Let G be a group s.t. $\forall g \in G, g^2 = e$. Show G is abelian.

Defn: A group $(G, \cdot)$ is a set G with a binary operation $\cdot$ s.t.

1. $\cdot$ is associative $(g_1 \cdot g_2) g_3 = g_1 (g_2 \cdot g_3)$
2. $\exists e \in G$ s.t. $g \cdot e = e \cdot g = g$ $\forall g \in G$
3. $\forall g \in G, \exists g^{-1}$ s.t. $g g^{-1} = g^{-1} g = e$

Defn: G is a group $H \subseteq G$ is a subgroup if H is a group under the same $\cdot$ as G.

Example: $O_n(\mathbb{R}) = \{ n \times n \text{ real matrices s.t. } A A^T = I \}$

$SO_n(\mathbb{R}) \subseteq O_n(\mathbb{R})$

$\{ \ldots \text{ in addition } \det A = 1 \}$

Q: Given $H \subseteq G$ how do we know if H is a subgroup?

Thm (Subgroup Test): $H \subseteq G$ is a subgroup iff $\forall g, h \in H$, $gh^{-1} \in H$, $H \neq \emptyset$.

Pf:

$\Rightarrow$ H is a subgroup. Then, let $g, h \in H$. Then $h^{-1} \in H$ and by closure, $gh^{-1} \in H$.

$\Leftarrow$ $H \subseteq G$ s.t. $\forall g, h \in H$, $gh^{-1} \in H$.

- let $g \in H$. Then $gg^{-1} \in H$, so $e \in H$.
- let $g \in H$. We know $e \in H$ so $e \cdot g^{-1} = g^{-1} \in H$.
- let $g, h \in H$. $h^{-1} \in H$, so $g(h^{-1})^{-1} = gh \in H$.

$\blacksquare$

S_3 = bijections $\{1, 2, 3\} \to$ itself

$\sigma_1 = \begin{matrix} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{matrix}$ $\sigma_4 = \begin{matrix} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{matrix}$ $\{\sigma_1, \sigma_2\}$

$\sigma_2 = \begin{matrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{matrix}$ $\sigma_5 = \begin{matrix} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{matrix}$

$\sigma_3 = \begin{matrix} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{matrix}$ $\sigma_6 = \begin{matrix} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{matrix}$ $\sigma_6 \{\sigma_1, \sigma_2\} = \{\sigma_6, \sigma_3\}$

$(23)(12)$
$(1\ 3\ 2)$

$\{\sigma_1, \sigma_2\} \sigma_6 = \{\sigma_6, \sigma_5\}$

$(12)(23)$
$= (1\ 2\ 3)$

Defn: $H \subseteq G$ be a subgroup. The number of cosets of H denoted $[G : H]$ is called the index of H.

Thm: G be a finite group. Then,

$\#G = \#H [G : H]$

Pf.

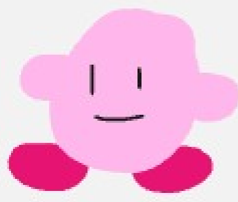
G is partitioned into equivalence classes under $g \sim g'$ iff $g(g')^{-1} \in H$. Equiv. classes are just cosets of H so $\# \bigcup$ cosets $\#G$. We know $\#gH = \#H$

$\Rightarrow \# \bigcup$ cosets $= \#H \cdot [G : H]$ $\blacksquare$

:)

Thm: $H, K \subseteq G$ be subgroups. $H \cap K$ is a subgroup.

Pf: Suffices to show $\forall h, k \in H \cap K$, $hk^{-1} \in H \cap K$. By Thm 1



Let $h, k \in H \cap K$

$k^{-1} \in H$, $k^{-1} \in K$

$\to k^{-1} \in H \cap K$

$hk^{-1} \in H$, $hk^{-1} \in K$

$hk^{-1} \in H \cap K$

$\blacksquare$

Defn: $H \subseteq G$ be a subgroup. Then let $g \in G$. We say $gH = \{ gh \mid h \in H \}$ is a left coset of G. $Hg = \{ hg \mid h \in H \}$ is a right coset of G.

$g \sim g'$ iff $g(g')^{-1} \in H$

Thm: $\#H = \#gH = \#Hg$.

Pf. We will only show $\#H = \#gH$. Let $f : H \to gH$ be defined as $f(h) = gh$.

- Injective: $gh = gh'$, then $g^{-1}gh = g^{-1}gh'$, $h = h'$ ✓
- Surjective: $x \in gH$, $x = gh$ for some $h \in H$. $f(h) = gh$ ✓.

$\blacksquare$

Defn: order of G = cardinality of G $\#G$ or $|G|$

order of $g \in G$ = smallest $n \in \mathbb{N}$ s.t. $g^n = \underbrace{g \cdot g \cdots g}_{n\text{-times}} = e$

denote $\#g$ or $|g|$

Thm You have the same number of left cosets as right cosets.

Pf

Let S be the set of left cosets of H, T be the set of right cosets of H.

$f : S \to T$ $f(gH) = f(Hg^{-1})$ is bijective. Check.

$\blacksquare$

Examples of Cosets:

$GL_n(\mathbb{R}) = \{ n \times n \text{ real invertible matrices} \}$

$SL_n(\mathbb{R}) = \{ n \times n \text{ real matrices s.t. determinant} = 1 \}$

Cosets are $\{ n \times n \text{ real matrices s.t. determinant} = c, c \in \mathbb{R} \}$

$C^\infty = \{ \text{real valued fns } f : \mathbb{R} \to \mathbb{R} \text{ s.t. continuously differentiable} \}$ under addition

$C = \{ f(x) = c, c \in \mathbb{R} \}$

Cosets of $C = f + C$, $f \in C^\infty$