MATH 221 Practice Midterm Answers — November, 2024, Duration: 80 minutes

*This test has **5 questions** on **X pages**, for a total of 55 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/55				

1. Give an example or say does not exist for each of the following:

2 Marks (a) A matrix A such that $\text{Nul}(A) = \text{Col}(A)$

Solution:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

There are more examples but this is the easiest

2 Marks (b) A subspace of $\mathbb{R}^n$ such that it has only one basis

Solution: $\{\mathbf{0}\}$, and this is in fact the only subspace with one basis, the basis being $\varnothing$

2 Marks (c) A linear transformation $T : V \mapsto W$ such that $T(\mathbf{0}) \neq \mathbf{0}'$ where $\mathbf{0} \in V$, $\mathbf{0}' \in W$

Solution: Does not exist

2 Marks (d) An onto function mapping the set of $n \times n-$matrices with real entries to $\mathbb{R}$

Solution: The determinant (almost anything works here, another answer would be mapping a matrix to its upper left entry)

2 Marks (e) An invertible $2 \times 2-$matrix with integer entries that has a non-integer determinant

Solution: Does not exist.

2. Determine whether T is one-to-one and/or onto for each of the following: (No need to justify)

4 Marks (a) $T : \mathbb{R} \mapsto \mathbb{R}^2$ where if $x \neq 0$, $T(x) = \langle \cos\left(\frac{2\pi}{x}\right), \sin\left(\frac{2\pi}{x}\right)\rangle$ and $T(0) = 1$

Solution: Not one-to-one and not onto

4 Marks (b) $T : \mathbb{R} \mapsto \mathbb{R}$ where $T(x)$ is a polynomial of odd degree that has complex roots.

Solution: Not one-to-one and onto

4 Marks (c) $T : M \mapsto M$, M is the set of invertible $n \times n-$matrices, $A, B \in M$, $T(B) = ABA^{-1}$

Solution: One-to-one and onto

3. Consider the following matrix A:

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 1 & 2 & k^2 & 2 \\ 2 & 0 & k & 12 \end{bmatrix}$$

3 Marks (a) Without row reducing the matrix, show that $\mathrm{Nul}(A) \neq \{\mathbf{0}\}$

Solution: Observe that this matrix is 3×4 so by rank theorem, the number of columns is equal to the rank + nullity. Notice the rank is at most 3 since the matrix can have at most 3 pivots, so the nullity must be at least 1. Hence, $\mathrm{Nul}(A) \neq \{\mathbf{0}\}$.

4 Marks (b) Find $k \in \mathbb{R}$ such that A is onto, or show that such k does not exist.

Solution: Row reducing this matrix gives

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & k^2 & -2 \\ 0 & 0 & 2k^2 + k & 0 \end{bmatrix}$$

so we want to guarantee that $2k^2 + k \neq 0$ as otherwise the bottom row would be all zeroes. Now, observe that this means $k(2k+1) \neq 0$ so $k \neq 0$ and $k \neq \frac{-1}{2}$ will guarantee that A is onto.

3 Marks

(c) Find $k \in \mathbb{R}$ such that rank$A$ is as small as possible, and state explicitly what rank$A$ is

Solution: From part (b), we see that A can be row reduced into the following

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & k^2 & -2 \\ 0 & 0 & 2k^2+k & 0 \end{bmatrix}$$

So A has at least 2 rows with pivots, meaning that the smallest possible rank is 2. Also from part (b), we know $k \neq 0$ and $k \neq \frac{-1}{2}$ will give the smallest possible rank for A.

4. Consider the following matrix A :

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{3 Marks}}$ (a) Compute A^{-1}

Solution:

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{-2}{5} & \frac{-4}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{3}{5} \end{bmatrix}$$

4 Marks

(b) Let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ denote the column vectors of A^{-1}. Rewrite the basis for $\text{Col}(A^{-1})$ as $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3}\})$ where T is a linear transformation such that $T(\mathbf{e_2}) = \mathbf{e_2}, T(\mathbf{e_3}) = \mathbf{e_3}$, and

$$T(\mathbf{v_1}) = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix} \qquad T(\mathbf{v_2}) = \begin{bmatrix} * \\ 2 \\ * \end{bmatrix} \qquad T(\mathbf{v_3}) = \begin{bmatrix} 1 \\ * \\ * \end{bmatrix}$$

Solution:

$$T(\mathbf{v_1} - \mathbf{v_2}) = T\left(\begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \\ \frac{-1}{5} \end{bmatrix} - \begin{bmatrix} \frac{1}{5} \\ \frac{-2}{5} \\ \frac{-1}{5} \end{bmatrix}\right) = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix} - \begin{bmatrix} * \\ 2 \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\implies T(\mathbf{v_1}) = \begin{bmatrix} * \\ 3 \\ 1 \end{bmatrix}, T(\mathbf{v_2}) = \begin{bmatrix} * \\ 2 \\ 1 \end{bmatrix}$$

$$T(2\mathbf{v_2} - \mathbf{v_3}) = T\left(\begin{bmatrix} \frac{2}{5} \\ \frac{-4}{5} \\ \frac{-2}{5} \end{bmatrix} - \begin{bmatrix} \frac{2}{5} \\ \frac{-4}{5} \\ \frac{3}{5} \end{bmatrix}\right) = \begin{bmatrix} * \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\implies T(\mathbf{v_2}) = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \end{bmatrix}, T(\mathbf{v_3}) = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\implies T(\mathbf{v_1}) = \begin{bmatrix} \frac{1}{2} \\ 3 \\ 1 \end{bmatrix}$$

So the final answer is $\left\{ \begin{bmatrix} \frac{1}{2} \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\}$

4 Marks

(c) Suppose the following are vectors written relative to the bases of $\mathbb{R}^3$ and $\mathrm{Col}(A)$ respectively.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Does $\{\mathbf{e_1}, \mathbf{v}, \mathbf{w}\}$ form a linearly independent set? What about $\{[\mathbf{e_1}]_{\mathcal{B}}, [\mathbf{v}]_{\mathcal{B}}, [\mathbf{w}]_{\mathcal{B}}\}$? Justify your answer.

Solution: It suffices to check that one of these sets are linearly independent/dependent as you can get from one set to another via A or A^{-1}, which are invertible linear transformations and hence preserve the number of pivots in the matrix formed with these sets as column vectors (and thus linear independence/dependence of the set). We check $\{\mathbf{e_1}, \mathbf{v}, \mathbf{w}\}$ so we convert $[\mathbf{w}]_{\mathcal{B}}$ back into standard coordinates. We know $A^{-1}\mathbf{w}$ will give us $[\mathbf{w}]_{\mathcal{B}}$ so

$$\begin{aligned} A([\mathbf{w}]_{\mathcal{B}}) &= A\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = \mathbf{w} \end{aligned}$$

But then this means the matrix whose column vectors are $\mathbf{e_1}, \mathbf{v}, \mathbf{w}$ would already be in REF and have 3 pivots, so the matrix is invertible and hence both $\{\mathbf{e_1}, \mathbf{v}, \mathbf{w}\}$ and $\{[\mathbf{e_1}]_{\mathcal{B}}, [\mathbf{v}]_{\mathcal{B}}, [\mathbf{w}]_{\mathcal{B}}\}$ are linearly independent.

5. Let D_4 be the possible matrices obtained by multiplying different powers of R and S together in different orders,

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\boxed{\text{4 Marks}}$ (a) Show that $SR^k = R^{-k}S$

Solution: R is a rotation matrix so R^k is just rotating counterclockwise by $\frac{\pi}{2}$ $k-$times and similarly for R^{-k}, it is just rotating clockwise by $\frac{\pi}{2}$ $k-$times. Now,

$$\begin{aligned}
SR^kS^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{k\pi}{2}\right) & -\sin\left(\frac{k\pi}{2}\right) \\ \sin\left(\frac{k\pi}{2}\right) & \cos\left(\frac{k\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{k\pi}{2}\right) & -\sin\left(\frac{k\pi}{2}\right) \\ \sin\left(\frac{k\pi}{2}\right) & \cos\left(\frac{k\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{k\pi}{2}\right) & \sin\left(\frac{k\pi}{2}\right) \\ \sin\left(\frac{k\pi}{2}\right) & -\cos\left(\frac{k\pi}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \cos\left(\frac{k\pi}{2}\right) & \sin\left(\frac{k\pi}{2}\right) \\ -\sin\left(\frac{k\pi}{2}\right) & \cos\left(\frac{k\pi}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \cos\left(\frac{-k\pi}{2}\right) & -\sin\left(\frac{-k\pi}{2}\right) \\ \sin\left(\frac{-k\pi}{2}\right) & \cos\left(\frac{-k\pi}{2}\right) \end{bmatrix} \\
&= R^{-k}
\end{aligned}$$

so $SR^kS^{-1} = R^{-k}$ and hence $SR^k = R^{-k}S$ as required.

4 Marks

(b) Suppose $B \in D_4$ is a matrix representing $T : \mathbb{R}^2 \mapsto V$ where the basis of V is as follows:

$$\left\{ \begin{bmatrix} * \\ -1 \end{bmatrix}, \begin{bmatrix} * \\ * \end{bmatrix} \right\}$$

Find all possible $B \in D_4$ (There are more than one different matrices depending on how you order the basis of V and depending on how you combine powers of S and R)

Solution: We want matrices of the following forms

$$\begin{bmatrix} * & * \\ -1 & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & -1 \end{bmatrix}$$

I claim that every matrix in D_4 is either R^k or SR^k for $0 \leq k \leq 3$. Now R is a matrix representing rotation by $\frac{\pi}{2}$ so R^k for $0 \leq k \leq 3$ will give distinct matrices in D_4. From part (a), we can rewrite every product of powers of S and R in the form $S^i R^j$ and notice $0 \leq i \leq 1$ since $S^2 = I$. This gives R^k, SR^k as the only possible matrices in D_4. Trivially, S is a possible B, but I is not a possible B. R does not have 1 in the bottom row so R is not a possible B. Notice

$$SR = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

so SR has -1 and 0 in the bottom row, SR is a possible B. $R^2 = -I$ so $-I$ is a possible B, and $-S$ does not have -1 in the bottom row, so $-S$ is not a possible B. $R^3 = -R$ which has -1 in the bottom row, so $-R$ is a possible B, and from before we know SR has -1 and 0 in the bottom row, so $-SR$ does not. Hence, the only possible matrices for B are

$$S, SR, -I, -R$$

4 Marks

(c) Let $n \in \mathbb{N}$. Show that $S \neq R^n$ and hence show that there cannot exist 2×2-matrices A, B that satisfy $SAB = R^k - S(A+I)$, $I - B = S$, $AS - I = R^\ell$ all at the same time by assuming that they can exist and deduce that $S = R^n$ for some n (which is a contradiction).

Solution: For showing $S \neq R^n$ for all $n \in \mathbb{N}$, notice $\det(S) = -1$ and $\det(R^k) = 1$. The determinants are different so S and R^k must be different. Now assume there exist 2×2-matrices A, B that satisfy $SAB = R^k - S(A+I)$, $I - B = S$, $AS - I = R^\ell$. Observe that $-R^k = R^{-k}$ since $-I$ is rotation by π and $-I \in D_4$. Then,

$$\begin{aligned}
SAB &= R^k - S(A+I) \\
SA(B - I + I) &= R^k - S(A+I) \\
SA(-S + I) &= R^k - S(A+I) \\
-SAS - SA &= R^k - SA + S \\
-SAS &= R^k + S \\
-SAS - S &= R^k \\
-S(AS - I) &= R^k \\
-SR^\ell &= R^k \\
-S &= R^{k-\ell} \\
S &= -R^{k-\ell} \\
S &= R^{\ell-k}
\end{aligned}$$

so $S = R^n$, but this contradicts $S \neq R^n$ so there cannot exist such A, B as required.