1 Problem of the Week

There are weekly problems assigned which will be discussed in reading groups. A solution will also be given in the reading group. You are not expected to necessarily complete them, but it will be very useful to look through it and have some ideas about how you might go about solving the problem even if you are very unsure.

Weekly Problem 1. Let G be a group with subgroups H, K. Define $HK = \{hk | h \in H, k \in K\}$. Show that $\#HK = \frac{\#H\#K}{\#H\cap K}$

2 More theorems on subgroups

Most of these are just useful theorems to know.

Theorem 1 (Subgroup test/Subgroup Criterion)**.** Let G be a group. Then, $H \subseteq G$ is a subgroup if and only if for all $g, h \in H$, $gh^{-1} \in H$.

Proof. For one direction, it is trivial. For the other direction, assume for all $g, h \in H$, $gh^{-1} \in H$. Associativity is inherited from $\cdot$ in G. Notice if $g \in H$, $gg^{-1} = e \in H$. Likewise, $g, e \in H$ implies $eg^{-1} = g^{-1} \in H$. To check closure, let $g, h \in H$. Since we have shown $h^{-1} \in H$, we know $g(h^{-1})^{-1} = gh \in H$, so H is a subgroup as required. □

Theorem 2. If H, K are subgroups of G, then $H \cap K$ is a subgroup of G.

Proof. Let H, K be subgroups of G. Then, let $h, k \in H \cap K$. Then, notice $hk^{-1} \in H$ since $h, k \in H$ and also $hk^{-1} \in K$ since $h, k \in K$ so the result follows from Theorem 1.1. □

3 Cosets

Definition 1. A left coset of H denoted gH is the set $\{gh | h \in H\}$ for some $g \in G$. Likewise, a right coset of H denoted Hg is the set $\{hg | h \in H\}$ for some $g \in G$.

Theorem 3. Given any $g \in G$, $\#H = \#gH = \#Hg$.

Proof. Define the map $f : H \mapsto gH$ by $f(h) = gh$. For injectivity, if $gh = gh'$ then multiplying g^{-1} to the left gives $h = h'$. For surjectivity, this is clear as given any $gh \in gH$, multiplying by g^{-1} gives h such that $f(h) = gh$. The proof for right cosets is done in the same fashion. □

Theorem 4. The number of left cosets of H is the same as the right.

Proof. Let S denote the set of left cosets and T denote the set of right cosets. Define the map $f : S \mapsto T$ by $f(gH) = f(Hg)$. The rest of the proof is left to the reader. □

Definition 2. The index of H, denoted $[G : H]$, is the number of distinct cosets of H.

Theorem 5 (Lagrange's theorem)**.** Let G be a finite group and H be a subgroup of G. Then, $\#H \mid \#G$ and $\#G = \#H[G : H]$.

Proof. Let G be a finite group and H be a subgroup of G. Notice we can map aH to H bijectively under the map $x \mapsto a^{-1}x$ which has the inverse $y \mapsto ay$, so $\#aH = \#H$. Notice the left cosets are the equivalence classes of the equivalence relation $x \sim y$ if and only if $x = yh$, so the left cosets form a partition on G and thus the union of distinct left cosets will be equal to G so $\#G = \#H[G : H]$ as required. $\square$

4 Exercises

These are exercises which will be worked through in the reading group. You are more than welcome to try them beforehand, but you are not expected to. Solutions will be posted after the reading section.

1. Complete the proof for Theorem 4.
2. Let G be a group and A, B be non-empty sets in G. Prove that if $\#A + \#B > \#G$ then $AB = G$. Notation: $AB = \{ab | a \in A, b \in B\}$.
3. Let G be a group with subgroups H, K. Show that if H, K are proper subgroups in G then $G \neq H \cup K$.
4. (a) Show that if G has prime order then $G = \left\{g^k | k \in \mathbb{Z}\right\}$ for some $g \neq e \in G$.
   (b) Show that for $g \in G$, $\#g \mid \#G$ and that $g^{\#G} = e$.
5. Show that if $\#G$ is even, there exists an element of order 2.
6. Prove that HK is a subgroup if and only if $HK = KH$.
7. Prove that if H, K are subgroups of G and $\#H, \#K > \sqrt{\#G}$, then $H \cap K \neq \{e\}$.