MATH 220 Practice Midterm 2 — Duration: 50 minutes

*This test has **5 questions** on **X pages**, for a total of 50 points.*

Disclaimer: This test is definitely harder than actual 220 midterm exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 50 minutes but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/50				

10 Marks

1. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$, such that there exists $z \in \mathbb{Z}$, such that $xy > z$ implies $x + y < z$".

Solution: The negation is "There exists $x \in \mathbb{Z}$ such that for all $y \in \mathbb{Z}$, there exists $z \in \mathbb{Z}$ such that $xy > z$ and $z \leq x + y$.

Proof. Let $x \in \mathbb{Z}$. Choose $y = 0$ and $z = -1$. Notice the hypothesis $xy > z$ is false, so the result holds. $\square$

(b) For all $x \in \mathbb{N}$, for all $y \in \mathbb{N}$ such that $x < y$, there exists $a, b \in \mathbb{Z}$ such that $ax + by < \left|\frac{x}{y}\right|$

Solution: The negation is "There exists $x \in \mathbb{N}$ such that there exists $y \in \mathbb{N}$ such that $x < y$ such that for all $a, b \in \mathbb{Z}$, $ax + by \geq \left|\frac{x}{y}\right|$

Proof. Let $x, y \in \mathbb{N}$. Choose $a, b = -1$ and hence the result holds. $\square$

10 Marks 2. Let $n, k \in \mathbb{N}$. Prove that if for all $m \in \mathbb{Z}$, $m^k \neq n$, then $n^{1/k}$ is irrational.

Solution:

Proof. Let $n, k \in \mathbb{N}$. We prove the contrapositive. Assume $n^{1/k}$ is rational, so $n^{1/k} = \frac{a}{b}$ for some coprime $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Notice $n = \frac{a^k}{b^k}$. Since n is an integer, we know $b = 1$ so $n = a^k$ and thus the results follows. $\square$

10 Marks

3. The Archimedean property of the reals guarantees that for all $x, y \in \mathbb{R}$ where $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

(a) Let $x, y \in \mathbb{R}^+$. Prove that there exists $m \in \mathbb{N}$ such that $(m-1)x \leq y < mx$.

Solution:

Proof. Let $x, y \in \mathbb{R}^+$. By the Archimedean property of the reals, we know there exists $m \in \mathbb{N}$ such that $mx > y$. Now choose m to be the smallest possible m such that $mx > y$. It follows that $(m-1)x \leq y$ as otherwise we would have $(m-1)x = mx$ which is impossible since $x \neq 0$. Thus, $(m-1)x \leq y < mx$ as required. $\square$

(b) Using (a), prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $q \in \mathbb{Q}$ such that $x < q < y$. (Hint: Consider $y - x$ and also notice $1 \in \mathbb{R}^+$)

Solution:

Proof. Let $x, y \in \mathbb{R}^+$. By the Archimedean property of the reals, we know there exists $n \in \mathbb{N}$ such that $n(y-x) > 1$. From part (a), we also know there exists $m \in \mathbb{N}$ such that $(m-1) \leq nx < m$. Hence,

$$\begin{aligned} 1 &< n(y-x) \\ 1 + nx &< ny \\ 1 + (m-1) \leq 1 + nx &< ny \\ m &< ny \\ nx < m &< ny \\ x < \frac{m}{n} &< y \end{aligned}$$

Since $n \in \mathbb{N}$, it follows $\frac{m}{n} \in \mathbb{Q}$ so there exists $q \in \mathbb{Q}$ such that $x < q < y$ as required. $\square$

(c) Assume that $\sqrt{2}$ is irrational. Using $\sqrt{2}$, prove that for all $x, y \in \mathbb{R}^+$ such that $x < y$, there exists $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$. You may assume that irrational numbers added to and multiplied by rational numbers are still irrational. (Hint: From part (b), we have that there exists $p, q \in \mathbb{Q}$ such that $x < p < q < y$)

Solution:

Proof. Let $x, y \in \mathbb{R}^+$ and assume that $\sqrt{2}$ is irrational. Notice, $\frac{\sqrt{2}}{2}$ is also irrational and $\frac{\sqrt{2}}{2} < 1$. From part (b), we know there exists $p, q \in \mathbb{Q}$ such that $x < p < q < y$. Choose $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then, $q - p > 0$ since $q > p$, so $\alpha > p$. But also, since $\frac{\sqrt{2}}{2} < 1$, $\alpha < p + (q - p) = q$. Since α is irrational and $x < p < \alpha < y$, we have that for all x, y, there exists $z \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < z < y$ as required. $\square$

10 Marks 4. (a) Let $A = \left\{-1, 2, \frac{1}{2}\right\}$. Prove by induction that if 1 is written as a product $1 = p_1 p_2 p_3 \dots p_n$ where $p_i \in A$, then n is even.

Solution:

Proof. Let A be as stated. We proceed with mathematical induction on the number of $p_i \in A$ that are being multiplied together.

- For the base case, notice $1 \notin A$ so $1 \neq p_1$. The base case holds.
- Assume that if for all i such that $1 \leq i \leq k$, $1 = p_1 p_2 p_3 \dots p_k$ is only possible when k is even. For $k+1$, we proceed with a proof by cases.
  - Assume $p_{k+1} = \frac{1}{p_i}$ for some i such that $1 \leq i \leq k$. Then, $p_i p_{k+1} = 1$ and thus we are left with $p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_k$, which has a length of $k-1$ and thus by our assumption, $k-1$ is even or 1 cannot be written as a product $1 = p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_k$. In both cases, we have $k+1$ is even or 1 cannot be written as a product $1 = p_1 p_2 \dots p_{k+1}$
  - Assume $p_{k+1} \neq \frac{1}{p_i}$ for all i such that $1 \leq i \leq k$. We split this into another proof by cases.
    * If $p_{k+1} = -1$, then we know for all i such that $1 \leq i \leq k$, $p_i \neq -1$ so $p_1 p_2 \dots p_{k+1} < 0$ and hence cannot be 1.
    * If $p_{k+1} = 2$, then for all $1 \leq i \leq k$, $p_i = 2$ or -1, so $|p_1 p_2 \dots p_{k+1}| \geq 2$ and thus $p_1 p_2 \dots p_{k+1} \neq 1$.
    * If $p_{k+1} = \frac{1}{2}$, then for all $1 \leq i \leq k$, $p_i = \frac{1}{2}$ or -1, so $0 < |p_1 p_2 \dots p_{k+1}| \leq \frac{1}{2}$ and thus $p_1 p_2 \dots p_{k+1} \neq 1$.

    Since we have that this product is equal to 1 in none of the cases, our inductive step still holds.

Since both the base case and the inductive step hold, by the principle of mathematical induction, the result holds for all $n \in \mathbb{N}$. $\square$

(b) Prove or disprove that the same applies for $B = \left\{-1, \pm 2, \pm\frac{1}{2}\right\}$.

Solution:

Disproof. Let B be as stated. Notice $1 = -1 \cdot -2 \cdot \frac{1}{2}$ so $1 = p_1 p_2 p_3$ where $p_1, p_2, p_3 \in B$ but 3 is not even, so the same does not apply for B. $\square$

10 Marks 5. (a) Prove that for all $\delta > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{\frac{\pi}{2}+2\pi n} < \delta$ and likewise, there exists $m \in \mathbb{N}$ such that $\frac{1}{\frac{-\pi}{2}+2\pi m} < \delta$ (Hint: Archimedean property of the reals)

Solution:

Proof. Let $\delta > 0$. Notice $2\pi, \frac{1}{\delta} - \frac{\pi}{2} \in \mathbb{R}$ and by the Archimedean property of the reals, we know there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} 2\pi n &> \frac{1}{\delta} - \frac{\pi}{2} \\ \frac{\pi}{2} + 2\pi n &> \frac{1}{\delta} \end{aligned}$$

so $\frac{1}{\frac{\pi}{2}+2\pi n} < \delta$ as required. Similarly, notice $2\pi, \frac{1}{\delta}+\frac{\pi}{2} \in \mathbb{R}$ and by the Archimedean property of the reals, we know there exists $m \in \mathbb{N}$ such that

$$\begin{aligned} 2\pi m &> \frac{1}{\delta} + \frac{\pi}{2} \\ \frac{-\pi}{2} + 2\pi m &> \frac{1}{\delta} \end{aligned}$$

so $\frac{1}{\frac{-\pi}{2}+2\pi m} < \delta$ as required. $\square$

(b) Hence prove that the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = L$ does not exist.

Solution:

Proof. Let $L \in \mathbb{R}$. Notice $L = 1$ or $L \neq 1$.

- If $L = 1$, let $\varepsilon = 1$, $\delta > 0$, and choose $x = \frac{1}{\frac{-\pi}{2} + 2\pi n}$ such that $x < \delta$. Notice $x = |x|$ and $x > 0$ so we have $0 < |x| < \delta$. Observe that $\left|\sin\left(\frac{1}{x}\right) - 1\right| = \left|\sin\left(\frac{\pi}{2} + 2\pi n\right) - 1\right| = |-1-1| = 2 \geq \varepsilon$.
- If $L \neq 1$, let $\varepsilon = |1 - L|$, $\delta > 0$ and choose $x = \frac{1}{\frac{\pi}{2} + 2\pi n}$ such that $x < \delta$, and notice $x = |x|$ and $x > 0$ so we have $0 < |x| < \delta$. Observe that $\left|\sin\left(\frac{1}{x}\right) - L\right| = \left|\sin\left(\frac{\pi}{2} + 2\pi n\right) - L\right| = |1 - L| \geq \varepsilon$

It follows that the limit $\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = L$ does not exist. □