MATH 221 Practice Midterm — November, 2024, Duration: 80 minutes

*This test has **5 questions** on **11 pages**, for a total of 55 points.*

First Name:	Last Name:
Student Number:	Section:
Signature:	

Question:	1	2	3	4	5
Points:					
Total:	/55				

1. Give an example or say does not exist for each of the following:

$\boxed{\text{2 Marks}}$ (a) A matrix A such that $\mathrm{Nul}(A) = \mathrm{Col}(A)$

$\boxed{\text{2 Marks}}$ (b) A subspace of $\mathbb{R}^n$ such that it has only one basis

$\boxed{\text{2 Marks}}$ (c) A linear transformation $T : V \mapsto W$ such that $T(\mathbf{0}) \neq \mathbf{0}'$ where $\mathbf{0} \in V$, $\mathbf{0}' \in W$

$\boxed{\text{2 Marks}}$ (d) An onto function mapping the set of $n \times n-$matrices with real entries to $\mathbb{R}$

$\boxed{\text{2 Marks}}$ (e) An invertible $2\times2-$matrix with integer entries that has a non-integer determinant

2. Determine whether T is one-to-one and/or onto for each of the following: (No need to justify)

$\boxed{\text{4 Marks}}$ (a) $T : \mathbb{R} \mapsto \mathbb{R}^2$ where if $x \neq 0$, $T(x) = \langle \cos\left(\frac{2\pi}{x}\right), \sin\left(\frac{2\pi}{x}\right)\rangle$ and $T(0) = 1$

$\boxed{\text{4 Marks}}$ (b) $T : \mathbb{R} \mapsto \mathbb{R}$ where $T(x)$ is a polynomial of odd degree that has complex roots.

$\boxed{\text{4 Marks}}$ (c) $T : M \mapsto M$, M is the set of invertible $n\times n-$matrices, $A, B \in M$, $T(B) = ABA^{-1}$

3. Consider the following matrix A:

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 1 & 2 & k^2 & 2 \\ 2 & 0 & k & 12 \end{bmatrix}$$

3 Marks (a) Without row reducing the matrix, show that $\text{Nul}(A) \neq \{\mathbf{0}\}$

4 Marks (b) Find $k \in \mathbb{R}$ such that A is onto, or show that such k does not exist.

3 Marks

(c) Find $k \in \mathbb{R}$ such that $\text{rank}A$ is as small as possible, and state explicitly what $\text{rank}A$ is

4. Consider the following matrix A :

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

3 Marks (a) Compute A^{-1}

4 Marks

(b) Let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ denote the column vectors of A^{-1}. Rewrite the basis for $\mathrm{Col}(A^{-1})$ as $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3}\})$ where T is a linear transformation such that $T(\mathbf{e_2}) = \mathbf{e_2}, T(\mathbf{e_3}) = \mathbf{e_3}$, and

$$T(\mathbf{v_1}) = \begin{bmatrix} * \\ * \\ 1 \end{bmatrix} \qquad T(\mathbf{v_2}) = \begin{bmatrix} * \\ 2 \\ * \end{bmatrix} \qquad T(\mathbf{v_3}) = \begin{bmatrix} 1 \\ * \\ * \end{bmatrix}$$

4 Marks

(c) Suppose the following are vectors written relative to the bases of $\mathbb{R}^3$ and $\text{Col}(A)$ respectively.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, [\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Does $\{\mathbf{e_1}, \mathbf{v}, \mathbf{w}\}$ form a linearly independent set? What about $\{[\mathbf{e_1}]_{\mathcal{B}}, [\mathbf{v}]_{\mathcal{B}}, [\mathbf{w}]_{\mathcal{B}}\}$? Justify your answer.

5. Let D_4 be the possible matrices obtained by multiplying different powers of R and S together in different orders,

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\boxed{\text{4 Marks}}$ (a) Show that $SR^k = R^{-k}S$

4 Marks

(b) Suppose $B \in D_4$ is a matrix representing $T : \mathbb{R}^2 \mapsto V$ where the basis of V is as follows:

$$\left\{ \begin{bmatrix} * \\ -1 \end{bmatrix}, \begin{bmatrix} * \\ * \end{bmatrix} \right\}$$

Find all possible $B \in D_4$ (There are more than one different matrices depending on how you order the basis of V and depending on how you combine powers of S and R)

4 Marks (c) Let $n \in \mathbb{N}$. Show that $S \neq R^n$ and hence show that there cannot exist 2×2-matrices A, B that satisfy $SAB = R^k - S(A+I)$, $I - B = S$, $AS - I = R^\ell$ all at the same time by assuming that they can exist and deduce that $S = R^n$ for some n (which is a contradiction).