MATH 220 Practice Midterm 2 — Duration: 50 minutes

*This test has **5 questions** on **10 pages**, for a total of 50 points.*

Disclaimer: This test is definitely harder than actual 220 midterm exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 50 minutes but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:					/50

10 Marks

1. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$, such that there exists $z \in \mathbb{Z}$, such that $xy > z$ implies $x + y < z$.

(b) For all $x \in \mathbb{N}$, for all $y \in \mathbb{N}$ such that $x < y$, there exists $a, b \in \mathbb{Z}$ such that $ax + by < \left|\frac{x}{y}\right|$

10 Marks 2. Let $n, k \in \mathbb{N}$. Prove that if for all $m \in \mathbb{Z}$, $m^k \neq n$, then $n^{1/k}$ is irrational.

10 Marks

3. The Archimedean property of the reals guarantees that for all $x, y \in \mathbb{R}$ where $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

   (a) Let $x, y \in \mathbb{R}^+$. Prove that there exists $m \in \mathbb{N}$ such that $(m-1)x \leq y < mx$.

   (b) Using (a), prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $q \in \mathbb{Q}$ such that $x < q < y$. (Hint: Consider $y - x$ and also notice $1 \in \mathbb{R}^+$)

(c) Assume that $\sqrt{2}$ is irrational. Using $\sqrt{2}$, prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $z \in \mathbb{R} \backslash \mathbb{Q}$ such that $x < z < y$.You may assume that irrational numbers added to and multiplied by rational numbers are still irrational. (Hint: From part (b), we have that there exists $p, q \in \mathbb{Q}$ such that $x < p < q < y$)

10 Marks 4. (a) Let $A = \left\{-1, 2, \frac{1}{2}\right\}$. Prove by induction that if 1 is written as a product $1 = p_1 p_2 p_3 \ldots p_n$ where $p_i \in A$, then n is even.

(b) Prove or disprove that the same applies for $B = \left\{-1, \pm 2, \pm \frac{1}{2}\right\}$.

10 Marks 5. (a) Prove that for all $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{\frac{\pi}{2}+2\pi n} < \delta$ and likewise, there exists $m \in \mathbb{N}$ such that $\frac{1}{\frac{-\pi}{2}+2\pi m} < \delta$ (Hint: Archimedean property of the reals)

(b) Hence prove that the limit $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = L$ does not exist.