MATH 220 Practice Finals 3 — Duration: 2.5 hours

*This test has **9 questions** on **18 pages**, for a total of 90 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9
Points:									
Total:									/90

10 Marks 1. Carefully define or restate each of the following:

(a) A is as subset of B

(b) Bézout's lemma

(c) A partition on a set X

(d) A composite number $n \in \mathbb{N}$

(e) The principle of mathematical induction

$\boxed{\text{10 Marks}}$ 2. For each of the following statements, write down its negation and prove or disprove the statement.

(a) For all $n \in \mathbb{N}$, for all $x \in \mathbb{N}$, for all $y \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $yk \equiv x \mod n$.

(b) For all people $e \in D_e$ where D_e is the set of all people, there exists a function $f : D_e \mapsto D_e$ such that f maps e to the biological grandmothers of e.

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) If $g \circ f$ is bijective, then f is injective.

(b) If $g \circ f$ is bijective, then f is surjective.

10 Marks

4. Let $A = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{Q}$ and $S = \{c_1a_1 + c_2a_2 + \ldots + c_na_n | c_i \in \mathbb{Z}\}$. Prove by induction that if A has $n \in \mathbb{N}$ elements, then there exists $q \in \mathbb{Q}$ such that $S = \{cq | c \in \mathbb{Z}\}$. (Hint: Find q in your inductive step)

10 Marks

5. Let $\mathbb{R}[x]$ denote the set of polynomials with real coefficients and define a set $I = \{(x^2+1)p(x) | p(x) \in \mathbb{R}[x]\}$. Let $\sim$ be a relation on $\mathbb{R}[x]$ defined as $f(x) \sim g(x)$ if and only if $f(x) - g(x) \in I$.

   (a) Prove $\sim$ is an equivalence relation.

(b) Let $f(x) \in \mathbb{R}[x]$ and $[f(x)]$ be its equivalence class under $\sim$. Prove that $[f(x)]$ must be of the form $\{a + bx + p(x) | p(x) \in I\}$ where $a, b \in \mathbb{R}$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

(c) Prove that if $f(x), g(x) \in \mathbb{R}[x]$ and $f(x)g(x) \in I$, then $f(x) \in I$ or $g(x) \in I$. (Hint: The polynomial division algorithm states that for all $f(x), g(x) \in \mathbb{R}[x]$ where $\deg g(x) = k$, $f(x) = q(x)g(x) + r(x)$ with $q(x), r(x) \in \mathbb{R}[x]$ such that $0 \leq \deg r(x) < k$)

$\boxed{\text{10 Marks}}$ 6. Prove or disprove each of the following:

(a) If A is an infinite set and $P \subseteq \mathcal{P}(A)$ is a finite partition of A, then for all $X \in P$, X is infinite.

(b) If A is an infinite set and $P \subseteq \mathcal{P}(A)$ is an infinite partition of A, then for all $X \in P$, X is finite.

10 Marks

7. (a) Prove that if $A_1, A_2, A_3 \ldots A_n$ and $B_1, B_2, B_3, \ldots B_n$ are non-empty sets such that for all $i \leq n \in \mathbb{N}$, $|A_i| \leq |B_i|$, then $|\prod_{i=1}^{n} A_i| \leq |\prod_{i=1}^{n} B_i|$ by constructing an explicit injection $f : \prod_{i=1}^{n} A_i \mapsto \prod_{i=1}^{n} B_i$

(b) Prove or disprove that the result holds when there exists B_i such that B_i is empty.

10 Marks

8. Let $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q} \\ -2x & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$

(a) Recall that $f(x)$ is continuous if for all $a \in \mathbb{R}$, $\lim_{x \to a} f(x) = f(a)$. Prove that $f(x)$ is discontinuous. (Hint: Use density of rationals/irrationals in the reals)

(b) We say $f(x)$ is everywhere discontinuous if for all $a \in \mathbb{R}$ there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$. Prove or disprove that $f(x)$ everywhere discontinuous.

$\boxed{\text{10 Marks}}$ 9. Let $f : (0, 1) \mapsto \mathbb{R}$ be defined as $f(x) = \frac{2x-1}{x-x^2}$.

(a) Prove that f is injective.

(b) Prove that f is surjective.

(c) Hence prove that $|(0,1)| = |\mathbb{R}|$.