Week 4 Jan 26

Exercise:

Let G be a group s.t. $\forall g \in G$, $g^2 = e$. Show G is abelian.

Last time:

Defn of monoid, group, submonoid, subgroup

Given $H \subseteq G$, how do we know H subgroup?

Thm 1. Subgroup test

G group, $H \leq G$ iff $gh^{-1} \in G \ \forall g, h \in G$

Pf:

$\Rightarrow$ easy
$\Leftarrow$ easy
} on notes

Some properties of subgroups:

$\underset{\text{subgroup}}{H, K} \subseteq G \implies H \cap K$ is a subgroup

Pf: Show by subgroup test.

Made with Goodnotes

Thm: $H \cup K$ subgroup iff $H \subseteq K$ or $H \supseteq K$

Pf: Exercise. $\blacksquare$

Def^n Cosets

$gH = \{gh \mid h \in H\}$ left coset, likewise for right coset

Thm: $\#H = \#gH = \#Hg$ $\forall g$

Pf: Suffice to show left since $\#H = \#Hg$ is the same.

Consider $f: H \to gH$ by $f(h) = gh$

$gh = gh' \Rightarrow h = h'$, f injective

$gh \in gH$, then $f(h) = gh$, f surjective. $\blacksquare$

Examples of cosets:

$GL_n(\mathbb{R})$ = set of $n \times n$ invertible real matrices

$SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$
A s.t. $\det A = 1$

Cosets: matrices with the same determinant

Set of continuously differentiable real valued fns C^∞

Subgroup: the constant functions
$C = \{f(x) = c,\ c \in \mathbb{R}\}$

Cosets: $f + C$, $f \in C^\infty$

Thm 4: same # of left and right cosets.

Pf: $S = \{gH\}$, $T = \{Hg\}$

Check $f: S \to T$ defined as
$f(gH) = Hg^{-1}$ is bijective. $\blacksquare$

Def^n $[G:H]$ is no. of cosets of H,

called index

Lagrange's Thm: $\#G = \#H[G:H]$

Pf: By thm 3, $\#gH = \#H$

Cosets form a partition on G,

union of distinct cosets $= G$ $\blacksquare$

Thm: $g(H \cap K) = gH \cap gK$

Pf: Show $\subseteq$ and $\supseteq$. $\blacksquare$

Weekly problem:

Consider $H \times K$ and $(h,k) \sim (h',k')$ iff $hk = h'k'$. Each equiv class has $\# H \cap K$.

$\# H \times K / \sim \; = \#H \#K / \# H \cap K$. $\blacksquare$