MATH 220 Practice Finals 1 Answers — Duration: 2.5 hours

*This test has **9 questions** on **20 pages**, for a total of 100 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:									/100

10 Marks

1. Carefully define or restate each of the following:

(a) The preimage of a set $U \subseteq B$ under the function $f : A \mapsto B$

Solution: $\{x \in A | f(x) \in U\}$

(b) The power set of X

Solution: $\{A | A \subseteq X\}$

(c) An equivalence class of x under the equivalence relation $\sim$, $x \in X$

Solution: $[x] = \{y \in X | x \sim y\}$

(d) $A = B$ where A, B are sets

Solution: $A \subseteq B$ and $B \subseteq A$

(e) The principle of mathematical induction

Solution: Let $\ell \in \mathbb{Z}$ and let $S = \{k \in \mathbb{Z} | n \geq \ell\}$. If $P(\ell)$ is true and $P(k)$ being true implies $P(k+1)$ being true for some $k \in S$, then $P(n)$ is true for all $n \in S$.

10 Marks

2. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) There exists a subset E of the irrational numbers such that E is denumerable.

Solution: The negation is "For all subsets E of the irrational numbers, E is finite or E is uncountable." The statement is true.

Proof. Take $A = \{n + \sqrt{2} | n \in \mathbb{N}\}$. $\sqrt{2}$ is irrational. Assume for the sake of contradiction that $n + \sqrt{2} \in A$ is rational. Then, $n + \sqrt{2} = \frac{p}{q}$ so $\sqrt{2} = \frac{p-qn}{q}$ which is rational, a contradiction. Hence, for all $x \in A$, x is irrational. Observer that $f : \mathbb{N} \mapsto A$ where $f(n) = n + \sqrt{2}$ is a bijective map with the inverse $f^{-1}(x) = x - \sqrt{2}$ so A is countably infinite. $\square$

(b) Let $\{a_n\}$ be the sequence $a_1 = \sqrt{2}$, $a_{n+1} = (a_n)^{\sqrt{2}}$. Then, for all $n \in \mathbb{N}$, a_n is irrational.

Solution: The negation is "Let $\{a_n\}$ be the sequence $a_1 = \sqrt{2}$, $a_{n+1} = (a_n)^{\sqrt{2}}$. Then, there exists $n \in \mathbb{N}$ such that a_n is rational" The statement is false.

Disproof. Let $\{a_n\}$ be as stated. Then, for $n = 3$, $(\sqrt{2})^{(\sqrt{2})^{\sqrt{2}}} = (\sqrt{2})^{(\sqrt{2}\cdot\sqrt{2})} = (\sqrt{2})^2 = 2 \in \mathbb{Q}$. $\square$

10 Marks

3. Let $p \in \mathbb{N}$. Show that if $2^p - 1$ is prime then p is prime. You may use the formula $x^n - y^n = (x-y)\sum_{i=0}^{n-1} x^{n-1-i}y^i$

Solution:

Proof. We prove the contrapositive. Let p be not prime. Then, $p = ab$ for some $a, b \in \mathbb{Z} \neq 1$ or $a, b = 1$. If $a, b = 1$ then we are done since $2 - 1 = 1$ which is not prime. Hence, assume $a \neq 1 \neq b$. Then,

$$\begin{aligned} 2^p - 1 = 2^{ab} - 1 &= (2^a - 1)\sum_{i=0}^{b-1}(2^a)^{b-1-i}1^i \\ &= (2^a - 1)\sum_{i=0}^{b-1} 2^{a(b-1-i)} \end{aligned}$$

Observe that $2^a - 1 \neq 1$ and $\sum_{i=0}^{b-1} 2^{a(b-1-i)} \geq 3$, so $2^p - 1$ is not prime as required. $\square$

10 Marks 4. Let A be a non-empty set. Prove or disprove each of the following:

(a) For all bijective functions $f : A \mapsto A$ and $g : A \mapsto A$, $f \circ g = g \circ f$

Solution:

Disproof. Let $A = \{1, 2, 3\}$, f, g be defined as follows:

$$f(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \\ 3 \mapsto 3 \end{cases}, g(n) = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 1 \end{cases}$$

Notice $f \circ g(1) = 1$ and $g \circ f(1) = 3$ so $f \circ g \neq g \circ f$. $\square$

(b) There exists a surjective but not injective function $f : A \mapsto A$ and an injective but not surjective function $g : A \mapsto A$ such that $f \circ g = g \circ f$

Solution:

Proof. Let $A = \mathbb{Z}$, $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ x+1 & \text{otherwise} \end{cases}$ and $g(x) = 1$. Then, observe that $f \circ g(x) = f(1) = 1$ and $g \circ f(x) = g(f(x)) = 1$ so $f \circ g = g \circ f$. $\square$

(c) Let $f : A \mapsto A$ and $g : A \mapsto A$ be bijective functions. Then, if $f \circ g = g \circ f$, $f^{-1} \circ g^{-1} = g^{-1} \circ f^{-1}$

Solution:

Proof. Assune f, g are bijective and $f \circ g = g \circ f$. Then, notice $(f \circ g)^{-1} = (g \circ f)^{-1}$ so $g^{-1} \circ f^{-1} = f^{-1} \circ g^{-1}$. □

10 Marks 5. Let $n \in \mathbb{N}$ be odd.

(a) Prove that $n^2 \equiv 1 \mod 8$ by cases.

Solution:

Proof. Let $n \in \mathbb{N}$ be odd, so $n = 2k+1$ for some $k \in \mathbb{N}$ and $n^2 = 4k^2 + 4k + 1$. k is odd or even.

- If k is odd, then $k = 2\ell + 1$ for some $\ell \in \mathbb{N}$, so

$$\begin{aligned} n^2 = 4k^2 + 4k + 1 &= 4(4\ell^2 + 4\ell + 1) + 4\ell + 4 + 1 \\ &= 16\ell^2 + 8\ell + 8 + 1 \\ &= 8(2\ell^2 + \ell + 1) + 1 \end{aligned}$$

so $n^2 \equiv 1 \mod 8$.

- If k is even, then $k = 2\ell$ for some $\ell \in \mathbb{N}$, so

$$\begin{aligned} n^2 = 4k^2 + 4k + 1 &= 4(4\ell^2) + 8\ell + 1 \\ &= 16\ell^2 + 8\ell + 1 \\ &= 8(2\ell^2 + \ell) + 1 \end{aligned}$$

so $n^2 \equiv 1 \mod 8$.

In both cases, $n^2 \equiv 1 \mod 8$ so the result follows. □

(b) Prove that $n^2 \equiv 1 \mod 8$ by induction.

Solution:

Proof. Let $n = 2k + 1$ for some $k \in \mathbb{N} \cup \{0\}$. We proceed with mathematical induction on k.

- For the base case, let $k = 0$. Trivially, $n^2 = 1 \equiv 1 \mod 8$.
- Assume that $(2\ell + 1)^2 \equiv 1 \mod 8$. Then, for $\ell + 1$,

$$\begin{aligned}(2\ell+3)^2 &= 4\ell^2 + 12\ell + 9 \\ &= (2\ell+1)^2 + 8\ell + 8 \\ &\equiv (2\ell+1)^2 \mod 8 \qquad \equiv 1 \mod 8\end{aligned}$$

  so $n^2 \equiv 1 \mod 8$

By the principle of mathematical induction, $n^2 \equiv 1 \mod 8$ for all $k \in \mathbb{N} \cup \{0\}$ so $n^2 \equiv 1 \mod 8$ for all odd n as required. $\square$

15 Marks

6. Prove or disprove that each of the following induces an equivalence relation on $\mathbb{Z}$. If they are, find all equivalence classes with 1 element only or prove that such equivalence class does not exist.

(a) $a \sim b$ if and only if $a \leq b$

Solution:

Disproof. $\sim$ is not symmetric. Notice if $a < b$, then $b > a$ so $b \nsim a$. $\square$

(b) $a \sim b$ if and only if $a + b = 3621q$ for some $q \in \mathbb{Z}$

Solution:

Disproof. $\sim$ is not transitive. Notice $1 \sim 3620$ and $3620 \sim 3622$, but $1 + 3622 = 3623$ so $1 \not\sim 3622$. $\square$

(c) $a \sim b$ if and only if $ax^2 + bx + 3621 = 0$ has a rational solution

Solution:

Disproof. $\sim$ is not reflexive. I claim $x^2+x+3621$ has no rational solutions. Assume for the sake of contradiction that y is a rational solution, so $y = \frac{p}{q}$ for some $p \in \mathbb{Z}, q \in \mathbb{N}$ such that p, q are coprime. Then, $p^2+pq+3621q^2 = 0$. Under mod 2, clearly p, q are not both even. If one of p, q is odd, then pq is even and one of $p^2, 3621q^2$ is even and the other is odd, so $p^2 + pq + 3621q^2 \equiv 1 \mod 2$, a contradiction. Likewise, if both p, q are odd, then $p^2, pq, 3621q^2$ are all odd so $p^2 + pq + 3621q^2 \equiv 1 \mod 2$, a contradiction. Hence, $x^2+x+3621$ has no rational solutions. It follows that $1 \not\sim 1$ so $\sim$ is not reflexive. □

(d) $a \sim b$ if and only if $a = \pm b$

Solution:

Proof. We prove that it is reflexive, symmetric, and transitive.

- For reflexivity, it is trivial.
- For symmetry, let $a, b \in \mathbb{Z}$ and assume $a \sim b$. $a = \pm b$ so $b = \pm a$.
- For transitivity,let $a, b, c \in \mathbb{Z}$ and assume $a \sim b$ and $b \sim c$. Then, $a = \pm b$ and $b = \pm c$ so $a = \pm c$.

The only equivalence class with 1 element is $[0]$ since 0 is the only element in $\mathbb{Z}$ such that $a = -a$. $\square$

10 Marks 7. Let A, B be non-empty sets. Prove or disprove each of the following:

(a) If $A \subseteq B$, then $|A| \leq |B|$

Solution:

Proof. Let $f : A \mapsto B$ be $f(x) = x$. Then, f is trivially an injection $A \mapsto B$ so $|A| \leq |B|$ as required. $\square$

(b) Let $f : A \mapsto B$ be a function and $U \subseteq B$. Then, $f(f^{-1}(U)) = U$.

Solution:

Disproof. Let $A = \{0\}, B = \{0,1\}, U = 1$ and $f : A \mapsto B$ be $f(x) = 0$. Then, $f^{-1}(U) = \varnothing$ so $f(f^{-1}(U)) = f(\varnothing) = \varnothing \neq U$. $\square$

(c) If $|A| = |\mathbb{R}|$ and $A \subseteq B$, then $|A| = |B|$.

Solution:

Disproof. Let $A = \{\{x\} \,|\, x \in \mathbb{R}\}$ and $B = \mathcal{P}(\mathbb{R})$. Notice $f : A \mapsto \mathbb{R}$ defined by $\{x\} \mapsto x$ is a bijection, so $|A| = |\mathbb{R}|$, and $A \subseteq B$. However, by Cantor's theorem, $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$ so $|A| < |B|$. $\square$

15 Marks

8. Recall that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous if for all $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = f(a)$.

   (a) Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is a function satisfying the following property:

   - For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$, $0 < |x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$

   Show that f is continuous.

Solution:

Proof. Assume $f : \mathbb{R} \mapsto \mathbb{R}$ satisfies the property above. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then, let δ be the same δ such that for all $x, y \in \mathbb{R}$, $0 < |x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$. Now assume $0 < |x - a| < \delta$. By the property, $|f(x) - f(a)| < \varepsilon$ so it follows that $\lim_{x\to a} f(x) = f(a)$. Hence, f is continuous. $\square$

(b) Give an example of a function $g : \mathbb{R} \mapsto \mathbb{R}$ that is continuous, but does not satisfy the property given in (a). Justify your answer.

Solution: An example would be $g(x) = x^2$.

Proof. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be defined as $g(x) = x^2$. We first show g is continuous. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$. Assume $0 < |x - a| < \delta$. Then,

$$\begin{aligned}
|x^2 - a^2| &= |x - a||x + a| \\
&< \delta|x + a| \\
&\leq \delta|x| + \delta|a| \\
&= \delta|x - a + a| + \delta|a| \\
&\leq \delta|x - a| + 2\delta|a| \\
&= \delta(\delta + 2|a|) \\
&< \frac{\varepsilon}{1 + 2|a|}(1 + 2|a|) = \varepsilon
\end{aligned}$$

This proves continuity of g. Now we show that g does not satisfy the property. Let $\varepsilon = 1$, $\delta > 0$, and choose $x = \frac{1}{\delta}, y = x + \frac{\delta}{2}$. Now,

$$\begin{aligned}
|x^2 - y^2| &= \left|x^2 - x^2 - \delta x - \frac{\delta^2}{4}\right| \\
&= \left|\delta x + \frac{\delta^2}{4}\right| \\
&= \left|1 + \frac{\delta^2}{4}\right| = 1 + \frac{\delta^2}{4} > \varepsilon
\end{aligned}$$

so it follows g does not satisfy the property. □

10 Marks

9. Let $\mathbb{Q}[x]$ denote the set of polynomials with rational coefficients, $f(x) \in \mathbb{Q}[x]$. Prove that there exists $p \in \mathbb{R}$ such that for all $f(x) \in \mathbb{Q}[x]$, $f(p) \neq 0$. (Hint: A countable union of countable sets is countable)

Solution:

Proof. Let $\mathbb{A} = \{a \in \mathbb{Q} | \exists f(x) \in \mathbb{Q}[x] \text{ s.t. } f(a) = 0\}$. We prove that $\mathbb{A}$ is countable. Now, let $k \geq 0 \in \mathbb{Z}$. For each distinct degree k polynomial in $\mathbb{Q}[x]$, construct a set S of solutions for the polynomial, and note that there are only countably many distinct polynomials of degree k with rational coefficients for all $k \geq 0 \in \mathbb{Z}$. Let A_k be union of all such sets S for distinct degree k polynomials in $\mathbb{Q}[x]$. Then, by the fundamental theorem of algebra, there are at most k solutions to $f(x) \in A_k$, so every S is finite. It follows that A_k is a countable union of finite sets, so A_k is countable. Notice $\mathbb{A} = \bigcup_{k \geq 0 \in \mathbb{Z}} A_k$ so $\mathbb{A}$ is a countable union of countable sets and hence $\mathbb{A}$ is countable. However, $\mathbb{A} \subseteq \mathbb{R}$ and $\mathbb{R}$ is uncountable so there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{A}$, which is equivalent to saying that there exists $p \in \mathbb{R}$ such that for all $f(x) \in \mathbb{Q}[x]$, $f(p) \neq 0$, so the result holds. $\square$