MATH 220 Practice Finals 4 — Duration: 2.5 hours

*This test has **9 questions** on **19 pages**, for a total of 100 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:	/100								

10 Marks

1. Carefully define or restate each of the following:

(a) A relation on A

(b) A function $f : A \mapsto B$

(c) The limit of a function $f : \mathbb{R} \mapsto \mathbb{R}$ as $x \to a$

(d) Euclidean division

(e) The principle of mathematical induction

10 Marks

2. (a) Prove that if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

(b) Prove that if $A \cap B = \varnothing$, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$

10 Marks

3. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all primes p, there exists a function $\varphi : \mathbb{Z} \mapsto \mathbb{Z}$ that is not the identity map such that for all $n \in \mathbb{N}$, $p \mid n$ if and only if $p \mid \varphi(n)$.

(b) There exists a prime $p \in \mathbb{Z}$ such that there exists $r, k \in \mathbb{Z}$ such that $0 \leq r \leq p-1$ and $k < p$ and $p \mid kr$.

10 Marks

4. Let A be a non-empty set and $f : A \mapsto A$ be a function. Prove or disprove each of the following:

   (a) Let $B \subseteq A$. Then, $f^{-1}(f(f^{-1}(B))) = f^{-1}(B)$

(b) Let $B \subseteq A$. Then, $f(f^{-1}(f(B))) = f(B)$

(c) If f is injective or surjective, then f is bijective.

$\boxed{\text{10 Marks}}$ 5. Prove or disprove each of the following:

(a) A countable union of countable sets is countable.

(b) A denumerable intersection of uncountable sets is countable.

10 Marks

6. Let $f : \mathbb{R} \mapsto \mathbb{R}\backslash\{0\}$. Show that if f is bijective then there exists $x, y \in \mathbb{R}$ such that $f(x+y) \neq f(x)f(y)$. (Hint: Use the fact that $-1 \cdot -1 = 1$)

$\boxed{\text{10 Marks}}$ 7. Let $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$

(a) Let $x, y \in \mathbb{Q}(\sqrt{2})$. Show that "$xRy$ if and only if $x - y \in \mathbb{Q}$" is an equivalence relation.

(b) Compute its equivalence classes. (Writing down $[x] = \{y \in \mathbb{Q}(\sqrt{2}) | x - y \in \mathbb{Q}\}$ is not enough, you must construct a set $A \subseteq \mathbb{Q}(\sqrt{2})$ and show that $y \in [x]$ if and only if $y \in A$.)

$\boxed{\text{10 Marks}}$ 8. What is $\lim_{x\to 1} \frac{x-\sqrt{x}}{3621x-e^{621}}$? Prove your result.

20 Marks 9. Let $I \subseteq \mathbb{Z}$ be a non-empty set with the property as follows:

- For all $x, y \in I$, $x + y \in I$
- For all $a \in \mathbb{Z}$ and for all $x \in I$, $ax \in I$.

(a) Show that $I = \{kx | x \in \mathbb{Z}\}$ for some $k \in \mathbb{Z}$. (Hint: If $I \neq \{0\}$, try showing that $I = \{kx | x \in \mathbb{Z}\}$ where k is the smallest positive integer in I using Euclidean division and showing that the remainder of any $y \in I$ must be 0)

(b) Let $I_i \subseteq \mathbb{Z}$ satisfy the following properties:

- For all $x, y \in I_i$, $x + y \in I_i$
- For all $a \in \mathbb{Z}$ and for all $x \in I_i$, $ax \in I_i$.

Consider $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ where every I_i satisfies the properties above, $i \in \mathbb{N} \cup \{0\}$. Show that there exists $N \in \mathbb{N}$ such that for all $n \geq N \in \mathbb{N} \cup \{0\}$, $I_N = I_n$. (Hint: Part (a) tells you that every I_i is just the set of all multiples of a certain integer)

(c) Let $\mathbb{Z}[x]$ denote the set of polynomials with integer coefficients. Let $J \subseteq \mathbb{Z}[x]$ satisfy the following properties:

- For all $f(x), g(x) \in J$, $f(x) + g(x) \in J$
- For all $p(x) \in \mathbb{Z}[x]$ and for all $f(x) \in J$, $p(x)f(x) \in J$.

Let R_i be the set of leading coefficients of degree i polynomials in J (so all possible $k \in \mathbb{Z}$ such that there exists $f(x) = kx^i + a_{i-1}x^{i-1} + \ldots + a_1x + a_0 \in J$). Show that R_i satisfies the properties in part (a) and for all $i \in \mathbb{N} \cup \{0\}$, $R_i \subseteq R_{i+1}$. You may assume $0 \in R_i$ and $0 \in J$.

(d) From part (a) and (b) we know $R_i = \{k_i x | x \in \mathbb{Z}\}$ for some $k_i \in \mathbb{Z}$. From part (b) and (c), there exists N such that for all $n \geq N \in \mathbb{N} \cup \{0\}$, $R_N = R_n$. Let $p_i(x)$ and S and be defined as follows

- $p_i(x) = k_0 + k_1 x + \ldots + k_i x^i, 0 \leq i \leq N$
- $S = \{f_0(x)p_0(x) + f_1(x)p_1(x) + \ldots + f_N(x)p_N(x) | f_i(x) \in \mathbb{Z}[x]\}$

Show by strong induction on the degree of the polynomial that for all $f(x) \in J$, $f(x) \in S$. You may assume S is closed under linear combinations, that is if $f(x), g(x) \in S$ and $c(x), d(x) \in \mathbb{Z}[x]$, $c(x)f(x) + d(x)g(x) \in S$. (Hint: Proof by cases in inductive step)