THE UNIVERSITY of ADELAIDE

Financial Modelling: Tools and Techniques

MATHS 3012 MATHS 4112
MATHS 7070 CORPFIN 7023

School of Mathematical Sciences
The University of Adelaide

July 13, 2021

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Contents

Preface

In this course we study *derivative* products (e.g., options, forward contracts and futures contracts) written on a variety of assets (e.g., stocks, interest rates, foreign currencies and bonds), known as the *underlying* asset. A derivative is a financial contract which derives its value from its underlying asset. We present general principles for calculating *rational* prices of derivatives over time. To calculate the derivative price, we use predictions about how the underlying asset price will vary in the future. Knowing how derivative prices are expected to vary over time allows traders in financial markets to contain losses and maximise profits.

The Greek alphabet

In this course we use a number of Greek letters. Below is a list of all upper and lower case Greek letters and their pronunciations.

A	α	alpha	B	β	beta	Γ	γ	gamma
Δ	δ	delta	E	ϵ	epsilon	Z	ζ	zeta
H	η	eta	Θ	θ	theta	I	ι	iota
K	κ	kappa	Λ	λ	lambda	M	μ	mu
N	ν	nu	Ξ	ξ	xi	O	o	omicron
Π	π	pi	P	ρ	rho	Σ	σ	sigma
T	τ	tau	Υ	υ	upsilon	Φ	ϕ	phi
X	χ	chi	Ψ	ψ	psi	Ω	ω	omega

1 One-step binomial asset pricing model

Contents

The one-step binomial asset pricing model is a mathematical model for determining values of a derivative (such as an option). Before we discuss this model we introduce *options*, a common derivative product.

1.1 Call options

Options are a type of contract concerning the buying or selling of an asset at some future time. The asset is called the *underlying* and examples of underlying assets include stocks, bonds and commodities. Buyers of options are called *holders* and sellers of options are called *writers*. To acquire an option and become a holder, a trader must pay the option writer a fee or *premium*.

Definition 1.1. A *call option* gives the holder *the right to buy* the underlying asset at a set price (the *strike price*) at a future time. However, the holder is *not obliged to buy* the underlying asset. If the holder does choose to buy the asset (that is, *exercise* the option) at the strike price, then the writer must sell.

Buying a call option is commonly described as 'taking a call' or 'going long'. A holder of a call option is 'long in the call' or has taken a 'long position'.

A call option contract contains:

- the expiry date (also called the maturity date);
- the strike price (also called the exercise price);
- the underlying (the product for sale and how many are to be sold, e.g., 100 shares from company XYZ);
- the option style (e.g., European, American, Asian);
- the premium (also called the option price or call price).

If the holder chooses not exercise the option prior to expiry, then the option expires and it is worthless.

Example 1.1. The Australian Securities Exchange (ASX, `https://www2.asx.com.au/`) lists current options. Find some options listed on the ASX and identify their expiry date, strike price, style, and underlying asset.

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For options written on shares (that is, the underlying is shares) and traded on the ASX, expiry dates are typically on the Thursday prior to the last working Friday of a month. Options expire on their expiry date at 5 pm Eastern Standard Time.

Definition 1.2. The holder of an *American option* has the right (but not the obligation) to exercise the option (that is, buy the underlying) on any date up to and including the expiry date.

Definition 1.3. The holder of a *European option* has the right (but not the obligation) to exercise the option (that is, buy the underlying) only on the expiry date.

The holder of an American call option can, prior to expiry,

(a) sell the call to someone else;

(b) exercise the call option;

(c) do nothing.

For the holder of a European call option, only (a) and (c) are possible prior to the expiry date.

An important feature of options is that there is no limit to a holder's potential profit, but the holder's potential loss is limited.

1.1.1 European call options

Example 1.2. Shares of company XYZ are currently trading at \$30. A trader buys a European call option written by XYZ on 100 XYZ shares for a strike price of \$30 per share. The expiry date is in three months and the premium of the option is \$2 per share. What is the trader's payoff at expiry if the share price stays at \$30? What is the trader's payoff at expiry if the share price increases to \$35? What is the trader's payoff at expiry if the share price increases to \$100? What is the trader's payoff at expiry if the share price drops below \$30? Plot the trader's payoff at expiry per share against share price. Also plot the trader's total profit per share against share price.

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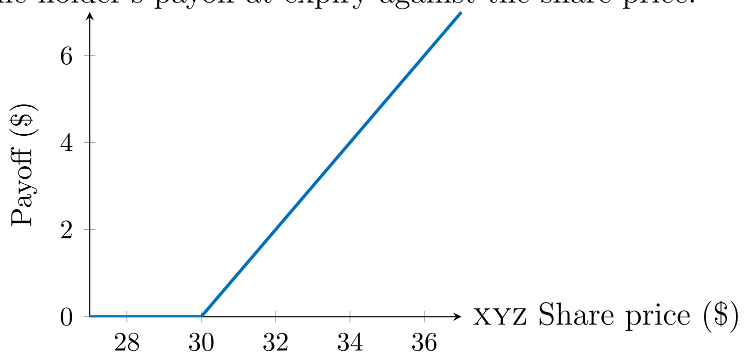
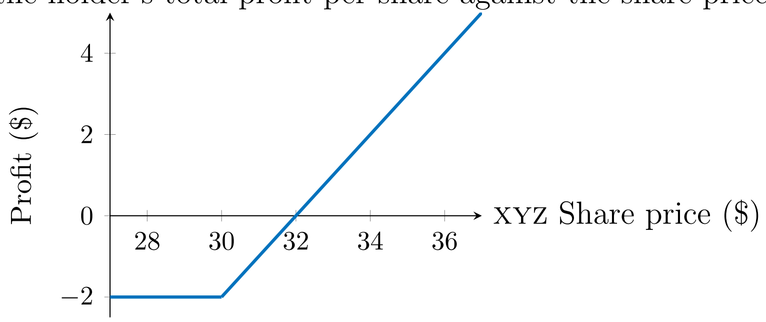
Plot the holder's payoff at expiry against the share price.



Plot the holder's total profit per share against the share price.



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Define $t = 0$ as the time the option is purchased and $t = T$ as the time at expiry. Define $S(t)$ as the underlying price (e.g., share price) at time t where $0 \le t \le T$. The underlying price at the time the option is purchased is $S(0)$ and at expiry it is $S(T)$. Define K as the strike price of the underlying. At expiry there are two cases to consider.

1. If the underlying price at expiry is equal to or less than the strike price, $S(T) \le K$, then the holder should not exercise the option. In this case the option has no value and is 'out of the money' (or, if $S(T) = K$ it is 'at the money').

2. If the underlying price at expiry is greater than the strike price, $S(T) > K$, then the holder should exercise the option. The payoff at expiry is $S(T) - K$ and the option is 'in the money'.

Define $C(t)$ as the value of a European call option at time t. At expiry the value of a European call option is the payoff the holder receives from exercising the option,

$$C(T) = \max[0, S(T) - K] \equiv (S(T) - K)^+ \,. \tag{1.1}$$

The value of an option at initial time $t = 0$ is $C(0)$ and this is the premium of the option. In this course, one of the main tasks is to use mathematical models to evaluate a rational value for the premium $C(0)$ using estimations of the underlying price $S(t)$ during the time interval $0 \le t \le T$.

Example 1.3. Sketch the graphs

(a) $y = (x - 5)^+$,

(b) $y = (x - 5)^+ - 0.2$,

(c) $y = (5 - x)^+$,

(d) $y = (5 - x)^+ - 0.2$.

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Example 1.4. Write two equations to describe the two plots in Example 1.2.

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1.1.2 American call options

Define $C_A(t)$ as the value of an American call option at some time t where $0 \le t \le T$. The value of an American call at expiry T is the same as a European call,

$$C_A(T) = (S(T) - K)^+ \,. \tag{1.2}$$

However, since the American call option may be exercised at any time during the life of the contract, we want to know the value $C_A(t)$ at all times between $t = 0$ and $t = T$. The value of an American call for $0 \leq t \leq T$ is

$$C_A(t) \geq (S(t) - K)^+ . \tag{1.3}$$

We will discuss American calls in more detail in Section 3.1.1 and prove the above formula for $C_A(t)$ in the binomial pricing model.

Remark 1.1. This course is concerned with developing mathematical models for derivative pricing. When developing mathematical models it is necessary to make several assumptions or approximations. One of the main reasons for this is that we don't want our models to become unnecessarily complicated; we only want as much detail as is necessary to make meaningful estimations and we do not want to obtain an equation which is extremely difficult to solve. One significant assumption is that a seller will always be able to find a buyer and a buyer will always be able to find a seller (this means there is no *bid-ask spread*, that is, the buyer and seller can always agree on a price). We also usually ignore some small costs of trading, such as small fees incurred when buying or selling shares. □

1.2 Trading options

Why is there a market for options? Why do some traders want to buy options while others want to sell them? The answer is that different types of traders operate in different ways and look for different opportunities.

Three types of traders are commonly identified in financial markets:

- speculators (risk takers);
- hedgers (risk avoiders);
- arbitrageurs (bargain hunters).

Here we discuss strategies used by speculators and hedgers, but will discuss arbitrageurs in Section 1.4. Typically, all traders make assumptions about the future behaviour of underlying prices. For example, a trader may assume that a stock price will soon increase. Such a trader is said to be *bullish*. In contrast, a trader who assumes a stock price will fall is said to be *bearish*. Once a trader has decided how prices will change in the future, this trader will buy or sell financial products to make a profit or contain a loss.

1.2.1 Leverage

Definition 1.4. *Leverage* is a method for increasing the potential return of an investment. A common example of leverage is buying assets with borrowed funds with the belief that, in the future, the asset value will become greater than the cost of borrowing, thus yielding a profit. While leverage has the potential for greater rewards, it typically has greater risk.

Speculators use the leverage strategy when purchasing options rather than stock.

Say the price of one XYZ company share is currently $S(0) = \$14.64$. Call options for XYZ company shares are also available for a strike price of \$14 per share and a premium of \$1 per share. You believe that at expiry time T the share price will rise to $S(T) = \$16.00$. You have \$1464 at your disposal. Consider two possible strategies and their potential outcomes, described below.

1. You buy 100 XYZ company shares for \$1464 at $t = 0$ (assuming there are at least 100 shares for sale) and sell the shares at $t = T$ (assuming there is a buyer).
   (a) Your prediction is realised. You make a profit of $\$100(16.00-14.64) = \$136$. The percentage profit is $100 \times 136/1464 = 9.29\%$.
   (b) Your prediction is not realised. Say the share price falls to \$13. The loss is $\$100 \times (14.64 - 13.00) = \$164$. The percentage loss is $100 \times 164/1464 = 11.20\%$.
2. You buy XYZ company options for \$1464 at $t = 0$ (assuming there are at least 1464 options for sale) and, if the option is exercised at $t = T$, immediately sell the shares (assuming there is a buyer).
   (a) Your prediction is realised. At time T you exercise the option and sell the 1464 shares on the open market. The total payoff is $\$1464(16.00 - 14.00) = \$2928$. Subtracting off the premium gives the total profit \$1464. The percentage profit is 100%.
   (b) Your prediction is not realised. Say the share price falls to \$13. You do not exercise the option and so you have no shares to sell. The loss is 100%.

Without leverage you are able to buy only 100 shares, but with leverage (that is, buying the options) you are able to buy 1464 shares and, if your prediction is correct, your profits increase dramatically. This is a possible strategy for a speculator in a bullish market.

Usually, one company has several different options for sale. Call options with higher strike prices usually have lower premiums. These cheaper options provide greater leverage. In the scenario described above, if XYZ company also sold options for \$0.20 with a strike price of \$15.50 then, with your funds of \$1464 you are able to buy 7320 options. If your prediction is realised the total payoff is \$7320(16.00 − 15.50) = \$3660 and the profit is \$2196, or 150%.

1.2.2 Hedging

Definition 1.5. *Hedging* is a method for reducing the risk of adverse price movements in assets.

Say you want to buy XYZ shares which are currently available for \$15, but you do not want to purchase them for another three months. You are worried that the share price will rise to \$16. You can guarantee the shares will be available for \$15 by buying XYZ call options with an expiry date of three months and strike price of \$15. If the share does indeed rise to \$16 after three months, then you exercise the option and buy the shares for \$15 each. If the share price actually falls to \$13 after three months, then you do not exercise the option and you buy the shares on the open market for \$13 per share.

When hedging, the payment of the premium for a call can be regarded as an *insurance payment* against a possible rise in share price. Hedgers typically buy options which have a strike price equal to the share price at time $t = 0$, that is, $K = S(0)$, which is an 'at the money' call. We will discuss hedging in more detail in Section 5.2 and show how writers use hedging to contain losses.

1.2.3 Selling call options

Selling a call is referred to as writing a call, since the seller writes the contract. Once a call is sold the writer is 'short in the call' or has taken a 'short position'. Writers receive a premium when an option is sold, but the transaction is usually unprofitable whenever the holder exercises the option.

For example, a writer sells a call on an XYZ share with strike price of \$30 and receives a premium of \$1. If the share price rises to \$32 on the expiry date the option is exercised and the loss to the writer is \$32 − \$30 − \$1 = \$1 per share.

When selling call options, there is no limit to a writer's potential loss, but the potential profit is limited.

Example 1.5. In Example 1.2 we plotted the holder's payoff at expiry and the total profit. Give equations for the writer's payoff at expiry and total profit for the same call option and plot the graphs.

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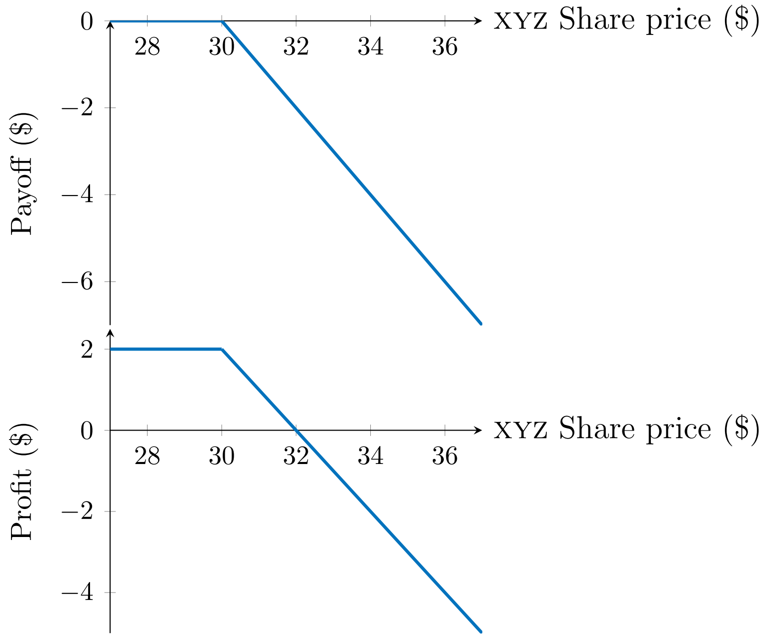

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1.3 Put options

Definition 1.6. A *put option* gives the holder *the right to sell* the underlying asset at a set price (the *strike price*) at a future time. However, the holder is *not obliged to sell* the underlying asset. If the holder does choose to sell the asset (that is, *exercise* the option), then the writer must buy.

Buying a put options is commonly described as ‘taking a put’. The holder of a put, like the holder of a call, has taken a ‘long position’. Similarly, the writer of a put, like the writer of a call, has taken a ‘short position’.

The terminology for put options is similar to call options, although for call options exercising the option means buying the underlying and for put options exercising the option means selling the underlying. Two styles of put options are American and European, as described in Definitions 1.2 and 1.3, respectively. That is, American options can be exercised at any time before expiry but European options can only be exercised on the expiry date.

As before, define underlying price $S(t)$ at time t for $0 \leq t \leq T$, with $t = 0$ the time the option is purchased and $t = T$ the expiry time. Define K as the strike price per share. There are two cases to consider.

1. If the underlying price at expiry is less than the strike price,

$S(T) < K$, then the holder should exercise the option. The payoff is $K - S(T)$, it is 'in the money'.

2. If the underlying price at expiry is equal to or greater than the strike price, $S(T) \geq K$, then the holder should not exercise the option. In this case the option has no value, it is 'out of the money' (or, if $S(T) = K$ it is 'at the money').

Define $P(t)$ as the value of the put option at time t. An option has some value if allows an underlying to be sold at a higher price than its current price. At expiry the value of a put option is

$$P(T) = \max[0, K - S(T)] = (K - S(T))^+ . \tag{1.4}$$

Unlike a call option, a put option's potential profit is limited. Maximum profit from a put is obtained when the price of the underlying at maturity drops to zero, $S(T) = 0$. Like a call option, the maximum potential loss of a put is the premium.

Example 1.6. A trader buys a European put option written on an XYZ share. The put is such that, in three months' time, the trader has the right to sell the XYZ share at \$30. The option premium is \$1. What is the trader's payoff at expiry if the share price decreases to \$25 by the expiry date? What is the trader's payoff at expiry if the share prices increases to \$35 by the expiry date? Plot the trader's payoff at expiry and total profit. Plot the writer's payoff at expiry and total profit.

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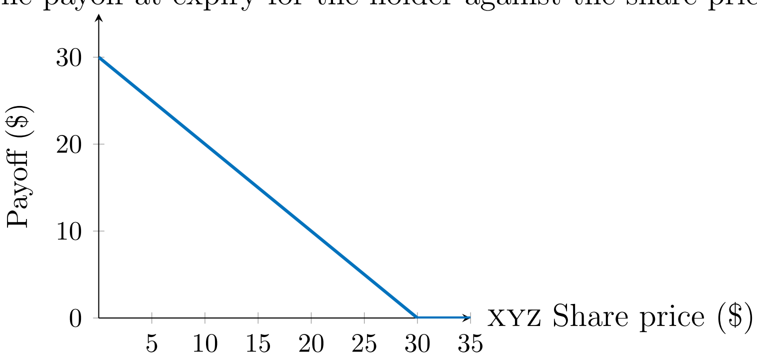
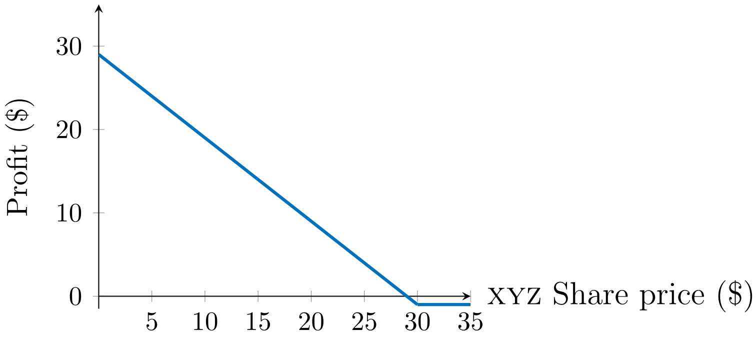
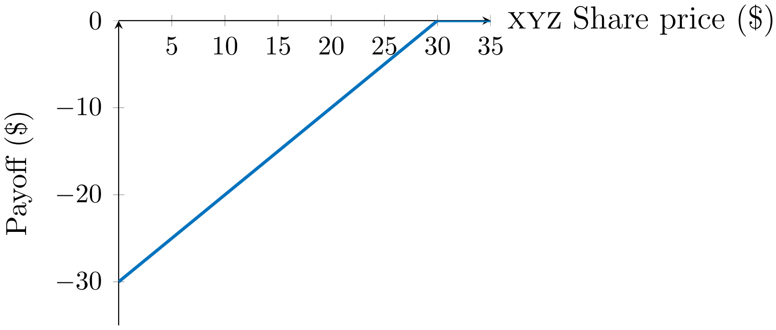
Plot the payoff at expiry for the holder against the share price.



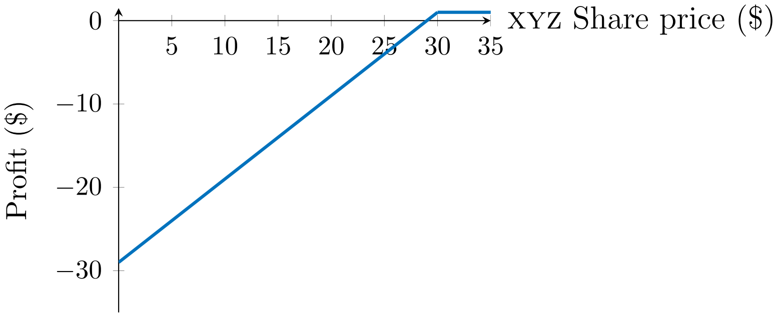
Plot the holder's profit per share against the share price.



Plot the payoff at expiry for the writer against the share price.



Plot the writer's profit per share against the share price.



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Example 1.7. In Example 1.3, four equations were plotted. State what each plot could describe in options trading.

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1.3.1 Short selling

Definition 1.7. A *short seller* borrows an asset and then sells it. At a later date the short seller must buy back the asset and return it to the original owner. Short selling is also known as shorting.

The short seller is bearish and expects the asset to lose value over the course of the borrowing contract. If all goes to plan, an asset of reduced value is returned to the owner. For example, say a short seller borrows 100 shares with share price $30, worth a total of \$3000. If the share price falls to \$25, then the short seller has to return 100 shares worth only \$2500, making a profit of \$500. If instead the share price rises to \$35, then the short seller has to return 100 shares worth \$3500, making a loss of \$500.

Short selling and buying put options are similar in that they are both bearish strategies. However, the potential losses are greater for short selling.

Example 1.8. Shares in some company are currently selling at \$30 each. A bearish trader can either buy 100 put options for \$100 with a strike price of \$30 per share, or short sell 100 shares after paying a fee of \$80. Compare the potential losses for both choices.

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When comparing different trading strategies, it is not as simple as looking at derivative payoffs at initial time $t = 0$ and final time $t = T$. In the previous example, the trader could reduce losses by investing the cash received from short selling, but we ignored this possibility. In the next section we will include interest rates in our calculations.

Remark 1.2. In finance, 'going long' is generally a *bullish* strategy and describes a trader who *owns assets* which are expected to increase in value. In contrast, 'going short' is generally a *bearish* strategy and describes a trader who *borrows assets* which are expected to decrease in value (such as in short selling). However, in the case of options, 'going long' always refers to the option holder, whether bullish (calls) or bearish (puts) since the holder *owns the option.* Also, 'going short' always refers to the option writer since the writer *does not own the option.* □

1.4 Arbitrage

Definition 1.8. *Arbitrage* is when a trader (or *arbitrageur*) buys and sells financial products and assets in order to take advantage of differences in price. Arbitrage produces a *risk-free profit.*

Arbitrage sometimes takes advantage of different prices for the same asset in different markets. A typical example is a trader buying company XYZ shares on the New York Stock Exchange for \$10 each and then selling them on the London Stock Exchange for \$11 each, making a guaranteed profit of \$1 per share.

Arbitrage is also possible when two financial products provide an equal payoff at one time, but an unequal payoff at another time, due to one of the products being under or overvalued. For example, if share prices are certain to rise at a quicker rate than the current interest rate, an arbitrageur will borrow money to buy shares. At a later date, the arbitrageur sells the shares and pays off the loan with interest. The money required to pay off the loan is less than the value of the shares so the arbitrageur makes a profit.

Example 1.9. A trader borrows \$100 from a bank at an interest rate of 5% per month. The trader also buys 100 European call options for a \$1 premium. The option has strike price \$20 and an expiry in one month. The share price is \$22 at expiry. At expiry the trader exercises the option, sells the shares on the open market and returns the \$100 to the bank with interest. What is the traders profit? Show that the trader has taken advantage of an arbitrage opportunity.

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Example 1.10. Say at $t = 0$ a share price is $S(0) = \$5$. Say at expiry $t = 1$ the share price becomes $S(1) = \$4.50$. The current interest rate is $r = \frac{1}{5}$ so the return on one dollar is $R = 1 + r = \frac{6}{5}$. Show how a profit can be made with a zero investment.

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In this course, our mathematical models are based on a *no arbitrage assumption*. No arbitrage means that all identical or similar assets always have the same price and we cannot make a risk-free profit through buying and selling these similar assets. No arbitrage means that all assets are priced at their *rational* value and there is no chance of finding a bargain.

Example 1.11. Say two different companies have shares $S_1(t)$ and $S_2(t)$. At the current time $t = 0$ the two share prices are equal, $S_1(0) = S_2(0)$, but there is absolute certainty that at $t = T$, $S_1(T) < S_2(T)$. Show that there is an arbitrage opportunity. Is there an arbitrage opportunity when $S_1(0) > S_2(0)$ and $S_1(T) = S_2(T)$? What about $S_1(0) = S_2(0)$ and $S_1(T) = S_2(T)$?

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Example 1.12. Suppose we know a stock price $S(t)$ at times $t = 0$ and $t = 1$. Then, the value of a call option with known strike price

K expiring at $t = 1$ is $C(1) = (S(1) - K)^+$. Assuming no arbitrage, at $t = 0$ what is the value of the option $C(0)$ (the option premium)?

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1.5 One-step binomial asset pricing model

The binomial asset pricing model provides a rational price for a derivative asset. This model is discrete in time, meaning the asset valuation is only determined at certain times, and not all times in some interval.

We begin with a one-step model and in Chapter 2 will develop a general N-step model.

1.5.1 Binomial pricing trees

Define the value of some derivative as $W(t)$ at time t. Define the current time as $t = 0$ and the expiry or maturity time as $t = 1$. At time $t = 0$ we want to trade this derivative and to do this we need to know $W(0)$. A rational value for $W(0)$ depends on how the price of the underlying asset is expect to change over time t. For example, a call option will have a high price if the underlying asset is almost certain to grow in value. In contrast, if the underlying asset price is almost certain to fall, the value of the call will be low.

Suppose the derivative has underlying asset with value $S(t)$ (such as share price). At time $t = 0$ the underlying has known value $S(0) \geq 0$. At time $t = 1$ we assume there are two possible values of the underlying, $S(1,\uparrow) \geq 0$ and $S(1,\downarrow) \geq 0$ where $S(1,\uparrow) > S(1,\downarrow)$ and both these values are known. We require $S(1,\uparrow) \neq S(t,\downarrow)$ because we want the underlying to be *risky*. The underlying asset is risky because we know it has one of two values, but we do not know which; this is how we take account of market volatility.

We plot a *binomial price tree* to illustrate the one underlying asset value at $t = 0$ and the two possible underlying asset values at $t = 1$.

$$
\begin{tikzcd}[arrows=dash]
t=0 & t=1 \\
 & S(1,\uparrow) \\
S(0) \arrow[ur] \arrow[dr] & \\
 & S(1,\downarrow)
\end{tikzcd}
$$

The three points on the binomial price tree are called *nodes*.

At time $t = 1$ there are two possible states of $S(1)$, the upstate represented by $\uparrow$ and the downstate represented by $\downarrow$. The term *binomial* implies that there are two states. Binomial models have many applications in many unrelated fields, for example tossing a coin is described by a binomial model because there are two possible states, heads and tails.

Since the derivative value is dependent on the underlying asset price the derivative value must also have two states at $t = 1$. We name these two states $W(1, \uparrow)$ and $W(1, \downarrow)$. We know the possible underlying values at expiry $t = 1$, $S(1, \uparrow)$ and $S(1, \downarrow)$, and therefore we know the derivatives values at $t = 1$, $W(1, \uparrow)$ and $W(1, \downarrow)$. We plot a price tree for the derivative value, indicating the unknown value at $t = 0$ which we want to determine.

$$\begin{array}{ccc} t=0 & & t=1 \\ & & W(1,\uparrow) = \text{known} \\ & \nearrow & \\ W(0) = \text{unknown} & & \\ & \searrow & \\ & & W(1,\downarrow) = \text{known} \end{array}$$

Say at time $t = 0$ we also have some cash investment or bonds. We know the interest rate and therefore this cash investment is *riskless*; this means there is only one value for the cash investment at time $t = 1$ (that is, the same value whether we end up in the upstate or the downstate). For interest rate $r \geq 0$ the return on each dollar is $R = 1 + r$ by time $t = 1$ and $R \geq 1$.

Notice that we have not defined the units of the times $t = 0$ and $t = 1$. These times can be in any units we require, for example, hours, days, weeks, months, years. The choice of units depends on the particular problem being considered.

1.5.2 Replicating portfolio

Assuming no arbitrage, if two different but similar portfolios provide the same cash flow, then these portfolios always have the same value.

We define our derivative as one portfolio, and imagine a second portfolio which is similar to this derivative portfolio. We define this second portfolio to have the same cash flow as the derivative portfolio. Since the two portfolios have the same cash flow, they always have the same value when we assume no arbitrage. This second portfolio is the *replicating portfolio* since it is intended to imitate or replicate the first portfolio.

We define a replicating portfolio of value $V(t)$ which is equal in value to our derivative $W(t)$. At time $t = 0$ we set up this portfolio to contain H_0 dollars in bank investments or bonds with return R, and H_1 number of the underlying asset (for example, the number of shares). Since we know the asset price to be $S(0)$ at $t = 0$ the initial value of the replicating portfolio is

$$V(0) = H_0 + H_1 S(0)\,. \tag{1.5}$$

At $t = 1$ each dollar of money (or bonds) increases to R and the asset value is either $S(1,\uparrow)$ or $S(1,\downarrow)$. Thus there are two possible values of the replicating portfolio,

$$\begin{aligned} V(1,\uparrow) &= H_0 R + H_1 S(1,\uparrow)\,, \\ V(1,\downarrow) &= H_0 R + H_1 S(1,\downarrow)\,. \end{aligned} \tag{1.6}$$

These two possible values are defined to be equal to the two possible values of the derivative portfolio at $t = 1$:

$$\begin{aligned} V(1,\uparrow) &= H_0 R + H_1 S(1,\uparrow) = W(1,\uparrow)\,, \\ V(1,\downarrow) &= H_0 R + H_1 S(1,\downarrow) = W(1,\downarrow)\,, \end{aligned} \tag{1.7}$$

and at $t = 0$:

$$W(0) = V(0) = H_0 + H_1 S(0)\,. \tag{1.8}$$

Draw the binomial price tree for $V(t)$ and $W(t)$ at $t = 0, 1$.

$$\begin{tikzcd}[arrows=-] t=0 & t=1 \\ & H_0R + H_1S(1,\uparrow) = W(1,\uparrow) \\ H_0 + H_1S(0) = W(0) \arrow[ur] \arrow[dr] & \\ & H_0R + H_1S(1,\downarrow) = W(1,\downarrow) \end{tikzcd}$$

We know the $t = 1$ value of the derivative portfolio, $W(1,\uparrow)$, $W(1,\downarrow)$, the three asset prices, $S(0)$, $S(1,\uparrow)$ and $S(1,\downarrow)$, and the return on the cash investment, R. So, we can use equation (1.7) to determine H_0 and H_1. Then, using equation (1.8) we can determine $V(0)$ and $W(0)$.

The constants H_0 and H_1 can be negative or positive. For $H_0 < 0$ we are borrowing money but for $H_0 > 0$ we are lending money. For $H_1 > 0$ we are buying the underlying asset but for $H_1 < 0$ we are short selling the underlying asset. For example, for options, the replicating portfolios are usually

- Calls: borrowing money and buying the underlying asset ($H_0 < 0$, $H_1 > 0$);
- Puts: lending money and short selling the underlying asset ($H_0 > 0$, $H_1 < 0$).

Note that buying calls and buying assets are both bullish strategies whereas buying puts and short selling are both bearish strategies. Thus the replicating portfolio mimics the option portfolio.

Remark 1.3. Our claim that two portfolios have the same value when they have the same cash flow is based on *the Law of One Price*. This law is a consequence of our no arbitrage assumption. The Law of one Price applies to portfolios which are similar with regards to various factors such as taxation, liquidity, credit risk and transaction costs. □

1.5.3 Solving the one-step model

The first step in solving the one-step binomial model is to determine H_0 and H_1. Then, we substitute these values into (1.8) to find $W(0)$.

Example 1.13. Assuming a no arbitrage position, calculate the value of a derivative $W(0)$ using stock prices and derivative values given in the following binomial price tree and $R = \frac{10}{9}$.

$$\begin{tikzcd}[column sep=small]
t=0 & t=1 \\
 & S(1,\uparrow) = \frac{20}{3},\, W(1,\uparrow) = 7 \\
S(0) = 5,\, W(0) = ? \arrow[ur, dash] \arrow[dr, dash] & \\
 & S(1,\downarrow) = \frac{40}{9},\, W(1,\downarrow) = 2
\end{tikzcd}$$

We determine H_0 and H_1 from substituting known parameters into equation (1.7):

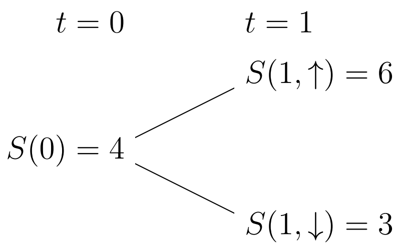
$$\begin{aligned}
H_0 \tfrac{10}{9} + H_1 \tfrac{20}{3} &= 7\,, \\
H_0 \tfrac{10}{9} + H_1 \tfrac{40}{9} &= 2\,.
\end{aligned}$$

Subtract the two equations to find $H_1 \frac{20}{9} = 5$, so $H_1 = 2.25$. Then, substitute H_1 into either of the above equations to find $H_0 = -7.2$. The replicating portfolio is buy 2.25 shares and borrow \$7.2. The value of the derivative is

$$W(0) = V(0) = H_0 + H_1 S(0) = 4.05\,.$$

□

Example 1.14. Assuming a no arbitrage position, calculate the time $t = 0$ value of a European call option with strike price $K = 4$, using underlying stock prices given in the following binomial price tree and $R = \frac{6}{5}$.



. . . . ☐

Example 1.15. Draw cash flow tables at times $t = 0$ and $t = 1$ for a portfolio which consists of the replicating portfolio plus short selling the derivative, as describe in Example 1.13. Show that this portfolio provides no arbitrage.

Consider the following cash flows.

$t = 0$	cash flow
short sell one derivative	$W(0) = 4.05$
borrow cash	$-H_0 = 7.20$
buy shares	$-H_1 S(0) = -11.25$
total:	0

$t = 1$	cash flow
buy derivative and return to owner	$W(1, \uparrow / \downarrow) = -7 \text{ or } -2$
pay back cash	$RH_0 = -8$
sell shares	$H_1 S(1, \uparrow / \downarrow) = 15 \text{ or } 10$
total:	0

In all cases the total cash flow is zero and therefore there is no arbitrage opportunity. ☐

Remember that the derivative portfolio and the replicating portfolio are only equal at $t = 0$ (that is, $W(0) = V(0)$) when we assume no arbitrage. If we allow arbitrage, then it is possible to have $W(0) > V(0)$ or $W(0) < V(0)$ and our one-step binomial model is not applicable. In general, there will be no arbitrage opportunity when

$$S(1, \downarrow) < RS(0) < S(1, \uparrow) . \tag{1.9}$$

Example 1.16. In Example 1.14, show that equation (1.9) is true for this no arbitrage case.

. . . . ☐

Example 1.17. Show that there are arbitrage opportunities when buying or selling shares if $S(1, \downarrow) > RS(0)$ or $S(1, \uparrow) < RS(0)$.

. . . . ☐

1.6 General pricing formula and risk neutral probabilities

For the one-step binomial pricing model we have three general formulas

$$W(0) = H_0 + H_1 S(0)\,, \tag{1.10}$$
$$W(1,\uparrow) = H_0 R + H_1 S(1,\uparrow)\,, \tag{1.11}$$
$$W(1,\downarrow) = H_0 R + H_1 S(1,\downarrow)\,. \tag{1.12}$$

We want to rearrange these three equations and write then in more convenient forms. Ultimately, we want to write the derivative value $W(0)$ in terms of the derivative values $W(1,\uparrow)$ and $W(1,\downarrow)$ so we do not have to bother calculating H_0 and H_1.

First find equations for H_1 and H_0. Subtract (1.11) and (1.12):

$$W(1,\uparrow) - W(1,\downarrow) = H_1[S(1,\uparrow) - S(1,\downarrow)]\,, \tag{1.13}$$

and rearrange to obtain

$$H_1 = \frac{W(1,\uparrow) - W(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)}\,. \tag{1.14}$$

Substitute (1.14) into (1.11) and rearrange to obtain

$$\begin{aligned} H_0 R &= W(1,\uparrow) - S(1,\uparrow)\frac{W(1,\uparrow) - W(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} \\ &= \frac{W(1,\uparrow)[S(1,\uparrow) - S(1,\downarrow)] - S(1,\uparrow)[W(1,\uparrow) - W(1,\downarrow)]}{S(1,\uparrow) - S(1,\downarrow)}\,, \\ H_0 &= \frac{S(1,\uparrow)W(1,\downarrow) - S(1,\downarrow)W(1,\uparrow)}{R[S(1,\uparrow) - S(1,\downarrow)]}\,. \end{aligned} \tag{1.15}$$

Substitute equations (1.14) and (1.15) into equation (1.10) to obtain

$$\begin{aligned} W(0) &= \frac{S(1,\uparrow)W(1,\downarrow) - S(1,\downarrow)W(1,\uparrow)}{R[S(1,\uparrow) - S(1,\downarrow)]} + S(0)\frac{W(1,\uparrow) - W(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} \\ &= \frac{S(1,\uparrow)W(1,\downarrow) - S(1,\downarrow)W(1,\uparrow) + RS(0)[W(1,\uparrow) - W(1,\downarrow)]}{R[S(1,\uparrow) - S(1,\downarrow)]} \\ &= \frac{1}{R}\left[W(1,\uparrow)\frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} + W(1,\downarrow)\frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)}\right]. \end{aligned} \tag{1.16}$$

We have now written $W(0)$ in terms of $W(1,\uparrow)$ and $W(1,\downarrow)$.

Define the *risk neutral probabilities* of states $\uparrow$ and $\downarrow$, respectively,

$$\pi = \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)}\,,$$

$$(1-\pi) = \frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)}\,. \tag{1.17}$$

Substitute these probabilities into equation (1.16) to obtain the *general pricing formula* for a derivative in a one-step binomial model

$$W(0) = \tfrac{1}{R}[\pi W(1,\uparrow) + (1-\pi)W(1,\downarrow)]. \tag{1.18}$$

The factor $\frac{1}{R}$ in the pricing formula is called the *discount factor*.

When we have no arbitrage and (1.9) is true, the risk neutral probabilities satisfy $0 < \pi < 1$. It can be shown that there is no arbitrage in our binomial model if and only if the general pricing formula and $0 < \pi < 1$ are true.

The probability π defines the likelihood of entering the upstate at $t = 1$, while probability $(1-\pi)$ defines the likelihood of entering the downstate at $t = 1$. Equation 1.18 is essentially a weighted average of the two $t = 1$ derivative values (weighted by their probabilities), and the discount factor adjusts the $t = 1$ average derivative value to give the $t = 0$ derivative value.

We now express the asset price $S(0)$ in terms of π and the two underlying prices at $t = 1$. Consider

$$\begin{aligned}
\pi S(1,\uparrow) + (1-\pi)S(1,\downarrow) &= S(1,\uparrow)\frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} \\
&\quad + S(1,\downarrow)\frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)} \\
&= \frac{S(1,\uparrow)RS(0) - S(1,\downarrow)RS(0)}{S(1,\uparrow) - S(1,\downarrow)} \\
&= RS(0)\,.
\end{aligned} \tag{1.19}$$

Therefore,

$$S(0) = \tfrac{1}{R}[\pi S(1,\uparrow) + (1-\pi)S(1,\downarrow)]\,. \tag{1.20}$$

Like the general pricing formula (1.18), this asset price equation is the weighted average of the asset values at $t = 1$, adjusted by the discount factor to give the $t = 0$ value $S(0)$. Notice that the derivative and its underlying asset have the same risk neutral probabilities π and $(1-\pi)$ and vary in a similar way over time.

Example 1.18. Use the same information as given in Example 1.13 to find the risk neutral probabilities. Then calculate $W(0)$ using the general pricing formula.

Example 1.13 gives $R = \frac{10}{9}$, $S(0) = 5$, $S(1,\uparrow) = \frac{20}{3}$, $S(1,\downarrow) = \frac{40}{9}$, $W(1,\uparrow) = 7$ and $W(1,\downarrow) = 2$. The risk neutral probabilities are

$$\pi = \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)}$$

$$
\begin{aligned}
&= \frac{\frac{10}{9}\times 5 - \frac{40}{9}}{\frac{20}{3}-\frac{40}{9}} = \frac{1}{2}\,, \\
(1-\pi) &= \frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)} \\
&= \frac{\frac{20}{3} - \frac{10}{9}\times 5}{\frac{20}{3}-\frac{40}{9}} = \frac{1}{2}\,.
\end{aligned}
$$

From the general pricing formula

$$
\begin{aligned}
W(0) &= \tfrac{1}{R}[\pi W(1,\uparrow) + (1-\pi)W(1,\downarrow)] \\
&= \tfrac{9}{10}[\tfrac{1}{2}\times 7 + \tfrac{1}{2}\times 2] = \tfrac{81}{20} \\
&= 4.05.
\end{aligned}
$$

□

Remark 1.4. In a *risk neutral* financial market we assume traders are neither *risk adverse* nor *risk seeking*. Traders are assumed to be only interested in *expected* returns on investments and we do not take account their feelings concerning risk. A risk neutral investor chooses the portfolio with the highest expected return. In contrast, a risk adverse investor is more likely to diversify across a number of products or choose products with almost guaranteed profit, even if the profit is low. On the other hand, a risk seeking investor is more likely to chase the maximum profit, even if the risk of loss is significant. Say I owe \$10 to an investor but I suggest we instead toss a coin and I will pay \$20 for heads and \$0 for tails. Assuming the coin is fair, a risk neutral investor sees no difference between these scenarios since they both pay on average \$10. What scenario will a risk adverse investor prefer? What about a risk seeking investor? What if the coin is not fair and is more likely to throw heads? Or more likely to throw tails? □

1.7 Cox–Ross–Rubinstein model

There are many different types of binomial pricing models. The Cox–Ross–Rubinstein (CRR) model is a well known binomial pricing model which uses the asset price notation

$$
S(0) = S > 0\,, \quad S(1,\uparrow) = uS\,, \quad S(1,\downarrow) = dS\,, \tag{1.21}
$$

where u and d are constants called the *up and down factors*. The binomial price tree is similar to before.

$$\begin{tikzcd}[row sep=small]
t=0 & t=1 \\
 & uS \\
S \arrow[ur, dash] \arrow[dr, dash] & \\
 & dS
\end{tikzcd}$$

In the CRR model the inequality (1.9) is equivalent to

$$0 < d < R < u\,. \tag{1.22}$$

In terms of u and d, the risk neutral probabilities are

$$\begin{aligned}
\pi &= \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} \\
&= \frac{RS - dS}{uS - dS} \\
&= \frac{R-d}{u-d}\,, \\
(1-\pi) &= \frac{S(1,\uparrow) - RS(0)}{S(1,\uparrow) - S(1,\downarrow)} \\
&= \frac{uS - RS}{uS - dS} \\
&= \frac{u-R}{u-d}\,.
\end{aligned} \tag{1.23}$$

In terms of u and d, the general pricing formula is

$$\begin{aligned}
W(0) &= \tfrac{1}{R}[\pi W(1,\uparrow) + (1-\pi)W(1,\downarrow)] \\
&= \frac{1}{R}\left[\frac{R-d}{u-d}W(1,\uparrow) + \frac{u-R}{u-d}W(1,\downarrow)\right].
\end{aligned} \tag{1.24}$$

Example 1.19. For a European call option with $C(1) = (S(1) - K)^+$ and $dS < K < uS$, assume the upstate risk neutral probability is π and determine $C(0)$.

. . . . $\square$

Example 1.20. For a European put option with $P(1) = (K - S(1))^+$ and $dS < K < uS$, assume the upstate risk neutral probability is π and determine $P(0)$.

. . . . $\square$

1.8 Put-call parity

The put-call parity is a relationship between the values of European puts and European calls with the same strike price K and expiry T.

At some time t the put-call relationship is

$$C(t) - P(t) = S(t) - \text{PV}_t(K) \tag{1.25}$$

where $\text{PV}_t(K)$ is called the *present value* of the strike price K at time t.

Definition 1.9. How much do I need to invest at time t to obtain amount P at some future time T? The answer to this question is $\text{PV}_t(P)$, the *present value* of P. For constant return R over the period t to T, $\text{PV}_t = P/R$ since investing P/R at t will provide $RP/R = P$ at time T.

In the one-step binomial model at $t = 0$ the present value of the strike price is $\text{PV}_0(K) = K/R$. At expiry, $\text{PV}_T(K) = K$. At $t = T$ the put-call relationship (1.25) is

$$C(T) - P(T) = S(T) - K\,. \tag{1.26}$$

At $t = T$ the put-call relationship for puts and calls with equal strike price and expiry T is *always true*. At $t = 0$ the put-call relationship is only true when there is no arbitrage opportunity.

We prove the put-call relationship at $t = T$. First consider $S(T) \geq K$ so the values of the put and call are

$$P(T) = (K - S(T))^+ = 0\,, \quad C(T) = (S(T) - K)^+ = S(T) - K\,, \tag{1.27}$$

(when $S(T) = K$ we have $P(T) = C(T) = 0$) and then

$$C(T) - P(T) = S(T) - K - 0\,, \tag{1.28}$$

which is identical to equation (1.26) Now consider $S(T) \leq K$ so the values of the put and call are

$$P(T) = (K - S(T))^+ = K - S(T)\,, \quad C(T) = (S(T) - K)^+ = 0\,, \tag{1.29}$$

and then

$$C(T) - P(T) = 0 - [K - S(T)]\,, \tag{1.30}$$

which is identical to equation (1.26). We have shown that the put-call relationship at $t = T$ is true for both $S(T) \geq K$ and $S(T) \leq K$ and therefore equation (1.26) is true in general when $t = T$.

Example 1.21. Show that when the put-call parity does not hold at $t = 0$ we have an arbitrage opportunity. Use $\text{PV}_0(K) = K/R$.

. . . . □

Remark 1.5. It is possible to calculate the present value PV_t at any time t, but usually we are interested in the value of some asset or cash flow at the current time $t = 0$. If no time t is given when discussing the present value, then assume the time is the current time $t = 0$ and the present value is PV_0. The present value is commonly used to compare different cash flows made at different times. For example, say we have the opportunity of earning either \$100 at end of the next two months (totalling \$200), or \$202 in two months. The present value can help us decide which is the better investment choice at the current time $t = 0$. □

1.8.1 Put-call parity in the CRR model

Using CRR notation for the one-step model we show below that $\text{PV}_0(K) = K/R$ when $dS < K < uS$.

In Examples 1.19 and 1.20 we showed that for $dS < K < uS$ and upstate risk neutral probability π, at $t = 0$ call and put option values satisfy

$$C(0) = \frac{\pi(uS - K)}{R}, \quad P(0) = \frac{(1-\pi)(K - dS)}{R}. \tag{1.31}$$

So,

$$\begin{aligned} C(0) - P(0) &= \frac{\pi(uS - K)}{R} - \frac{(1-\pi)(K - dS)}{R} \\ &= \frac{1}{R}[\pi uS + (1-\pi)dS - \pi K - (1-\pi)K] \\ &= \frac{\pi uS + (1-\pi)dS}{R} - \frac{K}{R} \\ &= S(0) - K/R, \end{aligned} \tag{1.32}$$

where we use equation (1.20) for the asset price $S(0)$. From equation (1.25) we see that $\text{PV}_0(K) = K/R$.

Example 1.22. For a share price $S(0) = 22$ and interest rate $r = 5\%$ so that $R = 1.05$, a put and a call both have a strike price of \$21 and one year to maturity. The cost of the call option is $C(0) = 2.00$ and the cost of the put option is $P(0) = 1.00$. Show, using put-call parity, that an arbitrage opportunity exists. Show the cash flow which gives this arbitrage opportunity.

. . . .

□

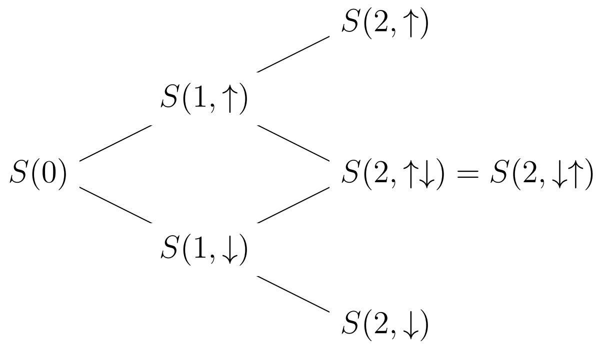
2 Multi-step binomial asset pricing models

Contents

In the previous chapter we developed the one-step binomial asset pricing model which described derivative values at time steps $t = 0, 1$. We now develop more general binomial asset pricing models which describe derivative values at three or more time steps.

2.1 Two-step model

We now consider values of assets and derivatives at times $t = 0, 1, 2$ in a two-step binomial asset pricing model. We plot a binomial price tree for asset prices.



We define $S(2,\uparrow\downarrow) = S(2,\downarrow\uparrow)$ so that the path taken does not matter (we get the same asset price at $t = 2$ if the asset price first goes up and then down, or down and then up). A tree for which the path does not matter is called a *recombining tree*.

There are a number of different ways to write the asset price. Sometimes, rather than using $\uparrow$ and $\downarrow$, we number the asset prices at each time step, starting from the lowest state on the tree,

$$\begin{aligned} S(1,1) &= S(1,\uparrow)\,, \qquad & S(2,2) &= S(2,\uparrow)\,, \\ S(1,0) &= S(1,\downarrow)\,, \qquad & S(2,1) &= S(2,\uparrow\downarrow) = S(2,\downarrow\uparrow)\,, \\ & & S(2,0) &= S(2,\downarrow)\,. \end{aligned} \tag{2.1}$$

So, $S(n,j)$ means the asset price at time $t = n$ after j increases in asset price (j number of $\uparrow$) and $(n-j)$ decreases in asset price ($n - j$ number of $\downarrow$).

Also, we often write $S(0,0)$ rather than $S(0)$, to give the $t = 0$ value the same form as the $t > 0$ values. Using CRR notation, as before, $S(0) = S$, $S(1,\uparrow) = uS$, $S(1,\downarrow) = dS$ and

$$\begin{aligned} S(2,\uparrow) &= u^2 S\,, \\ S(2,\uparrow\downarrow) &= S(2,\downarrow\uparrow) = udS\,, \\ S(2,\downarrow) &= d^2 S\,. \end{aligned} \tag{2.2}$$

We can have as many steps in the binomial model as we wish. Here we just consider recombining trees so that the path along the tree does not matter.

Example 2.1. For a three-step CRR model with $S = 4$, $u = 2$ and $d = \frac{1}{2}$, draw a binomial price tree and find all asset prices at $t = 1, 2, 3$.

. . . . $\square$

2.2 Multi-step model

For a general N-step binomial model we have $N+1$ times numbered $t = 0, 1, \ldots, N$. Each *node* (or point) on a N-step binomial price tree is represented by two numbers (n, j) where time is $t = n$ and $j = 0, 1, \ldots, n$ describes the vertical position or *state* of the node on the tree. We choose this method for labelling nodes because it is more compact than writing multiple $\uparrow$ and $\downarrow$.

For example, we draw the binomial price tree for asset prices in a four-step model, $N = 4$. Observe that at each time $t = n$ where $0 \le n \le N$ there are $n + 1$ nodes which are in different states $j = 0, 1, \ldots, n$. We use (n, j) to label nodes.

$$
\begin{array}{ccccc}
 & & & & S(4,4) \\
 & & & S(3,3) & \\
 & & S(2,2) & & S(4,3) \\
 & S(1,1) & & S(3,2) & \\
S(0,0) & & S(2,1) & & S(4,2) \\
 & S(1,0) & & S(3,1) & \\
 & & S(2,0) & & S(4,1) \\
 & & & S(3,0) & \\
 & & & & S(4,0)
\end{array}
$$

We can draw similar trees for derivative prices $W(n, j)$ with $n = 0, 1, \ldots, N$ and $0 \le j \le n$. Consider just one part of a multi-step model from time $t = n$ to time $t = n + 1$ starting from asset price $S(n, j)$.

$$
\begin{array}{ccc}
 & & S(n+1, j+1),\ W(n+1, j+1) \\
S(n, j),\ W(n, j) & & \\
 & & S(n+1, j),\ W(n+1, j)
\end{array}
$$

We treat this part of the multi-step model like it is a one-step model. Then, in the general pricing formula for the one-step binomial model (1.18), replace $W(0)$ with $W(n, j)$, $W(1, \uparrow)$ with $W(n + 1, j + 1)$, and $W(1, \downarrow)$ with $W(n + 1, j)$ to obtain a new general pricing formula

$$W(n, j) = \tfrac{1}{R}[\pi W(n + 1, j + 1) + (1 - \pi)W(n + 1, j)]\,. \qquad (2.3)$$

Similarly, in equation (1.20) replace $S(0)$ with $S(n, j)$, $S(1, \uparrow)$ with $S(n + 1, j + 1)$, and $S(1, \downarrow)$ with $S(n + 1, j)$ to obtain the general rule for the asset prices

$$S(n,j) = \tfrac{1}{R}[\pi S(n+1,j+1) + (1-\pi)S(n+1,j)]\,. \tag{2.4}$$

Example 2.2. A European call option with strike price $K = 3$ in a two step CRR model has $u = 2$ and $d = \frac{1}{2}$ for initial underlying share price $S = 4$ and return $R = \frac{5}{4}$. In Example 2.1 we calculated all share prices on the three-step binomial tree for the same u and d. Calculate the initial value (premium) of the option given that $t = 2$ is expiry.

. . . . □

Example 2.3. A European call option with strike price $K = 3$ in a three step CRR model has $u = 2$ and $d = \frac{1}{2}$. For initial underlying share price $S = 4$ and return $R = \frac{5}{4}$. In Example 2.1 we calculated all share prices on the three-step binomial tree for the same u and d. Calculate the initial value (premium) of the option given that $t = 3$ is expiry.

. . . . □

2.2.1 State prices

Say we know $W(2,j)$ for all j in some binomial model and just want to know $W(0,0)$. We do not have to evaluate all in between values $W(1,j)$. An alternative method is to write the general pricing formula (2.3) for $W(0,0)$ ($n = 0$ in equation (2.3)) and then substitute the general pricing formulas for $W(1,1)$ and $W(1,0)$ ($n = 1$ in equation (2.3)):

$$\begin{aligned} W(0,0) &= \tfrac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)] \\ &= \tfrac{1}{R}\left[\pi \tfrac{1}{R}[\pi W(2,2) + (1-\pi)W(2,1)]\right. \\ &\quad \left. +(1-\pi)\tfrac{1}{R}[\pi W(2,1) + (1-\pi)W(2,0)]\right] \\ &= \tfrac{\pi^2}{R^2}W(2,2) + 2\pi(1-\pi)\tfrac{1}{R^2}W(2,1) + \tfrac{(1-\pi)^2}{R^2}W(2,0)\,. \end{aligned} \tag{2.5}$$

In the above equation, define the coefficients of the $W(2,j)$ on the right hand side as *state prices* $\lambda(2,j)$ so,

$$\begin{aligned} W(0,0) &= \lambda(2,2)W(2,2) + \lambda(2,1)W(2,1) + \lambda(2,0)W(2,0) \\ &= \sum_{j=0}^{2} \lambda(2,j)W(2,j)\,, \end{aligned} \tag{2.6}$$

where

$$\lambda(2,2) = \tfrac{\pi^2}{R^2}\,, \quad \lambda(2,1) = 2\pi(1-\pi)\tfrac{1}{R^2}\,, \quad \lambda(2,0) = \tfrac{(1-\pi)^2}{R^2}\,. \tag{2.7}$$

If we again substitute the general pricing formula (2.3) into equation (2.5) to remove $W(2,2)$, $W(2,1)$ and $W(2,0)$ ($n=2$ in equation (2.3)), then we would obtain an equation for $W(0,0)$ in terms of $W(3,j)$ with $j = 0,1,2,3$. With more substitutions of the general price formula we can always find $W(0,0)$ as a function of $W(n,j)$ for any $t=n$ and $j=0,1,\ldots,n$.

For any time $t=n$ in an N-step binomial model with $0 \le n \le N$ the initial price of a derivative equals

$$\begin{aligned} W(0,0) &= \lambda(n,n)W(n,n) + \lambda(n,n-1)W(n,n-1) + \cdots \\ &\quad + \lambda(n,0)W(n,0) \\ &= \sum_{j=0}^{n} \lambda(n,j)W(n,j)\,, \end{aligned} \tag{2.8}$$

for some state prices $\lambda(n,j) \ge 0$. The $n=2$ state prices are given in (2.7). The $n=1$ state prices are found directly from the general pricing formula (1.18) or (2.3) with $n=0$,

$$\lambda(1,1) = \tfrac{\pi}{R}\,, \quad \lambda(1,0) = \tfrac{1-\pi}{R}\,. \tag{2.9}$$

We want to find a formula for any state price $\lambda(n,j)$. One way to calculate the state prices is the same way we found the $n=2$ state prices in equation (2.7); that is, by substituting the general pricing formula (2.3) until the desired value of n is obtained. Another way is to find a *recursion* relation for $\lambda(n,j)$. A recursion relation defines state price $\lambda(n,j)$ in terms of state prices $\lambda(n-1,j)$ from the previous time step.

Equation (2.8) is true for any $0 \le n \le N$, so we replace n with $n-1$:

$$W(0,0) = \sum_{j=0}^{n-1} \lambda(n-1,j)W(n-1,j)\,. \tag{2.10}$$

Now substitute the general pricing formula (2.3) into equation (2.10):

$$\begin{aligned} W(0,0) &= \sum_{j=0}^{n-1} \lambda(n-1,j)\tfrac{1}{R}[\pi W(n,j+1) + (1-\pi)W(n,j)] \\ &= \tfrac{\pi}{R}\sum_{j=0}^{n-1} \lambda(n-1,j)W(n,j+1) \\ &\quad + \tfrac{1-\pi}{R}\sum_{j=0}^{n-1} \lambda(n-1,j)W(n,j) \\ &= \tfrac{\pi}{R}\sum_{j=1}^{n} \lambda(n-1,j-1)W(n,j) \end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{1-\pi}{R} \sum_{j=0}^{n-1} \lambda(n-1,j) W(n,j) \\
&= \tfrac{\pi}{R}\lambda(n-1,n-1)W(n,n) + \tfrac{1-\pi}{R}\lambda(n-1,0)W(n,0) \\
&+ \sum_{j=1}^{n-1} [\tfrac{\pi}{R}\lambda(n-1,j-1) + \tfrac{1-\pi}{R}\lambda(n-1,j)]W(n,j)\,.
\end{aligned}
\tag{2.11}
$$

We have written $W(0,0)$ in terms of $W(n,j)$; this equation should be the same as equation (2.8).

Comparing coefficients of each $W(n,j)$ in equations (2.11) and equation (2.8) we find the recursion relation for the state prices

$$
\lambda(n,j) = \begin{cases} \tfrac{\pi}{R}\lambda(n-1,n-1) & \text{for } j = n\,, \\ \tfrac{1-\pi}{R}\lambda(n-1,j) + \tfrac{\pi}{R}\lambda(n-1,j-1) & \text{for } 0 < j < n\,, \\ \tfrac{1-\pi}{R}\lambda(n-1,0) & \text{for } j = 0\,. \end{cases}
\tag{2.12}
$$

This recursion formula is called *Jamshidian's forward induction formula*. Forward induction means that a value at some time $t = n$ is determined by values at the earlier time $t = n - 1$. The solution of this recursion relation is

$$
\lambda(n,j) = C_j^n \frac{\pi^j(1-\pi)^{n-j}}{R^n}\,, \quad C_j^n = \frac{n!}{j!(n-j)!}\,.
\tag{2.13}
$$

This general equation for the state price can be formally proved using mathematical induction, but we will not do this.

Example 2.4. For a CRR model with $u = 2$, $d = \frac{1}{2}$ and $R = \frac{5}{4}$, find $\lambda(3,j)$ for $j = 0, 1, 2, 3$. Then, use these state prices to calculate the premium of a European call option with $S = 4$ and $K = 3$. This premium should agree with the premium calculated in Example 2.3.

.... □

2.2.2 Arrow–Debreu securities

Definition 2.1. An *Arrow–Debreu security* is a type of derivative which pays a fixed price at some node (N,j) but pays zero for all other states at the same time N. For simplicity we will assume that the fixed price payment is \$1. For a \$1 payment at node (N,j), the value of an Arrow–Debreu security at initial time $t = 0$ is the *state price* $\lambda(N,j)$.

We will now show that $W(0,0)$, the value of an Arrow–Debreu security at $t = 0$, is the state price $\lambda(N,j)$ in a N-step binomial model. Consider the general equation (2.8) for a binomial model.

Say we have an Arrow–Debreu security where one $W(N,j) = 1$ in (2.8) and all the others values are zero. Therefore,

$$W(0,0) = \lambda(N,j)W(N,j) = \lambda(N,j)\,. \tag{2.14}$$

As an example, we now show this is true for Arrow–Debreu securities in a two-step binomial model.

Consider an Arrow–Debreu security in a two-step binomial model. At $t = 2$ the security has three possible values: $W(2,0)$, $W(2,1)$ and $W(2,2)$; one of these values equals \$1 and the other two are zero. We consider each possible case for this two-step model and calculate all $W(n,j)$ in the binomial price tree using the general pricing formula (2.3).

First consider $W(2,2) = 1$ and $W(2,1), W(2,0) = 0$. From equation (2.3)

$$\begin{aligned} W(1,1) &= \tfrac{1}{R}[\pi W(2,2) + (1-\pi)W(2,1)] = \tfrac{\pi}{R}\,,\\ W(1,0) &= \tfrac{1}{R}[\pi W(2,1) + (1-\pi)W(2,0)] = 0\,,\\ W(0,0) &= \tfrac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)] = \tfrac{\pi^2}{R^2}\,. \end{aligned} \tag{2.15}$$

The initial value of the security is $W(0,0) = \frac{\pi^2}{R^2} = \lambda(2,2)$, as shown in equation (2.7). Draw the two-step binomial price tree for $W(2,2) = 1$.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & W(2,2) = 1 \\
 & W(1,1) = \frac{\pi}{R} \arrow[ur, dash] \arrow[dr, dash] & \\
W(0,0) = \lambda(2,2) = \frac{\pi^2}{R^2} \arrow[ur, dash] \arrow[dr, dash] & & W(2,1) = 0 \\
 & W(1,0) = 0 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & W(2,0) = 0
\end{tikzcd}$$

Now consider $W(2,1) = 1$, $W(2,2), W(2,0) = 0$. From equation (2.3)

$$\begin{aligned} W(1,1) &= \tfrac{1}{R}[\pi W(2,2) + (1-\pi)W(2,1)] = \tfrac{1-\pi}{R}\,,\\ W(1,0) &= \tfrac{1}{R}[\pi W(2,1) + (1-\pi)W(2,0)] = \tfrac{\pi}{R}\,,\\ W(0,0) &= \tfrac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)]\\ &= \tfrac{1}{R}[\pi \tfrac{1-\pi}{R} + (1-\pi)\tfrac{\pi}{R}] = 2\pi(1-\pi)\tfrac{1}{R^2}\,. \end{aligned} \tag{2.16}$$

The initial value of the security is $W(0,0) = 2\pi(1-\pi)\frac{1}{R^2} = \lambda(2,1)$, as shown in equation (2.7). Draw the two-step binomial price tree for $W(2,1) = 1$.

$$\begin{array}{ccccc} & & & & W(2,2)=0\\ & & W(1,1)=\frac{1-\pi}{R} & & \\ W(0,0)=\lambda(2,1)=\frac{2\pi(1-\pi)}{R^2} & & & & W(2,1)=1 \\ & & W(1,0)=\frac{\pi}{R} & & \\ & & & & W(2,0)=0\end{array}$$

Finally consider $W(2,0) = 1$, $W(2,2), W(2,1) = 0$. From equation (2.3)

$$\begin{aligned} W(1,1) &= \tfrac{1}{R}[\pi W(2,2) + (1-\pi)W(2,1)] = 0\,, \\ W(1,0) &= \tfrac{1}{R}[\pi W(2,1) + (1-\pi)W(2,0)] = \tfrac{1-\pi}{R}\,, \\ W(0,0) &= \tfrac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)] = \tfrac{(1-\pi)^2}{R^2}\,. \end{aligned} \tag{2.17}$$

The initial value of the security is $W(0,0) = \frac{(1-\pi)^2}{R^2} = \lambda(2,0)$, as shown in equation (2.7). Draw the two-step binomial price tree for $W(2,0) = 1$.

$$\begin{array}{ccccc} & & & & W(2,2)=0\\ & & W(1,1)=0 & & \\ W(0,0)=\lambda(2,0)=\frac{(1-\pi)^2}{R^2} & & & & W(2,1)=0 \\ & & W(1,0)=\frac{1-\pi}{R} & & \\ & & & & W(2,0)=1\end{array}$$

Example 2.5. For a one-step binomial model, show that the premium of an Arrow–Debreu security is $W(0,0) = \lambda(1,1) = \frac{\pi}{R}$ or $W(0,0) = \lambda(1,0) = \frac{1-\pi}{R}$. The state prices $\lambda(1,1)$ and $\lambda(1,0)$ are shown in equation (2.9).

. . . . □

2.3 Distribution functions

2.3.1 Complementary distribution function

We present the complementary binomial distribution function in the context of a European call option using CRR notation. The complementary binomial distribution function provides another way to calculate the initial price of a derivative $W(0,0)$. Begin by substituting equation (2.13) into equation (2.8),

$$W(0,0) = \tfrac{1}{R^n} \sum_{j=0}^{n} C_j^n \pi^j (1-\pi)^{n-j} W(n,j)\,. \tag{2.18}$$

Consider a European call option with strike price K and expiry $t = N$. Use the CRR notation and assume $d^N S < K < u^N S$ so that at least one state causes the option to be exercised and at least one state is not exercised.

The value of the option at expiry in the jth state is

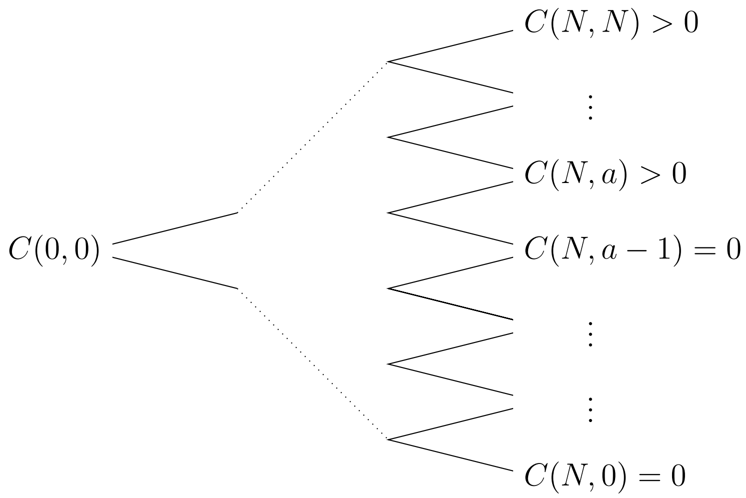
$$C(N,j) = (S(N,j) - K)^+ = (u^j d^{N-j} S - K)^+\,, \tag{2.19}$$

for example,

$$\begin{aligned} C(N,0) &= (d^N S - K)^+ = 0\,, \\ C(N,N) &= (u^N S - K)^+ = (u^N S - K)\,, \end{aligned} \tag{2.20}$$

since we know $d^N S < K < u^N S$.

There must be an integer a such that all states with $j \ge a$ provided an option with positive value and all states with $j < a$ provided a worthless option, $C(N,j) = 0$. Draw the binomial price tree for this European call option.



Using equation (2.18) with equation (2.19), the value of the option at $t = 0$ is

$$\begin{aligned} C(0,0) &= \tfrac{1}{R^N} \sum_{j=0}^{N} C_j^N \pi^j (1-\pi)^{N-j} [u^j d^{N-j} S - K]^+ \\ &= \tfrac{1}{R^N} \sum_{j=a}^{N} C_j^N \pi^j (1-\pi)^{N-j} (u^j d^{N-j} S - K) \end{aligned} \tag{2.21}$$

where we remove all states with $j < a$ because for these states the option is worthless. Break the sum in the above equation into two sums,

$$\begin{aligned} C(0,0) &= \tfrac{S}{R^N} \sum_{j=a}^{N} C_j^N \pi^j (1-\pi)^{N-j} u^j d^{N-j} \\ &\quad - \tfrac{K}{R^N} \sum_{j=a}^{N} C_j^N \pi^j (1-\pi)^{N-j} \\ &= S J_1 - \tfrac{K}{R^N} J_2 \,, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} J_1 &= \tfrac{1}{R^N} \sum_{j=a}^{N} C_j^N \pi^j (1-\pi)^{N-j} u^j d^{N-j} \\ &= \sum_{j=a}^{N} C_j^N \left(\tfrac{\pi u}{R}\right)^j \left(\tfrac{(1-\pi)d}{R}\right)^{N-j} , \\ J_2 &= \sum_{j=a}^{N} C_j^N \pi^j (1-\pi)^{N-j} \,. \end{aligned} \tag{2.23}$$

Define the *complementary binomial distribution function*

$$\Psi(a; N, p) = \sum_{j=a}^{N} C_j^N p^j (1-p)^{N-j}, \tag{2.24}$$

for some probability $0 < p < 1$. For the European call option problem discussed above, since we know $0 < \pi < 1$ (when we assume no arbitrage),

$$J_2 = \Psi(a; N, \pi) \,. \tag{2.25}$$

We also want to write J_1 in terms of Ψ, but what should we choose as the probability p? From equation (1.23), $\pi = \frac{R-d}{u-d}$ and $(1-\pi) = \frac{u-R}{u-d}$ so,

$$\begin{aligned} 1 - \frac{\pi u}{R} &= 1 - \frac{(R-d)u}{(u-d)R} \\ &= \frac{(u-d)R - (R-d)u}{(u-d)R} \\ &= \frac{-dR + du}{(u-d)R} \\ &= \frac{(u-R)d}{(u-d)R} \end{aligned}$$

$$= (1-\pi)\frac{d}{R}\,. \tag{2.26}$$

Therefore, we choose the probabilities p and $(1-p)$ in the complementary binomial distribution to be

$$p = \frac{\pi u}{R}\,, \quad 1-p = \frac{(1-\pi)d}{R}\,, \tag{2.27}$$

and thus

$$J_1 = \Psi(a; N, \tfrac{\pi u}{R})\,. \tag{2.28}$$

So the initial value of the European call option in terms of the complementary binomial distribution is

$$C(0,0) = S\Psi(a; N, \tfrac{\pi u}{R}) - \tfrac{K}{R^N}\Psi(a; N, \pi)\,. \tag{2.29}$$

2.3.2 Many-step binomial models

In a binomial problem there are two different states, such as heads and tails when tossing coins, or upstate and downstate derivative prices. If the probability of the first state happening is p, then the probability of the second state happening is $1-p$. In the case of a coin toss, usually $p = (1-p) = \frac{1}{2}$, but this might not be the case if a coin is weighted.

Say there are N times where one of the two states occurs, for example N coin tosses, or N branches in the share price. You then must have j of one case and $N-j$ of the other case for some $0 \le j \le N$. For example, j heads and $N-j$ tails, or j upstates and $N-j$ downstates. For a binomial problem, the probability of having j of one state and $N-j$ of the other is $C_j^N p^j (1-p)^{N-j}$. This probability is the probability for *all* states with j of one state and $N-j$ of the other—the ordering of the states does not matter. For example, for a coin toss problem with $p = \frac{1}{2}$, $N = 3$ and $j = 1$ the probability $C_2^3 p(1-p)^2 = \frac{3}{8}$ is the probability of throwing: tails, tails, heads; plus tails, heads, tails; plus heads, tails, tails.

A cumulative distribution function is the sum of several probabilities starting from the sate $j = 0$ to some state $j > 0$. For example, consider a coin toss problem where the probability of a single coin toss being a head is p, the probability of a single coin toss being a tail is $1-p$, and the coin tossed a total of $N = 3$ times. Say we want to find the probability of throwing *no more than* one head, in any order (this is equivalent to throwing *at least* two tails in any order). Also, say $p = \frac{1}{2}$. The probability of throwing no heads and three tails is $C_3^3 p^0 (1-p)^3 = \frac{1}{8}$ and the probability of throwing one head and two tails is $C_2^3 p(1-p)^2 = \frac{3}{8}$. The probability of throwing no more than one head is the cumulative distribution

$$F_{B(N,p)}(1) = F_{B(3,p)}(1) = \sum_{j=0}^{1} C_j^3 p^j (1-p)^{3-j} = \tfrac{1}{2}\,. \tag{2.30}$$

In general, in a total of N throws, to throw no more than $(a-1)$ heads (or at least $N-a+1$ tails) the probability is $F_{B(N,p)}(a-1)$. The probability of throwing any head and tail combination in N throws is the sum of all possible cases, so the cumulative distribution is

$$F_{B(N,p)}(N) = \sum_{j=0}^{N} C_j^3 p^j (1-p)^{N-j} = 1\,. \tag{2.31}$$

For the European call option discussed in Section 2.3.1, the cumulative distribution is very similar to the coin toss example above—just replace heads with the upstate. In N time steps with p being the probability of a upstate, the probability of having no more than $(a-1)$ upstates is the cumulative distribution

$$F_{B(N,p)}(a-1) = \sum_{j=0}^{a-1} C_j^N p^j (1-p)^{N-j}\,, \tag{2.32}$$

and the probability of any number of upstates or downstates is

$$F_{B(N,p)}(N) = \sum_{j=0}^{N} C_j^N p^j (1-p)^{N-j} = 1\,. \tag{2.33}$$

From equations (2.24), (2.32) and (2.33), the relationship between the binomial cumulative distribution and the complementary binomial distribution is

$$\Psi(a;N,p) = 1 - F_{B(N,p)}(a-1)\,. \tag{2.34}$$

In terms of the cumulative distribution the initial option value in equation (2.29) is

$$C(0,0) = S[1 - F_{B(N,\frac{\pi u}{R})}(a-1)] - \tfrac{K}{R^N}[1 - F_{B(N,\pi)}(a-1)]\,. \tag{2.35}$$

Figures 2.1–2.3 plot the probability $C_j^N p^j (1-p)^{N-j}$ and cumulative $F_{B(N,p)}(a)$ and complementary $\Psi(a;N,p)$ distributions for $p=\frac{1}{2}$ and $N=5,10,100$. Notice that, as N increases, the step sizes in the plots decrease and the plots become more smooth. If N became infinitely large, then the probability and distribution functions would become smooth. For these plots, because $p=\frac{1}{2}$, the probability is symmetric about $\frac{N}{2}$ and the cumulative and complementary distributions are mirror images about $\frac{N}{2}$. Observe also that $F_{B(N,p)}(a-1) + \Psi(a;N,p) = 1$.

Figure 2.4 plots the probability and cumulative and complementary distributions for $p=\frac{1}{3}$ and $N=100$. These two plots look very similar to the plots for $p=\frac{1}{2}$ and $N=100$ in Figure 2.3, but shifted to the left and symmetric about $\frac{N}{3}$.

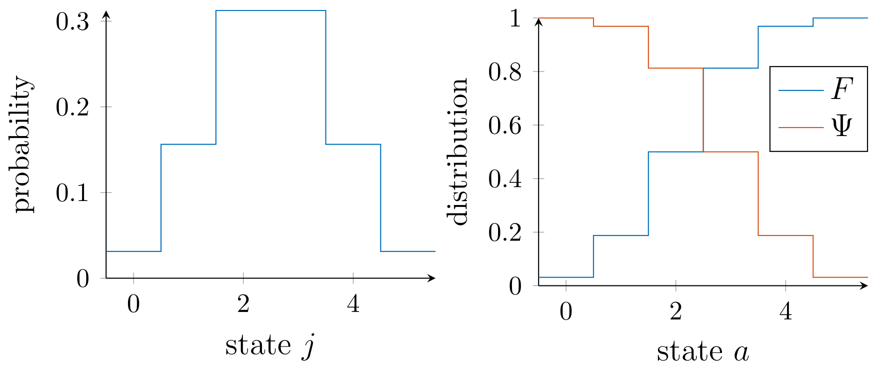


Figure 2.1: For $N = 5$, $p = \frac{1}{2}$. The left shows the probability $C_j^N p^j (1-p)^{N-j}$ and the right shows, in blue, the cumulative distribution $F_{B(N,p)}(a)$ and, in red, the complementary function $\Psi(a; N, p)$.

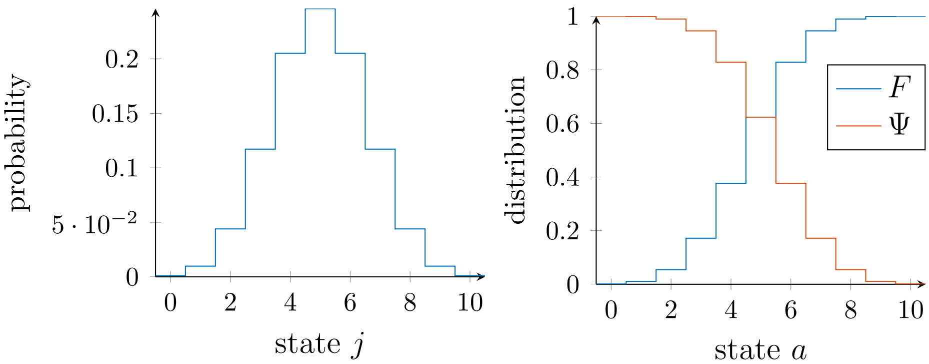


Figure 2.2: For $N = 10$, $p = \frac{1}{2}$. The left shows the probability $C_j^N p^j (1-p)^{N-j}$ and the right shows, in blue, the cumulative distribution $F_{B(N,p)}(a)$ and, in red, the complementary function $\Psi(a; N, p)$.

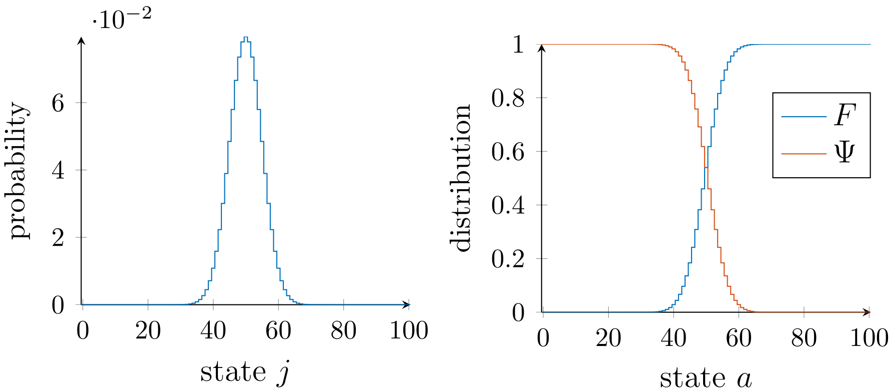


Figure 2.3: For $N = 100$, $p = \frac{1}{2}$. The left shows the probability $C_j^N p^j (1-p)^{N-j}$ and the right shows, in blue, the cumulative distribution $F_{B(N,p)}(a)$ and, in red, the complementary function $\Psi(a; N, p)$.

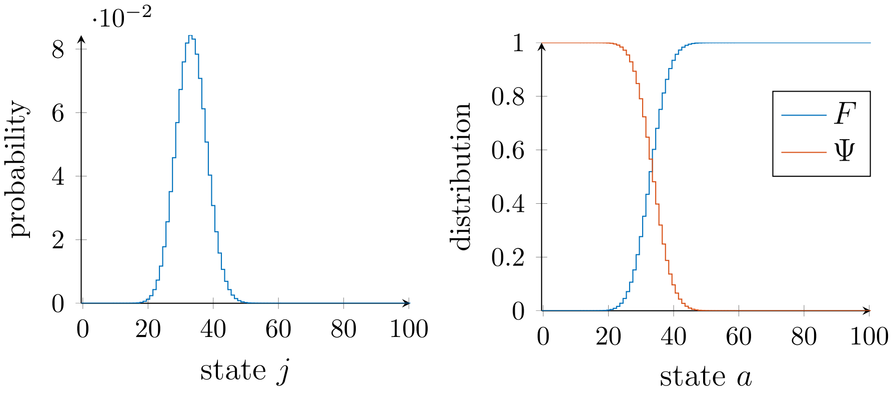


Figure 2.4: For $N = 100$, $p = \frac{1}{3}$. The left shows the probability $C_j^N p^j (1-p)^{N-j}$ and the right shows, in blue, the cumulative distribution $F_{B(N,p)}(a)$ and, in red, the complementary function $\Psi(a; N, p)$.

2.4 Black–Scholes model

The binomial asset pricing model is a discrete time model, meaning that we can only determine the derivative value at fixed time steps. The Black–Scholes (BS) model (or Black–Scholes–Merton model) is a continuous time model, meaning that we can use the BS model to determine a derivative value at any time t.

2.4.1 BS model for a call option

Consider an N-step binomial asset pricing model for a European call option with initial time $t = 0$ and expiry time $t = T$, such as that described in Section 2.3.1. To obtain the BS model we take the limit $N \to \infty$ while T remains constant; this means that each time step $\frac{T}{N}$ becomes infinitely small. In Section 2.3.2 we showed that as N increases, the binomial distribution functions $\Psi(a; N, p)$ and $F_{B(N,p)}(a)$ becomes smoother over state a. In the limit $N \to \infty$ the cumulative distribution functions are completely smooth (or, continuous) and we instead use the cumulative *normal* (or Gaussian) distribution

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \,, \tag{2.36}$$

(here, $\pi = 3.14\ldots$, not the probability). Some solutions of the normal cumulative distribution function are

$$\mathcal{N}(-\infty) = 0 \,, \quad \mathcal{N}(0) = \tfrac{1}{2} \,, \quad \mathcal{N}(\infty) = 1 \,. \tag{2.37}$$

The BS model defines the return R such that r is a continuous compounding interest rate. From initial time $t = 0$ to expiry time T the return is

$$R = e^{rT} \,. \tag{2.38}$$

Note that in the N-step model R is defined as the rate of return over one time step of size $\frac{T}{N}$, but here R is defined as the rate of return over the whole time interval T, which contains many infinitesimal time steps.

The BS model formula for calculating the value of a European call option at time $t = 0$ is

$$C(0) = S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)\,, \tag{2.39}$$

where S is the underlying asset price at time $t = 0$, and

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\,, \\ d_2 &= \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}\,. \end{aligned} \tag{2.40}$$

where 'ln' is the natural logarithm (if $\ln(y) = x$, then $y = e^x$) and σ is the *volatility* of the underlying asset. We will not derive equation (2.39), it is sufficient to know in a qualitative way how it relates to the N-step binomial model.

When describing interest rates (or rates of return), a time period must always be specified, such as per year, per month or per day. The same is true of volatility σ. With volatility we specify if it is, for example, the yearly volatility, monthly volatility or daily volatility. When substituting into the BS model (2.39) the interest rate r, volatility σ and the time to maturity T must all be in terms of *the same time unit* (e.g., year, month or day) and it may sometimes be necessary to convert from one time unit to another. In the BS model interest rates are continuously compounding so to convert from one time unit to another we use, for example,

$$r_{\text{month}} = \frac{r_{\text{year}}}{12} \qquad r_{\text{day}} = \frac{r_{\text{year}}}{365}\,. \tag{2.41}$$

To convert volatilities is similar, but with a square root:

$$\sigma_{\text{month}} = \frac{\sigma_{\text{year}}}{\sqrt{12}} \qquad \sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{365}}\,. \tag{2.42}$$

Section 2.5 discusses the volatility in more detail.

Example 2.6. At $t = 0$ a call option has $S = 100$, $K = 100$, $T = 1$ year, yearly volatility $\sigma = 15\%$ and $r = 5\%$ pa Find the $t = 0$ value of this call using the given parameters. Next, convert all year units into a month units and use these monthly parameters to calculate the $t = 0$ value of the call. Show that the two $t = 0$ call values are equal.

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Example 2.7. Do the following conversions:

(a) yearly volatility $\sigma = 15\%$ into monthly, daily volatility and 90 day volatility;

(b) monthly volatility $\sigma = 5\%$ into yearly volatility.

. . . . $\square$

2.4.2 BS model for a put option

The put-call parity, discussed in Section 1.8, relates the value of a European put option $P(t)$ to the value of a European call option $C(t)$ with the same strike price and expiry. From equation (1.25) at $t = 0$,

$$P(0) = C(0) - S + \mathrm{PV}_0(K)\,. \tag{2.43}$$

We use this equation to derive $P(0)$ in the BS model.

First we need present value $\mathrm{PV}_0(K) = K/R$ (the amount to be invested at time $t = 0$ to obtain amount K at time T). The rate of return is $R = e^{rT}$ and therefore $\mathrm{PV}_0(K) = K/R = Ke^{-rT}$. Substitute this present value and the value of the call (2.39) into equation (2.43),

$$\begin{aligned} P(0) &= S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) - S + Ke^{-rT} \\ &= S[\mathcal{N}(d_1) - 1] - Ke^{-rT}[\mathcal{N}(d_2) - 1]\,. \end{aligned} \tag{2.44}$$

The normal distribution function satisfies

$$1 - \mathcal{N}(x) = \mathcal{N}(-x)\,, \tag{2.45}$$

and thus, in the BS model, the value of a European put option at time $t = 0$ is

$$P(0) = -S\mathcal{N}(-d_1) + Ke^{-rT}\mathcal{N}(-d_2)\,. \tag{2.46}$$

Example 2.8. Using the data from Example 2.6, calculate the value of a put at time $t = 0$ in the BS model.

. . . . $\square$

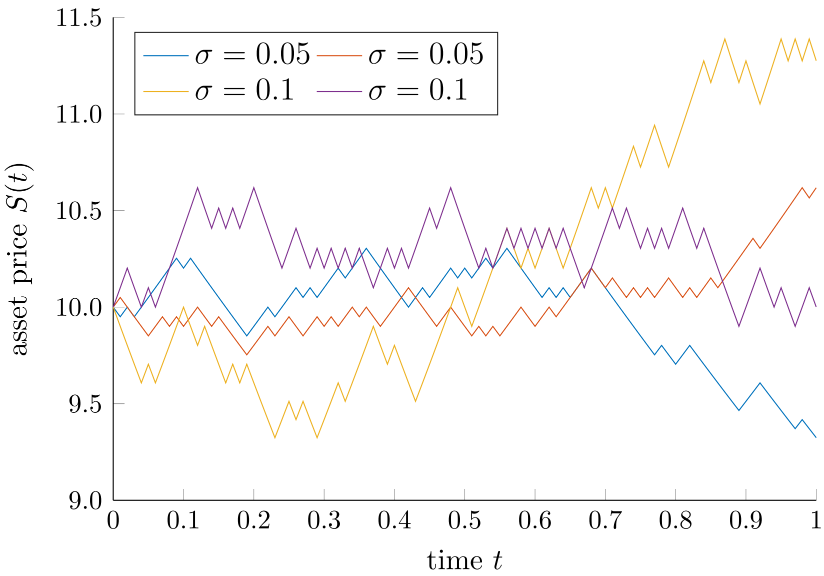
2.4.3 BS and CRR models

The BS model is derived from an N-step CRR model by defining times steps of length $\frac{T}{N}$, the return over *one* of the N times steps $R = e^{rT/N}$ (remember, this R is not the same as R in the BS model) and the up and down factors

$$u = e^{\sigma\sqrt{T/N}}\,, \quad d = 1/u = e^{-\sigma\sqrt{T/N}}\,. \tag{2.47}$$

As $N \to \infty$ the BS model can be obtained. This form of u and d show how the asset price is dependent on the volatility.

The plot below shows how the asset price varies over many time steps. In this plot $T = 1$ and $N = 100$ so each time step is $\frac{T}{N} = 0.01$ and the initial share price is $S = 10$. For volatility $\sigma = 0.05$ the up and down factors are $u = 1.005$ and $d = 0.995$. For volatility $\sigma = 0.1$ the up and down factors are $u = 1.010$ and $d = 0.990$. This random movement in asset price is called *geometric Brownian motion*. The mathematics describing geometric Brownian motion was developed by Einstein in 1905 to describe the random movement of particles suspended in fluids.



Example 2.9. In a two-step CRR model, use data from Example 2.6 to calculate the return per time step, the up and down factors, and the risk neutral probability π. Then, use this two-step model to calculate the premium of the European call. Compare this premium to that calculated in Example 2.6.

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2.5 Volatility

2.5.1 Implied volatility

A major problem with the BS model is that we do not know the value of the volatility σ. To determine the volatility we have a few options.

- Estimate σ from past data. But why should the past determine the future?
- Guess σ. But how do we do this with any confidence?

- Accept that the BS model only gives an estimation of an option value because σ can only be an estimation.

If we know an option's value at $t = 0$, then we can use this to estimate for volatility σ. We call this calculated volatility the *implied volatility.*

Example 2.10. Consider a European call option over stock. The current share price is $S = 10$, the strike price is $K = 8$, and the expiry is in $T = 0.5$ years. The annual continuous compounding interest rate is $r = 0.05$. The call option is currently trading on the market at \$2.50. Find the implied volatility σ to three decimal places.

. . . . □

Example 2.11. At 2 pm on 12 August 2010, BHP is trading at \$39.540. A call option is available for \$1.660, expiring on 28/10/2010, that is, in 0.21 years, with strike price \$40. The effective annual rate is 4.5% pa. Determine the implied volatility of BHP stock to three decimal places.

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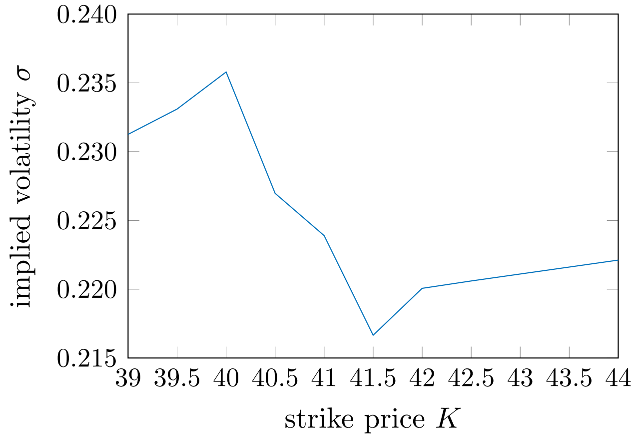
2.5.2 Volatility smiles and smirks

One company often sells several options with different strike prices and expiry times, but with the same underlying asset. If we worked out the implied volatility from all available options we would find several different implied volatilities for the same asset. If the BS model was a correct model of reality, then we should find only one volatility for each type of asset.

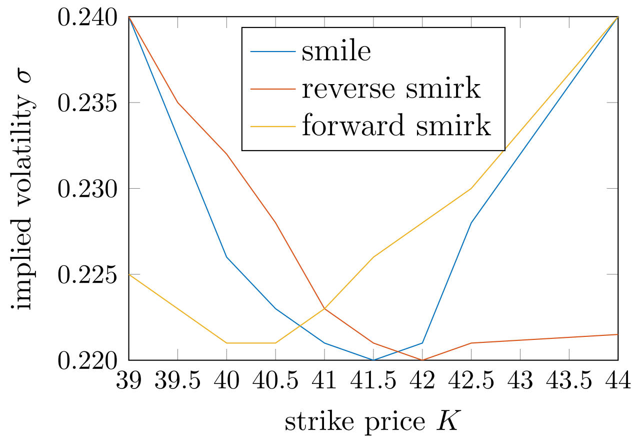
For example, consider the following nine call options for BHP stock, available at 2 pm on 12 August 2010 (the third one was discussed in Example 2.11) In this table, the column titled 'last' gives the premiums of each call.

code	expiry	strike	bid	offer	last	volume
BHPLG9	28/10/2010	39.00	2.130	2.235	2.140	14
BHPKM9	28/10/2010	39.50	1.845	1.960	1.885	10
BHPKQ9	28/10/2010	40.00	1.600	1.700	1.660	2
BHPKO9	28/10/2010	40.50	1.355	1.455	1.375	12
BHPKS9	28/10/2010	41.00	1.135	1.230	1.155	76
BHPKK9	28/10/2010	41.50	0.930	1.035	0.930	2
BHPKX9	28/10/2010	42.00	0.000	0.000	0.800	27
BHPXA9	28/10/2010	42.50	0.620	0.715	0.670	73
BHPSP9	28/10/2010	44.00	0.000	0.340	0.380	18

We calculate nine different implied volatilities for all nine calls and plot the implied volatility against the strike price; in the above table the 'last' column is the premium (these are the latest premiums). The shape of this plot is clearly not flat so the implied volatility is not constant.



Implied volatility plots commonly form shapes described as 'smiles' or 'smirks'. Typical examples are shown in the plot below. The shape of the implied volatility plot depends on a number of factors including the type of derivative, the type of underlying asset, the time to expiry, and even the mathematical model used to calculate the implied volatility.



A 'U' shaped *volatility smile* is observed for some options close to expiry; here, implied volatility is increased for options which are at the extreme ends of 'in the money' (small K for call option) and 'out of the money' (large K for call option). A more common shape is a *reverse volatility smirk* where small K produces larger implied volatilities than large K, typical of options not close to expiry. Our BHP example produced a reverse smirk. Another common shape is the *forward volatility smirk* where large K produces larger implied volatilities than small K, this is often seen for options in the commodities market.

2.6 Variable interest rates

In our previous discussion on the multi-step binomial model we assumed interest rates were constant over time steps $0 \le n \le N$. We now relax this assumption.

2.6.1 Generalised backward-induction pricing formula

Section 2.2 derived the general pricing formula for an N-step model, shown in equation (2.3). This pricing formula finds the value of a derivative $W(n,j)$ at time $t = n$ from two derivative values, $W(n+1,j)$ and $W(n+1,j+1)$, at time step $t = n+1$. This general pricing model assumes return R and risk neutral probability π remain constant over all times. We now allow R and π to vary.

Define interest rate $r(n,j)$ as the interest rate from node (n,j) to the nodes at the next time step $(n+1,j)$ and $(n+1,j+1)$. From this variable interest rate we define return $R(n,j) = 1 + r(n,j)$. The return $R(n,j)$ is the return on one dollar invested in a bank or bonds between node (n,j) and its following nodes $(n+1,j)$ and $(n+1,j+1)$.

Previously, in equation (1.17), we defined the risk neutral upstate probability

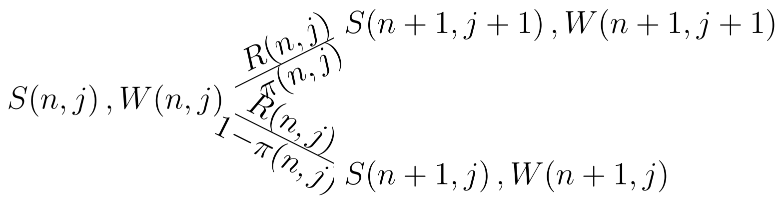
$$\pi = \frac{RS(0) - S(1,\downarrow)}{S(1,\uparrow) - S(1,\downarrow)} = \frac{RS(0,0) - S(1,0)}{S(1,1) - S(1,0)}. \tag{2.48}$$

This probability is defined only using asset prices at $t = 0$ and $t = 1$, but in our original N-step binomial pricing model it is applied to every time step between $t = 0$ and $t = N$. We now generalise the risk neutral probability so that it changes over different time steps and for different states. We do this by replacing fixed return R with variable return $R(n,j)$ and replacing $\uparrow$ with $j+1$ and $\downarrow$ with j. The variable risk neutral probability is

$$\pi(n,j) = \frac{R(n,j)S(n,j) - S(n+1,j)}{S(n+1,j+1) - S(n+1,j)}. \tag{2.49}$$

The probability $\pi(n,j)$ is the probability of the j state at time $t = n$ becoming the $j+1$ state at time $t = n+1$. The probability of the j state at $t = n$ becoming the j state at $t = n+1$ is $(1-\pi(n,j))$. As before we assume no arbitrage and require $0 < \pi(n,j) < 1$ for all n and j.

The binomial price tree from node (n,j) to the following two nodes is drawn below. This tree illustrates the variable return and variable risk neutral probability. Notice that the two branches following node (n,j) have the same $R(n,j)$.



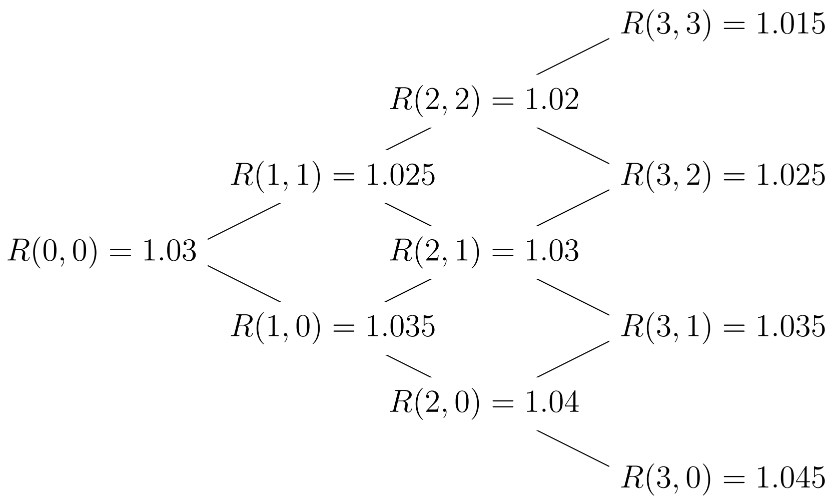
We write a new general pricing formula for an N-step binomial model by simply replacing the constant return R and constant risk neutral probability π in equation (2.3), with variable $R(n, j)$ and variable $\pi(n, j)$,

$$W(n, j) = R(n, j)^{-1}[\pi(n, j)W(n+1, j+1)+(1-\pi(n, j))W(n+1, j)]. \tag{2.50}$$

for $n = 1, 2, \ldots, N-1$ and $0 \le j \le n$. This is called the *generalised backward-induction pricing formula*. From equation (2.4) we also generalise the relationship between asset prices,

$$S(n, j) = R(n, j)^{-1}[\pi(n, j)S(n+1, j+1)+(1-\pi(n, j))S(n+1, j)]\,. \tag{2.51}$$

Example 2.12. Consider a European call written on some asset which in CRR notation has $S = 20$, $u = 1.1$ and $d = 1/u$. The strike price for this call is $K = 19$. The variable returns over each time step are given in the binomial pricing tree below. Find the premium of this call by constructing a four-step binomial model. Also calculate the premium of the call using a four-step binomial model, but with constant return $R = 1.03$ over each time step.



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2.6.2 Jamshidian's forward induction formula

In Section 2.2.1 we wrote a formula for the initial derivative value $W(0, 0)$ in terms of state prices $\lambda(N, j)$ and final derivative values

$W(N, j)$, shown in equation (2.8). We write the same formula for our new N-step binomial model,

$$W(0,0) = \sum_{j=0}^{N} \lambda(N,j) W(N,j)\,, \tag{2.52}$$

although, because of the variable return and variable probability, the terms in this equation will not equal the terms in equation (2.8).

Recall from Section 2.2.2 that the state prices $\lambda(N, j)$ are the value at $t = 0$ of the Arrow–Debreu security that pays one dollar at $t = N$ in some state j and zero dollars in any other state.

In Section 2.2.1, equation (2.12) describes the recursion relation (Jamshidian's forward induction formula) for the state prices $\lambda(n, j)$. We write a general version here by replacing R and π with our variable return and probability,

$$\lambda(n,j) = \begin{cases} \frac{\pi(n-1,n-1)}{R(n-1,n-1)}\lambda(n-1,n-1) & j = n\,, \\ \frac{1-\pi(n-1,j)}{R(n-1,j)}\lambda(n-1,j) + \frac{\pi(n-1,j-1)}{R(n-1,j-1)}\lambda(n-1,j-1) & \\ & 0 < j < n\,, \\ \frac{1-\pi(n-1,0)}{R(n-1,0)}\lambda(n-1,0) & j = 0\,, \end{cases} \tag{2.53}$$

and $\lambda(0,0) = 1$. Because $R(n,j)$ and $\pi(n,j)$ are variable, the solution to this recursion relation is complicated and is in general only solvable numerically.

3 The binomial model for other contracts

Contents

So far, we have only considered the valuation of European options with underlying assets such as stock. Here we use the binomial asset pricing model to determine the value of several different derivative products. We also consider valuations of derivatives on foreign exchange.

3.1 Valuing American options

European options can only be exercised on the expiry date. However, many options traded in Australia are American options which can be exercised any time prior to expiry. This flexibility in American options affects their valuation and means that American options are often worth more than European options with the same strike price and expiry.

In valuing American options we assume that owning the underlying asset does not provide an income. For example, if the underlying is shares, these shares do not pay dividends, or at least do not pay dividends in the time period being considered.

3.1.1 American call options

As discussed in Section 1.1.2, at expiry $t = T$ an American option is equivalent to a European option with the same underlying asset, strike and expiry. Thus,

$$C_A(T) = (S(T) - K)^+ . \tag{3.1}$$

To value the American option over times $0 \le t \le T$, compare it to a European option over the same time period, with the same strike price K and same underlying asset. The two options are identical, except that the American option can be exercised at any time $0 \le t \le T$. This extra flexibility makes the American option more valuable than the European option so

$$C_A(t) \ge C(t) . \tag{3.2}$$

We now show that prior to expiry, $0 \le t < T$, the value of an American call is

$$C_A(t) \ge (S(t) - K)^+ . \tag{3.3}$$

This states that any time prior to expiry an American call is worth at least its exercise value at that time, and therefore it is generally not optimal to exercise the option prior to expiry (you would obtain more or equal money if you sold the option). Remember, this assumes owning the underlying provides no income (e.g., no dividends).

Firstly we show that equation (3.3) is true when $S(t) > K$. Recall the present value $\text{PV}_t(K)$ is the amount which needs to be invested at time t to obtain K at time T; since interest rates are always positive we know that $\text{PV}_t(K) < K$. So, when $S(t) > K$,

$$S(t) - \text{PV}_t(K) > S(t) - K > 0 . \tag{3.4}$$

From put-call parity we know $C(t) - P(t) = S(t) - \text{PV}_t(K)$ and we also know that $P(t) \ge 0$. Therefore,

$$\begin{aligned} C_A(t) &\ge C(t) \\ &= S(t) - \text{PV}_t(K) + P(t) \quad \text{from put-call parity} \\ &\ge S(t) - \text{PV}_t(K) \quad \text{since } P(t) \ge 0 \\ &> S(t) - K \quad \text{since } \text{PV}_t(K) < K \\ &= (S(t) - K)^+ \quad \text{since } S(t) > K . \end{aligned} \tag{3.5}$$

This shows that $C_A(t) > (S(t) - K)^+$ when $S(t) > K$, that is, we have proved equation (3.3) for $S(t) > K$.

Finally we have to prove equation (3.3) for $S(t) \le K$. In this case, $(S(t) - K)^+ = 0$, but prior to expiry $C_A(t) \ge C(t) \ge 0$ and therefore $C_A(t) \ge (S(t) - K)^+$ and we have proved equation (3.3).

As stated above, if owning the underlying provides no income during the lifetime of an American option, then it is never worth exercising that American option prior to expiry. In this case, an American option is exercised at $t = T$ and has the same value as a European option at this time. Dividends are typically paid twice a year, so there are several short term American call options which fall between dividend payments and are exercised on expiry.

3.1.2 American put options

An American put option is always more valuable or of equal value to a European put, $P_A(t) \geq P(t)$ (for the same reasons American calls are more valuable than European calls), but there is no equation for American puts corresponding to equation (3.3). The difference between the value of an American and European put is the *early exercise premium*

$$\epsilon(t) = P_A(t) - P(t) \geq 0\,. \tag{3.6}$$

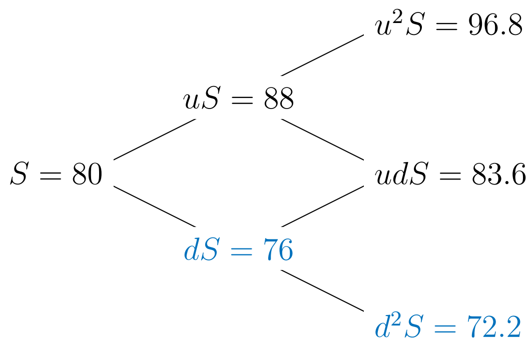
This $\epsilon(t)$ is the additional value of an American put which pays for the right to exercise early. Unlike American calls, it is sometimes profitable to exercise an American put prior to expiry.

To understand why and when a holder should exercise an American put we consider an example using the CRR notation in a two-step binomial model. Consider an American put option with strike price $K = \$80$ which may be exercised at any time step $n = 0, 1, 2$ with expiry $n = 2$. Choose $u = 1.1$, $d = 0.95$ and $R = 1.05$.

Firstly, calculate the risk neutral probabilities,

$$\pi = \tfrac{R-d}{u-d} = \tfrac{2}{3}\,, \quad (1-\pi) = \tfrac{1}{3}\,. \tag{3.7}$$

Draw the two-step binomial tree for the underlying asset prices $S(n, j)$ in this problem. The coloured nodes are where the underlying asset price is less than the strike, so exercising the put may be beneficial.



At expiry $n = 2$ the value of the option is the payoff $(K - S(2, j))^+$. Draw the binomial tree for the American put with the calculated values at expiry.

$$\begin{tikzcd}
& & P_A(2,2)=0 \\
& P_A(1,1) \arrow[ur, dash] \arrow[dr, dash] & \\
P_A(0,0) \arrow[ur, dash] \arrow[dr, dash] & & P_A(2,1)=0 \\
& P_A(1,0) \arrow[ur, dash] \arrow[dr, dash] & \\
& & P_A(2,0)=7.8
\end{tikzcd}$$

Now consider what happens at node $(1,1)$. At node $(1,1)$ we know that in the future the value of the option will be either $P_A(2,2)=0$ or $P_A(2,1)=0$. We also know that at node $(1,1)$ the strike price $K=\$80$ is less than the underlying value $S(1,1)=\$88$, so there is no possibility of exercising the put. Since we can never receive a payoff at $(1,1)$ or in the future, the option is worthless and $P_A(1,1)=0$.

At node $(1,0)$ the situation is different for two reasons. Firstly, because there is a possibility the option will be worth something in the future, we deduce it worth something at $n=1$. Secondly, the strike price $K=\$80$ is greater than the underlying $S(1,0)=\$76$ so there is the possibility of exercising the option. We have two ways of valuing the option, the general pricing formula (2.3) or the payoff obtained from exercising the option,

$$P_A(1,0) = \begin{cases} \frac{1}{R}[\pi P_A(2,1) + (1-\pi)P_A(2,0)] = 2.48\,, & \text{or} \\ (K - S(1,0))^+ = 4\,. \end{cases} \tag{3.8}$$

The value of the put is the highest number $P_A(1,0)=4$.

Finally, we calculate the value of the put at node $(0,0)$. Here, the strike price equals the share price so there is no reason to exercise the put. Thus the value of the put is calculated from the general pricing formula (2.3)

$$\begin{aligned} P_A(0,0) &= \tfrac{1}{R}[\pi P_A(1,1) + (1-\pi)P_A(1,0)] \\ &= 1.05^{-1} \times \tfrac{1}{3} \times 4 = 1.27\,. \end{aligned} \tag{3.9}$$

$$\begin{tikzcd}
& & P_A(2,2)=0 \\
& P_A(1,1)=0 \arrow[ur, dash] \arrow[dr, dash] & \\
P_A(0,0)=1.27 \arrow[ur, dash] \arrow[dr, dash] & & P_A(2,1)=0 \\
& P_A(1,0)=4 \arrow[ur, dash] \arrow[dr, dash] & \\
& & P_A(2,0)=7.8
\end{tikzcd}$$

We see from the above binomial price tree that the American put is worth the most at expiry $n = 2$ in node $(2, 0)$, so, in this case it is worth waiting until expiry to exercise the option (and hope that the node $(2, 0)$ is obtained).

How does this American put compare to a European put with the same strike price and underlying asset? For the European put, the values at $n = 2$ are calculated in the same way as the American put at $n = 2$. However, for earlier times we can only using the general pricing formula because we cannot exercise the option at these times. The results are given in the binomial tree below.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & P(2,2) = 0 \\
 & P(1,1) = 0 \arrow[ur, dash] \arrow[dr, dash] & \\
P(0,0) = 0.79 \arrow[ur, dash] \arrow[dr, dash] & & P(2,1) = 0 \\
 & P(1,0) = 2.48 \arrow[ur, dash] \arrow[dr, dash] & \\
 & & P(2,0) = 7.8
\end{tikzcd}$$

Note that we always have $P_A(t) \geq P(t)$ at all times t, and there are a few cases where the two put options are equal. At expiry $t = n = 2$ and we have $P_A(2, j) = P(2, j)$ for $j = 0, 1, 2$, and at node $(1, 1)$ both options are worthless, $P_A(1, 1) = P(1, 1) = 0$.

The early exercise premium at time $t = n = 0$ is

$$\epsilon(0) = P_A(0) - P(0) = 1.27 - 0.79 = 0.48\,. \tag{3.10}$$

This shows that is rational for a trader to pay \$0.48 more for this American put than a trader would for a similar European put.

Example 3.1. Consider an American and European call, both with strike price $K = \$80$ and with the same underlying asset as described above. The return is $R = 1.05$. Show that the two-step binomial model gives the same values for both options. Draw the binomial price tree.

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3.1.3 American put options and the binomial model

We saw in the previous section that the value of an American put option at some time t is the larger of the payoff obtained from exercising the option and the value determined from the general pricing formula (2.3). The value of an American option at node (n, j) is

$$P_A(n,j) = \begin{cases} W(n,j)\,, & \text{or} \\ (K - S(n,j))^+\,, \end{cases} \tag{3.11}$$

where $W(n, j)$ is calculated from either the general pricing formula (2.3) or the general backward-induction pricing formula (2.50) (with variable return $R(n, j)$ and risk neutral probability $\pi(n, j)$). The value of the American put $P_A(n, j)$ is the larger of the two right hand side equations in equation (3.11).

The algorithm for calculating the value of an American put option $P_A(n, j)$ is described below.

1. Find the value of the put at expiry $n = N$, $P_A(N, j) = (K - S(N, j))^+$.
2. Repeat Steps 3–4, for all time steps starting from time $n = N - 1$ and ending at time $n = 0$.
3. Calculate $W(n, j)$ using the value of the put at the later time step $(n + 1)$,

$$\begin{aligned} W(n,j) = \tfrac{1}{R(n,j)}[&\pi(n,j)P_A(n+1,j+1) \\ &+ (1-\pi(n,j))P_A(n+1,j)]\,. \end{aligned} \tag{3.12}$$

4. Determine the value of the put at node (n, j) using

$$P_A(n,j) = \max[(K - S(n,j))^+, W(n,j)]\,. \tag{3.13}$$

5. The premium of the American put is $P_A(0, 0)$.

This method for computing the value of an American put shows the flexibility of the binomial model for computing derivative products. The algorithm described above can be implemented in a spreadsheet, even when the number of steps is quite large. There is no Black–Scholes formula for American puts.

Example 3.2. With a ten-step binomial model, calculate the premium of a European put and call and an American put and call with strike price $K = 35$. In CRR notation, $S = 32$, $\sigma = 0.4$, $r = 0.04$ and $T = 0.5$; use equation (2.47) to calculate the up and down factors.

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□

3.1.4 Implied volatility

In Section 2.5 we discussed how to calculate the implied volatility σ using known derivative values and the BS model. The implied volatility can also be calculated from a binomial asset pricing model with CRR notation. Equation (2.47) shows how the volatility relates to the up and down factors of the CRR notation. This method also reveals volatility smiles and smirks.

Example 3.3. Consider Example 3.2, but now knowing option value $P_A(0, 0) = 7$ but not the volatility. Calculate the implied volatility.

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3.2 Barrier options

The put and call options which we have been discussing are often described as (plain) *vanilla* options while *exotic* options include barrier options, Asian options and average rate options. While vanilla options are typically traded on the stock exchange, exotic options are typically traded by banks over the counter. Here we only discuss barrier options. Barrier options have two forms: knock-out options and knock-in options.

Definition 3.1. A *knock-out option* ceases to exist when the underlying price goes beyond a certain value. There are two types of knock-out options. A *down-and-out option* ceases to exist when the underlying price goes below a given value. An *up-and-out option* ceases to exist when the underlying price goes above a given value.

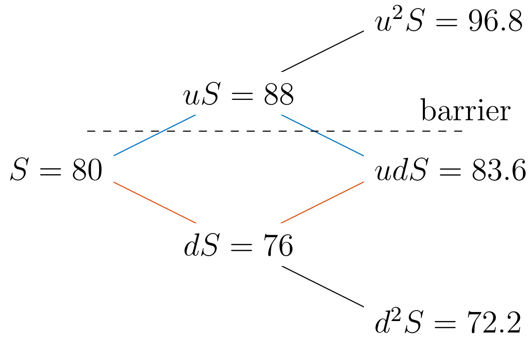
Definition 3.2. A *knock-in option* begins to exist when the underlying price goes beyond a certain value. There are two types of knock-out options. A *down-and-in option* begins to exist when the underlying price goes below a given value. An *up-and-in option* begins to exist when the underlying price goes above a given value.

These options can be either calls or puts and also European or American style.

The underlying price at which these options cease or begin to exist is the *barrier level.* The *barrier event* is when the underlying passes the barrier level. Barrier options are always cheaper than similar options with no barrier.

Barrier options can be valued in the same way as vanilla options using the binomial pricing model (2.3) or (2.50), however, the barrier level must be taken into account. A significant difference between barrier options and vanilla options is that barrier options are *path dependent* because of the barrier level.

For example, using a two-step binomial model, consider an up-and-out European call with barrier \$85, expiry $n = 2$, strike \$80 and underlying asset with $S = 80$, $u = 1.1$ and $d = 0.95$, and $R = 1.05$ (this asset was discussed in Section 3.1.2). The asset's binomial tree is drawn below and indicates the barrier level.



Nodes $(1,1)$ and $(2,2)$ are above the barrier so the option does not exist at these nodes. In addition, compare node paths $(0,0) \to (1,1) \to (2,1)$ (coloured blue) and $(0,0) \to (1,0) \to (2,1)$ (coloured red). Both paths begin and end below the barrier, but the first path goes above the barrier and the second path does not. The barrier option is worthless at the end of the first path (since it ceases to exist as soon as it goes over the barrier) but it is worth $(S(2) - K)^+ = 3.6$ at the end of the second path. Contrast this to the case of a vanilla European call which is path independent and both paths end with an option of value $(S(2) - K)^+ = 3.6$.

Example 3.4. With a ten-step binomial model, calculate the premium of an up-and-out European call and an up-and-out American put and call with barrier $B = 50$ and strike price $K = 35$. In CRR notation, $S = 32$, $\sigma = 0.4$, $r = 0.04$ and $T = 0.5$. Compare these premiums with those calculated in Example 3.2.

...... □

There is no put-call parity for barrier options, but there is a relationship known as *in-out parity*. In-out parity holds (assuming no arbitrage) in four different cases:

$$\begin{aligned} C(n,j) &= C_{\text{up-in}}(n,j) + C_{\text{up-out}}(n,j)\,, \\ C(n,j) &= C_{\text{down-in}}(n,j) + C_{\text{down-out}}(n,j)\,, \\ P(n,j) &= P_{\text{up-in}}(n,j) + P_{\text{up-out}}(n,j)\,, \\ P(n,j) &= P_{\text{down-in}}(n,j) + P_{\text{down-out}}(n,j)\,. \end{aligned} \tag{3.14}$$

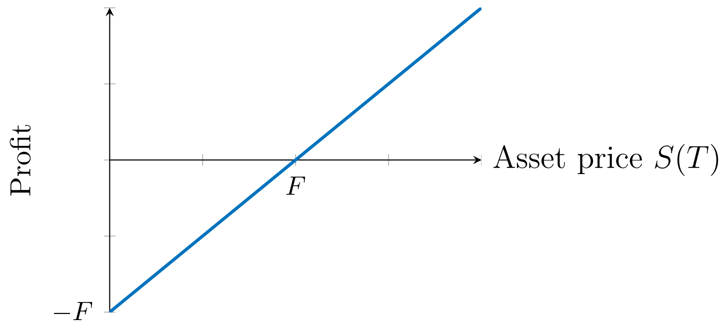
Note that in-out parity is only for *European* style options and that the two barrier options must be either 'up' or 'down' and have the same barrier level. In addition, all three options must have the same time to expiry, underlying asset and strike price.

Example 3.5. Using a ten-step binomial, calculate the premiums for an up-and-in European put and call with the same expiry, underlying asset and strike price as the options in Example 3.4. Compare them to the premiums calculated in Example 3.4. . . . . □

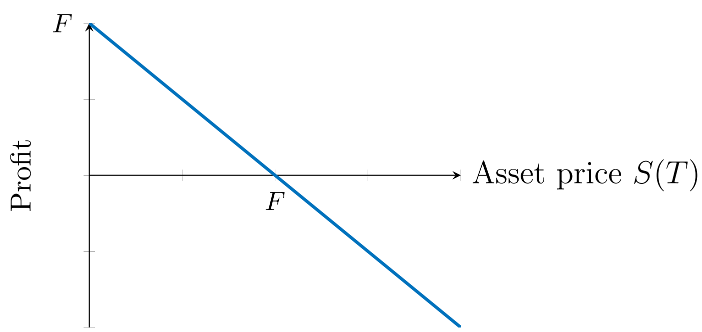
3.3 Forward contracts

Definition 3.3. A *forward contract* is an agreement between two parties where one party agrees to buy and the other agrees to sell an asset at a set price F at some future time T. The price F is called the *delivery* or *forward price* (it plays the same role as the option strike price K). A forward contract is binding and has zero premium.

In a forward contract, the buyer of the asset takes a long position and is bullish. The buyer is said to enter a 'long forward' contract. We plot the profit from a long forward contract at $t = T$ against the asset price $S(T)$.



The seller of the asset takes a short position and is bearish. The seller is said to enter a 'short forward' contract. We plot the profit from a short forward contract at $t = T$ against the asset price $S(T)$.



The graphs for long and short forward contracts show that one party in the contract will make a profit and the other will make an equal loss. For the long forward the profit at $t = T$ is $S(T) - F$

and for the short forward the profit is $F - S(T)$.

Unlike options, forward contracts have zero premium and so we do not need to use any mathematical models to derive the premium. However, we use the binomial model to calculate the forward price F. We now show that for the one-step binomial model the forward price should be $F = RS(0)$.

Consider the binomial pricing model (2.3) over just one step with asset prices given by equation (2.4). Nothing is paid for a forward contact at $t = 0$ so we have $W(0) = 0$. First consider a long forward. The value of a long forward at expiry $t = 1$ is the payoff $W(1,1) = S(1,1) - F$ and $W(1,0) = S(1,0) - F$. Substituting these values into the general pricing formula gives

$$\begin{aligned}
0 &= \tfrac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)] \\
0 &= \tfrac{1}{R}[\pi(S(1,1) - F) + (1-\pi)(S(1,0) - F)] \\
0 &= \tfrac{1}{R}[\pi S(1,1) + (1-\pi)S(1,0) - F] \\
0 &= S(0) - F/R \\
F &= RS(0)\,. \qquad (3.15)
\end{aligned}$$

Now we consider a short forward, We have $W(1,1) = F - S(1,1)$ and $W(1,0) = F - S(1,0)$ which we substitute into the general pricing formula,

$$\begin{aligned}
0 &= \tfrac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)] \\
0 &= \tfrac{1}{R}[\pi(F - S(1,1)) + (1-\pi)(F - S(1,0))] \\
0 &= \tfrac{1}{R}[F - \pi S(1,1) - (1-\pi)S(1,0)] \\
0 &= F/R - S(0) \\
F &= RS(0)\,. \qquad (3.16)
\end{aligned}$$

A large number of forward contracts use foreign currency as the underlying asset. Section 3.4.4 discusses forward contracts on foreign exchange.

Example 3.6. The one-step binomial model assumes no arbitrage so $F = S(0)R$ should not provide an arbitrage opportunity. Show that $F \neq S(0)R$ provides an arbitrage opportunity.

. . . . □

3.4 Derivatives on exchange rates

Australian exchange rates are typically quoted in *indirect form*, that is, what one Australian dollar buys in foreign currency. For example, the exchange rate: AU\$1/US\$0.78, gives the US dollar price of one Australian dollar (AU\$1 buys US\$0.78). An alternative approach is to quote exchange rates in *direct form*. In the direct form, the price of the foreign currency is given in terms of the domestic currency. In the above example, the direct version of the exchange rate is obtained by inverting the dollar values,

$$\text{US\$1/AU\$}\tfrac{1}{0.78} = \text{US\$1/AU\$1.28}\,. \tag{3.17}$$

Here we use the direct form (US\$1 buys AU\$1.28). We use $X(t)$ to denote the value at time t of \$1 of the foreign currency expressed in terms of the domestic currency. Here, '\$' does not necessarily represent a dollar but might represents any unit of foreign currency.

We show how to price a derivative written on an exchange rate, that is, the underlying asset is foreign currency. The values of these derivatives are always given in domestic currency.

Previously we denoted the underlying asset's value by $S(t)$, but here, to be clear we are talking about exchange rates, we use $X(t)$ (given in direct form). Just like the asset value $S(t)$ in the binomial model, at time step $t = 1$ we define two states for the exchange rate with values $X(1,\uparrow)$ and $X(1,\downarrow)$ and assume $X(1,\uparrow) > X(1,\downarrow)$. For example, the following binomial tree might describe the variation in the underlying asset $X(t)$ in a two-step model where the current $t = 0$ exchange rate is US\$1/AU\$1.28.

$$\begin{array}{ccc}
 & & X(2,\uparrow) = 1.3057 \\
 & X(1,\uparrow) = 1.2928 & \\
X(0) = 1.2800 & & X(2,\uparrow\downarrow) = 1.2800 \\
 & X(1,\downarrow) = 1.2673 & \\
 & & X(2,\downarrow) = 1.2548
\end{array}$$

When dealing with different currencies there are two different interest rates, the domestic interest rate r_d and the foreign interest rate r_f. For example, r_d is the interest rate in Australia and r_f is the interest rate in the US over one time step. Since there are different interest rates there are also two different returns, the domestic $R_d = 1 + r_d$, and the foreign, $R_f = 1 + r_f$, both over one time step. We have to modify the binomial asset pricing model to account for these two different returns.

We now look at a one-step binomial model for some general derivative. Consider a one-step binomial model where we know derivative values $W(1,\uparrow)$ and $W(1,\downarrow)$ in the domestic currency and want to find $W(0)$ in the domestic currency. As before we find a replicating portfolio which has the same value as the derivative at $t = 1$. We assume no arbitrage and thus the replicating portfolio has the same value as the derivative at $t = 0$.

At $t = 0$ define the replicating portfolio to consist of H_0 of the domestic currency and H_1 of the foreign currency. In domestic currency the value of the replicating portfolio at $t = 0$ is

$$V(0) = H_0 + H_1 X(0)\,. \tag{3.18}$$

At time $t = 1$ the portion of the portfolio in domestic currency has return R_d and the portion in foreign currency has return R_f over one time step. In addition, the exchange rate will change. As we want to build risk into our model, say the exchange rate can take one of two values: $X(1,\uparrow)$ or $X(1,\downarrow)$. Therefore the replicating portfolio at $t = 1$ has two possible values

$$\begin{aligned} V(1,\uparrow) &= H_0 R_d + H_1 R_f X(1,\uparrow)\,, \\ V(1,\downarrow) &= H_0 R_d + H_1 R_f X(1,\downarrow)\,. \end{aligned} \tag{3.19}$$

We equate the replicating portfolio and the derivative portfolio. At $t = 0$,

$$W(0) = V(0) = H_0 + H_1 X(0)\,, \tag{3.20}$$

and at $t = 1$,

$$\begin{aligned} W(1,\uparrow) = V(1,\uparrow) &= H_0 R_d + H_1 R_f X(1,\uparrow)\,, \\ W(1,\downarrow) = V(1,\downarrow) &= H_0 R_d + H_1 R_f X(1,\downarrow)\,. \end{aligned} \tag{3.21}$$

From simultaneous equations (3.21) we find

$$\begin{aligned} H_1 &= \frac{1}{R_f}\frac{W(1,\uparrow) - W(1,\downarrow)}{X(1,\uparrow) - X(1,\downarrow)}\,, \\ H_0 &= \frac{1}{R_d}\frac{X(1,\uparrow)W(1,\downarrow) - X(1,\downarrow)W(1,\uparrow)}{X(1,\uparrow) - X(1,\downarrow)}\,. \end{aligned} \tag{3.22}$$

Substitute these into equation (3.20) to find the derivative value at $t = 0$,

$$W(0) = \tfrac{1}{R_d}[\pi W(1,\uparrow) + (1-\pi) W(1,\downarrow)]\,, \tag{3.23}$$

where the risk neutral probabilities are

$$\begin{aligned} \pi &= \frac{\frac{R_d}{R_f}X(0) - X(1,\downarrow)}{X(1,\uparrow) - X(1,\downarrow)}\,, \\ 1-\pi &= \frac{X(1,\uparrow) - \frac{R_d}{R_f}X(0)}{X(1,\uparrow) - X(1,\downarrow)}\,. \end{aligned} \tag{3.24}$$

When there is no arbitrage $0 < \pi < 1$ and this requires

$$X(1,\downarrow) < \tfrac{R_d}{R_f}X(0) < X(1,\uparrow)\,. \tag{3.25}$$

The general pricing formula (3.23) looks exactly like our old general pricing formula (1.18) for the one-step model, but with return R changed to domestic return R_d. The exchange rate risk neutral probabilities are also similar to the asset price $S(t)$ risk neutral probabilities (1.17), but are dependent on both foreign and domestic returns.

We also construct an equation which relates exchange rates at $t = 0$ and $t = 1$,

$$\begin{aligned}\pi X(1,\uparrow) + (1-\pi)X(1,\downarrow) &= \frac{\frac{R_d}{R_f}X(0) - X(1,\downarrow)}{X(1,\uparrow) - X(1,\downarrow)}X(1,\uparrow) \\ &\quad + \frac{X(1,\uparrow) - \frac{R_d}{R_f}X(0)}{X(1,\uparrow) - X(1,\downarrow)}X(1,\downarrow) \\ &= \frac{\frac{R_d}{R_f}X(0)[X(1,\uparrow) - X(1,\downarrow)]}{X(1,\uparrow) - X(1,\downarrow)} \\ &= \frac{R_d}{R_f}X(0)\,. \end{aligned} \tag{3.26}$$

Or,

$$X(0) = \frac{R_f}{R_d}[\pi X(1,\uparrow) + (1-\pi)X(1,\downarrow)]\,. \tag{3.27}$$

This relation is similar to the relation (1.20) which compares asset prices at $t = 0$ and $t = 1$.

3.4.1 Exchange rate derivatives and the CRR model

The CRR notation is similar to before, with initial exchange rate $X = X(0)$, and up and down factors u and d defining the price variations in the underlying asset:

$$\begin{aligned} X(1,\uparrow) &= uX \\ X(1,\downarrow) &= dX\,. \end{aligned} \tag{3.28}$$

Substituting the above into the no arbitrage condition $0 < \pi < 1$ gives

$$0 < d < \frac{R_d}{R_f} < u\,. \tag{3.29}$$

and the risk neutral probabillities are

$$\pi = \frac{\frac{R_d}{R_f} - d}{u - d}\,, \quad (1-\pi) = \frac{u - \frac{R_d}{R_f}}{u - d}\,. \tag{3.30}$$

3.4.2 European calls on foreign exchange

A European call option on foreign exchange gives the holder the right, but not the obligation, to buy F in foreign currency at a fixed exchange rate k, called the *strike rate*, at the expiry time T. The foreign exchange amount F is called the *face value.*

At expiry T the value of the option in domestic currency is

$$C(T) = \max[0, F(X(T) - k)] = F(X(T) - k)^+ \,, \tag{3.31}$$

where $X(T)$ is the direct domestic exchange rate at time T.

Remark 3.1. Here k is the strike *rate*, not the strike *price* K. If the foreign exchange call option is exercised, then the amount paid (in domestic currency) for the foreign currency is Fk. The units of the strike rate k are AU\$/US\$, and since since F is measured in US\$, Fk must be in AU\$. □

Remark 3.2. The exchange rates X, like the strike rate k, have units AU\$/US\$. You can think of exchange rate X as how many Australian dollars per US\$1. Confusingly, the finance way of representing the direct exchange rate (e.g., US\$1/AU\$1.10 in the following example) is opposite to how a mathematician would write the units. □

Example 3.7. A European call option is bought with face value US\$1000 and strike rate US\$1/AU\$1.08. At expiry the exchange rate is US\$1/AU\$1.10. What is the value of the option at expiry?

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We now show how to apply the binomial pricing model for exchange rates (3.23) to a European call. Using CRR notation in the one-step binomial model, set $X(0) = X$, $X(1, \uparrow) = uX$ and $X(1, \downarrow) = dX$. When $X(1, \downarrow) < k < X(1, \uparrow)$, from equation (3.23) the call option at $t = 0$ has value

$$\begin{aligned} C(0) &= \tfrac{1}{R_d}[\pi F(X(1,\uparrow) - k)^+ + (1 - \pi)F(X(1,\downarrow) - k)^+] \\ &= \tfrac{1}{R_d}\pi F(X(1,\uparrow) - k) \\ &= \tfrac{\pi}{R_d}F(uX - k)\,. \end{aligned} \tag{3.32}$$

Typically the price of an option on foreign exchange is given as a percentage of the face value F. For example, a call option is priced at 5% (in units of AU\$/US\$). Then, for face value US\$100 000 the value of the option is $C(0) = \text{AU}(\$100\,000 \times 0.05) = \text{AU\$}5\,000$. If instead the face value is US\$1 000, then $C(0) = \text{AU\$}(1\,000 \times 0.05) = \text{AU\$}50$.

Example 3.8. An Australian company is bidding F (in USD) for a US denominated asset (for example, a warehouse to hold exports before sales to the US market), and the success or failure of the bid will not be known until time T. If the bid is successful, the amount to be paid is $FX(T)$ in AUD, but at the present time $t = 0$ the company does not know the AUD amount that will have to be paid because the exchange rate at time T is unknown. The company wants to limit its costs so buys a European call with premium $C(0)$ (as a percentage of face value), strike rate k, face value F and expiry date T. Explain how this option helps the company. Plot the company's payoff for the option.

. . . . □

3.4.3 European puts on foreign exchange

For European put options on foreign exchange the holder has the right to sell face value F of foreign currency at strike rate k at expiry T. The value of the put option at expiry is

$$P(T) = F(k - X(T))^+ . \tag{3.33}$$

The option is exercised when $k > X(T)$ and the holder sells F of foreign currency at larger exchange rate k. The option is not exercised when $k < X(T)$ and if the holder wishes to sell foreign currency will do so at the higher exchange rate $X(T)$.

Example 3.9. An Australian company exports to the US. When this company issues invoices to US companies it typically takes three months for the invoices to be paid. The company is owed US\$100 million and want to insure against a strengthening Australian dollar over the next three months so buys a European put option with face value $F =$ US\$100 million and strike rate k to expire in three months. How much money, in AU\$, can the company expect to receive in three months?

. . . . . . □

3.4.4 Forward contracts on foreign exchange

A large number of forward contracts use foreign exchange as the underlying asset. These contracts are commonly sold by banks over the counter. The holder of a long forward exchange rate contract is obliged to buy the face value F of foreign currency at a specified exchange rate k at time T, where k is now called the forward rate. At expiry T the value of the long forward is $F(X(T) - k)$ (which can be negative), and the short forward value is $-F(X(T) - k)$.

We show below that the forward rate of a forward contract is $k = \frac{R_d}{R_f} X(0)$ in a one-step model.

As discussed before, the premium of a forward contract is zero so in the one-step binomial model (3.23) we set $W(0) = 0$. Now substitute the upstate and downstate values of the long forward into equation (3.23) to obtain

$$0 = \tfrac{1}{R_d}[\pi F(X(1,\uparrow) - k) + (1-\pi)F(X(1,\downarrow) - k)]\,, \tag{3.34}$$

with risk neutral probability π. Rearrange equation (3.34) to obtain

$$\begin{aligned} 0 &= \tfrac{1}{R_d}[\pi X(1,\uparrow) + (1-\pi)X(1,\downarrow)]F - \tfrac{k}{R_d}F \\ 0 &= \tfrac{1}{R_f}X(0)F - \tfrac{k}{R_d}F \\ k &= \tfrac{R_d}{R_f}X(0)\,, \end{aligned} \tag{3.35}$$

where we made use of equation (3.27) to obtain the second line. This equation for the strike rate k is the exchange rate version of equation (3.16).

Remark 3.3. In Section 3.3 we introduced forward contracts and defined forward price F as the price to be paid for the underlying asset at expiry. However, for forward contracts on foreign exchange F is the face value, k is the forward rate and the amount paid for the foreign currency is Fk (measured in domestic currency). $\square$

4 Interest rates

Contents

Interest rate derivatives are the most commonly traded derivatives. An interest rate derivative is a contract giving the right to receive or the right to pay a future cash payment calculated using a fixed interest rate. These derivatives are typically traded over the counter and can be customised for different requirements, for example, to hedge a variable interest rate loan.

Here, our main concern is not to discuss these interest rate derivatives but to use an N-step binomial model to calculate future interest rates. This N-step model is based on a special type of bond called a zero-coupon bond. Once we have determined future interest rates, they can be used to value different financial products such as bonds, interest rate derivatives, and other derivatives whose values are described by the generalised backward-induction pricing formula (2.50).

4.1 Bonds

Bonds are often sold by governments or companies to raise money to finance new projects. A bond is essentially a loan made by an investor to a borrower, where the borrower is the government or company selling the bond. The bond must be paid back over a fixed period of time, and usually requires the borrower to make regular payments until the loan, plus interest, is repaid.

4.1.1 Coupon bonds

We first consider coupon bonds, where some principal amount P is borrowed and is paid back over n years. In making an investment decision concerning a bond, the principal P (also called the price of the bond) is important, but an investor will also consider the following.

- The face value F: the bond's final payment at maturity.
- Coupon rate c: the percentage of the face value F which must be paid annually.
- Current yield: a measure of the annual payout as a percentage of the bond price P.
- Yield to maturity r (or discount rate): the rate of return obtained from a bond, assuming that the bond is held until maturity.

To pay back the bond in n years, the borrower must make a coupon payment of cF at the end of each year, as well as a final payment of F.

Example 4.1. Consider a bond with face value \$20 000. The price of this bond is \$18 000 and the time to maturity is 5 years. The coupon rate is 5% pa. What is the bond's current yield?

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Assuming no arbitrage, the price of the bond P is equivalent to the sum of the present values of all coupon payments plus the face value paid at maturity. In other words, the amount P which the investor pays for the bond is equal to the current value of all future payments made to the investor. For n time steps to maturity, coupon rate c and face value F, the price of the bond is

$$P = \sum_{t=1}^{n} \frac{cF}{(1+r)^t} + \frac{F}{(1+r)^n}, \tag{4.1}$$

where the yield to maturity r discounts the coupon payments and face value to the present value.

Example 4.2. For the bond given in Example 4.1, show that the yield to maturity is 7.47%.

.... □

Example 4.3. Calculate the yield to maturity of a bond selling for \$950 which has a coupon rate of 7%, matures in 5 years and has face value \$1000.

.... □

4.1.2 Zero-coupon bonds

Zero-coupon bonds are a simple type of interest rate dependent asset. While zero-coupon bonds are not commonly traded, the values of these bonds are important in financial modelling and have several uses. For example, in this chapter we show how zero-coupon bond values relate to future variable interest rates and future cash flows.

Definition 4.1. A *zero-coupon bond* is a bond which makes no regular coupon payments, but only pays a single fixed price at maturity. They are also known as discount bonds because they are typically sold at a premium P which is significantly less than their maturity payoff F. We will assume a zero-coupon bond pays $F = \$1$ at maturity.

Define $P(t, T)$ as the value at time t of a zero-coupon bond with maturity T. The premium of this bond is its value at zero time $P(0, T)$. Since we assume this bond pays \$1 at maturity $t = T$, the maturity value is $P(T, T) = 1$. For compactness, we sometimes refer to a zero-coupon bond with maturity at time T as a T-zero.

By applying no arbitrage assumptions, Section 4.1.3 shows how zero-coupon bonds are related to rates of return and also how zero-coupon bonds are useful in comparing and valuing different future cash flows made at different times. However, in order to make effective use of zero-coupon bonds, we need to have methods for calculating the zero-coupon bond values $P(t, T)$ for different times t and with different maturity values T. Section 4.1.4 presents one method for calculating $P(t, T)$.

4.1.3 Current value of future cash flows

Zero-coupon bonds are often used to value future cash flows at the current time $t = 0$, or in other words, the present value PV_0 of a future cash flow. To do this, we first show how to calculate the $t = 0$ value of the zero-coupon bond $P(0, T)$ for a constant interest rate from time $t = 0$ to the bond's maturity $t = T$.

Consider a bank investment which is a replicating portfolio of a zero-coupon bond. The bank investment must have value $P(0,T)$ at time $t=0$ and value $P(T,T)=\$1$ at time $t=T$. For constant return R over $t=0$ to $t=T$, the initial value of the bank investment must be $P(0,T) = 1/R$ since this ensures that the final value of the investment is $R \times 1/R = \$1$. This derivation of $P(0,T)$ is based on a no arbitrage assumption as it assumes the payoffs from the bank investment at $t=0$ and $t=T$ are equivalent to the payoffs from the zero-coupon bond.

Over the time step $t=0$ to $t=T$ with constant return R, we draw the pricing 'tree' for this zero-coupon bond as a single straight line because it has only one final state (it always pays \$1 at maturity).

$$P(0,T) = 1/R \;\text{———}\; P(T,T) = 1$$

Example 4.4. The return over time step n to $n+1$ is $R(n)$. Consider a zero-coupon bond with maturity $n+1$. Assuming no arbitrage, show that the value of the zero-coupon bond at time n is

$$P(n, n+1) = \tfrac{1}{R(n)} \,. \tag{4.2}$$

......

□

To show how zero-coupon bonds may be used to determine the current $t=0$ value of a future payment, first consider a payment of C which is to be made at time T. We want to know the $t=0$ current value of this cash flow. At time T the value of this payment is $\text{PV}_T = C$. We invent a zero-coupon bond with maturity at time T so that, since $P(T,T) = \$1$,

$$\text{PV}_T = C = CP(T,T) \,. \tag{4.3}$$

Now, to find the value of this payment at any time $0 \le t \le T$, we adjust the variable time of the zero-coupon bond,

$$\text{PV}_t = CP(t,T) \,. \tag{4.4}$$

So, at the current time $t=0$ the value of the payment of C at time T is

$$\text{PV}_t = CP(0,T) \,. \tag{4.5}$$

We draw the value of the payoff at times $t=0$ and $t=T$ below.

$$CP(0,T) \;\text{———}\; CP(T,T) = C$$

A more complicated example is m cash flows $C_1, C_2, \ldots, C_m$ paid at different times $T_1 < T_2 < \cdots < T_m$. We want to find the

$t = 0$ current value of all these cash flows. We invent a set of m zero-coupon bonds which pay \$1 at different maturity times $T_1, T_2, \ldots, T_m$. In terms of zero-coupon bonds, the known payoffs are $C_1 P(T_1, T_1), C_2 P(T_2, T_2), \ldots, C_m P(T_m, T_m)$ made at times $T_1, T_2, \ldots, T_m$, respectively. Therefore, the value of all payoffs at $t = 0$ is

$$\text{PV}_0 = C_1 P(0, T_1) + C_2 P(0, T_2) + \cdots + C_m P(0, T_m) . \tag{4.6}$$

For $m = 3$ the values of the payoffs at times $t = 0, T_1, T_2, T_3$ are the three lines drawn below. The total payoff is the sum of all three lines.

$$\begin{array}{ccccc}
t=0 & & t=T_1 & t=T_2 & t=T_3 \\
C_3P(0,T_3) & \text{———————} & \text{———————} & \text{———————} & C_3P(T_3,T_3)=C_3 \\
C_2P(0,T_2) & \text{———————} & \text{———————} & C_2P(T_2,T_2)=C_2 & \\
C_1P(0,T_1) & \text{———————} & C_1P(T_1,T_1)=C_1 & &
\end{array}$$

While zero-coupon bonds are a useful tool for determining variable interest rates (as shown in Example 4.4) and cash flows at different times, zero-coupon bonds are uncommon, particularly for long maturity times, so their values usually need to be estimated using other data. There are various approaches that might be used to calculate zero-coupon bond values $P(0, T)$ for different maturities T. Section 4.1.4 shows one common approach, which is to use known information from coupon bonds.

4.1.4 Zero-coupon bonds valued from treasury data

The Australian Treasury sells coupon bonds with different maturities and coupon rates, and from knowing the details of these coupon bonds, we can calculate the values of zero-coupon bonds $P(0, T)$ with different maturities T.

Recall that in Section 4.1.3 we showed how cash flows can be represented by zero-coupon bonds. Therefore, we can invent a portfolio consisting of several zero-coupon bonds which provides the same cash flow as a portfolio consisting of one or several coupon bonds.

Example 4.5. Consider three coupon bonds with expiries $T = 1, 2, 3$, face values \$100 and paying coupons once a year, starting in one year's time. The coupon rates and prices are 6% and \$99.07 (one year bond), 7% and \$100 (two year bond), 8% and \$102.62 (three year bond). Calculate the zero-coupon bond current values $P(0, 1)$, $P(0, 2)$ and $P(0, 3)$.

We assume a portfolio of zero-coupon bonds can provide the same cash flow as the coupon bonds. For the one year coupon bond we spend \$99.07 at $t = 0$ to receive a cash flow of \$106 at $t = 1$. To receive a cash flow of \$106 at $t = 1$ from a zero-coupon bond you need 106 1-zeros. Therefore, at $t = 0$ you need to buy 106 1-zeros. Since all cash flows are equal, we equate the price of the coupon bond and the 1-zeros at $t = 0$:

$$99.07 = 106P(0, 1) .$$

For the two year coupon bond we spend \$100 at $t = 0$ and receive \$7 at $t = 1$ and \$107 at $t = 2$. To receive \$7 at $t = 1$ you need 7 1-zeros and to receive \$107 at $t = 2$ you need 107 2-zeros. Equate the prices of the coupon bond and zero-coupon bonds at $t = 0$:

$$100 = 7P(0, 1) + 107P(0, 2) .$$

Similarly, for the three year coupon bond,

$$102.62 = 8P(0, 1) + 8P(0, 2) + 108P(0, 3) .$$

We have three simultaneous equations for $P(0, 1)$, $P(0, 2)$ and $P(0, 3)$. Solving these equations gives $P(0, 1) = 0.9346$, $P(0, 2) = 0.8735$ and $P(0, 3) = 0.8162$. □

Say a coupon bond which expires at T_n, has face value F and provides coupons c at $T_1, T_2, \ldots, T_n$. The above example shows that the price P of the coupon bond can be written in terms of zero-coupon bond values:

$$P = \sum_{i=1}^{n} cFP(0, T_i) + FP(0, T_n) . \tag{4.7}$$

This equation is similar to (4.1) (a major difference is it does not require the yield to maturity to be constant) and is obtained by assuming a set of zero-coupon bonds provide the same cash flow as the coupon bond (this is a no arbitrage assumption). If we have several different coupon bonds with different P, c and F, then we can construct several equations like (4.7) and use simultaneous equations to solve for all $P(0, T_i)$. Often, because of missing data, it is not possible to uniquely solve for all $P(0, T_i)$.

Example 4.6. Treasury notes are short term bonds which do not pay coupons ($c = 0$), but we can still use these notes to calculate values of zero-coupon bonds. Say we have a 90 day treasury note with 12% pa discount rate (that is, yield to maturity). What is the price of a note which needs to be invested to produce \$1 000 000 at maturity? What is the corresponding zero-coupon bond value?

.... □

Example 4.7. Consider the three coupon bonds in Example 4.5. Say, rather than the three year coupon bond we now have a four year coupon bond with value \$105 and coupon rate 9%. Write down three equations in terms of present value zero-coupon bond values. Do these equations have a unique solution? Find as many zero-coupon bond values as possible.

.... □

We saw in the previous example that we might not have enough information to solve for all $P(0,T)$ uniquely. In addition, if the longest maturity of a coupon bond is N, then we can only calculate $P(0,T)$ for $T \leq N$. Also, we know that we should have $P(0,1) > P(0,2) > P(0,3) > \dots$, but we do not necessarily obtain this rule when we calculate these zero-coupon bond values from coupon bond values.

If we cannot calculate all required zero-coupon bond values from different coupon bond prices (4.7), then one alternative method is to use a approximate polynomial formula. For example, we could use a cubic equation

$$P(0,t) = a + bt + ct^2 + dt^3\,. \tag{4.8}$$

If we know four values of $P(0,t)$ (from coupon bond data, or otherwise), then we can find the constants a, b, c and d. We can easily find constant a since $P(0,0) = 1 = a$. Once we know all four constants, then we can use the cubic equation to calculate $P(0,t)$ for any t. This method is a fairly rough approximate and will not give good values for all $P(0,t)$. It will work best when the known $P(0,t)$ have t similar to the unknown $P(0,t)$

Example 4.8. In Example 4.5 we found $P(0,1) = 0.9346$, $P(0,2) = 0.8735$ and $P(0,3) = 0.8162$ and we always know $P(0,0) = 1$. These four zero-coupon bond values fit the cubic equation

$$P(0,t) = 1 - 0.0677t + 0.0024t^2 - 8.3333 \times 10^{-5}t^3\,.$$

Show that the above zero-coupon bonds do indeed satisfy the cubic equation. Use the cubic equation to calculate $P(0,4)$ and $P(0,10)$. Comment on the accuracy of these two zero-coupon bond values.

.... □

4.2 Zero-coupon bonds as the underlying asset

A zero-coupon bond may be the underlying asset of a derivative. In this case, we have two maturity times: that of the derivative and that of the zero-coupon bond.

4.2.1 Forward contracts on zero-coupon bonds

Here we discuss a forward contract which at maturity gives one party the right to buy (long forward) and the other party the right to sell (short forward) a particular zero-coupon bond for a fixed price (the forward price). That is, the underlying of the forward is a zero-coupon bond.

Specifically, consider a long forward entered into at time $t = 0$. The contract is the obligation to buy a zero-coupon bond with maturity $T + 1$ at time $t = 1$ for forward price F. We set $T \geq 1$. Note that the forward contract's maturity is $t = 1$ but the zero-coupon bond's maturity is $t = T+1$. So, when the forward matures, then the zero-coupon bond will mature in T time steps.

Assuming no arbitrage, we show that at $t = 1$ the forward price is

$$F = \frac{P(0, T + 1)}{P(0, 1)} . \tag{4.9}$$

The forward price is the price at which the zero-coupon bond must be bought when the forward matures at $t = 1$. Define R as the return over $t = 0$ to $t = 1$. We will make use of $R = 1/P(0, 1)$ (see Example 4.4) to describe the return from a bank investment over times $t = 0$ to $t = 1$. Also, recall that forward contracts have no premium.

Equation (4.9) is equivalent to

$$P(0, T + 1) - FP(0, 1) = 0 . \tag{4.10}$$

We assume that the above equation is not true, and show that this allows arbitrage. Therefore, when there is no arbitrage the above equation must be true.

First assume that

$$P(0, T + 1) - FP(0, 1) > 0 . \tag{4.11}$$

$t = 0$	cash flow
short sell a $(T + 1)$-zero	$P(0, T + 1)$
deposit cash	$-FP(0, 1)$
buy long forward	0
total:	$P(0, T + 1) - FP(0, 1) + 0 > 0$

$t = 1$	cash flow
with forward buy $(T + 1)$-zero, return to owner	$-F$
withdraw investment	F
total:	0

This scenario provides an arbitrage opportunity at $t = 0$ so cannot be true.

Now we consider the opposite case,

$$P(0, T+1) - FP(0,1) < 0\,. \tag{4.12}$$

$t = 0$	cash flow
borrow	$FP(0,1)$
buy $(T+1)$-zero	$-P(0,T+1)$
short forward	0
total:	$FP(0,1) - P(0,T+1) + 0 > 0$

$t = 1$	cash flow
repay loan	$-F$
with forward sell $(T+1)$-zero	F
total:	0

This scenario provides an arbitrage opportunity at $t = 0$ so cannot be true.

Therefore, assuming no arbitrage, the only possibility is $P(0, T + 1) - FP(0, 1) = 0$, which is equivalent to equation (4.9).

Example 4.9. Sometimes cash flow analyses, such as those above, can be performed with different portfolios. Show that in the above argument the lending and borrowing of cash can be replaced with zero-coupon bond purchases or short selling.

. . . . . . □

4.2.2 One-step binomial model

We now derive a one-step binomial model for a derivative with a $(T + 1)$-zero as the underlying asset. The zero-coupon bond has value $P(t, T + 1)$ at time t and we assume $T \geq 1$. Note that the derivative matures at $t = 1$, since this is a one-step model, but at time $t = 1$ the zero-coupon bond has not yet matured and will mature in T time steps.

In Section 1.6 we derived the general pricing formula (1.18). Here we use the same general pricing formula for the derivative,

$$W(0) = \tfrac{1}{R}[\pi W(1, \uparrow) + (1 - \pi) W(1, \downarrow)]\,, \tag{4.13}$$

where R is the return from $t = 0$ to $t = 1$. The risk neutral probabilities (1.17) π and $(1 - \pi)$ were originally defined in terms of asset prices $S(0)$, $S(1, \uparrow)$ and $S(1, \downarrow)$. Now the underlying asset is a zero-coupon bond with value $P(t, T + 1)$ at time t. In the risk neutral probabilities we simply replace $S(0)$ with $P(0,T+1)$ and $S(1,\uparrow/\downarrow)$ with $P(1,T+1,\uparrow/\downarrow)$:

$$\pi = \frac{RP(0,T+1) - P(1,T+1,\downarrow)}{P(1,T+1,\uparrow) - P(1,T+1,\downarrow)},$$
$$(1-\pi) = \frac{P(1,T+1,\uparrow) - RP(0,T+1)}{P(1,T+1,\uparrow) - P(1,T+1,\downarrow)}. \tag{4.14}$$

Similarly, in equation (1.20) replace $S(0)$ with $P(0,T+1)$ and $S(1,\uparrow/\downarrow)$ with $P(1,T+1,\uparrow/\downarrow)$:

$$P(0,T+1) = \tfrac{1}{R}[\pi P(1,T+1,\uparrow) + (1-\pi)P(1,T+1,\downarrow)]. \tag{4.15}$$

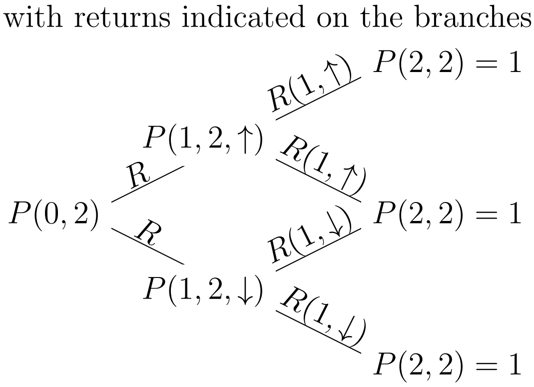
Since we are considering the one-step binomial model we assume we know the underlying values at $t = 0, 1$, that is we know the values $P(0,T+1)$, $P(1,T+1,\uparrow)$ and $P(1,T+1,\downarrow)$.

We now consider the case where the zero-coupon bond matures at $T+1 = 2$, that is, it matures one time step after the derivative. In Example 4.4 we showed how zero-coupon bond values relate to the return on a bank investment: $P(n,n+1) = 1/R(n)$. So, for $n = 0$ and $n = 1$ (in both the upstate and downstate):

$$R = \tfrac{1}{P(0,1)}, \quad R(1,\uparrow) = \tfrac{1}{P(1,2,\uparrow)}, \quad R(1,\downarrow) = \tfrac{1}{P(1,2,\downarrow)}. \tag{4.16}$$

Recall that for the underlying asset the upstate is defined to be larger than the downstate $P(1,2,\uparrow) > P(1,2,\downarrow)$, which means that $R(1,\uparrow) < R(1,\downarrow)$ so we have constructed a model with variable returns (notice that the upstate return is *less than* the downstate return). If we know the zero-coupon bond values in the above equations, then we know the return R from $t = 0$ to $t = 1$ and we also know the returns from $t = 1$ to $t = 2$ in both the upstate and downstate.

A pricing tree for a zero-coupon bond with maturity $T+1 = 2$ is drawn below, with returns indicated on the branches.



When we invest in a bank (or buy bonds) at $t = 0$ we know the return R at $t = 0$ and so we know what this investment will be worth at $t = 1$. Contrast this to an investment in stock where we know the value of the stock at $t = 0$ but we do not know for certain what the stock is worth at $t = 1$ (it is either in the upstate or downstate). This shows that investments which are reliant on interest rates are more predictable than investments in stock. Because we always know the return on a bank (or bond) investment, these investments are commonly referred to as *riskless*, at least over short time steps. However, interest rates can change and we cannot know for certain what the interest rate will be at some time further into the future. At $t = 0$ we cannot say for certain what the interest rate will be between times $t = 1$ and $t = 2$ and therefore the value of the investment at $t = 2$ is unknown (so over this time step we have returns $R(1,\uparrow)$ and $R(1,\downarrow)$).

From equation (4.15) we know that zero-coupon bond values are not independent, and since they are related to the returns via equation (4.16), we conclude that the returns are also not independent. Since $R(1,\uparrow)$ and $R(1,\downarrow)$ are not independent, assume that $R(1,\downarrow) = kR(1,\uparrow)$. Since $R(1,\uparrow) < R(1,\downarrow)$ we have $k > 1$. The parameter k is a measure of the interest rate volatility, with larger k indicating larger volatility. Substitute

$$P(1,\uparrow) = \tfrac{1}{R(1,\uparrow)} = \tfrac{k}{R(1,\downarrow)}\,, \quad P(1,\downarrow) = \tfrac{1}{R(1,\downarrow)}\,, \tag{4.17}$$

into equation (4.15) with $T = 1$ (that is, zero-coupon bond maturity $T + 1 = 2$) to obtain

$$P(0,2) = \frac{1}{R}\left[\frac{k\pi + (1-\pi)}{R(1,\downarrow)}\right], \tag{4.18}$$

which is rearranged to give the return

$$R(1,\downarrow) = \frac{1}{R}\left[\frac{k\pi + (1-\pi)}{P(0,2)}\right]. \tag{4.19}$$

Example 4.10. A European call with expiry $t = 1$ gives the option to buy a 2-zero for $K = \$0.95$. The zero-coupon bond has values $P(0,2) = \$0.9$, $P(1,2,\downarrow) = \$0.92$, $P(1,2,\uparrow) = \$0.97$ and we know $R = 1.05$. What is a rational price for the option premium? What are the upstate and downstate returns and interest rates at $t = 1$? Show that both these returns ensure that the 2-zero is worth \$1 at its maturity.

.... □

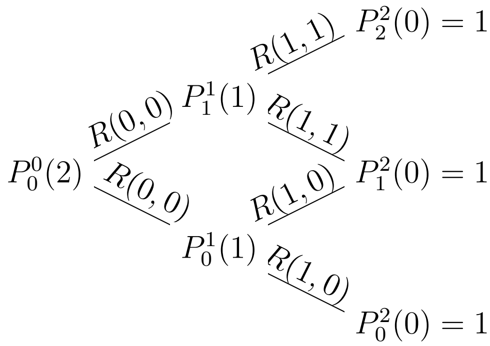
Example 4.11. Given $P(0,1) = 0.95$, $P(0,2) = 0.90$, $\pi = 0.5$ and $k = 1.04$, calculate all returns and interest rates.

.... □

4.2.3 Multi-step binomial models

Equation (4.16) shows how returns over different time steps are related to different zero-coupon bond values. Here we generalise this relationship between returns and zero-coupon bond values in a multi-step model.

We have been using $P(t,T)$ to describe the value of a T-zero at time t. We now introduce new notation which includes the state j. Define $P_j^n(T-n)$ as the value of a zero-coupon bond at node (n,j) which matures in $(T-n)$ time steps, and thus has maturity time $n+(T-n)=T$. When $n=T$ the zero-coupon bond is at maturity and $P_j^T(0)=1$. For example, $P_0^1(2)$ is the value of a zero-coupon bond at node $(1,0)$ which will mature in two time steps. As a second example, we redraw pricing tree from Section 4.2.2 which was for a zero-coupon bond with maturity at time 2 .



In Example 4.4, using a no arbitrage assumption, we showed that $R(n)=1/P(n,n+1)$. This formula relates the return $R(n)$ for a bank investment over times n to $n+1$ to the time n value of a zero-coupon bond $P(n,n+1)$ which matures at time $n+1$ (that is, it matures in one time step). Now introduce the state j so the return becomes $R(n,j)$ and the zero-coupon bond is $P_j^n(1)$. Therefore, the return $R(n,j)$ from node (n,j) to the next time step is

$$R(n,j)=\frac{1}{P_j^n(1)}\,. \tag{4.20}$$

We have already seen this formula in equation (4.16) for nodes $(0,0)$, $(1,0)$, and $(1,1)$ where $P(0,1)=P_0^0(1)$, $P(1,2,\downarrow)=P_1^1(1)$ and $P(1,2,\uparrow)=P_1^1(1)$.

Section 2.6 discussed an N-step binomial model with variable interest rates (or variable returns). Equation (4.20) gives us a method of calculating the variable returns, provided we know zero-coupon bond values $P_j^n(1)$ with maturity in one time step. Once we have the required returns over different time steps we apply the backward-induction pricing formula (2.50) to calculate derivative values over different time steps. Note that we can use these variable returns to calculate the values of many different derivatives; we are not restricted to derivatives written on zero-coupon bonds.

Example 4.12. A 2-zero has values $P_0^1(1) = \$0.96$ and $P_1^1(1) = \$0.98$ and the current return is $R = 1.03$. Calculate the returns $R(1,0)$ and $R(1,1)$. Using the variable returns, calculate the premium of a European call option with maturity in two time steps, strike $K = 20$ and values of the underlying stock determined from CRR notation $S = 20$, $u = 1.1$ and $d = 1/u$.

...... □

Remark 4.1. We only use the more complex notation $P_j^n(T-n)$ (and later, $F_j^n(t)$) when we need to specify both the node (n,j) and the maturity of the asset T. There are times where we do not need this more complex notation and so return to the simpler notation $P(n,T)$. For example, at initial node $(0,0)$ since there is only one state $j = 0$ we usually use the simpler notation $P(0,T)$ rather than $P_0^0(T)$. □

4.3 Ho–Lee model

To calculate return $R(n,j)$ using equation (4.20) we need to know $P_j^n(1)$, but at the current time $t = 0$ we do not usually know this zero-coupon bond value. The Ho–Lee model provides a method of calculating $P_j^n(1)$. The Ho–Lee model is based on a series of forward contracts begun at different nodes and with zero-coupon bonds as the underlying assets. The Ho–Lee model gives a formula for zero-coupon bond value $P_j^n(t)$, for any time n, state j and time to maturity t. Since we know $P_j^n(1) = 1/R(n,j)$, the Ho-Lee model is a method for determining future interest rates.

4.3.1 Forward price

In Section 4.2.1 we considered a forward contract entered into at $t = 0$ to buy, at time $t = 1$, a zero-coupon bond with maturity at time $T+1$. We showed that the zero-coupon bond should have forward price $F = P(0,T+1)/P(0,1)$. In our newer notation,

$$F = \frac{P_0^0(T+1)}{P_0^0(1)}. \tag{4.21}$$

This forward contract was entered into at node $(0,0)$.

Now consider a forward contract which is identical to the forward begun at $(0,0)$, except shifted to node (n,j). That is, consider a forward contract entered into at (n,j) to buy, at time $t = n+1$, a zero-coupon bond with maturity at $T+1+n$. The forward price is

$$F_j^n(T) = \frac{P_j^n(T+1)}{P_j^n(1)} \,. \tag{4.22}$$

In the notation $F_j^n(T)$, the time T describes the time steps between the forward's maturity and the zero-coupon bond's maturity. In other words, when the forward matures at $t = n+1$ the zero-coupon bond will mature in T time steps.

At (n, j) the zero-coupon bond is worth $P_j^n(T+1)$. At the next time step (when the forward contract matures) the bond is worth $P_j^{n+1}(T)$ or $P_{j+1}^{n+1}(T)$, depending on whether we move into the downstate or upstate, respectively. The Ho–Lee model is based on a number of assumptions concerning the values of this zero-coupon bond at different time steps and in different states. These assumptions lead to a general formula to price a zero-coupon bond at any node and with any maturity.

4.3.2 First assumption

The first assumption of the Ho–Lee model is that there exists perturbation functions $h^*(T)$ and $h(T)$ which satisfy $h^*(T) < 1 < h(T)$ for $T > 0$ and

$$P_j^{n+1}(T) = F_j^n(T)h^*(T) \,, \quad P_{j+1}^{n+1}(T) = F_j^n(T)h(T) \,. \tag{4.23}$$

Below we draw the binomial tree of the zero-coupon bonds values in terms of the forward prices obtained from equations (4.22) and (4.23).

$$\begin{tikzcd} & P_{j+1}^{n+1}(T) = F_j^n(T)h(T) \\ P_j^n(T+1) = F_j^n(T)P_j^n(1) \arrow[ur, dash] \arrow[dr, dash] & \\ & P_j^{n+1}(T) = F_j^n(T)h^*(T) \end{tikzcd}$$

As an example, consider the case $T = 0$. If $T = 0$, then the zero-coupon bond and the forward contract both mature at time $t = n+1$. From equation (4.22) with $T = 0$,

$$F_j^n(0) = \frac{P_j^n(1)}{P_j^n(1)} = 1 \,. \tag{4.24}$$

In other words, say you enter into a forward contract at (n, j), agreeing to buy at time $t = n+1$ something worth \$1 (the mature zero-coupon bond) then the rational forward price is $F_j^n(0) = 1$. When a zero-coupon bond matures, it does not matter what state the bond is in, all states pay \$1 so for the downstate $P_j^{n+1}(0) = 1$ and for the upstate $P_{j+1}^{n+1}(0) = 1$. Thus, $P_j^{n+1}(0) = F_j^n(0)$ and $h^*(0) = 1$. Similarly, $P_{j+1}^{n+1}(0) = F_j^n(0)$ and $h(0) = 1$.

Now we return to the general case with some $T > 0$. From equations (4.22) and (4.23) we obtain

$$
\begin{aligned}
P_j^{n+1}(T) &= \frac{P_j^n(T+1)}{P_j^n(1)} h^*(T)\,, \\
P_{j+1}^{n+1}(T) &= \frac{P_j^n(T+1)}{P_j^n(1)} h(T)\,.
\end{aligned}
\tag{4.25}
$$

These equations show how to write zero-coupon bonds values in terms of other zero-coupon bond values, without requiring the forward value $F_j^n(T)$.

We can use perturbation functions $h^*(T)$ and $h(T)$ to describe the risk neutral probabilities. Equation (2.49) gives the risk neutral probability in terms of asset value $S(n,j)$, so we replace $S(n,j)$ with $P_j^n(T+1)$ and the risk neutral probability is, for all (n,j) and $T > 0$,

$$
\begin{aligned}
\pi(n,j) &= \frac{R(n,j)P_j^n(T+1) - P_j^{n+1}(T)}{P_{j+1}^{n+1}(T) - P_j^{n+1}(T)} \\
&= \frac{\frac{1}{P_j^n(1)} P_j^n(T+1) - \frac{P_j^n(T+1)}{P_j^n(1)} h^*(T)}{\frac{P_j^n(T+1)}{P_j^n(1)} [h(T) - h^*(T)]} \\
&= \frac{1 - h^*(T)}{h(T) - h^*(T)}\,,
\end{aligned}
\tag{4.26}
$$

where we have substituted equations (4.20) and (4.25).

4.3.3 Second assumption

Although the first line of the above equation for $\pi(n,j)$ depends on node (n,j), the last line does not. The second assumption of the Ho–Lee model is that the risk neutral probability $\pi(n,j)$ is independent of node (n,j) so we can simply write it as π. If the risk neutral probability was not independent of node (n,j), then we would have to define the perturbations $h^*(T)$ and $h(T)$ as functions of (n,j).

4.3.4 Third assumption

The third assumption of the Ho–Lee model is that zero-coupon bond values are path independent; that is, the binomial pricing tree for some zero-coupon bond $P_j^n(T)$ is a recombining tree. This means, for example, that the path $(n,j) \to (n+1,j+1) \to (n+2,j+1)$ leads to the same zero-coupon bond value as the path $(n,j) \to (n+1,j) \to (n+2,j+1)$. Comparing these two different paths, we derive a relationship between the two perturbation functions $h^*(T)$ and $h(T)$.

Using equation (4.25) twice, the first path gives

$$P_{j+1}^{n+2}(T) = \frac{P_{j+1}^{n+1}(T+1)}{P_{j+1}^{n+1}(1)} h^*(T) = \frac{\frac{P_j^n(T+2)}{P_j^n(1)} h(T+1)}{P_{j+1}^{n+1}(1)} h^*(T) \,. \quad (4.27)$$

Also, set $T = 1$ in the second equation of (4.25),

$$P_{j+1}^{n+1}(1) = \frac{P_j^n(2)}{P_j^n(1)} h(1) \,, \quad (4.28)$$

and, after substituting into the first path and simplifying,

$$P_{j+1}^{n+2}(T) = \frac{P_j^n(T+2)}{P_j^n(2)} \frac{h(T+1)h^*(T)}{h(1)} \,. \quad (4.29)$$

Using equation (4.25) twice, the second path gives

$$P_{j+1}^{n+2}(T) = \frac{P_j^{n+1}(T+1)}{P_j^{n+1}(1)} h(T) = \frac{\frac{P_j^n(T+2)}{P_j^n(1)} h^*(T+1)}{P_j^{n+1}(1)} h(T) \,. \quad (4.30)$$

Also, set $T = 1$ in the first equation of (4.25),

$$P_j^{n+1}(1) = \frac{P_j^n(2)}{P_j^n(1)} h^*(1) \,, \quad (4.31)$$

and, after substituting into the second path and simplifying,

$$P_{j+1}^{n+2}(T) = \frac{P_j^n(T+2)}{P_j^n(2)} \frac{h^*(T+1)h(T)}{h^*(1)} \,. \quad (4.32)$$

Equate the two versions of $P_{j+1}^{n+2}(T)$ found from the first path (4.29) and the second path (4.32) to obtain, for any $T \geq 0$,

$$\frac{h(T+1)h^*(T)}{h(1)} = \frac{h^*(T+1)h(T)}{h^*(1)} \,. \quad (4.33)$$

4.3.5 The Ho–Lee model

From the risk neutral probability (4.26) and the relationship between perturbation functions $h^*(T)$ and $h(T)$ (4.33), we can derive formulas for the perturbation functions in terms of risk neutral probability π and a parameter δ. Together, π and δ are commonly referred to as the two parameters of the Ho–Lee model. The parameter δ is defined to satisfys $h^*(1) = \delta h(1)$ and since $h^*(T) < h(T)$ we have $\delta < 1$. We do not do this derivation but simply present the solution,

$$h(T) = \frac{1}{\pi + (1-\pi)\delta^T} \,,$$

$$h^*(T) = \frac{\delta^T}{\pi + (1-\pi)\delta^T}, \tag{4.34}$$

where

$$\delta = \frac{1 - \pi h(1)}{(1-\pi)h(1)}. \tag{4.35}$$

By substitution it can be shown that these formulas for $h(T)$ and $h^*(T)$ are a solution of equation (4.33).

We can also write the zero-coupon bond values in terms of perturbation function $h(n)$, parameter δ, and zero-coupon bond values at $t = 0$,

$$\begin{aligned} P_j^n(1) &= \frac{1}{R(n,j)} = \frac{P(0,n+1)}{P(0,n)} h(n)\delta^{n-j}, \\ P_j^n(T) &= \frac{P(0,n+T)}{P(0,n)} \frac{h(T)h(T+1)\cdots h(T+n-1)}{h(0)h(1)\cdots h(n-1)} \delta^{(n-j)T}. \end{aligned} \tag{4.36}$$

Recall that $P(0,T) = P_0^0(T)$ is the value of a zero-coupon bond at time $t = 0$ with maturity T. At maturity the value of a zero-coupon bond at any node (n, j) is

$$P_j^n(0) = \frac{P(0,n)}{P(0,n)} \frac{h(0)h(1)\cdots h(n-1)}{h(0)h(1)\cdots h(n-1)} \delta^{(n-j)\times 0} = 1, \tag{4.37}$$

as expected.

Equation (4.36) is the Ho–Lee model for calculating general zero-coupon bond values $P_j^n(T)$. To calculate a general zero-coupon bond value $P_j^n(T)$ we need to know several current time $t = 0$ zero-coupon bond values $P(0,n)$ for different n, and two parameters, δ and π. The zero-coupon bond values $P(0,n)$ required by the Ho–Lee model are generally called the *inputs* of the Ho–Lee model. Section 4.1.4 showed how to determine these inputs for the Ho–Lee model from treasury coupon bonds. If we know the Ho–Lee parameters π and δ, and inputs $P(0,n)$ for various n, we can derive all values of the zero-coupon bonds at different nodes $P_j^n(T)$ and all interest rates $R(n,j)$.

Example 4.13. Say $P(0,1) = 0.9091$, $P(0,2) = 0.8417$, $\pi = 0.4$ and $\delta = 0.8$. Find $r(1,1)$ using the Ho–Lee model. Discuss the problem with this interest rate.

. . . . □

A problem with the Ho–Lee model is that it may produce negative interest rates. To ensure interest rates are positive, we need to specify the valid range of δ. To ensure $r(n,j) \geq 0$ we require

$R(n,j) = 1 + r(n,j) \geq 1$ and therefore $P_j^n(1) = \frac{1}{R(n,j)} \leq 1$. It has been shown that to ensure $P_j^n(1) \leq 1$ we require

$$\delta \geq \delta_1 = \max_{1 \leq n \leq T-1} \left[\frac{\frac{P(0,n+1)}{P(0,n)} - \pi}{1-\pi} \right]^{1/n} . \tag{4.38}$$

Example 4.14. Consider a two-step Ho–Lee model with zero-coupon bond inputs $P(0,1) = 0.95$, $P(0,2) = 0.9$ and $P(0,3) = 0.83$, and parameters $\pi = 0.5$ and $\delta = 0.95$. Calculate all interest rates $r(n,j)$ and draw the binomial pricing tree for interest rates.

.... □

Example 4.15. (a) Write an expression for $h(T) - h^*(T)$ in terms of π and δ. Show that this expression decreases with δ, indicating that smaller δ implies greater volatility.

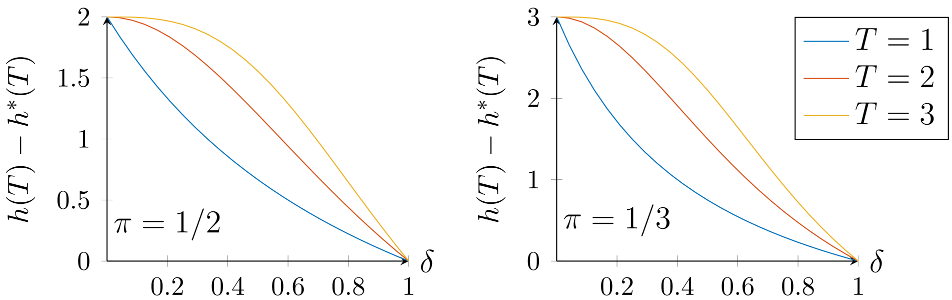
(b) Show that $1 + r(n, j+1) = \delta[1 + r(n,j)]$ and conclude that this leads to the same interpretation.

(a) Remember, $h^*(T) < 1 < h(T)$ so $h(T) - h^*(T) > 0$ and also $\delta < 1$. From equation (4.34),

$$\begin{aligned} h(T) - h^*(T) &= \frac{1}{\pi + (1-\pi)\delta^T} - \frac{\delta^T}{\pi + (1-\pi)\delta^T} \\ &= \frac{1 - \delta^T}{\pi + (1-\pi)\delta^T} \end{aligned}$$

Consider some points on this curve. When $\delta = 1$ we obtain $h(T) - h^*(T) = 0$, but when $\delta = 0$ we obtain $h(T) - h^*(T) = 1/\pi$. So, there is a greater difference between $h(T)$ and $h^*(T)$, and thus greater volatility, when $\delta = 0$. Thus smaller δ appear to give larger volatility.

To confirm our prediction, below we plot $h(T) - h^*(T)$ against δ for $\pi = \frac{1}{2}$ and $\pi = \frac{1}{3}$. The plots shows that the volatility is greatest for smallest δ.



(b) For interest rates, we know

$$1 + r(n,j) = R(n,j) = 1/P_j^n(1) .$$

$$1 + r(n, j+1) = R(n, j+1) = 1/P_{j+1}^{n}(1)\,.$$

So, dividing one by the other and using equation (4.36),

$$\begin{aligned}\frac{1 + r(n, j+1)}{1 + r(n, j)} &= \frac{P_j^n(1)}{P_{j+1}^n(1)} \\ &= \frac{\frac{P(0,n+1)}{P(0,n)} h(n)\delta^{n-j}}{\frac{P(0,n+1)}{P(0,n)} h(n)\delta^{n-(j+1)}} \\ &= \delta\,.\end{aligned}$$

So, $1 + r(n, j+1) = \delta[1 + r(n, j)]$. This relation is equivalent to $R(n, j+1) = \delta R(n, j)$. When $\delta \sim 1$ the two returns $R(n, j)$ and $R(n, j+1)$ are similar, but when $\delta \sim 0$ the two returns are as different as they can possibly be, indicating greater volatility.

□

5 Managing risk

Contents

While contracts such as forwards are binding, there is always the potential that the losing party will default on the contract. Here we discuss some methods for minimising risks in financial transactions.

5.1 Forwards and futures

5.1.1 Forward contracts

We have already encountered forward contracts in Sections 3.3, 3.4.4 (written on foreign exchange), 4.2.1 (written on zero-coupon bonds) and 4.3.1 (in the Ho–Lee model).

In a binomial model, define $W(n, j)$ as the value of the forward at node (n, j) with underlying value $S(n, j)$. The forward price is the amount for which the underlying asset must be bought or sold at expiry. For a forward contract initiated at node (n, j), define the forward price as $F(n, j)$ (note, this notation is different from the Ho–Lee model notation for the forward price $F_j^n(T)$).

Consider an N-step binomial model for a long forward contract initiated at node $(0, 0)$ and maturing at time step N. The forward price for this contract is $F(0,0)$. From Section 3.3, the value of the forward contract in state j at maturity time N is

$$W(N,j) = S(N,j) - F(0,0)\,. \tag{5.1}$$

In Section 4.1.2 we learned that a cash flow of C at time $t = T$ has value $CP(T,T) = C$ and the current value $t = 0$ of this cash flow is $CP(0,T)$, where $P(t,T)$ is the value at time t of a T-zero. Now we want to calculate how the value of the forward price varies over different time steps. To do this, we invent a zero-coupon bond with maturity at time $n = N$. At time N the forward price is $F(0,0)P(N,N) = F(0,0)$ and at initial time $n = 0$ the forward price is $F(0,0)P(0,N)$. Thus the initial value of the forward contract is

$$W(0,0) = S(0,0) - F(0,0)P(0,N)\,. \tag{5.2}$$

Since forward contracts have zero premium we must have $W(0,0) = 0$ and therefore the forward price is

$$F(0,0) = \frac{S(0,0)}{P(0,N)}\,. \tag{5.3}$$

At some general node (n,j), where $0 \le n \le N$ and $0 \le j \le n$, the value of the forward is

$$\begin{aligned} W(n,j) &= S(n,j) - F(0,0)P_j^n(N-n) \\ &= S(n,j) - \frac{S(0,0)}{P(0,N)}P_j^n(N-n)\,. \end{aligned} \tag{5.4}$$

Notice that at $n = 0$ we obtain $W(0,0) = 0$ and at $n = N$ we obtain equation (5.1).

Now consider a forward contract which is entered into at node (n,j) with maturity N. For this forward there are $N-n$ time steps until maturity. The forward price is now defined to be $F(n,j)$. At expiry node (N,j) the value of the forward is

$$\begin{aligned} W(N,j) &= S(N,j) - F(n,j) \\ &= S(N,j) - F(n,j)P_j^N(0)\,, \end{aligned} \tag{5.5}$$

where in the second line we have invented a zero-coupon bond with value $P_j^N(0) = 1$. At initial node (n,j) the forward value is

$$W(n,j) = S(n,j) - F(n,j)P_j^n(N-n)\,. \tag{5.6}$$

Since the premium is $W(n,j) = 0$ the forward price is

$$F(n,j) = \frac{S(n,j)}{P_j^n(N-n)}\,. \tag{5.7}$$

This equation for the forward price is a generalization of equation (5.3).

If the forward contract is entered into at (N, j) and immediately expires, then $W(N, j) = 0$ and, by substituting $n = N$ in equation (5.7),

$$F(N, j) = \frac{S(N, j)}{P_j^N(0)} = S(N, j)\,, \tag{5.8}$$

for all $0 \le j \le N$. This simply says that, in this case, the asset will be purchased for its current market price $S(N, j)$ and both long and short forward contracts make zero profit (a profit or loss requires that a produce is sold above or below its market price).

Remark 5.1. The forward contract discussed in this section has some general underlying asset with value $S(n, j)$. The zero-coupon bond with value $P_j^n(t)$ is not the underlying asset, it is simply a mathematical tool which allows us to calculate values of the forward price $F(n, j)$ at different time steps. In contrast, Section 4.2.1 considers a forward contract with a zero-coupon bond as the underlying. □

5.1.2 Futures contracts

Futures contracts are similar to forward contracts. One major difference is that forwards are sold over the counter and futures are exchange-traded. The other major difference is that futures are *margined* to minimise the risk of default. Like forward contracts, futures contracts have zero premium.

Definition 5.1. A *margin* is cash (or securities) which must be deposited with the exchange (or, for some contract types, with a broker) in order to cover a credit risk. The margin is usually a few percent of the value of the contract, but the actual amount is determined by the exchange, with different exchanges having different rules. As the value of the contract fluctuates, the value of the margin will also fluctuate.

Both parties in a futures contract must open a *margin account* with the exchange. While it is still possible for one of the parties to default and not pay the required amount into the margin account, the risks and potential losses in futures contracts are less than in forward contracts. In the following analysis we assume that neither party defaults on paying into the margin account.

Say a futures contract is entered into at node (n, j) and has maturity at time step N (in $N - n$ time steps). As before, we define the underlying's value at node (n, j) as $S(n, j)$. Define $G(n, j)$ as the price to be paid for the underlying at maturity, this is called the *future price* (this is similar to the forward price $F(n, j)$ in a forward contract). We will calculate rational values for both the margin account and the future price.

We show how to calculate $G(n, j)$ with a backward-induction formula (similar to equation (2.50)). To find $G(n, j)$ with this backward-induction formula, we need to know future prices for futures contracts begun at later nodes, $G(n+1, j+1)$ and $G(n+1, j)$. So, we need to know future prices at the final nodes (N, j). If the futures contract is entered into at (N, j) and immediately expires, then the future price is, for all $0 \leq j \leq N$, simply the value of the underlying asset:

$$G(N, j) = S(N, j)\,. \tag{5.9}$$

This is equivalent to the forward price $F(N, j)$ in equation (5.8).

Define $M(n, j)$ as the amount in the margin account for the long side of a futures contract at node (n, j). Like the underlying and the value of the future, the margin value varies at every node (n, j). We draw one step of the binomial pricing tree for the margin account.

$$\begin{array}{ccl} & & M(n+1, j+1) \\ & \nearrow & \\ M(n, j) & & \\ & \searrow & \\ & & M(n+1, j) \end{array}$$

The value of the margin account at time $n+1$ depends on the amount in the account at the previous time step n (which increases with return $R(n, j)$) and the change in the future price,

$$\begin{aligned} M(n+1, j+1) &= M(n, j)R(n, j) + [G(n+1, j+1) - G(n, j)]\,, \\ M(n+1, j) &= M(n, j)R(n, j) + [G(n+1, j) - G(n, j)]\,. \end{aligned} \tag{5.10}$$

In the backward-induction pricing formula (2.50), replace derivative value $W(n, j)$ with margin account value $M(n, j)$:

$$M(n, j) = \tfrac{1}{R(n,j)}[\pi(n, j)M(n+1, j+1) + (1 - \pi(n, j))M(n+1, j)]\,. \tag{5.11}$$

In the above equation $\pi(n, j)$ is the risk neutral upstate probability defined in the usual way, as in equation (2.49). Now substitute (5.10) into the above equation:

$$\begin{aligned} M(n, j) = {} & \frac{\pi(n, j)}{R(n, j)}\big[M(n, j)R(n, j) + [G(n+1, j+1) - G(n, j)]\big] \\ & + \frac{1 - \pi(n, j)}{R(n, j)}\big[M(n, j)R(n, j) + [G(n+1, j) - G(n, j)]\big] \end{aligned}$$

$$\begin{aligned}
&= M(n,j) + \frac{\pi(n,j)}{R(n,j)}[G(n+1,j+1) - G(n,j)] \\
&\quad + \frac{1-\pi(n,j)}{R(n,j)}[G(n+1,j) - G(n,j)]\,.
\end{aligned} \tag{5.12}$$

We rearrange this equation to obtain

$$\begin{aligned}
0 &= \pi(n,j)[G(n+1,j+1) - G(n,j)] \\
&\quad + (1-\pi(n,j))[G(n+1,j) - G(n,j)] \\
0 &= -G(n,j) + \pi(n,j)G(n+1,j+1) + (1-\pi(n,j))G(n+1,j)\,,
\end{aligned}$$

which gives the backward-induction formula for the future price

$$G(n,j) = \pi(n,j)G(n+1,j+1) + (1-\pi(n,j))G(n+1,j)\,. \tag{5.13}$$

Also, from equation (5.9), $G(N,j) = S(N,j)$. If we know $G(N,j)$ for all $0 \le j \le N$ then we use equation (5.13) to solve for all $G(n,j)$ and ultimately calculate $G(0,0)$. The backward-induction formula for $G(n,j)$ is similar to previously derived backward-induction formulas such as equations (2.50) and (2.51) but does not use the return $R(n,j)$.

We now look at how the margin account changes over several time steps. Return to equation (5.10) and subtract $M(n,j)$ from both sides:

$$\begin{aligned}
M(n+1,j+1) - M(n,j) &= M(n,j)[R(n,j)-1] \\
&\quad + [G(n+1,j+1) - G(n,j)] \\
&= M(n,j)r(n,j) \\
&\quad + [G(n+1,j+1) - G(n,j)]\,.
\end{aligned} \tag{5.14}$$

Similarly,

$$\begin{aligned}
M(n+1,j) - M(n,j) &= M(n,j)r(n,j) \\
&\quad + [G(n+1,j) - G(n,j)]\,.
\end{aligned} \tag{5.15}$$

Rather than looking at a general path, consider the example of a three-step model and the path $(0,0) \to (1,0) \to (2,1) \to (3,1)$. At each of the three nodes the margin account increases by

$$\begin{aligned}
M(1,0) - M(0,0) &= M(0,0)r(0,0) + G(1,0) - G(0,0)\,, \\
M(2,1) - M(1,0) &= M(1,0)r(1,0) + G(2,1) - G(1,0)\,, \\
M(3,1) - M(2,1) &= M(2,1)r(2,1) + G(3,1) - G(2,1)\,.
\end{aligned} \tag{5.16}$$

Add these three equations to give the change in the margin account between node $(0,0)$ and node $(3,1)$:

$$M(3,1) - M(0,0) = M(0,0)r(0,0) + M(1,0)r(1,0) \tag{5.17}$$

$$+ M(2,1)r(2,1) + G(3,1) - G(0,0)\,.$$

Rearrange the above equation to find the amount in the margin account at node $(3,1)$:

$$\begin{aligned} M(3,1) = M(0,0) &+ M(0,0)r(0,0) + M(1,0)r(1,0) \\ &+ M(2,1)r(2,1) + [G(3,1) - G(0,0)]\,. \end{aligned} \tag{5.18}$$

We see that the final amount in the margin account $M(3,1)$ equals the original amount in the margin account $M(0,0)$, plus interest earned over the three time steps, plus the change in the future price between nodes $(0,0)$ and $(3,1)$. The actual value of $M(0,0)$ is set by the Exchange which manages the margin account.

Now we consider the short side of a futures contract. Define $L(n,j)$ as the amount in the margin account for the short side of a futures contract at node (n,j). The value in this margin account is similar to equation (5.19), but the change in the value of the futures contract is *subtracted* from the account,

$$\begin{aligned} L(n+1,j+1) &= L(n,j)R(n,j) - [G(n+1,j+1) - G(n,j)]\,, \\ L(n+1,j) &= L(n,j)R(n,j) - [G(n+1,j) - G(n,j)]\,. \end{aligned} \tag{5.19}$$

We can analyse $L(n,j)$ the same way we just analysed $M(n,j)$, the only difference will be the sign change in front of the futures contract values.

5.1.3 Differences between forwards and futures

Example 5.1. Consider a two-step binomial model with stock as the underlying and variable interest rates. The stock prices are $S(0,0) = 8$, $S(1,1) = 10$, $S(1,0) = 6$, $S(2,2) = 12$, $S(2,1) = 8$ and $S(2,0) = 4$. The interest rates are $r(0,0) = 5\%$, $r(1,1) = 6\%$ and $r(1,0) = 4\%$. Calculate the forward price $F(0,0)$ and the future price $G(0,0)$. Show that $F(0,0) \neq G(0,0)$.

. . . . □

In the above example we calculated the forward value $F(0,0)$ and the future value $G(0,0)$ and found they are not equal. The difference between the two is small, but it is *not* due to numerical error. In general, $F(n,j) = G(n,j)$ for all nodes (n,j) when interest rates $r(n,j)$ (or returns $R(n,j)$) are *deterministic*. When interest rates are *stochastic* we might find $F(n,j) \neq G(n,j)$ at some nodes.

Definition 5.2. Interest rates $r(n,j)$ are *deterministic* when they are state independent (independent of state j). That is, for a given n, all interest rates $r(n,j)$ with $0 \le j \le n$ are equal. Interest rates are *stochastic* when they are state dependent (vary with state j). That is, for a given n, interest rates $r(n,j)$ with $0 \le j \le n$ are unequal.

When interest rates are deterministic we write $r(n,j) = r(n)$. Similarly, the returns are also deterministic and $R(n,j) = R(n)$. Note that deterministic interest rates do not need to be constant; they can be different at different times n.

We now show that $F(n,j) = G(n,j)$ when interest rates are deterministic. We know that at the final time step $S(N,j) = F(N,j) = G(N,j)$ and we know that $G(n,j)$ is calculated from the backward-induction formula (5.13). To show that $F(n,j) = G(n,j)$ for all (n,j) we need to show that $F(n,j)$ satisfies the same backward-induction formula as $G(n,j)$.

The forward price $F(n,j)$ is given by equation (5.7) and to calculate this equation we need to find $P_j^n(N-n)$. We find $P_j^n(N-n)$ using a backward-induction formula like equations (2.50) and (2.51). We know $P_j^N(0) = 1$ for all j and so, at the previous time step $N-1$ for any j,

$$\begin{aligned} P_j^{N-1}(1) &= \tfrac{1}{R(N-1)}[\pi(N-1,j)P_{j+1}^N(0) + (1-\pi(N-1,j))P_j^N(0)] \\ &= \tfrac{1}{R(N-1)}[\pi(N-1,j) + (1-\pi(N-1,j))] \\ &= \tfrac{1}{R(N-1)} \,. \end{aligned} \tag{5.20}$$

Notice that in the above equation the return $R(N-1)$ only depends on the time and not the state j because we assume interest rates are deterministic. Now consider $t = N-2$ for any j,

$$\begin{aligned} P_j^{N-2}(2) &= \tfrac{1}{R(N-2)}[\pi(N-2,j)P_{j+1}^{N-1}(1) \\ &\quad + (1-\pi(N-2,j))P_j^{N-1}(1)] \\ &= \tfrac{1}{R(N-1)R(N-2)} \,. \end{aligned} \tag{5.21}$$

If we continue calculating $P_j^n(N-n)$ by substituting into the backward-induction formula, then we will find, for all j and $0 \le n < N$,

$$P_j^n(N-n) = \frac{1}{R(N-1)R(N-2)\cdots R(n)} \,. \tag{5.22}$$

Substitute this form of $P_j^n(N-n)$ into the forward price formula (5.7) and manipulate this equation using asset formula (2.51) until it resembles the backward-induction formula (5.13) for $G(n,j)$:

$$\begin{aligned} F(n,j) &= R(N-1)R(N-2)\cdots R(n+1)R(n)S(n,j) \\ &= R(N-1)R(N-2)\cdots R(n+1) \end{aligned}$$

$$
\begin{aligned}
&\times [\pi(n,j)S(n+1,j+1) + (1-\pi(n,j))S(n+1,j)] \\
&= [\pi(n,j)R(N-1)R(N-2)\cdots R(n+1)S(n+1,j+1) \\
&\quad + (1-\pi(n,j))R(N-1)R(N-2)\cdots R(n+1)S(n+1,j)] \\
&= [\pi(n,j)F(n+1,j+1) + (1-\pi(n,j))F(n+1,j)]\,.
\end{aligned}
$$

So we obtain a backward-induction formula

$$F(n,j) = [\pi(n,j)F(n+1,j+1) + (1-\pi(n,j))F(n+1,j)]\,, \quad (5.23)$$

which is identical to the backward-induction formula (5.13) for $G(n,j)$. Since $F(N,j) = G(N,j)$ for all j and since $F(n,j)$ and $G(n,j)$ have the same backward-induction formula, we conclude $F(n,j) = G(n,j)$ at every node (n,j) when interest rates are deterministic.

5.2 Hedging

Section 1.2.2 briefly introduced hedging. We now discuss hedging in the binomial pricing model. We begin by considering an example scenario.

Say a writer sells a European call option for premium $C(0)$. This option is written on asset $S(t)$ with strike price K and has expiry T, and thus has value $(S(T)-K)^+$ at time T. If the writer does nothing to manage risk, then the writer's profit is $C(0) - (S(T) - K)^+$, which could be a very large loss if $S(T)$ is much larger than K. To make the potential loss smaller the writer might invest the premium $C(0)$; for example, if the continuously compounding interest rate is r then the writer's profit is $C(0)e^{rT} - (S(T) - K)^+$. Investing the premium does reduce the potential loss, but the loss can still be very large. A better way to minimise the potential loss is to use the premium to set up a *hedge* which is designed to have value $(S(T) - K)^+$ at expiry, thus completely covering the cost of the option even when $S(T)$ is large.

Notice that we discuss hedging options from the point of view of the writer. This is because the writer carries the greater risk in an option contract. In fact, the writer's risk is unlimited when selling calls because it is possible for the asset price to rise indefinitely. The holder's risk is less than the writer's risk because the holder chooses whether or not to exercise the option and is therefore only risking the relatively small option premium.

5.2.1 Dynamic hedging

We now show how a writer hedges or, more precisely, how a writer replicates a claim in order to minimise loss. 'Dynamic' hedging implies a hedge which is not constant but changes from time step to time step.

The writer starts with some amount of cash which is used to create a portfolio containing a bank investment and some amount of the asset S. The proportion of bank investment and assets in the portfolio is adjusted at each time step n, without adding or withdrawing cash from the portfolio. The aim is for the value of the portfolio to equal the value of the claim at expiry (such as $(S(T) - K)^+$ in the above call option example). When we have equalled the claim value we are said to have hedged, or replicated, the claim. This process is called *self-financing dynamic hedging*.

We now consider dynamic hedging in a N-step binomial model. We define the derivative that we are trying to hedge to have value $W(n, j)$ at (n, j) (such as the value of a European call option in the above example). As before, we can calculate $W(n, j)$ from the backward-induction formula (2.50). Define $H_0(n, j)$ as the value of the hedge bank investment at node (n, j) and $H_1(n, j)$ as the number of underlying assets held in the hedge at (n, j). Note that $H_0(n, j)$ and $H_1(n, j)$ are generalizations of the parameters of the replication portfolio discussed in Section 1.5.2. Remember that H_0 and H_1 can be negative or positive; $H_0 < 0$ for borrowing, $H_0 > 0$ for lending, $H_1 < 0$ for short selling and $H_1 > 0$ for buying.

We want the value of the hedge to match the value of the derivative at expiry. In the N-step binomial model we aim for the value of the hedge and the derivative to be equal at each node. So, at some node (n, j) we set

$$W(n, j) = H_0(n, j) + H_1(n, j)S(n, j)\,. \tag{5.24}$$

At the following time step $t = n + 1$ the investment increases by $R(n, j)$ and the asset has two possible values: $S(n + 1, j + 1)$ or $S(n + 1, j)$. For the derivative value and hedge value to remain equal we must have

$$\begin{aligned} W(n+1, j+1) =& H_0(n, j)R(n, j) + H_1(n, j)S(n+1, j+1)\,, \\ W(n+1, j) =& H_0(n, j)R(n, j) + H_1(n, j)S(n+1, j)\,. \end{aligned} \tag{5.25}$$

Subtract the first and second equation to obtain

$$H_1(n, j) = \frac{W(n+1, j+1) - W(n+1, j)}{S(n+1, j+1) - S(n+1, j)}\,. \tag{5.26}$$

The quantity of shares in the hedge $H_1(n, j)$ is called the *hedge ratio*. The value of the bank investment is obtained by rearranging equation (5.24),

$$H_0(n, j) = W(n, j) - H_1(n, j)S(n, j)\,. \tag{5.27}$$

Equations (5.26) and (5.27) tell us what we should hold in the hedge at each node (n, j).

5.2.2 Self-financing

Consider the first equation in (5.25), but replace n with $n-1$ and j with $j-1$ to obtain the derivative price at node (n,j):

$$W(n,j) = H_0(n-1,j-1)R(n-1,j-1) + H_1(n-1,j-1)S(n,j)\,. \tag{5.28}$$

Equation (5.24) gives another formula for $W(n,j)$. Equating the two formulas for $W(n,j)$ and rearranging gives

$$\begin{aligned}&[H_1(n,j) - H_1(n-1,j-1)]S(n,j)\\ &= -[H_0(n,j) - H_0(n-1,j-1)R(n-1,j-1)]\,.\end{aligned} \tag{5.29}$$

This equation shows that if we increase our stock (making $H_1(n,j) > H_1(n-1,j-1)$), then we must reduce the cash in the bank investment (making $H_0(n,j) < H_0(n-1,j-1)$). Similarly, if we decrease our stock, then we must increase our bank investment.

The condition in equation (5.29) is termed *self financing* because an increase in stock is paid for by a reduction in the bank investment and conversely, an increase in the bank investment is paid for by selling stock. The writer never has to add cash into the portfolio, but does have to transfer value between the bank investment and the asset.

Note that when the hedge is begun at node $(n,j) = (0,0)$ the premium of the option is, from equation (5.24)

$$W(0,0) = H_0(0,0) + H_1(0,0)S(0,0)\,. \tag{5.30}$$

That is, the hedge and the option premium are equal. When the writer receives the derivative premium a hedge can be constructed using the premium (and no additional cash), then, provided the hedge is managed correctly, when the derivative expires the value of the hedge should equal the value of the derivative.

Example 5.2. Suppose $S(0) = S = 80$, $u = 1.5$, $d = 0.5$, $R = 1.1$, using CRR notation in a three-step model. Let C be the value of a European call option with strike price 80 expiring at $n = 3$. Use a backward-induction formula with constant return to calculate all $C(n,j)$. Also calculate all $H_0(n,j)$ and $H_1(n,j)$ in a hedge.

. . . . $\square$

Example 5.3. In the previous example we calculated the market price for a call option $C(0,0) = \$34.08$ using the binomial model. Suppose that the call is selling for \$36.00. Show that there is an arbitrage opportunity (assuming the binomial model provides the correct prediction of the call value).

Solution: Short sell the call gaining \$36. Set up a hedge as described in the previous example, that is, borrow \$23.4410 (since $H_0(0,0) = -23.4410$) and buy 0.7190 of stock (since $H_1(0,0) = 0.7190$). The total payoff at $n = 0$ for short selling and setting up the hedge is $36 + 23.4410 - 0.7190 \times 80 = 1.9210$, that is, a payoff of \$1.9210. Now, maintain the hedge in the same way a writer would maintain the hedge. When the call expires the value of the hedge will be the same as the value of the call. Therefore, use the hedge to buy the call and return it to the original owner. The total payoff at $n = 3$ is zero.

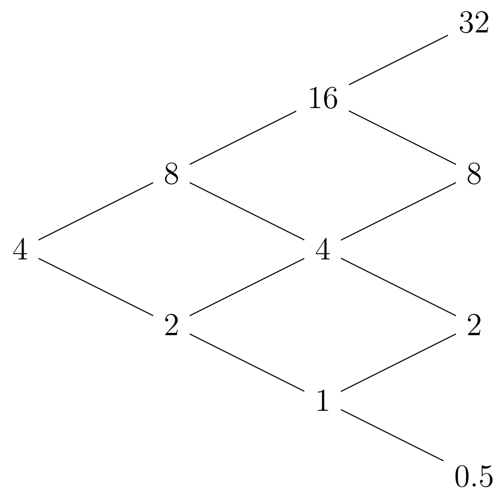
Over the three time steps the only nonzero payoff is \$1.92 at time $t = 0$, and by using a hedge all liabilities are met at $t = 3$. Thus we have arbitrage. □

Remark 5.2. Remember that any prediction of profit and successful hedging depends on the accuracy of the binomial model. If the binomial model is not correct, then there may be no profit and the hedge may not have sufficient funds. The binomial model is a good model of financial markets, but it is not perfect. □

5.2.3 Hedging for puts

The writer of a European put option is paid premium $P(0)$ at $t = 0$ and at expiry T the profit is (without a hedge but when investing the premium) $P(0)e^{rT} - (K - S(T))^+$. In this case the writer's financial risk is limited because the asset price cannot go below zero. However, compared to the premium, the loss can be quite large. Hedge equations (5.26) and (5.27) derived in the previous section are for general derivatives and so are applicable to European puts.

Example 5.4. Consider the three-step binomial model with interest rate $r = 25\%$ and stock prices shown in the following binomial tree. Consider a European put with strike price $K = 5$. Use risk neutral pricing to obtain the premium of a European put. Then calculate all H_0 and H_1 in a hedge and discuss what the writer must do at each node.



$\square$

5.3 The Greeks

In the previous section we saw how changes in underlying asset prices affect the value of the derivative and the writer's hedge portfolio. Understanding how different inputs affect the value of a derivative help a writer be better prepared when hedging. The Greeks describe the sensitivity of a derivative to various inputs, such as underlying asset price, expiry time and volatility. They are called 'Greeks' because most are symbolised by a Greek letter. We discuss these parameters for options using both the Black–Sholes model.

5.3.1 The delta of an option

The delta Δ of an option describes the sensitivity of an option's value to changes in the value of the underlying asset. For *continuous* models, such as the Black–Scholes model, this sensitivity is described by a derivative (a calculus derivative, not a finance derivative) of the option value with respect to the asset value.

For a call option in the Black–Scholes model

$$\Delta_C = \frac{\partial C}{\partial S}\,. \tag{5.31}$$

Recall from Section 2.4.1 that the $t = 0$ value of a European call is given by equation (2.39):

$$C(0) = S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)\,, \tag{5.32}$$

where S is the underlying asset price at time $t = 0$. From equation (2.40),

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\,,$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}\,, \tag{5.33}$$

and from equation (2.37) the normal cumulative distribution is

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy\,. \tag{5.34}$$

Substitute equation (5.32) into equation (5.31) and use the chain rule:

$$\begin{aligned}\Delta_C &= \mathcal{N}(d_1) + S\frac{\partial \mathcal{N}(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} - Ke^{-rT}\frac{\partial \mathcal{N}(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S} \\ &= \mathcal{N}(d_1) + \left[S\frac{\partial \mathcal{N}(d_1)}{\partial d_1} - Ke^{-rT}\frac{\partial \mathcal{N}(d_2)}{\partial d_2}\right]\frac{\partial d_1}{\partial S}\end{aligned} \tag{5.35}$$

where in the second line we use

$$d_2 = d_1 - \sigma\sqrt{T} \quad \text{which implies} \quad \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}\,. \tag{5.36}$$

We now use the identity (which we will not prove):

$$S\frac{\partial \mathcal{N}(d_1)}{\partial d_1} = Ke^{-rT}\frac{\partial \mathcal{N}(d_2)}{\partial d_2}\,, \tag{5.37}$$

which reduces equation (5.35) to

$$\Delta_C = \mathcal{N}(d_1)\,. \tag{5.38}$$

Thus we have derived a expression for the delta of a European call.

The delta of a European put can be derived in the same way as the delta of a European call, or from put-call parity. The delta of a European put is

$$\Delta_P = \frac{\partial P}{\partial S} = \mathcal{N}(d_1) - 1 = -\mathcal{N}(-d_1)\,, \tag{5.39}$$

since $\mathcal{N}(x) - 1 = -\mathcal{N}(-x)$ for any x.

For all x, the normal cumulative distribution $\mathcal{N}(x)$ is a number in the interval $[0, 1]$ and so the delta of the call also lies in $[0, 1]$ and is always positive. In contrast, the delta of the put lies between $[-1, 0]$ and is always negative. Positive Δ_C means that when S increases (or decreases) slightly, C also increases (or decreases). Negative Δ_P means that when S increases (or decreases) slightly, P decreases (or increases).

In general, any derivative $\partial y/\partial x$ tells us how y depends on x, or, more precisely, the rate of change of y with respect to x. Although (calculus) derivatives are only defined for continuous functions (such as the Black–Scholes model) approximations can be made for discrete functions (such as the N-step binomial model). Approximately,

$$\frac{\partial y}{\partial x} = \frac{\text{change in } y}{\text{change in } x} . \tag{5.40}$$

In the N-step model for an option the delta is approximated by

$$\Delta_C = \frac{\text{change in } C}{\text{change in } S} , \quad \Delta_P = \frac{\text{change in } P}{\text{change in } S} . \tag{5.41}$$

To calculate these changes for a call option, consider moving from node $(n+1, j)$ to node $(n+1, j+1)$; the asset price increases from $S(n+1, j)$ to $S(n+1, j+1)$ while the option price increases from $C(n+1, j)$ to $C(n+1, j+1)$ and thus

$$\Delta_C = \frac{C(n+1, j+1) - C(n+1, j)}{S(n+1, j+1) - S(n+1, j)} . \tag{5.42}$$

This Δ_C is the same as the hedge ratio $H_1(n, j)$ in equation (5.26) when $C = W$. Similarly Δ_P is the hedge ratio (5.26) when $P = W$.

Remark 5.3. Say we have some function $f(x_1, x_2, x_3)$. In general, the two derivatives $\partial f/\partial x_1$ and df/dx_1 are *not equal.* The former is the *partial derivative* and it describes the change in $f(x_1, x_2, x_3)$ due to changes in x_1, *while keeping variables x_2 and x_3 constant.* In contrast, the derivative df/dx_1 describes the change in $f(x_1, x_2, x_3)$ due to changes in x_1 while allowing x_2 and x_3 to change as x_1 changes. For example, when calculating $\partial C/\partial S$ we only allow S and C to change and fix all other variables such as the time to maturity, the interest rate and the volatility. □

5.3.2 Delta hedging

When delta hedging, an option writer buys Δ number of shares and also maintains a bank investment (either borrowing or lending money); this portfolio stays equal to the option value as the share price changes. In this way the writer can protect against potential losses, but it also limits the writer's potential profits. Since the delta is the same as the hedge ratio (5.26), delta hedging is similar to self-financing dynamic hedging. Note that the delta is not constant and will change over time. Thus a delta hedging trader will have to regularly adjust the hedge portfolio, similarly to self-financing dynamic hedging.

For example, say a call option is selling for $C(0)$ with share price S. To delta hedge the call, a writer sells the call for $C(0)$ and also buys $S\Delta_C$ shares, giving a cash flow of

$$H_0 = C(0) - S\Delta_C\,. \tag{5.43}$$

If H_0 flow is positive, then the writer will deposit the amount H_0 in a bank (or similar), but if H_0 is negative, then the writer must borrow the amount $|H_0|$ (for a call, H_0 is usually negative). In either case, the total cash flow will be zero, $C(0) - S\Delta_C - H_0 = 0$. The value of the hedge portfolio is the value of the cash H_0 plus the value of the Δ_C assets, that is, $H_0 + S\Delta_C$, and this hedge portfolio is equal to the value of the call $C(0)$.

We now show that the hedge portfolio stays equal to the call value when the share price changes. Define a change in share price as s (which might be negative or positive) so that the new share price is $S + s$. Define the new value of the option as $C(1)$ and use the delta to calculate it:

$$\begin{aligned} \Delta_C &= \frac{\text{change in } C}{\text{change in } S} = \frac{C(1) - C(0)}{s}\,, \\ C(1) &= C(0) + s\Delta_C\,. \end{aligned} \tag{5.44}$$

When calculating the delta, we assume that no other parameters change (because we use partial derivatives). Therefore, the time is constant and so the cash H_0 is also constant (there is no time for interest to accumulate). So, the delta hedge is now worth

$$H_0 + (S + s)\Delta_C = H_0 + S\Delta_C + s\Delta_C = C(0) + s\Delta_C\,, \tag{5.45}$$

which is equal to the new call value $C(1)$.

Now consider a put option selling for $P(0)$ with share price S. The writer sells the option and short sells shares (since $\Delta_P < 0$), giving a cash flow of

$$H_0 = P(0) - S\Delta_P\,. \tag{5.46}$$

The amount H_0 is then deposited or borrowed (for a put, H_0 is usually positive so the money is deposited in a bank), giving a total cash flow of $P(0) - S\Delta_P - H_0 = 0$. The value of the hedge portfolio is the value of the cash H_0 plus the value of the Δ_P assets, that is, $H_0 + S\Delta_P$, and this hedge portfolio is equal to the value of the put $P(0)$. Now, for a share price change to $S + s$ the change in put value is $P(1) = P(0) + s\Delta_P$. So, the delta hedge is now worth

$$H_0 + (S + s)\Delta_P = H_0 + S\Delta_P + s\Delta_P = P(0) + s\Delta_P\,, \tag{5.47}$$

which is equal to the new put value $P(1)$.

Deltas for calls are always positive and a writer can hedge a call by buying the underlying security (go short on the call and long on the underlying). Deltas for puts are always negative and a writer can hedged a put by short selling the underlying security (go short on the put and short on the underlying). It is also possible for a holder to delta hedge by going long on a call and short on the underlying, or going long on a put and long on the underlying. However, since the holder has less risk than the writer, hedging is usually not as important.

There are more complicated ways to delta hedge. For example, a writer delta hedging a call need not buy the underlying, but might instead buy a call option with a delta equal to the delta of the call being hedged. Or, the writer could buy two calls whose deltas sum to the delta of the call being hedged.

Example 5.5. Suppose the share price is \$10, the price of a call option is \$1 and the delta of the call option is 0.4. Show how a writer who has sold 500 of these calls can successfully delta hedge with the Black–Scholes model when the share price increases to \$10.50.

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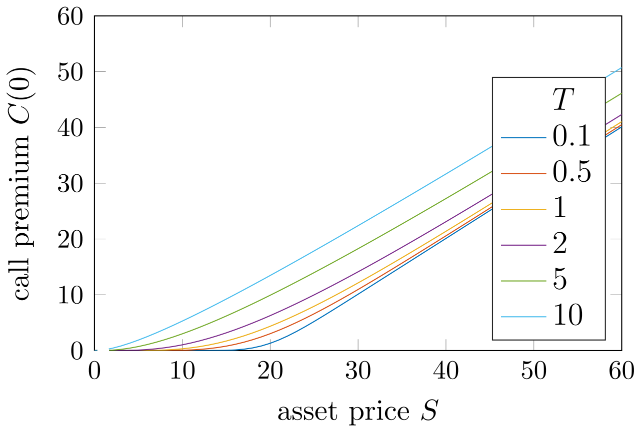
Example 5.6. Consider a 90 day call option with strike price $K = 60$ written on stock with current price $S = 60$. Assume that the risk free interest rate is $r = 8\%$ pa and the yearly volatility of S is $\sigma = 30\%$. Use the Black–Scholes model to price the option at $t = 0$ and calculate the delta. Contrast a writer who does not delta hedge with one that does delta hedge if after one day the share price is \$60, \$61 or \$59. Say both writers sell 1 000 options.

. . . . □

Example 5.7. The plots below show how option values $C(0)$ and $P(0)$, and their deltas, Δ_C and Δ_P, vary with asset price S and time to expiry T. Parameters chosen are strike price $K = 20$, interest rate $r = 0.05$ and volatility $\sigma = 0.5$. Discuss the different behaviour of the deltas over S and T.


60
50
40
30
20
10
0
call premium $C(0)$
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10
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60
asset price S
T
0.1
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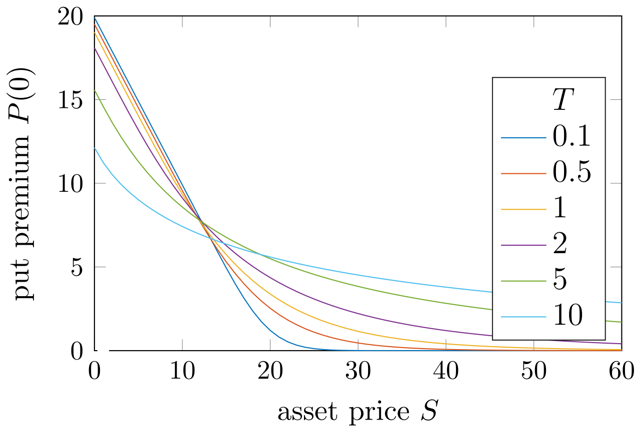
1
2
5
10


20
15
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put premium $P(0)$
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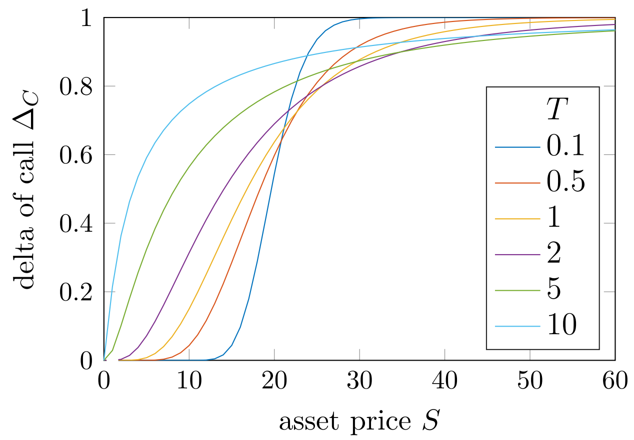
1
0.8
0.6
0.4
0.2
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delta of call Δ_C
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asset price S
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5
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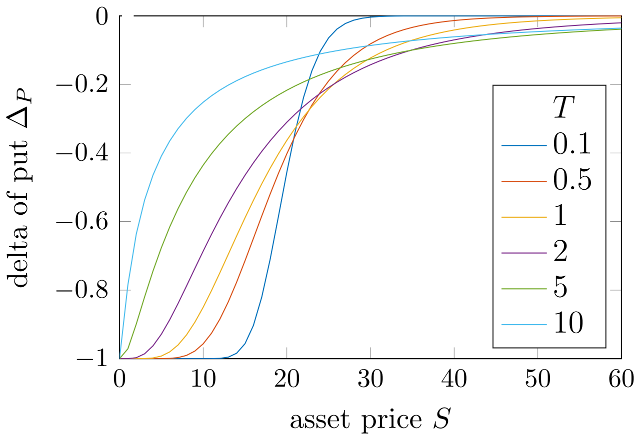

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5.3.3 The gamma of an option

The gamma Γ is the rate of change of delta Δ with respect to the value of the underlying asset S. For call and put options, respectively,

$$\Gamma_C = \frac{\partial \Delta_C}{\partial S}\,, \quad \Gamma_P = \frac{\partial \Delta_P}{\partial S}\,. \tag{5.48}$$

In the Black–Scholes model the gamma of call options equals the gamma of put options:

$$\Gamma_C = \Gamma_P = \frac{e^{-d_1^2/2}}{S\sigma\sqrt{2\pi T}}\,, \tag{5.49}$$

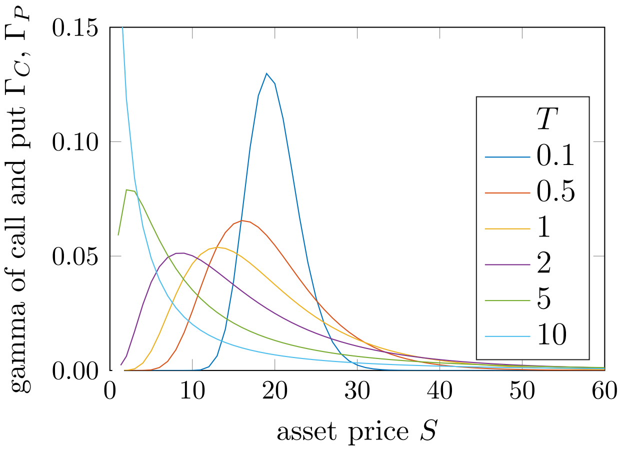
and this gamma is always positive (other models for other derivatives may have negative gamma).

We have shown that delta is essentially the hedge ratio, and thus gamma is the sensitivity of the hedge ratio to changes in the value of the underlying asset. Like delta, gamma is an important tool in hedging and a trader should consider both quantities. If gamma is zero, then the delta is constant (at least over a short time period), and the trader knows the hedge portfolio does not need adjusting. When the gamma is nonzero, the trader must prepare to adjust the hedge portfolio.

If the gamma is positive and the underlying asset price is increasing, then the delta is also increasing. In the case of a call, since the delta is always positive an increasing delta is increasing in magnitude. So, a writer who is hedging a call should prepare to buy more of the underlying asset. In the case of a put, since the delta is always negative an increasing delta is decreasing in magnitude. So a writer who is hedging a put should reduce the number of short-sold underlying assets in the hedge portfolio (that is, buy some of the underlying asset and return it to the original owner).

If the gamma is positive and the underlying asset price is decreasing, then the delta is also decreasing. In the case of a call, since the delta is always positive a decreasing delta is decreasing in magnitude, so a writer who is hedging should prepare to sell some of the underlying asset. In the case of a put, since the delta is always negative a decreasing delta is increasing in magnitude, so a writer who is hedging should prepare to short sell more of the underlying asset.

Example 5.8. The plot below shows how Γ_C and Γ_P vary with asset price S and time to expiry T. Parameters chosen are strike price $K = 20$, interest rate $r = 0.05$ and volatility $\sigma = 0.5$. Discuss the different behaviour over S and T.



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5.3.4 The theta of an option

The theta Θ of an option is the rate of change of the option value with respect to time t. Since time t increases as time to maturity T decreases, for calls and puts, respectively,

$$\Theta_C = \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial T}\,, \quad \Theta_P = \frac{\partial P}{\partial t} = -\frac{\partial P}{\partial T}\,. \tag{5.50}$$

In the Black–Scholes model

$$\begin{aligned} \Theta_C &= -\frac{S\sigma e^{-d_1^2/2}}{2\sqrt{2\pi T}} - rKe^{-rT}\mathcal{N}(d_2)\,, \\ \Theta_P &= -\frac{S\sigma e^{-d_1^2/2}}{2\sqrt{2\pi T}} + rKe^{-rT}\mathcal{N}(-d_2)\,. \end{aligned} \tag{5.51}$$

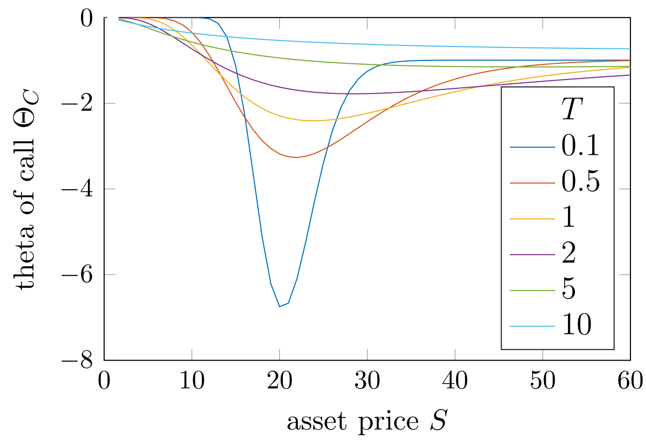
In the Black–Scholes model we can combine Δ, Γ and Θ (for calls or puts) into one formula:

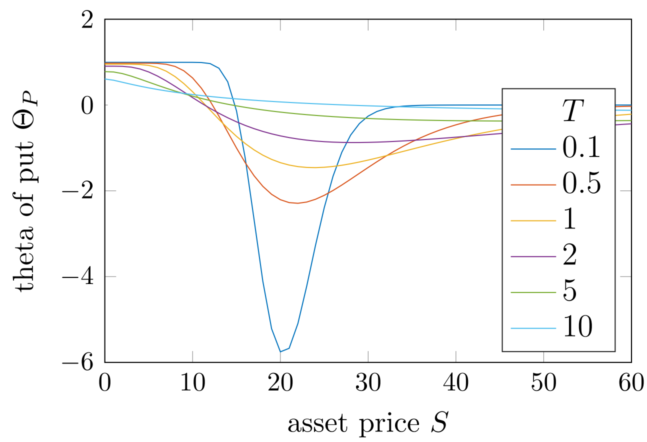
$$\Theta_C + rS\Delta_C + \frac{1}{2}\sigma^2 S(0)^2\Gamma_C - rC(0) = 0\,,$$

$$\Theta_P + rS\Delta_P + \frac{1}{2}\sigma^2 S(0)^2 \Gamma_P - rP(0) = 0\,, \tag{5.52}$$

and thus it is possible to calculate one of these three Greeks, provided we know the other two.

Example 5.9. The plot below shows how Θ_C and Θ_P vary with asset price S and time to expiry T. Parameters chosen are strike price $K = 20$, interest rate $r = 0.05$ and volatility $\sigma = 0.5$. Discuss the different behaviour over S and T.





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5.3.5 The vega of an option

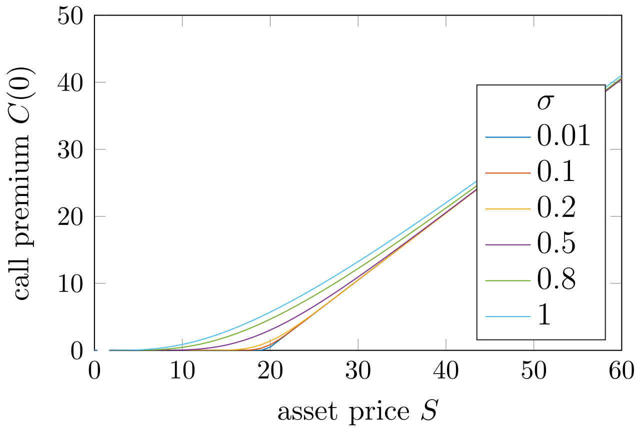
Vega is not a Greek letter but is symbolised by the Greek letter ν ('nu'). Vega is the rate of change of the option price with respect to the volatility σ. For calls and puts, respectively,

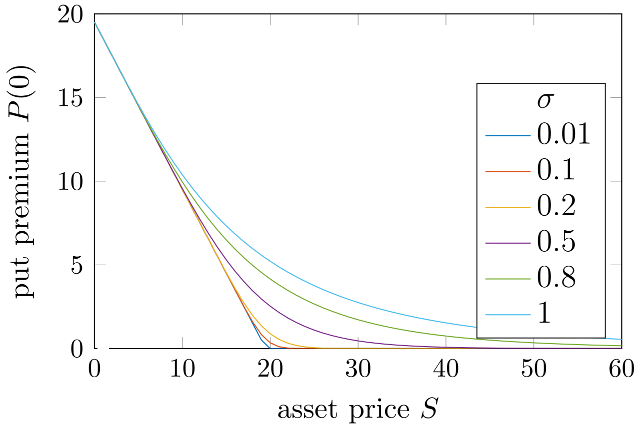
$$\nu_C = \frac{\partial C}{\partial \sigma}\,, \quad \nu_P = \frac{\partial P}{\partial \sigma}\,. \tag{5.53}$$

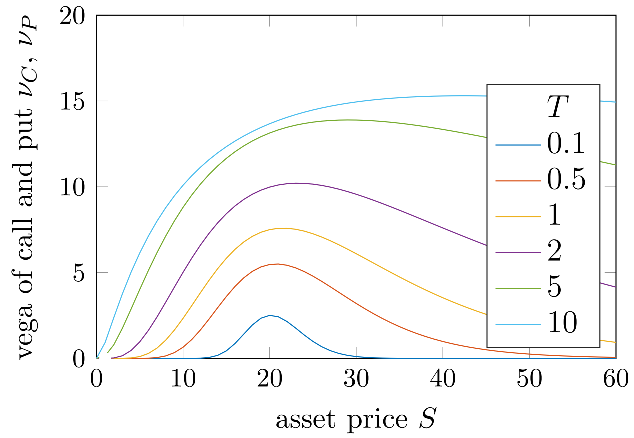
In the Black–Scholes model for the vega is the same for call and put options,

$$\nu_C = \nu_P = \frac{\partial C}{\partial \sigma} = \frac{S\sqrt{T}e^{-d_1^2/2}}{\sqrt{2\pi}}\,. \tag{5.54}$$

Example 5.10. The plots below shows how option values $C(0)$ and $P(0)$ vary with volatility σ and asset prices S when time to expiry is $T = 0.5$, and how their vega $\nu_C = \nu_P$ varies with asset price S and time to expiry T when volatility $\sigma = 0.5$. Other parameters chosen are strike price $K = 20$ and interest rate $r = 0.05$. Discuss the different behaviour of vega over S and T.







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□

5.3.6 The rho of an option

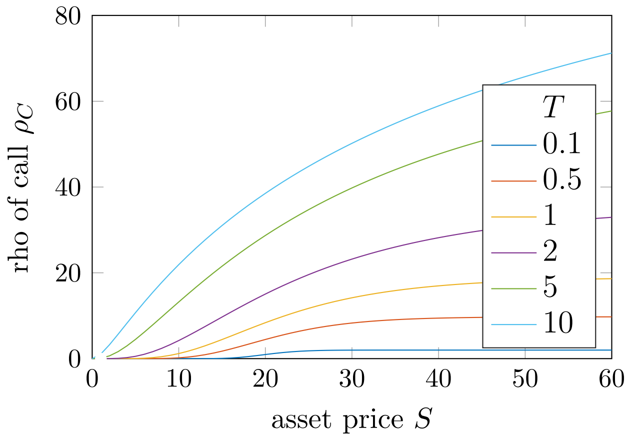
The rho ρ of an option is the rate of change of the option price with respect to interest rate r. For calls and puts, respectively,

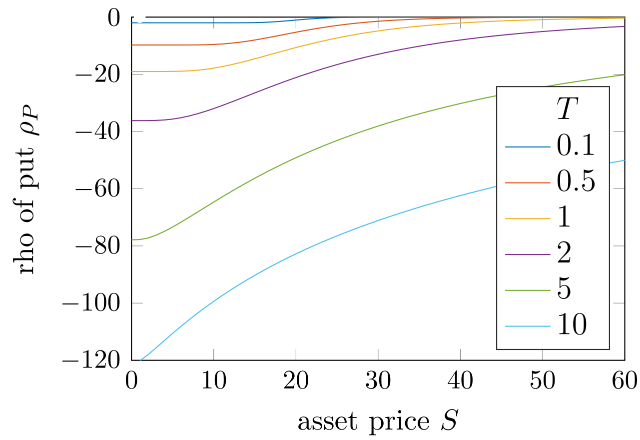
$$\rho_C = \frac{\partial C}{\partial r}\,, \quad \rho_P = \frac{\partial P}{\partial r}\,. \tag{5.55}$$

In the Black–Scholes model

$$\begin{aligned} \rho_C &= KTe^{-rT}\mathcal{N}(d_2)\,, \\ \rho_P &= -KTe^{-rT}\mathcal{N}(-d_2)\,. \end{aligned} \tag{5.56}$$

Example 5.11. The two plots below show how ρ_C and ρ_P vary with asset price S and time to expiry T. Parameters chosen are strike price $K = 20$, interest rate $r = 0.05$ and volatility $\sigma = 0.5$. Discuss the different behaviour over S and T.





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Appendix A Spreadsheet formulas

The functions and operations in this appendix can be used in Excel, LibreOffice or OpenOffice.

When using Windows, use F4 to toggle between \$ and no \$ about a cell number. To select a non-rectangular area, hold the control key when selecting cells.

When using a Mac, use command-T to toggle between \$ and no \$ about a cell number. To select a non-rectangular area, hold the command key when selecting cells.

The arithmetic operations addition, subtraction, multiplication and division are applied using `+`, `-`, `*` and `/`.

Powers such as x^y are applied using `x^y`.

Many spreadsheet functions are self-explanatory, such as `exp(x)` and `ln(x)` for the natural exponential and natural logarithm. The factorial function $n! = 1 \times 2 \times 3 \times \cdots \times n$ is `fact(n)`. Below we discuss a few more complicated functions.

`max(x,y)`

equals whichever value is higher, `x` or `y`.

For example, `max(1,2)=2` and `max(3,-1)=3`.

The `max` function is used when calculating exercise values of options, as in equations (1.2) and (1.4). For example, for a European call option with strike `K` and asset price at maturity `S(T)`, the exercise value is

`max(0,S(T)-K).`

Similarly, for a European put option the exercise value is

`max(0,K-S(T)).`

The `max` function is also used when calculating values of American options prior to expiry. For example, for an American put option with strike `K` and asset price `S(n,j)` at node `(n,j)`, the value of the option at node `(n,j)` for $n < N$ is

`max(W(n,j),K-S(n,j))`

where `W(n,j)` is calculated from an N-step binomial model, as discussed in Section 3.1.2. Similarly, for an American call option the value is

`max(W(n,j),S(n,j)-K)`

but Section 3.1.1 shows that exercising an American call prior to maturity is usually not optimal and therefore we can replace the above with `W(n,j)`.

`norm.s.dist(x,true)`

equals the cumulative normal distribution at `x`.

For example, `norm.s.dist(0,true)=0.5`.

The `norm.s.dist` function calculates the integral shown in equation (2.36) and is required for the Black–Scholes model, as discussed in Section 2.4.

The function `norm.s.dist(x,false)` calculates the probability of the normal distribution at `x`, but we do not require this in this course.

`if(condition,x,y)`

if `condition` is true, then equals `x`, otherwise equals `y`.

For example, if we have some parameter `z=2`, then

`if(z>1,x,y)=x` but `if(z<1,x,y)=y`.

The `if` function is used to calculate values of barrier options, discussed in Section 3.2. For example, for an up-and-out European option with barrier `B`, strike `K` and underlying asset value `S(n,j)` at node `(n,j)`, the value of the option at node `(n,j)` is

`if(W(n,j)>B,0,W(n,j))`

where `W(n,j)` is calculated from an N-step binomial model.

The condition in the `if` function often uses one of the following: `=` (equals), `<>` (not equal), `<=` (less than or equal to), `>=` (greater than or equal to), `<` (less than) or `>` (greater than).