MATH 220 Practice Finals 3 Answers — Duration: 2.5 hours

This test has ***9 questions*** *on* ***X pages****, for a total of 90 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:	/90								

10 Marks 1. Carefully define or restate each of the following:

(a) A is as subset of B

Solution: For all $x \in A$, $x \in B$

(b) Bézout's lemma

Solution: Let $a, b \in \mathbb{Z}$. Then, there exists $x, y \in \mathbb{Z}$ such that $ax+by = \gcd(a, b)$.

(c) A partition on a set X

Solution: $P \subseteq \mathcal{P}(X)$ is a partition on X if

- For all $A \in P$, $A \neq \varnothing$.
- For all $x \in X$, there exists $A \in P$ such that $x \in P$.
- For all $A_1, A_2 \in P$, $A_1 = A_2$ or $A_1 \cap A_2 = \varnothing$.

(d) A composite number $n \in \mathbb{N}$

Solution: We say $n \in \mathbb{N}$ is composite if there exists distinct primes p, q such that $p \mid n$ and $q \mid n/$

(e) The principle of mathematical induction

Solution: Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks

2. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all $n \in \mathbb{N}$, for all $x \in \mathbb{Z}$, for all $y \in \mathbb{Z}$, there exists $k \in \mathbb{Z}$ such that $yk \equiv x \mod n$.

Solution: The negation is "There exists $n \in \mathbb{N}$ such that there exist $x \in \mathbb{N}$ such that there exists $y \in \mathbb{N}$ such that for all $k \in \mathbb{N}, yk \not\equiv x \mod n$". The original statement is false. .

Disproof. Let $n = 4$, $x = 1$, $y = 2$ and let $k \in \mathbb{N}$. Notice for all $k \in \mathbb{N}$, $2k \equiv 2 \mod 4$ or $2k \equiv 0 \mod 4$ so the original statement is false. □

(b) For all people $e \in D_e$ where D_e is the set of all people, there exists a function $f : D_e \mapsto D_e$ such that f maps e to the biological grandmothers of e.

Solution: The negation is "There exists a person $e \in D_e$ such that for all functions $f : D_e \mapsto D_e$, f does not map e to the biological grandmothers of e. The original statement is false.

Disproof. Notice for all $e \in D_e$, e has two biological grandmothers. It follows that if f maps e to their biological grandmothers, then $f(e) = e_1 = e_2$ for some $e_1 \neq e_2$, a contradiction. Hence, there does not exist such f. $\square$

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) If $g \circ f$ is bijective, then f is injective.

Solution 1:

Proof. Let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$ and thus $g \circ f(a_1) = g \circ f(a_2)$. Since $g \circ f$ is injective, it follows that $a_1 = a_2$. □

Solution 2:

Proof. We prove the contrapositive. Assume f is not injective, so there exists $a_1, a_2 \in A$ where $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. Then, $g \circ f(a_1) = g \circ f(a_2)$ so $g \circ f$ is not injective and hence not bijective. □

(b) If $g \circ f$ is bijective, then f is surjective.

Solution:

Disproof. Let $A = \{0\}$, $B = \{0, 1\}$ and $C = \{0\}$, let $f(x) = 0$ and $g(x) = 0$. Notice $g \circ f$ maps 0, the only element in A, to 0, the only element in C, so $g \circ f$ is bijective. However, $f^{-1}(\{1\}) = \varnothing$ so f is not surjective. $\square$

10 Marks 4. Let $A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{Q}$ and $S = \{c_1a_1 + c_2a_2 + \ldots + c_na_n | c_i \in \mathbb{Z}\}$. Prove by induction that if A has $n \in \mathbb{N}$ elements, then there exists $q \in \mathbb{Q}$ such that $S = \{cq | c \in \mathbb{Z}\}$. (Hint: Find q in your inductive step)

Solution:

Proof. Let A and S be as stated. We proceed with mathematical induction on the number of elements in A.

- If A has 1 element a, then, $S = \{ca | c \in \mathbb{Z}\}$ and since $a \in \mathbb{Q}$, the base case holds.
- Assume when $A = A_k$ has k, that there exists $q \in \mathbb{Q}$ such that $S = \{cq | c \in \mathbb{Z}\}$. Consider $A = A_{k+1}$ and partition A into $A \setminus \{a\}$ and $\{a\}$ for some $a \in A_{k+1}$. Notice S consists of all the ways multiples of a can add to multiples of elements in $A \setminus \{a\}$, which are precisely elements in $\{c_1a + c_2q | c_1, c_2 \in \mathbb{Z}\}$. Since $a, q \in \mathbb{Q}$, $a = \frac{x_1}{y_1}$ and $q = \frac{x_2}{y_2}$ where $y_1, y_2 \neq 0$. Let $\frac{\gcd(y_2x_1, y_1x_2)}{y_1y_2} = u$. We claim that $\{cu | c \in \mathbb{Z}\} = S$.
  - Let $p \in \{cu | c \in \mathbb{Z}\} = S$. By Bézout's lemma, we know there exists integers α, β such that

$$\begin{aligned} p = cu &= \frac{c(\alpha y_2 x_1 + \beta y_1 x_2)}{y_1 y_2} \\ &= \frac{c\alpha y_2 x_1 + c\beta y_1 x_2}{y_1 y_2} \\ &= \frac{c\alpha x_1}{y_1} + \frac{c\beta x_2}{y_2} \\ &= c\alpha(a) + c\beta(q) \end{aligned}$$

    so $p \in \{c_1a + c_2q | c_1, c_2 \in \mathbb{Z}\} = Se$, $\{cu | c \in \mathbb{Z}\} \subseteq S$.
  - Let $p \in S$ so $p = c_1a + c_2q$ for some $c_1, c_2 \in \mathbb{Z}$. Then,

$$\begin{aligned} p &= \frac{c_1x_1}{y_1} + \frac{c_2x_2}{y_2} \\ &= \frac{c_1y_2x_1 + c_2y_1x_2}{y_1y_2} \end{aligned}$$

    Notice $\gcd(y_2x_1, y_1x_2) \mid y_2x_1$ and $\gcd(y_2x_1, y_1x_2) \mid y_1x_2$, so

$$y_2x_1 = \alpha \gcd(y_2x_1, y_1x_2)$$

$$y_1x_2 = \beta \gcd(y_2x_1, y_1x_2)$$

where $\alpha, \beta \in \mathbb{Z}$ and thus

$$\begin{aligned} p &= \frac{c_1\alpha \gcd(y_2x_1, y_1x_2) + c_2\beta \gcd(y_2x_1, y_1x_2)}{y_1y_2} \\ &= (c_1\alpha + c_2\beta)u \end{aligned}$$

so $p \in \{cu | c \in \mathbb{Z}\}$, $\{cu | c \in \mathbb{Z}\} \supseteq S$.

So there exists $q \in \mathbb{Q}$ such that $S = \{cq | c \in \mathbb{Z}\}$ as required.

Since the base case and the inductive step hold, by the principle of mathematical induction, the result follows. $\square$

10 Marks

5. Let $\mathbb{R}[x]$ denote the set of polynomials with real coefficients and define a set $I = \{(x^2+1)p(x) | p(x) \in \mathbb{R}[x]\}$. Let $\sim$ be a relation on $\mathbb{R}[x]$ defined as $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I$.

   (a) Prove $\sim$ is an equivalence relation.

Solution:

Proof. We prove reflexivity, symmetry, and transitivity in turn.

- For reflexivity, let $f(x) \in \mathbb{R}[x]$. Then, $f(x) - f(x) = 0 = (x^2+1) \cdot 0 \in I$ so $f(x) \sim f(x)$.
- For symmetry, let $f(x), g(x) \in \mathbb{R}[x]$ and assume $f(x) \sim g(x)$. Then, $f(x) - g(x) = (x^2+1)p(x)$ for some $p(x) \in \mathbb{R}[x]$ so $g(x) - f(x) = (x^2+1)(-p(x)) \in I$ so $g(x) \sim f(x)$.
- For transitivity, let $f(x), g(x), h(x) \in \mathbb{R}[x]$ and assume $f(x) \sim g(x)$ and $g(x) \sim h(x)$. Then, $f(x) - h(x) = f(x) - g(x) + g(x) - h(x) = (x^2+1)p(x) + (x^2+1)q(x) = (x^2+1)(p(x)+q(x)) \in I$ for some $p(x), q(x) \in \mathbb{R}[x]$ so $f(x) \sim h(x)$.

Hence, $\sim$ is an equivalence relation as required. $\square$

(b) Let $f(x) \in \mathbb{R}[x]$ and $[f(x)]$ be its equivalence class under $\sim$. Prove that $[f(x)]$ must be of the form $\{a + bx + p(x)|p(x) \in I\}$ where $a, b \in \mathbb{R}$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

Solution:

Proof. Let $f(x) \in \mathbb{R}[x]$. By the polynomial division algorithm, $f(x) = q(x)(x^2 + 1) + r(x)$ for some $q(x), r(x) \in \mathbb{R}[x]$ and $r(x)$ has degree 0 or 1. By definition, we know $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I$ which implies $f(x) - g(x)$ has the same polynomial remainder $r(x)$, namely, $r(x) = a + bx$ for some $a, b \in \mathbb{R}$ since $r(x)$ has degree 0 or 1. Also notice if $p(x) \in I$, we have $f(x) \sim f(x)+p(x)$ so $r(x)+p(x) \in [f(x)]$ for all $p(x) \in I$ and thus $r(x) + p(x) \in I$ for all $p(x) \in I$. Hence, $[f(x)] = \{a + bx + p(x)|p(x) \in I\}$ as required. $\square$

(c) Prove that if $f(x), g(x) \in \mathbb{R}[x]$ and $f(x)g(x) \in I$, then $f(x) \in I$ or $g(x) \in I$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

Solution:

Proof. We prove the contrapositive. Assume $f(x) \notin I$ and $g(x) \notin I$, so by the polynomial division algorithm $f(x) = q_1(x)p(x) + r_1(x)$ and $g(x) = q_2(x)p(x) + r_2(x)$ where $r_1(x)$ and $r_2(x)$ are of degree 1. Then,

$$\begin{aligned} f(x)g(x) &= (q_1(x)p(x) + r_1(x))(q_2(x)p(x) + r_2(x)) \\ &= q_1(x)p(x)q_2(x)p(x) + r_1(x)q_2(x)p(x) + q_1(x)p(x)r_2(x) + r_1(x)r_2(x) \\ &= k(x)p(x) + (a_1 + b_1x)(a_2 + b_2x) \\ &= k(x)p(x) + a_1a_2 + a_2b_1x + a_1b_2x + b_1b_2x^2 \end{aligned}$$

where one of $a_1, b_1 \in \mathbb{R}$ is non-zero and one of $a_2, b_2 \in \mathbb{R}$ is non-zero.

- If both b_1, b_2 are zero, both a_1, a_2 are non-zero so we have $f(x)g(x) = k(x)p(x) + a_1a_2$. $f(x)g(x)$ cannot be factored by $x^2 + 1$ so $f(x)g(x) \notin I$ as required.
- If b_1, b_2 are both non-zero, then a_1, a_2 are zero so $f(x)g(x) = k(x)p(x) + b_1b_2x^2$ which cannot be factored by $p(x)$ so $f(x)g(x) \notin I$.
- If b_1, a_2 are non-zero but a_1, b_2 are zero, then $f(x)g(x) = k(x)p(x) + a_2b_1x$ so $x^2 + 1$ cannot factor $f(x)g(x)$ and $f(x)g(x) \notin I$. Without loss of generality, the same holds for b_1, a_2 are zero but a_1, b_2 are non-zero.

This proves all cases so $f(x)g(x) \notin I$ and so this proves the contrapositive. $\square$

10 Marks

6. Prove or disprove each of the following:

   (a) If A is an infinite set and $P \subseteq \mathcal{P}(A)$ is a finite partition of A, then for all $X \in P$, X is infinite.

Solution:

Disproof. Let $A = \mathbb{N} \cup \{0\}$ and $P = \{\{0\}, \mathbb{N}\}$. Observe that A is infinite and P is a finite partition, but $X = \{0\} \in P$ is finite, so the statement is false. $\square$

(b) If A is an infinite set and $P \subseteq \mathcal{P}(A)$ is an infinite partition of A, then for all $X \in P$, X is finite.

Solution:

Disproof. Let $A = \mathbb{N}$ and P be the partition where all composite numbers form their own equivalence classes and the set of primes is one equivalence class. Since there are infinite composite numbers, P is an infinite partition. However, there are also infinite primes so there exists $X \in P$ such that X is infinite. $\square$

$\boxed{\text{10 Marks}}$ 7. (a) Prove that if $A_1, A_2, A_3 \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$ are non-empty sets such that for all $i \leq n \in \mathbb{N}$, $|A_i| \leq |B_i|$, then $|\prod_{i=1}^n A_i| \leq |\prod_{i=1}^n B_i|$ by constructing an explicit injection $f : \prod_{i=1}^n A_i \mapsto \prod_{i=1}^n B_i$

Solution:

Proof. Since for all non-empty sets $A_1, A_2, A_3 \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$, we have $|A_i| \leq |B_i|$, we know that for all A_i there exists an injection $f_i : A_i \mapsto B_i$. Then, I claim $f(a_1, a_2, \ldots, a_n) = (f_1(a_1), f_2(a_2), \ldots, f_n(a_n))$ would be an injection. This is clear since if

$$\begin{aligned} f(a_1, a_2, \ldots, a_n) &= (f_1(a_1), f_2(a_2), \ldots, f_n(a_n)) \\ &= (f_1(a'_1), f_2(a'_2), \ldots, f_n(a'_n)) \\ &= f(a'_1, a'_2, \ldots, a'_n) \end{aligned}$$

Then by the injectivity of f_i, we have $a_i = a'_i$ for all $0 \leq i \leq n$, so $(a_1, a_2, \ldots, a_n) = (a'_1, a'_2, \ldots, a'_n)$. Hence, f is an injection. $\square$

(b) Prove or disprove that the result holds when there exists B_i such that B_i is empty.

Solution:

Proof. Assume for all $A_1, A_2, A_3 \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$, we have $|A_i| \leq |B_i|$, and there exists B_i such that B_i is empty. Then, since $|A_i| \leq |B_i|$, $A_i = \varnothing$. It follows that $\prod_{i=1}^{n} A_i = \varnothing = \prod_{i=1}^{n} B_i$ so $|\prod_{i=1}^{n} A_i| \leq |\prod_{i=1}^{n} B_i|$ as required, $\square$

$\boxed{\text{10 Marks}}$ 8. Let $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -2x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$

(a) Recall that $f(x)$ is continuous if for all $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = f(a)$. Prove that $f(x)$ is discontinuous. (Hint: Use density of rationals/irrationals in the reals)

Solution:

Proof. Let $a = 1$ and choose $\varepsilon = 1$. Then, let $\delta > 0$ and choose x to be any irrational number between 1 and $\delta + 1$ so $0 < |x - 1| < \delta$. Then,

$$\begin{aligned} |f(x) - f(1)| &= |f(x) - 2| \\ &= |-2x - 2| \\ &= |-1||2x + 2| \\ &= 2x + 2 \geq 1 = \varepsilon \end{aligned}$$

So f is discontinuous as required. □

(b) We say $f(x)$ is everywhere discontinuous if for all $a \in \mathbb{R}$ there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$. Prove or disprove that $f(x)$ everywhere discontinuous.

Solution: $f(x)$ is not everywhere discontinuous.

Disproof. I claim $f(x)$ is continuous at $a = 0$. Let $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$ and assume $0 < |x| < \delta$. Then, $|f(x) - f(0)| = \left|\frac{\pm 2\varepsilon}{4}\right| = \frac{\varepsilon}{2} < \varepsilon$ so $f(x)$ is continuous at $a = 0$ and hence not everywhere discontinuous. $\square$

10 Marks 9. Let $f : (0, 1) \mapsto \mathbb{R}$ be defined as $f(x) = \frac{2x-1}{x-x^2}$.

(a) Prove that f is injective.

Solution:

Proof. Let f be as stated and assume $f(x_1) = f(x_2)$. Then,

$$\begin{aligned}
\frac{2x_1 - 1}{x_1 - {x_1}^2} &= \frac{2x_2 - 1}{x_2 - {x_2}^2} \\
(2x_1 - 1)(x_2 - {x_2}^2) &= (2x_2 - 1)(x_1 - {x_1}^2) \\
2x_1(x_2 - {x_2}^2) - (x_2 - {x_2}^2) &= 2x_2(x_1 - {x_1}^2) - (x_1 - {x_1}^2) \\
2x_2{x_1}^2 - 2x_1{x_2}^2 + x_1 - {x_1}^2 - x_2 + {x_2}^2 &= 0 \\
2x_2{x_1}^2 - {x_1}^2 + x_1 - 2x_1{x_2}^2 + {x_2}^2 - x_2 &= 0 \\
(x_1 - x_2)(2x_1x_2 - x_1 + 1) &= 0
\end{aligned}$$

So $x_1 = x_2$ or $2x_1x_2 - x_1 + 1 = 0$. Assume for the sake of contradiction that $x_1 \neq x_2$. Then, $2x_1x_2 - x_1 + 1 = 0$ so

$$\begin{aligned}
2x_1x_2 - x_1 &= -1 \\
x_1(2x_2 - 1) &= -1 \\
2x_2 - 1 &= \frac{-1}{x_1}
\end{aligned}$$

Now $x_1 > 0$, so $2x_2 - 1 < 0$. Notice RHS < -1 since $0 < x_1 < 1$, but LHS > -1, so LHS $\neq$ RHS, a contradiction. Hence, $x_1 = x_2$ and f is injective as required. □

(b) Prove that f is surjective.

Solution:

Proof. Let $y \in \mathbb{R}$. Then, we solve for $y = \frac{2x-1}{x-x^2}$ where $x \in (0,1)$ so we solve $yx^2 + (y-2)x - 1 = 0$. Either $y = 0$ or $y \neq 0$.

- If $y = 0$, notice if $x = \frac{1}{2}$ then $f(x) = 0$.
- If $y \neq 0$, observe that by the quadratic formula,

$$\begin{aligned} x &= \frac{y - 2 \pm \sqrt{(y-2)^2 + 4y}}{2y} \\ &= \frac{y - 2 \pm \sqrt{y^2 + 4}}{2y} \end{aligned}$$

  Now to check $x \in (0,1)$, it suffices to check $0 < y - 2 + \sqrt{y^2+4} < 2y$. Now observe that $\sqrt{(2-y)^2} = \sqrt{y^2 - 4y + 4} < \sqrt{y^2+4} < \sqrt{y^2+4y+4} = \sqrt{(y+2)^2}$ so

$$\begin{aligned} y - 2 + \sqrt{(2-y)^2} < y - 2 + \sqrt{y^2+4} &< y - 2 + \sqrt{(y+2)^2} \\ 0 < y - 2 + \sqrt{y^2+4} &< y - 2 + y + 2 = 2y \end{aligned}$$

  So $x = \frac{y-2+\sqrt{y^2+4}}{2y}$ maps to y as required.

Hence, f is surjective. □

(c) Hence prove that $|(0,1)| = |\mathbb{R}|$.

Solution:

Proof. From above, f is injective and surjective so f is a bijection between $(0,1)$ and $\mathbb{R}$ so $|(0,1)| = |\mathbb{R}|$. □