MATH 220 Practice Finals 2 Answers — Duration: 2.5 hours

*This test has **10 questions** on **18 pages**, for a total of 100 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:	/100									

10 Marks

1. Carefully define or restate each of the following:

(a) A rational number $q \in \mathbb{Q}$

Solution: $q \in \mathbb{Q}$ if there exists coprime $a, b \in \mathbb{Z}$ where $b \neq 0$ such that $q = \frac{a}{b}$.

(b) Bézout's lemma

Solution: For all $a, b \in \mathbb{Z}$, there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$

(c) The Fundamental Theorem of Arithmetic

Solution: Let $n \in \mathbb{N}$. Then, n can be uniquely factorised into a product of prime powers ${p_1}^{e_1}{p_2}^{e_2}\dots{p_n}^{e_n}$ up to order, where p_i are distinct primes and $e_i \in \mathbb{Z}$.

(d) A convergent sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

Solution: $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$ converges to $L \in \mathbb{R}$ if for all $\varepsilon > 0 \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - L| < \varepsilon$.

(e) The principle of mathematical induction

Solution: Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks

2. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$.

Solution: The negation is "There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $x + y < z$ and $x - y \leq z$". The original statement is false.

Disproof. Let $x = 0$ and $y \in \mathbb{R}$. Choose $z = |y| + 1$. Notice $\pm y \leq |y| < 1 + |y|$, so $z > x + y$ and $z > x - y$ so the statement is false. $\square$

(b) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$.

Solution: The negation is "For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that there exists $z \in \mathbb{R}$ such that $xy \leq z$". The original statement is false.

Disproof. Let $x \in \mathbb{R}$ and choose $y = z = 0$. Then, $xy = 0 = z$ so the statement is false. $\square$

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) For all $U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

Solution:

Proof. Let everything be as stated. We show each inclusion in turn.

- Assume $x \notin f^{-1}(g^{-1}(U))$, so $f(x) \notin g^{-1}(U)$ and $g(f(x)) \notin U$. It follows that $x \notin (g \circ f)^{-1}(U)$, so by contraposition, $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$
- Assume $x \in f^{-1}(g^{-1}(U))$. Then, $f(x) \in g^{-1}(U)$ and $g(f(x)) \in U$ so $(g \circ f)^{-1}(U) \subseteq f^{-1}(g^{-1}(U))$.

It follows that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. as required. $\square$

(b) For all $U \subseteq B$, $(g \circ f)^{-1}(g(U)) = f^{-1}(U)$

Solution:

Disproof. Let $A = \{1\} = C$, $B = \{0, 1\}$, $f(x) = 1$ and $g(x) = 1$. Then, notice for $U = \{0\}$, $f^{-1}((U)) = \varnothing$, but $(g \circ f)^{-1} g(U) = (g \circ f)^{-1}(\{1\}) = \{1\}$ so $(g \circ f)^{-1}(g(U)) \neq f^{-1}((U))$ and thus the statement is false. $\square$

10 Marks

4. Let $n \in \mathbb{N}$ be even and $K = \{0, 1, 2, \ldots, n-1\}$. Let $S = \{k \in K | 2k \equiv 0 \mod n\}$. Prove that $|S|$ is even.

Solution:

Proof. Assume for the sake of contradiction that $|S|$ is not even, so for there exists an odd number of elements $k \in K$ such that $2k \equiv 0 \mod n$. Consider $K - S$, which has an odd number of elements. Let $a \in K - S$. Since $2a \not\equiv 0 \mod n$, it follows that there exists $a^{-1} \in K - S$ such that $a + a^{-1} \equiv 0 \mod n$, but there are only an odd number of a so there exists $a_0 \in K - S$ such that $a_0 = {a_0}^{-1}$ which implies $2a_0 \equiv 0 \mod n$, a contradiction. Hence, $|S|$ is even as required. $\square$

10 Marks 5. Let $\sim$ be a relation defined on $\mathbb{R}$ by "$a \sim b$ if and only if $a - b \in \mathbb{Z}$".

(a) Prove that $\sim$ is an equivalence relation.

Solution:

Proof. Let $\sim$ be as stated. We prove in turn that $\sim$ is reflexive, symmetric, and transitive.

- For reflexivity, let $a \in \mathbb{R}$. $a - a = 0 \in \mathbb{Z}$ so $\sim$ is reflexive.
- For symmetry, let $a, b \in \mathbb{R}$ and assume $a \sim b$ so $a - b \in \mathbb{Z}$. Then, $b - a = -(a - b) \in \mathbb{Z}$ so $\sim$ is symmetric.
- For transitivity, let $a, b, c \in \mathbb{R}$ and assume $a \sim b$ and $b \sim c$. Then, $a - b = k$ and $b - c = \ell$ for some $k, \ell \in \mathbb{Z}$. Observe that $a - c = a - b + b - c = k + \ell \in \mathbb{Z}$ so $\sim$ is transitive.

Hence, $\sim$ is an equivalence relation as required. $\square$

(b) Let $\mathbb{R}/\sim$ be the set of equivalence classes under $\sim$. Construct a bijection between $\mathbb{R}/\sim$ and $[0, 1)$.

Solution: Let $f : [0, 1) \mapsto \mathbb{R}/\sim$ be defined as $f(x) = [x]$ where $[x]$ is the equivalence class in $\mathbb{R}$.

Proof. We prove in turn that f is injective and surjective.

- For injectivity, assume $x, y \in [0, 1)$ and $x \neq y$. Observe $x, y \in \mathbb{R}$. Also notice $0 < |x - y| < 1$ so $x - y \notin \mathbb{Z}$. Hence, $x \not\sim y$ and $[x] \neq [y]$/
- For surjectivity, let $[x] \in \mathbb{R}$. It suffices to construct some $r \in [0, 1)$ such that $r \sim x$. Either $x \in \mathbb{Z}$ or not.
  - If $x \in \mathbb{Z}$, choose $r = 0$.
  - If not, then there exists $k \in \mathbb{Z}$ such that $k < x < k + 1$. Notice then $0 < x - k < 1$, and $x - x - k = k \in \mathbb{Z}$, so $x \sim x - k$. It follows that $f(x - k) = [x - k] = [x]$ so f is surjective.

Hence, f is bijective. $\square$

10 Marks

6. (a) Let X be a non-empty set. Prove that any equivalence relation on X forms a partition on X.

Solution:

Proof. Let $\sim$ be an equivalence relation on X. By reflexivity, we have that for all $x \in X$, x lies in some equivalence class and thus every equivalence class is non-empty. For showing that equivalence classes are disjoint, suppose $x \in [a]$ and $x \in [b]$, so $x \sim a$ and $x \sim b$. By symmetry and transitivity, we have $a \sim x$ and $x \sim b$ so $[a] = [b]$. Thus, equivalence classes are disjoint. It follows that the equivalence classes of an equivalence relation partition a set. $\square$

(b) Prove that any partition on X corresponds to equivalence classes of an equivalence relation on X.

Solution:

Proof. Let $P \subseteq \mathcal{P}(X)$ be a partition of X, and define a relation $x \sim y$ if and only if $x, y \in A_i$ where $A_i \in P$.

- Since every x lies in some $A_i \in P$ by the definition of a partition, $x \sim x$.
- If $x, y \in A_i$, then surely $y, x \in A_i$ so $x \sim y$.
- Similarly, if $x, y \in A_i$ and $y, z \in A_i$, it follows that $x, z \in A_i$ since every element lies in precisely one A_i.

Hence, $\sim$ is an equivalence relation and by our choice of equivalence relation, we have that every equivalence class corresponds precisely to some $A_i \in P$. $\square$

10 Marks 7. Let $\{x_n\}$ be a sequence and we define a subsequence of $\{x_n\}$ as a sequence obtained from removing terms in $\{x_n\}$.

(a) Construct an example where $\{x_n\}$ is a sequence and two subsequences of $\{x_n\}$ both converge but converge to different limits, and justify your answer.

Solution: Choose $\{x_n\}$ to be the sequence defined as $x_n = 1$ if n is odd and -1 if n is even. Consider the subsequences $\{x_n\}_{\text{odd}}$ and $\{x_n\}_{\text{even}}$. Obviously, both converge since every single term in the sequences are the same. $\{x_n\}_{\text{odd}}$ converges to 1 and $\{x_n\}_{\text{even}}$ converges to -1 so they converge to different limits.

(b) Prove that if a sequence $\{x_n\}$ converges to L then every subsequence of $\{x_n\}$ converges to L.

Solution:

Proof. Assume $\{x_n\}$ converges to L. Let $\{y_n\}$ be a subsequence of $\{x_n\}$ and $\varepsilon > 0$. Choose $N_y \in \mathbb{N}$ to be any N_y such that $N_y \geq N_x$ and $x_{N_y} \in \{y_n\}$ where N_x is the choice for N for $\{x_n\}$ by the convergence of $\{x_n\}$. Since $\{y_n\}$ cannot terminate before y_{N_y}, and for all $n > N_y$, we have $y_n = x_m$ where $m > N_y$ and so $m > N_y \geq N_x$, it follows that $|y_n - L| = |x_m - L| < \varepsilon$ so $\{y_n\}$ converges to L as required. $\square$

10 Marks

8. Prove or disprove each of the following:

   (a) Let $A, B \subseteq C$. If $|C \backslash A| = |C \backslash B|$, then $|A| = |B|$.

Solution:

Disproof. Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \mathbb{N}$. Notice there exists an explicit bijection $f : C \backslash A \mapsto C \backslash B$ where $f(n) = n + 1$, but $|A| = 1 \neq 2 = |B|$ so the statement is false. $\square$

(b) Let $A_i \in X$. Then, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$.

Solution:

Proof. Let $A_i \in X$ and $x \in X$. We show each inclusion in turn.

- Let $x \in X \backslash (\bigcup_{i \in I} A_i)$, so $x \notin (\bigcup_{i \in I} A_i)$ and thus $x \notin A_i$ for all $i \in I$. It follows that $x \in A_i \backslash X$ for all i and thus $x \in \bigcap_{i \in I} (X \backslash A_i)$.
- Let $x \in \bigcap_{i \in I} (X \backslash A_i)$ so for all $i \in I$, $x \notin A_i$. It follows that $x \notin \bigcup_{i \in I} A_i$ so $x \in X \backslash (\bigcup_{i \in I} A_i)$.

Hence, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$. □

10 Marks

9. Let $B_r(x) \subseteq \mathbb{R}$ be the open interval $(x - r, x + r)$ where $r > 0$.

(a) Let $p \in \mathbb{R}$ and $q \in B_r(p)$. Show that there exists $B_{r_2}(q) \subseteq B_r(p)$.

Solution:

Proof. Let $q \in B_r(p)$. Choose $r_2 = r - |p - q|$ and $q \in B_{r_2}(q)$. Now, $r > |p - q|$ so $r_2 > 0$. Let $x \in B_{r_2}(q)$. By the triangle inequality, $|p - x| \leq |p - q| + |q - x| < |p - q| + r_2 = r$ so $B_{r_2}(q) \subseteq B_r(p)$. $\square$

(b) A set $E \subseteq \mathbb{R}$ is open if for all $p \in E$, there exists $B_r(p) \subseteq E$. Prove that E is open if and only if E is a union of open intervals.

Solution:

Proof. We prove each direction in turn.

- For one direction, suppose E is open. I claim $E = \bigcup_{x \in E} B_r(\mathbf{x})$ where $B_r(x)$ is some open interval centered at x contained in E. Notice this yields $\bigcup_{x \in E} B_r(x) \subseteq E$. On the other hand, if $x \in E$, we know $x \in B_r(\mathbf{x}) \subseteq E$ for some $r > 0$ by E is open, so $x \in \bigcup_{x \in E} B_r(x)$. This proves $E = \bigcup_{x \in E} B_r(x)$ so E is a union of open balls.
- For the other direction, suppose E is a union of open balls $\{B_i\}$, and let $x \in E$. By definition, $x \in B_0$ for some $B_0 \in \{B_i\}$ and from part (a), we know there exists $B_r(x) \subseteq B_0 \subseteq E$ so E is open as required.

$\square$

10 Marks 10. Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let S_n be the set of bijective functions $f : N \mapsto N$. Prove by induction that for all $n \geq 3 \in \mathbb{N}$, there exists $f, g \in S_n$ such that $f \circ g \neq g \circ f$.

Solution:

Proof. Let $n \geq 3$. We proceed with mathematical induction on n.

- For the base case, let f and g in S_3 be defined as follows:

$$f(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}, g(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{cases}$$

  Notice $f \circ g(1) = 1$ and $g \circ f(1) = 3$ so $f \circ g \neq g \circ f$.

- Assume there exists $f, g \in S_k$ such that $f \circ g \neq g \circ f$ for some k. For S_{k+1}, choose such $f, g \in S_k$ and create f', g' which is the same as f and g except that the k+1th element gets mapped to itself. Notice there exists some $i \in \{1, 2, \ldots k\}$ such that $f' \circ g'(i) = f \circ g(i) \neq g \circ f(i) = g' \circ f'(i)$ so it follows that $f' \circ g' \neq g' \circ f'$ and thus the result holds.

By the principle of mathematical induction, the result holds for all $n \geq 3$. □