MATH 220 Practice Finals 4 Answers — Duration: 2.5 hours

*This test has **9 questions** on **19 pages**, for a total of 100 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:	/100								

10 Marks

1. Carefully define or restate each of the following:

(a) A relation on A

Solution: A subset of $\mathcal{P}(A \times A)$

(b) A function $f : A \mapsto B$

Solution: A subset $f \subseteq \mathcal{P}(A \times B)$ such that for all $a \in A$, there exists $b \in B$ such that $(a, b) \in f$ and for all $b_1, b_2 \in B$, if $(a, b_1) = (a, b_2)$ then $b_1 = b_2$

(c) The limit of a function $f : \mathbb{R} \mapsto \mathbb{R}$ as $x \to a$

Solution: $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$

(d) Euclidean division

Solution: For all $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, a can be written uniquely as $a = qb + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < b$

(e) The principle of mathematical induction

Solution: Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks 2. (a) Prove that if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Solution:

Proof. Let $A \subseteq B$. Then, let $C \in \mathcal{P}(A)$ so $C \subseteq A$. It follows $C \subseteq B$, so $C \in \mathcal{P}(B)$. $\square$

(b) Prove that if $A \cap B = \varnothing$, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$

Solution:

Proof. We prove the contrapositive. Assume $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\varnothing\}$. We know $\varnothing \in \mathcal{P}(A) \cap \mathcal{P}(B)$, and hence $\mathcal{P}(A) \cap \mathcal{P}(B) \nsubseteq \{\varnothing\}$. Thus, $\{\varnothing\} \subset \mathcal{P}(A) \cap \mathcal{P}(B)$ and there exists $C \neq \varnothing \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then, $C \subseteq A$ and $C \subseteq B$. Since $C \neq \varnothing$, there exists $c \in C$ so $c \in A \cap B$ and $A \cap B \neq \varnothing$. $\square$

10 Marks

3. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all primes p, there exists a function $\varphi : \mathbb{Z} \mapsto \mathbb{Z}$ that is not the identity map such that for all $n \in \mathbb{N}$, $p \mid n$ if and only if $p \mid \varphi(n)$.

Solution: The negation is "There exists a prime $p \in \mathbb{Z}$ such that for all functions $\varphi : \mathbb{Z} \mapsto \mathbb{Z}$ that are not the identity map, there exists $n \in \mathbb{N}$ such that, $p \mid n$ and $p \nmid \varphi(n)$, or, $p \nmid n$ and $p \mid \varphi(n)$". The statement is true.

Proof. Let p be a prime and consider $\varphi(n) = n^2$.

- For one direction, assume $p \mid n$. Then, $p \mid n^2$.
- For the other direction, assume $p \mid n^2$. By Euclid's lemma, $p \mid n$.

so $p \mid n$ if and only if $p \mid \varphi(n)$. □

(b) There exists a prime $p \in \mathbb{Z}$ such that there exists $r, k \in \mathbb{Z}$ such that $0 \leq r \leq p-1$ and $k < p$ and $p \mid kr$.

Solution: The negation is "For all primes $p \in \mathbb{Z}$, for all $r, k \in \mathbb{Z}$, $r < 0$ or $r > p-1$ or $k \geq p$ or $p \nmid kr$". The statement is false.

Proof. Let $p \in \mathbb{Z}$ be a prime and $r, k \in \mathbb{Z}$. If $r < 0$ or $r > p-1$ we are done. Hence, assume $0 \leq r \leq p-1$. Now, if $k \geq p$ we are done. Hence, assume $k < p$. Then, we know $p \nmid r$ and $p \nmid k$. Assume for the sake of contradiction that $p \mid kr$ so $kr = p\ell$ for some $\ell \in \mathbb{Z}$. If $p \mid k$, we have a contradiction. I claim $p \mid r$. Notice, by Bézout's lemma, we have that there exists $x, y \in \mathbb{Z}$ such that $px + ky = \gcd(p, k) = 1$. Then,

$$prx + kry = r$$
$$p(rx + \ell y) = r$$

so $p \mid r$, a contradiction. Hence, $p \nmid kr$. $\square$

10 Marks

4. Let A be a non-empty set and $f : A \mapsto A$ be a function. Prove or disprove each of the following:

   (a) Let $B \subseteq A$. Then, $f^{-1}(f(f^{-1}(B))) = f^{-1}(B)$

Solution:

Proof. Let $f : A \mapsto A$ be a function and $B \subseteq A$. We prove each inclusion in turn.

- We prove the contrapositive. Let $x \notin f^{-1}(B)$. Then, $f(x) \notin f(f^{-1}(B))$ so $x \notin f^{-1}(f(f^{-1}(B)))$.
- Let $x \in f^{-1}(B)$. Then, $f(x) \in f(f^{-1}(B))$ so $x \in f^{-1}(f(f^{-1}(B)))$.

This proves both inclusions so $f^{-1}(f(f^{-1}(B))) = f^{-1}(B)$ as required. $\square$

(b) Let $B \subseteq A$. Then, $f(f^{-1}(f(B))) = f(B)$

Solution:

Proof. Let $f : A \mapsto A$ be a function and $B \subseteq A$. We prove each inclusion in turn.

- Let $x \in f(f^{-1}(f(B)))$. Then, there exists $y \in f^{-1}(f(B))$ such that $x = f(y) \in f(f^{-1}(f(B)))$, but by definition of $y \in f^{-1}(f(B))$, $f(y) \in f(B)$. Hence, $x = f(y) \in f(f^{-1}(f(B)))$.
- Let $x \in f(B)$. Then, there exists $y \in f^{-1}(f(B))$ such that $f(y) = x$. However, notice $x = f(y) \in f(f^{-1}(f(B)))$ so $x \in f(f^{-1}(f(B)))$.

$\square$

(c) If f is injective or surjective, then f is bijective.

Solution:

Disproof. Let $A = \mathbb{Z}$ and $f(n) = 2n$. Then, f is injective but any odd number has an empty preimage so f is not surjective. $\square$

10 Marks

5. Prove or disprove each of the following:

(a) A countable union of countable sets is countable.

Solution:

Proof. Let $A_1, A_2, \ldots$ be countable sets and $\bigcup_{i=1}^{\infty} A_i$ be their union. Then, write $A_i = \{a_{i1}, a_{i2}, \ldots\}$ and list the entries of $A_1, A_2, \ldots$ in an array as follows:

a_{11}	a_{12}	a_{13}	a_{14}	$\ldots$
a_{21}	a_{22}	a_{23}	a_{24}	$\ldots$
a_{31}	a_{32}	a_{33}	a_{34}	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Count the entries diagonally as $a_{11}, a_{12}, a_{21}, a_{31} \ldots$ and skipping any repeated entries gives a sequence of elements in $\bigcup_{i=1}^{\infty} A_i$ and hence an $f : \mathbb{N} \mapsto \bigcup_{i=1}^{\infty} A_i$. But based on this counting, we know that for all $a \in \bigcup_{i=1}^{\infty} A_i$ that there exists $n \in \mathbb{N}$ such that $f(n) = a$ so f is surjective. Hence, $|\mathbb{N}| \geq |\bigcup_{i=1}^{\infty} A_i|$ so $\bigcup_{i=1}^{\infty} A_i$ is countable. $\square$

(b) A denumerable intersection of uncountable sets is countable.

Solution:

Disproof. Notice for all $n \in \mathbb{N}$, the identity map is an injection $\left[\frac{1}{n}, \infty\right)$ into $\mathbb{R}$ and $e^x + \frac{1}{n}$ is an injection $\mathbb{R}$ into $\left[\frac{1}{n}, \infty\right)$ so by the Cantor-Schröder-Bernstein Theorem, there is a bijection between the two sets and hence any interval of the form $\left[\frac{1}{n}, \infty\right)$ is uncountable. Consider the intersection $\bigcap_{i \in \mathbb{N}}^{\infty} \left[\frac{1}{n}, \infty\right)$, which is $[1, \infty)$. From before, this is uncountable. $\square$

10 Marks 6. Let $f : \mathbb{R} \mapsto \mathbb{R}\backslash\{0\}$. Show that if f is bijective then there exists $x, y \in \mathbb{R}$ such that $f(x+y) \neq f(x)f(y)$. (Hint: Use the fact that $-1 \cdot -1 = 1$)

Solution:

Proof. We prove the contrapositive. Let $f : \mathbb{R} \mapsto \mathbb{R}\backslash\{0\}$ and assume for all $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$. f is surjective or not.

- If f is not surjective, we are done.
- Assume f is surjective. Then, there exists $h \in \mathbb{R}$ such that $f(h) = -1$. Then, notice $-1 \cdot -1 = f(h)f(h) = f(2h) = 1$. Now, $h \neq 0$ since otherwise $1 = f(0) = -1$ which contradicts f being a function. Let $x \in \mathbb{R}$. Notice $f(0 + x) = f(0)f(x) = f(x)$ so $f(0) = \frac{f(x)}{f(x)} = 1$. Then, $f(h) = 1 = f(0)$ but $h \neq 0$, so f is not injective.

It follows that f is not bijective so this proves the contrapositive and thus the original statement. $\square$

10 Marks

7. Let $\mathbb{Q}(\sqrt{2}) = \left\{a + b\sqrt{2} | a, b \in \mathbb{Q}\right\}$

(a) Let $x, y \in \mathbb{Q}(\sqrt{2})$. Show that "$xRy$ if and only if $x - y \in \mathbb{Q}$" is an equivalence relation.

Solution:

Proof. Let everything be as stated.

- For reflexivity, $x - x = 0 \in \mathbb{Q}$.
- For symmetry, let $x, y \in \mathbb{Q}(\sqrt{2})$ and assume $x - y \in \mathbb{Q}$. Then, $x - y = \frac{p}{q}$ where $p \in \mathbb{Z}, q \in \mathbb{N}$ and $y - x = \frac{-p}{q}$ so $y - x \in \mathbb{Q}$.
- For transitivity, let $x, y, q \in \mathbb{Q}(\sqrt{2})$ and assume $x - y, y - z \in \mathbb{Q}$, so $x - y = \frac{p_1}{q_1}, y - z = \frac{p_2}{q_2}$ where $p_1, p_2 \in \mathbb{Z}, q_1, q_2 \in \mathbb{N}$. Then, $x - y + y - z = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{q_2 p_1 + q_1 p_2}{q_1 q_2} = x - z$ so $x - z \in \mathbb{Q}$.

It follows that R is an equivalence relation as required. $\square$

(b) Compute its equivalence classes. (Writing down $[x] = \left\{y \in \mathbb{Q}(\sqrt{2}) | x - y \in \mathbb{Q}\right\}$ is not enough, you must construct a set $A \subseteq \mathbb{Q}(\sqrt{2})$ and show that $y \in [x]$ if and only if $y \in A$.)

Solution: $[x] = \{q + x | q \in \mathbb{Q}\}$

Proof. We show each inclusion in turn.

- Let $y \in [x]$, so $y - x \in \mathbb{Q}$. Then, $y - x = q$ for some $q \in \mathbb{Q}$, so $y = x + q$. Hence, $[x] \subseteq \{q + x | q \in \mathbb{Q}\}$
- Let $y \in A$. Then, $y = q + x$ for some $q \in \mathbb{Q}$ so $y - x = q$ and thus $[x] \supseteq \{q + x | q \in \mathbb{Q}\}$.

This proves both inclusions so $[x] = \{q + x | q \in \mathbb{Q}\}$ $\square$

10 Marks

8. What is $\lim_{x\to 1} \frac{x-\sqrt{x}}{3621x-e^{621}}$? Prove your result.

Solution: The limit is 0.

Proof. Let $\varepsilon > 0$ and choose $\delta = \min\{1, \varepsilon\}$. Let $0 < |x-1| < \delta$ which implies $0 < x < 2$ and hence $3621x < e^{621} - 1$. Then,

$$\begin{aligned}
\left|\frac{x-\sqrt{x}}{3621x-e^{621}}\right| &= \left|\frac{x^2-x}{(3621x^{621}-e^{621})(x+\sqrt{x})}\right| \\
&= \left|\frac{x(x-1)}{(3621x-e^{621})(x+\sqrt{x})}\right| \\
&\leq \left|\frac{x(x-1)}{x+\sqrt{x}}\right| \\
&= \frac{|x||x-1|}{|x+\sqrt{x}|} \\
&\leq |x-1| < \delta \leq \varepsilon
\end{aligned}$$

so the limit is 0 as required. □

20 Marks 9. Let $I \subseteq \mathbb{Z}$ be a non-empty set with the property as follows:

- For all $x, y \in I$, $x + y \in I$
- For all $a \in \mathbb{Z}$ and for all $x \in I$, $ax \in I$.

(a) Show that $I = \{kx | x \in \mathbb{Z}\}$ for some $k \in \mathbb{Z}$. (Hint: If $I \neq \{0\}$, try showing that $I = \{kx | x \in \mathbb{Z}\}$ where k is the smallest positive integer in I using Euclidean division and showing that the remainder of any $y \in I$ must be 0)

Solution:

Proof. Let $I \subseteq \mathbb{Z}$ have the properties above. We prove by cases.

- If $I = \{0\}$, then $I = \{kx | x \in \mathbb{Z}\}$ for $k = 0$ so we are done.
- Hence, assume $I \neq \{0\}$. Let k be the smallest positive integer in I. I claim $I = \{kx | x \in \mathbb{Z}\}$.
  - Let $y \in I$. Then, by Euclidean division, $y = kq + r$ for some $q, r \in \mathbb{Z}$ where $0 \leq r < k$. Since k is the smallest positive integer in I, it follows that $r = 0$ so $y = kq, y \in \{kx | x \in \mathbb{Z}\}$.
  - Let $y \in \{kx | x \in \mathbb{Z}\}$, so $y = kq$ for some $q \in \mathbb{Z}$. Then, since $k \in I$, by the second property, $y = qk \in I$.

  This proves both inclusions so $I = \{kx | x \in \mathbb{Z}\}$.

It follows that for all I with the properties listed, $I = \{kx | x \in \mathbb{Z}\}$ for some $k \in \mathbb{Z}$ as required. □

(b) Let $I_i \subseteq \mathbb{Z}$ satisfy the following properties:

- For all $x, y \in I_i$, $x + y \in I_i$
- For all $a \in \mathbb{Z}$ and for all $x \in I_i$, $ax \in I_i$.

Consider $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ where every I_i satisfies the properties above, $i \in \mathbb{N} \cup \{0\}$. Show that there exists $N \in \mathbb{N}$ such that for all $n \geq N \in \mathbb{N} \cup \{0\}$, $I_N = I_n$. (Hint: Part (a) tells you that every I_i is just the set of all multiples of a certain integer)

Solution:

Proof. Consider $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ where every I_i satisfies the properties above, $i \in \mathbb{N} \cup \{0\}$. Assume for the sake of contradiction that for all $N \in \mathbb{N} \cup \{0\}$, there exists $n \geq N \in \mathbb{N} \cup \{0\}$ such that $I_N \neq I_n$. From Part (a), we know that $I_i = \{k_i x | x \in \mathbb{Z}\}$ for some $k_i \in \mathbb{Z}$. Then, $I_i \subseteq I_{i+1}$ implies $k_i = k_{i+1} q$ so $k_{i+1} \mid k_i$. Let $I_j = \{k_j x | x \in \mathbb{Z}\}$ for some $k_j \in \mathbb{Z}$. Then, by our assumption, since for all $N \in \mathbb{N} \cup \{0\}$, there exists $n \geq N \in \mathbb{N}$ such that $I_N \neq I_n$, it follows that there exists infinitely many distinct k such that $k \mid k_1$.

- If $k_j \neq 0$, by the fundamental theorem of arithmetic, k_1 is uniquely factored into a product of primes and possibly -1. There are a finite number of numbers between 0 and k_j and thus k_j can only have finitely many divisors, a contradiction.
- Hence, assume all $k_i = 0$. Again, we have a contradiction since for all $n \geq 1$, $I_1 = I_n = \{0\}$.

All cases yield a contradiction. Hence, it follows that there exists $N \in \mathbb{N} \cup \{0\}$ such that for all $n \geq N \cup \{0\}$, $I_N = I_n$ as required. $\square$

(c) Let $\mathbb{Z}[x]$ denote the set of polynomials with integer coefficients. Let $J \subseteq \mathbb{Z}[x]$ satisfy the following properties:

- For all $f(x), g(x) \in J$, $f(x) + g(x) \in J$
- For all $p(x) \in \mathbb{Z}[x]$ and for all $f(x) \in J$, $p(x)f(x) \in J$.

Let R_i be the set of leading coefficients of degree i polynomials in J (so all possible $k \in \mathbb{Z}$ such that there exists $f(x) = kx^i + a_{i-1}x^{i-1} + \ldots + a_1x + a_0 \in J$). Show that R_i satisfies the properties in part (a) and for all $i \in \mathbb{N} \cup \{0\}$, $R_i \subseteq R_{i+1}$. You may assume $0 \in R_i$ and $0 \in J$.

Solution:

Proof. Let R_i be as stated. Let $a, b \in R_i$, so there exists polynomials $f(x), g(x) \in J$ of degree i with a, b as leading coefficients. Then, $f(x) + g(x)$ is a polynomial of degree i with leading coefficient a, b, and by the first property listed in (c), we know $f(x) + g(x) \in J$. Hence, $a + b \in R_i$. Likewise, $kf(x)$ where $k \in \mathbb{Z}$ gives another polynomial in J by the second property so for all $k \in \mathbb{Z}$, $ka \in R_i$. Hence, R_i satisfies the properties listed in (a). Finally, observe $xf(x)$ is a polynomial of degree $i+1$ with a as the leading coefficient, and by the second property we have $xf(x) \in J$. Hence, $R_i \subseteq R_{i+1}$. $\square$

(d) From part (a) and (b) we know $R_i = \{k_i x | x \in \mathbb{Z}\}$ for some $k_i \in \mathbb{Z}$. From part (b) and (c), there exists N such that for all $n \geq N \in \mathbb{N} \cup \{0\}$, $R_N = R_n$. Let $p_i(x)$ and S and be defined as follows

- $p_i(x) = k_0 + k_1 x + \ldots + k_i x^i, 0 \leq i \leq N$
- $S = \{f_0(x)p_0(x) + f_1(x)p_1(x) + \ldots + f_N(x)p_N(x) | f_i(x) \in \mathbb{Z}[x]\}$

Show by strong induction on the degree of the polynomial that for all $f(x) \in J$, $f(x) \in S$. You may assume S is closed under linear combinations, that is if $f(x), g(x) \in S$ and $c(x), d(x) \in \mathbb{Z}[x]$, $c(x)f(x) + d(x)g(x) \in S$. (Hint: Proof by cases in inductive step)

Solution:

Proof. Let $f(x) \in J$. We proceed with induction on the degree of $f(x)$.

- For the base case, assume $f(x)$ has degree 0. Then, $f(x) = k$ for some $k \in \mathbb{Z}$ and by the definition of R_0, $k \in R_0$ and thus $k \in S$. Then, $f(x) = k \in S$ as required.
- For the inductive hypothesis, assume that for all polynomials $g(x)$ with degree less than that of $f(x)$, that $g(x) \in S$. We know $\deg f(x) \leq N$ or $\deg f(x) > N$ and thus we prove each case in turn.
  - If $\deg f(x) \leq N$, then the first term of the polynomial has some $k \in \mathbb{Z}$ as the coefficient and since $\deg f(x) \leq N$, we know $k \in R_i$ for some $i \leq N \in \mathbb{N} \cup \{0\}$ so $k = a_i k_i$ for some $a_i \in \mathbb{Z}$. Then, $f(x) - a_i p_i$ has degree less than $\deg f(x)$ and by the inductive hypothesis, this polynomial is in S. $a_i p_i \in S$ so $f(x) - a_i p_i + a_i p_i = f(x) \in S$ by closure under linear combinations.
  - If $\deg f(x) > N$, once again the first term of the polynomial has some $k \in \mathbb{Z}$. Since for all $n \geq N, R_n = R_N$, it follows that $k \in R_N$ so $k = a_N k_N$. Notice, $a_N p_N \in S$ and by definition of S, $a_N p_N \cdot x^{N - \deg f(x)} \in S$. Then, $f(x) - a_N p_N \cdot x^{N - \deg f(x)}$ gives a polynomial with a lower degree than $f(x)$ and by the inductive hypothesis, $f(x) - a_N p_N \cdot x^{N - \deg f(x)} \in S$ and thus by closure under linear combinations, $f(x) - a_N p_N \cdot x^{N - \deg f(x)} + a_N p_N \cdot x^{N - \deg f(x)} = f(x) \in S$.

By the principle of mathematical induction, it follows that for all polynomials $f(x) \in J$, $f(x) \in S$.

□