1 Problem of the Week

Proof. It suffices to show that G has a right identity and a right inverse for every $x \in G$. Let $x \in G$ and consider the set ${}_xG = \{xg | g \in G\}$ and the map $\varphi : G \mapsto_x G$ defined by $\varphi(g) = xg$. The map is injective since $xg = xg'$ implies $g = g'$ by cancellation. Since ${}_xG \subseteq G$ and we have an injection $G \mapsto_x G$, we know ${}_xG = G$ so by pigeonhole principle φ is bijective and there exists $h \in G$ such that $xh = x$. I claim such h is a right identity. Consider the map $G_x = \{gx | g \in G\}$ and the map $\psi : G \mapsto G_x$ defined by $\psi(g) = gx$. By the same argument as before, ψ is bijective and we know for every $g \in G$, there exists g' such that $g = g'x$. But then, $gh = g'xh = g'x = g$, so h is a right identity. Considering ${}_xG$ once again gives that $h = xg$ for some $g \in G$, but x is arbitrary so every $x \in G$ has a right inverse. By the lemma, G is a group as required. □

2 Exercises

These are exercises which will be worked through in the reading group. You are more than welcome to try them beforehand, but you are not expected to. Solutions will be posted after the reading section.

1. (a) Prove Lemma 1.1

   Proof. Assume G is a semigroup with a right identity e and every $x \in G$ has a right inverse y. I claim e is also a left inverse. Let $x \in G$ have a right inverse y and assume y has a right inverse z. Then, $e \cdot x = (xy)(xe) = xy(xy)z = xyz = x$ so e is also a left identity. Notice $yx = yxya = ya = e$ so y is also a left inverse. Hence, every x has an inverse. Thus, G is a group as required. □

   (b) Show that Lemma 1.1 is false if we assume G has a right identity and every x has left inverses by constructing a counterexample.

   A counterexample would be the following: Consider the set $A = \{0, 1\}$ and $G = \{f, g\}$ where the functions $f : A \mapsto A, g : A \mapsto A$ are defined as $f(x) = 0$ and $g(x) = 1$. Notice $g \circ g = g$ and $f \circ g = f$ so g is a right identity, and $g \circ f = g$ means f is a left inverse for f, so every element has a left inverse. However, G is not a group since f has no right inverse as $f \circ f = f$ and $f \circ g = f$.

2. (a) Show that the statement in the weekly problem is false if we assume only one of the cancellation laws hold.

   Take the same counterexample from 1b. Notice $f \circ g = f \neq g = g \circ g$ and $f \circ f = f \neq g = g \circ f$ so right cancellation holds. However, $g \circ f = g = g \circ g$ so left cancellation does not hold. G is not a group as explained in 1b.

   (b) Show that the statement in the weekly problem is false if we assume G is infinite by constructing a counterexample.

   Take $\mathbb{N}$ under $+$.

3. (a) Show that if G is a group, then for all a, b the equations $ax = b$ and $ya = b$ have unique solutions.

   Proof. Let G be a group, $a, b \in G$. Assume x, x' is such that $ax = b = ax'$. Multiplying a^{-1} to the left yields $x = a^{-1}b = x'$ so $x = x'$ and thus the equation $ax = b$ has a unique solution. The steps are the exact same for $ya = b$ but with right multiplication by a^{-1} instead. $\square$

   (b) Show that if G is a semigroup such that for all a, b the equations $ax = b$ and $ya = b$ have unique solutions, then G is a group.

   Proof. Let G be a semigroup such that for all $a, b \in G$, the equations $ax = b$ and $ya = b$ have unique solutions. Then, notice the equation $ax = a$ has a unique solution e. I claim e is a right identity. To this end we show e is the unique solution to the equation $gx = g$ for any $g \in G$. We first show that $e \cdot a = a$. Notice $aea = a(ea) = aa$ so ea is a solution to $ax = a^2$, but also notice $a \cdot a = a^2$ so a is also a solution to the equation $ax = a^2$ and thus $ea = a$ by uniqueness. Let $g \in G$ and assume h is a solution to the equation $gx = g$. Then, $gha = ga$ so ha is the unique solution to the equation $gx = ga$. Notice $g(ae) = ga$ so $a = ae = ha$ by uniqueness of the solution to $gx = ga$, but from before, e is the unique solution to $ya = a$ so $h = e$ and thus $h = e$, e is a right identity. Also note that for any $a \in G$, $ax = e$ has a unique solution so every a has a right inverse. By Lemma 1.1 G is a group as required.

   Alternatively, without invoking Lemma 1.1, one can also do a similar process and show that e is a left identity so e is an identity on G, and show that the unique solution to $ax = e$ is also a solution to $ya = e$. This shows every a has an inverse and thus G is a group. $\square$