MATH 223 Practice 1 — September, 2024, Duration: 2.5 hours

*This document has **6 questions** on **30 pages**, for a total of 80 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5	6
Points:						
Total:	/80					

12 Marks

1. Carefully define each of the following:

   (a) A subspace U of a vector space V over F

   (b) A linearly independent set of vectors $\{v_1, v_2, \ldots, v_n\} \subset V$ (You may assume V is finite dimensional and over a field F)

   (c) A linear transformation $T : U \mapsto V$ where U, V are over the field F

   (d) The null space of a matrix A

   (e) Similar matrices $A \sim B$

   (f) An inner product $\langle \cdot, \cdot \rangle : V \times V \mapsto F$ where V is a vector space over $F = \mathbb{R}$ or $\mathbb{C}$

20 Marks 2. This following section will ask you to prove some basic results about vector spaces.

(a) Prove that for any finite dimensional vector space V, W with the same dimension n over the same field F, there exists a bijective linear transformation $T : V \mapsto W$.

This page is for your work and solutions if needed

(b) Let A be an $n \times n-$matrix with complex entries. Prove that $1 \leq$ geo. multi. of $\lambda \leq$ alg. multi. of λ where λ is an eigenvalue of A. (Hint: Jordan normal form)

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(c) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues for eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ for some $n \times n-$matrix A. Prove that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent.

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(d) Prove that a matrix A has an eigenvalue $\lambda = 0$ if and only if $\det(A) = 0$.

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$\boxed{\text{12 Marks}}$ 3. Let $O_2(\mathbb{R})$ denote the set of $n \times n-$matrices that preserve the norm of vectors and angle between vectors, identically, the set of matrices such that $A^T = A^{-1}$.

(a) Prove that for all $A \in O_2(\mathbb{R})$, $\det(A) = \pm 1$.

(b) Let $SO_2(\mathbb{R})$ be the set of matrices in $O_n(\mathbb{R})$ such that they have a determinant of 1. Prove that $SO_2(\mathbb{R})$ is the set of rotation matrices for $\mathbb{R}^2$. (Hint: Consider what any $A \in SO_2(\mathbb{R})$ does to e_1 and e_2)

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(c) Prove that for all $A \in O_2(\mathbb{R}) \backslash SO_2(\mathbb{R})$, one of the eigenvalues of A is 1 and the other is -1.

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10 Marks

4. Let $GL_n(\mathbb{R})$ denote the set of $n \times n$ invertible matrices.

   (a) Let $\sim$ be a relation on $GL_n(\mathbb{R})$ where $A \sim B$ if and only if $\det(A) = \det(B)$. Prove that $\sim$ is an equivalence relation.

(b) Prove that $O_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ and $O_n(\mathbb{R})$ is closed under matrix multiplication. Prove or disprove that for all $A \in GL_n(\mathbb{R})$ and for all $B \in O_n(\mathbb{R})$, $ABA^{-1} \in O_n(\mathbb{R})$.

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12 Marks

5. Let $\langle A\rangle = \left\{B : B = A^k, \det(A) \neq 0, k \in \mathbb{Z}\right\}$ where A is an $n \times n$-matrix and assume $\langle A\rangle$ is closed under matrix multiplication.

   (a) Prove that for all $B, C \in \langle A\rangle$, $BC = CB$. Prove that if $\langle A\rangle$ is finite, $\langle A\rangle \subseteq O_n(\mathbb{R})$.

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(b) Prove that if m is the smallest positive integer such that $B^m = I$ for all $B \in \langle A \rangle$, then $\langle A \rangle$ has m elements.

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(c) Prove that if $\langle A \rangle$ is finite with k elements, then every $\langle B \rangle \subseteq \langle A \rangle$ has d elements such that $d \mid k$, and that there could only be one $\langle B \rangle$ for each divisor of k. You may use the fact that $\langle A \rangle = \left\{ I, A, A^2, \ldots, A^{k-1} \right\}$ and that $|\langle A^s \rangle| = \frac{n}{\gcd(n,s)}$ for all $s \neq 0 \in \mathbb{Z}$. (Note $\langle B \rangle$ also has to be closed under multiplication)

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14 Marks

6. Let A be a $n \times n$ permutation matrix, so every column of A has one entry that is 1 and every row of A has one entry that is 1, and the matrix is 0 everywhere else, and let A_n be the set of $n \times n$ permutation matrices.

   (a) Prove that if $n \geq 2$, then there exists a permutation matrix $A \neq I \in A_n$ such that $A^k = I$ for some $k \in \mathbb{N}$.

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(b) Let S_n be the set of bijections from $\{1, 2, 3, \dots, n\}$ to itself. Construct a bijection $\varphi : S_n \mapsto A_n$ between the set of permutation matrices such that for all $\sigma, \tau \in S_n$, $\varphi(\sigma \circ \tau) = \varphi(\sigma)\varphi(\tau)$.

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(c) Prove that if G is a non-empty finite set of $m \times m-$matrices with real entries such that for all $B, C \in G$, $BC^{-1} \in G$, then, there exists $n \in \mathbb{N}$ such that one can find an injection $f : G \mapsto A_n$ where for all $B, C \in G$, $f(BC) = f(B)f(C)$.

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