Week 3 Jan 19

Exercise: Show that $GL_n(\mathbb{R}) = \{A_{n\times n} \text{ s.t. } \det A \neq 0\}$ is a group under matrix multiplication

Prelims:

Euclidean division:

$a, b \in \mathbb{Z}, \Rightarrow \exists!\, q, r \in \mathbb{Z}$ s.t. $a = qb + r$, $0 \leq r < b$

Bézout's Lemma:

$a, b \in \mathbb{Z}$, $\exists\, x, y \in \mathbb{Z}$ s.t. $ax + by = \gcd(a, b)$

Equiv relation definition (see notes)

Examples:

$A, B$ $n\times n$ matrices over a field $F$ (or just $\mathbb{R}$)

$A \sim B \iff \exists$ invertible $P$ s.t. $A = PBP^{-1}$

$f, g$ continuously differentiable functions

$f \sim g \iff f = g + C$ for $C \in \mathbb{R}$

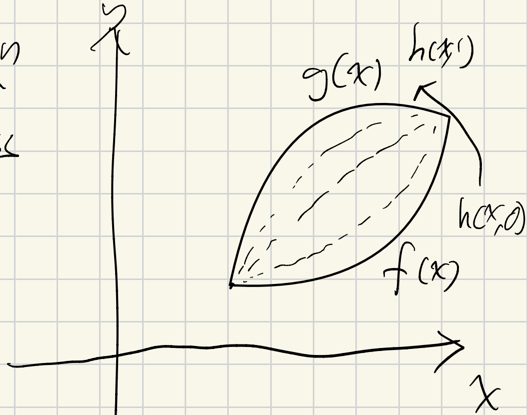
or equivalently, $f' = g'$

Topological spaces $X, Y$, (homotopic relation)

$f, g$ continuous functions $f, g: X \to Y$

$f \sim g \iff \exists$ continuous $h: X \times [0,1] \to Y$

s.t. $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ $\forall x \in X$




Every cont. function $X \to \mathbb{R}^n$ is in the same equiv class from $\sim$ before

(Omit pf in reading session)

Pf: $\mathbb{R}^n$ is contractible, I claim the identity map $i:\mathbb{R}^n \to \mathbb{R}^n$ is homotopic to $p:\mathbb{R}^n \to \mathbb{R}^n$ where $p(x)=\vec{0}$.
To this end, let $h: \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ be defined as $h(x,y) = x - yx$, $y \in [0,1]$, $x \in \mathbb{R}^n$.
For $h$ is continuous, suffice to show $\forall B_\varepsilon(x) \subseteq \mathbb{R}^n$, $h^{-1}(B_\varepsilon(x))$ is open. Observe if $\vec{0} \notin B_\varepsilon(x)$, $h^{-1}(B_\varepsilon(x)) = \emptyset$ (which is open) and if $B_\varepsilon(x)$ is not centered at $\vec{0}$, we can choose a smaller $\varepsilon' > 0$ s.t. $B_{\varepsilon'}(\vec{0}) \subseteq B_\varepsilon(x)$ and suffices to show $h^{-1}(B_{\varepsilon'}(\vec{0}))$ is open since then $h^{-1}(B_\varepsilon(x)) = \emptyset \cup h^{-1}(B_{\varepsilon'}(\vec{0})) = h^{-1}(B_{\varepsilon'}(\vec{0})))$
Furthermore, same holds if $\varepsilon > 1$, so we check $B_\varepsilon(\vec{0})$ centered at $\vec{0}$ for some $1 > \varepsilon > 0$. Now, $h^{-1}(B_\varepsilon(\vec{0})) = B_1(\vec{0}) \times [0, 1-\varepsilon)$, $B_1(\vec{0})$ open in $\mathbb{R}^n$, $[0,1-\varepsilon) = [0,1] \cap \underbrace{(0,1-\varepsilon)}_{\text{open in } \mathbb{R}}$ so $B_1(\vec{0}) \times [0,1-\varepsilon)$ is open in $\mathbb{R}^n \times [0,1]$.
Hence, $h$ is continuous.
Now, $h(x,0) = x - 0x = x$, so $h(x,0) = i(x)$. On the other hand, $h(x,1) = x - x = \vec{0} = p(x)$, so $i \sim p$. Let $X$ be top. space and $f,g: X \to \mathbb{R}^n$ be continuous. $f \sim f$ since homotopic relation is reflexive,

$f = f \circ i \sim f \circ p(x) = \overbrace{p_X(x)}^{p_X : X \to \mathbb{R} \text{ defined as } p_X(x) = \vec{0}} = g \circ p(x) \sim g \circ i = g$ since $i \sim p(x)$, so $f \sim g$. ∎

By compatibility of homotopy & function composition

# Quotient sets

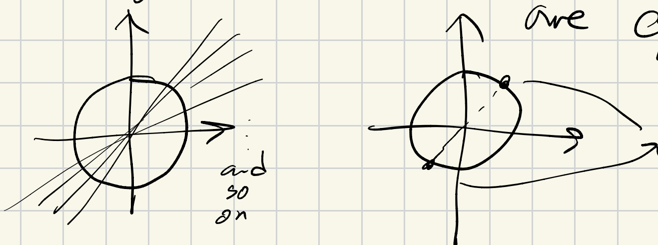
$X$ a set, $\sim$ an equiv. relation, $X/\sim$ the set of equiv classes.

Natural map $\pi : X \longrightarrow X/\sim$ where $\pi(x) = [x]_\sim$

— $\pi$ is surjective

Ex. points $\in \mathbb{R}^2$, $p \sim q \iff$ a line $\lambda$ passing $p, q$, and origin

Set of all 1-dimensional subspaces of $\mathbb{R}^2$, can be thought of as circle $S^1$ but antipodal points are equivalent, i.e.

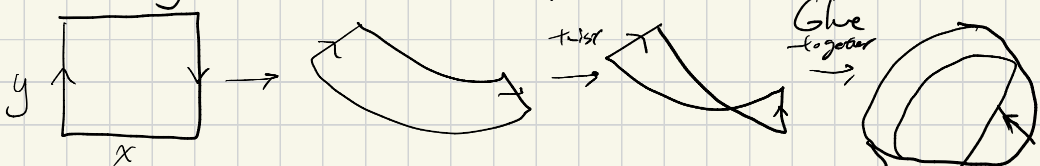



"same"

$\forall x, y \in S^1$,

$xRy \iff x = \pm y$

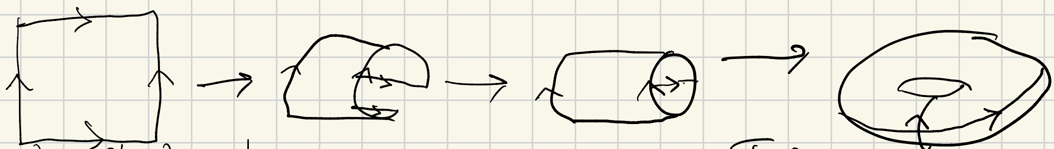
## Gluing sides of a square




$[0,1]\times[0,1]$

Points of form $(x,y)$, $(0,y) \sim (1, 1-y)$

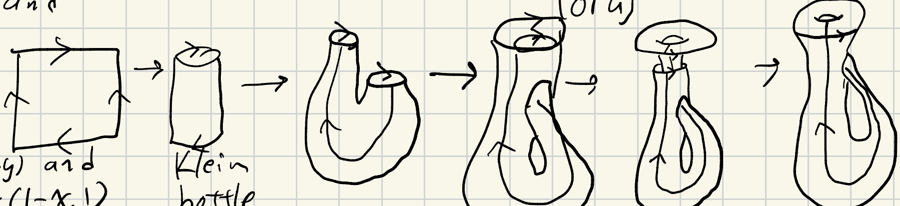
Möbius strip



$(0,y) \sim (1,y)$ and

$(x,0) \sim (x,1)$

Torus




$(0,y) \sim (1,y)$ and

$(x,0) \sim (1-x,1)$

Defn

A binary operation is a function $\cdot : X \to X$ $X$ a set

Defn

$(X, \cdot)$, $\cdot$ is associative, called semigroup

E.g. set of even numbers under $\times$

$\{f : A \to A\}$ where $A$ is a set under $\circ$

Defn

Semigroup + $\underbrace{\text{identity}}_{a \cdot 1 = 1 \cdot a = a \ \forall a}$ $1$ called a monoid.

Example: $\mathbb{Z}$ under $\times$, set of integers mod $n$ under $\times$,

$n \times n$ matrices under matrix multiplication

Group:

Monoid + 2 sided inverses. Called abelian if commutative

E.g. $S_n$ "set of bijective functions $X \to X$"

$\mathbb{H}$ — quaternions under $\times$

$\left\{ \begin{pmatrix} 1 & \cdots & \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ — unitriangular matrices

$\{\mathbb{Z}^*$ (aka $\mathbb{Z} \setminus \{0\}$ under $\times\}$

Def^n

Subgroup — group inside a group with same operation

Notation: $G$ — group. $H$ — subgroup.

$M$ — monoid, $\cdot$ operation, $H \leq G$ sometimes

$g^n = \underbrace{g \cdot \ldots \cdot g}_{n \text{ times}}$

E.g. Subgroups of $\mathbb{Z}$ under $(+)$:

$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$, and they are the only subgroups

Pf: Let $H \leq \mathbb{Z}$. Choose minimal $k > 0 \in H$.

Claim: $H = \{kx \mid x \in \mathbb{Z}\}$. Let $m \in H$. By Euclidean div., $m = qk + r$ for some $q, r \in \mathbb{Z}$, $0 \leq r < k$

By closure, $m - qk = r \in H$, by minimality, $r = 0$, so $m = qk$.

$m$ arbitrary so $H = \{kx \mid x \in \mathbb{Z}\}$. If $H = \{0\}$, $H = \{0 \cdot x \mid x \in \mathbb{Z}\}$

$\blacksquare$

Orthogonal group $O_n(\mathbb{R}) = \{AA^T = I\}$

$O_2(\mathbb{R})$ = rotational matrices, reflectional matrices

Thm: $\forall A \in O_n(\mathbb{R})$, $\det(A) = \pm 1$.

Pf: $1 = \det(AA^T) = \det(A)\det(A^T) = (\det(A))^2$ $\blacksquare$

Thm:
$\forall A \in O_2(\mathbb{R})$, if $A$ is a reflection, exactly one of $-1, 1$ as the eigenvalues
(Omit pf in reading session)

Pf: If $\lambda_1, \lambda_2$ are real, $A$ preserves distance between vectors, so $\lambda_1, \lambda_2 = \pm 1$. Consider orientation of $A(e_1)$ and $A(e_2)$. Must have one negative eigenvalue, say $\lambda_1$, since one of $e_1, e_2$ got flipped, and the other did not. $A$ does not scale vectors so $\lambda_1 = -1$, $\lambda_2 = 1$.
Claim: No complex eigenvalues.
Contradiction. By fundamental thm of algebra,
$\lambda_1 = z$, $\lambda_2 = \bar{z}$. $\quad z \cdot \bar{z} = \det(A) \geq 0 > -1 \Rightarrow\!\Leftarrow$
$\blacksquare$

Weekly problem:
Lemma 1.1 says enough to find right identity + right inverse. right identity first since inverses depend on identity.

We know:
- $G$ finite
- $ac = bc \Rightarrow a = b$
  $ca = cb \Rightarrow a = b$

G finite and want to somehow get a right identity from G. Try multiplying things. If we have $x \in G$, $xg = x$, we are happy since $x$ can be a candidate as a right identity. Try fixing $x$, can we somehow get $x$ from multiplying $g \in G$ to right of $x$ by $xg$?

$$G = \{g_1, g_2, g_3, \dots, g_n\}$$

$$_xG = \{xg_1, xg_2, xg_3, \dots, xg_n\}$$

By closure, $_xG \subseteq G$, but is $G \subseteq {_xG}$?

We know $x = g_i$, we want to show $x \in {_xG}$ so if $G \subseteq {_xG}$, we are good to go.

Show $\varphi(g) = xg$ is injective. ✓
holds by cancellation

$xg = x$ for some $g$. Claim: $g$ is right identity

Every $a \in G = bx$ for some $b \in G$ since

$Gx = \{g_1 x, g_2 x, \ldots, g_n x\} = G$ by cancellation.

$$ag = bxg = bx = a,$$ $g$ is right identity.

For inverses, $\exists\, a \in G$ s.t. $\cancel{ax} = e$ since $xG = G$. ✓

$\blacksquare$