MATH 220 Practice Midterm 1 — Duration: 50 minutes
*This test has **5 questions** on **8 pages**, for a total of 50 points.*

Disclaimer: This test is definitely harder than actual 220 midterm exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 50 minutes but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:					/50

10 Marks 1. For each of the following statements, write down its negation and prove or disprove the statement.

(a) For all $x \in \mathbb{R}$, there exists $q \in \mathbb{Q}$ such that for all $r < q \in \mathbb{Q}$, $r + q < x$".

Solution: The negation is "There exists $x \in \mathbb{R}$ such that for all $q \in \mathbb{Q}$, there exists $r < q \in \mathbb{Q}$ such that $r + q \geq x$.

Proof. Let $x \in \mathbb{R}$. Pick $q = \frac{\lfloor x \rfloor}{2} - 1 \in \mathbb{Q}$. Notice for all $r < q \in \mathbb{Q}$, $r + q < 2q = \lfloor x \rfloor - 2 \leq x - 2 < x$ so the result holds. $\square$

(b) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $xy < z$ then $x < 0$ or $x^2 + y^2 < z$

Solution: The negation is "There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $xy < z$ and $x \geq 0$ and $x^2 + y^2 \geq z$".

Disproof. Choose $x = 2$ and let $y \in \mathbb{R}$. Choose $z = 4 + y^2$. Notice $x \geq 0$ and $z \leq 4 + y^2 = x^2 + y^2$. Now, notice $y^2 - 2y + 4 = (y-1)^2 + 2 \geq 2 > 0$, so

$$\begin{aligned} y^2 - 2y + 4 &> 0 \\ y^2 + 4 &> 2y \\ z = x^2 + y^2 &> xy \end{aligned}$$

and thus $xy < z$. This proves the negation and thus disproves the original statement. $\square$

10 Marks 2. Let $p \in \mathbb{N}$ and assume $p > 1$. Prove that if there exists $x \in \mathbb{Z}$ with $x \not\equiv 0 \mod p$ such that for all $y \in \mathbb{Z}$, $xy \not\equiv 1 \mod p$, then p is not prime.

Solution:

Proof. We prove the contrapositive. Let $p \in \mathbb{N}$ be prime. Let $x \in \mathbb{Z}$ and assume $x \not\equiv 0 \mod p$. Then, we know $x \equiv r \mod p$ for some $1 \leq r < p$. By Bézout's lemma, we know there exists $\alpha, \beta \in \mathbb{Z}$ such that $\alpha r + \beta p = \gcd(r, p) = 1$ so $\alpha r = 1 - \beta p$. It follows that $\alpha r \equiv 1 \mod p$ so choosing such $\alpha = y$ would yield $xy \equiv 1 \mod p$ as required. $\square$

10 Marks 3. Let $a, b \in \mathbb{Z}$ where $a, b \neq 0$ and $S = \{ax + by | x, y \in \mathbb{Z}, ax + by > 0\}$. You may not use Bézout's lemma for this section.

(a) Prove that S is non-empty.

Solution:

Proof. Notice $a \in S$ so S is non-empty. $\square$

(b) Prove that the minimal element $d = as + bt \in S$ for some $s, t \in \mathbb{Z}$ divides both a and b. (Hint: Euclidean division of a by d and b by d)

Solution:

Proof. Let S be as stated and let d be the minimal element of S. By Euclidean division, we know there exists $m, r \in \mathbb{Z}$ such that $a = md + r$, $0 \leq r < d$. Notice $r = a - md = a - mas - mbt = a(1 - ms) + b(-mt)$ so $r \in S \cup \{0\}$. By assumption, we have d is the minimal element in S, but $0 \leq r < d$ so $r = 0$. Likewise, there exists $n, r \in \mathbb{Z}$ such that $b = nd + r$, $0 \leq r < d$. It follows that $r = a(-ns) + b(1 - nt)$ so $r \in S \cup \{0\}$, and by d is the minimal element, we have $r = 0$. It follows that $a = md$ and $b = nd$ so $d \mid a$ and $d \mid b$. $\square$

(c) Prove that if $c \mid a$ and $c \mid b$, then $c \leq d$.

Solution:

Proof. Let S and d be as stated and assume $c \mid a$ and $c \mid b$, so $a = ck$ and $b = c\ell$ for some $k, \ell \in \mathbb{Z}$. Notice $d = as + bt = c(sk + t\ell)$ so $c \mid d$. By assumption, since $d \in S$, $d > 0$, so $c \leq d$ as required. $\square$

10 Marks

4. Let p be a prime. Prove by induction that for all $n \in \mathbb{Z}$, $p \mid n^p - n$. (Hint: You will need to split into different cases for induction, and use the binomial theorem)

Solution:

Proof. We proceed with a proof by cases.

- For $n \geq 0 \in \mathbb{N}$, we proceed with induction on n.
  - For the base case, it holds trivially when $n = 0$ since $p \mid 0^p - 0 = 0$.
  - Assume that the statement holds for $n = k$. Then, observe that

$$\begin{aligned}(k+1)^p - k - 1 &= k^p + 1 + \sum_{i=1}^{k} \binom{k}{i} k^{k-i} - k - 1 \\ &= k^p - k + ps \\ &= pr + ps \\ &= p(r+s)\end{aligned}$$

    so $p \mid (k+1)^p - k - 1$ and the inductive step holds.

  Since the base case and the inductive step hold, by the principle of mathematical induction, the result holds for all $n \geq 0$.

- For $n < 0$, we proceed with induction on $m = -n$, and we show $p \mid (-m)^p + m$ for all $m \in \mathbb{N}$.
  - For the base case, notice when $m = 1 = -n$, $p = 2$ or $p \neq 2$.
    * If $p = 2$, $p \mid (-1)^2 + 1 = 2$
    * If $p \neq 2$, $p \mid (-1)^p + 1 = 0$

    So the base case holds.
  - Assume that the statement holds for $m = k$. Then,

$$\begin{aligned}(-k-1)^p + k + 1 &= (-k)^p + (-1)^p + \sum_{i=1}^{k} \binom{k}{i} (-k)^{k-i} (-1)^k + k + 1 \\ &= (-k)^p + (-1)^p + k + 1 + ps \\ &= (-1)^p + 1 + pr + ps\end{aligned}$$

    Now $p = 2$ or $p \neq 2$
    * If $p = 2$, $(-1)^p + 1 + pr + ps = 2 + 2(r+s) = 2(1+r+s)$.
    * If $p \neq 2$, $(-1)^p + 1 + pr + ps = -1 + 1 + pr + ps = pr + ps = p(r+s)$.

    In all cases, $p \mid (-k-1)^p + k + 1$ so the inductive step holds.

Since the base case and the inductive step hold, by the principle of mathematical induction, the result holds for all $m > 0$ and hence all $-n > 0$, which is the same as all .

As such, it holds for all $n \in \mathbb{Z}$ as required. $\square$

10 Marks

5. Prove or disprove that the sequence $(x_n)_{n\in\mathbb{N}} = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \ldots (2n)}$ converges. (Hint: First prove an inequality between this sequence and $\frac{1}{\sqrt{3n+1}}$ holds for all $n \in \mathbb{N}$)

Solution:

Proof. We first prove that for all $n \in \mathbb{N}$, $\frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \ldots (2n)} \leq \frac{1}{\sqrt{3n+1}}$. We proceed with induction.

- For the base case, $\frac{1}{2} = \frac{1}{\sqrt{4}} = \frac{1}{\sqrt{3+1}}$ so the base case holds.
- Assume that the result holds for $n = k$. Then, for $k+1$, by assumption,

$$\begin{aligned}
\frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdot \ldots (2n)(2n+2)} &\leq \frac{(2n+1)}{\sqrt{3n-1}(2n+2)} \\
&= \sqrt{\frac{(2n+1)^2}{(3n+1)(2n+2)^2}} \\
&= \sqrt{\frac{(2n+2)^2}{12n^3+28n^2+20n+4}} \\
&\leq \sqrt{\frac{(2n+2)^2}{12n^3+28n^2+19n+4}} \\
&= \sqrt{\frac{1}{(3n+4)}}
\end{aligned}$$

  so the inductive step holds

Since the base case and inductive step hold, this proves that $n \in \mathbb{N}$, $\frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \ldots (2n)} \leq \frac{1}{\sqrt{3n+1}}$. Now, let $\varepsilon > 0$ and choose $N = \left\lceil \frac{1}{3\varepsilon^2} \right\rceil \in \mathbb{N}$. Notice, for all $n > N$,

$$\begin{aligned}
\left| \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \ldots (2n)} \right| &\leq \left| \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2N-1)}{2\cdot 4\cdot 6\cdot \ldots (2N)} \right| \\
&\leq \left| \frac{1}{\sqrt{3N+1}} \right| \\
&< \frac{1}{\sqrt{3N}} \\
&= \frac{1}{\sqrt{3\left\lceil \frac{1}{3\varepsilon^2} \right\rceil}} \\
&\leq \frac{1}{\sqrt{\frac{3}{3\varepsilon^2}}} = \varepsilon
\end{aligned}$$

So $(x_n)_{n\in\mathbb{N}}$ converges as required. □