MATH 220 Practice Finals 2 —Duration: 2.5 hours

*This test has **10 questions** on **18 pages**, for a total of 100 points.*

Disclaimer: This test is definitely harder than actual 220 final exams. Treat it more like extra homework and take time to think through the problems, the duration is technically 2.5 hours but you will likely not be able to finish most of the test so give yourself more time if needed, your performance in this practice test is not a good indicator of success nor failure in the course.

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:	/100									

10 Marks

1. Carefully define or restate each of the following:

   (a) A rational number $q \in \mathbb{Q}$

   (b) Bézout's lemma

   (c) The Fundamental Theorem of Arithmetic

   (d) A convergent sequence $(x_n)_{n\in\mathbb{N}} : \mathbb{N} \mapsto \mathbb{R}$

   (e) The principle of mathematical induction

10 Marks

2. For each of the following statements, write down its negation and prove or disprove the statement.

   (a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, if $x + y < z$, then $x - y > z$.

(b) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, for all $z \in \mathbb{R}$, $xy > z$.

10 Marks 3. Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions. Prove or disprove each of the following:

(a) For all $U \subseteq C$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$.

(b) For all $U \subseteq B$, $(g \circ f)^{-1}(g(U)) = f^{-1}(U)$

$\boxed{\text{10 Marks}}$ 4. Let $n \in \mathbb{N}$ be even and $K = \{0, 1, 2, \ldots, n-1\}$. Let $S = \{k \in K | 2k \equiv 0 \mod n\}$. Prove that $|S|$ is even.

10 Marks

5. Let $\sim$ be a relation defined on $\mathbb{R}$ by "$a \sim b$ if and only if $a - b \in \mathbb{Z}$".

   (a) Prove that $\sim$ is an equivalence relation.

(b) Let $\mathbb{R}/\sim$ be the set of equivalence classes under $\sim$. Construct a bijection between $\mathbb{R}/\sim$ and $[0, 1)$.

10 Marks

6. (a) Let X be a non-empty set. Prove that any equivalence relation on X forms a partition on X.

(b) Prove that any partition on X corresponds to equivalence classes of an equivalence relation on X.

10 Marks

7. Let $\{x_n\}$ be a sequence and we define a subsequence of $\{x_n\}$ as a sequence obtained from removing terms in $\{x_n\}$.

   (a) Construct an example where $\{x_n\}$ is a sequence and two subsequences of $\{x_n\}$ both converge but converge to different limits, and justify your answer.

(b) Prove that if a sequence $\{x_n\}$ converges to L then every subsequence of $\{x_n\}$ converges to L.

$\boxed{\text{10 Marks}}$ 8. Prove or disprove each of the following:

(a) Let $A, B \subseteq C$. If $|C \backslash A| = |C \backslash B|$, then $|A| = |B|$.

(b) Let $A_i \in X$. Then, $X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i)$.

10 Marks

9. Let $B_r(x) \subseteq \mathbb{R}$ be the open interval $(x - r, x + r)$ where $r > 0$.

(a) Let $p \in \mathbb{R}$ and $q \in B_r(p)$. Show that there exists $B_{r_2}(q) \subseteq B_r(p)$.

(b) A set $E \subseteq \mathbb{R}$ is open if for all $p \in E$, there exists $B_r(p) \subseteq E$. Prove that E is open if and only if E is a union of open intervals.

$\boxed{\text{10 Marks}}$ 10. Let $N = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$, and let S_n be the set of bijective functions $f : N \mapsto N$. Prove by induction that for all $n \geq 3 \in \mathbb{N}$, there exists $f, g \in S_n$ such that $f \circ g \neq g \circ f$.