MATH 221 Practice Finals Answers — November, 2024, Duration: 150 minutes

*This test has **10 questions** on **X pages**, for a total of 80 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6	7	8	9	10
Points:										
Total:										/80

4 Marks

1. Solve the following system of linear equations and express the solution in parametric form

$$\begin{cases} 2x + 3y &= z - w \\ 2w + z &= -x \\ y + 3z &= 1 \end{cases}$$

Solution: We reorganize the system of equations into the following:

$$\begin{cases} w + 2x + 3y - 1z &= 0 \\ 2w + 1x + 0y + 1z &= 0 \\ 0w + 0x + 1y + 3z &= 1 \end{cases}$$

Then we organize it into an augmented matrix and row reduce it to get the solutions. The solutions are given by

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ \frac{7}{4} \\ \frac{-3}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -4 \\ 7 \\ -3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

There are other possible solutions of course, notably, as long as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} - \mathbf{x}_p = \mathbf{x}_c - \mathbf{x}_p \in \operatorname{Span}\left(\left\{\begin{bmatrix} -4 \\ 7 \\ -3 \\ 4 \end{bmatrix}\right\}\right)$$

(and the w, x, y, z rows match the entries -4,7,-3,4 and the rows for $\mathbf{x}_p - \mathbf{x}_c$), then the solution works.

6 Marks

2. Find all solutions to the system $A\mathbf{x} = \mathbf{b}$ given the following:

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

where $\mathbf{b} \neq \mathbf{0}$, $A^2\mathbf{v} = \mathbf{b}, \mathbf{u}_1, \mathbf{u}_2 \in \text{Nul}(A)$ and justify how you came to the solution. (You may need to express your solution in terms of A or A^{-1} in combination with the vectors provided)

Solution: We know the solutions to $A\mathbf{x} = \mathbf{b}$ are given in the form $\mathbf{x}_c + \mathbf{x}_p$ where $\mathbf{x}_c \in \text{Nul}(A)$ and $\mathbf{x}_p$ is any solution that solves $A\mathbf{x} = \mathbf{b}$. Observe that $\text{Nul}(A) \subseteq \mathbb{R}^3$ and $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent so $\text{Nul}(A)$ has dimension 2 or 3. Now if $\text{Nul}(A)$ is 3-dimensional, then A would be the zero matrix since it would have rank 0 by the rank theorem and thus $A^2\mathbf{v} = \mathbf{0}$, but this contradicts our assumption that $A^2\mathbf{v} = \mathbf{b}$ and $\mathbf{b} \neq \mathbf{0}$. Hence, $\text{Nul}(A)$ has dimension 2 and is spanned by $\mathbf{u}_1, \mathbf{u}_2$. Now, from $A^2\mathbf{v} = \mathbf{b}$ we know $A(A\mathbf{v}) = \mathbf{b}$ so $A\mathbf{v}$ is a possible $\mathbf{x}_p$, so it follows that all possible solutions are going to be of the form $C_1\mathbf{u}_1 + C_2\mathbf{u}_2 + A\mathbf{v}$ where $C_1, C_2 \in \mathbb{R}$, which is

$$C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + A \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)$$

3. Let $a_1, a_2, a_3 \in \mathbb{R}$ and assume $a_1 \neq 0$. Let A be a matrix defined as follows:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_1a_2 & a_2a_3 & a_1a_3 \\ 0 & 0 & a_1a_2a_3 \end{bmatrix}$$

2 Marks

(a) Show that for any square matrix, if its nullity is zero then it must be onto.

Solution: Let A be square and assume its nullity is zero. Then, by the rank theorem, we have its rank is equal to its number of columns so it is one-to-one. Since A is square, A is also onto.

4 Marks

(b) Find all instances where A has a nullity of 1.

Solution: This is equivalent to finding all instances where A has an REF that only has one pivot. Row reducing A gives

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_1a_2 - a_1a_2 & a_2a_3 - a_2a_2 & a_1a_3 - a_2a_3 \\ 0 & 0 & a_1a_2a_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_2(a_3 - a_2) & a_1a_3 - a_2a_3 \\ 0 & 0 & a_1a_2a_3 \end{bmatrix}$$

By assumption, $a_1 \neq 0$. Now we split into cases.

- Assume $a_2 = 0$. Then,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_2(a_3 - a_2) & a_1a_3 - a_2a_3 \\ 0 & 0 & a_1a_2a_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & a_1a_3 \\ 0 & 0 & 0 \end{bmatrix}$$

  And we must have $a_1a_3 \neq 0$, and thus we must have $a_3 \neq 0$.

- Now assume $a_2 \neq 0$. Then, we must have $a_3 = 0$ so that $a_1a_2a_3 = 0$. Then,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_2(a_3 - a_2) & a_1a_3 - a_2a_3 \\ 0 & 0 & a_1a_2a_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & -a_2{}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  And $-a_2{}^2 \neq 0$ since we assume $a_2 \neq 0$ so $a_2 \neq 0, a_3 = 0$ does indeed give a matrix A with nullity 1.

Thus, the instances for nullity$(A) = 1$ are when exactly one of a_2, a_3 is zero.

4. State the definition for each of the following.

2 Marks (a) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent for $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$.

Solution: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$ implies $c_1 = 0, c_2 = 0, \ldots, c_n = 0$.

2 Marks (b) B is a basis for a vector space V

Solution: B is linearly independent and spans V

2 Marks (c) $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is a linear transformation

Solution: For all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^m$ and $c, d \in \mathbb{R}$, $T(c\mathbf{v}_1 + d\mathbf{v}_2) = cT(\mathbf{v}_1) + dT(\mathbf{v}_2)$

2 Marks (d) The null space of A where A is a matrix

Solution: The solution set to $A\mathbf{x} = \mathbf{0}$ where $\mathbf{0}$ is in the codomain of the linear transformation that A represents

2 Marks (e) The orthogonal complement of V which is a subspace of $\mathbb{R}^n$

Solution: The subspace spanned by all the vectors such that its dot product with any vector from V is 0

5. Let f be a function that maps rational numbers to the set of $2 \times 2-$matrices with real entries defined as follows:

$$f(x) = \begin{bmatrix} \cos(2\pi x) & -\sin(2\pi x) \\ \sin(2\pi x) & \cos(2\pi x) \end{bmatrix}$$

2 Marks

(a) Is f onto? No need to justify.

Solution: No

3 Marks

(b) Find all x such that $f(x) = I$ and hence conclude that f is not one-to-one.

Solution: Observe that $\cos(2\pi x) = 1$ and $\sin(2\pi x) = 0$ if and only if $x \in \mathbb{Z}$, so any integer x will work. There is more than one integer, so there is more than one x such that $f(x) = I$ and thus f is not one-to-one.

$\boxed{\text{3 Marks}}$ (c) Show that given any rational number x, there exists $n \in \mathbb{N}$ such that $(f(x))^n = I$. (Hint: f has the property that $f(x+y) = f(x)f(y)$)

Let $x \in \mathbb{Q}$, so $x = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then, $(f(x))^q = f(qx) = f(p)$ but $p \in \mathbb{Z}$ and from part (b) we have $f(p) = I$ so the result holds.

6. Give an example or say does not exist for each of the following:

2 Marks (a) An invertible matrix A that cannot be diagonalized into PDP^{-1} where P, D consist of real entries

Solution: Rotation matrix of $\theta \neq \frac{\pi n}{2}$, $n \in \mathbb{Z}$ since P, D consist of complex entries. Many options work, this is just one.

2 Marks (b) A vector in $\mathbb{R}^n$ that is orthogonal to the zero vector

Solution: Any $\mathbf{v} \in \mathbb{R}^n$

2 Marks (c) An onto but not one-to-one linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ that maps vectors with only integer entries to vectors with only integer entries

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Many options work, this is just one.

2 Marks (d) Diagonalizable matrices A, B such that $A + B$ cannot be diagonalized

Solution:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

Many options work, this is just one.

2 Marks (e) Orthogonal matrices that are not one-to-one

Solution: Does not exist. The column vectors are orthogonal and so form a linearly independent set, this implies every column has a pivot in REF.

7. Let $O_2(\mathbb{R})$ be the set of invertible $2 \times 2-$matrices with real entries such that $AA^T = I$ (In otherwords, $A^{-1} = A^T$). Furthermore, let $SO_2(\mathbb{R})$ denote the set of rotational matrices.

2 Marks (a) Show that any $A \in O_2(\mathbb{R})$ has determinant ± 1

Solution: Let $A \in O_2(\mathbb{R})$. Then, $AA^T = I$ so $(\det(A))^2 = \det(A)\det A^T = 1$ so $\det(A) = \pm 1$

2 Marks (b) Show that any $R \in SO_2(\mathbb{R})$ is in $O_2(\mathbb{R})$ by verifying that $RR^T = I$

Solution:

$$\begin{aligned}
\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \left(\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \right)^T &= \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \\
&= \begin{bmatrix} \cos^2 x + \sin^2 x & 0 \\ 0x & \cos^2 x + \sin^2 x \end{bmatrix} \\
&= I
\end{aligned}$$

4 Marks

(c) Show that if $A \in O_2(\mathbb{R})$ and $A \notin SO_2(\mathbb{R})$, then A does not have complex eigenvalues. (Hint: $\det(A) = -1$, try assuming otherwise and deduce a contradiction)

Solution: Let $A \in O_2(\mathbb{R}) \backslash SO_2(\mathbb{R})$ so $\det(A) = -1$. Assume for the sake of contradiction that the eigenvalues of A are complex. Then, by the fundamental theorem of algebra, the roots of the characteristic polynomial, i.e. eigenvalues of A, are complex conjugates $z, \bar{z}$. Notice, $z\bar{z} \geq 0$, and we know the product of eigenvalues will yield the determinant, but the determinant is -1, a contradiction. Hence, A has no complex eigenvalues.

8. Consider the following matrix A

$$\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

3 Marks

(a) Compute the eigenvalues of A.

Solution: Compute λ such that $\det(A - \lambda I) = 0$.

$$\begin{aligned} \det(A - \lambda I) &= (2-\lambda)(-2-\lambda) + 3 \\ &= \lambda^2 - 4 + 3 \\ &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1) \end{aligned}$$

4 Marks

(b) Can A be diagonalized? If yes, diagonalize it by expressing it in the form PDP^{-1}. If no, explain why.

Solution: Clearly A can be diagonalized since it has 2 distinct eigenvalues and A is 2×2. We compute $\text{Nul}(A - \lambda I)$.

- For $\lambda = 1$,

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

  So the eigenvector for $\lambda = 1$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- For $\lambda = -1$,

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

  So eigenvector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and by the formula for inverses of 2×2 invertible real matrices,

$$\begin{aligned} P^{-1} &= \frac{1}{\det P} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \frac{1}{(3 \cdot 1) - (1 \cdot 1)} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Hence, we have

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix}$$

3 Marks

(c) Show that every matrix B that is similar to A has the property that $B^2 = I$

Solution: Let B be similar to A. A is similar to D, so B is also similar to D. Then, $B^2 = RD^2R^{-1}$ for some invertible R but $D^2 = I$ so $B^2 = RR^{-1} = I$.

3 Marks

9. (a) Suppose $D, P \neq I$ is an $n \times n$ invertible matrix such that $P^5 = I$, and $D^2 = PDP^{-1}$. Find the smallest $k \in \mathbb{N}$ such that $D^k = I$.

Solution: Observe that

$$\begin{aligned}
(PD^2P^{-1})^2 &= PD^2P^{-1} = D^4 \\
&= P^2DP^{-2} \\
(P^2D^2P^{-2})^2 &= P^2D^4P^{-2} = D^8 \\
&= P^3DP^{-3} \\
(P^3D^2P^{-3})^2 &= P^2D^8P^{-2} = D^{16} \\
&= P^4DP^{-4} \\
(P^4D^2P^{-4})^2 &= P^2D^{16}P^{-2} = D^{32} \\
&= P^5DP^{-5} = D
\end{aligned}$$

so $D^{31} = I$. $D \neq I$ so $k \neq 1$. Now since k is the smallest natural number such that $D^k = I$, $D^\ell = I$ if and only if $k \mid \ell$, but 31 is prime so the only possible k is so $k = 31$

3 Marks

(b) Suppose A is an $n \times n$ real matrix, λ is a real eigenvalue, and assume A is 3×3. Show that $(\text{Nul}(A - \lambda I))^{\perp} \neq \mathbb{R}^3$.

Solution: Suppose A is an $n \times n$ real matrix, λ be a real eigenvalue. Then, $\text{Nul}(A - \lambda I) \neq \{\mathbf{0}\}$ since A has at least one eigenvector corresponding to λ. The eigenvector is in $\mathbb{R}^3$ so $\text{Nul}(A - \lambda I)$ is a subspace of $\mathbb{R}^3$. Since $\text{Nul}(A - \lambda I)^{\perp} + \text{Nul}(A - \lambda I) = \mathbb{R}^3$ but $\text{Nul}(A - \lambda I)$ is non-empty, it follows that $\text{Nul}(A - \lambda I)^{\perp}$ has dimension lesser than 3 so $\text{Nul}(A - \lambda I)^{\perp} \neq \mathbb{R}^3$.

2 Marks

(c) Let A be an $n \times n$ real matrix. Show that dimension of $(\text{Row}(A))^{\perp}$ must be the same as the dimension of $\text{Nul}(A^T)$.

Solution: Suppose A is an $n \times n$ real matrix. We know the dimension of the row space and the column space are the same so $(\text{Row}(A))^{\perp}$ and $(\text{Col}(A))^{\perp}$ have the same dimension, but $(\text{Col}(A))^{\perp} = \text{Nul}(A^T)$ so the result holds.

10. Let $Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ where $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is orthonormal, and let A be a $3 \times 3-$matrix defined as follows:

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

3 Marks

(a) Show that $QQ^T = I$ (This is equivalent to $Q^T = Q^{-1}$)

Solution: We show $Q^TQ = I$ instead since that implies $Q^T = Q^{-1}$ which implies $QQ^T = I$. For that, the $ij-$th entry in Q^TQ is just $\mathbf{v}_i \cdot \mathbf{v}_j$ and since the column vectors are orthonormal, for any $i \neq j$ we get $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ and $\mathbf{v}_i \cdot \mathbf{v}_i = 1$. Thus, $Q^TQ = I$ and $Q^T = Q^{-1}$ so $Q^{-1}Q = I = QQ^{-1} = QQ^T$.

3 Marks

(b) Compute a matrix Q where $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are orthonormal vectors obtained (in order) by applying the Gram-Schmidt process on $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ (in order) where $A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \end{bmatrix}$

Solution: One can either normalize the vectors at each stage of the Gram-Schmidt process, or apply the Gram-Schmidt process first and then normalize them afterwards. Normalization can be done via $\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$ where $\hat{\mathbf{u}}$ is the normalized vector. The solution can be checked online and one possible answer is the following:

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix}$$

4 Marks

(c) Let Q be the same matrix as obtained in part (b). Find a matrix R such that $A = QR$.

Solution: From (a) we know $Q^T = Q^{-1}$ so then $Q^T A = Q^{-1}A = Q^{-1}QR = R$. Thus,

$$R = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 0 & \frac{5\sqrt{3}}{3} & 0 \\ 0 & \frac{-\sqrt{6}}{6} & \frac{\sqrt{6}}{2} \end{bmatrix}$$