0) = f(1) = 0 \in \mathbb{C}$ so f is not injective. Also notice there does not exist $x \in \mathbb{R}$ such that $f(x) = 0$ since $e^{2\pi i x} \neq 0$ for all $x \in \mathbb{R}$. Hence, f is not surjective. Finally, let $x, y \in \mathbb{R}$. Observe that $f(x+y) = e^{2\pi i (x+y)} = e^{2\pi i x + 2\pi i y} = e^{2\pi i x} e^{2\pi i y} = f(x)f(y)$ so $f(x+y) = f(x)f(y)$ as required. $\square$

(b) Let R be a relation on $\mathbb{R}$ be defined as xRy if and only if $x = y + z$ where $z \in f^{-1}(\{1\})$. Prove that R is an equivalence relation.

Solution:

Proof. We prove reflexivity, symmetry, and transitivity in turn.

- Notice $x = x + 0$ and $0 \in f^{-1}(\{1\})$ so xRx as required.
- Assume xRy. Then, notice $y = x - z$ and since $f(-z) = e^{-2\pi iz} = \frac{1}{f(z)} = 1$, we have $-z \in f^{-1}(\{1\})$ and so yRx.
- Assume xRy and yRw. Then, $x = y + z_1$ and $y = w + z_2$ so $x = w + z_2 + z_1$. Now notice $f(z_2 + z_1) = f(z_2)f(z_1) = 1 \cdot 1 = 1$ so $z_2 + z_1 \in f^{-1}(\{1\})$ and thus xRw

It follows that R is an equivalence relation. $\square$

(c) Find all equivalence classes under R. Show that the operation $[a]+[b]=[a+b]$ is well defined for $a, b \in \mathbb{R}$.

Solution:

Proof. There are infinite equivalence classes. In particular, for all $x \in [0,1)$, there exists an equivalence class of the form $[x]=\{k+x|k \in \mathbb{Z}\}$. For showing the operation is well defined, we claim that xRy if and only if $f(x)=f(y)$. Assume $x=y+z$ where $z \in f^{-1}(\{1\})$. Then, $f(x)=f(y+z)=f(y)f(z)=f(y)$ so $f(y)=f(x)$. On the other hand, assume $f(x)=f(y)$. Notice for all $k \in \mathbb{Z}$, $f(k)=1$. Then, $f(x)f(z_1)=f(x)1=f(y)1=f(y)f(z_2)=f(y+z_2)$ where $z_1, z_2 \in f^{-1}(\{1\})$ so $x=y+z_2-z_1$ and since $z_2-z_1 \in f^{-1}(\{1\})$, xRy. Let $a, b \in \mathbb{R}$. Since for all $y \in [a+b]$, $f(y)=f(a+b)$, it suffices to show that for all $x \in [a]+[b]$, $f(x)=f(a+b)$. Notice since $x \in [a]+[b]$, $x=a+z_1+b+z_2$ where $z_1, z_2 \in f^{-1}(\{1\})$ so $f(x)=f(a+z_1+b+z_2)=f(a)f(z_1)f(b)f(z_2)=f(a)f(b)=f(a+b)$ so $x \in [a+b]$. Hence, the operation is well defined. $\square$

(d) Let $\mathbb{R}/R$ denotes the set of equivalence classes under R. Find a bijective map $g : \mathbb{R}/R \mapsto \mathrm{Im}(f)$ such that $g([x] + [y]) = g([x])g([y])$. Prove your result.

Solution:

Proof. We claim the map $g([x]) = e^{2\pi ix}$ is such bijective map.

- For injectivity, assume $g([x]) = g([y])$, so $e^{2\pi ix} = e^{2\pi iy}$. It follows that $f(x) = f(y)$ so $[x] = [y]$.
- For surjectivity, let $z \in \mathrm{Im}(f)$, so there exists $x \in \mathbb{R}$ such that $z = f(x)$. It follows that $g([x]) = e^{2\pi ix} = f(x) = z$ so $[x] \in g^{-1}(\{z\})$

Finally, notice $g([x] + [y]) = e^{2\pi(x+y)} = e^{2\pi ix+2\pi iy} = e^{2\pi ix}e^{2\pi iy} = g([x])g([y])$ so $g([x] + [y]) = g([x])g([y])$ and thus g is such a map as required. $\square$

10 Marks

6. (a) Let X be a non-empty set. Prove that any equivalence relation on X forms a partition on X.

Solution:

Proof. Let $\sim$ be an equivalence relation on X. By reflexivity, we have that for all $x \in X$, x lies in some equivalence class and thus every equivalence class is non-empty. For showing that equivalence classes are disjoint, suppose $x \in [a]$ and $x \in [b]$, so $x \sim a$ and $x \sim b$. By symmetry and transitivity, we have $a \sim x$ and $x \sim b$ so $[a] = [b]$. Thus, equivalence classes are disjoint. It follows that the equivalence classes of an equivalence relation partition a set. $\square$

(b) Prove that any partition on X corresponds to equivalence classes of an equivalence relation on X.

Solution:

Proof. Let $P \subseteq \mathcal{P}(X)$ be a partition of X, and define a relation $x \sim y$ if and only if $x, y \in A_i$ where $A_i \in P$.

- Since every x lies in some $A_i \in P$ by the definition of a partition, $x \sim x$.
- If $x, y \in A_i$, then surely $y, x \in A_i$ so $x \sim y$.
- Similarly, if $x, y \in A_i$ and $y, z \in A_i$, it follows that $x, z \in A_i$ since every element lies in precisely one A_i.

Hence, $\sim$ is an equivalence relation and by our choice of equivalence relation, we have that every equivalence class corresponds precisely to some $A_i \in P$. $\square$

10 Marks 7. Prove that if $a \neq 0$, $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$.

Solution:

Proof. Notice $a > 0$, or $a < 0$.

- Let $\varepsilon > 0$ and assume $a > 0$. Choose $\delta = \min(\{b, (a^2 - a)\varepsilon\})$ where $0 < b < a$. Assume $0 < |x - a| < \delta$, so $|x - a| < b$ which implies $a - b < |x|$. Then, notice $\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| < \frac{\delta}{a^2-ab} = \varepsilon$ so $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ as required.
- Let $\varepsilon > 0$ and assume $a < 0$. Choose $\delta = \min(\{-b, |a^2 - ba|\varepsilon\})$ where $0 < -b < -a$. Assume $0 < |x - a| < \delta$, so $|x - a| < -b$ which implies $|a - b| < |x|$. Then, notice $\left|\frac{1}{x} - \frac{1}{a}\right| = \left|\frac{a-x}{ax}\right| < \frac{\delta}{|a^2-ba|} = \varepsilon$ so $\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ as required.

$\square$

10 Marks 8. Prove or disprove each of the following:

(a) Let $A, B \subseteq C$. If $|C \backslash A| = |C \backslash B|$, then $|A| = |B|$.

Solution:

Disproof. Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \mathbb{N}$. Notice there exists an explicit bijection $f : C \backslash A \mapsto C \backslash B$ where $f(n) = n + 1$, but $|A| = 1 \neq 2 = |B|$ so the statement is false. $\square$

(b) Let $A_i \in X$. Then, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$.

Solution:

Proof. Let $A_i \in X$ and $x \in X$. We show each inclusion in turn.

- Let $x \in X \backslash (\bigcup_{i \in I} A_i)$, so $x \notin (\bigcup_{i \in I} A_i)$ and thus $x \notin A_i$ for all $i \in I$. It follows that $x \in A_i \backslash X$ for all i and thus $x \in \bigcap_{i \in I} (X \backslash A_i)$.
- Let $x \in \bigcap_{i \in I} (X \backslash A_i)$ so for all $i \in I$, $x \notin A_i$. It follows that $x \notin \bigcup_{i \in I} A_i$ so $x \in X \backslash (\bigcup_{i \in I} A_i)$.

$X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$. Hence, $\square$

10 Marks

9. Let $B_r(x) \in \mathbb{R}^n$, called an open ball of radius r, be defined as $\{y \in \mathbb{R}^n | \|x - y\| < r\}$ for some $r > 0$, note that $\|x-y\|$ refers to the Euclidean norm $\|x-y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

   (a) Let $p \in \mathbb{R}^n$ and $q \in B_r(p)$. Show that there exists $B_{r_2}(q) \subseteq B_r(p)$. (Hint: The triangle inequality works for $\|x - y\| \leq \|x - z\| + \|z - y\|$ in $\mathbb{R}^n$ too)

Solution:

Proof. Let $q \in B_r(p)$. Choose $r_2 = r - \|p-q\|$ and $q \in B_{r_2}(q)$. Now, $r > \|p-q\|$ so $r_2 > 0$. Let $x \in B_{r_2}(q)$. By the triangle inequality, $\|p - x\| \leq \|p - q\| + \|q - x\| < \|p - q\| + r_2 = r$ so $B_{r_2}(q) \subseteq B_r(p)$. $\square$

(b) A set $E \subseteq \mathbb{R}^n$ is open if for all $p \in E$, there exists $B_r(p) \subseteq E$. Prove that E is open if and only if E is a union of open balls.

Solution:

Proof. We prove each direction in turn.

- For one direction, suppose E is open. I claim $E = \bigcup_{x \in E} B_r(x)$ where $B_r(x)$ is some open ball centered at x contained in E. Notice this yields $\bigcup_{x \in E} B_r(x) \subseteq E$. On the other hand, if $x \in E$, we know $x \in B_r(x) \subseteq E$ for some $r > 0$ by E is open, so $x \in \cup_{x \in E} B_r(x)$. This proves $E = \cup_{x \in E} B_r(x)$ so E is a union of open balls.
- For the other direction, suppose E is a union of open balls $\{B_i\}$, and let $x \in E$. By definition, $x \in B_{i0}$ for some $B_{i0} \in \{B_i\}$ and from part (a), we know there exists $B_r(x) \subseteq B_{i0} \subseteq E$ so E is open as required.

$\square$

10 Marks 10. Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let S_n be the set of bijective functions $f : N \mapsto N$.

(a) Prove by induction that for all $n \in \mathbb{N}$, $|S_n| = n!$.

Solution:

Proof. Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let S_n be the set of bijective functions $f : N \mapsto N$. We proceed with mathematical induction on n.

- For the base case, notice when $n = 1$, the only bijective map is $f(n) = 1$ so $S_n = \{f\}$ and $|S_n| = 1$.
- Assume that for $n = k$, $|S_k| = k$. For $|S_{k+1}|$, notice taking every map in S_k and fixing an element yields $k!$ bijections and hence elements in S_{k+1}, and fixing and repeating this process with a different element in $N = \{1, 2, 3, \ldots, k + 1\}$ will give $(k + 1)k!$ possible bijections from N to itself and hence $(k + 1)!$ bijections. It follows that $|S_{k+1}| = (k + 1)!$ as required.

By the principle of mathematical induction, $|S_n| = n!$ for all $n \in \mathbb{N}$. □

(b) Prove by induction that for all $n \geq 3 \in \mathbb{N}$, there exists $f, g \in S_n$ such that $f \circ g \neq g \circ f$.

Solution:

Proof. Let $n \geq 3$. We proceed with mathematical induction on n.

- For the base case, let f and g in S_3 be defined as follows:

$$f(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}, g(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{cases}$$

  Notice $f \circ g(1) = 1$ and $g \circ f(1) = 3$ so $f \circ g \neq g \circ f$.

- Assume there exists $f, g \in S_k$ such that $f \circ g \neq g \circ f$ for some k. For S_{k+1}, choose such $f, g \in S_k$ and create f', g' which is the same as f and g except that the $k + 1$th element gets mapped to itself. Notice there exists some $i \in \{1, 2, \ldots k\}$ such that $f' \circ g'(i) = f \circ g(i) \neq g \circ f(i) = g' \circ f'(i)$ so it follows that $f' \circ g' \neq g' \circ f'$ and thus the result holds.

By the principle of mathematical induction, the result holds for all $n \geq 3$. $\square$