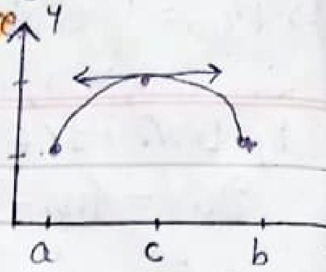
U1. Differential calculus :-

* 1> continuity – बिना हाथ उठाये draw किया हुआ curve.
 2> differentiablity – बिना corner वाला curve

1 Rolle's theorem –



1> f(x) is continuos on interval [a,b]
2> f(x) is differentiable on interval (a,b)
3> f(a) = f(b)

then By Rolle's theorem, this exist $c \in (a,b)$ such that
$f'(c) = 0$ (we have tangent parallel to x axis)

example –

$f(x) = \log(x^2+2) - \log(3)$: over $[-1,1]$

1> Conti. (log f^n defined all over $[-1,1]$ it is cont.)

2> diff. $f'(x) = \frac{2x}{x^2+2}$ is defined over $(-1,1)$ f(x) is diff

3> $f(-1) = f(1)$

$f(-1) = \log((-1)^2+2) - \log 3 = \log 3 - \log 3 = 0$

$f(1) = \log(1^2+2) - \log 3 = \log - \log 3 = 0$

$f(-1) = f(1)$

By Rolle's th^m $\exists c$ such that $f'(c) = 0$ –(1)

$f'(x) = \frac{2x}{x^2+2}$ $\therefore f'(c) = \frac{2c}{c^2+2} = 0$ – by (1) $\boxed{c \Rightarrow 0}$

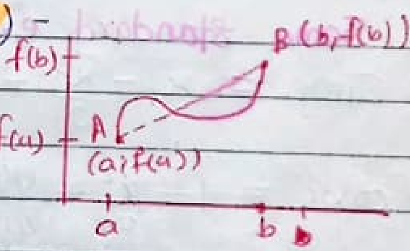
2. Lagrange's Mean value th^m (LMVT) –

Assumptions –

1> f(x) is continuous over [a,b]
2> f(x) is diffe. over (a,b)

Then By LMVT, $\exists c \in (a,b)$ such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$



$\text{slope} = \frac{f(b)-f(a)}{b-a}$

3. Cauchy's Mean value theorem (CMVT)

1> f(x), g(x) contin. over [a,b]
2> f(x), g(x) diff. over (a,b)
3> $g'(x) \neq 0 \ \forall x \in (a,b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

* Verify LMVT for f(x)

1) $\log x$ is conti over $[1, e]$

 $e = 2.718$

2) $f'(x) = \frac{1}{x}$ exists over $(1, e)$

$\therefore f(x) = \log x$ is diff over $(1, e)$

By LMVT, $\exists\, c \in (1, e)$ such that

$a = 1,\ b = e \quad f(x) = \log x \quad f'(x) = 1/x$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$1/c = \frac{\log e - \log 1}{e - 1} = \frac{1 - 0}{e - 1} \Rightarrow c = e - 1 = 2.718 - 1 = 1.7$$

* Imp -

Rolle thm is special case of LMVT &

LMVT is special case of CMVT

CMVT can't be deduced from LMVT

Rolles thm is deduced from CMVT

Mean value thm is also known as Rolls thm

4) Maclaurin's Series expansions (Maclaurin's thm)

let $f(x)$ be differentiable infinitely or any n of time

then $f(x) = f(0)x^0 + \frac{f'(0)x^1}{1!} + \frac{f''(0)x^2}{2!} + \dots + \frac{f^n(0)x^n}{n!} + \dots$

Some standard expansions -

1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$

2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$

3) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$

4) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

5) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$

6) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$

7) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

8) $\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$

derivative

[illegible] log function में कोई factorial नहीं है

9) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$

10) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$

11) $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \dots$

12) $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$

13) $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

example -

First two terms in expansion of $\tan^{-1}(1+x)$ by Mac thm

$\Rightarrow f(x) = \tan^{-1}(1+x) \qquad f'(x) = \frac{1}{1+(1+x)^2}$

$f(0) = \tan^{-1}(1+0) = \pi/4$

$f'(0) = \frac{1}{1+(1+0)^2} = \frac{1}{2}$

$f(x) = f(0) + x f'(0) + \dots = \pi/4 + \frac{1}{2}x + \dots$

5) Taylor's series expansions

Consider $f(x)$ is infinitely differentiable -

consider x & h are 2 variable $\Rightarrow$ ~~In power of x~~

In power of x

$f(x+h) = f(h)x^0 + \frac{f'(h)x^1}{1!} + \frac{f''(h)x^2}{2!} + \dots$

In power of h

$f(x+h) = f(x)h^0 + \frac{f'(x)h^1}{1!} + \frac{f''(x)h^2}{2!} + \dots$

* If $\boxed{h=0}$ it becomes Maclaurins series expansion

$f(x+h) = f(x) + \frac{f'(h)x^1}{1!} + \frac{f''(h)x^2}{2!} + \dots$

$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots$

Q expansion of $f(x)$ in ascending power of $(x-a)$ by Taylor's theorem is
$\Rightarrow$ $f(x)$ in powers of $(x-a)$
we have $f(x+h) = f(h) + \frac{f'(h)}{1!}x + \frac{f''(h)}{2!}x^2$
replace x by $x-a$ & h by a we have $f(x)$

$$f(x) = f(x-a+a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

Q first two terms in expansion of $\log\cos(x+\pi/4)$ by Taylor's thm in ascending power of x is
$\Rightarrow$ $f(x+\pi/4) = \log(\cos(x+\pi/4))$

$$f(x+h) = f(h) + \frac{f'(h)}{1!}x + \frac{f''(h)}{2!}x^2 + \dots$$

$$f(x+\pi/4) = f(\pi/4) + \frac{f'(\pi/4)}{1!}x + \frac{f''(\pi/4)}{2!}x^2 + \dots$$

$$f(\pi/4) = \log\cos\pi/4 = \log 1/\sqrt{2} = \log 2^{-1/2} = -\frac{1}{2}\log 2$$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x \cdot f'(\pi/4) = -\tan\pi/4 = -1$$

$$f(x+\pi/4) = -\frac{1}{2}\log 2 - x +$$

$$= \log 1/\sqrt{2} - x + \dots$$

6) Indeterminate forms

There are 7 indeterminate forms. (~~but in my~~
1) $\frac{0}{0}$ 2) $\frac{\infty}{\infty}$ 3) $0\times\infty$ 4) $\infty-\infty$ 5) 0^0 6) ∞^0 7) 1^∞

L'Hospital's Rule - If we have $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form then we just take derivative of numerator & dinominator separately
Note Please dont evaluate the derivative of $\frac{f(x)}{g(x)}$ by $\frac{u}{v}$ rule

ex- $\lim_{x\to 0} \frac{1-\cos x}{x} \Rightarrow \frac{0}{0}$

* forms 0^0 ∞^∞ 1^∞ take log for problem solving

0^0 $\log 0^0 = 0\times\log 0 = 0\times\infty$

∞^0 $\log\infty^0 = 0\times\log\infty = 0\times\infty$

1^∞ $\log 1^\infty = \infty\log 1 = \infty\times 0$

Main fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

...ourier Series $\Rightarrow f(x) = \frac{a_0}{2} + a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$

...on — $2\pi \Rightarrow$ sin x, cos x, sec x and cosec x

— $\pi \Rightarrow$ tan x and cot x

...riod of function.

$\frac{1}{n} = \frac{2\pi}{2} = \pi$ (T = Time period, n = coeffi of x)

...iod

Imp

2π

(0, 2π)

1) $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

2) $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

3) $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

(−π, π)

f(x) ⇒ even

$\boxed{b_n = 0}$

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

f(x) ⇒ odd

$\boxed{a_0 = a_n = 0}$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

$\neq 2\pi \Rightarrow (c, c+2l)$

$a_0 = \frac{1}{2l}\int_c^{c+2l} f(x)\,dx$

$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

(3) Dirichlet's conditions —

1) f(x) ⇒ periodic : $c \le x \le c+2\pi$
2) f(x) should have finite n. of finite discontinuous
3) f(x) should have finite n. of maximas & mini.

Imp points.

⇒ 1) fourier series representation of periodic fun f(x) with period 2π which satisfies Dirichlets conditions

→ $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\pi x + b_n \sin n\pi x)$

2) fourier series of an odd periodic fun. contains only sin terms & other (odd harmonic, even harmonic, cosine ⇒ even)

* Imp

If all satisfy Dirichlets condition then

1) $f(x) = -f(-x) \Rightarrow \boxed{\text{odd fun}} \Rightarrow$ only have sine terms

2) $\psi(x) = \underset{(a_0)}{f(x)} - \underset{(\text{odd})}{f(-x)} \Rightarrow$ odd fun ⇒ sine terms & constant terms

3) $\psi(x) = \underset{(a_0)}{f(x)} + \underset{\text{even}}{f(-x)} \Rightarrow$ cos terms & constant

Some combination -

$f \Rightarrow$ even & $g \to$ even $\Rightarrow f \cdot g \Rightarrow$ even

$f \Rightarrow$ even & $g \Rightarrow$ odd $\Rightarrow f \cdot g \Rightarrow$ odd

$f \Rightarrow$ odd & $g \Rightarrow$ even $\Rightarrow f \cdot g \Rightarrow$ odd

$f \Rightarrow$ odd & $g \Rightarrow$ odd $\Rightarrow f \cdot g \Rightarrow$ even

Q for even fun $f(x)$. define interval - $-\pi \le x \le \pi$ & $f(x+2\pi) = f(x)$ Fourier series is

$\Rightarrow f(x+2\pi) = f(x) \Rightarrow$ even $(b_n = 0)$

$-\pi \le x \le \pi \Rightarrow c < x < c+2\pi$ $\quad c = -\pi$

$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$f(x) = a_0 + a_n$

★ **Leibnitz's Rule — Generalise form rule**

($'$ $\Rightarrow$ derivative) (suffix, 1, 2 $\Rightarrow$ integ)

$\int uv = uv_1 - u'v_2 + u''v_3$

★ **half range series — (0, L)**

1) cosine series $\Rightarrow$ consider $f(x)$ = even $\Rightarrow \boxed{b_n = 0}$

2) sine series $\Rightarrow$ consider $f(x)$ = odd $\Rightarrow \boxed{a_0 = a_n = 0}$

$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

★ **harmonic Analysis -**

1) $f(x) = \frac{a_0}{2} \sum_{n=1} \left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$

$= \frac{a_0}{2} + \underbrace{a_1\cos\frac{\pi x}{L} + b_1\sin\frac{\pi x}{L}}_{\text{first harmonic}}, + \underbrace{a_2\cos\frac{2\pi x}{L} + b_2\sin\frac{2\pi x}{L}}_{\text{second harmonic}},$

$a_0 = \frac{2}{N}\sum y$, $a_n = \frac{2}{N}\sum y\cos nx$,

$b_n = \frac{2}{N}\sum y\sin nx$

✓ Amplitude of first harmonic $A_1 = \sqrt{a_1^2 + b_1^2}$

Q for the certain data $a_0 = 15, a_1 = 0.373, b_1 = 1.004$ then amplitude of 1st harmonic is

$\Rightarrow$

$A = \sqrt{a_1^2 + b_1^2} = \sqrt{(0.373)^2 + (1.004)^2} = \sqrt{1.147}$

$= 1.07$

U3 - Partial Differentiation

dependent variable — $y = f(x)$ — independent variable

- Unimaginable things

If we go beyond 3 variable no matter dep or indep geometry becomes unimaginable

Notations -- $y = f(x)$ $\quad f'(x) = \frac{dy}{dx}$ $\quad z = f(x,y)$

$z = f(x,y)$. $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$ $\left(\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x \right.$

$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$

higher order partial derivative & mixed partial derivative

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ mixed $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$z_{xyy} = \frac{d}{dx}\left(\frac{d}{dy}\left(\frac{dz}{dy}\right)\right)$

$z = f(x,y)$ $\quad \frac{\partial^2 z}{\partial y^2} = \frac{d^2 f}{dy^2} = z_{yy} = f_{yy}$

mixed partial diff.

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = z_{xy} = f_{xy} = \frac{\partial}{\partial x}(z_y)$

2) Homogeneous function & Degree of homogeneity

let $z = f(x,y)$ is said to be homogeneous function if $f(tx,ty) = t^n f(x,y)$ & n is known as degree of homogeneity of function

ex- $f(x,y) = x^2+y^2$

$f(tx,ty) = (tx)^2 + (ty)^2 = t^2x^2 + t^2y^2 = t^2(x^2+y^2)$

$= t^2 f(x,y)$.

$\therefore f(tx,ty) = t^2 f(x,y)$

$\therefore f(x,y)$ is homo. fun.

& deg of homogeneity $\underline{\underline{2}}$.

* Variable to be treated as constant

$\left(\frac{dz}{dx}\right)_y \rightarrow$ express z in terms of x & y

differentiate z w.r.t x treating y as constant

$x^2 = au+bv$ $\quad y = au-bv$ $\quad \left(\frac{du}{dx}\right)_y \cdot \left(\frac{dx}{du}\right)_y \cdot \left(\frac{dx}{dy}\right)_x \left(\frac{dy}{dx}\right)_x$

$x^2 = au+bv$ — (1) $\quad$ x u v

$+\ y = au-bv$ — (2) $\quad$ y u v $\quad$ uny chahiye

$\left(\frac{du}{dx}\right)_y \quad u = (x, y)$

निकाल पायेगा

★ ~~Homogeneous functions~~ ⇒ $f(tx, ty) = t^n f(x,y)$

~~$f(x,y) = x^2 + y^2 + 2x'y'$~~ (degree of function is same)

★ **Euler's thm** - होना चाहिए

If $z = f(x,y)$ is homogeneous function in x & y of degree 'n' then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

Deduction from Eulers thm.

(1) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

(2) If $u = f(x,y)$ is not homogeneous fun then but $f(u)$ is homogeneous fun in x & y of degree n

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

ex. $U = \sin^{-1}(x^2+y^2) \Rightarrow \sin^{-1}(\lambda^2(x^2+y^2))$ ⇒ Not homo.

$f(u) = \sin u = x^2+y^2$ → homo in x & y deg 2

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u} = 2\tan u$

(3) If $u = f(x,y)$ is not homogeneous but $f(u)$ is homogeneous in x & y of degree n $\left(g(u) = \frac{n f(u)}{f'(u)}\right)$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

★ **Composite function**

$u \to x, y \to r, \theta$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

★ **Differentiation of Implicit function**

If $f(x,y) = 0$ is implicit function then $\frac{dy}{dx} = \frac{-p}{q}$ $\left(p = \frac{\partial f}{\partial x} \quad q = \frac{\partial f}{\partial y}\right.$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{q^3}\begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix}$$

$p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Total derivative -

Q) If $ax^2+by^2+cz^2 = 1$ & $lx+my+nz = 0$ PT

$$\frac{dx}{bny-cmz} = \frac{dy}{clz-anx} = \frac{dz}{amx-bly}$$

⇒

let $f_1: ax^2+by^2+cz^2-1 = 0$

$f_2: lx+my+nz = 0$

$df_1 = \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz$

$df_2 = \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz$

$0 = (2ax)dx + (2by)dy + (2cz)dz$ — (1)

$0 = (l)dx + (m)dy + (n)dz$ — (2)

By applying crammer Rule (1) & (2)

$$\frac{dx}{\begin{vmatrix} 2by & 2cz \\ m & n \end{vmatrix}} = \frac{-dy}{\begin{vmatrix} 2ax & 2cz \\ l & n \end{vmatrix}} = \frac{dz}{\begin{vmatrix} 2ax & 2by \\ l & m \end{vmatrix}}$$

Unit-4 · Application of Partial Differentiation
(Jacobian)

* If u & v are function of two independent variables x & y then determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u v w.r.t. x, y

$J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Similarly jacobian of uvw wrt. x, y z

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

Properties -

① $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$ $(J \cdot J' = 1)$

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

② If u, v are functionally related to x, y then $\frac{\partial(u,v)}{\partial(x,y)} = 0$

If u, v, w are functionally related then x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Q $U = x^2+y^2$, $v = xy$, $J = ?$

$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2$

Jacobian of Implicit function

If $f_1 : (u,v,x,y) = 0$ & $f_2 : (u,v,x,y) = 0$ are two implicit fun

1) $\dfrac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \dfrac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)/\partial(u,v)}$

2) $\dfrac{\partial(x,y)}{\partial(u,v)} = (-1)^2 \dfrac{\partial(f_1,f_2)/\partial(u,v)}{\partial(f_1,f_2)/\partial(x,y)}$

3) $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \dfrac{\partial(f_1,f_2,f_3)}{\partial(f_1,f_2,f_3)} \Big/ \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Q. If $u^3+v^3 = x+y$; $u^2+v^2 = x^3+y^3$ then

PT $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}\dfrac{y^2-x^2}{uv(u-v)}$

$f_1 \quad u^3+v^3-x-y=0$

$f_2 \quad u^2+v^2-x^3-y^3=0$

$\dfrac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \dfrac{\partial(f_1,f_2)/\partial(x,y)}{\partial(f_1,f_2)\ \partial(u,v)} = (1)\dfrac{N}{D}$

$N = \dfrac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} = 3y^2-3x^2 = 3(y^2-x^2)$

$D = \dfrac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} f_{1u} & f_{1v} \\ f_{2u} & f_{2v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6uv(u-v)$

$\dfrac{\partial(u,v)}{\partial(x,y)} = (1)\dfrac{3(y^2-x^2)}{2 \cdot 3uv(u-v)} = \dfrac{1}{2}\dfrac{y^2-x^2}{uv(u-v)}$

Functional Dependance

If $u = f_1(x,y)$ & $v = f_2(x,y)$ then u & v are functionally dependent if $\boxed{\dfrac{\partial(u,v)}{\partial(x,y)} = 0}$

Partial derivation of Implicit fun using Jacobians

If $f_1(u,v,x,y) = 0$ & $f_2(u,v,x,y) = 0$ are imp functions

$\dfrac{\partial u}{\partial x} = (-1)\dfrac{\partial(f_1,f_2)}{\partial(f_1,f_2)} \Big/ \dfrac{\partial(x,v)}{\partial(u,v)}$

$\dfrac{\partial v}{\partial y} = -\dfrac{\partial(f_1,f_2)/\partial(u,y)}{\partial(f_1,f_2)/\partial(u,v)}$

$\dfrac{\partial x}{\partial u} = -\dfrac{\partial(f_1,f_2)}{\partial(f_1,f_2)} \Big/ \dfrac{\partial(u,y)}{\partial(x,y)}$

$\dfrac{\partial v}{\partial x} = -\dfrac{\partial(f_1,f_2)/\partial(u,x)}{\partial(f_1,f_2)/\partial(u,v)}$

Q.) If $u^2+xv^2-uxy=0$, $v^2-xy^2+2uv+u^2=0$ find $\dfrac{\partial u}{\partial x}$

$\Rightarrow f_1 \quad u^2+xv^2-uxy=0$, $f_2 \quad v^2-xy^2+2uv+u^2=0$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial(f_1,f_2)}{\partial(f_1,f_2)} \Big/ \dfrac{\partial(x,v)}{\partial(u,v)} = \dfrac{-N}{D}$

$N = \dfrac{\partial(f_1,f_2)}{\partial(x,v)} = \begin{vmatrix} f_{1x} & f_{1v} \\ f_{2x} & f_{2v} \end{vmatrix} = \begin{vmatrix} v^2-uy & 2xv \\ -y^2 & 2v+2u \end{vmatrix} = (v^2-uy)(2v+2u) + 2xy^2v$

$D = \dfrac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} f_{1u} & f_{1v} \\ f_{2u} & f_{2v} \end{vmatrix} = \begin{vmatrix} 2u-xy & 2xv \\ 2v+2u & 2v+2u \end{vmatrix}$

$= (2u-2xy)(2v-2u) - (2v+u)(2v\,u)$

$\dfrac{\partial u}{\partial x} = \dfrac{-(v^2-uy)(2v+2u)+2xy^2v}{2(u+v)(2u-xy-2vxy)}$

Errors & Approximations

If $u = f(x,y)$

$\delta u = \dfrac{\partial u}{\partial x}\cdot \delta x + \dfrac{\partial u}{\partial y}\cdot \delta y$

where $\delta u, \delta x, \delta y$ are errors in u, x, y respectively

$\dfrac{\delta u}{u}, \dfrac{\delta x}{x}, \dfrac{\delta y}{y}$ are relative error in u, x, y

$\dfrac{\delta u}{u} \times 100$, $\dfrac{\delta y}{y} \times 100$, $\dfrac{\delta z}{z} \times 100$ are percentage error in u, x, y.

Maxima & minima $z = f(x,y)$

Step 1 find partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

Consider $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

Solve this 2 eq. for x, y.

& let (a_1, b_1) & (a_2, b_2) are roots after solving

Step-2 Calculate

$r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

& substitute $(x,y) = (a, b_1)$: $(x, y) = (a_2, b_2)$

Step 3.		Remark	max, min value
$(rt-s^2)_{(a,b_1)}$	$r_{(a,b_1)}$		
1) $(rt-s^2) > 0$	$r < 0$	$f(x,y)$ has max. value	$f(a,b_1) = f_{max}$
2) $(rt-s^2) > 0$	$r > 0$.	$f(x,y)$ has min. value	$f(a,b_1) = f_{min}$
3) $(rt-s^2) < 0$	-	f has neither max nor min	(a,b_1) is saddle point
4) $(rt-s^2) = 0$	-	No conclusion	further tests. is ness.

Langrange methode to determine multiplier

$F = u + \lambda \phi$ $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$

from this equation calculate x, y, z)

Step. use ~~eom~~ $F = u + \lambda \phi$.

step-2 use $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial z} = 0$

then compare.

Unit-5 {Matrix}

1) Upper triangular matrix
(all element below the principal diagonal are zero. ex $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

2) lower triangular matrix
all the element above principal diagonal are zero. ex $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

3) Diagonal matrix.
A square matrix in which the entries outside the principal diagonal are all zero ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4) Scaler matrix- A diagonal matrix in which all diagonal elements are equal. ex $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

5) Symmetric matrix - A Square matrix $A = A^T$

6) Skew symmetric matrix
$A = -A^T$

7) Transpose matrix
$R \longleftrightarrow C$

8) Idempotent - A square matrix A is said to be if
$A^2 = A$

9) Involuntary matrix - a square matrix A is called
$A^2 = I$

10) Nilpotent matrix -
A square matrix A is called nilpotent matrix if ∃ +ve integer m such that $A^m = 0$

Row echelon form of a matrix

Condition

① leading element se pahle jitne bhi zero hai usse jada se usse next wale row me leading element se pahle hone chahiye ex.

(हर रो का leading element 1 होना चाहिए. $\begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(& zero element row हमेशा last मे होनी चाहिए.

Reduced row echelon form

1) वो Row echelon form होना चाहिए.

2) जो leading element होते है उसके ऊपर & निचे zero होने चाहिए.

ex: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

* Submatrix - A matrix B which is obtained from matrix A by deleting any rows or/columns from the original matrix A is known as submatrix of A

* minor of matrix - It is the determinant of a square submatrix of matrix

Rank of Matrix - $\rho(A)$

1) The rank of matrix A is the number of non-zero rows in row echelon form of matrix A
2) If $A = (a_{ij})_{m \times n}$ then $\rho(A) \le m/n$.
3) The rank of zero (null) matrix is 0
4) The rank of identity matrix I_n is n

ex. find rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ -3 & 1 & 5 & 2 \\ -2 & 3 & 9 & 2 \end{bmatrix}$$

make in row echelon form

Apply $R_2 + 3R_1$, $R_3 + 2R_1$

Apply $R_3 - R_2$

Apply $R_2 \times (1/7)$

$$A \sim \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 17/7 & 2/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

m = Total n. of rows

K = Total n. of row which contain all zero elements

$\rho(A) = m - K$

$= 3 - 1 = 2$

Normal form of matrix

normal form

$[I_r]$ $[I_r, 0]$ $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Step-1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

① $a_{11} = 1$ by using elem trans.

② Get zero below $a_{11} = 1$ using "

$\begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

③ Get zero $\begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$

④ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & * & * \end{bmatrix}$ → P_1

Inverse of matrix By using adjoint methode

① A → square matrix

$|A| \ne 0$ then A^{-1} exist

$A^{-1} = \frac{1}{|A|} \text{adj} A$

⇒ step -

① find $|A|$

② AT & C.F. with sign

Adj A = put

then $A^{-1} = \frac{1}{|A|} \text{adj} A$

S of Li. Alg. equation

Augmented matrix (A, B)

$$(A,B) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & & a_{2n} & : & b_2 \\ a_{m1} & a_{m2} & \cdots & a_{mn} & : & b_m \end{bmatrix}$$

Non-homogeneous equation

$AX = B$.

Condition for consistency of Non-Homo equation

m = total n. of equation

n = total n. of unknowns

Case① $m \ne n$

* $\rho(A) = \rho(A,B)$

⇒ consistent

* $\rho(A) \ne \rho(A,B)$

⇒ inconsistent & no solution

* Unique solution

$\rho(A) = \rho(A,B) = n$

$\rho(A) = \rho(A,B) <$

Case-2 $m = n > 3$

① $\rho(A) = \rho(A,B) = n$

⇒ consistent

② $\rho(A) \ne \rho(A,B)$ ⇒ inconsistent

$\rho(A) = \rho(A,B) = r < n$.

⇒ consistent & ∞ n. of sol.

Case-3. $m = n = 3$

① $|A| \ne 0$ ⇒ consistent then A^{-1} exist

② $|A| = 0$ then inconsistent

$\rho(A) = \rho(A,B) < n$ ⇒ sys. infinite n. of solution

$\rho(A) \ne \rho(A,B)$ ⇒ inconsistent & no solution

Homogeneous equation

$AX = 0$.

m = Total n. of equation

n = total n. of unknowns

Case ① $m \ne n$, $m = n > 3$

Unique solution

$\rho(A) = \rho(A,2)$

$\rho(A) = \rho(A,2) = r < n$

infinite n. of sol or non-trivial sol.

Case 2 $m = n = 3$

$X = A^{-1} Z$ ⇒ trivial sol

$|A| = 0$ then

$\rho(A) = \rho(A,2)$ ⇒

infinite n. of sol. (non-tri)

* linearly dependent

$C_1 x_1 + C_2 x_2 + \cdots C_n x_n = 0$

Orthogonal transformation

* Orthogonal matrix

$AA^T = I$

* If A is orthogonal matrix then $|A| = \pm 1$

determinant

* If A is an orthogonal matrix then

$A^{-1} = A^T$

Unit 6

Eigen values & vector, Diagonalization

1) Eigen value - Any non-zero vector x is said to be a characteristics vector (or eigen vector) of matrix A, if there exist a number λ such that
$Ax = \lambda x$
Also then λ is said to be a characteristics root or eigen value of the matrix A corresponding to the characteristics vector x.

Properties of Eigen values

1) Trace of A - The sum of entries on the main diagonal of or n×n matrix A is called " " .
Trace of A = $a_{11} + a_{22} + a_{33} \cdots + a_{nn}$

2) The sum of the eigen value of matrix is the sum of the element of the princip of diagonal Trace of A
Trace of A = $a_{11} + a_{22} + \cdots a_{nn} = \lambda_1 + \lambda_2 + \cdots \lambda_n$

3)* The eigen values of an upper or lower triangular matrix are the element on its main diagonal ex $\begin{bmatrix} 4 & 5 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow \lambda = 4, 2, 1$

4) The product of the eigen value of matrix equals the determinant of matrix
$\therefore \lambda_1 \times \lambda_2 \times \lambda_3 \times \cdots \lambda_n = |A|$

5) If $\lambda_1 \cdots \lambda_2 \cdot \lambda_n$ are eigen values of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \cdots \frac{1}{\lambda_n}$ are eigen values of A^{-1}

6) The matrix KA has the eigen value $K\lambda_1, K\lambda_2, K\lambda_n$

7) The matrix A^m (m = non-ve integer) has eigen value $\lambda_1^m \cdots \lambda_n$

8) Spectral shift - The matrix (A−KI) has the eigen value $\lambda_1 - k, \lambda_2 - k, \cdots \lambda_n - k$.

9) The eigen value of a symmetric matrix are real

** 10) The eigen value of A & A^T are same.

11) The inverse A^{-1} exist iff $\lambda_j \neq 0$, $j = 1, 2 \cdots n$

Properties of eigen vector

1) Orthogonal eigen vectors – Two eigen vector x_1 & x_2 are said to be orthogonal if $X_1 X_2 = 0$

ex $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $X_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ $X_1 X_2 = [1\,2\,3]\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 0$

Then $X_1\, X_2 \Rightarrow$ orthogonal

* for square matrix of order 2×2 $(\lambda_1 + \lambda_2 = a_{11} + a_{22})$ or $(\lambda_1 \cdot \lambda_2 = |A|)$

(1) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A|$ निकाल के रखो

(2) $\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = 0$ then λ निकल आयेगा

or direct

(3) (characteristic equation) $\lambda^2 - S_1\lambda + |A| = 0$ $S_1 = a_{11} + a_{22}$

* for square matrix of order 3×3

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ पहले $|A|$ = निकाल के रखो

$(\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33})$
$\& \ \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$

(3) characteristic equation

$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0}$

$S_1 = a_{11} + a_{22} + a_{33}$

$S_2 =$ minor of a_{11} + m. of a_{22} + m of a_{33}

(2) find eigen value & vector

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$ $|A| = 6$

$S_1 = 1 - 2 - 3 = -4$

$S_2 = 6 - 3 - 2 = 1$

characteristic eq

$\lambda^3 + 4\lambda^2 + \lambda - 6 = 0$

Roots are $(1, -2, -3)$

$\lambda = 1, -2, -3$ are eigen val.

let $x = [x_1\ x_2\ x_3]$ vector for eigen value λ.

use $[A - \lambda I]X = 0$

$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & -2-\lambda & 6 \\ 0 & 0 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(1) put $\lambda = 1$

$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0(x_1) + 2(x_2) + 3(x_3) = 0$

$2x_2 + 3x_3 = 0$ —(1)

$-3x_2 + 6x_3 = 0$ —(2)

$-4x_3 = 0 \Rightarrow x_3 = 0$

$x_2 = 0$

or use cramer rule

$a_1 x_1 + b_1 y + c_1 z = 0$
$a_2 x_2 + b_2 x_2 + c_2 z = 0$

$\dfrac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \dfrac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \dfrac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

* find the spectrum of a matrix

$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ find (1) $A = 6$

$S_1 = 4 + 1 + 1 = 6$

$S_2 = \cancel{6+6=12}$, $1 + 6 + 4 = 11$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\Rightarrow \lambda = 1, 3, 2$ are eigen values

spectrum of A is $(1, 3, 2)$ (used calc)

* Caley – hamilton thm –

* find characteristic equation of A

for $A_{2\times2} \Rightarrow \lambda^2 - S_1\lambda + |A| = 0$

$A_{3\times3} \Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

By Cayley – Hamilton thm

$A^2 - S_1 A + |A| = 0$

$A^3 - S_1 A^2 + S_2 A - |A| = 0$

* for $A_{2\times2}$ to find A^3 we multiply eq (1) by A

$A^2 = S_1 A - |A|$

$A^3 = S_1 A^2 - A(|A|)$

for $A_{3\times3}$ find A^4

$A^4 = S_1 A^3 - S_2 A^2 + A(|A|)$

Diagonalization of a matrix.

If a square matrix A of order n has linearly independent eigen vector then there exist a non-singular matrix P

$D = P^{-1}AP$ P = modal matrix & diagonalize matrix A

Spectral matrix

Power of matrix (A^n)

$A^n = PD^nP^{-1}$

Quadratic form $Q(x) = x^TAx$

Quadratic form for 2×2

ex - $x_1^2 - 6x_1x_2 - 3x_2^2$.

(दो variable है। then matrix 2×2 है।

$\Rightarrow$

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \frac{1}{2} \text{ of coeff of } x_1x_2 \\ \frac{1}{2} \text{ of coeff of } x_1x_2 & \text{coeff of } x_2^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & -3 \end{bmatrix}$$

Q find the matrix of quadratic form (3 variable then 3×3

$3x^2 - 8xy + y^2 - 16xz + 16yz + 5z^2$

$$A = \begin{bmatrix} \text{coeff of } x^2 & \frac{1}{2} \text{ of coeff of } xy & \frac{1}{2} \text{ of coeff of } xz \\ \frac{1}{2} \text{ of coeff of } xy & \text{coeff of } y^2 & \frac{1}{2} \text{ of coeff of } yz \\ \frac{1}{2} \text{ of coeff of } xz & \frac{1}{2} \text{ of coeff of } yz & \text{coeff of } z^2 \end{bmatrix}$$

$\Rightarrow$

$$A = \begin{bmatrix} 3 & -4 & -8 \\ -4 & 1. & 8 \\ -8 & 8 & 5 \end{bmatrix}$$