MATH 220 Practice Finals 1 — Duration: 2.5 hours

*This test has **9 questions** on **20 pages**, for a total of 100 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:									/100

10 Marks

1. Carefully define or restate each of the following:

(a) The preimage of a set $U \subseteq B$ under the function $f : A \mapsto B$

(b) The power set of X

(c) An equivalence class of x under the equivalence relation $\sim$, $x \in X$

(d) $A = B$ where A, B are sets

(e) The principle of mathematical induction

10 Marks

2. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) There exists a subset E of the irrational numbers such that E is denumerable.

(b) Let $\{a_n\}$ be the sequence $a_1 = \sqrt{2}$, $a_{n+1} = (a_n)^{\sqrt{2}}$. Then, for all $n \in \mathbb{N}$, a_n is irrational.

$\boxed{\text{10 Marks}}$ 3. Let $p \in \mathbb{N}$. Show that if $2^p - 1$ is prime then p is prime. You may use the formula $a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^{n-1-i}b^i$

10 Marks 4. Let A be a non-empty set. Prove or disprove each of the following:

(a) For all bijective functions $f : A \mapsto A$ and $g : A \mapsto A$, $f \circ g = g \circ f$

(b) There exists a surjective but not injective function $f : A \mapsto A$ and an injective but not surjective function $g : A \mapsto A$ such that $f \circ g = g \circ f$

(c) Let $f : A \mapsto A$ and $g : A \mapsto A$ be bijective functions. Then, if $f \circ g = g \circ f$, $f^{-1} \circ g^{-1} = g^{-1} \circ f^{-1}$

10 Marks

5. Let $n \in \mathbb{N}$ be odd.

   (a) Prove that $n^2 \equiv 1 \mod 8$ by cases.

(b) Prove that $n^2 \equiv 1 \mod 8$ by induction.

15 Marks

6. Prove or disprove that each of the following induces an equivalence relation on $\mathbb{Z}$. If they are, find all equivalence classes with 1 element only or prove that such equivalence class does not exist.

   (a) $a \sim b$ if and only if $a \leq b$

(b) $a \sim b$ if and only if $a + b = 3621q$ for some $q \in \mathbb{Z}$

(c) $a \sim b$ if and only if $ax^2 + bx + 3621 = 0$ has a rational solution

(d) $a \sim b$ if and only if $a = \pm b$

$\boxed{\text{10 Marks}}$ 7. Let A, B be non-empty sets. Prove or disprove each of the following:

(a) If $A \subseteq B$, then $|A| \leq |B|$

(b) Let $f : A \mapsto B$ be a function and $U \subseteq B$. Then, $f(f^{-1}(U)) = U$.

(c) If $|A| = |\mathbb{R}|$ and $A \subseteq B$, then $|A| = |B|$.

15 Marks

8. Recall that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous if for all $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = f(a)$.

(a) Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying the following property:

- For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $0 < |x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$

Show that f is continuous.

(b) Give an example of a function $g : \mathbb{R} \mapsto \mathbb{R}$ that is continuous, but does not satisfy the property given in (a). Justify your answer.

10 Marks

9. Let $\mathbb{Q}[x]$ denote the set of polynomials with rational coefficients, $f(x) \in \mathbb{Q}[x]$. Prove that there exists $p \in \mathbb{R}$ such that for all $f(x) \in \mathbb{Q}[x]$, $f(p) \neq 0$. (Hint: A countable union of countable sets is countable)