MATH 221 Practice Finals — November, 2024, Duration: 150 minutes

*This test has **10 questions** on **19 pages**, for a total of 80 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:										/80

$\boxed{\text{4 Marks}}$ 1. Solve the following system of linear equations and express the solution in parametric form

$$\begin{cases} 2x + 3y & = z - w \\ 2w + z & = -x \\ y + 3z & = 1 \end{cases}$$

6 Marks

2. Find all solutions to the system $A\mathbf{x} = \mathbf{b}$ given the following:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

where $\mathbf{b} \neq \mathbf{0}$, $A^2\mathbf{v} = \mathbf{b}, \mathbf{u}_1, \mathbf{u}_2 \in \mathrm{Nul}(A)$ and justify how you came to the solution. (You may need to express your solution in terms of A or A^{-1} in combination with the vectors provided)

3. Let $a_1, a_2, a_3 \in \mathbb{R}$ and assume $a_1 \neq 0$. Let A be a matrix defined as follows:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_1a_2 & a_2a_3 & a_1a_3 \\ 0 & 0 & a_1a_2a_3 \end{bmatrix}$$

2 Marks (a) Show that for any square matrix, if its nullity is zero then it must be onto.

4 Marks (b) Find all instances where A has a nullity of 1.

4. State the definition for each of the following.

$\boxed{\text{2 Marks}}$ (a) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent for $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$.

$\boxed{\text{2 Marks}}$ (b) B is a basis for a vector space V

$\boxed{\text{2 Marks}}$ (c) $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is a linear transformation

$\boxed{\text{2 Marks}}$ (d) The null space of A where A is a matrix

$\boxed{\text{2 Marks}}$ (e) The orthogonal complement of V which is a subspace of $\mathbb{R}^n$

5. Let f be a function that maps rational numbers to the set of $2 \times 2-$matrices with real entries defined as follows:

$$f(x) = \begin{bmatrix} \cos(2\pi x) & -\sin(2\pi x) \\ \sin(2\pi x) & \cos(2\pi x) \end{bmatrix}$$

2 Marks (a) Is f onto? No need to justify.

3 Marks (b) Find all x such that $f(x) = I$ and hence conclude that f is not one-to-one.

3 Marks

(c) Show that given any rational number x, there exists $n \in \mathbb{N}$ such that $(f(x))^n = I$. (Hint: f has the property that $f(x+y) = f(x)f(y)$)

6. Give an example or say does not exist for each of the following:

$\boxed{\text{2 Marks}}$ (a) An invertible matrix A that cannot be diagonalized into PDP^{-1} where P, D consist of real entries

$\boxed{\text{2 Marks}}$ (b) A vector in $\mathbb{R}^n$ that is orthogonal to the zero vector

$\boxed{\text{2 Marks}}$ (c) An onto but not one-to-one linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ that maps vectors with only integer entries to vectors with only integer entries

$\boxed{\text{2 Marks}}$ (d) Diagonalizable matrices A, B such that $A + B$ cannot be diagonalized

$\boxed{\text{2 Marks}}$ (e) Orthogonal matrices that are not one-to-one

7. Let $O_2(\mathbb{R})$ be the set of invertible $2 \times 2-$matrices with real entries such that $AA^T = I$ (In otherwords, $A^{-1} = A^T$). Furthermore, let $SO_2(\mathbb{R})$ denote the set of rotational matrices.

$\boxed{\text{2 Marks}}$ (a) Show that any $A \in O_2(\mathbb{R})$ has determinant ± 1

$\boxed{\text{2 Marks}}$ (b) Show that any $R \in SO_2(\mathbb{R})$ is in $O_2(\mathbb{R})$ by verifying that $RR^T = I$

4 Marks

(c) Show that if $A \in O_2(\mathbb{R})$ and $A \notin SO_2(\mathbb{R})$, then A does not have complex eigenvalues. (Hint: $\det(A) = -1$, try assuming otherwise and deduce a contradiction)

8. Consider the following matrix A

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$\boxed{\text{3 Marks}}$ (a) Compute the eigenvalues of A.

4 Marks

(b) Can A be diagonalized? If yes, diagonalize it by expressing it in the form PDP^{-1}. If no, explain why.

3 Marks (c) Show that every matrix B that is similar to A has the property that $B^2 = I$

$\boxed{\text{3 Marks}}$ 9. (a) Suppose $D, P \neq I$ is an $n \times n$ invertible matrix such that $P^5 = I$, and $D^2 = PDP^{-1}$. Find the smallest $k \in \mathbb{N}$ such that $D^k = I$.

3 Marks

(b) Suppose A is an $n \times n$ real matrix, λ is a real eigenvalue, and assume A is 3×3. Show that $(\mathrm{Nul}(A - \lambda I))^{\perp} \neq \mathbb{R}^3$.

2 Marks

(c) Let A be an $n \times n$ real matrix. Show that dimension of $(\mathrm{Row}(A))^{\perp}$ must be the same as the dimension of $\mathrm{Nul}(A^T)$.

10. Let $Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ where $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is orthonormal, and let A be a $3 \times 3-$matrix defined as follows:

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

3 Marks (a) Show that $QQ^T = I$ (This is equivalent to $Q^T = Q^{-1}$)

3 Marks

(b) Compute a matrix Q where $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are orthonormal vectors obtained (in order) by applying the Gram-Schmidt process on $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ (in order) where $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \end{bmatrix}$

$\boxed{\text{4 Marks}}$ (c) Let Q be the same matrix as obtained in part (b). Find a matrix R such that $A = QR$.